Exercise 12.1

Solve the following Linear Programming Problems graphically:

1. Maximise Z = $3x + 4y$
subject to the constraints: $x + y \le 4, x \ge 0, y \ge 0$.

Sol. Maximise Z = $3x + 4y$ ...(*i*)

subject to the constraints:

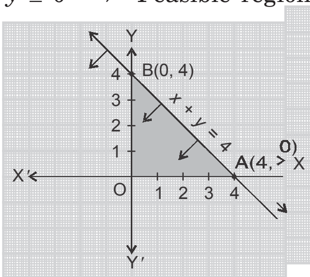
$$x + y \le 4 \quad ...(ii)$$

$$x \ge 0,\ y \ge 0 \quad ...(iii)$$

Step I. Constraint (*iii*) namely $x \ge 0, y \ge 0$ $\Rightarrow$ Feasible region is in first quadrant.

Table of values for line $x + y = 4$ corresponding to constraint (*ii*)

x	0	4
y	4	0



So let us draw the line joining the points (0, 4) and (4, 0).

Now let us test for origin ($x = 0$, $y = 0$) in constraint (*ii*) $x + y \le 4$. This gives us $0 \le 4$ which is true. Therefore region for constraint (*ii*) is on the origin side of the line.

The shaded region in the figure is the feasible region determined by the system of constraints (*ii*) and (*iii*). The feasible region OAB is bounded.

Step II. The coordinates of the corner points O, A and B are (0, 0), (4, 0) and (0, 4) respectively.

Step III. Now we evaluate Z at each corner point.

Corner Point	Z = $3x + 4y$	
O(0, 0)	0	
A(4, 0)	12	
B(0, 4)	16 = M	← Maximum

Hence, by Corner Point Method, the maximum value of Z is 16 attained at the corner point B(0, 4).$\Rightarrow$Maximum Z = 16 at (0, 4).

2. Minimise Z = – 3x + 4y
subject to $x + 2y \leq 8$, $3x + 2y \leq 12$, $x \geq 0$, $y \geq 0$.

Sol. Minimise Z = – 3x + 4y ...(*i*)

subject to: $x + 2y \leq 8$ ...(*ii*), $3x + 2y \leq 12$ ...(*iii*), $x \geq 0$, $y \geq 0$..(*iv*)

Step I. Constraint (*iv*) namely $x \geq 0$, $y \geq 0$ $\Rightarrow$ Feasible region is in first quadrant.

Table of values for line $x + 2y = 8$ of constraint (*ii*)

x	0	8
y	4	0

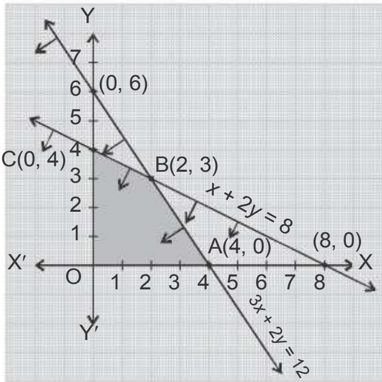
Let us draw the line joining the points (0, 4) and (8, 0).

Now let us test for origin (0, 0) in constraint (*ii*) which gives $0 \leq 8$ which is true.

$\therefore$ Region for constraint (*ii*) is on the origin side of the line.

Table of values for line $3x + 2y = 12$ of constraint (*iii*)

x	0	4
y	6	0



Let us draw the line joining the points (0, 6) and (4, 0).

Now let us test for origin (0, 0) in constraint (*iii*) which gives $0 \leq 12$ and which is true.

$\therefore$ Region for constraint (*iii*) is also on the origin side of the line. The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*iv*). The feasible region OABC is bounded.

Step II. The coordinates of the corner points O, A and C are (0, 0), (4, 0) and (0, 4) respectively.

Now let us find corner point B, intersection of lines

$$x + 2y = 8 \quad \text{and} \quad 3x + 2y = 12$$

Subtracting $2x = 4 \Rightarrow x = \frac{4}{2} = 2$.

Putting $x = 2$ in first equation $2 + 2y = 8$

$\Rightarrow 2y = 6 \Rightarrow y = 3$

$\therefore$ Corner point B is (2, 3)

Step III. Now let us evaluate Z at each corner point.

Corner Point	Z = – 3x + 4y	
O(0, 0)	0	
A(4, 0)	– 12 = m	← Minimum
B(2, 3)	6	
C(0, 4)	16	

Hence, by Corner Point Method, the minimum value of Z is – 12 attained at the point A(4, 0).

$\Rightarrow$ Minimum Z = – 12 at (4, 0).

3. Maximise Z = 5*x* + 3*y*
subject to $3x + 5y \le 15$, $5x + 2y \le 10$, $x \ge 0$, $y \ge 0$.

Sol. Maximise $Z = 5x + 3y$ ...(*i*)

subject to:

$3x + 5y \le 15$ ...(*ii*)

$5x + 2y \le 10$ ...(*iii*)

$x \ge 0, y \ge 0$ ...(*iv*)

Step I. Constraint (*iv*) namely $x \ge 0$ and $y \ge 0$

$\Rightarrow$ Feasible region is in first quadrant.

Table of values for line $3x + 5y = 15$ of constraint (*ii*)

x	0	5
y	3	0

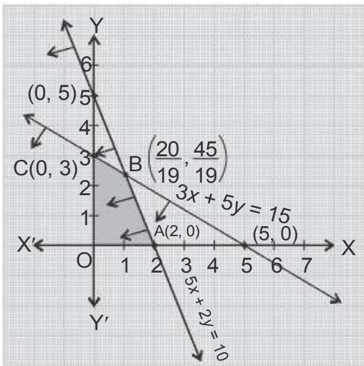
Let us draw the line joining the points (0, 3) and (5, 0).

Let us test for origin (0, 0) in constraint (*ii*) which gives $0 \le 15$ and which is true.

$\therefore$ Region for constraint (*ii*) contains the origin.

Table of values for line $5x + 2y = 10$ of constraint (*iii*).

x	0	2
y	5	0



Let us draw the line joining the points (0, 5) and (2, 0).

Let us test for origin (0, 0) in constraint (*iii*) which gives $0 \le 10$ and which is true.

$\therefore$ Region for constraint (*iii*) also contains the origin.

The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) and (*iv*). The feasible region OABC is bounded.

Step II. The coordinates of the corner points O, A and C are (0, 0), (2, 0) and (0, 3) respectively.

Now **let us find corner point B;** intersection of lines

$3x + 5y = 15$ and $5x + 2y = 10$

Ist eqn. × 2 – IInd eqn. × 5 gives $-\ 19x = -20 \Rightarrow x = \dfrac{20}{19}$

Putting $x = \dfrac{20}{19}$ in first eqn. $\Rightarrow$ $\dfrac{60}{19} + 5y = 15$

$$\Rightarrow \qquad 5y = 15 - \frac{60}{19} = \frac{285 - 60}{19} = \frac{225}{19}$$

$\Rightarrow \; y = \dfrac{45}{19}$. Therefore corner point $B\left(\dfrac{20}{19}, \dfrac{45}{19}\right)$.

Step III. Now we evaluate Z at each corner point.

Corner Point	$Z = 5x + 3y$	
$O(0, 0)$	0	
$A(2, 0)$	10	
$B\left(\dfrac{20}{19}, \dfrac{45}{19}\right)$	$\dfrac{100+135}{19} = \dfrac{235}{19} = M$	← Maximum
$C(0, 3)$	9	

Hence, by Corner Point Method, the maximum value of Z is $\dfrac{235}{19}$ attained at the corner point $B\left(\dfrac{20}{19}, \dfrac{45}{19}\right)$.

$\Rightarrow$ Maximum $Z = \dfrac{235}{19}$ at $\left(\dfrac{20}{19}, \dfrac{45}{19}\right)$.

4. Minimise $Z = 3x + 5y$

such that $x + 3y \geq 3$, $x + y \geq 2$, $x, y \geq 0$.

Sol. Minimise $Z = 3x + 5y$ ...(*i*)

such that: $x + 3y \geq 3$ ...(*ii*), $x + y \geq 2$ ...(*iii*), $x, y \geq 0$ ...(*iv*)

Step I. The constraint (*iv*) $x, y \geq 0 \Rightarrow$ Feasible region is in first quadrant.

Table of values for line $x + 3y = 3$ of constraint (*ii*)

x	0	3
y	1	0

Let us draw the line joining the points (0, 1) and (3, 0).

Now let us test for origin ($x = 0, y = 0$) in constraint (*ii*) $x + 3y \geq 3$, which gives us $0 \geq 3$ and which is not true.

$\therefore$ Region for constraint (*ii*) does not contain the origin *i.e.,* the region for constraint (*ii*) is **not** the origin side of the line.

Table of values for line $x + y = 2$ of constraint (*iii*)

x	0	2
y	2	0

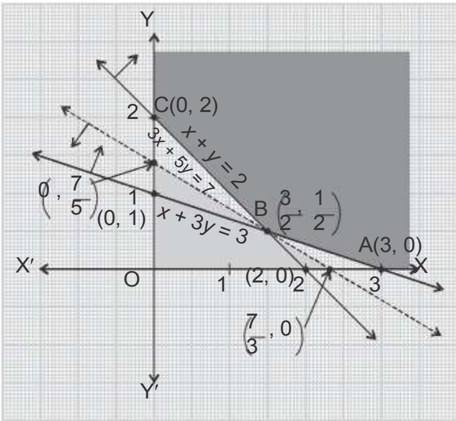
Let us draw the line joining the points (0, 2) and (2, 0).

Now let us test for origin ($x = 0, y = 0$) in constraint (*iii*), $x + y \geq 2$, which gives us $0 \geq 2$ and which is not true.

$\therefore$ Region for constraint (*iii*) does not contain the origin *i.e.,* is **not** the origin side of the line.

The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*iv*). The feasible region is unbounded.

Step II. The coordinates of the corner points A and C are (3, 0) and (0, 2) respectively.



Now let us find corner point B, the point of intersection of lines

$$x + 3y = 3 \qquad \text{and} \qquad x + y = 2$$

Subtracting, $2y = 1 \Rightarrow y = \frac{1}{2}$.

Putting $y = \frac{1}{2}$ in $x + y = 2$, we have $x = 2 - y = 2 - \frac{1}{2} = \frac{3}{2}$

$\therefore$ Corner point B is $\left(\frac{3}{2}, \frac{1}{2}\right)$.

Step III. Now, we evaluate Z at each corner point.

Corner Point	$Z = 3x + 5y$	
A(3, 0)	9	
$B\left(\frac{3}{2}, \frac{1}{2}\right)$	$\frac{9}{2} + \frac{5}{2} = 7 = m$	← Smallest
C(0, 2)	10	

From this table, we find that 7 is the smallest value of Z at the corner $B\left(\frac{3}{2}, \frac{1}{2}\right)$. Since the feasible region is unbounded, 7 may or may not be the minimum value of Z.

Step IV. To decide this, we graph the inequality $Z < m$

i.e., $\quad 3x + 5y < 7$.

Table of values for line $3x + 5y = 7$ corresponding to constraint $3x + 5y < 7$

x	0	$\frac{7}{3}$
y	$\frac{7}{5}$	0

Let us draw the dotted line joining the

points $\left(0, \frac{7}{5}\right)$ and $\left(\frac{7}{3}, 0\right)$. This line is to be shown dotted as constraint involves < and not ≤, so boundary of line is to be excluded.
Let us test for origin ($x = 0, y = 0$) in constraint $3x + 5y < 7$, we have $0 < 7$ which is true. Therefore region for this constraint is on the origin side of the line $3x + 5y = 7$.
We observe that the half-plane determined by $Z < m$ has no point in common with the feasible region. Hence $m = 7$ is the minimum value of Z attained at the point $B\left(\frac{3}{2}, \frac{1}{2}\right)$.

$\Rightarrow$ Minimum $Z = 7$ at $\left(\frac{3}{2}, \frac{1}{2}\right)$.

5. Maximise $Z = 3x + 2y$

subject to $x + 2y \le 10$, $3x + y \le 15$, $x, y \ge 0$.

Sol. Maximise $Z = 3x + 2y$ ...(*i*)

subject to:

$x + 2y \le 10$ ...(*ii*), $3x + y \le 15$ ...(*iii*), $x, y \ge 0$ ...(*iv*)

Step I. Constraint (*iv*) $x, y \ge 0$ $\Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $x + 2y = 10$ corresponding to constraint (*ii*)

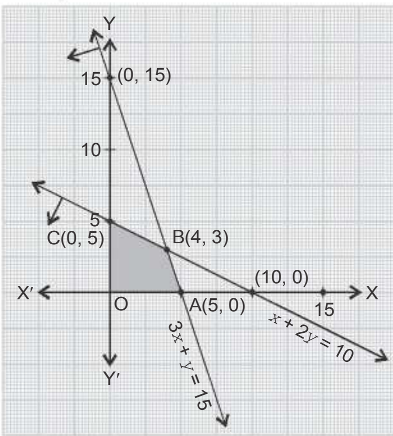
x	0	10
y	5	0

Let us draw the line joining the points (0, 5) and (10, 0).
Let us test for origin ($x = 0, y = 0$) in constraint (*ii*), we have $0 \le 10$ which is true.
$\therefore$ Region for constraint (*ii*) is on the origin side of this line.

Table of values for line $3x + y = 15$ corresponding to constraint (*iii*)

x	0	5
y	15	0

Let us draw the line joining the points (0, 15) and (5, 0).
Let us test for origin ($x = 0, y = 0$) in constraint (*iii*), we have $0 \le 15$ which is true.
$\therefore$ Region for constraint (*iii*) is on the origin side of this line.
The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*iv*). The feasible region OABC is bounded.



Step II. The coordinates of the corner points O, A and C are (0, 0), (5, 0) and (0, 5) respectively.

Now let us find corner point B, intersection of the lines

$$x + 2y = 10$$

and $$3x + y = 15$$

First equation – 2 × second equation gives

$-5x = 10 - 30 \Rightarrow -5x = -20 \Rightarrow x = 4$

Putting $x = 4$ in $x + 2y = 10$, we have

$4 + 2y = 10 \Rightarrow 2y = 6 \Rightarrow y = 3$

$\therefore$ Corner point B is B(4, 3).

Step III. Now we evaluate Z at each corner point.

Corner Point	$Z = 3x + 2y$	
O(0, 0)	0	
A(5, 0)	15	
B(4, 3)	18 = M	← Maximum
C(0, 5)	10	

Hence, by Corner Point Method, the maximum value of Z is 18 attained at the point B(4, 3).

$\Rightarrow$ Maximum Z = 18 at (4, 3).

6. Minimise Z = $x + 2y$
subject to $2x + y \ge 3$, $x + 2y \ge 6$, $x, y \ge 0$.
Show that the minimum of Z occurs at more than two points.

Sol. Minimise $Z = x + 2y$ ...(*i*)

subject to:

$2x + y \ge 3$ ...(*ii*), $x + 2y \ge 6$ ...(*iii*), $x, y \ge 0$ ...(*iv*)

Step I. Constraint (*iv*) $x, y \ge 0 \Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $2x + y = 3$ corresponding to constraint (*ii*).

x	0	$\frac{3}{2}$
y	3	0

Let us draw the line joining the points (0, 3) and $\left(\frac{3}{2}, 0\right)$.

Now let us test for origin ($x = 0, y = 0$) in constraint (*ii*) $2x + y \ge 3$, we have $0 \ge 3$ which is not true.

$\therefore$ The region of constraint (*ii*) is on that side of the line which does not contain the origin *i.e.,* the region other than the origin side of the line.

Table of values for the line $x + 2y = 6$ corresponding to constraint (*ii*).

x	0	6
y	3	0

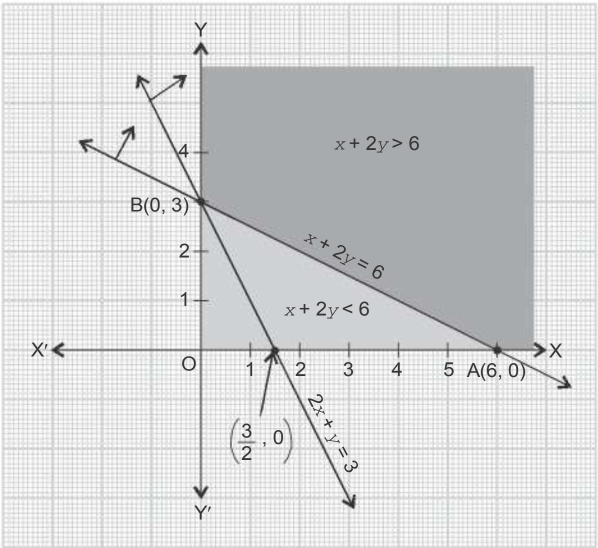
Let us draw the line joining the points (0, 3) and (6, 0).

Now let us test for origin ($x = 0, y = 0$) in constraint (iii) $x + 2y \geq 6$, we have $0 \geq 6$ which is not true.

$\therefore$ Region for constraint (iii) is the region other than the origin side of the line *i.e.,* region not containing the origin.

The shaded region in the figure is the feasible region determined by the system of constraints from (ii) to (iv). The feasible region is unbounded.

Step II. The coordinates of the corner points A and B are (6, 0) and (0, 3) respectively.



Step III. Now, we evaluate Z at each corner point.

Corner Point	$Z = x + 2y$	
A(6, 0)	6	$= m$ ← Smallest
B(0, 3)	6	

From this table, we find that 6 is the smallest value of Z at each of the two corner points. Since the feasible region is unbounded, 6 may or may not be the minimum value of Z.

Step IV. To decide this, we graph the inequality $Z < m$ *i.e.,* $x + 2y < 6$.

The line $x + 2y = 6$ for this constraint $Z < m$ ($\Rightarrow x + 2y < 6$) is the same as the line AB for constraint (iii).

Let us test for origin ($x = 0, y = 0$) for this constraint, we have $0 < 6$ which is true.

Therefore region for this constraint is the (half-plane on) origin side of this line.

Points on the line segment AB are included in the feasible region and not in the half-plane determined by $x + 2y < 6$.

We observe that the half-plane determined by $Z < m$ has no point in common with the feasible region. Hence $m = 6$ is the minimum

value of Z attained at each of the points A(6, 0) and B(0, 3).
$\Rightarrow$ Minimum Z = 6 at (6, 0) and (0, 3).

Remark. In fact, Z = 6 at all points on the line segment AB for example $\left(1, \frac{5}{2}\right)$, (2, 2), $\left(3, \frac{3}{2}\right)$ etc.

7. Minimise and Maximise Z = 5*x* + 10*y* subject to $x + 2y \le 120$, $x + y \ge 60$, $x - 2y \ge 0$, $x, y \ge 0$.

Sol. Minimise and Maximise $Z = 5x + 10y$ ...(*i*)

subject to: $x + 2y \le 120$ ...(*ii*)

$x + y \ge 60$ ...(*iii*), $x - 2y \ge 0$ ...(*iv*), $x, y \ge 0$ ...(*v*)

Step I. Constraint (*v*) $x, y \ge 0$ $\Rightarrow$ Feasible region is in first quadrant.

Table of values for line $x + 2y = 120$ of constraint (*ii*)

x	0	120
y	60	0

Let us draw the line joining the points (0, 60) and (120, 0).
Let us test for origin ($x = 0$, $y = 0$) in constraint (*iii*) $x + 2y \le 120$ we have $0 \le 120$ which is true.

$\therefore$ Region for constraint (*ii*) is on the origin side of the line

$$x + 2y = 120.$$

Table of values for line $x + y = 60$ of constraint (*iii*)

x	0	60
y	60	0

Let us draw the line joining the points (0, 60) and (60, 0).
Let us test for origin ($x = 0, y = 0$) in constraint (*iii*) $x + y \ge 60$, we have $0 \ge 60$ which is not true.

$\therefore$ Region for constraint (*iii*) is the half-plane on the non-origin side of the line $x + y = 60$ (*i.e.,* on the side of the line opposite to the origin side).

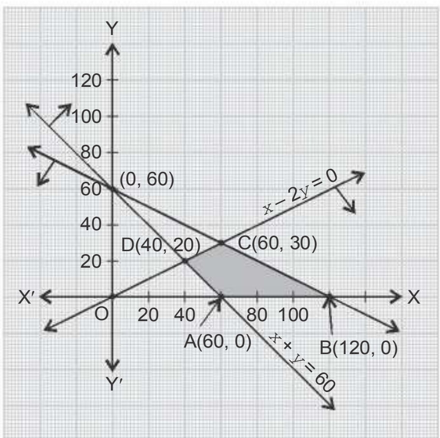
Table of values for line $x - 2y = 0$ of constraint (*iv*)

x	0	0	60
y	0	0	30

($\because$ The line $x - 2y = 0$ is passing through the origin, so we have taken still another point (60, 30) on the line).

Let us draw the line joining the points (0, 0) and (60, 30).
Let us test for (60, 0) (a point other than origin) in constraint (*iv*), we have $60 \ge 0$ which is true.

$\therefore$ Region for constraint (*iv*) is the half-plane on that side of the line which containing the point (60, 0).



The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*v*). The feasible region ABCD is bounded.

Step II. The coordinates of the corner points A and B are (60, 0) and (120, 0) respectively.

Corner point C is the intersection of the line $x - 2y = 0$ *i.e.,* $x = 2y$ and $x + 2y = 120$. Putting $x = 2y$ in $x + 2y = 120$, we have $2y + 2y = 120 \Rightarrow 4y = 120$
$\Rightarrow y = 30$ and therefore $x = 2y = 60$.
$\therefore$ Corner point C (60, 30).

Similarly for corner point D, putting $x = 2y$ in $x + y = 60$, we have $2y + y = 60 \Rightarrow 3y = 60 \Rightarrow y = 20$ and therefore $x = 2y = 40$. Therefore corner point D is (40, 20).

Step III. Now, we evaluate Z at each corner point.

Corner Point	$Z = 5x + 10y$	
A(60, 0)	300 = m	$\leftarrow$ Minimum
B(120, 0)	600	
C(60, 30)	300 + 300 = 600 = M	$\leftarrow$ Maximum
D(40, 20)	400	

Hence, by Corner Point Method,
Minimum Z = 300 at (60, 0)
Maximum Z = 600 at B(120, 0) and C(60, 30) and hence maximum at all the points on the line segment BC joining the points (120, 0) and (60, 30).

8. Minimise and Maximise $Z = x + 2y$
subject to $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$; $x, y \geq 0$.

Sol. Minimise and Maximise $Z = x + 2y$ ...(*i*)
subject to:

$$x + 2y \geq 100 \quad ...(ii)$$
$$2x - y \leq 0 \quad ...(iii)$$
$$2x + y \leq 200 \quad ...(iv)$$
$$x, y \geq 0 \quad ...(v)$$

Step I. The constraint (*v*) $x, y \geq 0 \Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $x + 2y = 100$ for constraint (*ii*).

x	0	100
y	50	0

Let us draw the line joining the points (0, 50) and (100, 0).
Let us test for origin ($x = 0$, $y = 0$) in constraint (*ii*) $x + 2y \geq 100$, we have $0 \geq 100$ which is not true.
$\therefore$ Region for constraint (*i*) is that half-plane which does not contain the origin.

Table of values for the line $2x - y = 0$ *i.e.,* $2x = y$ of constraint (*iii*).

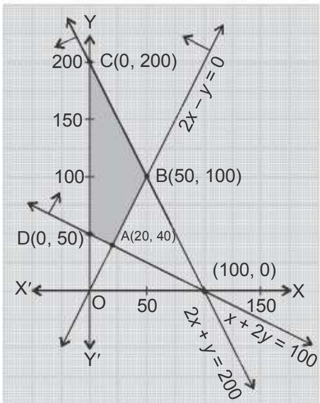
x	0	20
y	0	40

Let us draw the line joining the points (0, 0) and (20, 40).
Because this line passes through the origin, so we shall have the test for some point say (100, 0) other than the origin.
Putting $x = 100$ and $y = 0$ in constraint (*iii*) $2x - y \le 0$, we have $200 \le 0$ which is not true.
$\therefore$ Region for constraint (*iii*) is the half plane on the side of the line which does not contain the point (100, 0).

Table of values for the line $2x + y = 200$ of constraint (*iv*).

x	0	100
y	200	0

Let us draw the line joining the points (0, 200) and (100, 0).
Let us test for origin ($x = 0$, $y = 0$) in constraint (*iv*) $2x + y \le 200$, we have $0 \le 200$ which is true. Therefore region for constraint (*iv*) is the half-plane on origin side of the line.
The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*v*). The feasible region ABCD is bounded.



Step II. The coordinates of the two corner points are C(0, 200) and D(0, 50).
Corner point A is the intersection of boundary lines $x + 2y = 100$ and $2x - y = 0$ *i.e.,* $y = 2x$.
Solving them, putting $y = 2x$, $x + 4x = 100$
$\Rightarrow$ $5x = 100$ $\Rightarrow$ $x = 20$.
$\therefore$ $y = 2x = 2 \times 20 = 40$.
Therefore corner point A(20, 40).
Corner point B is the intersection of the boundary lines $2x + y = 200$ and $2x - y = 0$ *i.e.,* $y = 2x$.
Solving them, putting $y = 2x$, $2x + 2x = 200 \Rightarrow 4x = 200$
$\Rightarrow$ $x = 50$ and therefore $y = 2x = 100$. Therefore corner point B is (50, 100).

Step III. Now, we evaluate Z at each corner point.

Corner Point	$Z = x + 2y$	
A(20, 40)	$100 = m$	$\leftarrow$ Minimum
B(50, 100)	250	
C(0, 200)	$400 = M$	$\leftarrow$ Maximum
D(0, 50)	$100 = m$	$\leftarrow$ Minimum

By Corner Point Method,
Minimum Z = 100 at all the points on the line segment joining the points (20, 40) and (0, 50).
(See Step III, Example 7, Page 770.
Maximum Z = 400 at (0, 200).

9. Maximise Z = – x + 2y, subject to the constraints:
$x \geq 3$, $x + y \geq 5$, $x + 2y \geq 6$, $y \geq 0$.

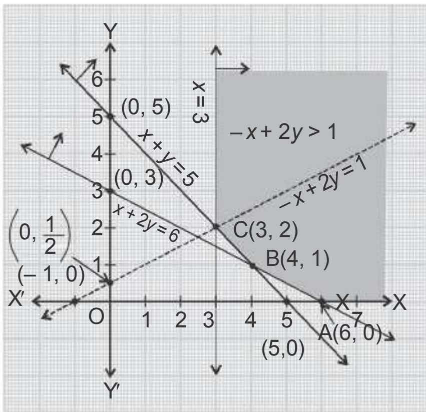
Sol. Maximise $Z = -x + 2y$ ...(i)
subject to the constraints:
$x \geq 3$ ...(ii), $x + y \geq 5$ ...(iii), $x + 2y \geq 6$ ...(iv), $y \geq 0$ ...(v)
Step I. Constraint (v), $y \geq 0$ $\Rightarrow$ Positive side of y-axis
$\Rightarrow$ Feasible region is in first and second quadrants.
Region for constraint (ii) $x \geq 3$.
We know that graph of the line $x = 3$ is a vertical line parallel to y-axis at a distance 3 from origin along OX.
$\therefore$Region for $x \geq 3$ is the half-plane on right side of the line $x = 3$.
Table of values for line $x + y = 5$ of constraint (iii)

x	0	5
y	5	0

Let us draw the line joining the points (0, 5) and (5, 0).
Let us test for origin (0, 0) in constraint (ii).
Putting $x = 0$ and $y = 0$ in $x + y \geq 5$, we have $0 \geq 5$ which is not true.
$\therefore$ Region for constraint (iii) is the half plane on the non-origin side of the line $x + y = 5$.
Table of values for the line $x + 2y = 6$ of constraint (iii)

x	0	6
y	3	0

Let us test for origin (0, 0) in constraint (iv) $x + 2y \geq 6$, we have $0 \geq 6$ which is not true.
$\therefore$ Region for constraint (iv) is again the half plane on the non-origin side of the line $x + 2y = 6$.
The shaded region in the figure is the feasible region determined by the system of constraints from (ii) to (v). The feasible region is unbounded.



Step II. The coordinates of the corner point A are (6, 0).
Corner point B is the intersection of the boundary lines
$x + y = 5$ and $x + 2y = 6$
Let us solve them for x and y.
Subtracting the two equations $2y - y = 6 - 5$ or $y = 1$.
Putting $y = 1$ in $x + y = 5$, we have $x + 1 = 5$ or $x = 4$. Therefore, vertex B is (4, 1).
Corner point C is the intersection of the boundary lines $x + y = 5$ and $x = 3$.

Solving for x and y; putting $x = 3$ in $x + y = 5$; $3 + y = 5$ or $y = 2$. Therefore corner point C is (3, 2).

Step III. Now, we evaluate Z at each corner point.

Corner Point	$Z = -x + 2y$	
A(6, 0)	-6	
B(4, 1)	-2	
C(3, 2)	$1 = M$	← Maximum

From this table, we find that 1 is the maximum value of Z at (3, 2).

Step IV. Since the feasible region is unbounded, 1 may or may not be the maximum value of Z. To decide this, we graph the inequality $Z > M$ *i.e.,* $-x + 2y > 1$.

Table of values for the line $-x + 2y = 1$ corresponding to constraint $Z > M$ *i.e.,* $-x + 2y > 1$.

x	0	-1
y	$\frac{1}{2}$	0

Let us draw the **dotted** line joining the points $\left(0, \frac{1}{2}\right)$ and $(-1, 0)$. The line is to be shown dotted because boundary of the line is to be excluded as equality sign is missing in the constraint $Z > M$. We observe that the half-plane determined by $Z > M$ has points in common with the feasible region. Therefore, $Z = -x + 2y$ has no maximum value subject to the given constraints.

10. Maximise $Z = x + y$,
subject to $x - y \le -1$, $-x + y \le 0$, $x, y \ge 0$.

Sol. Maximise $Z = x + y$ ...(*i*)

subject to:

$x - y \le -1$ ...(*ii*), $-x + y \le 0$ ...(*iii*), $x, y \ge 0$ ...(*iv*)

Step I. Constraint (*iv*) $x, y \ge 0$.

$\Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $x - y = -1$ of constraint (*ii*)

x	0	-1
y	1	0

Let us draw the straight line joining the points (0, 1) and (– 1, 0).

Let us test for origin (0, 0) in constraint (*ii*) $x - y \le -1$, we have $0 \le -1$ which is not true.

Therefore region for constraint (*ii*) is the region on that side of the line which is away from the origin (as shown shaded in the figure)

Table of values for the line $-x + y = 0$ *i.e.,* $y = x$ of constraint (*iii*)

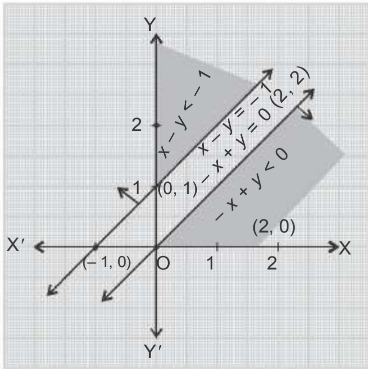
x	0	2
y	0	2

Let us draw the line joining the points (0, 0) and (2, 2).

Let us test for the point (2, 0) (say) [and not origin as line passes through (0, 0)] in constraint (*iii*) $-x + y \le 0$, we have $-2 \le 0$ which is true.

$\therefore$ Region for constraint (*iii*) is towards the point (2, 0) side of the line (shown shaded in the figure).

From the figure, we observe that there is no point common in the two shaded regions. Thus, the problem has no feasible region and hence no feasible solution *i.e.,* no maximum value of Z.



Exercise 12.2

1. Reshma wishes to mix two types of food P and Q in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food P costs ₹ 60/kg and Food Q costs ₹ 80/kg. Food P contains 3 units/kg of vitamin A and 5 units/kg of vitamin B while food Q contains 4 units/kg of vitamin A and 2 units/kg of vitamin B. Determine the minimum cost of the mixture.

Sol. Step I. Mathematical formulation of L.P.P.

Suppose Reshma mixes x kg of food P and y kg of food Q. The given data is condensed in the following table:

Type of Food	Quantity (kg)	Cost (₹/kg)	Vitamin A (units/kg)	Vitamin B (units/kg)
P	x	60	3	5
Q	y	80	4	2

Cost of mixture (in ₹) = $60x + 80y$

Let $Z = 60x + 80y$

We have the following mathematical model for the given problem:

Minimise $Z = 60x + 80y$ ...(*i*)

subject to the constraints:

$3x + 4y \geq 8$ (Vitamin A constraint) ...(*ii*)

[**Given:** Vitamin A content of foods X and Y is at least (*i.e.*, $\geq$) 8 units]

$5x + 2y \geq 11$ (Vitamin B constraint) ...(*iii*)

[**Given:** Vitamin B content of foods X and Y is at least (*i.e.*, $\geq$) 11 units]

$x, y \geq 0$ [$\because$ Quantities of food can't be negative] ...(*iv*)

Step II. The constraint (*iv*), $x, y \geq 0$.

$\Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $3x + 4y = 8$ of constraint (*ii*)

x	0	$\frac{8}{3}$
y	2	0

Let us draw the line joining the points (0, 2) and $\left(\frac{8}{3}, 0\right)$.

Let us test for origin (x = 0, y = 0) in constraint (ii) $3x + 4y \geq 8$, we have $0 \geq 8$ which is not true.

$\therefore$ The region for constraint (ii) is the half plane on non-origin side of the line $3x + 4y = 8$ *i.e.,* the region does not contain the origin.

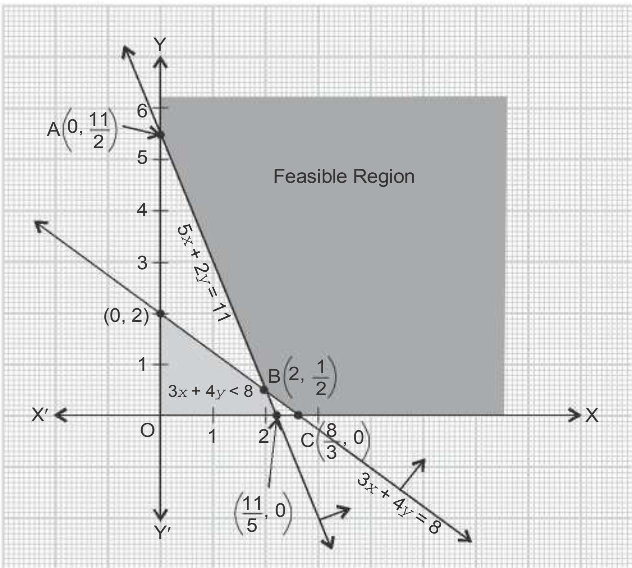
Now table of values for the line $5x + 2y = 11$ of constraint (iii).

x	0	$\frac{11}{5}$
y	$\frac{11}{2}$	0

Let us draw the line joining the points $\left(0, \frac{11}{2}\right)$ and $\left(\frac{11}{5}, 0\right)$.

Let us test for origin (x = 0, y = 0) in constraint (iii) $5x + 2y \geq 11$, we have $0 \geq 11$ which is not true.

$\therefore$ Region for constraint (iii) is on the non-origin side of the line *i.e.,* does not contain the origin.



The shaded region in the figure is the feasible region determined by the system of constraints from (ii) to (iv). The feasible region is unbounded.

Step III. The coordinates of the corner points A and C are A$\left(0, \frac{11}{2}\right)$ and C$\left(\frac{8}{3}, 0\right)$ respectively.

Corner point B; is the point of intersection of the lines

$3x + 4y = 8$ and $5x + 2y = 11$

Solve for x and y: First equation – 2 × second equation gives $3x + 4y - 10x - 4y = 8 - 22$

$\Rightarrow \quad -7x = -14 \quad \Rightarrow \quad x = 2$

Putting $x = 2$ in $3x + 4y = 8$, we have, $6 + 4y = 8 \quad \Rightarrow \quad 4y = 2$

$\Rightarrow \quad y = \frac{2}{4} = \frac{1}{2}$. Therefore vertex $B\left(2, \frac{1}{2}\right)$.

Step IV. Now, we evaluate Z at each corner point.

Corner Point	$Z = 60x + 80y$	
$A\left(0, \frac{11}{2}\right)$	440	
$B\left(2, \frac{1}{2}\right)$	160	$= m$ ← Minimum
$C\left(\frac{8}{3}, 0\right)$	160	

From this table, we find that 160 is the minimum value of Z at each of the two corner points $B\left(2, \frac{1}{2}\right)$ and $C\left(\frac{8}{3}, 0\right)$.

Step V. Since the feasible region is unbounded, 160 may or may not be the minimum value of Z. To decide this, we graph the inequality $Z < m$

i.e., $60x + 80y < 160$ or $3x + 4y < 8$

Table of values for the line $3x + 4y = 8$ for this constraint $Z < m$.

x	0	$\frac{8}{3}$
y	2	0

The line joining these two points (0, 2) and $\left(\frac{8}{3}, 0\right)$ has already been drawn for the line of constraint (*ii*).

Let us test for origin ($x = 0, y = 0$) in constraint $Z < m$

i.e., $3x + 4y < 8$, we have $0 < 8$ which is true.

$\therefore$ Region for constraint $Z < m$ in the origin side of the line $3x + 4y = 8$.

Of course points on the line segment BC are included in the feasible region ($\because$ of constraint (*ii*)) and not in the half-plane determined by $Z < m$ *i.e.,* $3x + 4y < 8$. We observe that the half-plane determined by $Z < m$ has no point in common with the feasible region. Hence $m = 160$ is the minimum value of Z attained at each of the points $B\left(2, \frac{1}{2}\right)$ and $C\left(\frac{8}{3}, 0\right)$. Therefore, minimum cost = ₹ 160 at all points lying on the segment joining $\left(2, \frac{1}{2}\right)$ and $\left(\frac{8}{3}, 0\right)$.

2. One kind of cake requires 200 g of flour and 25 g of fat, and another kind of cake requires 100 g of flour and 50 g of fat. Find the maximum number of cakes which can be made from 5 kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes.

Sol. Step I. Mathematical Formulation of L.P.P.

Let x be the number of cakes of first kind and y, the number of cakes of other kind. The given data is condensed in the following table:

Kind of cake	Number of cakes	Flour (gm/cake)	Fat (gm/cake)
I	x	200	25
II	y	100	50

Total number of cakes = $x + y$ Let $Z = x + y$

We have the following mathematical model for the given problem:

Maximise $Z = x + y$ ...(*i*)

subject to the constraints:

$200x + 100y \le 5000$

(**Given:** (Maximum) amount of flour available for both types of cakes is 5 kg = 5000 gm)

Dividing by 100,

or $2x + y \le 50$ (Flour constraint) ...(*ii*)

$25x + 50y \le 1000$

(Fat constraint)

(**Given:** (Maximum) amount of fat available for both types of cakes is 1 kg = 1000 gm)

Dividing by 25,

or $x + 2y \le 40$ (Fat constraint) ...(*iii*)

$x, y \ge 0$ ...(*iv*)

($\because$ Number of cakes can't be negative)

Step II. The constraint (*iv*) $x, y \ge 0$.

$\Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $2x + y = 50$ of constraint (*ii*)

x	0	25
y	50	0

Let us draw the line joining the points (0, 50) and (25, 0).

Let us test for origin (0, 0) ($x = 0$ and $y = 0$) in constraint (*ii*) $2x + y \le 50$, we have $0 \le 50$ which is true.

$\therefore$ Region for constraint (*ii*) is towards the origin side of the line.

Table of values for the line $x + 2y = 40$ of constraint (*iii*)

x	0	40
y	20	0

Let us draw the line joining the points (0, 20) and (40, 0).

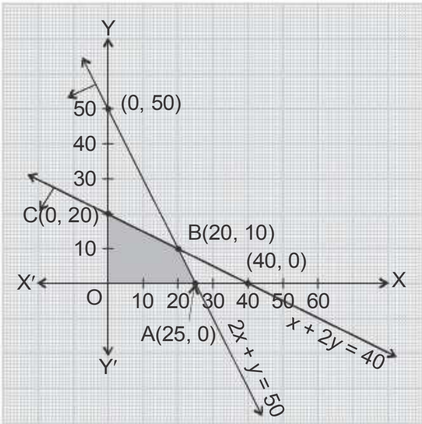
Let us test for origin ($x = 0, y = 0$) in constraint (*iii*) $x + 2y \le 40$, we have $0 \le 40$ which is true.

$\therefore$Region for constraint (*iii*) is also towards the origin side of the line.

The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*iv*). The feasible region is bounded.

Step III. The coordinates of the corner points O, A and C are (0, 0), (25, 0) and (0, 20) respectively.

Corner point B: It is the point of intersection of the boundary lines $2x + y = 50$ and $x + 2y = 40$



Let us solve them for x and y.

First equation – 2 × second equation gives

$2x + y - 2x - 4y = 50 - 80 \quad \Rightarrow \quad -3y = -30 \quad \Rightarrow \quad y = 10.$

Putting $y = 10$ in $2x + y = 50$

$\Rightarrow 2x + 10 = 50 \quad \Rightarrow \quad 2x = 40 \quad \Rightarrow \quad x = 20$

Therefore corner point B is (20, 10).

Step IV. Now we evaluate Z at each corner point.

Corner Point	$Z = x + y$	
O(0, 0)	0	
A(25, 0)	25	
B(20, 10)	30 = M	← Maximum
C(0, 20)	20	

By Corner Point Method, the maximum value of Z is 30 attained at the point B(20, 10).

Hence, maximum number of cakes = 30, 20 of first kind and 10 of second kind.

3. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftman's time in its making while a cricket bat takes 3 hours of machine time and 1 hour of craftman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time.

(*i*) What number of rackets and bats must be made if the factory is to work at full capacity?

(*ii*) If the profit on a racket and on a bat is ₹ 20 and ₹ 10 respectively, find the maximum profit of the factory when it works at full capacity.

Sol. Step I. Mathematical Formulation of L.P.P.

Suppose x is the number of tennis rackets and y is the number of cricket bats to be made in a day. The given data is condensed in the following table:

Item	Number	Machine Time (hours/item)	Craftman's Time (hours/item)	Profit (₹)
Tennis Racket	x	1.5	3	20
Cricket Bat	y	3	1	10

Total number of items = $x + y$ and total profit = $20x + 10y$

Let $Z = x + y$ and $P = 20x + 10y$

We have the following mathematical model for the given problem:

Maximise $Z = x + y$ and $P = 20x + 10y$ ...(*i*)

subject to the constraints:

$$1.5x + 3y \le 42 \quad \text{or} \quad \frac{3}{2}x + 3y \le 42$$

[**Given:** Number of machine hours available is not more than 42 hours *i.e.,* ≤ 42]

Dividing by 3 and multiplying by 2,

$x + 2y \le 28$ (Machine time constraint) ..(*ii*)

$3x + y \le 24$ (Craftman's time constraint) ...(*iii*)

[**Given:** Number of craftman's hours is not more than 24 hours *i.e.,* ≤ 24]

$x, y \ge 0$

($\because$ Number of tennis rackets and cricket bats can't be negative) ...(*iv*)

Step II. The constraint (*iv*) $x, y \ge 0 \Rightarrow$ Feasible region is in first quadrant.

Table of values of equation $x + 2y = 28$ of constraint (*ii*)

x	0	28
y	14	0

Let us draw the straight line joining the points (0, 14) and (28, 0).

Let us test for origin ($x = 0, y = 0$) in constraint (*ii*)

i.e., $x + 2y \le 28$; we have $0 \le 28$ which is true.

$\therefore$ Region for constraint (*ii*) is the region towards the origin side of the line $x + 2y = 28$.

Table of values of equation $3x + y = 24$ of constraint (*iii*)

x	0	8
y	24	0

Let us draw the line joining the points (0, 24) and (8, 0).

Let us test for origin ($x = 0, y = 0$) in constraint (*iii*) $3x + y \le 24$, we have $0 \le 24$ which is true.

$\therefore$ Region for constraint (*iii*) is the region towards the origin side of the line.

The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*iv*). The feasible region is bounded.

Step III. The coordinates of the corner points O, A and C are (0, 0), (8, 0) and (0, 14) respectively.

Corner point B: It is the point of intersection of the boundary lines

$$x + 2y = 28 \quad \text{and} \quad 3x + y = 24.$$

First eqn. – 2 × second eqn. gives

$$x + 2y - 2(3x + y) = 28 - 2 \times 24$$

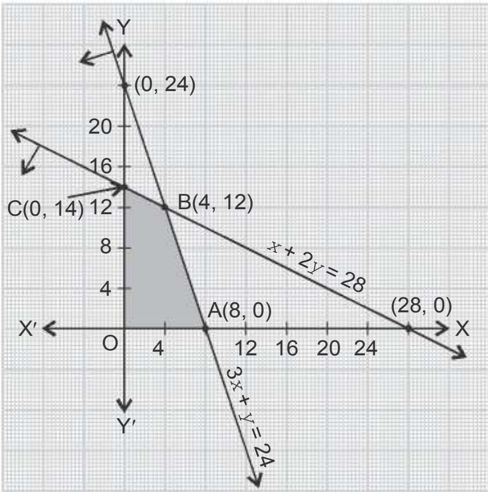
$\Rightarrow \quad x + 2y - 6x - 2y = 28 - 48 \quad \Rightarrow \quad -5x = -20$

$\Rightarrow \quad x = 4.$

Putting $x = 4$ in $x + 2y = 28$, $4 + 2y = 28$

$\Rightarrow \quad 2y = 24 \quad \Rightarrow \quad y = 12$

$\therefore$ Corner point B is (4, 12).



Step IV. (*i*) Now, we evaluate Z at each corner point.

Corner Point	$Z = x + y$	
O(0, 0)	0	
A(8, 0)	8	
B(4, 12)	16 = M	← Maximum
C(0, 14)	14	

By Corner Point Method, maximum Z = 16 at (4, 12).

(*ii*) Now, we evaluate P at each corner point.

Corner Point	$P = 20x + 10y$	
O(0, 0)	0	
A(8, 0)	160	
B(4, 12)	200 = M	← Maximum
C(0, 14)	140	

By Corner Point Method, maximum P = 200 at (4, 12).

Hence, the factory should make 4 tennis rackets and 12 cricket bats to make use of full capacity and then the profit is also maximum equal to ₹ 200.

4. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of ₹ 17.50 per package on nuts and ₹ 7.00 per package on bolts. How many packages of each should be produced each day so as to maximise his profit, if he operates his machines for at the most 12 hours a day?

Sol. Sol. Step I. Mathematical Formulation of L.P.P.

Suppose the manufacturer produces x packages of nuts and y packages of bolts each day. The given data is condensed in the following table:

Item	Number of packages	Number of hours per package		Profit (₹/package)
		on Machine A	on Machine B	
Nuts	x	1	3	17.50
Bolts	y	3	1	7.00

Total profit (in ₹) = $17.5x + 7y$

Let $\quad Z = 17.5x + 7y$

We have the following mathematical model for the given problem.

Maximise $Z = 17.5x + 7y$ ...(*i*)

subject to the constraints:

$x + 3y \le 12$ (Machine A constraint) ...(*ii*)

(**Given:** He operates his machine A for **at most** 12 hours *i.e.,* $\le$ 12 hours)

$3x + y \le 12$ (Machine B constraint) ...(*iii*)

(**Given:** He operates his machine B also for **at the most** 12 hours *i.e.,* $\le$ 12 hours)

$x, y \ge 0$ ...(*iv*)

($\because$ Number of packages of nuts and bolts can't be negative)

Constraint (*iv*) $x, y \ge 0$

$\Rightarrow$ Feasible region is in first quadrant.

Step-II. Table of values for the line $x + 3y = 12$ of constraint (*ii*)

x	0	12
y	4	0

Let us draw the straight line joining the points (0, 4) and (12, 0).

Let us test for origin ($x = 0$, $y = 0$) in constraint (*ii*).

$x + 3y \le 12$, we have $0 \le 12$ which is true.

$\therefore$ Region for constraint (*ii*) is the region on the origin side of the line $x + 3y = 12$.

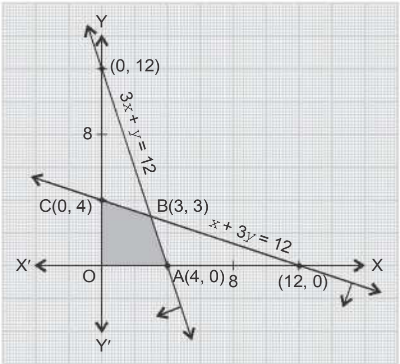
Table of values for the line $3x + y = 12$ of constraint (*iii*)

x	0	4
y	12	0

Let us draw the straight line joining the points (0, 12) and (4, 0).

Let us test for origin ($x = 0$, $y = 0$) in constraint (*iii*) $3x + y \le 12$, we have $0 \le 12$ which is true.

$\therefore$ Region for constraint (*iii*) is also on the origin side of the line $3x + y = 12$.



The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*iv*). The feasible region is bounded.

Step III. The coordinates of the corner points O, A and C are (0, 0), (4, 0) and (0, 4) respectively.

Corner point B: It is the point of intersection of the boundary lines

$$x + 3y = 12 \quad \text{and} \quad 3x + y = 12$$

Solving them for x, y:

Ist Eqn. – 3 × second Eqn. gives

$$x + 3y - 3(3x + y) = 12 - 36$$

$$\Rightarrow \quad x + 3y - 9x - 3y = -24 \quad \Rightarrow \quad -8x = -24$$

$$\Rightarrow \qquad x = \frac{-24}{-8} = 3$$

Putting $x = 3$ in $x + 3y = 12$, $3 + 3y = 12$

$$\Rightarrow \quad 3y = 9 \quad \Rightarrow \quad y = \frac{9}{3} = 3$$

∴ Corner point B is (3, 3).

Step IV. Now, we evaluate Z at each corner point.

Corner Point	$Z = 17.5x + 7y$	
O(0, 0)	0	
A(4, 0)	70	
B(3, 3)	73.5 = M	← Maximum
C(0, 4)	28	

By Corner Point Method, maximum Z = 73.5 at (3, 3).

Hence, the profit is maximum equal to ₹ 73.50 when 3 packages of nuts and 3 packages of bolts are manufactured.

5. A factory manufactures two types of screws, A and B. Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to

manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of ₹ 7 and screws B at a profit of ₹ 10. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximise his profit? Determine the maximum profit.

Sol. Step I. Mathematical Formulation of L.P.P.

Suppose the factory owner produces x packages of screw A and y packages of screw B in a day. The given data is condensed in the following table:

Type of screw	Number of packages	Time in minutes per item		Profit (₹/item)
		on automatic machine	on hand operated machine	
A	x	4	6	7
B	y	6	3	10

Total profit = $7x + 10y$

Let $\quad Z = 7x + 10y$

We have the following mathematical model for the given problem.

Maximise $Z = 7x + 10y$ ...(*i*)

subject to the constraints:

$$4x + 6y \le 240$$

[$\because$ Each machine *i.e.,* automatic machine is also available for atmost *i.e.,* ≤ 4 hours *i.e.,* $4 \times 60 = 240$ minutes]

or $\quad 2x + 3y \le 120$ (Automatic machine constraint) ...(*ii*)

$\quad 6x + 3y \le 240$

(Same argument as given above for constraint (*ii*))

or $\quad 2x + y \le 80$ ...(*iii*)

(Hand operated machine constraint)

$\quad x, y \ge 0$ ...(*iv*)

($\because$ Number of screws A and B can't be negative)

Step II. Table of values for the line $2x + 3y = 120$ of constraint (*ii*)

x	0	60
y	40	0

Let us draw the straight line joining the points (0, 40) and (60, 0).

Let us test for origin (put $x = 0$, $y = 0$) in constraint (*ii*) $2x + 3y \le 120$, we have $0 \le 120$ which is true.

$\therefore$ Region for constraint (*ii*) is on the origin side of the line

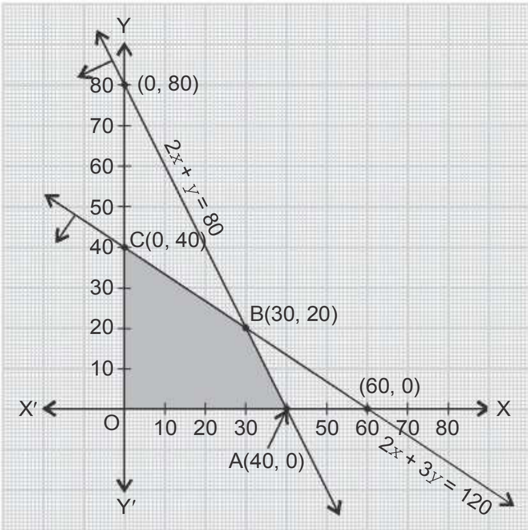
$$2x + 3y = 120.$$

Table of values for the line $2x + y = 80$ of constraint (*iii*)

x	0	40
y	80	0

Let us draw the straight line joining the points (0, 80) and (40, 0). Let us test for origin (put $x = 0, y = 0$) in constraint (*iii*) $2x + y \leq 80$, we have $0 \leq 80$ which is true.

$\therefore$ Region for constraint (*iii*) is also towards the origin side of the line $2x + y = 80$.



The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*iv*). The feasible region is bounded.

Step III. The coordinates of the corner points O, A and C are (0, 0), (40, 0) and (0, 40) respectively.

Corner Point B: It is the point of intersection of boundary lines

$$2x + 3y = 120 \quad \text{and} \quad 2x + y = 80$$

Let us solve them for x and y. Subtracting $2y = 40$

$\Rightarrow \quad y = 20$

Putting $y = 20$ in $2x + 3y = 120$; $2x + 60 = 120$

$\Rightarrow \quad 2x = 60 \quad \Rightarrow \quad x = 30.$

Therefore corner point B is (30, 20).

Step IV. Now, we evaluate Z at each corner point.

Corner Point	$Z = 7x + 10y$	
O(0, 0)	0	
A(40, 0)	280	
B(30, 20)	410 = M	$\leftarrow$ Maximum
C(0, 40)	400	

By Corner Point Method, maximum Z = 410 at (30, 20).

Hence, the profit is maximum equal to ₹ 410 when 30 packages of screws A and 20 packages of screws B are produced in a day.

6. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is ₹ 5 and that from a shade is ₹ 3. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximise his profit?

Sol. Step I. Mathematical formulation of L.P.P.

Suppose the manufacturer produces x pedestal lamps and y wooden shades. The given data is condensed in the following table:

Item	Number	Time on grinding/ cutting machine (hrs/item)	Time on sprayer (hrs/item)	Profit (₹/item)
Pedestal lamps	x	2	3	5
Wooden shades	y	1	2	3

Total profit = $5x + 3y$

Let $Z = 5x + 3y$

We have the following mathematical model for the given problem:

Maximise $Z = 5x + 3y$ ...(*i*)

subject to the constraints:

$2x + y \leq 12$ (Grinding/cutting machine constraint) ...(*ii*)

[**Given:** Cutting/grinding machine is available for at the most (*i.e.*, $\leq$) 12 hours]

$3x + 2y \leq 20$ (Sprayer constraint) ...(*iii*)

[**Given:** The sprayer is available for at the most 20 hours *i.e.*, $\leq$ 20]

$x, y \geq 0$ ...(*iv*) ($\because$ Number of pedestal lamps and wooden shades can't be negative)

Step II. The constraint (*iv*) $x, y \geq 0$ $\Rightarrow$ The feasible region is in first quadrant.

Table of values for the line $2x + y = 12$ of constraint (*ii*)

x	0	6
y	12	0

Let us draw the line joining the points (0, 12) and (6, 0).

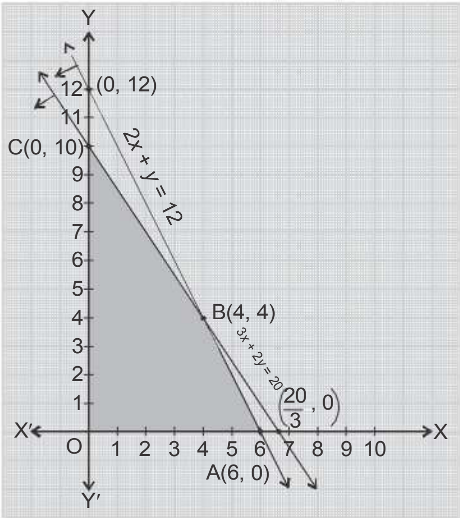
Let us test for origin ($x = 0$, $y = 0$) in constraint (*ii*) $2x + y \leq 12$, we have $0 \leq 12$ which is true.

$\therefore$ Region for constraint (*ii*) is on the origin side of the line $2x + y = 12$.

Table of values for the line $3x + 2y = 20$ of constraint (*iii*)

x	0	$\frac{20}{3}$
y	10	0

Let us draw the line joining the points (0, 10) and $\left(\frac{20}{3}, 0\right)$.



Let us test for origin ($x = 0, y = 0$) in constraint (*iii*) $3x + 2y \leq 20$, we have $0 \leq 20$ which is true.

$\therefore$ Region for constraint (*iii*) is on the origin side of the line

$$3x + 2y = 20.$$

The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*iv*). The feasible region is bounded.

Step III. The coordinates of the corner points O, A and C are (0, 0), (6, 0) and (0, 10) respectively.

Corner point B: It is the point of intersection of boundary lines

$$2x + y = 12$$

and $$3x + 2y = 20$$

$2 \times$ First eqn. – Second eqn. gives

$$4x + 2y - 3x - 2y = 24 - 20 \quad \Rightarrow \quad x = 4.$$

Putting $x = 4$ in $2x + y = 12$, we have $8 + y = 12$

$\Rightarrow \quad y = 4.$

$\therefore$ Corner point B is (4, 4).

Step IV. Now, we evaluate Z at each corner point.

Corner Point	$Z = 5x + 3y$	
O(0, 0)	0	
A(6, 0)	30	
B(4, 4)	32 = M	← Maximum
C(0, 10)	30	

By Corner Point Method, maximum Z = 32 at (4, 4).

Hence, the profit is maximum when 4 pedestal lamps and 4 wooden shades are manufactured. Maximum profit is ₹ 32.

7. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours for assembling. The profit is ₹ 5 each for type A and ₹ 6 each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximise the profit?

(*Important*)

Sol. Step I. Mathematical formulation of L.P.P.

Suppose the company manufactures x souvenirs of type A and y souvenirs of type B. The given data is condensed in the following table:

Type	Number	Time for cutting (min/item)	Time for assembling (min/item)	Profit (₹/item)
A	x	5	10	5
B	y	8	8	6

Total profit = $5x + 6y$

Let $Z = 5x + 6y$

We have the following mathematical model for the given problem:

Maximise $Z = 5x + 6y$ ...(*i*)

subject to the constraints:

$5x + 8y \le 200$ (Cutting constraint) ...(*ii*)

[**Given:** (Maximum) time available for cutting is 3 hours, 20 minutes = $3 \times 60 + 20 = 200$ minutes]

$10x + 8y \le 240$ (Assembling constraint) ...(*iii*)

[**Given:** (Maximum) Time available for assembly is 4 hours $= 4 \times 60 = 240$ minutes]

$x, y \ge 0$ ...(*iv*)

($\because$ Number of souvenirs can't be negative)

Step II. Constraint (*iv*) $x, y \ge 0 \Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $5x + 8y = 200$ of constraint (*ii*)

x	0	40
y	25	0

Let us draw the line joining the points (0, 25) and (40, 0).

Let us test for origin ($x = 0, y = 0$) in constraint (*ii*) $5x + 8y \le 200$ we have $0 \le 200$ which is true.

$\therefore$ Region for constraint (*ii*) is on the origin side of the line $5x + 8y = 200$.

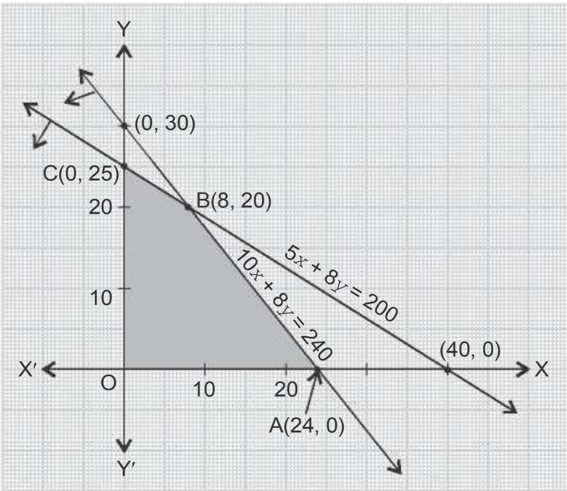
Table of values for the line $10x + 8y = 240$ of constraint (*iii*)

x	0	24
y	30	0

Let us draw the line joining the points (0, 30) and (24, 0).

Let us test for origin ($x = 0, y = 0$) in constraint (*iii*) $10x + 8y \le 240$, we have $0 \le 240$ which is true.

$\therefore$ Region for constraint (*iii*) is on the origin side of the line $10x + 8y = 240$.



The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*iv*). The feasible region is bounded.

Step III. The coordinates of the corner points O, A and C are (0, 0), (24, 0) and (0, 25) respectively.

Corner point B: It is the point of intersection of the boundary lines

$$5x + 8y = 200 \quad \text{and} \quad 10x + 8y = 240$$

Subtracting, $-5x = -40 \Rightarrow x = \dfrac{-40}{-5} = 8.$

Putting $x = 8$ in $5x + 8y = 200$, we have

$$40 + 8y = 200 \Rightarrow 8y = 160 \Rightarrow y = \frac{160}{8} = 20$$

$\therefore$ Corner point B(8, 20).

Step IV. Now, we evaluate Z at each corner point.

Corner Point	Z = 5x + 6y	
O(0, 0)	0	
A(24, 0)	120	
B(8, 20)	160 = M	← Maximum
C(0, 25)	150	

By Corner Point Method, maximum Z = 160 at (8, 20).

Hence, the profit is maximum when 8 souvenirs of type A and 20 souvenirs of type B are manufactured.

Maximum profit = ₹ 160.

8. A merchant plans to sell two types of personal computers a desktop model and a portable model that will cost ₹ 25,000 and ₹ 40,000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than ₹ 70 lakhs and if his profit on the desktop model is ₹ 4500 and on portable model is ₹ 5000.

Sol. Step I. Mathematical Formulation of L.P.P.

Suppose the merchant stocks x units of desktop model and y units of portable model. The given data is condensed in the following table.

Type of Model	Number of units	Cost (₹/unit)	Profit (₹/unit)
Desktop	x	25000	4500
Portable	y	40000	5000

Total profit = $4500x + 5000y$

Let $\quad Z = 4500x + 5000y$

We have the following mathematical model for the given problem:

Maximise profit $Z = 4500x + 5000y$ ...(*i*)

subject to the constraints:

$x + y \leq 250$ (Demand constraint) ...(*ii*)

[**Given:** Total monthly demand of computers will not exceed 250 *i.e.,* ≤ 250]

$25000x + 40000y \leq 70,00,000$

[**Given:** He does not want to invest more than ₹ 70 lakhs = ₹ 70 × 100,000]

Dividing every term by 5000,

or $\quad 5x + 8y \leq 1400$ (Investment constraint) ...(*iii*)

$x, y \geq 0$ ...(*iv*)

($\because$ Number of computers can't be negative)

Step II. Constraint (*iv*) $x, y \geq 0 \Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $x + y = 250$ of constraint (*ii*)

x	0	250
y	250	0

Let us draw the line joining the points (0, 250) and (250, 0).
Let us test for origin ($x = 0$, $y = 0$) in constraint (*ii*) $x + y \le 250$, we have $0 \le 250$ which is true.

$\therefore$ Region for constraint (*ii*) is on the origin side of the line $x + y = 250$.

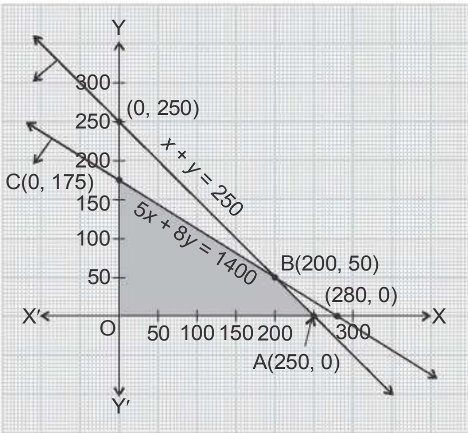
Table of values for the line $5x + 8y = 1400$ of constraint (*iii*)

x	0	280
y	175	0

Let us draw the line joining the points (0, 175) and (280, 0).
Let us test for origin (0, 0) in constraint (*iii*), $5x + 8y \le 1400$, we have $0 \le 1400$ which is true.

$\therefore$ Region for constraint (*iii*) is on the origin side of the line

$$5x + 8y = 1400.$$



The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*iv*). The feasible region is bounded.

Step III. The coordinates of the corner points O, A and C are (0, 0), (250, 0) and (0, 175) respectively.

Corner point B: It is the point of intersection of boundary lines:

$$x + y = 250 \quad \text{and} \quad 5x + 8y = 1400$$

Second Eqn. – 5 × Ist equation gives

$$5x + 8y - 5x - 5y = 1400 - 1250$$

or $\quad 3y = 150 \Rightarrow y = \dfrac{150}{3} = 50$

Putting $y = 50$ in $x + y = 250$,

we have $x + 50 = 250 \Rightarrow x = 200$

$\therefore$ Corner point B is (200, 50).

Step IV. Now, we evaluate Z at each corner point.

Corner Point	$Z = 4500x + 5000y$	
O(0, 0)	0	
A(250, 0)	11,25,000	
B(200, 50)	11,50,000 = M	← Maximum
C(0, 175)	8,75,000	

By Corner Point Method, maximum Z = 11,50,000 at (200, 50).

Hence, the merchant should stock 200 units of desktop model and 50 units of portable model for a maximum profit of ₹ 11,50,000.

9. A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foods F_1 and F_2 are avialable. Food F_1 costs ₹ 4 per unit and food F_2 costs ₹ 6 per unit. One unit of food F_1 contains 3 units of vitamin A and 4 units of minerals. One unit of food F_2 contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem. Find the minimum cost for diet that consists of mixture of these two foods and also meets the minimal nutritional requirements.

Sol. Step I. Mathematical formulation of L.P.P.

Suppose the diet contains x units of food F_1 and y units of food F_2. The given data is condensed in the following table:

Type of Food	Number of units	Cost (₹/unit)	Vitamin A (units)	Minerals (units)
F_1	x	4	3	4
F_2	y	6	6	3

Total cost = $4x + 6y$

Let $Z = 4x + 6y$

We have the following mathematical model for the given problem.

Minimise $Z = 4x + 6y$ ...(*i*)

subject to the constraints:

$3x + 6y \geq 80$ (Vitamin A constraint) ...(*ii*)

[**Given:** At least *i.e.,* $\geq$ 80 units of vitamin A]

$4x + 3y \geq 100$ (Mineral constraint) ...(*iii*)

[**Given:** At least *i.e.,* $\geq$ 100 units of minerals]

$x, y \geq 0$

($\because$ Units of vitamins and minerals can't be negative) ...(*iv*)

Step II. The constraint (*iv*) $x, y \geq 0$.

$\Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $3x + 6y = 80$ of constraint (*ii*)

x	0	$\frac{80}{3}$
y	$\frac{40}{3}$	0

Let us draw the line joining the points $\left(0, \frac{40}{3}\right)$ and $\left(\frac{80}{3}, 0\right)$.

Let us test for origin ($x = 0, y = 0$) in constraint (ii) $3x + 6y \geq 80$, we have $0 \geq 80$ which is not true.

$\therefore$ Region for constraint (ii) is the half-plane not containing the origin *i.e.,* region on the non-origin side of the line $3x + 6y = 80$.

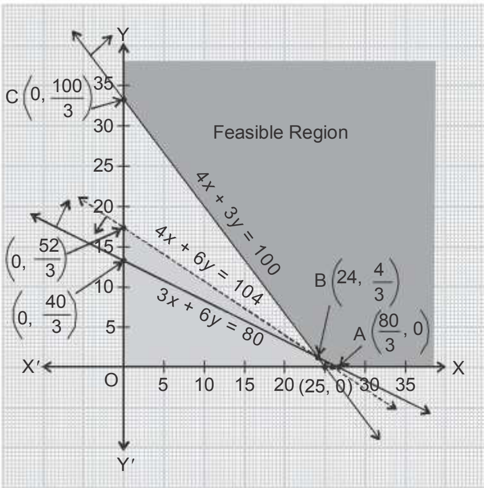
Table of values for the line $4x + 3y = 100$ of constraint (iii)

x	0	25
y	$\frac{100}{3}$	0

Let us draw the line joining the points $\left(0, \frac{100}{3}\right)$ and (25, 0).

Let us test for origin ($x = 0, y = 0$) in constraint (iii) $4x + 3y \geq 100$, we have $0 \geq 100$ which is not true.

$\therefore$ Region for constraint (iii) is the half-plane again on the non-origin side of the line $4x + 3y = 100$.



The shaded region in the figure is the feasible region determined by the system of constraints from (ii) to (iv).

The feasible region is unbounded.

Step III. The coordinates of the corner points A and C are $\left(\frac{80}{3}, 0\right)$ and $\left(0, \frac{100}{3}\right)$ respectively.

To find corner point B: Corner point B is the point of intersection of the boundary lines

$$3x + 6y = 80 \quad \text{and} \quad 4x + 3y = 100$$

First Eqn. – 2 × Second eqn. gives

$$3x + 6y - 8x - 6y = 80 - 200$$

or $$-5x = -120 \Rightarrow x = \frac{-120}{-5} = 24$$

Putting $x = 24$ in $3x + 6y = 80$, we have

$$72 + 6y = 80 \Rightarrow 6y = 8 \Rightarrow y = \frac{8}{6} = \frac{4}{3}$$

$\therefore$ Corner point B is $\left(24, \frac{4}{3}\right)$.

Step IV. Now, we evaluate Z at each corner point.

Corner Point	$Z = 4x + 6y$	
$A\left(\frac{80}{3}, 0\right)$	$\frac{320}{3}$	
$B\left(24, \frac{4}{3}\right)$	$104 = m$	← Smallest
$C\left(0, \frac{100}{3}\right)$	200	

From this table, we find that 104 is the smallest value of Z at the corner $B\left(24, \frac{4}{3}\right)$.

Step V. Since the feasible region is unbounded, 104 may or may not be the minimum value of Z. To decide this, we graph the inequality $Z < m$ *i.e.,* $4x + 6y < 104$.

Table of values for the line $4x + 6y = 104$ (of constraint $Z < m$ *i.e.,* $4x + 6y < 104$)

x	0	26
y	$\frac{52}{3}$	0

Let us draw the dotted line joining the points $\left(0, \frac{52}{3}\right)$ and (26, 0).

[(26, 0) not being marked in the graph because it is very close to the point $\left(\frac{80}{3}, 0\right)$ = (26.7, 0) already marked and (26, 0) is slightly to the left of $\left(\frac{80}{3}, 0\right)$]

The line is shown dotted because equality sign is absent in the constraint $Z < m$.

Let us test for origin ($x = 0, y = 0$) in constraint $Z < m$ *i.e.,* $4x + 6y < 104$, we have $0 < 104$ which is true.

$\therefore$ Region for constraint $Z < m$ *i.e.,* $4x + 6y < 104$ is the origin side of the line $4x + 6y = 104$

We observe that the half plane determined by Z < m has no point in common with the feasible region. Hence $m = 104$ is the minimum value of Z attained at the point $B\left(24, \frac{4}{3}\right)$.

$\therefore$ Minimum cost is ₹ 104 when 24 units of food F_1 are mixed with $\frac{4}{3}$ units of food F_2.

10. There are two types of fertilisers F_1 and F_2. F_1 consists of 10% nitrogen and 6% phosphoric acid and F_2 consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that she needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If F_1 costs ₹ 6/kg and F_2 costs ₹ 5/kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?

Sol. Step I. Mathematical formulation of L.P.P.

Suppose the farmer uses x kg of fertiliser F_1 and y kg of fertiliser F_2. The given data is condensed in the following table.

Fertiliser	Quantity (kg)	Nitrogen content	Phosphoric acid content	Cost (₹/kg)
F_1	x	10%	6%	6
F_2	y	5%	10%	5

Total cost = $6x + 5y$

Let Z = $6x + 5y$

We have the following mathematical model for the given problem:

Minimise Z = $6x + 5y$ ...(i)

subject to the constraints:

$$\frac{10}{100}x + \frac{5}{100}y \ge 14$$

[**Given:** She needs at least *i.e.,* $\ge$ 14 kg of nitrogen for her crops]

Multiplying by 100 and dividing by 5,

$$2x + y \ge 280 \quad \text{(Nitrogen constraint)} \quad ...(ii)$$

$$\frac{6}{100}x + \frac{10}{100}y \ge 14$$

[**Given:** She needs at least 14 kg of phosphoric acid for her crops]

Multiplying by 100 and dividing by 2,

$$3x + 5y \ge 700 \quad \text{(Phosphoric acid constraint)} \quad ...(iii)$$

$$x, y \ge 0 \quad ...(iv)$$

($\because$ Quantity of Nitrogen and Phosphoric acid can't be negative)

Step II. Constraint (iv) $x, y \ge 0$.

$\Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $2x + y = 280$ of constraint (*ii*)

x	0	140
y	280	0

Let us draw the line joining the points (0, 280) and (140, 0).
Let us test for origin ($x = 0, y = 0$) in constraint (*ii*) $2x + y \geq 280$, we have $0 \geq 280$ which is not true.

$\therefore$ Region for constraint (*ii*) is the half-plane not containing the origin *i.e.,* region on the non-origin side of the line $2x + y = 280$.

Table of values for the line $3x + 5y = 700$ corresponding to constraint (*iii*)

x	0	$\frac{700}{3}$
y	140	0

Let us draw the line joining the points (0, 140) and $\left(\frac{700}{3}, 0\right)$.

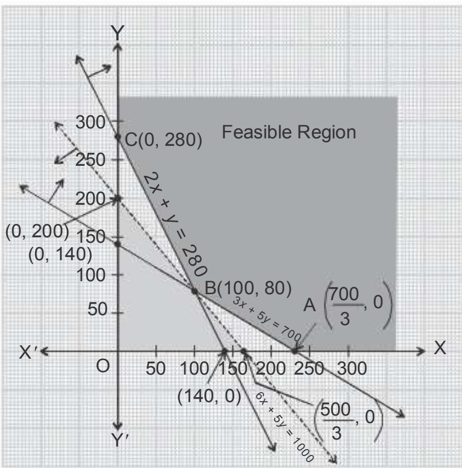
Let us test for origin ($x = 0, y = 0$) in constraint (*iii*) $3x + 5y \geq 700$, we have $0 \geq 700$ which is not true.

$\therefore$ Region for constraint (*iii*) is again on the non-origin side of the line $3x + 5y = 700$.

The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*iv*). The feasible region is unbounded.

Step III. The coordinates of the corner points. A and C are $\left(\frac{700}{3}, 0\right)$ and (0, 280) respectively.

To find corner point B: Let us solve the equations of bounding lines $2x + y = 280$ and $3x + 5y = 700$ for x and y.



Second eqn. $-$ 5 $\times$ first eqn. gives

$$3x + 5y - 10x - 5y = 700 - 1400$$

$$\Rightarrow \quad -7x = -700 \Rightarrow x = \frac{-700}{-7} = 100$$

Putting $x = 100$ in $2x + y = 280$, we have

$200 + y = 280 \Rightarrow y = 80$ $\therefore$ Corner point B is (100, 80).

Step IV. Now, we evaluate Z at each corner point.

Corner Point	$Z = 6x + 5y$	
$A\left(\frac{700}{3}, 0\right)$	1400	
B(100, 80)	$1000 = m$	← Smallest
C(0, 280)	1400	

From this table, we find that 1000 is the smallest value of Z at the corner B(100, 80). Since the feasible region is unbounded, 1000 may or may not be the minimum value of Z.

Step V. To decide this, we graph the inequality $Z < m$

i.e., $6x + 5y < 1000$.

Table of values for the line $6x + 5y = 1000$ (for constraint $Z < m$ *i.e.,* $6x + 5y < 1000$)

x	0	$\frac{500}{3}$
y	200	0

Let us draw the dotted line joining the points (0, 200) and $\left(\frac{500}{3}, 0\right)$.

The line is drawn dotted because equality sign is absent in the constraint $Z < m$.

We observe that the half-plane determined by $Z < m$ has no point in common with the feasible region. Hence, $m = 1000$ is the minimum value of Z attained at the point B(100, 80).

$\therefore$ Minimum cost is ₹ 1000 when the farmer uses 100 kg of fertiliser F_1 and 80 kg of fertiliser F_2.

11. The corner points of the feasible region determined by the following system of linear inequalities:

$2x + y \le 10$, $x + 3y \le 15$, $x, y \ge 0$ are (0, 0), (5, 0), (3, 4) and (0, 5). Let $Z = px + qy$, where $p, q > 0$. Condition on p and q so that the maximum of Z occurs at both (3, 4) and (0, 5) is

(A) $p = q$ (B) $p = 2q$ (C) $p = 3q$ (D) $q = 3p$.

Sol. We evaluate Z at each corner point.

Corner Point	$Z = px + qy$, $p > 0, q > 0$	
(0, 0)	0	
(5, 0)	$5p$	
(3, 4)	$3p + 4q$ = M	← Maximum
(0, 5)	$5q$ = M	← Maximum

$\because$ Maximum of Z occurs at both (3, 4) and (0, 5) (given)

$\therefore \quad 3p + 4q = 5q$

$\therefore \quad q = 3p$

Hence, the correct option is (D).

MISCELLANEOUS EXERCISE

1. (Refer to Example 9, NCERT Page 521). How many packets of each food should be used to maximise the amount of vitamin A in the diet? What is the maximum amount of vitamin A in the diet?

Sol. (NCERT Page 521), we find that Z is maximum at the point (40, 15). Hence, the amount of vitamin A under the constraints given in the problem will be maximum if 40 packets of food P and 15 packets of food Q are used in the special diet.

The maximum amount of vitamin A will be 285 units.

2. A farmer mixes two brands P and Q of cattle feed. Brand P costing ₹ 250 per bag, contains 3 units of nutritional element A, 2.5 units of element B and 2 units of element C. Brand Q costing ₹ 200 per bag contains 1.5 units of nutritional element A, 11.25 units of element B, and 3 units of element C. The minimum requirements of nutrients A, B and C are 18 units, 45 units and 24 units respectively. Determine the number of bags of each brand which should be mixed in order to produce a mixture having a minimum cost per bag? What is the minimum cost of the mixture per bag?

Sol. Step I. Mathematical Formulation of L.P.P.

Suppose the farmer mixes x bags of brand P and y bags of brand Q. The given data is condensed in the following table.

Brand	Number of bags	Cost (₹/bag)	Element A (units/bag)	Element B (units/bag)	Element C (units/bag)
P	x	250	3	2.5	2
Q	y	200	1.5	11.25	3

Total cost $= 250x + 200y$

Let $\quad Z = 250x + 200y$

We have the following mathematical model for the given problem:

Minimise $Z = 250x + 200y$ ...(i)

subject to the constraints:

$$3x + 1.5y \geq 18$$

[**Given:** Minimum requirement of nutritional element A is 18 units *i.e.,* $\geq$ 18 units]

or $\quad 3x + \frac{15}{10}y \geq 18$

Multiplying by 10 and dividing by 15,

or $\quad 2x + y \geq 12$ (Nutritional element A constraint)...(ii)

$$2.5x + 11.25y \geq 45$$

[**Given:** Minimum requirement of nutritional element B is, 45 units *i.e.*, ≥ 45 units]

or $\frac{25}{10}x + \frac{1125}{100}y \geq 45$

Multiplying by 100 and dividing by 125,

or $2x + 9y \geq 36$ ...(*iii*)

(Nutritional element B constraint)

$2x + 3y \geq 24$ (Nutritional element C constraint) ...(*iv*)

[**Given:** Minimum requirement of nutritional element C is 24 units *i.e.*, ≥ 24 units]

$x, y \geq 0$ ($\because$ Number of bags can't be negative)...(*v*)

Step II. Constraint (*v*) $x, y \geq 0$

$\Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $2x + y = 12$ of constraint (*ii*)

x	0	6
y	12	0

Draw the straight line joining the points (0, 12) and (6, 0).

Let us test for origin ($x = 0, y = 0$) in constraint $2x + y \geq 12$, we have $0 \geq 12$ which is not true.

$\therefore$ Region for constraint (*ii*) $2x + y \geq 12$ is the half-plane not containing the origin *i.e.*, region on the non-origin side of the line $2x + y = 12$.

Table of values for the line $2x + 9y = 36$ for constraint (*iii*)

x	0	18
y	4	0

Let us draw the line joining the points (0, 4) and (18, 0).

Let us test for origin ($x = 0, y = 0$) in constraint (*iii*) $2x + 9y \geq 36$, we have $0 \geq 36$ which is not true.

$\therefore$ Region for constraint (*iii*) is the region on the non-origin side of the line $2x + 9y = 36$.

Table of values for the line $2x + 3y = 24$ for constraint (*iv*)

x	0	12
y	8	0

Draw the line joining the points (0, 8) and (12, 0).

Let us test for origin ($x = 0, y = 0$) in constraint (*iii*) $2x + 3y \geq 24$, we have $0 \geq 24$ which is not true.

$\therefore$ Region for constraint (*iii*) $2x + 3y \geq 24$ is again the region on the non-origin side of the line $2x + 3y = 24$.

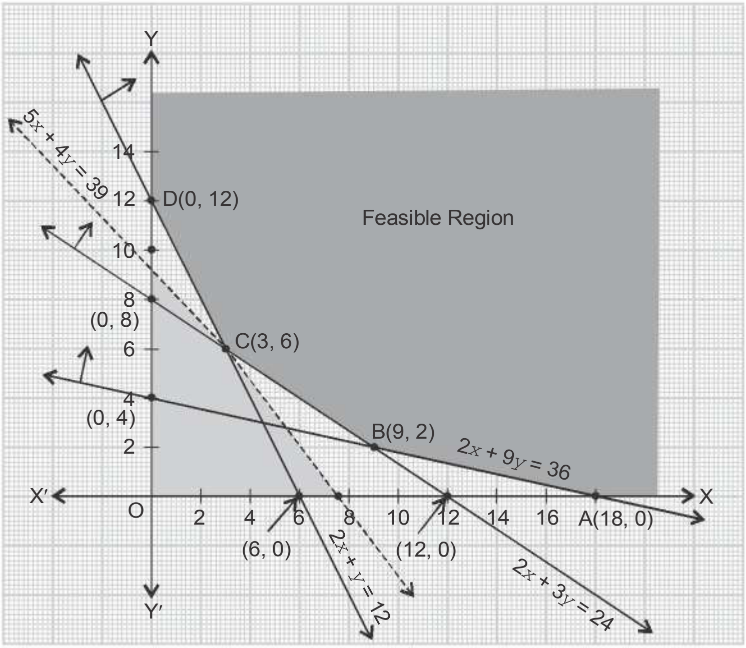
The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*v*). The feasible region is unbounded.

Step III. The coordinates of the corner points A and D are (18, 0) and (0, 12) respectively.

Corner point B: It is the point of intersection of the lines

$2x + 3y = 24$ and $2x + 9y = 36$

Subtracting $-6y = -12 \Rightarrow y = \frac{-12}{-6} = 2$



Putting $y = 2$ in $2x + 3y = 24$, we have

$$2x + 6 = 24 \Rightarrow 2x = 18 \Rightarrow x = 9$$

$\therefore$ Corner point B is (9, 2).

Corner point C: It is the point of intersection of the lines

$$2x + y = 12 \quad \text{and} \quad 2x + 3y = 24$$

Subtracting $-2y = -12 \Rightarrow y = \frac{-12}{-2} = 6$

Putting $y = 6$ in $2x + y = 12$, we have

$$2x + 6 = 12 \Rightarrow 2x = 6 \Rightarrow x = 3$$

$\therefore$ Corner point C is (3, 6).

Step IV. Now, we evaluate Z at each corner point.

Corner Point	$Z = 250x + 200y$	
A(18, 0)	4500	
B(9, 2)	2650	
C(3, 6)	$1950 = m$	← Smallest
D(0, 12)	2400	

From this table, we find that 1950 is the smallest value of Z at the corner C(3, 6). Since the feasible region is unbounded, 1950 may or may not be the minimum value of Z.

Step V. To decide this, we graph the inequality $Z < m$

i.e., $250x + 200y < 1950$ or $5x + 4y < 39$.

Table of values for the line $5x + 4y = 39$ corresponding to constraint $Z < m$ *i.e.,* $5x + 4y < 39$.

x	0	$\frac{39}{5} = 7.8$
y	$\frac{39}{4} = 9.75$	0

Let us draw the dotted line joining the points (0, 9.75) and (7.8, 0). The line is to be shown dotted because equality sign is absent in the constraint Z < m *i.e.,* in $5x + 4y < 39$.

Let us test for origin ($x = 0, y = 0$) in this constraint, we have $0 < 39$ which is true.

$\therefore$ Region for constraint Z < m *i.e.,* $5x + 4y < 39$ is towards the origin side of the line.

We observe that the half plane determined by Z < m has no point in common with the feasible region. Hence m = 1950 is the minimum value of Z attained at the point C(3, 6).

$\therefore$ Minimum cost is ₹ 1950 when 3 bags of brand P and 6 bags of brand Q are mixed.

3. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below:

Food	**Vitamin A**	**Vitamin B**	**Vitamin C**
X	**1**	**2**	**3**
Y	**2**	**2**	**1**

One kg of food X costs ₹ 16 and one kg of food Y costs ₹ 20. Find the least cost of the mixture which will produce the required diet?

Sol. Step I. Mathematical Formulation of L.P.P.

Let the dietician mix x kg of food X and y kg of food Y. The given data is condensed in the following table.

Food	Quantity (kg)	Vitamin A (units/kg)	Vitamin B (units/kg)	Vitamin C (units/kg)	Cost (₹/kg)
X	x	1	2	3	16
Y	y	2	2	1	20

Total cost = $16x + 20y$

Let Z = $16x + 20y$

We have the following mathematical model for the given problem:

Minimise Z = $16x + 20y$ ...(*i*)

subject to the constraints:

$x + 2y \geq 10$ (Vitamin A constraint) ...(*ii*)

[**Given:** The mixture contains at least 10 units (*i.e.,* ≥ 10) of vitamin A]

$2x + 2y \geq 12$

[**Given:** The mixture contains at least 12 units (*i.e.,* ≥ 12) of vitamin B]

or $x + y \geq 6$ (Vitamin B constraint) ...(*iii*)

$3x + y \geq 8$ (Vitamin C constraint) ...(*iv*)

[**Given:** The mixture contains at least 8 units (*i.e.,* ≥ 8) of vitamin C]

$x, y \geq 0$ ($\because$ Quantities of food can't be negative) ...(v)
The constraint (v), $x, y \geq 0$ $\Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $x + 2y = 10$ of constraint (ii).

x	0	10
y	5	0

Let us draw the line joining the points (0, 5) and (10, 0).
Let us test for origin ($x = 0, y = 0$) in constraint (ii) $x + 2y \geq 10$, we have $0 \geq 10$ which is not true.
$\therefore$ Region for constraint (ii) is the half-plane not containing the origin *i.e.,* region on the non-origin side of the line $x + 2y = 10$.

Table of values for the line $x + y = 6$ of constraint (iii).

x	0	6
y	6	0

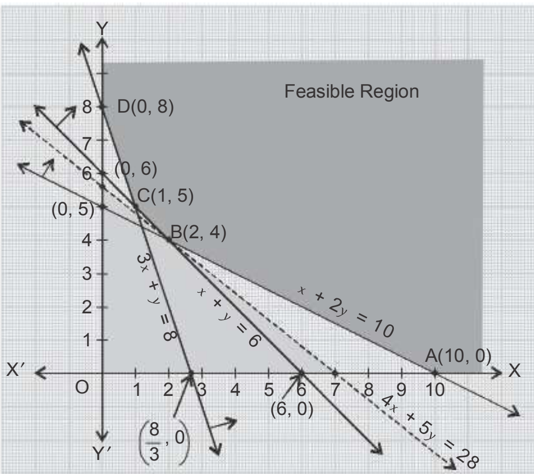
Let us draw the line joining the points (0, 6) and (6, 0).
Let us test for origin ($x = 0, y = 0$) in constraint $x + y \geq 6$, we have $0 \geq 6$ which is not true.
$\therefore$ Region for constraint (iii) is the half-plane not containing the origin *i.e.,* region on the non-origin side of the line $x + y = 6$.

Table of values for the line $3x + y = 8$ of constraint (iv).

x	0	$\frac{8}{3}$
y	8	0

Let us draw the line joining the points (0, 8) and $\left(\frac{8}{3}, 0\right)$.

Let us test for origin ($x = 0, y = 0$) in constraint (iv) $3x + y \geq 8$, we have $0 \geq 8$ which is not true.
$\therefore$ Region for constraint (iv) also is on the non-origin side of the line $3x + y = 8$.



The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*v*). The feasible region is unbounded.

Step III. The coordinates of the corner points A and D are (10, 0) and (0, 8) respectively.

Corner point B: It is the point of intersection of bounding lines

$$x + 2y = 10 \qquad \text{and} \qquad x + y = 6$$

Subtracting $y = 4$

Putting $y = 4$ in $x + 2y = 10$, $x + 8 = 10 \Rightarrow x = 2$

∴ Corner point B is (2, 4).

Corner point C: It is the point of intersection of bounding lines

$$x + y = 6 \qquad \text{and} \qquad 3x + y = 8$$

Subtracting $-2x = -2$ or $x = \frac{-2}{-2} = 1$

Putting $x = 1$ in $x + y = 6$, $1 + y = 6 \Rightarrow y = 5$

∴ Corner point C is (1, 5).

Step IV. Now, we evaluate Z at each corner point.

Corner Point	$Z = 16x + 20y$	
A(10, 0)	160	
B(2, 4)	$112 = m$	← Smallest
C(1, 5)	116	
D(0, 8)	160	

From this table, we find that 112 is the smallest value of Z at the corner B(2, 4). Since the feasible region is unbounded, 112 may or may not be the minimum value of Z.

Step V. To decide this, we graph the inequality $Z < m$ *i.e.,* $16x + 20y < 112$ or $4x + 5y < 28$.

Table of values for the line $4x + 5y = 28$ (of constraint $Z < m$ *i.e.,* $4x + 5y < 28$).

x	0	7
y	$\frac{28}{5} = 5.6$	0

Let us draw the dotted line joining the points (0, 5.6) and (7, 0). The line is drawn dotted because equality sign is absent in the constraint $Z < m$ *i.e.,* $4x + 5y < 28$.

Let us test for origin ($x = 0$, $y = 0$) in constraint $4x + 5y < 28$, we have $0 < 28$ which is true.

∴ Region for constraint $Z < m$ *i.e.,* $4x + 5y < 28$ is on the origin side of the line $4x + 5y = 28$.

We observe that the half-plane determined by $Z < m$ has no point in common with the feasible region. Hence, $m = 112$ is the minimum value of Z attained at the point B(2, 4).

∴ Minimum cost of the mixture is ₹ 112 when 2 kg of food X and 4 kg of food Y are mixed.

4. A manufacturer makes two types of toys A and B. Three machines are needed for this purpose and the time (in minutes) required for each toy on the machines is given below:

Types of Toys	**Machines**		
	I	**II**	**III**
A	**12**	**18**	**6**
B	**6**	**0**	**9**

Each machine is available for a maximum of 6 hours per day. If the profit on each toy of type A is ₹ 7.50 and that on each toy of type B is ₹ 5, show that 15 toys of type A and 30 of type B should be manufactured in a day to get maximum profit.

Sol. Step I. Mathematical formulation of L.P.P.

Let the manufacturer make x toys of type A and y toys of type B. The given data is condensed in the following table.

Types of toy	Number of toys	Time (min/toy) on machines			Profit (₹/toy)
		I	II	III	
A	x	12	18	6	7.50
B	y	6	0	9	5

Total profit = $7.50x + 5y$

Let Z = $7.50x + 5y$

We have the following mathematical model for the given problem:

Maximise Z = $7.50x + 5y$ ...(*i*)

subject to the constraints:

$12x + 6y \le 360$

[**Given:** Each of machines I, II, III is available for a maximum of 6 hours = 6×60 = 360 minutes]

or $2x + y \le 60$ (Machine I constraint) ...(*ii*)

$18x + 0y \le 360$

or $x \le 20$ (Machine II constraint) ...(*iii*)

$6x + 9y \le 360$

or $2x + 3y \le 120$ (Machine III constraint) ...(*iv*)

$x, y \ge 0$ ...(*v*)

($\because$ Number of toys can't be negative)

Step II. Constraint (*v*) $x, y \ge 0$.

$\Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $2x + y = 60$ of constraint (*ii*).

x	0	30
y	60	0

Let us draw the line joining the points (0, 60) and (30, 0).

Let us test for origin ($x = 0, y = 0$) in constraint (*ii*) $2x + y \le 60$, we have $0 \le 60$ which is true. Therefore region for constraint (*ii*) is on the origin side of the line $2x + y = 60$.

Region for constraint (*iii*) $x \le 20$

We know that graph of the line $x = 20$ is a vertical line (parallel to y-axis) at a distance of 20 units along OX.

$\therefore$ Region for $x \le 20$ is the region on the left side of the line $x = 20$.

Table of values for the line $2x + 3y = 120$ of constraint (*iv*).

x	0	60
y	40	0

Let us draw the line joining the points (0, 40) and (60, 0).

Let us test for origin ($x = 0, y = 0$) in constraint (*iv*) $2x + 3y \le 120$, we have $0 \le 120$ which is true.

$\therefore$ Region for constraint (*iv*) is on the origin side of the line $2x + 3y = 120$.

The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*v*). The feasible region is bounded.

Step III. The coordinates of the corner points O, A and D are (0, 0), (20, 0) and (0, 40) respectively.

Corner point B: It is the point of intersection of bounding lines

$2x + y = 60$ and $x = 20$

Putting $x = 20$ in $2x + y = 60$, we have $40 + y = 60$ or $y = 20$.

$\therefore$ Corner point B is (20, 20).

Corner point C: It is the point of intersection of bounding lines

$2x + y = 60$ and $2x + 3y = 120$

Subtracting $-2y = -60$ or $y = \dfrac{-60}{-2} = 30$

Putting $y = 30$ in $2x + y = 60$, we have

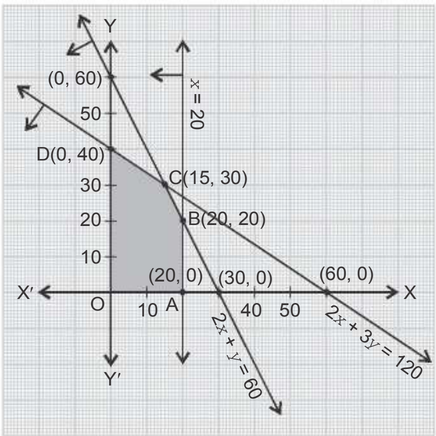
$2x + 30 = 60 \Rightarrow 2x = 30 \Rightarrow x = 15$.

$\therefore$ Corner point C is (15, 30).

Step IV. Now, we evaluate Z at each corner point.

Corner Point	Z = 7.50x + 5y	
O(0, 0)	0	
A(20, 0)	150	
B(20, 20)	250	
C(15, 30)	262.50 = M	← Maximum
D(0, 40)	200	

By Corner Point Method, maximum Z = 262.50 at (15, 30).



$\therefore$ For maximum profit, 15 toys of type A and 30 toys of type B should be manufactured.

5. An aeroplane can carry a maximum of 200 passengers. A profit of ₹ 1000 is made on each executive class ticket and a profit of ₹ 600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the profit for the airline. What is the maximum profit?

Sol. Step I. Let us formulate the L.P.P. mathematically.

Let the number of executive class tickets sold be x and the number of economy class tickets sold be y.

The aeroplane can carry a maximum of 200 passengers.

$\Rightarrow \quad x + y \le 200$

At least 20 seats are reserved for executive class $\Rightarrow x \ge 20$

Number of passengers in economy class is at least 4 times the number of passengers in executive class.

$\Rightarrow \quad y \ge 4x$

Profit from x executive class tickets at the rate of ₹ 1000 per ticket = ₹ $1000x$

Profit from y economy class tickets at the rate of ₹ 600 per ticket = ₹ $600y$.

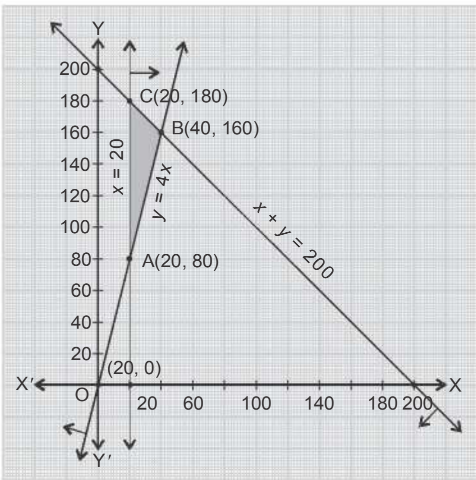
Let the total profit (in ₹) be denoted by P, then $P = 1000x + 600y$

$\therefore$ We have to **maximise** $P = 1000x + 600y$

subject to constraint $x + y \le 200$ $x \ge 20$, $y \ge 4x$.

Also $x \ge 0$ and $y \ge 0$ [$\because$ Number of tickets can't be negative.]

Step II. The reader is suggested to draw the graphs of constraints $x + y \le 200$ and $x \ge 20$ for himself or herself and compare them with the adjoining figure.



We, here graph the constraint $y \ge 4x$.

The corresponding equation is $y = 4x$.

Put $y = 0$, $\therefore x = 0$.

$\therefore$ The line $y = 4x$ passes through the origin (0, 0).

Put $x = 20$, $y = 80$

$\therefore$ Point is (20, 80).

$\therefore$ The graph of line $y = 4x$ is the line passing through the origin (0, 0) and point (20, 80).

Test for the point (1, 0).

Put $x = 1$ and $y = 0$ in $y \geq 4x$, $0 \geq 4$ which is not true.

$\therefore$ The region for $y \geq 4x$ does not contain the point (1, 0) (and also does not contain the point (20, 0) because on putting $x = 20$ and $y = 0$ in $y \geq 4x$ we have $0 \geq 80$ which is not true). This point is being mentioned as it happens to be a point on the graph) and is as shown by arrows in the figure.

The feasible region is the region bounded by the triangle ABC.

Step III. The corner points of the bounded feasible region are A, B and C. Corner (vetex) A is the point of intersection of the lines $x = 20$ and $y = 4x$.

Putting $x = 20$, $y = 4 \times 20 = 80$

$\therefore$ Vertex A is (20, 80)

Corner (or vertex) B is the point of intersection of the lines $y = 4x$ and $x + y = 200$.

Putting $y = 4x$, $x + 4x = 200$ or $5x = 200$

$\therefore$ $x = 40$ and therefore $y = 4x = 4(40) = 160$

$\therefore$ Vertex B is (40, 160)

Corner (or vertex) C is the point of intersection of the lines $x = 20$ and $x + y = 200$.

Putting $x = 20$, $20 + y = 200$

$\therefore$ $y = 180$

$\therefore$ Vertex C is (20, 180)

Step IV. Objective function is $P = 1000x + 600y$.

At A (20, 80); P = 1000(20) + 600(80) = 68000

At B (40, 160); P = 1000(40) + 600(160) = 136000

At C (20, 180); P = 1000(20) + 600(180) = 128000

We see that P is maximum at B where $x = 40$, $y = 160$.

$\therefore$ The airline should sell 40 executive class tickets and 160 economy class tickets to maximise profit.

Also, maximum profit = The value of P at B = ₹ 136000.

6. Two godowns A and B have grain capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, D, E and F whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table:

Transportation Cost per quintal (in ₹)		
From / To	**A**	**B**
D	**6**	**4**
E	**3**	**2**
F	**2.50**	**3**

How should the supplies be transported in order that the transportation cost is minimum? What is the minimum cost?

Sol. Step I. Mathematical formulation of L.P.P.

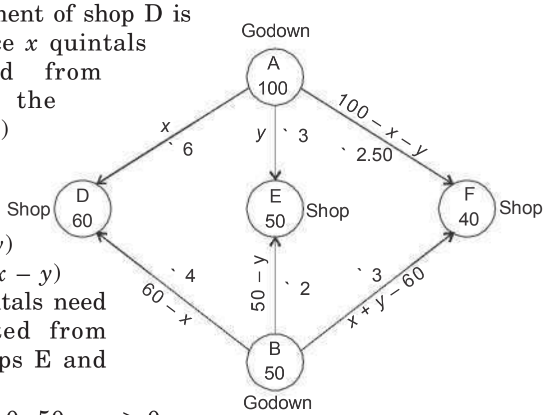
Let x quintals and y quintals of grain be transported from godowns A to ration shops D and E respectively. Then $100 - (x + y)$ quintals will be transported to ration shop F.

Clearly, $x \geq 0, y \geq 0$ and $100 - x - y \geq 0$ ($\Rightarrow 100 \geq x + y$)

($\because$ Amounts (in Quintals) of grain can't be negative)

i.e., $x \geq 0, y \geq 0$ and $x + y \leq 100$

Now, the requirement of shop D is 60 quintals. Since x quintals are transported from godown A, the remaining $(60 - x)$ quintals need to be transported from godown B. Similarly, $(50 - y)$ and $40 - (100 - x - y)$ $= x + y - 60$ quintals need to be transported from godown B to shops E and F respectively.



Clearly, $60 - x \geq 0$, $50 - y \geq 0$

(*i.e.*, $60 \geq x$, $50 \geq y$)

and $x + y - 60 \geq 0$

i.e., $x \leq 60$, $y \leq 50$ and $x + y \geq 60$

Total transportation cost Z is given by

$$Z = 6x + 3y + \frac{5}{2}(100 - x - y) + 4(60 - x) + 2(50 - y) + 3(x + y - 60)$$

$$= \frac{5}{2}x + \frac{3}{2}y + 410 = \frac{1}{2}(5x + 3y + 820)$$

We have the following mathematical model for the given problem:

$$\text{Minimise } Z = \frac{1}{2}(5x + 3y + 820) \quad ...(i)$$

subject to the constraints:

$$x \geq 0, y \geq 0 \quad ...(ii)$$
$$x + y \leq 100 \quad ...(iii)$$
$$x \leq 60 \quad ...(iv)$$
$$y \leq 50 \quad ...(v)$$
$$x + y \geq 60 \quad ...(vi)$$

Step II. Constraint (*ii*) $x \geq 0, y \geq 0$.

$\Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $x + y = 100$ of constraint (*iii*).

x	0	100
y	100	0

Let us draw the straight line joining the points (0, 100) and (100, 0).

Let us test for origin ($x = 0, y = 0$) in constraint (*ii*) $x + y \leq 100$, we have $0 \leq 100$ which is true.

∴ Region for constraint (*ii*) is on the origin side of the line $x + y = 100$.

Region for constraint (*iv*) $x \leq 60$

We know that graph of the line $x = 60$ is a vertical line (parallel to y-axis) at a distance of 60 units along OX.

∴ Region for constraint $x \leq 60$ is the region on the left side of the line $x = 60$.

Region for constraint (*v*) $y \leq 50$

We know that graph of the line $y = 50$ is a horizontal line (parallel to x-axis) at a distance of 50 units along OY.

∴ Region for constraint $y \leq 50$ is **below** the line $y = 50$.

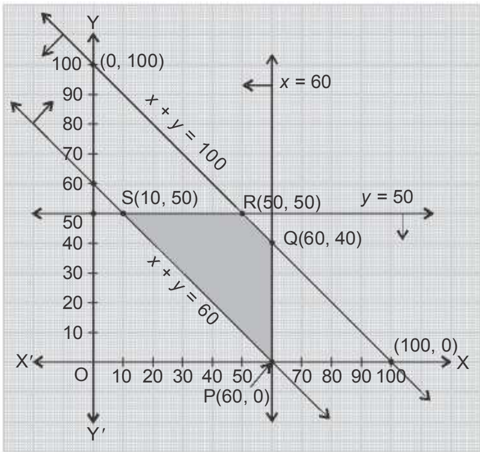
Finally, Table of values for the line $x + y = 60$ of constraint (*vi*).

x	0	60
y	60	0

Let us draw the line joining the points (0, 60) and (60, 0).

Let us test for origin ($x = 0$, $y = 0$) in constraint (*vi*) $x + y \geq 60$, we have $0 \geq 60$ which is not true.

∴ Region for constraint (*vi*) is the half plane not containing the origin *i.e.*, region on the non-origin side of the line $x + y = 60$.



The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*vi*). The feasible region is bounded.

Step III. The coordinates of the corner point P are (60, 0).

Corner point Q: It is the point of intersection of bounding lines

$$x = 60 \quad \text{and} \quad x + y = 100$$

Putting $x = 60$, $60 + y = 100 \Rightarrow y = 100 - 60 = 40$

∴ Corner point Q is (60, 40).

Corner point R: It is the point of intersection of bounding lines

$$y = 50 \quad \text{and} \quad x + y = 100$$

Putting $y = 50$, $x + 50 = 100 \Rightarrow x = 100 - 50 = 50$

∴ Corner point R is (50, 50).

Corner point S: It is the point of intersection of bounding lines

$$y = 50 \quad \text{and} \quad x + y = 60$$

Putting $y = 50$, $x + 50 = 60 \Rightarrow x = 10$

∴ Corner point S is (10, 50).

Step IV. Now, we evaluate Z at each corner point.

Corner Point	$Z = \frac{1}{2}(5x + 3y + 820)$
P(60, 0)	560
Q(60, 40)	620
R(50, 50)	610
S(10, 50)	$510 = m$

← Minimum

By Corner Point Method, minimum Z = 510 at (10, 50).

Hence, the transportation cost is minimum, equal to ₹ 510, when the supplies are transported as under:

From / To	D	E	F
A	10	50	40
B	50	0	0

$(\because \quad x = 10, y = 50)$

7. An oil company has two depots A and B with capacities of 7000 L and 4000 L respectively. The company is to supply oil to three petrol pumps, D, E and F whose requirements are 4500L, 3000L and 3500L respectively. The distances (in km) between the depots and the petrol pumps is given in the following table:

Distance (in km.)		
From / To	**A**	**B**
D	**7**	**3**
E	**6**	**4**
F	**3**	**2**

Assuming that the transportation cost of 10 litres of oil is ₹ 1 per km, how should the delivery be scheduled in order that the transportation cost is minimum? What is the minimum cost?

Sol. Step I. Mathematical formulation of L.P.P.

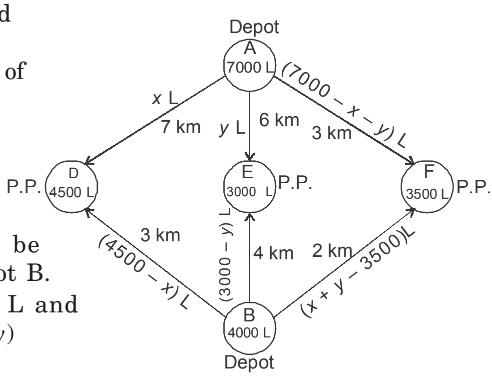
Let x L and y L of oil be transported from depot A to petrol pumps D and E respectively. Then $\{7000 - (x + y)\}$ L will be transported to petrol pump F.

Clearly, $x \geq 0$, $y \geq 0$ and $7000 - x - y \geq 0$ $(\Rightarrow 7000 \geq x + y)$

($\because$ Amounts of petrols (in litres) can't be negative)

i.e., $x \geq 0$, $y \geq 0$ and $x + y \leq 7000$

Now, the requirement of petrol pump D is 4500 L. Since, x L are transported from depot A, the remaining $(4500 - x)$ L need to be transported from depot B. Similarly, $(3000 - y)$ L and $3500 - (7000 - x - y) = (x + y - 3500)$



need to be transported from depot B to petrol pumps E and F respectively.

Clearly, $4500 - x \geq 0$, $3000 - y \geq 0$ (*i.e.*, $4500 \geq x$, $3000 \geq y$)

and $x + y - 3500 \geq 0$

i.e., $x \leq 4500$, $y \leq 3000$, $x + y \geq 3500$

Cost of transportation of 10 litres of oil is ₹ 1 per km

$\Rightarrow$ Cost of transportation of 1 litre of oil is ₹ $\frac{1}{10}$ per km.

Total transportation cost Z is given by

$$Z = \frac{1}{10}[7x + 6y + 3(7000 - x - y) + 3(4500 - x) + 4(3000 - y) + 2(x + y - 3500)]$$

$$= \frac{1}{10}(3x + y + 39500)$$

We have the following mathematical model for the given problem:

Minimise $Z = \frac{1}{10}(3x + y + 39500)$ ...(*i*)

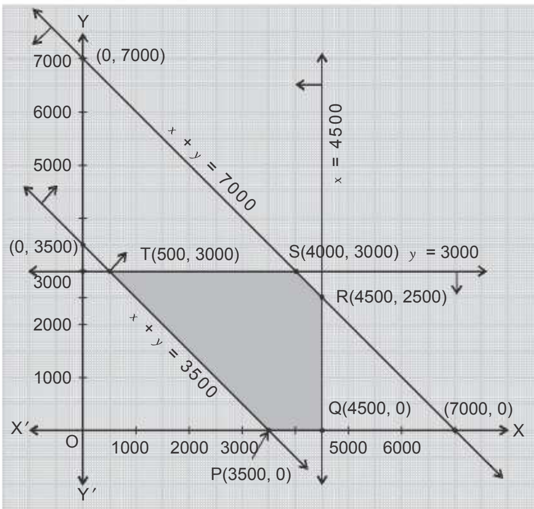
subject to the constraints:

$x \geq 0, y \geq 0$ ...(*ii*), $x + y \leq 7000$ ...(*iii*), $x \leq 4500$ ...(*iv*)

$y \leq 3000$ ...(*v*), $x + y \geq 3500$ ...(*vi*)

Step II. Step II of this question Q. No. 7 is very similar to step II of Q. No. 6 and is being left as an exercise for the reader.

The reader after drawing his or her graphs and regions should compare with the adjoining figure.



The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*vi*). The feasible region is bounded.

Step III. The coordinates of the corner points P and Q are (3500, 0) and (4500, 0) respectively.

Corner point R: It is the point of intersection of bounding lines $x = 4500$ and $x + y = 7000$

Putting $x = 4500$, $4500 + y = 7000$ $\Rightarrow$ $y = 7000 - 4500 = 2500$

$\therefore$ Corner point R is (4500, 2500).

Similarly corner points S and T are (4000, 3000) and (500, 3000) respectively.

(This is being left as an exercise for the reader).

Step IV. Now, we evaluate Z at each corner point.

Corner Point	$Z = \frac{1}{10}(3x + y + 39500)$	
P(3500, 0)	5000	
Q(4500, 0)	5300	
R(4500, 2500)	5550	
S(4000, 3000)	5450	
T(500, 3000)	$4400 = m$	$\leftarrow$ Minimum

By Corner Point Method, minimum Z = 4400 at (500, 3000).

Hence, the transportation cost is minimum, equal to ₹ 4400, when the supplies are transported as under:

From / To	D	E	F
A	500L	3000L	3500L
B	4000L	0L	0L

$(\because \; x = 500,$
$y = 3000)$

8. A fruit grower can use two types of fertilizer in his garden, brand P and brand Q. The amounts (in kg) of nitrogen, phosphoric acid, potash, and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid, at least 270 kg of potash and at most 310 kg of chlorine.

kg per bag		
	Brand P	**Brand Q**
Nitrogen	**3**	**3.5**
Phosphoric acid	**1**	**2**
Potash	**3**	**1.5**
Chlorine	**1.5**	**2**

If the grower wants to minimise the amount of nitrogen added to the garden, how many bags of each brand should be used? What is the minimum amount of nitrogen added in the garden?

Sol. Step I. Mathematical formulation of L.P.P.

Let the fruit grower use x bags of brand P and y bags of brand Q. The given data is condensed in the following table.

Brand of fertilizer	Number of bags	Amount in kg per bag			
		Nitrogen	Phosphoric Acid	Potash	Chlorine
P	x	3	1	3	1.5
Q	y	3.5	2	1.5	2

Amount of nitrogen $= 3x + 3.5y$

Let $Z = 3x + 3.5y$

We have the following mathematical model for the given problem:

Minimise $Z = 3x + 3.5y$ ...(i)

subject to the constraints:

$x + 2y \geq 240$ (Phosphoric acid constraint) ...(ii)

[**Given:** The garden needs at least ($i.e.,$ $\geq$) 240 kg of phosphoric acid]

$$3x + 1.5y \geq 270 \quad \text{or} \quad 3x + \frac{3}{2}y \geq 270$$

[**Given:** The garden atleast 270 kg of potash]

Dividing by 3 and multiplying by 2,

or $2x + y \geq 180$ (Potash constraint) ...(iii)

$$1.5x + 2y \leq 310 \quad \text{or} \quad \frac{3}{2}x + 2y \leq 310$$

[**Given:** The garden needs at the most $i.e.,$ $\leq$ 310 kg of chlorine]

Multiplying by 2, $3x + 4y \leq 620$.

or $3x + 4y \leq 620$ (Chlorine constraint) ...(iv)

$x, y \geq 0$ ...(v)

($\because$ Amounts of phosphoric acid, potash and chlorine can't be negative)

Step II. The region for constraint (v), $x, y \geq 0$

$\Rightarrow$ Feasible region is in first quadrant.

Table of values for the line $x + 2y = 240$ of constraint (ii)

x	0	240
y	120	0

Let us draw the line joining the points (0, 120) and (240, 0).

Let us test for origin ($x = 0, y = 0$) in constraint (ii), $x + 2y \geq 240$, we have $0 \geq 240$ which is not true.

$\therefore$ Region for constraint (ii) is on the non-origin side of the line $x + 2y = 240$ $i.e.,$ region is half plane on the above side of the line $x + 2y = 240$.

Table of values for the line $2x + y = 180$ for constraint (iii)

x	0	90
y	180	0

Let us draw the line joining the points (0, 180) and (90, 0).

Let us test for origin ($x = 0, y = 0$) in constraint (iii) $2x + y \geq 180$, we have $0 \geq 180$ which is not true.

$\therefore$ Again region for constraint (iii) is also the half-plane not containing the origin $i.e.,$ on the non-origin side of the line $2x + y = 180$.

Table of values for the line $3x + 4y = 620$ for constraint (iv)

x	0	$\frac{620}{3} = 200.7$
y	155	0

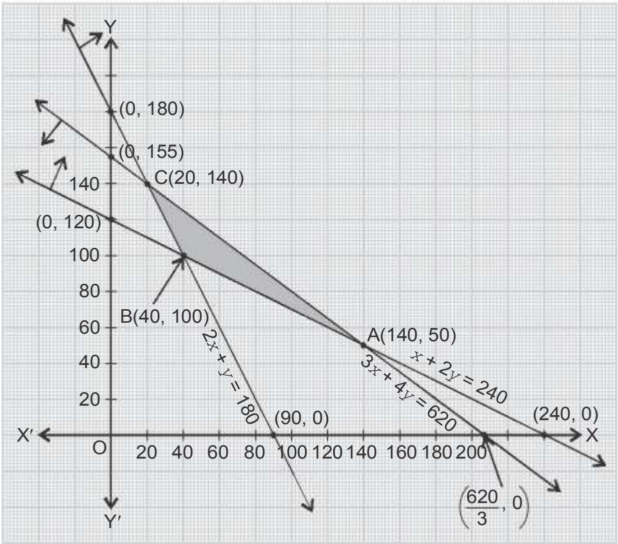
Let us draw the line joining the points (0, 155) and (200.7, 0).

Let us test for origin ($x = 0, y = 0$) in $3x + 4y \leq 620$, we have $0 \leq 620$ which is true.

$\therefore$ Region for constraint (*iv*) is on the origin side of the line

$$3x + 4y = 620.$$

The shaded region in the figure is the feasible region determined by the system of constraints from (*ii*) to (*v*). The feasible region is bounded.



Step III. Let us find the corner points A, B and C.

Corner point A: It is the point of intersection of the lines

$$x + 2y = 240 \quad \text{and} \quad 3x + 4y = 620$$

Second Eqn. – 3 × First equation gives

$$3x + 4y - 3x - 6y = 620 - 720$$

$$\Rightarrow \quad -2y = -100 \Rightarrow y = \frac{-100}{-2} = 50$$

Putting $y = 50$ in $x + 2y = 240$, we have

$$x + 100 = 140 \quad \Rightarrow \quad x = 140$$

$\therefore$ Corner point A is (140, 50).

Corner point B: It is the point of intersection of bounding lines

$$x + 2y = 240 \quad \text{and} \quad 2x + y = 180$$

First Eqn. – 2 × Second equation gives

$$x + 2y - 4x - 2y = 240 - 360$$

$$\Rightarrow \quad -3x = -120 \quad \Rightarrow \quad x = 40$$

Putting $x = 40$ in $x + 2y = 240$, we have

$$40 + 2y = 240 \quad \Rightarrow \quad 2y = 200 \quad \Rightarrow \quad y = 100$$

$\therefore$ Corner point B is (40, 100).

Corner point C: It is the point of intersection of bounding lines

$$2x + y = 180 \quad \text{and} \quad 3x + 4y = 620$$

Second Eqn. – 4 × First equation gives

$$3x - 8x = 620 - 720 \quad \Rightarrow \quad -5x = -100 \quad \Rightarrow \quad x = 20$$

Putting $x = 20$ in $2x + y = 180$, we have $40 + y = 180 \Rightarrow y = 140$

$\therefore$ Corner point C is (20, 140).

Step IV. Now, we evaluate Z at each corner point.

Corner Point	$Z = 3x + 3.5y$	
A(140, 50)	595	
B(40, 100)	$470 = m$	← Minimum
C(20, 140)	550	

By Corner Point Method, minimum Z = 470 at (40, 100).

$\therefore$ Minimum amount of nitrogen = 470 kg when 40 bags of brand P and 100 bags of brand Q are used.

9. Refer to Question 8. If the grower wants to maximise the amount of nitrogen added to the garden, how many bags of each brand should be added? What is the maximum amount of nitrogen added?

Sol. From the above Table of Step IV in solution of question 8, we find that Z = 595 is maximum at (140, 50).

$\therefore$ Maximum amount of nitrogen = 595 kg when 140 bags of brand P and 50 bags of brand Q are used.

10. A toy company manufactures two types of dolls, A and B. Market tests and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is at most half of that for dolls of type A. Further, the production level of dolls of type A can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of ₹ 12 and ₹ 16 per doll respectively on dolls A and B, how many of each should be produced weekly in order to maximise the profit? Solve it graphically.

Sol. Step I. Mathematical Formulation of L.P.P.

Let x dolls of type A and y dolls of type B be produced to have the maximum profit.

Given: Company makes profit of ₹ 12 and ₹ 16 per doll respectively on doll A and B.

$\Rightarrow$ Objective function is

Profit $\quad Z = 12x + 16y$

Constraint on number of dolls

Given: Combined production level of dolls should not exceed 1200 dolls per day.

$\Rightarrow \quad x + y \le 1200 \quad$ ...(*i*)

Again given demand for dolls of type B is at most half that for dolls of type A. At most $\Rightarrow \le$

$\Rightarrow \quad y \le \dfrac{x}{2} \quad$ ...(*ii*)

Again given: production level of dolls of type A can exceed three times the production of dolls of other type (B) by **at most** 600 units.

$\Rightarrow \quad x \le 3y + 600$

$\Rightarrow \quad x - 3y \le 600 \quad$ ...(*iii*)

Also $x \ge 0$, $y \ge 0$ because number of dolls can't be negative.

Step II. To draw the graphs for regions of all constraints and locate the common feasible region.

Constraint (*i*) is $x + y \leq 1200$

Replacing $\leq$ by =, $x + y = 1200$

x	0	1200
y	1200	0

(0, 1200)
(1200, 0)

$\therefore$ Graph of $x + y = 1200$ is the straight line joining the points (0, 1200) and (1200, 0).

Let us test for origin in (*i*),

Put $x = 0$ and $y = 0$ in (*i*), $0 \leq 1200$ which is true.

$\therefore$ Region given by (*i*) is towards the origin and is being shown by horizontal lines.

Constraint (*ii*) is $y \leq \frac{x}{2}$

Let us draw graph of $y = \frac{x}{2}$

x	0	400
y	0	200

$\therefore$ Graph of $y = \frac{x}{2}$ is the straight line joining (0, 0) and (400, 200).

Let us test for (1200, 0) in (*ii*), $0 \leq 600$ which is true.

$\therefore$ Region given by (*ii*) is towards the point (1200, 0), shown by vertical lines.

Constraint (*iii*) is $x - 3y \leq 600$

Let us draw the graph of $x - 3y = 600$

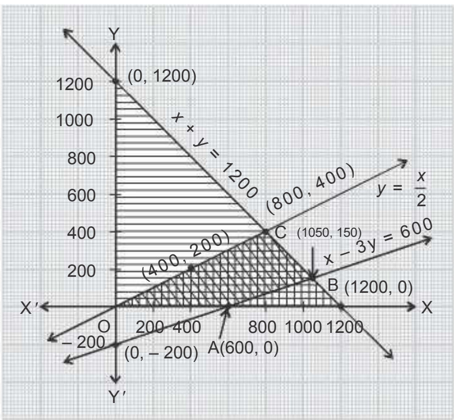
x	0	600
y	– 200	0

$\therefore$ Graph of $x - 3y = 600$ is the straight line joning the points (0, – 200) and (600, 0).

Let us test for origin (0, 0) in (*iii*).

Put $x = 0$ and $y = 0$ in (*iii*). $0 \leq 600$ which is true.

$\therefore$ Region given by (*iii*) is towards the origin shown by slanting lines.



The common feasible region is bounded by quadrilateral OABC.

Step III. The vertices of this feasible region are

O(0, 0); A(600, 0)

B, point of intersection of the lines:

$$x - 3y = 600$$

and $$x + y = 1200$$

Subtracting

$$-4y = -600$$

$\therefore$ $$y = \frac{600}{4} = 150$$

Putting $y = 150$ in $x + y = 1200$,

$$x + 150 = 1200$$

$\Rightarrow$ $$x = 1200 - 150 = 1050$$

$\therefore$ Corner point B(1050, 150)

Corner point C is point of intersection of lines:

$$y = \frac{x}{2}$$

and $$x + y = 1200$$

Solving $$x + \frac{x}{2} = 1200 \Rightarrow 2x + x = 2400$$

$\Rightarrow$ $$3x = 2400 \Rightarrow \frac{2400}{3} = 800$$

$\therefore$ $$y = \frac{x}{2} = \frac{800}{2} = 400$$

$\therefore$ Corner point C is (800, 400)

Step IV. Values of objective (profit) function Z at corner points are:

Corner point	Value of objective function $Z = 12x + 16y$
O(0, 0)	$Z = 12(0) + 16(0) = 0$
A(600, 0)	$Z = 12(600) + 16(0) = 7200$
B(1050, 150)	$Z = 12(1050) + 16(150)$ $= 12600 + 2400 = 15000$
C(800, 400)	$Z = 12(800) + 16(400)$ $= 9600 + 6400$ $=$ ₹ $16000 \rightarrow$ M

$\therefore$ Maximum profit is ₹ 16000 when $x = 800$, $y = 400$.