Analogues of Bosons and Coherent States in Non-Hermitian Quantum Mechanics

Alex Kazachek

Abstract

Coherent states originated in quantum optics as a model for the ray of light emitted by a laser. Their unique mathematical and physical properties have since made them a staple in many areas outside quantum optics as well, from quantum cryptography to geometric quantization. Traditionally, these states are defined using bosonic creation and annihilation operators, which describe introducing or removing bosons, such as photons, from a state. Bosonic operators are derived under the assumption that the Hamiltonian, an equation dictating the energy of a quantum system, is Hermitian. In the emerging field of non-Hermitian quantum mechanics, however, this assumption is dropped and thus there are no such operators. To adapt the notion of coherent states here, the more abstract pseudobosonic operators are used instead. Two sets of states, jointly known as bicoherent states, may then be constructed. Together they exhibit properties similar to coherent states, and under Hermicity the two sets collapse into one set of coherent states. Here, we motivate the definition of pseudobosonic operators and examine the properties of bicoherent states. We also explicitly derive them for shifted bosonic operators.

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1 Introduction

Quantum mechanics is an interesting subject in that it may be approached from two perspectives – that of a mathematician or of a physicist [Lan22]. As a subfield of physics, its theory and results ultimately aim to model and predict real world phenomena. To this end, a number of prominent physicists, such as Dirac, Schrödinger, and Heisenberg, used brilliant heuristics and intuition in guiding their development of its foundations and seminal results. And, though sometimes wanting mathematical rigour, their predicted results continuously agreed with experimental results. A theory with informal backing producing formal results piqued several mathematicians, such as Hilbert and von Neumann. In their efforts to develop a rigorous framework surrounding these heuristics and symbolic expressions, great strides were made in mathematics, particularly in functional analysis and noncommutative geometry.

We will jaunt alongside some of these mathematical strides, namely the theory of unbounded operators and operator-valued integration. Our ultimate destination, bicoherent states, will require this mathematical precision. Bicoherent states generalize coherent states, a set of quantum states whose quantum mechanical description is as close to classical mechanics as possible. These were introduced in a somewhat artificial sense by Schrödinger [Sch26], and later made concrete by Glauber in quantum optics to model certain rays of light [Gla63]. This was soon abstracted to a group-theoretic construction by Perelomov [Per72], and coherent states have held a central position in mathematical physics since. For instance, they are closely related to the theory of geometric quantization [Raw77; Odz92; Hal94], the task of ascribing a quantum mechanical theory to a given classical theory. On the applied side, such states lie at the heart of models for quantum computation [Ral+03], an implementation for present-day state-of-the-art quantum computers [Zho+20]. They appear in quantum key distribution, the task of securely transmitting messages over insecure quantum channels [Hut+95]. Recently, they have also been explored as a way of encoding data in the context of quantum machine learning [SP21].

In each quantum mechanical system, an operator known as the Hamiltonian encapsulates its total energy. In particular, the eigenvalues of the Hamiltonian dictate what energy levels a system is allowed to take. The prototypical Hamiltonian is the one for a quantum harmonic oscillator, with more complicated systems often using it as its starting point. More or less, it is the equation $H = X^2 + P^2$, where X is the position of the oscillating particle and P is its momentum. This equation is intimately related to quantum optics, and thus coherent states, and so we will later discuss it in detail and derive its eigenvalues $2n + 1$ for non-negative integers n.

Of note are that its eigenvalues are real, positive, and discrete, reflecting the physicality of the situation. This should be true for all Hamiltonians H, and to ensure this it is often taken as an assumption that H is Hermitian (self-adjoint). However, Bender and Boettcher experimentally demonstrated the Hamiltonian $H = P^2 + X^2 + iX^3$, though manifestly not Hermitian, still had real, positive, and discrete eigenvalues [BB98]. This observation demonstrated the assumption of Hermicity may be unnecessary, and kickstarted the field of non-Hermitian quantum mechanics. They conjectured the true necessary assumption was instead connected to PT-symmetry – the stability of H under the joint reversal of parity ($X \mapsto -X$ and $P \mapsto -P$) and time ($P \mapsto -P$ and $i \mapsto -i$). Shortly after, Mostafazadeh proved non-Hermitian Hamiltonians with such eigenvalues are pseudo-Hermitian, meaning $H = U^{-1}H^*U$ for a unitary U [Mos02]. PT-symmetry and pseudo-Hermicity were recently linked by Zhang et al., showing the former implies the latter in finite dimensions [ZQX20].

The analogues of coherent states in non-Hermitian quantum systems are bicoherent states. Originally introduced by Trifonov [Tri08], they were then studied by Bagarello [Bag10] outside the setting of a non-Hermitian Hamiltonian. His discussion focused on the creation and annihilation operators, a celebrated pair of operators whose origins are in finding the eigenvalues of the quantum harmonic oscillator. Since photons are a type of particle known as a boson, in the context of coherent states these operators are sometimes known as bosonic operators. The idea is to find a generalization of these operators – pseudobosonic operators – such that they may construct bicoherent states the same way bosonic operators construct coherent states. This is our ultimate destination, and to get there we will first need some math.

2 Some Functional Analysis

At a very high level, functional analysis is the study of linearity. At perhaps an even higher one, quantum mechanics is the study of the probabilities of the outcomes of measurements. Probability is a linear theory, and this means quantum mechanics is too. As a result, functional analysis turns out to serve as an excellent foundation. One of the first foundations, initially developed by Heisenberg [Hei25], then fleshed out with the help of Born and Jordan [BJ25; BHJ26] and independent work by Dirac [Dir25], described measurements by matrices. Mathematical developments now let us view these measurements as operators on a linear space, and employ the full language of functional analysis.

2.1 Preliminary Review and Notation

Let us define the basic objects of functional analysis and establish our notation. A good introductory reference is [Mac09].

Definition 1 (Banach Space)**.** *Let X be a vector space endowed with a norm $\|\cdot\|\colon X \to [0,\infty)$. This* *defines a metric $d\colon X \times X \to [0,\infty)$ through $d(x,y) = \|x-y\|$. If X is complete under d, we say it is a Banach space.*

Definition 2 (Hilbert Space). *Let X be a complex vector space with an inner product $\langle\cdot,\cdot\rangle\colon X\times X \to \mathbb{C}$. This defines a norm by $\|x\| = \sqrt{\langle x,x\rangle}$. If X is Banach under this norm, we call X a Hilbert space.*

In principle there is nothing wrong with a Hilbert space over $\mathbb{R}$, but the theory typically turns out to be more inconvenient and less physically relevant. Also, we will take antilinearity in the second slot of the inner product, meaning for $x, y \in X$ a Hilbert space and $\alpha \in \mathbb{C}$ we have $\langle x, \alpha y\rangle = \bar{\alpha}\langle x,y\rangle$. Let us now introduce a famous inequality and the two spaces we will use most frequently.

Lemma 1 (Cauchy-Schwarz). *Let X be a Hilbert space. Then, for all $x, y \in X$, we have the inequality*

$$|\langle x,y\rangle|^2 \le \langle x,x\rangle\langle y,y\rangle = \|x\|^2\|y\|^2.$$

Proposition 1 (L^p Space). *Let X be a space with measure μ. For $p \in [1,\infty)$ define*

$$L^p(X,\mu) = \left\{ f\colon X \to \mathbb{C} : \|f\| = \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} < \infty \right\}.$$

Then, this is a complex Banach space upon taking equivalence classes of μ-almost everywhere equal functions. Moreover, with $p = 2$ this is a Hilbert space with inner product

$$\langle f,g\rangle = \int_X f\bar{g}\,\mathrm{d}\mu.$$

Proposition 2 (Continuous Function Space). *On a compact metric space (M,d) let $C(M)$ be continuous functions $M \to \mathbb{C}$. This is a complex Banach space when endowed with the norm $\|f\| = \sup_{x\in M}|f(x)|$.*

The true value of a Hilbert space lies in orthogonality and the notion of an orthonormal basis. Such a basis makes computation much easier than in a Banach space. We will also make heavy use of orthogonal complements (and specifically how they interact with density) when working with unbounded operators.

Definition 3 (Orthonormal Basis). *Let X be a Hilbert space. A subset $E \subseteq X$ is said to be an orthonormal set if for all distinct $e, f \in E$ we have $\|e\| = 1$ and $\langle e,f\rangle = 0$. If E is maximal (in the sense of set inclusion) we say it is an orthonormal basis. If X is separable, every orthonormal basis is countable and for an arbitrary $x \in X$ we have*

$$x = \sum_{e\in E}\langle x,e\rangle e.$$

Lemma 2 (Orthogonal Complement). *Let X be a Hilbert space and $S \subseteq X$ a linear subspace. Define the orthogonal complement of S by*

$$S^\perp = \{x \in X : \langle x,s\rangle = 0 \text{ for all } s \in S\}.$$

Then, $S^\perp$ is always closed. Also, $S^\perp = \{0\}$ if and only if S is dense in X.

Let us now introduce bounded operators (and remark on the special case of unitarity), and a useful technical lemma.

Definition 4 (Bounded Linear Operator). *Let $T\colon X \to Y$ be a linear map between Banach spaces. Define the operator norm*

$$\|T\| = \sup_{0 \neq x \in X} \frac{\|Tx\|}{\|x\|}.$$

If $\|T\| < \infty$ then T is continuous and we say it is a bounded linear operator. The set of all such operators is denoted $\mathcal{B}(X, Y)$, and is Banach under the above norm. If $X = Y$, we just write $\mathcal{B}(X)$.

Lemma 3. *Let $T \in \mathcal{B}(X, Y)$ for Banach spaces X, Y. Suppose T is bounded from below, meaning there exists some $c \in \mathbb{R}$ so $\|Tx\| \geq c\|x\|$ for all $x \in X$. Then, T is injective. If T further has dense range, then in fact it is surjective and its set-theoretic inverse T^{-1} belongs to $\mathcal{B}(Y, X)$.*

Definition 5 (Unitary Operators). *Let X, Y be Hilbert spaces and $T \in \mathcal{B}(X, Y)$. Then, if T is surjective and preserves inner products, meaning $\langle x, y\rangle = \langle Ux, Uy\rangle$ for all $x, y \in X$, we say T is unitary.*

The term functional analysis comes from the word functional, which is a linear scalar-valued map on the underlying space. Technically, this is just a special case of a linear operator, since $\mathbb{C}$ and $\mathbb{R}$ are Banach spaces (even if boring ones). But, these turn out to be so useful they receive their own special notation.

Definition 6 (Bounded Linear Functional). *Let $\ell\colon X \to \mathbb{F}$ be a linear map on a Banach space over the field $\mathbb{F}$. If ℓ is continuous, we say it is a bounded linear functional. The set of all such ℓ is denoted $\mathcal{F}(X)$.*

Theorem 1 (Riesz-Fréchet). *Let $\ell \in \mathcal{F}(X)$ for a Hilbert space X. Then, there exists some $z \in X$ so $\ell(x) = \langle x, z\rangle$ for all $x \in X$.*

Note too that $\langle \cdot, z\rangle \in \mathcal{F}(X)$ for a fixed z, so this theorem fully characterizes functionals on a Hilbert space. Frequently, z will be called something along the lines of a reproducing vector.

2.2 Unbounded Operators

We have just defined bounded linear operators above. However, it turns out essentially every important operator in quantum mechanics, though linear, is unbounded (meaning $\|Tx_n\| \to \infty$ for some sequence of unit vectors x_n, so $\|T\| = \infty$). Even worse, the Hellinger–Toeplitz theorem (which we will not discuss) tells us these operators cannot even be defined at every point of our space! To make sense of all this, we need to discuss a more general (and much less well-behaved) class of operators.

Throughout, we will let $\mathcal{H}$ be a generic Hilbert space and T an unbounded operator (which we will define in a moment). The primary references for this section are [RS81] and [Hal13].

Definition 7 (Unbounded Operator). *An unbounded operator on $\mathcal{H}$ is a linear operator $T\colon \operatorname{dom} T \to \mathcal{H}$, where $\operatorname{dom} T$ is a dense linear subspace of $\mathcal{H}$. The set of all such operators will be denoted $\mathcal{L}(\mathcal{H})$.*

Note that bounded operators are, unfortunately, also unbounded operators since $\mathcal{H}$ is of course dense and linear in itself. Sometimes unbounded operators are instead called densely defined or not necessarily bounded, to avoid this clash of terminology. But instead of changing our terminology, we will just leave it at this warning.

Definition 8 (Extension). *For $S, T \in \mathcal{L}(\mathcal{H})$, we say S extends T if $\operatorname{dom} T \subseteq \operatorname{dom} S$ and $S|_{\operatorname{dom} T} \equiv T$.*

Definition 9 (Closed). *An operator $T \in \mathcal{L}(\mathcal{H})$ is closed if its graph*

$$\Gamma(T) = \{(x, Tx) : x \in \operatorname{dom} T\}$$

is a closed subset of $\mathcal{H} \times \mathcal{H}$. We say T is closable it admits a closed extension $\bar{T}$, its closure, which satisfies $\Gamma(\bar{T}) = \overline{\Gamma(T)}$.

Being closable is not at all a trivial condition. It is very much possible for $\overline{\Gamma(T)}$ to not correspond to the graph of any operator. Thankfully, the specific operators we will consider will always be closable.

Definition 10 (Adjoint)**.** *Let $T \in \mathcal{L}(\mathcal{H})$ and define*

$$\operatorname{dom} T^* = \{y \in \mathcal{H} : \textit{there exists } z \in \mathcal{H} \textit{ so } \langle Tx, y\rangle = \langle x, z\rangle \textit{ for all } x \in \operatorname{dom} T\}.$$

On $\operatorname{dom} T^*$ define the function T^* by $T^* y = z$.

It is easy to see that an equivalent way to define the domain $\operatorname{dom} T^*$ are the $y \in \mathcal{H}$ so that $x \mapsto \langle Tx, y\rangle \in \mathcal{F}(\operatorname{dom} T)$. Then, z is given by Riesz-Fréchet.

Lemma 4. *For $T \in \mathcal{L}(\mathcal{H})$ we have $(\operatorname{ran} T)^\perp = \ker T^*$.*

Proof. Let $z \in (\operatorname{ran} T)^\perp$. Then, for any $x \in \operatorname{dom} T$ we have $\langle Tx, z\rangle = 0$. Note though that $\langle x, 0\rangle = 0$, meaning $T^* z = 0$ and so $z \in \ker T^*$. On the other hand, if $x \in \ker T^*$ then $\langle z, T^* x\rangle = 0$ for all $z \in \mathcal{H}$. This is true in particular for all $z \in \operatorname{dom} T$, whence $\langle Tz, x\rangle = 0$ and $x \in (\operatorname{ran} T)^\perp$. □

Proposition 3. *Let $T \in \mathcal{L}(\mathcal{H})$. Then, T^* is closed, and T is closable (in fact $\bar{T} = T^{**}$) if and only if T^* is an unbounded operator (specifically, if $\operatorname{dom} T^*$ is dense).*

Proof. It is straightforward to verify that T^* is linear. For this, let $p, q \in \operatorname{dom} T^*$. We know that for arbitrary $x \in \operatorname{dom} T$ we have $\langle Tx, p\rangle = \langle x, T^* p\rangle$ and $\langle Tx, q\rangle = \langle x, T^* q\rangle$. Bilinearity lets us verify $\langle Tx, p + q\rangle = \langle x, T^* p + T^* q\rangle$, meaning $p + q \in \operatorname{dom} T^*$. This simultaneously verifies that both $\operatorname{dom} T^*$ and T^* are linear. Thus, T^* being unbounded hinges solely on the density of $\operatorname{dom} T^*$.

But first, let us prove T^* is closed. It is easy to prove $\mathcal{H} \times \mathcal{H}$ is a Hilbert space when endowed with the inner product

$$\langle (x, y), (a, b)\rangle = \langle x, a\rangle + \langle y, b\rangle.$$

So, on $\mathcal{H} \times \mathcal{H}$ define $U : (x, y) \mapsto (-y, x)$, which we readily see is unitary. Now, let $(x, y) \in U(\Gamma(T))^\perp$. Then, for all $z \in \operatorname{dom} T$ we have

$$0 = \langle (-y, x), (z, Tz)\rangle = -\langle y, z\rangle + \langle x, Tz\rangle.$$

Thus, $\langle Tz, x\rangle = \langle z, y\rangle$ for all such z, and so $(x, y) \in \Gamma(T^*)$. Given any element of $\Gamma(T^*)$ we may follow these implications backwards immediately, from which we conclude $\Gamma(T^*) = U(\Gamma(T))^\perp$. But, orthogonal complements are always closed by LEMMA 2, and so T^* is indeed closed.

By definition, unitary maps preserve inner products, so $U(E)^\perp = (U(E))^\perp$ for any linear subspace E. In particular, this holds for $E = \Gamma(T^*)$. So,

$$(U(\Gamma(T^*)))^\perp = (U(U(\Gamma(T))^\perp)^\perp = (U^2(\Gamma(T))^\perp)^\perp = \overline{\Gamma(T)}.$$

Now, suppose $\operatorname{dom} T^*$ is dense, so its adjoint T^{**} is well-defined. Replicating the previous paragraph, we conclude $\Gamma(T^{**}) = \overline{\Gamma(T)}$, and so we may take $\bar{T} = T^{**}$. Thus, T is closable.

On the other hand, say $\operatorname{dom} T^*$ is not dense. Then, let $z \in \operatorname{dom} T^*$ and, since it is non-empty by LEMMA 2, we can find $0 \neq x \in (\operatorname{dom} T^*)^\perp$. Then,

$$\langle (x, 0), (z, T^* z)\rangle = \langle\langle x, z\rangle + \langle 0, T^* z\rangle = 0.$$

So, $(0, x) \in U(\Gamma(T^*)^\perp)$. But, we also have $(0, 0) \in U(\Gamma(T^*)^\perp)$. We know from above

$$U(\Gamma(T^*)^\perp) = (U(\Gamma(T^*)))^\perp = \overline{\Gamma(T)},$$

but clearly then $\overline{\Gamma(T)}$ cannot be the graph of any operator (as it would be multivalued). Thus, T is not closable. □

Definition 11 (Symmetric). *We say $T \in \mathcal{L}(\mathcal{H})$ is symmetric if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \operatorname{dom} T$.*

Proposition 4. *If $T \in \mathcal{L}(\mathcal{H})$ is symmetric, then it is closable.*

Proof. Let $x, y \in \operatorname{dom} T$. Then, since $\langle Tx, y\rangle = \langle x, Ty\rangle$, we see $\operatorname{dom} T \subseteq \operatorname{dom} T^*$ (since we may define $T^*y = Ty$). Since we know from PROPOSITION 3 that T^* is closed, we see T has a closed extension by its adjoint. □

Definition 12 (Self-Adjoint). *A symmetric and closed $T \in \mathcal{L}(\mathcal{H})$ is self-adjoint if* $\operatorname{dom} T^* = \operatorname{dom} T$. *If T is not closed but closable, we say T is essentially self-adjoint if $\bar{T}$ is self-adjoint.*

We will discuss what the word essential means in a moment. But first, let us remark on the subtle but very important distinction between self-adjoint and symmetric operators. If T is bounded, then these actually are the same thing. However, we will later develop the spectral theory of bounded operators. And, though we will not do it ourselves, it is possible to extend that theory to unbounded operators too – so long as they are self-adjoint. Simply being symmetric is not enough, hence why such a distinction is made.

Corollary 1. *If $T \in \mathcal{L}(\mathcal{H})$ is symmetric, then $T^* = \bar{T}^*$.*

Proof. By PROPOSITION 4 we know T is closable, and then by PROPOSITION 3 get that $\bar{T} = T^{**}$. Now, the adjoint is always closed by PROPOSITION 3, meaning $T^* = \bar{T^*}$. Reapplying the proposition here tells us $\bar{T^*} = T^{***}$. Linking these three equalities,

$$T^* = \overline{T^*} = T^{***} = \bar{T}^*$$

as claimed. □

Suppose we further knew $\bar{T}$ is self-adjoint. This corollary then says $T^* = \bar{T}$. So, although the adjoint of T is not T itself, we come pretty close – it is the closure $\bar{T}$. And in fact, we rarely actually care about T, and are only concerned about $\bar{T}$! This is because, for all intents and purposes, we are always actually working with the closure, in the following precise sense.

Proposition 5. *If T is essentially self-adjoint, then $\bar{T}$ is the unique self-adjoint extension of T.*

Proof. Let S be a self-adjoint extension of T. By PROPOSITION 3, S is closed, and since by definition $\bar{T}$ is the smallest closed extension, we know $\operatorname{dom} \bar{T} \subseteq \operatorname{dom} S$. Now, by definition of the adjoint, for any $y \in \operatorname{dom} S^*$ we can find $z \in \mathcal{H}$ so for any $x \in \operatorname{dom} S$ we have $\langle x, z\rangle = \langle Sx, y\rangle$. In particular, this is true for all $x \in \operatorname{dom} \bar{T}$, where $Sx = \bar{T}x$, and so $\langle x, z\rangle = \langle \bar{T}x, y\rangle$. But, this exactly means $y \in \operatorname{dom} \bar{T}^*$. Since $\bar{T}$ is self-adjoint by assumption, we have

$$\operatorname{dom} S^* \subseteq \operatorname{dom} \bar{T}^* = \operatorname{dom} \bar{T} \subseteq \operatorname{dom} S,$$

and since $S = S^*$ we in fact have equality everywhere above. Thus, $S = \bar{T}$. □

So, say we have an operator which is essentially self-adjoint on a number of plausible domains (that is, the operator will agree on the intersections of these domains, but maybe some are larger or smaller). The above proposition tells us it does not actually matter which of these domains we choose, since they will all correspond to the same closure. The question now is how to determine if an operator is essentially self-adjoint in the first place.

Theorem 2. *A symmetric operator $T \in \mathcal{L}(\mathcal{H})$ is essentially self-adjoint if and only if both $T^* \pm i\mathbb{1}$ are injective on* $\operatorname{dom} T^*$.

Proof. Suppose first T is essentially self-adjoint, meaning $\bar{T}$ is self-adjoint. Then, $T^* = \bar{T}^* = \bar{T}$ by COROLLARY 1. Suppose we had some $x \in \operatorname{dom} \bar{T}$ so $\bar{T}x = \pm ix$. Then,

$$\mp i\langle x, x\rangle = \langle x, \pm ix\rangle = \langle x, Tx\rangle = \langle Tx, x\rangle = \pm i\langle x, x\rangle.$$

Thus, we must have $x = 0$. Since $\operatorname{dom} \bar{T} = \operatorname{dom} T^*$, this proves $T \pm i\mathbb{1}$ is injective.

Instead now suppose we have the injectivity of $T^* \pm i\mathbb{1}$. Let $z \in \operatorname{ran}(\bar{T} \pm i\mathbb{1})^\perp$. Then, for any $x \in \operatorname{dom} \bar{T}$ we have

$$0 = \langle(\bar{T} \pm i\mathbb{1})x, z\rangle = \langle \bar{T}x, z\rangle - \langle x, \mp iz\rangle.$$

However, this implies $z \in \operatorname{dom} T^*$ since $T^*z = \mp iz$. As $T^* \pm i\mathbb{1}$ is injective, this means $z = 0$, and so we conclude $\operatorname{ran}(\bar{T} \pm i\mathbb{1})$ is dense by LEMMA 2.

In fact, it is even closed. To see this, witness for any $x \in \operatorname{dom} \bar{T}$ we have

$$\|(\bar{T} \pm i\mathbb{1})x\|^2 = \langle(\bar{T} \pm i\mathbb{1})x, (\bar{T} \pm i\mathbb{1})x\rangle = \|\bar{T}x\|^2 + \|x\|^2.$$

Now, take some sequence $x_n \in \operatorname{dom} \bar{T}$ so $(T \pm i\mathbb{1})x_n$ converges, say to z. The above equality tells us x_n converges too (say to x) and Tx_n does as well (say to y). However, since $\Gamma(\bar{T})$ is closed, we know $y = Tx$, and so $z = (\bar{T} \pm i\mathbb{1})x$. That is, $\operatorname{ran}(\bar{T} \pm i\mathbb{1})$ is closed, and therefore is equal to $\mathcal{H}$.

Now, let $y \in \operatorname{dom} T^*$. We know that we can find some $x \in \operatorname{dom} \bar{T}$ so $(\bar{T} \pm i\mathbb{1})x = (T^* \pm i\mathbb{1})y$. Since $\bar{T}$ is symmetric, we know T^* extends $\bar{T}$ always, and thus $(T^* \pm i\mathbb{1})(y - x) = 0$. However, this means $x = y$, and so in particular $y \in \operatorname{dom} \bar{T}$. However, this means $\operatorname{dom} \bar{T} = \operatorname{dom} T^*$, and therefore $\bar{T}$ is self-adjoint. □

2.3 Operator-Valued Measures and Spectral Theory

Spectral theory is an incredibly rich and deep subfield of functional analysis. At its simplest, it is studying the eigenvalues of matrices. We saw previously how this can be related to the energy levels of a Hamiltonian, but the applications extend far beyond quantum mechanics. Spectral theory is used extensively in the study of differential equations and graph theory, for instance.

The classical spectral theorem over $\mathbb{C}^n$ says a self-adjoint matrix can be diagonalized and this diagonal will be its eigenvalues. Both parts of the theorem are difficult to generalize to operators. In some special cases (like for compact operators) the notion of an infinite-dimensional matrix still makes sense, but in general there is no such representation. On top of this, the definition of an eigenvalue is not so clear – operators may fail to be invertible yet still be injective. So, $T - \lambda\mathbb{1}$ might not be invertible but there is no x so $Tx = \lambda x$. Evidently, some entirely new way of understanding this theorem will be required.

We will cover the spectral theory of bounded (self-adjoint) operators only, although the case for unbounded operators is very similar – just much messier. So, let $\mathcal{H}$ be a generic Hilbert space as before, but $T \in \mathcal{B}(\mathcal{H})$ specifically. This section primarily references [BS87], although both [RS81] and [Hal13] are used occasionally.

Definition 13 (Operator-Valued Measure)**.** *Let (X, Ω) be a measurable space. Then, $E\colon \Omega \to \mathcal{B}(\mathcal{H})$ is an operator-valued measure if $E(\varnothing) = 0$ and it is countably additive, meaning for pairwise disjoint $A_i \in \Omega$ we have*

$$E\left(\dot{\bigcup_i} A_i\right) = \sum_i E(A_i).$$

Moreover, with $C, D \in \Omega$ being arbitrary,

- *if $E(C \cap D) = E(C)E(D)$ we say E is spectral;*
- *if $E(X) = \mathbb{1}$ and $E(C)$ is positive we say E is a probability measure;*
- *and if $E(C)$ is self-adjoint we say E is self-adjoint.*

Definition 14 (Projection-Valued Measure)**.** *A projection-valued measure E on a measurable space (X, Ω) is a self-adjoint spectral operator-valued measure.*

Note that for any $C \in \Omega$ we have $E(C) = E(C \cap C) = E(C)^2$, meaning $E(C)$ is a projection since it is idempotent and self-adjoint.

So, we know what an operator-valued measure is. However, we still do not know how we might construct them or what exactly it means to integrate with respect to them (or why they are useful!). The key is that operator-valued measures live in correspondence with ordinary measures, as a consequence of the following theorem.

Theorem 3 (Riesz–Markov–Kakutani)**.** *Let M be a compact metric space and $A \in \mathcal{F}(C(M))$. Then, there exists some complex measure μ on the Borel σ-algebra of M such that*

$$Af = \int_M f \,\mathrm{d}\mu$$

for all $f \in C(M)$.

To construct the functional we are alluding to we first need to recall some facts about (bounded!) operators and their spectra.

Definition 15 (Spectrum)**.** *For an operator $T \in \mathcal{B}(\mathcal{H})$, its spectrum is*

$$\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda\mathbb{1})^{-1} \notin \mathcal{B}(\mathcal{H})\}.$$

Note that the spectrum is not necessarily just eigenvalues. There even exist operators with an uncountable spectrum, and yet no eigenvalues. Though, we will decompose the spectrum into different parts later, and eigenvalues will be one of these.

Proposition 6. *Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then, $\sigma(T) \subseteq [0, \infty)$, and in particular it is closed and bounded above by $\|T\|$.*

Proof. We will first show $\sigma(T)$ is bounded. Let $x \in \mathcal{H}$ and let $\lambda = a + bi \in \mathbb{C}$. Note that

$$\|(T - \lambda\mathbb{1})x\|^2 = \|(T - a\mathbb{1})x\|^2 + b^2\|x\|^2 \geq b^2\|x\|^2.$$

Suppose $b \neq 0$, meaning $T - \lambda\mathbb{1}$ is bounded below and thus injective. The same is true for $T - \bar{\lambda}\mathbb{1}$. Being self-adjoint and injective means

$$(\operatorname{ran} T - \lambda\mathbb{1})^{\perp} = \left(\operatorname{ran}(T - \bar{\lambda}\mathbb{1})^*\right)^{\perp} = \ker T - \bar{\lambda}\mathbb{1} = \{0\}$$

due to LEMMA 4. But, this tells us $\operatorname{ran} T - \lambda\mathbb{1}$ is dense. And, once more from being bounded below, LEMMA 3 tells us it is invertible. So, $\lambda \notin \sigma(T)$, meaning $\sigma(T) \subseteq \mathbb{R}$. Non-negativity follows immediately from symmetry.

Now, suppose $\lambda \in \sigma(T)$ but

$$\lambda \notin \overline{\{\langle Tx, x\rangle : \|x\| = 1\}}.$$

This means we can find some $\varepsilon > 0$ such that for all $x \neq 0$ we have

$$\varepsilon \leq \left|\frac{\langle Tx, x\rangle}{\|x\|^2} - \lambda\right| \implies \varepsilon\|x\|^2 \leq \|\langle (T - \lambda\mathbb{1})x, x\rangle\|.$$

An application of Cauchy-Schwarz shows $\varepsilon\|x\| \leq \|(T - \lambda\mathbb{1})x\|$. Again, $T - \lambda\mathbb{1}$ is bounded below, so using the same reasoning as above (noting $\lambda = \bar{\lambda}$) we derive a contradiction. For unit vectors x we note

$$|\langle Ax, x\rangle|^2 \leq \|Ax\|^2 \leq \|A\|^2,$$

giving us the claimed bound. The proof of closure is omitted. □

Theorem 4 (Continuous Functional Calculus)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then, there is a unique map $\Phi_T : C(\sigma(T)) \to \mathcal{B}(\mathcal{H})$ such that for all $f, g \in C(\sigma(T))$ and $z \in \mathbb{C}$ we have*

- $\Phi_T(fg) = \Phi_T(f)\Phi_T(g)$
- $\Phi_T(zf) = z\Phi_T(f)$
- $\Phi_T(\bar{f}) = \Phi_T(f)^*$
- $\Phi_T(1) = \mathbb{1}$
- $\Phi_T(\mathrm{Id}) = T$

We also have the spectral mapping

$$\sigma(\Phi_T(f)) = \{f(\lambda) : \lambda \in \sigma(T)\}.$$

Lastly, $\|\Phi_T(f)\| = \|f\|$ and if $f_n \to f$ then $\Phi_T(f_n) \to \Phi_T(f)$.

The continuous functional calculus has a great deal of application. It makes rigorous things like square-roots or absolute values of operators, so we often suggestively write $f(T) = \Phi_T(f)$. Our use for it will be to define operator-valued integration, through an alchemy of all our previous results.

Fix $x, y \in \mathcal{H}$ and a self-adjoint $T \in \mathcal{B}(\mathcal{H})$. For $f \in C(\sigma(T))$ define the functional $L^T_{xy} f = \langle f(T)x, y\rangle$ on $C(\sigma(T))$, which we readily see is continuous (due to the convergence of the continuous functional calculus). Since we know $\sigma(T)$ is compact from PROPOSITION 6, we may then apply Riesz–Markov–Kakutani to obtain a measure μ^T_{xy} so

$$L^T_{xy} f = \int_{\sigma(T)} f \, \mathrm{d}\mu^T_{xy}.$$

In fact, with this measure fixed, we can then use L^T_{xy} to define $f(T)$ for a much larger class of functions. This observation comprises the spectral theorem.

Definition 16 (Bounded Borel Functions)**.** *Let M be a metric space. Then, $\mathscr{B}_0(M)$ is the space of bounded Borel functions $M \to \mathbb{C}$.*

Borel means for every $C \in \Omega$, where Ω is the Borel σ-algebra of $\mathbb{C}$, that $f^{-1}(C)$ is open for all $f \in \mathscr{B}_0(M)$.

Bounded means $\|f\| = \sup_{x \in M} |f(x)| < \infty$ for all $f \in \mathscr{B}_0(M)$.

Theorem 5 (Spectral Theorem)**.** *Let Ω be the Borel σ-algebra on $\mathbb{C}$. For a fixed self-adjoint $T \in \mathcal{B}(\mathcal{H})$, define $E_T : \Omega \to \mathcal{L}(\mathcal{H})$ by $E_T(C) = \chi_C(T)$, where χ_C is the characteristic function on $C \in \Omega$. Then, E_T is a projection-valued probability measure, called the resolution of the identity of T.*

Moreover, for any $f \in \mathscr{B}_0(\sigma(T))$ we define the operator

$$I_T(f) = \int_{\sigma(T)} f \, \mathrm{d}E_T$$

which uniquely satisfies

$$\langle I_T(f)x, y\rangle = \int_{\sigma(T)} f \, \mathrm{d}\mu_{xy}^T$$

for all $x, y \in \mathcal{H}$. This operator is bounded, and in particular $\|I_T(f)\| \leq \|f\|$. Moreover, for all $f, g \in \mathscr{B}_0(\mathbb{C})$ we have

- $I_T(f + g) = I_T(f) + I_T(g)$
- $I_T(fg) = I_T(f)I_T(g)$
- $I_T(f)^* = I_T(\bar{f})$

Note that when $f \equiv 1$ we have

$$I_T(f) = \int_{\sigma(T)} \mathrm{d}E_T,$$

which for all $x, y \in \mathcal{H}$ satisfies

$$\langle I_T(f)x, y\rangle = \int_{\sigma(T)} \mathrm{d}\mu_{xy}^T = L_{xy}^T(1) = \langle x, y\rangle,$$

and so $I_T(f) = \mathbb{1}$. This is why we call E_T a resolution of the identity. Similarly, when $f = \mathrm{Id}$ we have the useful identity

$$I_T(f) = \int_{\sigma(T)} \lambda \, \mathrm{d}E_T(\lambda) = T.$$

This is reminiscent of a diagonalization of T, and why it is called the spectral theorem.

These two observations also allude to the fact that, essentially by construction, the spectral theorem extends the continuous functional calculus (in the sense that for continuous f we have $\Phi_T(f) = I_T(f)$). For this reason it is sometimes known as the Borel functional calculus.

Of course, to be worthy of the title of a spectral theorem it should give the ordinary spectral theorem as a special case. Indeed, it does, as we will now show.

Definition 17 (Point and Continuous Spectrum)**.** *For $T \in \mathcal{L}(\mathcal{H})$, define its point spectrum*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue}\}$$

and continuous spectrum

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : T - \lambda \mathbb{1} \text{ has dense but not surjective range}\} \setminus \sigma_p(T).$$

Proposition 7. *Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then, $\sigma(T) = \sigma_p(T) \cup \sigma_c(T)$.*

Proof. It is obvious that $\sigma_p(T) \cup \sigma_c(T) \subseteq \sigma(T)$. So, suppose $\lambda \in \sigma(T)$ but $\lambda \notin \sigma_p(T) \cup \sigma_c(T)$. Since λ is the spectrum, but cannot be an eigenvalue and $T - \lambda\mathbb{1}$ cannot have dense range, it must be that $T - \lambda\mathbb{1}$ is not invertible because it fails to have dense range. So, we can find some $0 \neq y \in (\operatorname{ran} T - \lambda\mathbb{1})^{\perp}$. However, this means for all $x \in \mathcal{H}$ we have

$$0 = \langle (T - \lambda\mathbb{1})x, y\rangle = \langle Tx, y\rangle - \lambda\langle x, y\rangle.$$

Since T is self-adjoint, we have

$$0 = \langle x, Ty\rangle + \langle x, -\bar{\lambda}y\rangle = \langle x, (T - \bar{\lambda}\mathbb{1})y\rangle.$$

From PROPOSITION 2 we know $\lambda \in \mathbb{R}$, and so the arbitrariness of x in fact means $(T - \lambda\mathbb{1})y = 0$. But then, $y \in \sigma_p(T)$, which gives contradiction. □

Note that we defined the point and continuous spectrum for unbounded operators, and then showed for bounded self-adjoint ones that this is all of the spectrum. This was not a mistake. For unbounded T there may be points $\lambda \in \sigma(T)$ which does not belong to the point or continuous spectrum – meaning $T - \lambda\mathbb{1}$ will not even have dense range. Such λ are said to belong to the residual spectrum. It is particularly ill-behaved, and it is hard to formulate any reasonable or consistent notion of a spectral theory in its presence. The absence of a residual spectrum is why (essentially!) self-adjoint operators are so favoured.

Proposition 8. *Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint and let E_T be its resolution of the identity. Say $\lambda \in \sigma(T)$. Then, $E_T((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0$ for all $\varepsilon > 0$.*

Proof. Suppose this were not the case, finding $\varepsilon > 0$ so $E_T((\lambda - \varepsilon, \lambda + \varepsilon)) = 0$. This means

$$0 = \langle E_T((\lambda - \varepsilon, \lambda + \varepsilon))x, x\rangle = \int_{(\lambda-\varepsilon,\lambda+\varepsilon)} \mathrm{d}\mu_{xx}^T$$

for all $x \in \mathcal{H}$. Equivalently, $\mu_{xx}^T \equiv 0$ over this interval. Define now

$$f(t) = \begin{cases} \frac{1}{t-\lambda} & |t - \lambda| \geq \varepsilon \\ 0 & \text{elsewhere.} \end{cases}$$

By construction, $f(t)(t - \lambda) \equiv 1$ μ_{xx}^T-almost everywhere. Therefore,

$$\int_{\sigma(T)} f(t)(t - \lambda)\,\mathrm{d}\mu_{xx}^T(t) = \int_{\sigma(T)} \mathrm{d}\mu_{xx}^T(t) = \langle x, x\rangle.$$

This holding for all x implies

$$\int_{\sigma(T)} f(t)(t - \lambda)\,\mathrm{d}E_T(t) = \mathbb{1}.$$

Operator-valued integrals are multiplicative, however, and so

$$\mathbb{1} = I_T(f(t)(t-\lambda)) = I_T(f(t))I_T((t-\lambda)) = I_T((t-\lambda))I_T(f(t)).$$

Note now that $I_T((t-\lambda)) = T - \lambda\mathbb{1}$, meaning that in fact $I_T(f)$ is its inverse. This contradicts the assumption that $\lambda \in \sigma(T)$, meaning no such ε can exist. □

Proposition 9. *Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint and let E_T be its resolution of the identity. If $\lambda \in \sigma_p(T)$, then $E_T(\{\lambda\} = P_\lambda$, where P_λ is the projection onto the corresponding eigenspace.*

Proof. Let $f \in \mathscr{B}_0(\sigma(T))$. Observe now that that for all $0 \neq x \in \mathcal{H}$ we have

$$\langle I_T(f)x, I_T(f)x\rangle = \langle I_T(f), I_T(\bar{f})^* x\rangle = \langle I_T(\bar{f})I_T(f)x, x\rangle = \langle I_T(|f|^2)x, x\rangle.$$

Thus, we know

$$\|I_T(f)x\|^2 = \int_{\sigma(T)} |f|^2 \,\mathrm{d}\mu_{xx}^T.$$

Note this implies μ_{xx}^T is not the zero measure, since taking $f \equiv 1$ gives

$$\|x\|^2 = \int_{\sigma(T)} \mathrm{d}\mu_{xx}^T = \mu_{xx}^T(\sigma(T)).$$

Now, let $\lambda \in \sigma_p(T)$. Then,

$$\|(T - \lambda\mathbb{1})x\|^2 = \int_{\sigma(T)} (t-\lambda)^2 \,\mathrm{d}\mu_{xx}^T.$$

Since $(t-\lambda)^2 > 0$ only when $t \neq \lambda$, we see $\|(T-\lambda\mathbb{1})x\| = 0$ if and only if $\mu_{xx}^T(\sigma(T) \setminus \{\lambda\}) = 0$. As μ_{xx}^T is non-zero, this implies $\mu_{xx}^T(\{\lambda\}) > 0$. Of course, $\|(T-\lambda\mathbb{1})x\| = 0$ precisely means that x is an eigenvector with eigenvalue λ.

Now, by definition we have $E_T(\{\lambda\}) = \chi_{\{\lambda\}}(T)$. Therefore,

$$\langle E_T(\{\lambda\})x, x\rangle = \int_{\sigma(T)} \chi_{\{\lambda\}} \,\mathrm{d}\mu_{xx}^T = \mu_{xx}^T(\{\lambda\}).$$

We thus know

$$\langle(\mathbb{1} - E_T(\{\lambda\}))x, x\rangle = \|x\|^2 - \langle E_T(\{\lambda\})x, x\rangle = \mu_{xx}^T(\sigma(T) \setminus \{\lambda\}).$$

Witness that $E_T(\{\lambda\})x = x$ if and only if $\mu_{xx}^T(\sigma(T) \setminus \{\lambda\}) = 0$, and so $E_T(\{\lambda\}) = P_\lambda$, as desired. □

These two propositions show us why exactly this is called the spectral theorem. For any self-adjoint $T \in \mathcal{B}(\mathcal{H})$, we have

$$T = \int_{\sigma(T)} \lambda \,\mathrm{d}E_T(\lambda) = \sum_{\lambda \in \sigma_p(T)} \lambda P_\lambda + \int_{\sigma_c(T)} \lambda \,\mathrm{d}E_T(\lambda).$$

If the continuous spectrum is empty, such as when T is compact (or even better, has finite rank), this reduces to the classical spectral theorem.

Later, we will have to speak of resolving the identity in a weak sense. To understand the motivation, first witness that we may speak a resolution of the identity in the absence of any particular operator. This is just some projection-valued probability measure E such that

$$\mathbb{1} = \int_{\mathbb{R}} \mathrm{d}E.$$

Equivalently,

$$\langle x, y\rangle = \left\langle \left(\int_{\mathbb{R}} \mathrm{d}E\right) x, y \right\rangle$$

for all $x, y \in \mathcal{H}$.

Now, let $\alpha \in \mathbb{R}$ be an eigenvalue, in the sense that $E(\{\alpha\})$ is, without loss of generality, a rank-one projection onto some vector $z_\alpha \in \mathcal{H}$. Then,

$$\mu_{xy}(\{\alpha\}) = \langle E(\{\alpha\})x, y\rangle = \langle \langle x, z_\alpha\rangle z_\alpha, y\rangle = \langle x, z_\alpha\rangle\langle z_\alpha, y\rangle.$$

The above is just a real-valued function of α. So, under suitable conditions we may integrate it and obtain

$$\int_{\mathbb{R}} \mu_{xy}(\{\alpha\})\,\mathrm{d}\alpha = \int_{\mathbb{R}} \langle x, z_\alpha\rangle\langle z_\alpha, y\rangle\,\mathrm{d}\alpha.$$

Heuristically, we would expect something along the lines of

$$\int_{\mathbb{R}} \mu_{xy}(\{\alpha\})\,\mathrm{d}\alpha = \int_{\mathbb{R}} \mu_{xy} = \langle x, y\rangle.$$

If we have countably-many α we wish to integrate over, this is true in the sense that we may always construct a self-adjoint operator whose resolution will give us such an E. We just pick our favourite orthonormal basis and consider the operator which scales it by the eigenvalues α. Then, its spectrum will consist entirely of points, so all the integrals reduce to summations, and it is easy to verify this holds.

The issue is that self-adjoint operators only have countably many eigenvectors, so speaking of uncountably-many eigenvalues (i.e. indexing α over all of $\mathbb{R}$) simply makes no sense in the context of resolving the identity. This is neglecting to even mention that the fact that we might want the α to be complex. This is where we use the heuristic, defining a resolution in a weak sense.

Definition 18 (Weak Resolution of the Identity). *Let $\{z_\alpha\}_{\alpha\in\mathbb{C}}$ be a collection of vectors in $\mathcal{H}$. Then, if for all $x, y \in \mathcal{H}$ we have*

$$\langle x, y\rangle = \int_{\mathbb{C}} \langle x, z_\alpha\rangle\langle z_\alpha, y\rangle\,\mathrm{d}\alpha,$$

we say the collection forms a weak resolution of the identity. Here, $\mathrm{d}\alpha$ is the 2-dimensional Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$.

3 The Operators of Quantum Mechanics

With the mathematical theory covered, we are now ready to discuss the quantum theory. The main setting will be a separable infinite-dimensional Hilbert space, of which $L^2(\mathbb{R})$ is an excellent example. Of course, we might naturally ask – why? In a pragmatic sense, it is due to convenience. Having an orthonormal basis makes computation simple, and even more so when it is countable. Physical measurements are also very naturally modelled as self-adjoint operators, interpreting their eigenvalues as probabilities of different outcomes.

However, there is some deeper inevitability. The Haag-Kastler axioms [HK64] are a foundation for quantum mechanics which, at a high level, aims to describe how measurements ought to interact with each other. The structures used in this foundation are C^*-algebras, and the Gelfand-Naimark theorem says locally these are spaces of bounded operators over some Hilbert space. Fortunately, we will not need to delve into these foundations. For us, pragmatism is sufficient motivation.

3.1 Position and Momentum

The space $L^2(\mathbb{R})$ naturally describes the quantum mechanics of the simplest system: a single particle in a one-dimensional space.

Definition 19 (Wave Function)**.** *The wave function of a particle is a map $\psi\colon \mathbb{R} \to \mathbb{C}$ such that*

$$\int_{\mathbb{R}} |\psi(x)|^2 \,\mathrm{d}x = 1.$$

Equivalently, ψ is a unit function in $L^2(\mathbb{R})$. For this reason, we may actually take the equivalence classes of rays in $L^2(\mathbb{R})$ as uniquely representing a particle. Thus, it is sometimes said quantum mechanics is done in a projective Hilbert space.

This looks an awful lot like a probability density, and that is because it is. At the atomic scale, where quantum mechanics becomes applicable, we can no longer describe particles as point-masses. Instead, it is better of think of them as something like probabilistic point-masses, existing in a superposition of several possible positions. In $L^2(\mathbb{R})$, each position of the particle (whose wave function is) ψ is somewhere along the real line. The probability distribution describing its position is $|\psi(x)|^2$ for $x \in \mathbb{R}$. Until we measure its position, we cannot speak of it any definite sense – the best we can do is a probability.

With this, we move on to the two fundamental properties of a particle: its position and momentum. The measurement of these properties is respectively described by the operators

$$X\colon \psi(x) \mapsto x\psi(x) \text{ and } P\colon \psi(x) \mapsto -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\psi(x).$$

Here, $\hbar \approx 1.055 \cdot 10^{-34}\,\mathrm{J} \cdot \mathrm{s}$ is the (reduced) Planck constant and is something which must be experimentally determined. Though it has physical meaning, this is irrelevant to us since we will later change units so $\hbar = 1$.

Notice that, although we called them measurements, neither X nor P return a number. Instead, they return a new wave function. We understand this again in the sense of a superposition. We may apply a measurement, though choose to not observe its value. Until we do, the best we can do is describe the outcomes of the measurement as probabilities associated with the original wave function.

From our not-so-subtle foreshadowing we may suspect these operators are unbounded, and indeed this is true. For example, define $\psi_n = \chi_{[n,n+1]}$. Then, $\|\psi_n\| = 1$ but clearly $\|X\psi_n\| \to \infty$ as $n \to \infty$. So, let us formally define these operators and their domains now. We will start with the position operator, noting the proof actually extends to the general case for any multiplication operator in any L^p space.

Proposition 10 (Position Operator)**.** *Define the set*

$$D = \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} x^2 f(x)\,\mathrm{d}x < \infty \right\}.$$

Then, D is dense and the operator $X\colon D \to L^2(\mathbb{R})$ defined by $f(x) \mapsto xf(x)$ is essentially self-adjoint.

Proof. We will first show D is dense in $L^2(\mathbb{R})$. To see this, suppose $f \in (\operatorname{dom} X)^\perp$. Note that $g(x) = f(x)/(x^2+1) \in L^2(\mathbb{R})$ by monotonicty, and so $\langle f, g\rangle = 0$. So,

$$0 = \int_{\mathbb{R}} \frac{|f(x)|^2}{x^2+1}\,\mathrm{d}x.$$

Since $1/(x^2+1) > 0$ and $|f| > 0$, we must have that $f = 0$ for this integral to hold. So, $(\operatorname{dom} X)^\perp = \{0\}$, giving density by LEMMA 2.

Note that a symmetric operator A is injective if and only if it has dense range, since $\ker A = (\operatorname{ran} A)^\perp$ by LEMMA 4. So, to invoke THEOREM 2, we will show that X has dense range (since its symmetry is obvious). However, this is straightforward to show. By the Pythagorean theorem, $|x \pm i|^2 = |x|^2 + 1$, and so by monotonicity we have $f(x)/(x \pm i) \in L^2(\mathbb{R})$ for any $f \in L^2(\mathbb{R})$. However, this means

$$(X \pm i\mathbb{1})\left(\frac{f(x)}{x \pm i}\right) = \frac{xf(x)}{x \pm i} + \frac{\pm i f(x)}{x \pm i} = f(x).$$

Thus, $\operatorname{ran}(X \pm i\mathbb{1}) = L^2(\mathbb{R})$, and so we are done. □

Up next we have the position operator. It is a little harder to define its domain, since differentiating a function in L^p usually makes no sense (as the functions are actually equivalence classes, hence not defined pointwise). Moreover, there are a number of plausible domains. What differentiability condition do we want? Once, twice, continuously, smoothly? Will we consider ordinary differentiation, or should we admit weak derivatives? Fortunately, this is a non-issue.

Proposition 11 (Test Functions)**.** *Let $k \geq 0$ be an integer or $k = \infty$. We say a function is k-times continuously differentiable if $f^{(k)}$ is continuous for integer k, and if f is smooth for $k = \infty$.*

Define $C_c^k(\mathbb{R})$ to be compactly-supported k-times continuously differentiable functions $\mathbb{R} \to \mathbb{C}$. When $k = \infty$, we call these test functions. Then, $C_c^k(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ for all k.

Lemma 5. *Let f be locally absolutely-integrable on $\mathbb{R}$, meaning $f|_K \in L^1(\mathbb{R})$ for all compact $K \subseteq \mathbb{R}$. Then, if*

$$\int_{\mathbb{R}} f\varphi'\,\mathrm{d}x = 0$$

for all $\varphi \in C_c^1(\mathbb{R})$, f is almost everywhere constant.

Proof. Fix any $\psi \in C_c^\infty(\mathbb{R})$ so $\int_{\mathbb{R}} \psi\,\mathrm{d}x = 1$. For an arbitrary $\omega \in C_c^\infty(\mathbb{R})$ define

$$h(y) = \omega(y) - \psi(y)\int_{\mathbb{R}} \omega\,\mathrm{d}x.$$

It is clear h is continuous with compact support. Moreover,

$$\int_{\mathbb{R}} h\,\mathrm{d}x = \int_{\mathbb{R}} \left(\omega - \psi \int_{\mathbb{R}} \omega\,\mathrm{d}x\right)\mathrm{d}x = \int_{\mathbb{R}} \omega\,\mathrm{d}x \left(1 - \int_{\mathbb{R}} \psi\,\mathrm{d}x\right) = 0.$$

It is straightforward to verify that the primitive of h (given by the fundamental theorem of calculus) belongs to $C_c^1(\mathbb{R})$.

So, with φ being the primitive of h, we have

$$0 = \int_{\mathbb{R}} f\varphi' \, \mathrm{d}x = \int_{\mathbb{R}} f\left(\omega - \psi \int_{\mathbb{R}} \omega \, \mathrm{d}x\right)$$

by hypothesis. We rearrange this to find

$$\int_{\mathbb{R}} \left(f - \int_{\mathbb{R}} f\psi \, \mathrm{d}x\right)\omega \, \mathrm{d}x = 0.$$

Since ω was arbitrary, we may take it to be non-zero, which forces $f \equiv \int_{\mathbb{R}} f\psi \, \mathrm{d}x$ almost everywhere. This is our desired constant. □

Now, due to PROPOSITION 5, all we need to do is show P is essentially self-adjoint on $C_c^\infty(\mathbb{R})$, where P is just the normal derivative. If any other version of the momentum operator agrees with ours on this domain, we know they will actually correspond to the same closure.

Proposition 12 (Momentum Operator). *Define P to be the linear operator defined on the dense domain $C_c^\infty(\mathbb{R}) \subseteq L^2(\mathbb{R})$ by*

$$P\colon \psi(t) \mapsto -i\hbar \left.\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|_t.$$

Then, P is essentially self-adjoint.

Proof. Let $\varphi, \psi \in C_c^\infty(\mathbb{R})$. Find $c > 0$ such that $[-c, c] \supseteq \operatorname{supp}\varphi \cap \operatorname{supp}\psi$. Then,

$$\langle P\varphi, \psi\rangle = -i\hbar \int_{-c}^{c} \left(\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_t\right)\overline{\psi(t)} \, \mathrm{d}t = -i\hbar\varphi(t)\overline{\psi(t)}\Big|_{-c}^{c} + i\hbar \int_{-c}^{c} \varphi(t)\overline{\left.\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|_t} \, \mathrm{d}t$$

by integration by parts. However, as we chose c to lie outside the supports of φ and ψ, the product term vanishes. So,

$$\langle P\varphi, \psi\rangle = 0 + \int_{-c}^{c} \varphi(t)\overline{\left(-i\hbar \left.\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|_t\right)} \, \mathrm{d}t = \langle \varphi, P\psi\rangle.$$

Thus, we know P is symmetric.

To get the rest, we will use THEOREM 2. Now, suppose $\psi \in \ker(P^* + i\mathbb{1})$. In particular, $\psi \in \operatorname{dom}(P^*)$, and so for any $f \in C_c^\infty(\mathbb{R})$ we have

$$\langle P^*\psi, f\rangle = \langle \psi, Pf\rangle = -i\hbar \int_{\mathbb{R}} \psi(t)\overline{\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_t} \, \mathrm{d}t.$$

Now, note that

$$-i\langle \psi, f\rangle = -i \int_{\mathbb{R}} \psi(t)\overline{f(t)} \, \mathrm{d}t$$

and since $P^*\psi = -i\psi$, we therefore have

$$0 = \int_{\mathbb{R}} \left(\hbar\overline{\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_t} - \overline{f(t)}\right)\psi(t) \, \mathrm{d}t.$$

In particular, $f(t)e^{-t/\hbar} \in C_c^\infty(\mathbb{R})$, so an application of the product rule yields

$$0 = \int_{\mathbb{R}} \Big(\hbar e^{-t/\hbar}\overline{f'(t)} + e^{-t/\hbar}\overline{f(t)} - e^{-t/\hbar}\overline{f(t)}\Big)\psi(t)\,\mathrm{d}t = \hbar \int_{\mathbb{R}} \overline{f'(t)}e^{-t/\hbar}\psi(t)\,\mathrm{d}t.$$

But, this is true for all test functions f and so $e^{-t/\hbar}\psi(t)$ is constant almost everywhere by LEMMA 5 (note $C_c^\infty(\mathbb{R}) \subseteq L^1(\mathbb{R})$). Thus, $\psi(t) = ce^{t/\hbar}$ for some $c \in \mathbb{C}$. Since we must also have $\psi \in L^2(\mathbb{R})$, necessarily then $c = 0$, and so $\psi = 0$.

Thus, $P^* + i\mathbb{1}$ is injective on $\mathrm{dom}(P^*)$. The result for $P^* - i\mathbb{1}$ holds identically, and so we conclude P is essentially self-adjoint. □

The multiplication (i.e. composition) and sum of X and P is easily defined, and is exactly what we expect. But, we must be aware that this only works out nicely because $C_c^\infty(\mathbb{R}) \subseteq \mathrm{dom}\, X$. In general, the multiplication and sum of unbounded operators is rather unpleasant to think about. For example, even if the common domain of two operators is dense, and even if both are self-adjoint, their sum may fail to even be essentially self-adjoint.

3.2 Creation and Annihilation

Let us now discuss the renowned quantum harmonic oscillator. Its Hamiltonian is

$$H = \frac{1}{2m}\Big(P^2 + (m\omega X)^2\Big),$$

where m is the mass of the particle and ω is the frequency of its oscillation. Let us introduce an operator and its adjoint, respectively annihilation and creation, and briefly ignore their domains:

$$a = \frac{m\omega X + iP}{\sqrt{2\hbar m\omega}} \text{ and } a^* = \frac{m\omega X - iP}{\sqrt{2\hbar m\omega}}.$$

It is easy to formally verify that

$$H = \hbar\omega\left(a^*a + \frac{1}{2}\mathbb{1}\right),$$

so to understand the Hamiltonian it (probably) suffices to understand a^*a.

Proposition 13 (Canonical Commutation Relation)**.** *Let X be a Banach space and $T, S \in \mathcal{L}(X)$. We define their commutator $[T, S] = TS - ST$ on the domain $\mathrm{dom}(TS) \cap \mathrm{dom}(ST)$.*

The creation and annihilation operator satisfy the canonical commutation relation $[a, a^*] = \mathbb{1}$, in the sense that $[a, a^*]f = f$ for all $f \in \mathrm{dom}[a, a^*]$.

This relationship is so important that it is actually used as the definition of creation and annihilation operators in abstract spaces, as we will see later. Now, let us answer the burning question – what exactly is the domain of a^*a? For simplicity, we will now work in units where $1 = \hbar$ and assume $\omega = m = 1$.

Proposition 14 (Schwartz Space)**.** *The Schwartz space, or space of rapidly decreasing functions, is defined by*

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} \left| x^m f^{(n)}(x) \right| < \infty \text{ for all } m, n \in \mathbb{N} \right\}.$$

It is dense in $L^2(\mathbb{R})$.

Theorem 6. *The position and momentum operators*

$$X\colon f(t) \mapsto t f(t) \text{ and } \psi(t) \mapsto -i \frac{\mathrm{d}\psi}{\mathrm{d}x}\bigg|_t$$

defined on $\mathcal{S}(\mathbb{R})$ *are essentially self-adjoint. Moreover, they both fix the Schwartz space, meaning* $\operatorname{ran} X$ *and* $\operatorname{ran} P$ *are both subsets of* $\mathcal{S}(\mathbb{R})$.

The usefulness of this new version of position and momentum is immediate. We now know the commutator $[a^*, a]$ and $a^*a = N$ both have the same dense domain – Schwartz functions. We write N for number, and we will see the reason why later.

The reason we previously defined the momentum operator on test functions in THEOREM 10 is because they are relatively straightforward to work with. Unfortunately, it is considerably more difficult to work with the momentum operator on Schwartz functions, requiring the theory of tempered distributions. This is why we do not prove this new version. But, because of PROPOSITION 5, we know both of these definitions correspond to the same closure.

Theorem 7. *Consider the Hermite polynomials, recursively defined by* $H_0(x) = 1$ *and*

$$H_{n+1}(x) = \frac{1}{\sqrt{2}}\left(2xH_n(x) - \frac{\mathrm{d}}{\mathrm{d}x}H_n(x)\right).$$

With $\psi_0(x) = \sqrt{\pi}\exp\left(-x^2/2\right)$, *for integers* $n \geq 1$ *define* $\psi_n(x) = H_n(x)\psi_0(x)$. *Then,* ψ_n *form an orthonormal basis of* $L^2(\mathbb{R})$ *(once normalized) and are called number states.*

Proposition 15. *Consider the number operator* $N = a^*a$. *Then, the number states* ψ_n *are eigenvectors of* N *with corresponding eigenvalues* n. *Moreover,*

- $\langle \psi_n, \psi_m \rangle = 0$ *for* $m \neq n$
- $\|\psi_n\|^2 = n!$
- $\psi_n = (a^*)^n \psi_0$
- $a\psi_{n+1} = (n+1)\psi_n$
- $a^*\psi_n = \psi_{n+1}$

Proof. It is easy to verify the ψ_n are Schwartz, and so belong to $\operatorname{dom} N$. Now, observe that

$$a^*\psi_n = \frac{X - iP}{\sqrt{2}}\psi_n = \frac{1}{\sqrt{2}}\left(x\psi_n(x) - \frac{\mathrm{d}}{\mathrm{d}x}\psi_n(x)\right).$$

Expanding the definition of ψ_n and using the product rule, we get

$$a^*\psi_n = \frac{1}{\sqrt{2}}\left(xH_n(x)\psi_0(x) - \hbar\left[H_n(x)\frac{\mathrm{d}}{\mathrm{d}x}\psi_0(x) + \psi_0(x)\frac{\mathrm{d}}{\mathrm{d}x}H_n(Cx)\right]\right).$$

Now, since

$$\frac{\mathrm{d}}{\mathrm{d}x}\psi_0(x) = -x\sqrt{\pi}\exp\left(-\frac{x^2}{2}\right) = -x\psi_0(x),$$

we have

$$a^*\psi_n = \frac{1}{\sqrt{2}}\left(2xH_n(x) - \frac{\mathrm{d}}{\mathrm{d}x}H_n(Cx)\right)\psi_0 = \psi_{n+1}.$$

A similar computation will show $a\psi_{n+1} = (n+1)\psi_n$. Combining these with the canonical commutation relation, we see

$$N = a^*a = aa^* - \mathbb{1}$$

and so

$$N\psi_n = (aa^* - \mathbb{1})\psi_n = a(a^*\psi_n) - \psi_n = a\psi_{n+1} - \psi_n = n\psi_n.$$

Also witness now that

$$a\psi_0 = \left(\frac{X - iP}{\sqrt{2}}\right)\psi_0 = \frac{1}{\sqrt{2}}\left(x\psi_0(x) + \frac{\mathrm{d}}{\mathrm{d}x}\psi_0(x)\right) = 0$$

which gives a lower bound on the eigenvalues.

For orthogonality, observe that

$$n\langle\psi_n, \psi_m\rangle = \langle N\psi_n, \psi_m\rangle = \langle a^*a\psi_n, \psi_m\rangle = \langle\psi_n, a^*a\psi_m\rangle = m\langle\psi_n, \psi_m\rangle,$$

using the definition of the adjoint twice. So, we must have $\langle\psi_n, \psi_m\rangle = 0$ if $m \neq n$. To get the norm, we use induction. Note that ψ_0 clearly has norm $1 = 0!$. Then, if $\|\psi_n\|^2 = n!$, we have

$$\|\psi_{n+1}\|^2 = \langle\psi_{n+1}, \psi_{n+1}\rangle = \langle a^*\psi_n, a^*\psi_n\rangle = \langle aa^*\psi_n, \psi_n\rangle.$$

Once more using the commutator,

$$aa^*\psi_n = (a^*a + \mathbb{1})\psi_n = n\psi_n + \psi_n,$$

and so

$$\|\psi_{n+1}\|^2 = (n+1)\langle\psi_n, \psi_n\rangle = (n+1)n!,$$

as needed. □

Recall now that the Hamiltonian in question is

$$H = a^*a + \frac{1}{2}\mathbb{1}.$$

The previous proposition gives us its entire spectral decomposition: $H\psi_n = (n + 1/2)\psi_n$. The smallest energy level the system may attain is $1/2$, which occurs in state ψ_0. We call this the ground state. We also see a^* raises the energy, since $a^*\psi_n$ is at an energy level 1 higher than ψ_n. This is why a^* and a are sometimes respectively called the raising and lower operators, with the pair as a whole known as the ladder operators.

4 (Pseudo-)Bosons and (Bi-)Coherent States

Perhaps the most infamous result in quantum mechanics is the Heisenberg uncertainty principle. It gives a bound on much information one can glean by measuring a system.

Definition 20 (Expectation and Variance)**.** *Let $A \in \mathcal{L}(\mathcal{H})$ be symmetric. Define its expectation under $\psi \in \operatorname{dom} A$ by $\langle A \rangle_\psi = \langle A\psi, \psi \rangle$ and variance by*

$$\left(\Delta_\psi A\right)^2 = \left\langle (A - \langle A \rangle_\psi \mathbb{1})\psi, (A - \langle A \rangle_\psi \mathbb{1})\psi \right\rangle.$$

Theorem 8 (Heisenberg)**.** *Let $A, B \in \mathcal{L}(\mathcal{H})$ be symmetric and $\psi \in \operatorname{dom}(AB) \cap \operatorname{dom}(BA)$. Then,*

$$\left(\Delta_\psi A\right)^2 \left(\Delta_\psi B\right)^2 \geq \frac{1}{4} \left| \langle [A, B] \rangle_\psi \right|^2.$$

In the particular case of X and P in $L^2(\mathbb{R})$ this becomes $\left(\Delta_\psi X\right)\left(\Delta_\psi P\right) \geq 1/2$. Therefore, as the variance of a measurement in either position or momentum vanishes, the variance of the other measurement diverges. There is a fundamental uncertainty at the quantum level, and we are never able to fully describe the state of a system (when our measurements do not commute).

A natural question to ask then is which ψ have minimal uncertainty, meaning they saturate the inequality. Schrödinger [Sch26] first studied the minimum uncertainty states of the quantum harmonic oscillator. There was not much follow up, until independent work by Glauber [Gla63], Klauder [Kla63], and Sudarshan [Sud63] connected these states to quantum optics. Rays of light emitted by a laser admit a Hamiltonian which takes the same form as a quantum harmonic oscillator. In this setting, the energy levels correspond to the number of photons in said ray – with creation and annihilation now adding or removing a photon. The states with minimum uncertainty correspond to a coherent ray, those where the frequencies of the waves comprising the ray are all identical and their phase differences are constant.

This terminology, though optical in its origins, is now used to refer to the minimum uncertainty states of a Hamiltonian in general. Arguably, their study is now a field in and of itself. The original construction was rapidly generalized, and is now understood through (Lie) group-theoretic constructions and representation theory [Per72; Gil72]. Their use has also proliferated. In mathematical physics they are deeply involved in the process of geometric quantization [Raw77; Odz92], especially through the Segal-Bargmann transform [Hal94]. Continuing with their origin in quantum optics, they are also at the heart of optical quantum computation [Ral+03] and continuous-variable quantum information [BL05], and related fields such as quantum cryptography [Hut+95].

4.1 Coherent States

Before we proceed, we will make a notational change: let us replace the ψ_n from THEOREM 7 with $\psi_n/\sqrt{n!}$. Thus, our new ψ_n form an orthonormal basis following PROPOSITION 15.

Lemma 6. *The set of (finite!) linear combinations $S = \operatorname{span}\{\psi_n\}$ is dense in $\mathcal{H}$. Moreover, for any $z, w \in \mathbb{C}$ the operators e^{za} and e^{za^*} are well-defined unbounded operators $S \to S$, given by their respective exponential series. They satisfy the properties*

- $e^{za} e^{wa} = e^{(z+w)a}$
- $e^{za^*} e^{wa} = e^{-zw} e^{wa} e^{za^*}$

- $(e^{za})^* = e^{\bar{z}a^*}$

Theorem 9 (Baker–Campbell–Hausdorff). *Let $z, w \in \mathbb{C}$. Then,*

$$e^{za+wa^*} = e^{\frac{zw}{2}} e^{za} e^{wa^*}$$

are well-defined and equivalent operators on S.

This theorem and the preceding lemma are very easy to prove using formal manipulation, but are extraordinarily tedious to prove when properly considering domains and convergence [Bag22]. Indeed, there are a number of questions. Here are just a few:

- Does $\sum_{k=0}^{n} \frac{(za)^k}{k!}$ converge? In what sense?
- Is the domain of e^{za} the same as $\bigcap_{k=0}^{\infty} \operatorname{dom}(za)^k$?
- Does e^{wa^*} map into the domain of e^{za}?

Thankfully, we may now freely perform these manipulations as long as we understand they are only necessarily true on S.

We make a note too that this is not the traditional presentation of Baker–Campbell–Hausdorff. In fact, that is actually a theorem about exponentials in a Lie algebra and not specific to operators at all. However, in the context of operators $A, B \in \mathcal{B}(\mathcal{H})$, it means

$$e^A e^B = \exp\left(A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]] + \cdots\right).$$

For creation and annihilation operators, the canonical commutation relation allows us to prove most of these commutators vanish, which is related to the simplified form above.

Definition 21 (Displacement Operator). *For $\alpha \in \mathbb{C}$, define the displacement operator over S by*

$$D(\alpha) = e^{\alpha a^* - \bar{\alpha} a}.$$

Corollary 2 (Kermack-McCrae).

$$D(\alpha) = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^*} e^{-\bar{\alpha} a} = e^{\frac{|\alpha|^2}{2}} e^{-\bar{\alpha} a} e^{\alpha a^*}.$$

The displacement operator has the remarkable property that, although $\exp(za + wa^*)$ is in general unbounded, this is not the case for D.

Proposition 16. *For all $\alpha \in \mathbb{C}$, $D(\alpha)$ is bounded and unitary.*

Proof. Using both Kermack-McCrae identities, we get

$$D(\alpha)D(-\alpha) = \left[e^{\frac{|\alpha|^2}{2}} e^{\alpha a^*} e^{-\bar{\alpha} a}\right]\left[e^{-\frac{|\alpha|^2}{2}} e^{\bar{\alpha} a} e^{-\alpha a^*}\right] = e^{\alpha a^*} e^{-\bar{\alpha} a} e^{\bar{\alpha} a} e^{-\alpha a^*}.$$

It is straightforward to verify, again as an operator over S, that $e^{za} e^{wa} = e^{(z+w)a}$ for all $z, w \in \mathbb{C}$. The same is true for a^*, so applying this twice shows $D(\alpha)D(-\alpha) = \mathbb{1}$ as operators on S. The same is true for $D(-\alpha)D(\alpha)$, and so we see $D(\alpha)$ is invertible on S.

Now, observe that

$$(D(\alpha))^* = \left(e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^*} e^{-\bar{\alpha} a}\right)^* = e^{-\frac{|\alpha|^2}{2}} e^{-\alpha a^*} e^{\bar{\alpha} a} = D(-\alpha).$$

That is, $(D(\alpha))^{-1} = (D(\alpha))^*$. So, $D(\alpha)$ is an isometry on S, and as a consequence is bounded on S. However, being bounded on a dense set implies the existence of a unique continuous extension, and so we are done. □

We are now ready to define coherent states. The series given is a straightforward application of the Kermack-McCrae identities and PROPOSITION 15.

Definition 22 (Glauber Coherent State)**.** *We call*

$$\varphi_\alpha = D(\alpha)\psi_0 = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \psi_n$$

a Glauber coherent state with amplitude α.

We know that, historically, these states were characterized by saturating the Heisenberg uncertainty principle. Though this has a nice physical interpretation, it is hard to work with computationally as a starting point. It is also not entirely clear how this property would be generalized. Instead, we will use something else, recalling our discussion of spectral theory.

Proposition 17. *With $\alpha, \beta \in \mathbb{C}$, we have*

$$\langle \psi_\alpha, \psi_\beta \rangle = e^{-\frac{|\alpha|^2+|\beta|^2}{2}} e^{\alpha\bar{\beta}}.$$

In particular, coherent states all have unit norm and are never orthogonal.

Proof. We have

$$\begin{aligned} \langle \psi_\alpha, \psi_\beta \rangle &= \left\langle e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \psi_n, e^{-\frac{|\beta|^2}{2}} \sum_{m=0}^{\infty} \frac{\beta^m}{\sqrt{m!}} \psi_m \right\rangle \\ &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left\langle \psi_n, e^{-\frac{|\beta|^2}{2}} \sum_{m=0}^{\infty} \frac{\beta^m}{\sqrt{m!}} \psi_m \right\rangle \end{aligned}$$

Number states are orthonormal, as shown in PROPOSITION 15, and thus the above reduces to

$$e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left\langle \psi_n, e^{-\frac{|\beta|^2}{2}} \frac{\beta^n}{\sqrt{n!}} \psi_n \right\rangle = e^{-\frac{|\alpha|^2+|\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha\bar{\beta})^n}{n!} \langle \psi_n, \psi_n \rangle.$$

But, $\langle \psi_n, \psi_n \rangle = 1$, so we are left with an exponential series which gives the desired result. □

Proposition 18. *The collection of subnormalized coherent states $\{\psi_\alpha / \sqrt{\pi}\}$ weakly resolves the identity.*

Proof. Without loss of generality we need only show

$$\langle \psi_n, \psi_m \rangle = \frac{1}{\pi} \int_{\mathbb{C}} \langle \psi_n, \psi_\alpha \rangle \langle \psi_\alpha, \psi_m \rangle \, \mathrm{d}\alpha$$

for any integers $m, n \geq 0$. We have

$$\langle \psi_\alpha, \psi_n \rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} \langle \psi_n, \psi_m \rangle = e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^m}{m!}$$

due to PROPOSITION 15, and so

$$\int_{\mathbb{C}} \langle \psi_n, \psi_\alpha \rangle \langle \psi_\alpha, \psi_m \rangle \, \mathrm{d}\alpha = \int_{\mathbb{C}} e^{-|\alpha|^2} \bar{\alpha}^m \alpha^n \, \mathrm{d}\alpha.$$

We will take the value of this integral for granted – namely, that it evaluates to $\pi \delta_{nm}$. With this, we are done. □

This correcting factor of $1/\sqrt{\pi}$ is intimately related to the fact that coherent states are not orthogonal. Sometimes, this property of coherent states is referred to as overcompleteness. A natural follow-up is to ask if there is a countable subset of coherent states which still satisfies this identity (sometimes called a complete subset). The answer is yes [Per71; Bar+71].

There is one last property of coherent states which we will reflect on and prove later. It is in fact another way to characterize them.

Proposition 19. *For all $\alpha \in \mathbb{C}$, $a\psi_\alpha = \alpha\psi_\alpha$.*

4.2 Bosonic Operators and Shifts

Photons are a type of particle known as a boson, and for this reason the creation and annihilation operators are sometimes known as bosonic operators when discussing coherent states. Their key property is that they satisfy the canonical commutation relation. So key is it, in fact, that it is used to define bosonic operators in any abstract Hilbert space $\mathcal{H}$.

Definition 23 (Bosonic Operators)**.** *Let $c \in \mathcal{L}(\mathcal{H})$. Suppose that there exists some $D \subseteq \operatorname{dom} c \cap \operatorname{dom} c^*$ dense in $\mathcal{H}$ which is fixed by both c and c^*. Also suppose that $[c, c^*] = \mathbb{1}$ over D and that there exists some $\varphi_0 \in D$ so $c\varphi_0 = 0$, the ground state. Then, we call c^* and c bosonic creation and annihilation operators and $c^*c = N$ their number operator.*

Theorem 10 (Number States)**.** *Let c, c^* be bosonic operators over $\mathcal{H}$ with domain D and ground state φ_0. Without loss of generality suppose $\|\varphi_0\| = 1$ and define*

$$\varphi_n = \frac{1}{\sqrt{n!}} (c^*)^n \varphi_0.$$

Then, the number states φ_n are an orthonormal basis of $\mathcal{H}$ and $S = \operatorname{span}\{\varphi_n\}$ is dense in D. Moreover,

- $c^*\varphi_n = \sqrt{n+1}\varphi_{n+1}$
- $c\varphi_n = \sqrt{n}\varphi_{n-1}$
- $c^*c\varphi_n = n\varphi_n$

The proof of this theorem is the exact same as the latter half of the proof of PROPOSITION 15. Indeed, the hardest part is actually defining the φ_n so they would satisfy the hypotheses, which was the work of THEOREM 7. An interesting remark is that such operators cannot exist in any finite-dimensional space.

Proposition 20. *Bosonic operators are necessarily unbounded.*

Proof. Let c be a bosonic annihilation operator with number states φ_n. Then, via THEOREM 10,

$$\|c\| = \sup_{0\neq x\in \operatorname{dom} c} \frac{\|cx\|}{\|x\|} \geq \sup_{n\in\mathbb{N}} \sqrt{\langle c\varphi_n, c\varphi_n\rangle} = \sup_{n\in\mathbb{N}} \sqrt{\langle \varphi_n, c^* c\varphi_n\rangle} = \sup_{n\in\mathbb{N}} n.$$

Therefore, $\|c\| = \infty$, meaning $c \notin \mathcal{B}(\mathcal{H})$. □

For any bosonic operators c and c^*, there are a number of transformations which create new bosonic operators. A simple one is shifting, as for any $z \in \mathbb{C}$ the pair $c + z\mathbb{1}$ and $c^* + \bar{z}\mathbb{1}$ are also bosonic operators (and we will verify this later). Another is the Bogoliubov–Valatin transformation $c \mapsto uc + vc^*$ and $c^* \mapsto \bar{v}c + \bar{u}c^*$ for $u, v \in \mathbb{C}$ where $|u|^2 - |v|^2 = 1$, which was originally introduced to study superconductivity [Bog58; Val58]. There is an overarching framework here, in that these are actually morphisms of particular spaces known as CCR C^*-algebras [Sla72].

Let us examine the shifted bosonic operator in a more general case where we shift c and c^* by (potentially) unequal amounts.

Proposition 21 (Shifted Bosonic Operators)**.** *Let c be the annihilation operator in $L^2(\mathbb{R})$ and ψ_0 its ground state. Fix some $\alpha, \beta \in \mathbb{C}$ and define $a = c + \alpha\mathbb{1}$ and $b = c^* + \beta\mathbb{1}$ on $\mathcal{S}(\mathbb{R})$.*

Then, $[a, b] = \mathbb{1}$ on $\mathcal{S}(\mathbb{R})$. Moreover, with $\tilde{\varphi}_0 = D(-\alpha)\psi_0$ and $\varphi_0 = D(-\bar{\beta})\psi_0$, we have $a\tilde{\varphi}_0 = 0$ and $b^*\varphi_0 = 0$.

Proof. Observe that

$$[a, b] = ab - ba = (c + \alpha\mathbb{1})(c^* + \beta\mathbb{1}) - (c^* + \beta\mathbb{1})(c + \alpha\mathbb{1}).$$

We see

$$(c + \alpha\mathbb{1})(c^* + \beta\mathbb{1}) = cc^* + \alpha c^* + \beta c + \alpha\beta\mathbb{1}$$

and

$$(c^* + \beta\mathbb{1})(c + \alpha\mathbb{1}) = c^*c + \alpha c^* + \beta c + \alpha\beta\mathbb{1}.$$

Therefore, $[a, b] = cc^* - c^*c$. However, from the canonical commutation relation we know this is the identity on, in particular, $\mathcal{S}(\mathbb{R})$.

Now, we have

$$\tilde{\varphi}_0 = D(-\alpha)\psi_0 = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{\sqrt{n!}} \psi_n.$$

Following PROPOSITION 15, we have

$$c\left(e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{\sqrt{n!}} \psi_n\right) = e^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{(-\alpha)^n}{\sqrt{(n-1)!}} \psi_{n-1}$$

since $c\psi_n = \sqrt{n}\psi_{n-1}$ (recall we changed the normalization). But, note that

$$e^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{(-\alpha)^n}{\sqrt{(n-1)!}} \psi_{n-1} = -\alpha e^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{(-\alpha)^{n-1}}{\sqrt{(n-1)!}} \psi_{n-1} = -\alpha D(-\alpha)\psi_0.$$

Therefore,

$$(c + \alpha \mathbb{1})\tilde{\varphi}_0 = -\alpha\varphi_0 + \alpha\varphi_0 = 0,$$

as claimed. This follows identically for φ_0. □

The value of this proposition is in elaborating on the exact role the adjoint plays in the definition of bosonic operators. When $\bar{\alpha} = \beta$ we have $a^* = b$ and $\varphi_0 = \tilde{\varphi}_0$. However, when $\bar{\alpha} \neq \beta$ we split our original bosonic operators into two sets of bosonic operators: the pairs a, a^* and b, b^*. Each pair has their own unique ground state and independently satisfies the canonical commutation relation. However, the fact a and b also satisfy the relation means these two pairs interact with each other.

Theorem 11. *Let a, b be shifted annihilation and creation operators on $L^2(\mathbb{R})$ with respective ground states $\tilde{\varphi}_0$ and φ_0. Suppose $\|\varphi_0\| = 1 = \|\tilde{\varphi}_0\|$ and consider their respective number states*

$$\tilde{\varphi}_n = \frac{1}{\sqrt{n!}}(a^*)^n\tilde{\varphi}_0 \text{ and } \varphi_n = \frac{1}{\sqrt{n!}}b^n\varphi_0.$$

Define $N = ba$ and $\tilde{N} = N^* = a^*b^*$. Then,

- $\langle \varphi_n, \tilde{\varphi}_m \rangle = \delta_{mn}$
- $a\varphi_n = \sqrt{n}\varphi_{n-1}$
- $b^*\tilde{\varphi}_n = \sqrt{n}\tilde{\varphi}_{n-1}$
- $N\varphi_n = n\varphi_n$
- $\tilde{N}\tilde{\varphi}_n = n\tilde{\varphi}_n$

So, the two number states of the two sets of bosonic operators are intertwined. The states b creates are annihilated by a and the states a^* creates are annihilated by b^*. Again, we do not provide a proof, as it is essentially just the same algebraic manipulation from PROPOSITION 15.

When $\bar{\alpha} = \beta$, these number states are basically the original ones up to some constant displacement. For this reason, they are sometimes called displaced number states. They inherit more or less the same properties as the original number states [Wün91].

4.3 Quasi Bases and Bicoherent States

The conclusion from the previous section is that much of the structure of bosonic operators stems solely from the canonical commutation relation and not necessarily the adjoint. This leads immediately into the following generalization.

Definition 24 (Pseudobosonic Operators)**.** *Let $a, b \in \mathcal{L}(\mathcal{H})$. Suppose there exists some $D \subseteq \operatorname{dom} a \cap \operatorname{dom} b$ dense in $\mathcal{H}$ fixed by both a and b. Further suppose $[a, b] = \mathbb{1}$ on D and that they both have a ground state, some $\varphi_0, \tilde{\varphi}_0 \in D$ so $a\tilde{\varphi}_0 = 0$ and $b\varphi_0 = 0$. Then, the pair a and b are called pseudobosonic operators.*

The structure of THEOREM 11 immediately applies to pseudobosonic operators. Such operators arise naturally in non-Hermitian quantum mechanics. One example is in the case of the Swanson Hamiltonians, one of the first classes of non-Hermitian Hamiltonians shown to possess a real, positive, and discrete spectrum [Swa04]. With c, c^* being the standard ladder operators in $L^2(\mathbb{R})$, these Hamiltonians take the form [PNP11]

$$H = \frac{1}{\cos 2\vartheta}\left(ba - \frac{1}{2}\right) \text{ where } \vartheta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$$

for the pseudobosonic operators

$$a = \cos\vartheta c + i\sin\vartheta c^* \text{ and } b = i\sin\vartheta c + \cos\vartheta c^*.$$

There are also pseudobosonic analogues of the Bogoliubov–Valatin transforms [Cal11].

A natural question to ask is how the notion of coherent states carries over to pseudobosonic operators. This question was first asked and answered, though informally, by Trifonov [Tri08]. The formal details were then filled in by Bagarello, defining what are now known as regular bicoherent states [Bag10]. A more general construction which has more physical relevance was introduced in [Bag13], and is the approach we present.

Definition 25 (G-Quasi Bases)**.** *Let $G \subseteq \mathcal{H}$ and say $e_n, f_n \in G$ are biorthogonal, meaning $\langle e_n, f_m\rangle = \delta_{mn}$. Then, the pair $\{e_n\}$ and $\{f_n\}$ are called G-quasi bases if*

$$\langle x, y\rangle = \sum_{n=0}^{\infty} \langle x, e_n\rangle\langle f_n, y\rangle = \sum_{n=0}^{\infty} \langle x, f_n\rangle\langle e_n, y\rangle$$

for all $x, y \in G$.

Definition 26 (Factorial of a Sequence)**.** *Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence. Define $\alpha_0! = 1$ and $\alpha_n! = \alpha_1 \cdots \alpha_n$.*

Definition 27 (Open Disk)**.** *Let $z \in \mathbb{C}$ and $\rho > 0$. The open disk about z with radius ρ is denoted $D(z, \rho)$.*

The following theorem [BGS17] then establishes the existence of bicoherent states, with their namesake being explained shortly. Note that, although the theorem statement is quite long, most of it is merely establishing hypotheses. This is done to unify the various notions of bicoherent states originally idiosyncratic to specific Hamiltonians or assumptions on domains. It applies to a setting even more abstract than pseudobosonic operators.

Theorem 12. *Let $\tilde{\varphi}_n$ and φ_n be G-quasi bases for some dense $G \subseteq \mathcal{H}$. Suppose there exist two operators $a, b \in \mathcal{L}(\mathcal{H})$ with $G \subseteq \operatorname{dom} a \cap \operatorname{dom} b^*$ such $a\tilde{\varphi}_n = \alpha_n\tilde{\varphi}_{n-1}$ and $b\varphi_n = \alpha_n\varphi_{n-1}$ for a (strictly!) increasing sequence $\alpha_n \in \mathbb{R}$ with $\alpha_0 = 0$. Set $\alpha = \sup_{n\in\mathbb{N}} \alpha_n$ (which may be infinite).*

Further suppose there exist constants $A, \tilde{A}, r, \tilde{r} > 0$ and two positive sequences M_n and $\tilde{M}_n$ such that the limits

$$\lim_{n\to\infty} \frac{\tilde{M}_n}{\tilde{M}_{n+1}} = \tilde{M} \quad \textit{and} \quad \lim_{n\to\infty} \frac{M_n}{M_{n+1}} = M$$

are well-defined (though possibly infinite), and moreover $\|\tilde{\varphi}_n\| \leq \tilde{A}\tilde{r}^n\tilde{M}_n$ and $\|\varphi_n\| \leq Ar^nM_n$. Define the normalizer

$$N(z) = \left(\sum_{n=0}^{\infty} \frac{z^{2n}}{(\alpha_n!)^2}\right)^{-\frac{1}{2}},$$

which converges in $D(0, R)$, where

$$R = \alpha \min\left(1, \frac{\tilde{M}}{\tilde{r}}, \frac{M}{r}\right).$$

Then, the points defined on $D(0,R)$ by

$$\tilde{\Phi}(z) = N(|z|)\sum_{n=0}^{\infty}\frac{z^n}{\alpha_n!}\tilde{\varphi}_n \quad \textit{and} \quad \Phi(\alpha) = N(|z|)\sum_{n=0}^{\infty}\frac{z^n}{\alpha_n!}\varphi_n$$

are called a pair of bicoherent states. They have the property $a\tilde{\Phi}(z) = z\tilde{\Phi}(z)$ and $b^*\Phi(z) = z\Phi(z)$.

Suppose further there exists a measure μ so

$$\int_0^R \rho^{2n}\,\mathrm{d}\mu(\rho) = \frac{(\alpha_n!)^2}{2\pi}$$

for all integers $n \geq 0$. In polar coordinates $z = \rho e^{i\vartheta}$ define the normalized area form $\mathrm{d}\nu(z,\bar{z}) = N(\rho)^{-2}\,\mathrm{d}\mu(\rho)\,\mathrm{d}\vartheta$. Then, for all $f, g \in G$ we have

$$\langle f, g\rangle = \int_{D(0,R)}\langle f, \tilde{\Phi}(z)\rangle\langle\Phi(z), g\rangle\,\mathrm{d}\nu(z,\bar{z}) = \int_{D(0,R)}\langle f, \Phi(z)\rangle\langle\tilde{\Phi}(z), g\rangle\,\mathrm{d}\nu(z,\bar{z}).$$

Proof. We will first examine the normalizer N. With $w = |z|^2$ and $c_n = 1/(\alpha_n!)^2$ we have $N(z) = \sum_{n=0}^{\infty} c_n w^n$. We conduct the ratio test, finding

$$\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left(\frac{\alpha_{n+1}!}{\alpha_n!}\right)^2 = \lim_{n\to\infty}\alpha_{n+1}^2 = \alpha^2.$$

So, $N(z)$ converges for all $|w| < \alpha^2$, or when $|z| < \alpha$.

Moving on to the bicoherent states, observe

$$\left\|\sum_{n=0}^{\infty}\frac{z^n}{\alpha_n!}\tilde{\varphi}_n\right\| \leq \sum_{n=0}^{\infty}\left|\frac{z^n}{\alpha_n!}\right|(Ar^nM_n) = A\sum_{n=0}^{\infty}\frac{M_n}{\alpha_n!}(r|z|)^n.$$

A ratio test again tells us this series converges whenever $r|z| < M\alpha$. This carries through identically for the series in $\tilde{\varphi}$, and taking the minimum radii between these three series gives us the desired R.

For the eigenvalue equations, we have

$$a\tilde{\Phi}(z) = N(|z|)\sum_{n=0}^{\infty}\frac{z^n}{\alpha_n!}(a\tilde{\varphi}_n) = N(|z|)\sum_{n=0}^{\infty}\frac{z^n}{\alpha_{n-1}!}\varphi_{n-1}.$$

Pulling out the factor of z and reindexing (since $\alpha_0 = 0$ we have $a\varphi_0 = 0$), we are done.

Now, for $z \in D(0,R)$ and $g \in G$ we have

$$\langle\Phi(z), g\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\alpha_n!}\langle\varphi_n, g\rangle.$$

The same is true for $\tilde{\Phi(z)}$ and $f \in G$, meaning

$$\int_{D(0,R)}\langle f, \tilde{\Phi}(z)\rangle\langle\Phi(z), g\rangle\,\mathrm{d}\nu(z,\bar{z}) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\langle f, \tilde{\varphi}_n\rangle\langle\varphi_m, g\rangle}{\alpha_n!\alpha_m!}\int_{D(0,R)}N(|z|)^2 z^n\bar{z}^m\,\mathrm{d}\nu(z,\bar{z}).$$

Switching to polar coordinates and rewriting the measure yields

$$\int_{D(0,R)} N(|z|)^2 z^n \bar{z}^m \,\mathrm{d}\nu(z,\bar{z}) = \int_0^R \rho^{n+m} \,\mathrm{d}\mu(\rho) \int_0^{2\pi} e^{i\vartheta(n-m)} \,\mathrm{d}\vartheta = 2\pi\delta_{nm} \int_0^R \rho^{n+m} \,\mathrm{d}\mu(\rho).$$

Thus,

$$\int_{D(0,R)} \langle f, \tilde{\Phi}(z)\rangle \langle \Phi(z), g\rangle \,\mathrm{d}\nu(z) = \sum_{n=0}^{\infty} \frac{2\pi\langle f, \tilde{\varphi}_n\rangle\langle \varphi_n, g\rangle}{(\alpha_n!)^2} \int_0^R \rho^{2n} \,\mathrm{d}\mu(\rho).$$

However, by hypothesis this integral equals $(\alpha_n!)^2/2\pi$ and we have G-quasi bases. So, this indeed is equal to $\langle f, g\rangle$, as needed. □

Recall that THEOREM 11 provides the interpretation of pseudobosonic operators as two sets of bosonic operators linked through a joint canonical commutation relation. This is not entirely precise in the general case (i.e. outside of shifted bosonic operators) since a, a^* and b, b^* are rarely bosonic operators themselves (e.g. the φ_n are not orthogonal). However, it allows us to morally view a and b^* as annihilation-like operators. Recall now from PROPOSITION 19 that coherent states are eigenvectors of the annihilation operator. Indeed, these bicoherent states now are two collections of states, each eigenvectors for one of these annihilation-like operators.

Moreover, in THEOREM 11 we sacrificed orthogonality in one set of number states for biorthogonality between two sets of number states. For these bicoherent states, the sacrificial analogue is the weak resolution of identity from PROPOSITION 18. Instead they somehow weakly resolve it together, in what is sometimes called biovercompleteness. It is in these two senses that bicoherent states naturally generalize coherent states.

Let us discuss the bicoherent states of shifted bosonic operators on $L^2(\mathbb{R})$, heavily referencing [Bag22]. We will begin our final stretch by fixing notation. Let c, c^* be the standard annihilation and creation operators in $L^2(\mathbb{R})$ with a ground state ψ_0 and number states ψ_n, as discussed in PROPOSITION 15. Let a, b be shifts of c, c^* by the respective amounts $\alpha, \beta \in \mathbb{C}$, with corresponding ground states $\tilde{\varphi}_0, \varphi_0$ and number states $\tilde{\varphi}_n, \varphi_n$, as discussed in PROPOSITION 21 and THEOREM 11. Recall as well the notation $S = \operatorname{span}\{\psi_n\}$.

Lemma 7. *Define* $N_0 = e^{\frac{|\alpha|^2+|\beta|^2}{4} - \frac{\alpha\beta}{2}}$ *and* $\gamma = \beta - \bar{\alpha}$. *Then,*

$$\tilde{\varphi}_n = \frac{N_0}{\sqrt{n!}} D(-\alpha)(c^* + \gamma\mathbb{1})^n \psi_0$$

and

$$\varphi_n = \frac{\bar{N}_0}{\sqrt{n!}} D(-\bar{\beta})(c^* - \gamma\mathbb{1})^n \psi_0.$$

Lemma 8. *For all integers* $n \geq 0$, $\|\tilde{\varphi}_n\|$ *and* $\|\varphi_n\|$ *are both bounded above by* $|N_0| e^{\frac{1}{2}n|\gamma|^2}$, *where* $N_0 = e^{\frac{|\alpha|^2+|\beta|^2}{4} - \frac{\alpha\beta}{2}}$ *and* $\gamma = \beta - \bar{\alpha}$.

Proof. Using the representation of $\tilde{\varphi}_n$ from LEMMA 7, write

$$\begin{aligned}\|\tilde{\varphi}_n\|^2 &= \frac{|N_0|^2}{n!} \langle D(-\alpha)(c^* + \gamma\mathbb{1})^n \psi_0, D(-\alpha)(c^* + \gamma\mathbb{1})^n \psi_0\rangle \\ &= \frac{|N_0|^2}{n!} \langle (c^* + \gamma\mathbb{1})^n \psi_0, (c^* + \gamma\mathbb{1})^n \psi_0\rangle,\end{aligned}$$

since D is unitary via PROPOSITION 16. Using the binomial theorem and properties from THEOREM 11 we find

$$(c^* + \gamma \mathbb{1})^n \psi_0 = \sum_{k=0}^{n} \binom{n}{k} \left(\gamma^k \mathbb{1}\right)\left((c^*)^{n-k}\psi_0\right) = \sum_{k=0}^{n} \binom{n}{k} \gamma^k \sqrt{(n-k)!}\,\psi_{n-k}.$$

Substituting this inner product and using the orthogonality of number states from PROPOSITION 15, we conclude

$$\|\tilde{\varphi}_n\|^2 = \frac{|N_0|^2}{n!} \sum_{k=0}^{n} \binom{n}{k}^2 \bar{\gamma}^{2k}(n-k)! = n!|N_0|^2 \sum_{k=0}^{n} \frac{|\gamma|^{2k}}{(k!)^2(n-k)!}.$$

Now, observe for all non-negative integers $k \leq n$ we have

$$\frac{1}{(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{n!} \leq \frac{n^k}{n!}$$

and, using the fact $(k!)^2 \geq k!$, we get

$$\sum_{k=0}^{n} \frac{|\gamma|^{2k}}{(k!)^2(n-k)!} \leq \frac{1}{n!} \sum_{k=0}^{n} \frac{\left(n|\gamma|^2\right)^k}{k!}.$$

Continuing the sum to the exponential series, we conclude $\|\tilde{\varphi}_0\|^2 \leq |N_0|^2 e^{n|\gamma|^2}$, as desired. The same technique works for φ_n. □

Theorem 13. *Associated with a and b are a pair of bicoherent states defined everywhere on $\mathbb{C}$ by*

$$\tilde{\Phi}(z) = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \tilde{\varphi}_n$$

and

$$\Phi(z) = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \varphi_n.$$

In particular, they satisfy $a\tilde{\Phi}(z) = z\tilde{\Phi}(z)$ and $b^*\Phi(z) = z\Phi(z)$, as well as

$$\langle f, g\rangle = \frac{1}{\pi}\int_{\mathbb{C}} \langle f, \tilde{\Phi}(z)\rangle\langle \Phi(z), g\rangle e^{-|z|^2}\,\mathrm{d}z = \frac{1}{\pi}\int_{\mathbb{C}} \langle f, \Phi(z)\rangle\langle \tilde{\Phi}(z), g\rangle e^{-|z|^2}\,\mathrm{d}z$$

for all $f, g \in S$. Here, $\mathrm{d}z$ is the Lebesgue measure over $\mathbb{C} \cong \mathbb{R}^2$.

Proof. We will establish their existence and properties through THEOREM 12. First, the density of S as well as $\tilde{\varphi}_n$ and φ_n being S-quasi bases will be taken as given. Now, using the notation of theorem, we have $\alpha_n = \sqrt{n!}$ through the raising and lowering properties in THEOREM 11. This means the normalizer is

$$N(z) = \left(\sum_{n=0}^{\infty} \frac{z^{2n}}{n!}\right)^{-\frac{1}{2}} = e^{-\frac{1}{2}z^2}.$$

Now, set $\gamma = \beta - \bar{\alpha}$. We only need one set of shared constants, namely $r = e^{-\frac{1}{2}|\gamma|^2}$ and $A = |N_0|$. Also define the series $M_n = 1$, whose ratios clearly tend to 1. As a direct consequence of LEMMA 8 we have $\|\tilde{\varphi}_n\|^2 \leq Ar^n M_n$ and $\|\varphi_n\|^2 \leq Ar^n M_n$. So, not only do the bicoherent series converge, but since $\alpha_n \to \infty$ they indeed do so globally ($R = \infty$).

Now, for integers $k \geq 0$ we examine the integral

$$\frac{1}{\pi}\int_0^\infty \rho^{2k}\left(e^{-\rho^2}\rho\,\mathrm{d}\rho\right) = \frac{1}{\pi}\int_0^\infty (\rho^2)^{k+1-1}\rho e^{-\rho^2}\,\mathrm{d}r.$$

This is the gamma function up to the change of variables $t = \rho^2$, hence the integral evaluates to $\Gamma(k+1)/2 = k!/2$. Our desired measure is clear, and thus (in polar coordinates) we have

$$\mathrm{d}\nu(z,\bar{z}) = \frac{1}{\pi}\left(e^{-\frac{1}{2}|z|^2}\right)^{-2} e^{-|z|^2}\rho\,\mathrm{d}\rho\,\mathrm{d}\vartheta = \frac{1}{\pi}\rho\,\mathrm{d}\rho\,\mathrm{d}\vartheta.$$

However, up to the constant factor so this is the usual Lebesgue measure, and so we are done. □

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