Math 3120 – Assignment 4

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Problem 1.

List the distinct Sylow 2- and 3-subgroups for the given group.

Part A.

$$Z_3 \times Z_3.$$

Solution. We know $|Z_3 \times Z_3| = 9$. However, 9 is odd and so we cannot write it in the form $9 = 2^n m$ where $n > 0$ and $2 \nmid m$. So, the only Sylow 2-subgroup is the trivial group. On the other hand, we have $9 = 3^2 \cdot 1$, and so all Sylow 3-subgroups have order 9, hence the only such subgroup is $Z_3 \times Z_3$ itself.

Part B.

$$D_{12}.$$

Solution. We know
$$|D_{12}| = 12 = 2^2(3) = 3(4).$$
We first consider n_2. We must have $n_2 \mid 3$ and $n_2 \equiv_2 1$, hence our options are $n_2 = 1$ or $n_2 = 3$. Consider first $\langle r^3, s\rangle$ where r is rotation and s is reflection:

$$\langle r^3, s\rangle = \{r^3, r^6 = 1, s, r^3 s\}.$$

Observe that this group is of order 4, and hence is a Sylow 2-subgroup. Therefore, all other Sylow 2-subgroups are conjugates, if they exist. We have

$$r\langle r^3, s\rangle r^{-1} = \langle r^3, r^2 s\rangle = \{r^3, 1, r^2 s, r^5 s\}$$

and
$$r^2\langle r^3, s\rangle r^{-2} = \langle r^3, r^4 s\rangle = \{r^3, 1, r^4 s, s\}.$$
As we have 3 distinct subgroups, and $n_2 \leq 3$, we are done.

For Sylow 3-subgroups, we must have $n_3 \mid 4$ and $n_3 \equiv_3 1$. So, $n_3 = 1$ or $n_3 = 4$. Observe that
$$\langle r^2\rangle = \{r^2, r^4, 1\}$$

is one such Sylow 3-subgroup. We see that for arbitrary $r^i s^j \in D_{12}$, we have

$$r^i s^j \langle r^2 \rangle (r^i s^j)^{-1} = \langle r^i s^j r^2 s^j r^{-i} \rangle = \langle r^2 \rangle$$

hence it is normal, so $n_3 = 1$ as it is the only Sylow 3-subgroup.

PART C.

$$S_3 \times S_3.$$

Solution. We see

$$|S_3 \times S_3| = 36 = 2^2(9) = 3^2(4).$$

Observe that

$$\langle ((1 \quad 2), 1), (1, (1 \quad 2)) \rangle = \{((1 \quad 2), 1), (1, 1), (1, (1 \quad 2)), ((1 \quad 2), (1 \quad 2))\}.$$

So, the above is a Sylow 2-subgroup by its order. Recall that conjugation in S_n corresponds exactly to permutations of the elements in the cycles. That is,

$$\langle ((1 \quad 2), 1), (1, (1 \quad 2)) \rangle^{S_3 \times S_3} = \{\langle ((a \quad b), 1), (1, (x \quad y)) \rangle : a \neq b, x \neq y\}$$

fully describes all of the Sylow 2-subgroups as they are given by conjugation.

For Sylow 3-subgroups, recall that $A_3 \trianglelefteq S_3$, and that $|A_3| = 3$. Thus, we have $A_3 \times A_3 \trianglelefteq S_3 \times S_3$, and $|A_3 \times A_3| = 9$ so it is a Sylow 3-subgroup. However, normality means $n_3 = 1$, hence this is the only such one.

PART D.

$$A_4.$$

Solution. We have

$$|A_n| = \frac{4!}{2} = 12 = 2^2(3) = 3(4).$$

We know from assignment 2 problem 8.a that V_4 in A_4 is a normal subgroup, and as $|V_4| = 4$, we conclude it is the only Sylow 2-subgroup.

For Sylow 3-subgroups, we know that every 3-cycle is even, hence lying in A_4. Moreover, 3-cycles have order 3, and hence by the fact that Sylow subgroups arising from conjugation, and cycle-type is preserved under conjugation, we conclude

$$\mathrm{Syl}_3(A_4) = \langle (1 \quad 2 \quad 3) \rangle^{A_4} = \{\langle (a \quad b \quad c) \rangle : a, b, c \text{ distinct}\}.$$

PART E.

$$S_4.$$

Solution. We see

$$|S_4| = 4! = 24 = 2^3(3) = 3(8).$$

Identically as concluded above in 1.D, we have that

$$\mathrm{Syl}_3(S_4) = \langle (1 \quad 2 \quad 3) \rangle^{S_4} = \{\langle (a \quad b \quad c) \rangle : a, b, c \text{ distinct}\}.$$

For Sylow 2-subgroups, we examine D_8 in S_4 (since $|D_8| = 8$). This embedding can be found by geometrically viewing D_8 as actions on the vertices of a square. With the labelling as per figure 1, we specifically have

$$D_8 \hookrightarrow S_4 \text{ by } r \mapsto (1 \quad 2 \quad 3 \quad 4) \text{ and } s \mapsto (1 \quad 3).$$

Therefore, by order,

$$\langle (1 \quad 2 \quad 3 \quad 4), (1 \quad 3) \rangle \in \mathrm{Syl}_2(S_4)$$

as $D_8 = \langle r, s \rangle$. We can then find other Sylow 3-subgroups by considering different labellings of the square, namely by replacing the bottom vertex with 2 or 4. Note that, regardless of the bottom vertex's label, swapping the left and right vertices is immaterial as it does not change the square itself, as those are already contained in the symmetries of some labelling (as it is just reflection). As we know $n_2 \mid 3$ and $n_2 \equiv_2 1$, we know $n_2 = 1$ or $n_2 = 3$, these 3 labellings tell us we have found all Sylow 2-subgroups. Specifically,

$$\begin{aligned}\mathrm{Syl}_2(S_4) = \{&\langle (1 \quad 2 \quad 3 \quad 4)(1 \quad 3) \rangle,\\ &\langle (1 \quad 3 \quad 2 \quad 4), (1 \quad 2) \rangle,\\ &\langle (1 \quad 2 \quad 4 \quad 3), (1 \quad 4) \rangle\}.\end{aligned}$$

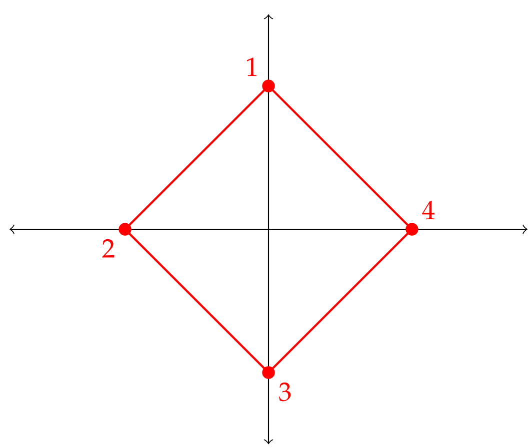

Figure 1: Example labelling of a square.

PROBLEM 2.

For an arbitrary group G with the given order, prove it has normal Sylow p-subgroup for some p dividing the order.

PART A.

$$|G| = 56.$$

Proof. We decompose the order as $56 = 2^3(7) = 7(8)$. Consider n_7. We know $n_7 \mid 8$ and $n_7 \equiv_7 1$, hence $n_7 = 1$ or $n_7 = 8$. If $n_7 = 1$, we are done as the unique Sylow 7-subgroup is normal.

Say instead $n_7 = 8$. Observe that as 7 is prime, neither of the 8 distinct Sylow 7-subgroups share any elements (except for the identity). So, we have discovered $8 \cdot 6 = 48$ distinct non-identity elements in G. However, we also know there is at least one Sylow 2-subgroup, which has order 8. Any one of these is disjoint from each Sylow 7-subgroup (except for the identity) as $8 > 7$ and $7 \nmid 8$. Hence, we have discovered at least 7 more non-identity elements, per Sylow 2-subgroup, meaning we now have $48 + 7 = 55$ such distinct elements. However, accounting for the identity, we have now found 56 elements - that is, we have found all of G. So, we must have $1 \leq n_2 \leq 1$, hence $n_2 = 1$, and so the Sylow 2-subgroup is normal in G. □

PART B.

$$|G| = 312.$$

Proof. We factor the order to see $312 = 13(24)$. We know $n_{13} \mid 24$ and $n_{13} \equiv_{13} 1$. However, the only way for this to be true is if $n_{13} = 1$, hence there exists a normal Sylow 13-subgroup. □

PART C.

$$|G| = 351.$$

Proof. We see

$$351 = 3^3(13) = 13(27).$$

We know $n_{13} \mid 27$ and $n_{13} \equiv_{13} 1$, hence $n_{13} = 1$ or $n_{13} = 27$. Likewise, $n_3 \mid 13$ and $n_3 \equiv_3 1$, so $n_3 = 1$ or $n_3 = 13$. If $n_3 = 1$ or $n_{13} = 1$, we are done, as then there exists a normal Sylow 3- or 13-subgroup.

Suppose then the opposite, so $n_3 = 13$ and $n_{13} = 27$. As 13 is prime, the logic of 2.a gives us $27 \cdot 12 = 324$ distinct non-identity elements, and hence $351 - 324 = 27$ elements unaccounted for. However, Sylow 3-subgroups have order 27, which means that actually $n_3 = 1$, contrary to the assumption. So, this case is not possible. □

part d.

$$|G| = 105.$$

Proof. We see
$$105 = 3(35) = 5(21) = 7(15).$$
We consider n_7. We know $n_7 \mid 15$ and $n_7 \equiv_7 1$. So, either $n_7 = 1$ or $n_7 = 15$. For n_5, we see $n_5 \mid 21$ and $n_5 \equiv_5 1$, hence $n_5 = 1$ or $n_5 = 21$. If either n_5 or n_7 is equal to 1, we are done, so suppose this is not the case. That is, suppose $n_5 = 21$ and $n_7 = 15$. As every Sylow 5-subgroup has order 5 and each Sylow 7-subgroup has order 7, both prime, repeating the same steps in 2.a gives us

$$21 \cdot 4 + 15 \cdot 6 = 174$$

distinct non-identity elements. However, $174 > 105$, which is absurd, hence there is at least a normal Sylow 7- or 5-subgroup. □

part e.

$$|G| = 200.$$

Proof. We factor the order as $200 = 5^2(8)$. We know that $n_5 \mid 8$ and $n_5 \equiv_5 1$, hence we must have $n_5 = 1$, and so the unique Sylow 5-subgroup is normal. □

problem 3.

How many elements of order 7 exist in a simple group of order 168?

Solution. We decompose the order as $168 = 7(24)$. We know $n_7 \mid 24$ and $n_7 \equiv_7 1$. Thus, $n_7 = 1$ or $n_7 = 8$. However, our group is simple, hence there exist no normal subgroups, so we must have $n_7 = 8$. In the same manner as in 2.a, we are left with $8 \cdot 6 = 48$ distinct non-identity elements (each belonging to some Sylow 7-subgroup). As 7 is prime, and these elements are not the identity, they each themselves generate a Sylow 7-subgroup, and hence have order 7. Moreover, any element with order 7 generates a Sylow 7-subgroup, and so is contained within these 48 elements. Thus, there are 48 elements with order 7.

PROBLEM 4.

For $p = 2, 3$, and 5, find $n_p(G)$ for the given group.

PART A.

$$G = A_5.$$

Solution. We know that

$$|A_5| = \frac{5!}{2} = 60 = 2^2(15) = 3(20) = 5(12).$$

We note that for all p, $n_p \neq 1$ as A_5 is simple. Now, consider $p = 2$. We see $n_2 \mid 15$ and $n_2 \equiv_2 1$, so $n_2 = 3, 5$, or $n_2 = 15$. Then, for n_3 we know $n_3 \mid 20$ and $n_3 \equiv_3 1$, hence $n_3 = 4$ or $n_3 = 10$. Lastly, for n_5, since $n_5 \mid 12$ and $n_5 \equiv_5 1$, we have $n_5 = 6$. Of note, the Sylow 5-subgroups have order 5, and as 5 is prime by the technique of 2.a we have $6 \cdot 4 = 24$ non-identity elements found amongst the Sylow 5-subgroups.

As $3, 4 < 5$, we know that no Sylow 5-subgroup will be contained in a Sylow 2- or 3-subgroup. So, there are $60 - 24 = 36$ elements which can belong to them. As $3 < 4$, no Sylow 2-subgroup will be contained in a Sylow 3-subgroup, and since $3 \nmid 4$ the Sylow 3-subgroups will not be contained in the Sylow 2-subgroups. That is, all the remaining Sylow subgroups share only the identity. Now, note that every 3-cycle (which will belong to A_5) generates a Sylow 3-subgroup, so there are

$$|\begin{pmatrix}1 & 2 & 3\end{pmatrix}^{A_5}| = \frac{5 \cdot 4 \cdot 3}{3} = 20$$

different 3-cycles. If we take any 3-cycle and generate its subgroup, it will contain itself, its square (another 3-cycle), and the identity. Thus, each 3-cycle has a pair which generates the same cyclic subgroup, so in fact there are $20/2 = 10$ different subgroups of order 3 in A_5. That is, $n_3 = 10$. So, we have $10 \cdot 2 = 20$ new non-identity elements, giving a total of $36 - 20 = 16$ elements which belong to Sylow 2-subgroups.

For Sylow 2-subgroups, consider first V_4 in A_4 (see 1.d), given explicitly by

$$V_4 \cong \{1, \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}, \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix}, \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix}\}.$$

Note that as A_4 can be identified in A_5, so can V_4. So, this is a Sylow 2-subgroup (as $|V_4| = 4$). As Sylow subgroups are conjugates of each other, we seek to find the conjugates of V_4. In particular, recall that conjugates in S_n are transpositions of the elements inside the cycles, so we go through and find the conjugates of $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}$ that do not already lie in V_4:

$$\begin{gathered}
\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 5\end{pmatrix}\\
\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}4 & 5\end{pmatrix}\\
\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}4 & 5\end{pmatrix}\\
\begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}4 & 5\end{pmatrix}.
\end{gathered}$$

We know that each of these represents a distinct conjugation of V_4 as they each share some 2-cycle, so they cannot lie in the same conjugate of V_4. There are no other ways to accomplish this, hence we have found each Sylow 2-subgroup: V_4 and its 4 conjugates. To summarize,

$$n_2 = 5 \qquad n_3 = 10 \qquad n_5 = 6.$$

PART B.

$$G = S_5.$$

Solution. We have

$$S_5 = |5!| = 120 = 2^3(15) = 3(40) = 5(24).$$

We see $n_2 \mid 15$ and $n_2 \equiv_2 1$, so either $n_2 = 1, 3, 5$, or $n_2 = 15$. Then, $n_3 \mid 40$ and $n_3 \equiv_3 1$, so $n_3 = 1, 4, 10$, or $n_3 = 40$. Lastly, $n_5 \mid 24$ and $n_5 \equiv_5 1$, so $n_5 = 1$ or $n_5 = 6$. Recall that if $Q \in \mathrm{Syl}_p(A_5)$ and then there is some $P \in \mathrm{Syl}_p(S_5)$ so $Q \leq P$. As $A_5 \leq S_5$, this means $n_p(A_5) \leq n_p(S_5)$, so we in fact know

$$n_2 = 5 \text{ or } n_2 = 15 \qquad n_3 = 10 \text{ or } n_3 = 40 \qquad n_5 = 6.$$

Recall then that the Sylow 3-subgroups of A_5 were simply 3-cycles, which themselves are in S_5, so in fact $n_3(S_5) = 10$ as well. Thus, with reasoning from 2.a applied to 5 and 3 both being prime, we obtain

$$n_5 \cdot 4 + n_3 \cdot 2 = 6 \cdot 4 + 10 \cdot 2 = 44$$

distinct non-identity elements in S_5. Now, consider the best possible case for the Sylow 2-subgroups: that they are all disjoint from each other (less the identity). If we have $n_2 = 5$, this gives us $5 \cdot 7 = 35$ new non-identity elements in S_5, meaning

$$|S_5| = 1 + 35 + 44 = 80 = 120$$

which is not true. Hence, we must have $n_2 = 15$.

PROBLEM 5.

Prove that if $G_1 \leq G_2 \leq \cdots$ is a chain of simple groups, then

$$G = \bigcup_{i=1}^{\infty} G_i$$

is also simple.

Proof. Observe that if the chain is eventually constant (that is, there is some natural n so $G_i = G_{i+1}$ for each $i \geq n$), we are done as $G = G_n$ which is simple.

So, we can suppose, without loss of generality, that every inclusion is proper (that is, $G_1 < G_2 < \cdots$).

Now, say G were not simple, and let $1 \neq H \lhd G$. We know that $H \supsetneq \bigcup_{i=1}^{n} G_i$ for each $n \in \mathbb{N}$, for suppose H was equal to or contained in the union. Then, since $g^{-1}hg \in H$ for all $g \in G$, it is true in particular for all $g \in G_{n+1}$. However, $H \subseteq G_n < G_{n+1}$, so then $1 \neq H \lhd G_{n+1}$, giving contradiction as G_{n+1} is simple. So, we must have that

$$H \supseteq \lim_{n\to\infty} \bigcup_{i=1}^{n} G_i = G$$

however this can only be true if $H = G$, while we assumed H to be proper. This is contradictory, so in fact G must be simple. □

PROBLEM 6.

Let $n \geq 5$.

PART A.

Prove A_n does not have a proper subgroup of index less than n.

Proof. Suppose the contrary to obtain a proper subgroup H so $|A_n : H| =: m < n$. We know A_n acts on the set of left cosets of H by left-multiplication, $\sigma \cdot \tau H = (\sigma\tau)H$ where $\sigma, \tau \in A_n$. If the kernel were trivial, then by an argument identical to that of Cayley's theorem, we would conclude A_n embeds into $S_{|A_n:H|} = S_m$. However, this would mean

$$|A_n| = \frac{n!}{2} \mid |S_m| = m!$$

which is not true for $n \geq 5$, giving contradiction. So, the kernel is non-trivial.

However, the kernel is a normal subgroup, and as A_n is simple for $n \geq 5$, we must have that the kernel is all of A_n. Then, every permutation in A_n stabilizes every left coset of H, and so in particular H itself. However, $1 \in H$, and as $\sigma H = H$ for all $\sigma \in A_n$, we have that $\sigma \in H$, and so $H = A_n$. However, this means that the subgroup is not proper, giving contradiction, hence no such subgroup exists. □

PART B.

Find all normal subgroups of S_n.

Solution. We have immediately that S_n and 1 are normal in S_n. Moreover, by parity, we know that conjugation of an even number of transpositions will result in another even number of transpositions, meaning that A_n is also normal in S_n.

Now, suppose there existed H, a normal subgroup distinct from the above. We know $H \cap A_n \trianglelefteq A_n$ as they are both normal. However, A_n is simple (for $n \geq 5$), so $H \cap A_n = 1$ or $H \cap A_n = A_n$. Since $H \neq 1$, the first case implies that H

consists entirely of odd permutations (and the trivial permutation). However, parity shows that conjugating an odd permutation with an even permutation gives an even permutation, so H cannot be normal. Moreover, since $H \neq A_n$, the second case implies that $A_n \subsetneq H$, but this would imply that $1 < |S_n : H| < 2$, for recall $H \neq S_n$ and that $|A_n| = n!/2$. Therefore, both cases are impossible, so no such H exists, and we are left with only the initial three normal subgroups.

part c.

Prove A_n is the only proper subgroup of index less than n in S_n.

Proof. Suppose we had some $1 < H < S_n$ so $|S_n : H| =: m < n$. Identically to the proof of 6.a, we deduce that if S_n were to embed into $S_{|S_n:H|} = S_m$, we would have contradiction as this would imply $n! \mid m!$ but $m < n$. So, the kernel of the action of left-multiplication on the set of left cosets of H is non-trivial. If the kernel were to be all of S_n, we would again get contradiction as it would imply $H = S_n$, similar to 6.a. As kernels are normal, 6.b implies the kernel is A_n. Then, every $\sigma \in A_n$ stabilizes every left coset, and so H in particular, and so $\sigma H = H$ always. As $1 \in H$, we have $H \supseteq A_n$. This inclusion, along with the fact that

$$1 < |S_n : H| \leq 2$$

means we must have $H = A_n$. □

problem 7.

part a.

Give a non-normal subgroup of $Q_8 \times Z_4$.

Solution. Consider $\langle (i, 1) \rangle$. Observe that

$$(j, 0)^{-1}(i, 1)(j, 0) = (-j, 0)(k, 1) = (-i, 1).$$

However,

$$\langle (i, 1) \rangle = \{(i, 1), (-1, 2), (-i, 3), (1, 0)\rangle \not\ni (-i, 1)$$

so the group is not normal.

part b.

Show that all subgroups of $G = Q_8 \times E_{2^n}$ are normal.

Proof. Suppose $H \leq G$. Let $(x, y) \in H$ and $(q, z) \in G$. Recall that $E_{2^n} = \prod_{i=1}^{n} Z_2$, so we can consider the i-th component of z and y (call them z_i and y_i, respectively). As Z_2 is Abelian, we will have $z_i^{-1} y_i z_i = y_i$, and as i was arbitrary, $z^{-1} y z = y$.

Consider now x and q. If $x = 1$, then $q^{-1} x q = 1 = x$. If $x = -1$, then we have $q^{-1} x q = -1 = x$. If $x \in \{\pm i\}$, then from assignment 3 problem 3.b we know that

$q^{-1}xq = x^q \in \{\pm i\}$, and likewise if $x \in \pm\{i, j\}$. That is, $q^{-1}xq \in \{\pm x\}$ always. So, we have

$$(q, z)^{-1}(x, y)(q, z) \in \{(x, y), (-x, y)\}.$$

However, $y = -y$ in Z_2, and $(-x, -y) = (x, y)^{-1}$. So,

$$(q, z)^{-1}(x, y)(q, z) \in \{(x, y), (x, y)^{-1}\} \in H$$

hence $H \trianglelefteq G$. □

problem 8.

part a.

Let I be a (non-empty) indexing set with G_i a group for each $i \in I$. Prove

$$H = \coprod_{i \in I} G_i \trianglelefteq \prod_{i \in I} G_i = G.$$

Proof. Let $h \in H$ and $g \in G$, and h_i and g_i denote the i-th term in the (not necessarily countable) sequences h and g, respectively. We know that there exists some $i_0 \in I$ such that for each $i \geq i_0$, we have $h_i = 1$. So, for $i \geq i_0$, we have

$$g_i^{-1} h_i g_i = g^{-1}(1) g_i = 1.$$

That is, almost every term of $g^{-1}hg$ will be the identity, hence $g^{-1}hg \in H$, and so $H \trianglelefteq G$. □

part b.

Let (p_i) be the sequence of primes, and $G_i = Z_{p_i}$. Prove $H = \coprod_{i \in \mathbb{N}} G_i$ is the torsion subgroup of $G = \prod_{i \in \mathbb{N}} G_i$.

Proof. It is clear that G is Abelian as each Z_{p_i} is, and operations are done component-wise. Now, consider an arbitrary $h \in H$ and let h_i be the i-th component. We know there exists some i_0 such that for all $i > i_0$, $h_i = 1$. Therefore, for all such i, $h_i^n = 1$ for all $n \in \mathbb{N}$. Now, let $d = \operatorname{lcm}(p_1, \ldots, p_{i_0})$. Consider some i so $1 \leq i \leq i_0$. We can write $d = p_i n$ for some $n \in \mathbb{N}$ by definition of lcm, and then we have $h_i^d = (h_i^{p_i})^n = 1$ (as $h_i \in Z_{p_i}$). That is, we have showed $h^d = 1$, and hence $h \in T(G)$. As h was arbitrary, $H \subseteq T(G)$.

Now, take some $t \in T(G)$, and let $n = |t|$. Suppose $t \notin H$. Find some prime $p_i > n$. Then, as t is not in the direct sum, we can find some $j \geq i$ so the component $t_j \neq 1$. Observe that $p_j \geq p_i > n$. However, this means $t_j^n \neq 1$ since $t_j \in Z_{p_j}$ and $t_j \neq 1$. However, this contradicts the fact that $|t| = n$. So, we must have $t \in H$, and as t was arbitrary, we know $T(G) = H$, as desired. □