# Exercise 5.1

**1. Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.**

**Sol. Given:** $f(x) = 5x - 3$ ...(*i*)

**Continuity at $x = 0$**

$\lim_{x \to 0} f(x) = \lim_{x \to 0} (5x - 3)$ (By (*i*))

Putting $x = 0$, $= 5(0) - 3 = 0 - 3 = -3$

Putting $x = 0$ in (*i*), $f(0) = 5(0) - 3 = -3$

$\therefore \lim_{x \to 0} f(x) = f(0)\ (= -3)$ $\therefore$ $f(x)$ is continuous at $x = 0$.

**Continuity at $x = -3$**

$\lim_{x \to -3} f(x) = \lim_{x \to -3} (5x - 3)$ (By (*i*))

Putting $x = -3$, $= 5(-3) - 3 = -15 - 3 = -18$

Putting $x = -3$ in (*i*), $f(-3) = 5(-3) - 3 = -15 - 3 = -18$

$\therefore \lim_{x \to -3} f(x) = f(-3)(= -18)$

$\therefore$ $f(x)$ is continuous at $x = -3$.

**Continuity at $x = 5$**

$\lim_{x \to 5} f(x) = \lim_{x \to 5} (5x - 3)$ (By (*i*))

Putting $x = 5$, $5(5) - 3 = 25 - 3 = 22$

Putting $x = 5$ in (*i*), $f(5) = 5(5) - 3 = 25 - 3 = 22$

$\therefore \lim_{x \to 5} (5x - 3) = f(5)\ (= 22)$ $\therefore$ $f(x)$ is continuous at $x = 5$.

**2. Examine the continuity of the function**

**$f(x) = 2x^2 - 1$ at $x = 3$.**

**Sol. Given:** $f(x) = 2x^2 - 1$ ...(*i*)

**Continuity at $x = 3$**

$\lim_{x \to 3} f(x) = \lim_{x \to 3} (2x^2 - 1)$ [By (*i*)]

Putting $x = 3$, $= 2.3^2 - 1 = 2(9) - 1 = 18 - 1 = 17$

Putting $x = 3$ in (*i*), $f(3) = 2.3^2 - 1 = 18 - 1 = 17$

$\therefore \lim_{x \to 3} f(x) = f(3)\ (= 17)$ $\therefore$ $f(x)$ is continuous at $x = 3$.

**3. Examine the following functions for continuity:**

**(*a*) $f(x) = x - 5$** **(*b*) $f(x) = \dfrac{1}{x - 5}$, $x \neq 5$**

**(*c*) $f(x) = \dfrac{x^2 - 25}{x + 5}$, $x \neq -5$** **(*d*) $f(x) = |x - 5|$.**

**Sol.** (*a*) **Given:** $f(x) = x - 5$ ...(*i*)

The domain of $f$ is R

($\because$ $f(x)$ is real and finite for all $x \in$ R)

Let $c$ be any real number (*i.e.,* $c \in$ domain of $f$).

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x - 5) \quad \text{[By } (i)\text{]}$$

Putting $x = c$, $= c - 5$

Putting $x = c$ in (*i*), $f(c) = c - 5$

$\therefore$ $\lim_{x \to c} f(x) = f(c)$ $(= c - 5)$

$\therefore$ $f$ is continuous at every point $c$ in its domain (here R).

Hence $f$ is continuous.

**Or**

Here $f(x) = x - 5$ is a polynomial function. We know that every polynomial function is continuous (see note below).

Hence $f(x)$ is continuous (in its domain R)

**Very important Note.** The following functions are continuous (for all $x$ in their domain).

1. Constant function
2. Polynomial function.
3. Rational function $\dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomial functions of $x$ and $g(x) \neq 0$.
4. Sine function ($\Rightarrow$ $\sin x$).
5. $\cos x$.
6. $e^x$.
7. $e^{-x}$.
8. $\log x$ $(x > 0)$.
9. Modulus function.

(*b*) **Given:** $f(x) = \dfrac{1}{x-5}$, $x \neq 5$ ...(*i*)

**Given:** The domain $f$ is R – ($x \neq 5$) *i.e.,* R – {5}

($\because$ For $x = 5$, $f(x) = \dfrac{1}{x-5} = \dfrac{1}{5-5} = \dfrac{1}{0} \to \infty$

$\therefore$ $5 \notin$ domain of $f$)

Let $c$ be any real number such that $c \neq 5$

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x-5} \quad \text{[By } (i)\text{]}$$

Putting $x = c$, $= \dfrac{1}{c-5}$

Putting $x = c$ in (*i*), $f(c) = \dfrac{1}{c-5}$

$\therefore$ $\lim_{x \to c} f(x) = f(c) \left(= \dfrac{1}{c-5}\right)$

$\therefore$ $f(x)$ is continuous at every point $c$ in the domain of $f$.

Hence $f$ is continuous.

**Or**

Here $f(x) = \dfrac{1}{x-5}$, $x \neq 5$ is a rational function

$\left( = \dfrac{\text{Polynomial 1 of degree 0}}{\text{Polynomial } (x-5) \text{ of degree 1}} \right)$ and its denominator *i.e.,* $(x - 5) \neq 0$ ($\because$ $x \neq 5$). We know that every rational function is continuous (By Note below Solution of Q. No. 3(*a*)). Therefore $f$ is continuous (in its domain R – {5}).

(*c*) $f(x) = \dfrac{x^2 - 25}{x+5}$, $x \neq -5$

Here $f(x) = \dfrac{x^2 - 25}{x+5}$, $x \neq -5$ is a rational function and denominator $x + 5 \neq 0$ ($\because$ $x \neq -5$).

(In fact $f(x) = \dfrac{x^2 - 25}{x+5}$, $(x \neq -5) = \dfrac{(x+5)(x-5)}{x+5}$ $= x - 5$, $(x \neq -5)$ is a polynomial function). We know that every rational function is continuous. Therefore $f$ is continuous (in its domain R – {– 5}).

**Or**

*Proceed as in Method I of Q. No. 3(b).*

(*d*) **Given:** $f(x) = |x - 5|$

Domain of $f(x)$ is R ($\because$ $f(x)$ is real and finite for all real $x$ in $(-\infty, \infty)$)

Here $f(x) = |x - 5|$ is a modulus function.

We know that every modulus function is continuous.

(By Note below Solution of Q. No. 3(*a*)). Therefore $f$ is continuous in its domain R.

**4. Prove that the function $f(x) = x^n$ is continuous at $x = n$ where $n$ is a positive integer.**

**Sol.** **Given:** $f(x) = x^n$ where $n$ is a positive integer. ...(*i*)

Domain of $f(x)$ is R ($\because$ $f(x)$ is real and finite for all real $x$)

Here $f(x) = x^n$, where $n$ is a positive integer.

We know that every polynomial function of $x$ is a continuous function. Therefore, $f$ is continuous (in its whole domain R) and hence continuous at $x = n$ also.

**Or**

$$\lim_{x \to n} f(x) = \lim_{x \to n} x^n \qquad \text{[By } (i)\text{]}$$

Putting $x = n$, $= n^n$

Again putting $x = n$ in $(i)$, $f(n) = n^n$

$\therefore \lim_{x \to n} f(x) = f(n)\ (= n^n)$ $\qquad \therefore$ $f(x)$ is continuous at $x = n$.

**5. Is the function $f$ defined by**

$$f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$$

**continuous at $x = 0$?, At $x = 1$?, At $x = 2$?**

**Sol.** **Given:** $f(x) = \begin{cases} x, & \text{if } x \le 1 & ...(i) \\ 5, & \text{if } x > 1 & ...(ii) \end{cases}$

(Read Note **(on continuity)** before the solution of Q. No. 1 of this exercise)

**Continuity at $x = 0$**

Left Hand Limit $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x$ [By $(i)$]

($x \to 0^- \Rightarrow$ $x <$ slightly less than 0 $\Rightarrow$ $x < 1$)

Putting $x = 0$, $= 0$

Right hand limit $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x$ [By $(i)$]

($x \to 0^+ \Rightarrow x$ is slightly greater than 0 say $x = 0.001$ $\Rightarrow$ $x < 1$)

Putting $x = 0$, $\lim_{x \to 0^+} f(x) = 0$ $\qquad \therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0$

$\therefore \lim_{x \to 0} f(x)$ exists and $= 0 = f(0)$

($\because$ Putting $x = 0$ in $(i)$, $f(0) = 0$)

$\therefore$ $f(x)$ is continuous at $x = 0$.

**Continuity at $x = 1$**

Left Hand Limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x$ [By $(i)$]

Putting $x = 1$, $= 1$

Right Hand Limit $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 5$

Putting $x = 1$, $\lim_{x \to 1^+} f(x) = 5$

$\therefore \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$ $\qquad \therefore \lim_{x \to 1} f(x)$ does not exist.

$\therefore$ $f(x)$ is discontinuous at $x = 1$.

**Continuity at $x = 2$**

Left Hand Limit $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 5$ [By $(ii)$]

($x \to 2-$ $\Rightarrow x$ is slightly $< 2 \Rightarrow x = 1.98$ (say) $\Rightarrow x > 1$)

Putting $x = 2$, $= 5$

Right Hand Limit $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 5$ [By ($ii$)]

($x \to 2+$ $\Rightarrow$ $x$ is slightly $> 2$ and hence $x > 1$ also)

Putting $x = 2$, $= 5$

$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ $(= 5)$

$\therefore \lim_{x \to 2} f(x)$ exists and $= 5 = f(2)$

(Putting $x = 2 > 1$ in ($ii$), $f(2) = 5$)

$\therefore$ $f(x)$ is continuous at $x = 2$

**Answer.** $f$ is continuous at $x = 0$ and $x = 2$ but not continuous at $x = 1$.

**Find all points of discontinuity of $f$, where $f$ is defined by (Exercises 6 to 12)**

**6.** $f(x) = \begin{cases} 2x + 3, & x \le 2 \\ 2x - 3, & x > 2 \end{cases}$.

**Sol. Given:** $f(x) = 2x + 3$, $x \le 2$ ...($i$)

$= 2x - 3$ $x > 2$ ...($ii$)

*To find points of discontinuity of f (in its domain)*

Here $f(x)$ is defined for $x \le 2$ *i.e.,* on $(-\infty, 2]$

and also for $x > 2$ *i.e.,* on $(2, \infty)$

$\therefore$ Domain of $f$ is $(-\infty, 2] \cup (2, \infty) = (-\infty, \infty) = \mathrm{R}$

By ($i$), for all $x < 2$ ($x = 2$ being partitioning point can't be mentioned here) $f(x) = 2x + 3$ is a polynomial and hence continuous.

By ($ii$), for all $x > 2$, $f(x) = 2x - 3$ is a polynomial and hence continuous. Therefore $f(x)$ is continuous on $\mathrm{R} - \{2\}$.

**Let us examine continuity of $f$ at partitioning point $x = 2$**

Left Hand Limit $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 3)$ [By ($i$)]

Putting $x = 2$, $= 2(2) + 3 = 4 + 3 = 7$

Right Hand Limit $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 3)$ [By ($ii$)]

Putting $x = 2$, $= 2(2) - 3 = 4 - 3 = 1$

$\therefore \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

$\therefore \lim\limits_{x \to 2} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 2$ (only).

**7.** $\boldsymbol{f(x) = \begin{cases} |x| + 3, & \text{if} \quad x \le -3 \\ -2x, & \text{if} \quad -3 < x < 3 \\ 6x + 2, & \text{if} \quad x \ge 3 \end{cases}}$.

**Sol. Given:** $f(x) = \begin{cases} |x| + 3, & \text{if} \quad x \le -3 & \ldots(i) \\ -2x, & \text{if} \quad -3 < x < 3 & \ldots(ii) \\ 6x + 2, & \text{if} \quad x \ge 3 & \ldots(iii) \end{cases}$

Here $f(x)$ is defined for $x \le -3$ *i.e.,* $(-\infty, -3]$ and also for $-3 < x < 3$ and also for $x \ge 3$ *i.e.,* on $[3, \infty)$.

$\therefore$ Domain of $f$ is $(-\infty, -3] \cup (-3, 3) \cup [3, \infty) = (-\infty, \infty) = \mathbb{R}$.

By $(i)$, for all $x < -3$, $f(x) = |x| + 3 = -x + 3$

($\because$ $x < -3$ means $x$ is negative and hence $|x| = -x$)

is a polynomial and hence continuous.

By $(ii)$, for all $x$ $(-3 < x < 3)$ $f(x) = -2x$ is a polynomial and hence continuous.

By $(iii)$, for all $x > 3$, $f(x) = 6x + 2$ is a polynomial and hence continuous. Therefore, $f(x)$ is continuous on $\mathbb{R} - \{-3, 3\}$.

From $(i)$, $(ii)$ and $(iii)$ we can observe that $x = -3$ and $x = 3$ are partitioning points of the domain R.

**Let us examine continuity of $\boldsymbol{f}$ at partitioning point $\boldsymbol{x = -3}$**

Left Hand Limit $= \lim\limits_{x \to -3^-} f(x) = \lim\limits_{x \to -3^-} (|x| + 3)$ [By $(i)$]

($\because$ $x \to -3^- \Rightarrow x < -3$)

$= \lim\limits_{x \to -3^-} (-x + 3)$

($\because$ $x \to -3^- \Rightarrow x < -3$ means $x$ is negative and hence $|x| = -x$)

Put $x = -3$, $= 3 + 3 = 6$

Right Hand Limit $= \lim\limits_{x \to -3^+} f(x) = \lim\limits_{x \to -3^+} (-2x)$ [By $(ii)$]

($\because$ $x \to -3^+ \Rightarrow x > -3$)

Putting $x = -3$, $= -2(-3) = 6$

$\therefore \lim\limits_{x \to -3^+} f(x) = \lim\limits_{x \to -3^+} f(x)$ $(= 6)$

$\therefore \lim\limits_{x \to -3} f(x)$ exists and $= 6$

Putting $x = -3$ in $(i)$, $f(-3) = |-3| + 3 = 3 + 3 = 6$

$\therefore \lim\limits_{x \to -3} f(x) = f(-3)$ $(= 6)$

$\therefore$ $f(x)$ is continuous at $x = -3$.

**Now let us examine continuity of $f$ at partitioning point $x = 3$**

Left Hand Limit $= \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-2x)$ [By (*ii*)]

$(\because \ x \to 3^- \Rightarrow x < 3)$

Putting $x = 3$, $= -2(3) = -6$

Right Hand Limit $= \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x + 2)$ [By (*iii*)]

$(\because \ x \to 3^+ \Rightarrow x > 3)$

Putting $x = 3$, $= 6(3) + 2 = 18 + 2 = 20$

$\therefore \quad \lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

$\therefore \quad \lim_{x \to 3} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 3$ (only).

**8.** $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$.

**Sol. Given:** $f(x) = \dfrac{|x|}{x}$ if $x \neq 0$

[*i.e.*, $= \dfrac{x}{x} = 1$ if $x > 0$ $(\because$ For $x > 0$, $|x| = x)$

and $= -\dfrac{x}{x} = -1$ if $x < 0$ $(\because$ For $x < 0, |x| = -x)$

*i.e.*, $f(x) = 1$ if $x > 0$ ...(*i*)

$= -1$ if $x < 0$ ...(*ii*)

$= 0$ if $x = 0$ ...(*iii*)

Clearly domain of $f(x)$ is R $(\because$ $f(x)$ is defined for $x > 0$, for $x < 0$ and also for $x = 0)$

By (*i*), for all $x > 0$, $f(x) = 1$ is a constant function and hence continuous.

By (*ii*), for all $x < 0$, $f(x) = -1$ is a constant function and hence continuous.

Therefore $f(x)$ is continuous on R – {0}.

**Let us examine continuity of $f$ at the partitioning point $x = 0$**

Left Hand Limit $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -1$ [By (*ii*)]

$(\because \ x \to 0^- \Rightarrow x < 0)$

Put $x = 0$, $= -1$

Right Hand Limit $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1$ [By (*i*)]

$(\because \ x \to 0^+ \Rightarrow x > 0)$

Put $x = 0$, $= 1$

$\therefore \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\therefore \quad \lim_{x \to 0} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 0$ (only).

**Note.** It may be noted that the function given in Q. No. 8 is called a **signum function.**

**9.** $\boldsymbol{f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}}$.

**Sol. Given:**

$$f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 = \dfrac{x}{-x} = -1 \quad \text{if } x < 0 \quad ...(i) \\ & (\because \text{ For } x < 0, |x| = -x) \\ -1 & \text{if } x \geq 0 \quad ...(ii) \end{cases}$$

Here $f(x)$ is defined for $x < 0$ *i.e.,* on $(-\infty, 0)$ and also for $x \geq 0$ *i.e.,* on $[0, \infty)$.

$\therefore$ Domain of $f$ is $(-\infty, 0) \cup [0, \infty) = (-\infty, \infty) = \text{R}$.

From $(i)$ and $(ii)$, we find that

$$f(x) = -1 \text{ for all real } x \ (< 0 \text{ as well as } \geq 0)$$

Here $f(x) = -1$ is a constant function.

We know that every constant function is continuous.

$\therefore$ $f$ is continuous (for all real $x$ in its domain R)

Hence no point of discontinuity.

**10.** $\boldsymbol{f(x) = \begin{cases} x + 1, & \text{if } x \geq 1 \\ x^2 + 1, & \text{if } x < 1 \end{cases}}$.

**Sol. Given:** $\begin{cases} x + 1, & \text{if } x \geq 1 \quad ...(i) \\ x^2 + 1, & \text{if } x < 1 \quad ...(ii) \end{cases}$

Here $f(x)$ is defined for $x \geq 1$ *i.e.,* on $[1, \infty)$ and also for $x < 1$ *i.e.,* on $(-\infty, 1)$.

Domain of $f$ is $(-\infty, 1) \cup [1, \infty) = (-\infty, \infty) = \text{R}$

By $(i)$, for all $x > 1$, $f(x) = x + 1$ is a polynomial and hence continuous.

By $(ii)$, for all $x < 1$, $f(x) = x^2 + 1$ is a polynomial and hence continuous. Therefore $f$ is continuous on $\text{R} - \{1\}$.

**Let us examine continuity of $f$ at the partitioning point $x = 1$.**

Left Hand Limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1)$ [By $(ii)$]

$(\because \ x \to 1^- \Rightarrow x < 1)$

Putting $x = 1$, $= 1^2 + 1 = 1 + 1 = 2$

Right Hand Limit = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1)$ [By ($i$)]

($\because$ $x \to 1^+ \Rightarrow x > 1$)

Putting $x = 1$, $= 1 + 1 = 2$

$\therefore$ $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$ (= 2)

$\therefore$ $\lim_{x \to 1} f(x)$ exists and = 2

Putting $x = 1$ in ($i$), $f(1) = 1 + 1 = 2$

$\therefore$ $\lim_{x \to 1} f(x) = f(1)$ (= 2)

$\therefore$ $f(x)$ is continuous at $x = 1$ also.

$\therefore$ $f$ is be continuous on its whole domain (R here).

Hence no point of discontinuity.

**11.** $f(x) = \begin{cases} x^3 - 3, & \text{if } x \le 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$.

**Sol. Given:** $f(x) = \begin{cases} x^3 - 3, & \text{if } x \le 2 & ...(i) \\ x^2 + 1, & \text{if } x > 2 & ...(ii) \end{cases}$

Here $f(x)$ is defined for $x \le 2$ *i.e.,* on

$(-\infty, 2]$ and also for $x > 2$ *i.e.,* on $(2, \infty)$.

$\therefore$ Domain of $f$ is $(-\infty, 2] \cup (2, \infty) = (-\infty, \infty) = R$

By ($i$), for all $x < 2$, $f(x) = x^3 - 3$ is a polynomial and hence continuous.

By ($ii$), for all $x > 2$, $f(x) = x^2 + 1$ is a polynomial and hence continuous.

$\therefore$ $f$ is continuous on R – {2}.

**Let us examine continuity of $f$ at the partitioning point $x = 2$.**

Left Hand Limit = $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^3 - 3)$ [By ($i$)]

($\because$ $x \to 2^- \Rightarrow x < 2$)

Putting $x = 2$, $= 2^3 - 3 = 8 - 3 = 5$

Right Hand Limit = $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 1)$ [By ($ii$)]

($\because$ $x \to 2^+ \Rightarrow x > 2$)

Putting $x = 2$, $= 2^2 + 1 = 4 + 1 = 5$

$\therefore$ $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ (= 5)

$\therefore$ $\lim_{x \to 2} f(x)$ exists and = 5

Putting $x = 2$ in ($i$), $f(2) = 2^3 - 3 = 8 - 3 = 5$

$\therefore$ $\lim_{x \to 2} f(x) = f(2)$ (= 5)

$\therefore$ $f(x)$ is continuous at $x = 2$ (also).

Hence no point of discontinuity.

**12.** $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$.

**Sol. Given:** $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 & ...(i) \\ x^2, & \text{if } x > 1 & ...(ii) \end{cases}$

Here $f(x)$ is defined for $x \le 1$ *i.e.,* on $(-\infty, 1]$ and also for $x > 1$ *i.e.,* on $(1, \infty)$.

$\therefore$ Domain of $f$ is $(-\infty, 1] \cup (1, \infty) = (-\infty, \infty) = \text{R}$

By $(i)$, for all $x < 1$, $f(x) = x^{10} - 1$ is a polynomial and hence continuous.

By $(ii)$, for all $x > 1$, $f(x) = x^2$ is a polynomial and hence continuous.

$\therefore$ $f(x)$ is continuous on $\text{R} - \{1\}$.

**Let us examine continuity of $f$ at the partitioning point $x = 1$.**

Left Hand Limit $= \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x^{10} - 1)$ [By $(i)$]

$(\because\ x \to 1^- \Rightarrow x < 1)$

Putting $x = 1$, $= (1)^{10} - 1 = 1 - 1 = 0$

Right Hand Limit $= \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} x^2$ [By $(ii)$]

Putting $x = 1$, $= 1^2 = 1$

$\therefore$ $\lim\limits_{x \to 1^-} f(x) \ne \lim\limits_{x \to 1^+} f(x)$

$\therefore$ $\lim\limits_{x \to 1} f(x)$ does not exist.

Hence the point of discontinuity is $x = 1$ (only).

**13. Is the function defined by**

$$f(x) = \begin{cases} x + 5 & \text{if } x \le 1 \\ x - 5 & \text{if } x > 1 \end{cases}$$

**a continuous function?**

**Sol. Given:** $f(x) = \begin{cases} x + 5, & \text{if } x \le 1 & ...(i) \\ x - 5, & \text{if } x > 1 & ...(ii) \end{cases}$

Here $f(x)$ is defined for $x \le 1$ *i.e.,* on $(-\infty, 1]$ and also for $x > 1$ *i.e.,* on $(1, \infty)$

$\therefore$ Domain of $f$ is $(-\infty, 1] \cup (1, \infty] = (-\infty, \infty) = \text{R}$.

By $(i)$, for all $x < 1$, $f(x) = x + 5$ is a polynomial and hence continuous.

By $(ii)$, for all $x > 1$, $f(x) = x - 5$ is a polynomial and hence continuous.

$\therefore$ $f$ is continuous on R – {1}.

**Let us examine continuity at the partitioning point $x = 1$.**

Left Hand Limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 5)$ [By ($i$)]

Putting $x = 1$, $= 1 + 5 = 6$

Right Hand Limit $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x - 5)$ [By ($ii$)]

Putting $x = 1$, $= 1 - 5 = -4$

$\therefore \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

$\therefore \lim_{x \to 1} f(x)$ does not exist.

Hence $f(x)$ is discontinuous at $x = 1$.

$\therefore$ $x = 1$ is the only point of discontinuity.

**Discuss the continuity of the function, $f$, where $f$ is defined by**

**14.** $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$.

**Sol. Given:**

$$f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 & \text{...}(i) \\ 4, & \text{if } 1 < x < 3 & \text{...}(ii) \\ 5, & \text{if } 3 \le x \le 10 & \text{...}(iii) \end{cases}$$

From ($i$), ($ii$) and ($iii$), we can see that $f(x)$ is defined in $[0, 1] \cup (1, 3) \cup [3, 10]$ *i.e.,* $f(x)$ is defined in $[0, 10]$.

$\therefore$ Domain of $f(x)$ is $[0, 10]$.

From ($i$), for $0 \le x < 1$, $f(x) = 3$ is a constant function and hence is continuous for $0 \le x < 1$.

From ($ii$), for $1 < x < 3$, $f(x) = 4$ is a constant function and hence is continuous for $1 < x < 3$.

From ($iii$), for $3 < x \le 10$, $f(x) = 5$ is a constant function and hence is continuous for $3 < x \le 10$.

Therefore, $f(x)$ is continuous in the domain $[0, 10] - \{1, 3\}$.

**Let us examine continuity of $f$ at the partitioning point $x = 1$.**

Left Hand Limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3$ [By ($i$)]

($\because$ $x \to 1^- \Rightarrow x < 1$)

Putting $x = 1$; $= 3$

Right Hand Limit $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4$ [By ($ii$)]

($\because$ $x \to 1^+ \Rightarrow x > 1$)

Putting $x = 1$, $= 4$

$\therefore \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

$\therefore \quad \lim_{x \to 1} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 1$.

**Let us examine continuity of $f$ at the partitioning point $x = 3$.**

Left Hand Limit $= \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 4$ [By $(ii)$]

$(\because \; x \to 3^- \Rightarrow x < 3)$

Putting $x = 3$, $= 4$

Right Hand Limit $= \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 5$ [By $(iii)$]

$(\because \; x \to 3^+ \Rightarrow x > 3)$

Putting $x = 3$; $= 5$

$\therefore \quad \lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

$\therefore \quad \lim_{x \to 3} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 3$ also.

$\therefore \quad x = 1$ and $x = 3$ are the two points of discontinuity of the function $f$ in its domain [0, 10].

**15.** $\boldsymbol{f(x) = \begin{cases} 2x, & \text{if} \quad x < 0 \\ 0, & \text{if} \quad 0 \le x \le 1 \\ 4x, & \text{if} \quad x > 1 \end{cases}}$**.**

**Sol.** The domain of $f$ is $\{x \in \text{R} : x < 0\} \cup \{x \in \text{R} : 0 \le x \le 1\} \cup \{x \in \text{R} : x > 1\} = \text{R}$

$x = 0$ and $x = 1$ are partitioning points for the domain of this function.

**For all $x < 0$,** $f(x) = 2x$ is a polynomial and hence continuous.

**For $0 < x < 1$,** $f(x) = 0$ is a constant function and hence continuous.

**For all $x > 1$,** $f(x) = 4x$ is a polynomial and hence continuous.

**Let us discuss continuity at partitioning point $x = 0$.**

**At $x = 0$,** $f(0) = 0$ $[\because f(x) = 0 \text{ if } 0 \le x \le 1]$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 2x$ $[\because x \to 0^- \Rightarrow x < 0 \text{ and } f(x) = 2x \text{ for } x < 0]$

$= 2 \times 0 = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 0$ $[\because x \to 0^+ \Rightarrow x > 0 \text{ and } f(x) = 0 \text{ if } 0 \le x \le 1]$

$= 0$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0$

Thus $\lim_{x \to 0} f(x) = 0 = f(0)$ and hence $f$ is continuous at 0.

**Let us discuss continuity at partitioning point $x = 1$.**

**At $x = 1$,** $f(1) = 0$ $[\because f(x) = 0 \text{ if } 0 \le x \le 1]$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 0 \quad [x \to 1- \Rightarrow x < 1 \text{ and } f(x) = 0 \text{ if } 0 \le x \le 1]$$
$$= 0$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4x \quad [x \to 1+ \Rightarrow x > 1 \text{ and } f(x) = 4x \text{ for } x > 1]$$
$$= 4 \times 1 = 4$$

The left and right hand limits of $f$ at $x = 1$ do not coincide *i.e.*, are not equal.

$\therefore \lim\limits_{x \to 1} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 1$.

Thus $f$ is continuous at every point in the domain except $x = 1$. Hence, $f$ is not a continuous function and $x = 1$ is the only point of discontinuity.

**16.** $\boldsymbol{f(x) = \begin{cases} -2, & \text{if} \quad x \le -1 \\ 2x, & \text{if} \quad -1 < x \le 1 \\ 2, & \text{if} \quad x > 1 \end{cases}}$.

**Sol. Given:** $f(x) = \begin{cases} -2, & \text{if} \quad x \le -1 & \ldots(i) \\ 2x, & \text{if} \quad -1 < x \le 1 & \ldots(ii) \\ 2, & \text{if} \quad x > 1 & \ldots(iii) \end{cases}$

From $(i)$, $(ii)$ and $(iii)$ we can see that $f(x)$ is defined for

$\{x : x \le -1\} \cup \{x : -1 < x \le 1\} \cup \{x : x > 1\}$

*i.e.,* for $(-\infty, -1] \cup (-1, 1] \cup (1, \infty) = (-\infty, \infty) = \text{R}$

$\therefore$ Domain of $f(x)$ is R.

From $(i)$, for $x < -1$, $f(x) = -2$ is a constant function and hence is continuous for $x < -1$.

From $(ii)$, for $-1 < x < 1$, $f(x) = 2x$ is a polynomial function and hence is continuous for $-1 < x < 1$.

From $(iii)$, for $x > 1$, $f(x) = 2$ is a constant function and hence is continuous for $x > 1$.

Therefore $f(x)$ is continuous in $\text{R} - \{-1, 1\}$.

**Let us examine continuity of $f$ at the partitioning point $x = -1$.**

Left Hand Limit $= \lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} (-2)$ [By $(i)$]

$(\because \ x \to -1^- \Rightarrow x < -1)$

Putting $x = -1$, $= -2$

Right Hand Limit $= \lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} 2x$ (By $(ii)$]

$(\because \ x \to -1^+ \Rightarrow x > -1)$

Putting $x = -1$, $= 2(-1) = -2$

$\therefore \lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^+} f(x)$ $(= -2)$ $\therefore \lim\limits_{x \to -1} f(x)$ exists and $= -2$.

Putting $x = -1$ in $(i)$, $f(-1) = -2$

$\therefore \quad \lim_{x \to -1} f(x) = f(-1) \ (= -2) \ \therefore \quad f(x)$ is continuous at $x = -1$.

**Let us examine continuity of $f$ at the partitioning point $x = 1$**

Left Hand Limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x)$ [By (*ii*)]

$(\because \quad x \to 1^- \Rightarrow x < 1)$

Putting $x = 1$, $= 2(1) = 2$

Right Hand Limit $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2$ [By (*iii*)]

$(\because \quad x \to 1^+ \Rightarrow x > 1)$

Putting $x = 1$, $= 2$

$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \ (= 2) \quad \therefore \quad \lim_{x \to 1} f(x)$ exists and $= 2$.

Putting $x = 1$ in (*ii*), $f(1) = 2(1) = 2$

$\therefore \quad \lim_{x \to 1} f(x) = f(1) \ (= 2) \quad \therefore \quad f(x)$ is continuous at $x = 1$ also.

Therefore $f$ is continuous for all $x$ in its domain R.

**17. Find the relationship between $a$ and $b$ so that the function $f$ defined by**

$$f(x) = \begin{cases} ax + 1, & \text{if} \quad x \le 3 \\ bx + 3, & \text{if} \quad x > 3 \end{cases}$$

**is continuous at $x = 3$.**

**Sol. Given:** $f(x) = \begin{cases} ax + 1 & \text{if} \quad x \le 3 & \text{...}(i) \\ bx + 3 & \text{if} \quad x > 3 & \text{...}(ii) \end{cases}$

and $f(x)$ is continuous at $x = 3$.

Left Hand Limit $= \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax + 1)$ [By (*i*)]

$(x \to 3^- \Rightarrow x < 3)$

Putting $x = 3$, $= 3a + 1$ ...(*iii*)

Right Hand Limit $= \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (bx + 3)$ [By (*ii*)]

$(\because \quad x \to 3^+ \Rightarrow x > 3)$

Putting $x = 3$, $= 3b + 3$ ...(*iv*)

Putting $x = 3$ in (*i*), $f(3) = 3a + 1$ ...(*v*)

Because $f(x)$ is continuous at $x = 3$ (given)

$\therefore \quad \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$

Putting values from (*iii*), (*iv*) and (*v*) we have

$3a + 1 = 3b + 3 \quad (= 3a + 1)$

$\therefore \quad 3a + 1 = 3b + 3$ [$\because$ First and third members are equal]

$\Rightarrow \quad 3a = 3b + 2$

Dividing by 3, $a = b + \frac{2}{3}$.

**18. For what value of $\lambda$ is the function defined by**

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if} \quad x \le 0 \\ 4x + 1, & \text{if} \quad x > 0 \end{cases}$$

**continuous at $x = 0$? What about continuity at $x = 1$?**

**Sol. Given:** $$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if} \quad x \le 0 \quad \ldots(i) \\ 4x + 1, & \text{if} \quad x > 0 \quad \ldots(ii) \end{cases}$$

**Given:** $f(x)$ is continuous at $x = 0$. To find $\lambda$.

Left Hand Limit $= \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \lambda(x^2 - 2x)$ [By $(i)$]

$(\because \ x \to 0^- \Rightarrow x < 0)$

Putting $x = 0$, $= \lambda(0 - 0) = 0$

Right Hand Limit $= \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (4x + 1)$ [By $(ii)$]

$(\because \ x \to 0^+ \Rightarrow x > 0)$

Putting $x = 0$, $= 4(0) + 1 = 1$

$\therefore \lim\limits_{x \to 0^-} f(x) \ (= 0) \ne \lim\limits_{x \to 0^+} f(x) \ (= 1)$

$\therefore \lim\limits_{x \to 0} f(x)$ does not exist whatever $\lambda$ may be

$(\because$ Neither left limit nor right limit involves $\lambda)$

$\therefore$ For no value of $\lambda$, $f$ is continuous at $x = 0$.

**To examine continuity of $f$ at $x = 1$**

Left Hand Limit $= \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (4x + 1)$ [By $(ii)$]

$(x \to 1^- \Rightarrow x$ is slightly $< 1$ say $x = 0.99 > 0)$

**Put $x = 1$,** $= 4 + 1 = 5$

Right Hand Limit $= \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (4x + 1)$ [By $(ii)$]

$(x \to 1^+ \Rightarrow x$ is slightly $> 1$ say $x = 1.1 > 0)$

Put $x = 1$, $= 4 + 1 = 5$

$\therefore \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) \ (= 5)$

$\therefore \lim\limits_{x \to 1} f(x)$ exists and $= 5$

Putting $x = 1$ in $(ii)$ $(\because \ 1 > 0)$, $f(1) = 4 + 1 = 5)$

$\therefore \lim\limits_{x \to 1} f(x) = f(1) \ (= 5)$

$\therefore f(x)$ is continuous at $x = 1$ (for all real values of $\lambda$).

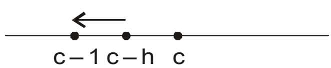
**19. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to $x$.**

**Sol. Given:** $g(x) = x - [x]$

Let $x = c$ be any integer (*i.e.*, $c \in$ Z (= I))

Left Hand Limit $= \lim\limits_{x \to c^-} g(x) = \lim\limits_{x \to c^-} (x - [x])$

Put $x = c - h$, $h \to 0^+$




$= \lim\limits_{h \to 0^+} (c - h - [c - h])$

$$= \lim_{h \to 0^+} (c - h - (c - 1))$$

[$\because$ If $c \in$ Z and $h \to 0^+$, then $[c - h] = c - 1$]

$$= \lim_{h \to 0^+} (c - h - c + 1) = \lim_{h \to 0^+} (1 - h)$$

Put $h = 0$, $= 1 - 0 = 1$

Right Hand Limit $= \lim_{x \to c^+} g(x) = \lim_{x \to c^+} (x - [x])$

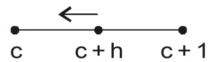
Put $x = c + h$, $h \to 0^+$

$$= \lim_{h \to 0^+} (c + h - [c + h]) \quad = \lim_{h \to 0^+} (c + h - c)$$

($\because$ If $c \in$ Z and $h \to 0^+$, then $[c + h] = c$)

$$= \lim_{h \to 0^+} h$$

Put $h = 0$; $\quad = 0$




$\therefore \lim_{x \to c^-} g(x) \neq \lim_{x \to c^+} g(x)$

$\therefore \lim_{x \to c} g(x)$ does not exist and hence $g(x)$ is discontinuous at $x = c$ (any integer).

$\therefore$ $g(x) = x - [x]$ is discontinuous at all integral points.

**Very Important Note.** If two functions $f$ and $g$ are continuous in a common domain D,

then (*i*) $f + g$ (*ii*) $f - g$ (*iii*) $fg$ are continuous in the same domain D.

(*iv*) $\dfrac{f}{g}$ is also continuous at all points of D except those where $g(x) = 0$.

**20. Is the function $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?**

**Sol. Given:** $f(x) = x^2 - \sin x + 5 = (x^2 + 5) - \sin x$

$$= g(x) - h(x) \qquad ...(i)$$

where $g(x) = x^2 + 5$ and $h(x) = \sin x$

We know that $g(x) = x^2 + 5$ is a polynomial function and hence is continuous (for all real $x$)

Again $h(x) = \sin x$ being a sine function is continuous (for all real $x$)

$\therefore$ By (*i*) $f(x) = x^2 - \sin x + 5 = g(x) - h(x)$

being the difference of two continuous functions is also continuous for all real $x$ (see Note above) and hence continuous at $x = \pi (\in$ R) also.

**Or**

**Given:** $f(x) = x^2 - \sin x + 5$ ...(*i*)

**To examine continuity at $x = \pi$**

$$\lim_{x \to \pi} f(x) = \lim_{x \to \pi} (x^2 - \sin x + 5) \qquad \text{[By } (i)]$$

Putting $x = \pi$, $\quad = \pi^2 - \sin \pi + 5$

$= \pi^2 + 5$

$[\because \ \sin \pi = \sin 180° = \sin (180° - 0°) = \sin 0° = 0]$

Again putting $x = \pi$ in $(i)$, $f(\pi) = \pi^2 - \sin \pi + 5$

$= \pi^2 - 0 + 5 = \pi^2 + 5$

$\therefore \quad \lim_{x \to \pi} f(x) = f(\pi)$

$\therefore$ $f(x)$ is continuous at $x = \pi$.

**21. Discuss the continuity of the following functions:**

**(*a*) $f(x) = \sin x + \cos x$ (*b*) $f(x) = \sin x - \cos x$**

**(*c*) $f(x) = \sin x \cdot \cos x$.**

**Sol.** We know that $\sin x$ is a continuous function for all real $x$

Also we know that $\cos x$ is a continuous function for all real $x$ (see solution of Q. No. 22($i$) below)

$\therefore$ By Note at the end of solution of Q. No. 19,

($i$) their sum function $f(x) = \sin x + \cos x$ is also continuous for all real $x$.

($ii$) their difference function $f(x) = \sin x - \cos x$ is also continuous for all real $x$.

($iii$) their product function $f(x) = \sin x \cdot \cos x$ is also continuous for all real $x$.

**Note.** To find $\lim_{x \to c} f(x)$, we can also start with putting $x = c + h$ where $h \to 0$ (and not only $h \to 0^+$)

$\therefore \quad \lim_{x \to c} f(x) = \lim_{h \to 0} f(c + h)$.

(Please note that this method of finding the limits makes us find both $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ simultaneously).

**22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.**

**Sol.** ($i$) Let $f(x)$ be the cosine function

*i.e.,* $f(x) = \cos x$ ...($i$)

Clearly, $f(x)$ is real and finite for all real values of $x$ *i.e.,* $f(x)$ is defined for all real $x$. Therefore domain of $f(x)$ is R.

Let $x = c \in$ R.

$\lim_{x \to c} f(x) = \lim_{x \to c} \cos x$

Put $x = c + h$ where $h \to 0$

$= \lim_{h \to 0} \cos (c + h) = \lim_{h \to 0} (\cos c \cos h - \sin c \sin h)$

Putting $h = 0$, $= \cos c \cos 0 - \sin c \sin 0$

$= \cos c \, (1) - \sin c \, (0)$

$= \cos c$

$\therefore \quad \lim_{x \to c} f(x) = \cos c$

Putting $x = c$ in $(i)$, $f(c) = \cos c$

$\therefore \quad \lim\limits_{x \to c} f(x) = f(c)$ $(= \cos c)$

$\therefore \quad f(x)$ is continuous at (every) $x = c \in R$

$\therefore \quad f(x) = \cos x$ is continuous on R.

$(ii)$ Let $f(x)$ be cosecant function

*i.e.,* $f(x) = \text{cosec } x = \dfrac{1}{\sin x}$

$f(x)$ is not finite *i.e.,* $\to \infty$

when $\sin x = 0$ *i.e.,* when $x = n\pi$, $n \in Z$.

$\therefore$ Domain of $f(x) = \text{cosec } x$ is $D = R - \{x = n\pi;\ n \in Z\}$.

$(\because\ f(x)$ is real and finite $\forall\ x \in D)$.

Now $f(x) = \text{cosec } x = \dfrac{1}{\sin x} = \dfrac{g(x)}{h(x)}$ ...$(i)$

Now $g(x) = 1$ being constant function is continuous on domain D and $h(x) = \sin x$ is non-zero and continuous on Domain D.

Therefore by $(i)$, $f(x) = \text{cosec } x \left( = \dfrac{1}{\sin x} = \dfrac{g(x)}{h(x)} \right)$ is continuous on domain $D = R - \{x = n\pi,\ n \in Z\}$

(Also read **Note** at the end of solution of Q. No. 19).

$(iii)$ Let $f(x)$ be the secant function

*i.e.,* $f(x) = \sec x = \dfrac{1}{\cos x}$ $f(x)$ is not finite *i.e.,* $\to \infty$

When $\cos x = 0$ *i.e.,* when $x = (2n + 1)\dfrac{\pi}{2}$, $n \in Z$.

$\therefore$ Domain of $f(x) = \sec x$ is

$$D = R - \{x = (2n + 1)\frac{\pi}{2};\ n \in Z\}$$

Now $f(x) = \sec x = \dfrac{1}{\cos x} = \dfrac{g(x)}{h(x)}$ ...$(i)$

Now $g(x) = 1$ being constant function is continuous on domain D and $h(x) = \cos x$ is non-zero and continuous on domain D.

Therefore by $(i)$, $f(x) = \sec x \left( = \dfrac{1}{\cos x} = \dfrac{g(x)}{h(x)} \right)$ is continuous on domain $D = R - \{x : x = (2n + 1)\dfrac{\pi}{2};\ n \in Z\}$.

$(iv)$ Let $f(x)$ be the cotangent function *i.e.,* $f(x) = \cot x = \dfrac{\cos x}{\sin x}$.

$f(x)$ is not finite *i.e.,* $\to \infty$

When $\sin x = 0$ *i.e.,* when $x = n\pi$, $n \in Z$.

$\therefore$ Domain of $f(x) = \cot x$ is

$$D = R - \{x = n\pi;\ n \in Z\}$$

Now $f(x) = \cot x = \dfrac{\cos x}{\sin x} = \dfrac{g(x)}{h(x)}$ ...(*i*)

Now $g(x) = \cos x$ being cosine function is continuous on D and is non-zero on D.

Therefore by (*i*), $f(x) = \cot x \left(= \dfrac{\cos x}{\sin x} = \dfrac{g(x)}{h(x)}\right)$ is continuous on domain $D = R - \{x : x = n\pi,\ n \in Z\}$.

**23. Find all points of discontinuity of *f*, where**

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ x + 1, & \text{if } x \ge 0 \end{cases}.$$

**Sol.** The domain of $f = \{x \in R : x < 0\} \cup \{x \in R : x \ge 0\} = R$

$x = 0$ is the partitioning point of the domain of the given function.

**For all $x < 0$,** $f(x) = \dfrac{\sin x}{x}$ (given)

Since $\sin x$ and $x$ are continuous for $x < 0$ (in fact, they are continuous for all $x$) and $x \ne 0$

$\therefore f$ is continuous when $x < 0$

**For all $x > 0$,** $f(x) = x + 1$ is a polynomial and hence continuous.

$\therefore f$ is continuous when $x > 0$.

**Let us discuss the continuity of $f(x)$ at the partitioning point $x = 0$.**

**At $x = 0$,** $f(0) = 0 + 1 = 1$ $[\because f(x) = x + 1 \text{ for } x \ge 0]$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin x}{x}$$

$$\left[\because x \to 0^- \Rightarrow x < 0 \text{ and } f(x) = \frac{\sin x}{x} \text{ for } x < 0\right]$$

$$= 1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 1)$$

$$\left[\because x \to 0^+ \Rightarrow x > 0 \text{ and } f(x) = x + 1 \text{ for } x > 0\right]$$

$$= 0 + 1 = 1$$

Since $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = 1 \quad \therefore \lim\limits_{x \to 0} f(x) = 1$

Thus $\lim\limits_{x \to 0} f(x) = f(0)$ and hence $f$ is continuous at $x = 1$.

Now $f$ is continuous at every point in its domain and hence $f$ is a continuous function.

**24. Determine if $f$ defined by**

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0 \end{cases}$$

**is a continuous function?**

**Sol.** For all $x \neq 0$, $f(x) = x^2 \sin \dfrac{1}{x}$ being the product function of two continuous functions $x^2$ (polynomial function) and $\sin \dfrac{1}{x}$ (a sine function) is continuous for all real $x \neq 0$.

**Now let us examine continuity at $x = 0$.**

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin \frac{1}{x}$$

Putting $x = 0$ $\qquad = 0 \times$ A finite quantity between $-1$ and $1 = 0$

$$\left[\because \sin \frac{1}{x} (= \sin \theta) \text{ always lies between } -1 \text{ and } 1\right]$$

Also $\qquad f(x) = 0$ at $x = 0$ *i.e.*, $f(0) = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$, therefore function $f$ is continuous at

$x = 0$ (also).

Hence $f(x)$ continuous on domain R of $f$.

**25. Examine the continuity of $f$, where $f$ is defined by**

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if} \quad x \neq 0 \\ -1, & \text{if} \quad x = 0 \end{cases}.$$

**Sol. Given:**

$$f(x) = \begin{cases} \sin x - \cos x & \text{if} \quad x \neq 0 \quad \quad ...(i) \\ -1 & \text{if} \quad x = 0 \quad \quad ...(ii) \end{cases}$$

From (*i*), $f(x)$ is defined for $x \neq 0$ and from (*ii*) $f(x)$ is defined for $x = 0$.

$\therefore$ Domain of $f(x)$ is $\{x : x \neq 0\} \cup \{0\} = \text{R}$.

From (*i*), for $x \neq 0$, $f(x) = \sin x - \cos x$ being the difference of two continuous functions $\sin x$ and $\cos x$ is continuous for all $x \neq 0$.

Hence $f(x)$ is continuous on $\text{R} - \{0\}$.

**Now let us examine continuity at $x = 0$.**

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (\sin x - \cos x)$$

[By (*i*) as $x \to 0$ means $x \neq 0$]

Putting $x = 0$, $\qquad = \sin 0 - \cos 0 = 0 - 1 = -1$

From (*ii*) $\qquad f(x) = -1$ when $x = 0$

*i.e.*, $\qquad f(0) = -1$

$$\therefore \quad \lim_{x \to 0} f(x) = f(0) \; (= -1)$$

$\therefore$ $f(x)$ is continuous at $x = 0$ (also).

Hence $f(x)$ is continuous on domain R of $f$.

**Find the values of $k$ so that the function $f$ is continuous at the indicated point in Exercises 26 to 29.**

**26.** $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$ **at** $x = \dfrac{\pi}{2}$.

**Sol.** Left Hand Limit = $\lim\limits_{x \to \frac{\pi}{2}} f(x) = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{k\cos x}{\pi - 2x}$

Put $x = \dfrac{\pi}{2} - h$ where $h \to 0^+$

$$= \lim_{h \to 0^+} \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim_{h \to 0^+} \frac{k\sin h}{\pi - \pi + 2h}$$

$$= \lim_{h \to 0^+} \frac{k\sin h}{2h} = \frac{k}{2} \times \lim_{h \to 0^+} \frac{\sin h}{h} = \frac{k}{2} \times 1 = \frac{k}{2} \qquad ...(i)$$

Right Hand Limit = $\lim\limits_{x \to \frac{\pi^+}{2}} f(x) = \lim\limits_{x \to \frac{\pi^+}{2}} \dfrac{k\cos x}{\pi - 2x}$

Put $x = \dfrac{\pi}{2} + h$ where $h \to 0^+$

$$= \lim_{h \to 0^+} \frac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = \lim_{h \to 0^+} \frac{-k\sin h}{\pi - \pi - 2h} = \lim_{h \to 0^+} \frac{-k\sin h}{-2h}$$

$$= \frac{k}{2} \times \lim_{h \to 0^+} \frac{\sin h}{h} = \frac{k}{2} \times 1 = \frac{k}{2} \qquad ...(ii)$$

Also $f\left(\dfrac{\pi}{2}\right) = 3$ ...(*iii*) $\because$ $f(x) = 3$ when $x = \dfrac{\pi}{2}$ (given)

Because $f(x)$ is continuous at $x = \dfrac{\pi}{2}$ (given)

$$\therefore \lim_{x \to \frac{\pi^-}{2}} f(x) = \lim_{x \to \frac{\pi^+}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

Putting values from (*i*), (*ii*), and (*iii*), $\dfrac{k}{2} = 3$ or $k = 6$.

**27.** $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$ **at $x = 2$.**

**Sol. Given:** $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 & \text{...}(i) \\ 3, & \text{if } x > 2 & \text{...}(ii) \end{cases}$

**Given:** $f(x)$ is continuous at $x = 2$.

Left Hand Limit $= \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} kx^2$ [By $(i)$]

($\because$ $x \to 2^-$ $\Rightarrow$ $x$ is $< 2$)

Put $x = 2$, $= k(2)^2 = 4k$

Right Hand Limit $= \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} 3$ [By $(ii)$]

($\because$ $x \to 2^+$ $\Rightarrow$ $x > 2$)

Putting $x = 2$, $= 3$

Putting $x = 2$ in $(i)$ $f(2) = k(2)^2 = 4k$.

Because $f(x)$ is continuous at $x = 2$ (given),

therefore $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = f(2)$

Putting values, $4k = 3 = 3$ $\Rightarrow$ $k = \dfrac{3}{4}$.

**28.** $f(x) = \begin{cases} kx + 1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$ **at $x = \pi$.**

**Sol. Given:** $f(x) = \begin{cases} kx + 1, & \text{if } x \le \pi & \text{...}(i) \\ \cos x, & \text{if } x > \pi & \text{...}(ii) \end{cases}$

**Given:** $f(x)$ is continuous at $x = \pi$.

Left Hand Limit $= \lim\limits_{x \to \pi^-} f(x) = \lim\limits_{x \to \pi^-} (kx + 1)$ [By $(i)$]

($\because$ $x \to \pi^-$ $\Rightarrow$ $x < \pi$)

Putting $x = \pi$, $= k\pi + 1$

Right Hand Limit $= \lim\limits_{x \to \pi^+} f(x) = \lim\limits_{x \to \pi^+} \cos x$ [By $(ii)$]

($\because$ $x \to \pi^+$ $\Rightarrow$ $x > \pi$)

Putting $x = \pi$, $= \cos \pi = \cos 180° = \cos (180° - 0)$

$= - \cos 0 = - 1$

Putting $x = \pi$ in $(i)$, $f(\pi) = k\pi + 1$

But $f(x)$ is continuous at $x = \pi$ (given), therefore

$$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi)$$

Putting values $k\pi + 1 = - 1 = k\pi + 1$

$\Rightarrow$ $k\pi + 1 = - 1$ [$\because$ First and third members are same]

$\Rightarrow$ $k\pi = - 2$ $\Rightarrow$ $k = -\dfrac{2}{\pi}$.

**29.** $f(x) = \begin{cases} kx + 1, & \text{if } x \le 5 \\ 3x - 5, & \text{if } x > 5 \end{cases}$ **at $x = 5$.**

**Sol. Given:** $f(x) = \begin{cases} kx + 1 & \text{if } x \le 5 & \text{...}(i) \\ 3x - 5 & \text{if } x > 5 & \text{...}(ii) \end{cases}$

**Given:** $f(x)$ is continuous at $x = 5$.

Left Hand Limit $= \lim\limits_{x \to 5^-} f(x) = \lim\limits_{x \to 5^-} (kx + 1)$ [By (*i*)]

Putting $x = 5$, $= k(5) + 1 = 5k + 1$

Right Hand Limit $= \lim\limits_{x \to 5^+} f(x) = \lim\limits_{x \to 5^+} (3x - 5)$ [By (*ii*)]

Putting $x = 5$, $= 3(5) - 5 = 15 - 5 = 10$

Putting $x = 5$ in (*i*), $f(5) = 5k + 1$

But $f(x)$ is continuous at $x = 5$ (given)

$\therefore \lim\limits_{x \to 5^-} f(x) = \lim\limits_{x \to 5^+} f(x) = f(5)$

Putting values $5k + 1 = 10 = 5k + 1$

$\Rightarrow 5k + 1 = 10 \Rightarrow 5k = 9 \Rightarrow k = \dfrac{9}{5}$.

**30. Find the values of *a* and *b* such that the function defined by**

$$f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}.$$

**is a continuous function.**

**Sol. Given:**

$$f(x) = \begin{cases} 5 & \text{if } x \le 2 & \ldots(i) \\ ax + b & \text{if } 2 < x < 10 & \ldots(ii) \\ 21 & \text{if } x \ge 10 & \ldots(iii) \end{cases}$$

From (*i*), (*ii*) and (*iii*), $f(x)$ is defined for $\{x \le 2\} \cup \{2 < x < 10\} \cup \{x \ge 10\}$ *i.e.,* for $(-\infty, 2] \cup (2, 10) \cup [10, \infty)$ *i.e.,* for $(-\infty, \infty)$ *i.e.,* on R. $\therefore$ Domain of $f(x)$ is R.

**Given:** $f(x)$ is a continuous function (of course on its domain here R), therefore $f(x)$ is also continuous at partitioning points $x = 2$ and $x = 10$ of the domain.

Because $f(x)$ is continuous at partitioning point $x = 2$, therefore

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) \quad \ldots(iv)$$

Now $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} 5$ [By (*i*)]

($\because x \to 2^- \Rightarrow x < 2$)

Putting $x = 2$, $= 5$

Again $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (ax + b)$ [By (*ii*)]

($\because x \to 2^+ \Rightarrow x > 2$)

Putting $x = 2$, $= 2a + b$

Putting $x = 2$ in (*i*), $f(2) = 5$.

Putting these values in eqn. (*iv*), we have

$$5 = 2a + b = 5 \Rightarrow 2a + b = 5 \quad \ldots(v)$$

Again because $f(x)$ is continuous at partitioning point $x = 10$, therefore $\lim\limits_{x \to 10^-} f(x) = \lim\limits_{x \to 10^+} f(x) = f(10)$ ...(*vi*)

Now $\lim\limits_{x \to 10^-} f(x) = \lim\limits_{x \to 10^-} (ax + b)$ [By (*ii*)]

$(x \to 10^- \Rightarrow x < 10)$

Putting $x = 10$, $= 10a + b$

Again $\lim\limits_{x \to 10^+} f(x) = \lim\limits_{x \to 10^+} 21$ [By (*iii*)]

$(\because x \to 10^+ \Rightarrow x > 10)$

Putting $x = 10$; $= 21$

Putting $x = 10$ in Eqn. (*iii*), $f(10) = 21$

Putting these values in eqn. (*vi*), we have

$10a + b = 21 = 21$

$\Rightarrow \quad 10a + b = 21$ ...(*vii*)

Let us solve eqns. (*v*) and (*vii*) for $a$ and $b$.

Eqn. (*vii*) – eqn. (*v*) gives $8a = 16 \Rightarrow a = \frac{16}{8} = 2$

Putting $a = 2$ in (*v*), $4 + b = 5 \quad \therefore \quad b = 1$.

$\therefore \quad a = 2, b = 1$.

**Very Important Result: Composite function of two continuous functions is continuous.**

We know by definition that $(fog)x = f(g(x))$

and $(gof)x = g(f(x))$

**31. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.**

**Sol. Given:** $f(x) = \cos(x^2)$ ...(*i*)

$f(x)$ has a real and finite value for all $x \in$ R.

$\therefore$ Domain of $f(x)$ is R.

Let us take $g(x) = \cos x$ and $h(x) = x^2$.

Now $g(x) = \cos x$ is a cosine function and hence is continuous.

Again $h(x) = x^2$ is a polynomial function and hence is continuous.

$\therefore \quad (goh)x = g(h(x)) = g(x^2)$ $[\because h(x) = x^2]$

$= \cos(x^2)$ (Changing $x$ to $x^2$ in $g(x) = \cos x$)

$= f(x)$ (By (*i*)) being the composite function of two continuous functions is continuous for all $x$ in domain R.

**Or**

Take $g(x) = x^2$ and $h(x) = \cos x$.

Then $(hog)x = h(g(x)) = h(x^2)$

$= \cos(x^2) = f(x)$.

**32. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.**

**Sol.** $f(x) = |\cos x|$ ...(*i*)

$f(x)$ has a real and finite value for all $x \in$ R.

$\therefore$ Domain of $f(x)$ is R.

Let us take $g(x) = \cos x$ and $h(x) = |x|$
We know that $g(x)$ and $h(x)$ being cosine function and modulus function are continuous for all real $x$.
Now $(goh)x = g(h(x)) = g(|x|) = \cos |x|$ being the composite function of two continuous functions is continuous (but $\neq f(x)$)
Again $(hog)x = h(g(x)) = h(\cos x)$

$$= |\cos x| = f(x) \qquad \text{[By } (i)\text{]}$$

[Changing $x$ to $\cos x$ in $h(x) = |x|$, we have $h(\cos x) = |\cos x|$]
Therefore $f(x) = |\cos x|$ $(= (hog)x)$ being the composite function of two continuous functions is continuous.

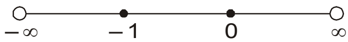
**33. Examine that sin | *x* | is a continuous function.**

**Sol.** Let $f(x) = \sin x$ and $g(x) = |x|$
We know that $\sin x$ and $|x|$ are continuous functions.
$\therefore$ $f$ and $g$ are continuous.
Now $\quad (fog)(x) = f\{g(x)\} = \sin \{g(x)\} = \sin |x|$
We know that composite function of two continuous functions is continuous.
$\therefore$ $fog$ is continuous. $\qquad$ Hence, $\sin |x|$ is continuous.

**34. Find all points of discontinuity of *f* defined by**

$$\boldsymbol{f(x) = |x| - |x + 1|.}$$

**Sol. Given:** $\quad f(x) = |x| - |x + 1| \qquad ...(i)$
This $f(x)$ is real and finite for every $x \in \mathbb{R}$.
$\therefore$ $f$ is defined for all $x \in \mathbb{R}$ *i.e.,* domain of $f$ is R.
Putting each expression within modulus equal to 0
*i.e.,* $x = 0$ and $x + 1 = 0$ *i.e.,* $x = 0$ and $x = -1$.




Marking these values of $x$ namely $-1$ and $0$ (in proper ascending order) on the number line, domain R of $f$ is divided into three sub-intervals $(-\infty, -1]$, $[-1, 0]$ and $[0, \infty)$.
On the sub-interval $(-\infty, -1]$ *i.e.,* for $x \leq -1$, (say for $x = -2$ etc.) $x < 0$ and $(x + 1)$ is also $< 0$ and therefore

$$|x| = -x \text{ and } |x + 1| = -(x + 1)$$

Hence $(i)$ becomes $f(x) = |x| - |x + 1|$

$$= -x - (-(x + 1)) = -x + x + 1$$

*i.e.,* $f(x) = 1$ for $x \leq -1 \qquad ...(ii)$

On the sub-interval $[-1, 0]$ *i.e.,* for $-1 \leq x \leq 0$ $\left(\text{say for } x = \frac{-1}{2}\right)$

$x < 0$ and $(x + 1) > 0$ and therefore $|x| - x$ and $|x + 1| = x + 1$.
Hence $(i)$ becomes $f(x) = |x| - |x + 1|$

$$= -x - (x + 1) = -x - x - 1$$

$$= -2x - 1$$

*i.e.,* $f(x) = -2x - 1$ for $-1 \le x \le 0$ ...(*iii*)

On the sub-interval $[0, \infty)$ *i.e.,* for $x \ge 0$,

$x \ge 0$ and also $x + 1 > 0$ and therefore

$$|x| = x \text{ and } |x + 1| = x + 1$$

Hence (*i*) becomes $f(x) = |x| - |x + 1| = x - (x + 1)$

$$= x - x - 1 = -1$$

*i.e.,* $f(x) = -1$ for $x \ge 0$ ...(*iv*)

From (*ii*), for $x < -1$, $f(x) = 1$ is a constant function and hence is continuous for $x < -1$.

From (*iii*), for $-1 < x < 0$, $f(x) = -2x - 1$ is a polynomial function and hence is continuous for $-1 < x < 0$.

From (*iv*), for $x > 0$, $f(x) = -1$ is a constant function and hence is continuous for $x > 0$.

$\therefore$ $f$ is continuous in $R - \{-1, 0\}$.

**Let us examine continuity of *f* at partitioning point $x = -1$.**

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} 1 \quad \text{[By (ii)]}$$

$$(\because \; x \to -1^- \Rightarrow x < -1)$$

Putting $x = -1$, $= 1$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (-2x - 1) \quad \text{(By (iii))}$$

$$(\because \; x \to -1^+ \Rightarrow x > -1)$$

Putting $x = -1$, $= -2(-1) - 1 = 2 - 1 = 1$

$$\therefore \quad \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) \; (= 1)$$

$\therefore$ $\lim\limits_{x \to -1} f(x)$ exists and $= 1$.

Putting $x = -1$ in (*ii*) or (*iii*), $f(-1) = 1$

$$\therefore \quad \lim_{x \to -1} f(x) = f(-1) \; (= 1)$$

$\therefore$ $f$ is continuous at $x = -1$ also.

**Let us examine continuity of *f* at partitioning point $x = 0$.**

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-2x - 1) \quad \text{(By (iii))}$$

$$(\because \; x \to 0^- \Rightarrow x < 0)$$

Putting $x = 0$, $= -2(0) - 1 = -1$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (-1) \quad \text{[By (iv)]}$$

$$(\ldots \to 0^+ \Rightarrow x > 0)$$

Putting $x = 0$, $= -1$ $\therefore \lim_{x \to 0^-} f(x) = \underset{x \to 0^+}{Lt} f(x)$ $(= -1)$

$\therefore \lim_{x \to 0} f(x)$ exists and $= -1$

Putting $x = 0$ in (*iii*) or (*iv*), $f(0) = -1$

$\therefore \lim_{x \to 0} f(x) = f(0)$ $(= -1)$

$\therefore$ $f$ is continuous at $x = 0$ also.

$\therefore$ $f$ is continuous on the domain R.

$\therefore$ There is no point of discontinuity.

**Second Solution**

We know that every modulus function is continuous for all real $x$.
Therefore $|x|$ and $|x + 1|$ are continuous for all real $x$.
Also, we know that difference of two continuous functions is continuous.

$\therefore$ $f(x) = |x| - |x + 1|$ is also continuous for all real $x$.

$\therefore$ There is no point of discontinuity.

# Exercise 5.2

**Differentiate the functions w.r.t. $x$ in Exercises 1 to 8.**

**1. sin ($x^2$ + 5).**

**Sol.** Let $y = \sin (x^2 + 5)$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx} \sin (x^2 + 5) = \cos (x^2 + 5) \frac{d}{dx}(x^2 + 5)$$

$$\left[\because \frac{d}{dx} \sin f(x) = \cos f(x) \frac{d}{dx} f(x)\right]$$

$$= \cos (x^2 + 5) \,.\, (2x + 0)$$

$$\because \left[\frac{d}{dx} x^n = n\, x^{n-1} \text{ and } \frac{d}{dx}(c) = 0\right]$$

$$= 2x \cos (x^2 + 5).$$

**Caution.** $\sin (x^2 + 5)$ is not the product of two functions. It is composite function: sine of $(x^2 + 5)$.

**2. cos (sin $x$).**

**Sol.** Let $y = \cos (\sin x)$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx} \cos (\sin x) = -\sin (\sin x) \frac{d}{dx} \sin x$$

$$\left[\because \frac{d}{dx} \cos f(x) = -\sin f(x) \frac{d}{dx} f(x)\right]$$

$$= -\sin (\sin x) \,.\, \cos x = -\cos x \sin (\sin x).$$

**3. sin ($ax + b$).**

**Sol.** Let $y = \sin (ax + b)$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx} \sin (ax + b) = \cos (ax + b) \frac{d}{dx} (ax + b)$$

$$= \cos (ax + b) \left[a \frac{d}{dx}(x) + \frac{d}{dx}(b)\right]$$

$$= \cos (ax + b) [a(1) + 0]$$

$$= a \cos (ax + b).$$

**Note.** It may be noted that letters $a$ to $q$ of English Alphabet are treated as constants (similar to 3, 5 etc.) as per convention.

**4. sec (tan $\sqrt{x}$).**

**Sol.** Let $y = \sec (\tan \sqrt{x})$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx} \sec (\tan \sqrt{x})$$

$$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\frac{d}{dx}(\tan\sqrt{x})$$

$$\left[\because \frac{d}{dx}\sec f(x) = \sec f(x)\tan f(x)\frac{d}{dx}f(x)\right]$$

$$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2(\sqrt{x})\frac{d}{dx}\sqrt{x}$$

$$\left[\because \frac{d}{dx}f(x) = \sec^2 f(x)\frac{d}{dx}f(x)\right]$$

$$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2\sqrt{x}\,\frac{1}{2\sqrt{x}}$$

$$\left[\because \frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}\right]$$

**5.** $\dfrac{\sin(ax+b)}{\cos(cx+d)}$.

**Sol.** Let $y = \dfrac{\sin(ax+b)}{\cos(cx+d)}$

$$\therefore \frac{dy}{dx} = \frac{\cos(cx+d)\frac{d}{dx}\sin(ax+b) - \sin(ax+b)\frac{d}{dx}\cos(cx+d)}{\cos^2(cx+d)}$$

$$\left[\because \text{By Quotient Rule } \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{(\text{DEN.})\frac{d}{dx}(\text{NUM}) - \text{NUM}\frac{d}{dx}(\text{DEN})}{(\text{DEN})^2}\right]$$

$$= \frac{\cos(cx+d)\cos(ax+b)\frac{d}{dx}(ax+b) - \sin(ax+b)(-\sin(cx+d))\frac{d}{dx}(cx+d)}{\cos^2(cx+d)}$$

$$= \frac{a\cos(cx+d)\cos(ax+b) + c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$$

$$\left[\because \frac{d}{dx}(ax+b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a\frac{d}{dx}(x) + 0 = a\,.\,1 = a\right.$$

$$\left.\text{Similarly } \frac{d}{dx}(cx+d) = c\right]$$

**6.** $\mathbf{\cos x^3 \sin^2(x^5)}$.

**Sol.** Let $y = \cos x^3 \sin^2(x^5) = \cos x^3 (\sin x^5)^2$

$$\therefore \frac{dy}{dx} = \cos x^3\frac{d}{dx}(\sin x^5)^2 + (\sin x^5)^2\frac{d}{dx}\cos x^3$$

$$\left[\because \text{ By Product Rule } \frac{d}{dx}(uv) = \text{I}\,\frac{d}{dx}(\text{II}) + \text{II}\,\frac{d}{dx}(\text{I})\right]$$

$$= \cos x^3 \,.\, 2(\sin x^5)\ \frac{d}{dx}\ \sin x^5 + (\sin x^5)^2\,(-\sin x^3)\ \frac{d}{dx}\ x^3$$

$$= \cos x^3 \,.\, 2(\sin x^5)\ \cos x^5\ (5x^4) + \sin^2 x^5\ (-\sin x^3)\ 3x^2$$

$$\left[\because\ \frac{d}{dx} \sin x^5 = \cos x^5 \frac{d}{dx} x^5 = \cos x^5\,(5x^4)\right]$$

$$= 10x^4 \cos x^3 \sin x^5 \cos x^5 - 3x^2 \sin^2 x^5 \sin x^3$$

$$= x^2 \sin x^5\,[10x^2 \cos x^3 \cos x^5 - 3 \sin x^5 \sin x^3].$$

**7.2** $\boldsymbol{\sqrt{\cot(x^2)}}$ **.**

**Sol.** Let $y = 2\sqrt{\cot(x^2)} = 2\,(\cot(x^2))^{1/2}$

$$\therefore\quad \frac{dy}{dx} = 2\,.\,\frac{1}{2}\ (\cot x^2)^{1/2\,-\,1}\ \frac{d}{dx}\ (\cot(x^2))$$

$$\left|\ \because\ \frac{d}{dx}(f(x))^n = n(f(x))^{n-1} \frac{d}{dx} f(x)\right.$$

$$= (\cot x^2)^{-1/2} \left(-\operatorname{cosec}^2(x^2)\,\frac{d}{dx}\,x^2\right)$$

$$\left|\ \because\ \frac{d}{dx} \cot f(x) = -\operatorname{cosec}^2(f(x))\,\frac{d}{dx} f(x)\right.$$

$$= \frac{-\operatorname{cosec}^2(x^2)}{\sqrt{\cot x^2}}\ (2x) = \frac{-2x \operatorname{cosec}^2(x^2)}{\sqrt{\cot(x^2)}}.$$

**8.** $\boldsymbol{\cos(\sqrt{x})}$**.**

**Sol.** Let $y = \cos(\sqrt{x})$

$$\therefore\quad \frac{dy}{dx} = \frac{d}{dx}\ \cos(\sqrt{x}) = -\sin\sqrt{x}\ \frac{d}{dx}\ \sqrt{x}$$

$$\left[\because\ \frac{d}{dx} \cos f(x) = -\sin f(x)\,\frac{d}{dx} f(x)\right]$$

$$= -\sin\sqrt{x}\ \ \frac{1}{2\sqrt{x}}\ \left[\because\ \frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}\right]$$

**9. Prove that the function $f$ given by $f(x) = |x - 1|$, $x \in R$ is not differentiable at $x = 1$.**

**Sol. Definition. A function $f(x)$ is said to be differentiable at a point $x = c$ if $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists**

(and then this limit is called $f'(c)$ *i.e.,* value of $f'(x)$ or $\dfrac{dy}{dx}$ at $x = c$)

Here $f(x) = |x - 1|, x \in R$ ...(*i*)

**To prove:** $f(x)$ is not differentiable at $x = 1$.

Putting $x = 1$ on (*i*), $f(1) = |1 - 1| = |0| = 0$

Left Hand Derivative = $Lf'(1) = \lim\limits_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1}$

$$= \lim_{x \to 1^-} \frac{|x-1| - 0}{x - 1} = \lim_{x \to 1^-} \frac{-(x-1)}{x-1}$$

$[\because x \to 1^- \Rightarrow x < 1 \Rightarrow x - 1 < 0 \Rightarrow |x - 1| = -(x - 1)]$

$$= \lim_{x \to 1^-} (-1) = -1 \quad ...(ii)$$

Right Hand derivative = $Rf'(1) = \lim\limits_{x \to 1^+} \dfrac{f(x) - f(1)}{x - 1}$

$$= \lim_{x \to 1^+} \frac{|x-1| - 0}{x-1} = \lim_{x \to 1^+} \frac{(x-1)}{x-1}$$

$(\because x \to 1^+ \Rightarrow x > 1 \Rightarrow x - 1 > 0 \Rightarrow |x - 1| = x - 1)$

$$= \lim_{x \to 1^+} 1 = 1 \quad ...(iii)$$

From (*ii*) and (*iii*), $Lf'(1) \neq Rf'(1)$

$\therefore$ $f(x)$ is not differentiable at $x = 1$.

**Note.** In problems on limits of Modulus function, and bracket function (*i.e.,* greatest Integer Function), we have to find both left hand limit and right hand limit (we have used this concept quite few times in Exercise 5.1).

**10. Prove that the greatest integer function defined by**

$$f(x) = [x],\ 0 < x < 3$$

**is not differentiable at $x = 1$ and $x = 2$.**

**Sol. Given:** $f(x) = [x], 0 < x < 3$ ...(*i*)

**Differentiability at $x = 1$**

Putting $x = 1$ in (*i*), $f(1) = [1] = 1$

Left Hand derivative = $Lf'(1) = \lim\limits_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{[x] - 1}{x - 1}$

Put $x = 1 - h$, $h \to 0^+$

$= \lim_{h \to 0^+} \frac{[1-h]-1}{1-h-1}$ $= \lim_{h \to 0^+} \frac{0-1}{-h} = \lim_{h \to 0^+} \frac{1}{h}$

[We know that as $h \to 0^+$, $[c - h] = c - 1$ if $c$ is an integer. Therefore $[1 - h] = 1 - 1 = 0$]

Put $h = 0$, $= \frac{1}{0} = \infty$ does not exist.

$\therefore$ $f(x)$ is not differentiable at $x = 1$.

(We need not find $Rf'(1)$ as $Lf'(1)$ does not exist).

**Differentiability at $x = 2$**

Putting $x = 2$ in $(i)$, $f(2) = [2] = 2$

Left Hand derivative $= Lf'(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{[x]-2}{x-2}$

Put $x = 2 - h$ as $h \to 0^+$

$$= \lim_{h \to 0^+} \frac{[2-h]-2}{2-h-2} = \lim_{h \to 0^+} \frac{1-2}{-h} = \lim_{h \to 0^+} \frac{-1}{-h}$$

(For $h \to 0^+$, $[2 - h] = 2 - 1 = 1$)

$$= \lim_{h \to 0^+} \frac{1}{h} = \frac{1}{0} = \infty \text{ does not exist.}$$

$\therefore$ $f(x)$ is not differentiable at $x = 2$.

**Note.** For $h \to 0^+$, $[c + h] = c$ if $c$ is an integer.

# Exercise 5.3

**Find $\frac{dy}{dx}$ in the following Exercises 1 to 15.**

**1. $2x + 3y = \sin x$.**

**Sol. Given:** $2x + 3y = \sin x$

Differentiating both sides w.r.t. $x$, we have

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}\sin x$$

$$\therefore 2 + 3\frac{dy}{dx} = \cos x \Rightarrow 3\frac{dy}{dx} = \cos x - 2 \quad \therefore \frac{dy}{dx} = \frac{\cos x - 2}{3}.$$

**2. $2x + 3y = \sin y$.**

**Sol. Given:** $2x + 3y = \sin y$

Differentiating both sides w.r.t. $x$, we have

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}\sin y \quad \therefore\ 2 + 3\frac{dy}{dx} = \cos y\frac{dy}{dx}$$

$$\Rightarrow -\cos y\frac{dy}{dx} + 3\frac{dy}{dx} = -2 \Rightarrow -\frac{dy}{dx}(\cos y - 3) = -2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\cos y - 3}.$$

**3. $ax + by^2 = \cos y$.**

**Sol. Given:** $ax + by^2 = \cos y$

Differentiating both sides w.r.t. $x$, we have

$$\frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx}(\cos y) \quad \therefore a + b\,.\,2y\frac{dy}{dx} = -\sin y\frac{dy}{dx}$$

$$\Rightarrow 2by\frac{dy}{dx} + \sin y\frac{dy}{dx} = -a$$

$$\Rightarrow \frac{dy}{dx}(2by + \sin y) = -a \Rightarrow \frac{dy}{dx} = \frac{-a}{2by + \sin y}$$

**4. $xy + y^2 = \tan x + y$.**

**Sol. Given:** $xy + y^2 = \tan x + y$

Differentiating both sides w.r.t. $x$, we have

$$\frac{d}{dx}(xy) + \frac{d}{dx}y^2 = \frac{d}{dx}\tan x + \frac{d}{dx}y$$

Applying product rule,

$$x\frac{d}{dx}y + y\frac{d}{dx}x + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow \quad x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow \quad x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x - y$$

$$\Rightarrow \quad (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y \quad \therefore \quad \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}.$$

**5. $x^2 + xy + y^2 = 100$.**

**Sol. Given:** $x^2 + xy + y^2 = 100$

Differentiating both sides w.r.t. $x$,

$$\frac{d}{dx}x^2 + \frac{d}{dx}xy + \frac{d}{dx}y^2 = \frac{d}{dx}(100)$$

$$\therefore \quad 2x + \left(x\frac{d}{dx}y + y\frac{d}{dx}x\right) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x + 2y)\frac{dy}{dx} = -2x - y \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{(2x + y)}{x + 2y}.$$

**6. $x^3 + x^2y + xy^2 + y^3 = 81$.**

**Sol. Given:** $x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating both sides w.r.t. $x$,

$$\frac{d}{dx}x^3 + \frac{d}{dx}x^2y + \frac{d}{dx}xy^2 + \frac{d}{dx}y^3 = \frac{d}{dx}81$$

$$\therefore \quad 3x^2 + \left(x^2\frac{dy}{dx} + y\,.\,\frac{d}{dx}x^2\right) + x\frac{d}{dx}y^2 + y^2\frac{d}{dx}(x) + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 3x^2 + x^2\frac{dy}{dx} + y\,.\,2x + x\,.\,2y\frac{dy}{dx} + y^2.1 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx}(x^2 + 2xy + 3y^2) = -3x^2 - 2xy - y^2$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(3x^2 + 2xy + y^2)}{x^2 + 2xy + 3y^2}.$$

**7. $\sin^2 y + \cos xy = \pi$.**

**Sol. Given:** $\sin^2 y + \cos xy = \pi$

Differentiating both sides w.r.t. $x$,

$$\frac{d}{dx}(\sin y)^2 + \frac{d}{dx}\cos xy = \frac{d}{dx}(\pi)$$

$$\therefore \quad 2(\sin y)^1\frac{d}{dx}\sin y - \sin xy\frac{d}{dx}(xy) = 0$$

$$\Rightarrow \quad 2\sin y\cos y\frac{dy}{dx} - \sin xy\left(x\frac{dy}{dx} + y\,.\,1\right) = 0$$

$$\Rightarrow \quad \sin 2y \frac{dy}{dx} - x \sin xy \frac{dy}{dx} - y \sin xy = 0$$

$$\Rightarrow \quad (\sin 2y - x \sin xy) \frac{dy}{dx} = y \sin xy$$

$$\therefore \quad \frac{dy}{dx} = \frac{y \sin xy}{\sin 2y - x \sin xy}.$$

**8.** $\sin^2 x + \cos^2 y = 1.$

**Sol. Given:** $\sin^2 x + \cos^2 y = 1$

Differentiating both sides w.r.t. $x$,

$$\frac{d}{dx} (\sin x)^2 + \frac{d}{dx} (\cos y)^2 = \frac{d}{dx} (1)$$

$$\therefore \quad 2 (\sin x)^1 \frac{d}{dx} \sin x + 2 (\cos y)^1 \frac{d}{dx} \cos y = 0$$

$$\Rightarrow \quad 2 \sin x \cos x + 2 \cos y \left( - \sin y \frac{dy}{dx} \right) = 0$$

$$\Rightarrow \quad 2 \sin x \cos x - 2 \sin y \cos y \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \sin 2x - \sin 2y \frac{dy}{dx} = 0$$

$$\Rightarrow \quad - \sin 2y \frac{dy}{dx} = - \sin 2x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}.$$

**9.** $y = \sin^{-1} \left( \frac{2x}{1+x^2} \right).$

**Sol. Given:** $y = \sin^{-1} \left( \frac{2x}{1+x^2} \right)$

To simplify the given Inverse T-function, put $x = \tan \theta$.

$$\therefore \quad y = \sin^{-1} \left( \frac{2 \tan \theta}{1 + \tan^2 \theta} \right) = \sin^{-1} (\sin 2\theta) = 2\theta$$

$$\Rightarrow \quad y = 2 \tan^{-1} x \qquad (\because \; x = \tan \theta \; \Rightarrow \; \theta = \tan^{-1} x)$$

$$\therefore \quad \frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2} = \frac{2}{1+x^2}.$$

**10.** $y = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right), \dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}$.

**Sol. Given:** $y = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right), \dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}$

To simplify the given Inverse T-function, put $x = \tan\theta$.

$$\therefore \quad y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta) = 3\theta$$

$$\Rightarrow \quad y = 3\tan^{-1} x \qquad (\because\ x = \tan\theta \Rightarrow \theta = \tan^{-1} x)$$

$$\therefore \quad \frac{dy}{dx} = 3 \cdot \frac{1}{1+x^2} = \frac{3}{1+x^2}.$$

**11.** $y = \cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right), 0 < x < 1$.

**Sol. Given:** $y = \cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right), 0 < x < 1$

To simplify the given Inverse T-function, put $x = \tan\theta$.

$$\therefore \quad y = \cos^{-1}\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right) = \cos^{-1}(\cos 2\theta)$$

$$= 2\theta = 2\tan^{-1} x \quad (\because\ x = \tan\theta \Rightarrow \theta = \tan^{-1} x)$$

$$\therefore \quad \frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2} = \frac{2}{1+x^2}.$$

**12.** $y = \sin^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right), 0 < x < 1$.

**Sol. Given:** $y = \sin^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right)$

To simplify the given Inverse T-function, **put $x = \tan\theta$.**

$$\therefore \quad y = \sin^{-1}\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right) = \sin^{-1}(\cos 2\theta)$$

$$= \sin^{-1}\sin\left(\frac{\pi}{2} - 2\theta\right) = \frac{\pi}{2} - 2\theta$$

$$\Rightarrow \quad y = \frac{\pi}{2} - 2\tan^{-1} x \qquad (\because\ x = \tan\theta \Rightarrow \theta = \tan^{-1} x)$$

$$\therefore \quad \frac{dy}{dx} = 0 - 2 \cdot \frac{1}{1+x^2} = \frac{-2}{1+x^2}.$$

**13.** $\boldsymbol{y = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right), -1 < x < 1.}$

**Sol. Given:** $y = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$

To simplify the given Inverse T-function **put $\boldsymbol{x = \tan\theta}$.**

$$\therefore \quad y = \cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\sin 2\theta)$$

$$= \cos^{-1}\cos\left(\frac{\pi}{2} - 2\theta\right) = \frac{\pi}{2} - 2\theta$$

$$\Rightarrow \quad y = \frac{\pi}{2} - 2\tan^{-1}x \quad (\because x = \tan\theta \;\Rightarrow\; \theta = \tan^{-1}x)$$

$$\therefore \quad \frac{dy}{dx} = 0 - 2\cdot\frac{1}{1+x^2} = \frac{-2}{1+x^2}.$$

**14.** $\boldsymbol{y = \sin^{-1}(2x\sqrt{1-x^2}), -\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}.}$

**Sol. Given:** $y = \sin^{-1}(2x\sqrt{1-x^2})$

**Put $\boldsymbol{x = \sin\theta}$**

To simplify the given Inverse T-function,

**put $\boldsymbol{x = \sin\theta}$** (For $\sqrt{a^2 - x^2}$, put $x = a\sin\theta$)

$$\therefore \quad y = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$$

$$= \sin^{-1}(2\sin\theta\sqrt{\cos^2\theta}) = \sin^{-1}(2\sin\theta\cos\theta)$$

$$y = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1}x$$

$$[\because x = \sin\theta \;\Rightarrow\; \theta = \sin^{-1}x]$$

$$\therefore \quad \frac{dy}{dx} = 2\cdot\frac{1}{\sqrt{1-x^2}}.$$

**15.** $\boldsymbol{y = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right) \; 0 < x < \dfrac{1}{\sqrt{2}}.}$

**Sol. Given:** $y = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$

To simplify the given inverse T-function, **put $\boldsymbol{x = \cos\theta}$.**

$$\therefore \quad y = \sec^{-1}\left(\frac{1}{2\cos^2\theta - 1}\right) = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$

$$= \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1}x \quad (\because x = \cos\theta \;\Rightarrow\; \theta = \cos^{-1}x)$$

$$\therefore \quad \frac{dy}{dx} = 2\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{-2}{\sqrt{1-x^2}}.$$

# Exercise 5.4

**Differentiate the following functions 1 to 10 w.r.t. $x$**

**1.** $\dfrac{e^x}{\sin x}$.

**Sol.** Let $y = \dfrac{e^x}{\sin x}$

$$\therefore \quad \frac{dy}{dx} = \frac{(\text{DEN})\dfrac{d}{dx}(\text{NUM}) - (\text{NUM})\dfrac{d}{dx}(\text{DEN})}{(\text{DEN})^2}$$

$$= \frac{\sin x \dfrac{d}{dx} e^x - e^x \dfrac{d}{dx}\sin x}{\sin^2 x} = \frac{\sin x \,.\, e^x - e^x \cos x}{\sin^2 x}$$

$$= e^x \frac{(\sin x - \cos x)}{\sin^2 x}.$$

**2.** $e^{\sin^{-1} x}$.

**Sol.** Let $y = e^{\sin^{-1} x}$

$$\therefore \quad \frac{dy}{dx} = e^{\sin^{-1} x} \frac{d}{dx} \sin^{-1} x \qquad \left[\because \frac{d}{dx} e^{f(x)} = e^{f(x)} \frac{d}{dx} f(x)\right]$$

$$= e^{\sin^{-1} x} \,.\, \frac{1}{\sqrt{1 - x^2}}.$$

**3.** $e^{x^3}$.

**Sol.** Let $y = e^{x^3} = e^{(x^3)}$

$$\therefore \quad \frac{dy}{dx} = e^{(x^3)} \frac{d}{dx} x^3 \qquad \left[\because \frac{d}{dx} e^{f(x)} = e^{f(x)} \frac{d}{dx} f(x)\right]$$

$$= e^{(x^3)}\, 3x^2 = 3x^2\, e^{(x^3)}.$$

**4.** $\sin(\tan^{-1} e^{-x})$.

**Sol.** Let $y = \sin(\tan^{-1} e^{-x})$

$$\therefore \quad \frac{dy}{dx} = \cos(\tan^{-1} e^{-x}) \frac{d}{dx}(\tan^{-1} e^{-x})$$

$$\left[\because \frac{d}{dx} \sin f(x) = \cos f(x) \frac{d}{dx} f(x)\right]$$

$$= \cos(\tan^{-1} e^{-x}) \frac{1}{1 + (e^{-x})^2} \frac{d}{dx} e^{-x}$$

$$\left[\because \frac{d}{dx} \tan^{-1} f(x) = \frac{1}{1 + (f(x))^2} \frac{d}{dx} f(x)\right]$$

$$= \cos(\tan^{-1} e^{-x}) \frac{1}{1+e^{-2x}} e^{-x} \frac{d}{dx}(-x)$$

$$= -\frac{e^{-x}\cos(\tan^{-1} e^{-x})}{1+e^{-2x}} \qquad \left[\because \frac{d}{dx}(-x) = -1\right]$$

**5. log (cos $e^x$).**

**Sol.** Let $y = \log(\cos e^x)$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos e^x} \frac{d}{dx}(\cos e^x)\left[\because \frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right]$$

$$= \frac{1}{\cos e^x}(-\sin e^x)\frac{d}{dx}e^x \qquad \left[\because \frac{d}{dx}\cos f(x) = -\sin f(x)\frac{d}{dx}f(x)\right]$$

$$= -(\tan e^x)\, e^x = -e^x(\tan e^x)$$

**6. $e^x + e^{x^2} + ... + e^{x^5}$.**

**Sol.** Let $y = e^x + e^{x^2} + ... + e^{x^5}$

$$= e^x + e^{x^2} + e^{x^3} + e^{x^4} + e^{x^5}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}e^x + \frac{d}{dx}e^{x^2} + \frac{d}{dx}e^{x^3} + \frac{d}{dx}e^{x^4} + \frac{d}{dx}e^{x^5}$$

$$= e^x + e^{x^2}\frac{d}{dx}x^2 + e^{x^3}\frac{d}{dx}x^3 + e^{x^4}\frac{d}{dx}x^4$$

$$+ e^{x^5}\frac{d}{dx}x^5 \quad \left[\because \frac{d}{dx}e^{f(x)} = e^{f(x)}\frac{d}{dx}f(x)\right]$$

$$= e^x + e^{x^2}\,.\,2x + e^{x^3}\,.\,3x^2 + e^{x^4}\,.\,4x^3 + e^{x^5}\,5x^4$$

$$= e^x + 2x\,e^{x^2} + 3x^2\,e^{x^3} + 4x^3\,e^{x^4} + 5x^4\,e^{x^5}.$$

**7. $\sqrt{e^{\sqrt{x}}}$, $x > 0$.**

**Sol.** Let $y = \sqrt{e^{\sqrt{x}}} = (e^{\sqrt{x}})^{1/2}$

$$\therefore \frac{dy}{dx} = \frac{1}{2}(e^{\sqrt{x}})^{-1/2}\frac{d}{dx}e^{\sqrt{x}}\left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$$

$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}}\, e^{\sqrt{x}}\frac{d}{dx}\sqrt{x} \qquad \left[\because \frac{d}{dx}e^{f(x)} = e^{f(x)}\frac{d}{dx}f(x)\right]$$

$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}}\, e^{\sqrt{x}}\frac{1}{2\sqrt{x}}\left[\because \frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}\right]$$

$$= \frac{e^{\sqrt{x}}}{4\sqrt{x}\,\sqrt{e^{\sqrt{x}}}}.$$

**8. log (log $x$), $x > 1$.**

**Sol.** Let $y = \log(\log x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\log x} \frac{d}{dx} (\log x) \qquad \left[\because \frac{d}{dx} \log f(x) = \frac{1}{f(x)} \frac{d}{dx} f(x)\right]$$

$$= \frac{1}{\log x} \frac{1}{x} = \frac{1}{x \log x}.$$

**9.** $\dfrac{\cos x}{\log x}$**,** $x > 0$**.**

**Sol.** Let $y = \dfrac{\cos x}{\log x}$

$$\therefore \quad \frac{dy}{dx} = \frac{(\text{DEN}) \frac{d}{dx} (\text{NUM}) - (\text{NUM}) \frac{d}{dx} (\text{DEN})}{(\text{DEN})^2}$$

$$= \frac{\log x \frac{d}{dx} (\cos x) - \cos x \frac{d}{dx} \log x}{(\log x)^2}$$

$$= \frac{\log x (-\sin x) - \cos x \cdot \frac{1}{x}}{(\log x)^2}$$

$$= \frac{-\left(\sin x \log x + \frac{\cos x}{x}\right)}{(\log x)^2} = -\frac{(x \sin x \log x + \cos x)}{x (\log x)^2}.$$

**10.** $\cos (\log x + e^x)$**,** $x > 0$**.**

**Sol.** Let $y = \cos (\log x + e^x)$

$$\therefore \quad \frac{dy}{dx} = -\sin (\log x + e^x) \frac{d}{dx} (\log x + e^x)$$

$$\left[\because \frac{d}{dx} \cos f(x) = -\sin f(x) \frac{d}{dx} f(x)\right]$$

$$= -\sin (\log x + e^x) \cdot \left(\frac{1}{x} + e^x\right)$$

$$= -\left(\frac{1}{x} + e^x\right) \sin (\log x + e^x).$$

# Exercise 5.5

**Note. Logarithmic Differentiation.**
The process of differentiating a function after taking its logarithm is called **logarithmic differentiation.**
This process of differentiation is very useful in the following situations:
(*i*) The given function is of the form $(f(x))^{g(x)}$
(*ii*) The given function involves complicated (as per our thinking) products (or and) quotients (or and) powers.

**Remark 1.** $\log \dfrac{(a^m\, b^n\, c^p)}{d^q\, l^k}$

$$= m \log a + n \log b + p \log c - q \log d - k \log l$$

**Remark 2.** $\log (u + v) \neq \log u + \log v$
and $\log (u - v) \neq \log u - \log v$.

**Differentiate the following functions given in Exercises 1 to 5 w.r.t. *x*.**

**1. $\cos x \cos 2x \cos 3x$.**

**Sol.** Let $y = \cos x \cos 2x \cos 3x$ ...(*i*)

Taking logs on both sides, we have (see Note, (*ii*) page 261)

$$\log y = \log (\cos x \cos 2x \cos 3x)$$
$$= \log \cos x + \log \cos 2x + \log \cos 3x$$

Differentiating both sides w.r.t. $x$, we have

$$\frac{d}{dx} \log y = \frac{d}{dx} \log \cos x + \frac{d}{dx} \log \cos 2x + \frac{d}{dx} \log \cos 3x$$

$$\therefore \quad \frac{1}{y} \frac{dy}{dx} = \frac{1}{\cos x} \frac{d}{dx} \cos x + \frac{1}{\cos 2x} \frac{d}{dx} \cos 2x$$

$$+ \frac{1}{\cos 3x} \frac{d}{dx} \cos 3x \left[\because \frac{d}{dx} \log f(x) = \frac{1}{f(x)} \frac{d}{dx} f(x)\right]$$

$$= \frac{1}{\cos x} (-\sin x) + \frac{1}{\cos 2x} (-\sin 2x) \frac{d}{dx} (2x)$$

$$+ \frac{1}{\cos 3x} (-\sin 3x) \frac{d}{dx} 3x$$

$$= -\tan x - (\tan 2x)\, 2 - \tan 3x\, (3)$$

$$\therefore \quad \frac{dy}{dx} = -y (\tan x + 2 \tan 2x + 3 \tan 3x).$$

Putting the value of $y$ from (*i*),

$$\frac{dy}{dx} = -\cos x \cos 2x \cos 3x (\tan x + 2 \tan 2x + 3 \tan 3x).$$

**2.** $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$.

**Sol.** Let $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} = \left[\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]^{1/2}$ ...(*i*)

Taking logs on both sides, we have

$$\log y = \frac{1}{2} [\log (x-1) + \log (x-2) - \log (x-3)$$
$$- \log (x-4) - \log (x-5)] \text{ (By Remark I above)}$$

Differentiating both sides w.r.t. $x$, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1}\frac{d}{dx}(x-1) + \frac{1}{x-2}\frac{d}{dx}(x-2) - \frac{1}{x-3}\frac{d}{dx}(x-3)\right.$$
$$\left. - \frac{1}{x-4}\frac{d}{dx}(x-4) - \frac{1}{x-5}\frac{d}{dx}(x-5)\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}y\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

Putting the value of $y$ from (*i*),

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right].$$

**3.** $(\log x)^{\cos x}$.

**Sol.** Let $y = (\log x)^{\cos x}$ ...(*i*) [Form $(f(x))^{g(x)}$]

Taking logs on both sides of (*i*), we have (see Note (*i*) page 261)

$$\log y = \log (\log x)^{\cos x} = \cos x \log (\log x)$$
$$[\because \log m^n = n \log m]$$

$$\therefore \quad \frac{d}{dx} \log y = \frac{d}{dx} [\cos x \,.\, \log (\log x)]$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \cos x \frac{d}{dx} \log (\log x) + \log (\log x) \frac{d}{dx} \cos x$$
[By Product Rule]

$$= \cos x \frac{1}{\log x} \frac{d}{dx} \log x + \log (\log x)(-\sin x)$$

$$= \frac{\cos x}{\log x} \cdot \frac{1}{x} - \sin x \log (\log x)$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{\cos x}{x \log x} - \sin x \log(\log x)\right].$$

Putting the value of $y$ from $(i)$,

$$\frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x \log x} - \sin x \log(\log x)\right].$$

**Very Important Note.**

To differentiate $y = (f(x))^{g(x)} \pm (l(x))^{m(x)}$

or $\quad y = (f(x))^{g(x)} \pm h(x)$

or $\quad y = (f(x))^{g(x)} \pm k$ where $k$ is a constant;

**Never start with taking logs of both sides,** put one term $= u$ and the other $= v$

$$\therefore \quad y = u \pm v$$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

Now find $\frac{du}{dx}$ and $\frac{dv}{dx}$ by the methods already learnt.

$$\therefore \quad \frac{dv}{dx} = \frac{d}{dx} 2^{\sin x} = 2^{\sin x} \log 2 \frac{d}{dx} \sin x$$

$$\left[\because \frac{d}{dx} a^{f(x)} = a^{f(x)} \log a \frac{d}{dx} f(x)\right]$$

$$\Rightarrow \quad \frac{dv}{dx} = 2^{\sin x} (\log 2) \cos x = \cos x \,.\, 2^{\sin x} \log 2 \qquad ...(iii)$$

Putting values from $(ii)$ and $(iii)$ in $(i)$,

$$\frac{dy}{dx} = x^x (1 + \log x) - \cos x \,.\, 2^{\sin x} \log 2.$$

**5.** $\mathbf{(x + 3)^2 (x + 4)^3 (x + 5)^4}$**.**

**Sol.** Let $y = (x + 3)^2 (x + 4)^3 (x + 5)^4 \qquad ...(i)$

Taking logs on both sides of eqn. $(i)$ (see Note $(ii)$ page 261) we have $\log y = 2 \log (x + 3) + 3 \log (x + 4) + 4 \log (x + 5)$ (By Remark I page 262)

$$\therefore \quad \frac{d}{dx} \log y = 2 \frac{d}{dx} \log (x + 3) + 3 \frac{d}{dx} \log (x + 4) + 4 \frac{d}{dx} \log (x + 5)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = 2 \,.\, \frac{1}{x+3}\frac{d}{dx}(x + 3) + 3\frac{1}{x+4}\frac{d}{dx}(x+4) + 4 \,.\, \frac{1}{x+5}\frac{d}{dx}(x + 5)$$

$$= \frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}$$

$$\therefore \quad \frac{dy}{dx} = y\left(\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right)$$

Putting the value of $y$ from $(i)$,

$$\frac{dy}{dx} = (x+3)^2 (x+4)^3 (x+5)^4 \left(\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right).$$

**Differentiate the following functions given in Exercises 6 to 11 w.r.t. $x$.**

**6.** $\left(x + \frac{1}{x}\right)^x + x^{\left(1 + \frac{1}{x}\right)}$.

**Sol.** Let $y = \left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Putting $\left(x + \frac{1}{x}\right)^x = u$ and $x^{\left(1+\frac{1}{x}\right)} = v$,

We have $\quad y = u + v \quad \therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...$(i)$

Now $\quad u = \left(x + \frac{1}{x}\right)^x$

Taking logarithms, $\log u = \log \left(x + \frac{1}{x}\right)^x = x \log\left(x + \frac{1}{x}\right)$[Form $uv$]

Differentiating w.r.t. $x$, we have

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{x + \frac{1}{x}} \frac{d}{dx}\left(x + \frac{1}{x}\right) + \log\left(x + \frac{1}{x}\right) . 1$$

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{x + \frac{1}{x}} \cdot \left(1 - \frac{1}{x^2}\right) + \log\left(x + \frac{1}{x}\right) . 1$$

$$\left[\because \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx} x^{-1} = (-1)\, x^{-2} = \frac{-1}{x^2}\right]$$

$$\Rightarrow \qquad \frac{du}{dx} = u\left[\frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right)\right]$$

$$= \left(x + \frac{1}{x}\right)^x \left[\frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right)\right] \qquad ...(ii)$$

Also $\quad v = x^{\left(1+\frac{1}{x}\right)}$

Taking logarithms, $\log v = \log x^{\left(1+\frac{1}{x}\right)} = \left(1 + \frac{1}{x}\right) \log x$

Differentiating w.r.t. $x$, we have

$$\frac{1}{v} \cdot \frac{dv}{dx} = \left(1 + \frac{1}{x}\right) \cdot \frac{1}{x} + \log x \cdot \left(-\frac{1}{x^2}\right)$$

$$\left[\because \frac{d}{dx}\frac{1}{x} = \frac{d}{dx} x^{-1} = (-1)\, x^{-2} = \frac{-1}{x^2}\right]$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left[\frac{1}{x}\left(1+\frac{1}{x}\right)-\frac{1}{x^2}\log x\right]$$

$$= x^{\left(1+\frac{1}{x}\right)}\left[\frac{1}{x}\left(1+\frac{1}{x}\right)-\frac{1}{x^2}\log x\right] \quad ...(iii)$$

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from $(ii)$ and $(iii)$ in $(i)$, we have

$$\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]+ x^{\left(1+\frac{1}{x}\right)}\left[\frac{1}{x}\left(1+\frac{1}{x}\right)-\frac{1}{x^2}\log x\right]$$

**7. $(\log x)^x + x^{\log x}$.**

**Sol.** Let $y = (\log x)^x + x^{\log x}$

$= u + v$ where $u = (\log x)^x$ and $v = x^{\log x}$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(i)$$

Now $u = (\log x)^x$ $\quad [(f(x))^{g(x)}]$

$\therefore$ $\log u = \log (\log x)^x = x \log (\log x)$ $[\because \log m^n = n \log m]$

$$\therefore \quad \frac{d}{dx}\log u = \frac{d}{dx}[x \log (\log x)]$$

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}x \text{ (By product rule)}$$

$$= x \cdot \frac{1}{\log x}\frac{d}{dx}\log x + \log(\log x) \cdot 1$$

$$= x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log(\log x)$$

$$\therefore \quad \frac{du}{dx} = u\left[\frac{1}{\log x}+\log(\log x)\right] = (\log x)^x\left(\frac{1}{\log x}+\log(\log x)\right)$$

$$= (\log x)^x \frac{(1+\log x \log(\log x))}{\log x}$$

$$= (\log x)^{x-1}(1+\log x\log(\log x)) \quad ...(ii)$$

Again $v = x^{\log x}$ $\quad$ [Form $(f(x))^{g(x)}$]

$\therefore$ $\log v = \log x^{\log x} = \log x \cdot \log x$ $\quad [\because \log m^n = n \log m]$

$= (\log x)^2$

$$\therefore \quad \frac{d}{dx}\log v = \frac{d}{dx}(\log x)^2 \qquad \therefore \quad \frac{1}{v}\frac{dv}{dx} = 2(\log x)^1\frac{d}{dx}\log x$$

$$\left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$$

$$= 2\log x \cdot \frac{1}{x}$$

$\therefore \quad \frac{dv}{dx} = v\left(\frac{2}{x}\log x\right) = x^{\log x} \cdot \frac{2}{x}\log x$

$= 2x^{\log x - 1}\log x$ ...(*iii*)

Putting values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*), we have

$\frac{dy}{dx} = (\log x)^{x-1}(1 + \log x \log(\log x)) + 2x^{\log x - 1}\log x.$

**8.** $\mathbf{(\sin x)^x + \sin^{-1}\sqrt{x}}$ **.**

**Sol.** Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$= u + v$ where $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(*i*)

Now $u = (\sin x)^x$ [Form $(f(x))^{g(x)}$]

$\therefore \log u = \log(\sin x)^x = x \log \sin x$

$\therefore \quad \frac{d}{dx}(\log u) = \frac{d}{dx}(x \log \sin x)$

$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log \sin x + \log \sin x \frac{d}{dx}x$

$= x \cdot \frac{1}{\sin x}\frac{d}{dx}\sin x + (\log \sin x) \cdot 1$

$= x\frac{1}{\sin x}\cos x + \log \sin x = x \cot x + \log \sin x$

$\therefore \quad \frac{du}{dx} = u(x \cot x + \log \sin x) = (\sin x)^x (x \cot x + \log \sin x)$...(*ii*)

Again $v = \sin^{-1}\sqrt{x}$

$\therefore \frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}\sqrt{x} \quad \left| \because \frac{d}{dx}\sin^{-1} f(x) = \frac{1}{\sqrt{1-(f(x))^2}}\frac{d}{dx}f(x)\right.$

$= \frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}} \left[\because \frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}\right]$

or $\quad \frac{dv}{dx} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x(1-x)}} = \frac{1}{2\sqrt{x-x^2}}$ ...(*iii*)

Putting values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*),

$\frac{dy}{dx} = (\sin x)^x (x \cot x + \log \sin x) + \frac{1}{2\sqrt{x-x^2}}.$

**9.** $\mathbf{x^{\sin x} + (\sin x)^{\cos x}}$**.**

**Sol.** Let $y = x^{\sin x} + (\sin x)^{\cos x}$

$= u + v$ where $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ...(*i*)

Now $u = x^{\sin x}$ [Form $(f(x))^{g(x)}$]

$\therefore \log u = \log x^{\sin x} = \sin x \log x$

$\therefore \quad \dfrac{d}{dx} \log u = \dfrac{d}{dx} (\sin x \log x)$

$\Rightarrow \quad \dfrac{1}{u} \dfrac{du}{dx} = \sin x \dfrac{d}{dx} \log x + \log x \dfrac{d}{dx} \sin x$

$= \sin x \cdot \dfrac{1}{x} + (\log x) \cos x \quad = \dfrac{\sin x}{x} + \cos x \log x$

$\therefore \quad \dfrac{du}{dx} = u \left( \dfrac{\sin x}{x} + \cos x \log x \right)$

$= x^{\sin x} \left( \dfrac{\sin x}{x} + \cos x \log x \right)$ ...(*ii*)

Again $v = (\sin x)^{\cos x}$ [Form $f(x)^{g(x)}$]

$\therefore \quad \log v = \log (\sin x)^{\cos x} = \cos x \log \sin x$

$\therefore \dfrac{d}{dx} (\log v) = \dfrac{d}{dx} [\cos x \log \sin x]$

$\Rightarrow \quad \dfrac{1}{v} \dfrac{dv}{dx} = \cos x \dfrac{d}{dx} \log \sin x + \log \sin x \dfrac{d}{dx} \cos x$

$= \cos x \dfrac{1}{\sin x} \dfrac{d}{dx} (\sin x) + \log \sin x (-\sin x)$

$= \cot x \cdot \cos x - \sin x \log \sin x$

$\therefore \quad \dfrac{dv}{dx} = v (\cos x \cot x - \sin x \log \sin x)$

$= (\sin x)^{\cos x} (\cos x \cot x - \sin x \log \sin x)$ ...(*iii*)

Putting values of $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*),

we have $\dfrac{dy}{dx} = x^{\sin x} \left( \dfrac{\sin x}{x} + \cos x \log x \right)$

$+ (\sin x)^{\cos x} (\cos x \cot x - \sin x \log \sin x)$

**10.** $\boldsymbol{x^{x \cos x} + \dfrac{x^2 + 1}{x^2 - 1}}$**.**

**Sol.** Let $y = x^{x \cos x} + \dfrac{x^2 + 1}{x^2 - 1}$

Putting $x^{x \cos x} = u$ and $\dfrac{x^2 + 1}{x^2 - 1} = v$

We have $y = u + v$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ...(*i*)

Now $u = x^{x \cos x}$

Taking logarithms, $\log u = \log x^{x \cos x} = x \cos x \log x$

Differentiating w.r.t. $x$, we have

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x \cos x \log x)$$

$$= \frac{d}{dx}(x) \cdot \cos x \log x + x \frac{d}{dx}(\cos x) \cdot \log x + x \cos x \frac{d}{dx}(\log x)$$

$$\left[\because \frac{d}{dx}(uvw) = \frac{du}{dx} vw + u\frac{dv}{dx} \cdot w + uv\frac{dw}{dx}\right]$$

$$= 1 \cos x \log x + x(-\sin x)\log x + x \cos x \cdot \frac{1}{x}$$

$$\Rightarrow \quad \frac{du}{dx} = u\,[\cos x \log x - x \sin x \log x + \cos x]$$

$$= x^{x \cos x}\,[\cos x \log x - x \sin x \log x + \cos x] \qquad ...(ii)$$

Also $v = \dfrac{x^2 + 1}{x^2 - 1}$. Using quotient rule, we have

$$\frac{dv}{dx} = \frac{(x^2 - 1)\frac{d}{dx}(x^2 + 1) - (x^2 + 1) \cdot \frac{d}{dx}(x^2 - 1)}{(x^2 - 1)^2}$$

$$= \frac{(x^2 - 1) \cdot 2x - (x^2 + 1) \cdot 2x}{(x^2 - 1)^2} = \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2 - 1)^2}$$

$$= -\frac{4x}{(x^2 - 1)^2} \qquad ...(iii)$$

Putting the values of $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*), we have

$$\frac{dy}{dx} = x^{x \cos x}\,[\cos x \log x - x \sin x \log x + \cos x] - \frac{4x}{(x^2 - 1)^2}.$$

**11. $(x \cos x)^x + (x \sin x)^{1/x}$.**

**Sol.** Let $y = (x \cos x)^x + (x \sin x)^{1/x}$

Putting $(x \cos x)^x = u$ and $(x \sin x)^{1/x} = v$,

We have $y = u + v$ $\quad \therefore \quad \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ $\qquad ...(i)$

Now $u = (x \cos x)^x$

Taking logarithms, $\log u = \log (x \cos x)^x = x \log (x \cos x)$

$= x (\log x + \log \cos x)$

Differentiating w.r.t. $x$, we have

$$\frac{1}{u} \cdot \frac{du}{dx} = x\left[\frac{1}{x} + \frac{1}{\cos x} \cdot (-\sin x)\right] + (\log x + \log \cos x) \cdot 1$$

$\Rightarrow \quad \dfrac{du}{dx} = u\,[1 - x \tan x + \log (x \cos x)]$

$[\because \;\; \log x + \log \cos x = \log (x \cos x)]$

$= (x \cos x)^x\,[1 - x \tan x + \log (x \cos x)]$ ...(*ii*)

Also $\quad v = (x \sin x)^{1/x}$

Taking logarithms, $\log v = \log (x \sin x)^{1/x} = \dfrac{1}{x} \log (x \sin x)$

$= \dfrac{1}{x}\,(\log x + \log \sin x)$

Differentiating w.r.t. $x$, we have

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{1}{x}\left[\frac{1}{x} + \frac{1}{\sin x} \cdot \cos x\right] + (\log x + \log \sin x)\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log (x \sin x)}{x^2}\right]$$

$$= (x \sin x)^{1/x} \cdot \left[\frac{1 + x \cot x - \log (x \sin x)}{x^2}\right] \qquad \text{...}(iii)$$

Putting the values of $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*), we have

$$\frac{dy}{dx} = (x \cos x)^x\,[1 - x \tan x + \log (x \cos x)] + (x \sin x)^{1/x}\left[\frac{1 + x \cot x - \log (x \sin x)}{x^2}\right].$$

**Find $\dfrac{dy}{dx}$ of the functions given in Exercises 12 to 15:**

**12. $x^y + y^x = 1$.**

**Sol. Given :** $x^y + y^x = 1$

$\Rightarrow \quad u + v = 1 \quad$ where $\quad u = x^y$ and $\quad v = y^x$

$\therefore \quad \dfrac{d}{dx}(u) + \dfrac{d}{dx}(v) = \dfrac{d}{dx}(1)$

*i.e.,* $\quad \dfrac{du}{dx} + \dfrac{dv}{dx} = 0$ ...(*i*)

Now $\quad u = x^y \qquad [(\text{Variable})^{\text{variable}} = (f(x))^{g(x)}]$

$\therefore \quad \log u = \log x^y = y \log x$

$\therefore \quad \dfrac{d}{dx} \log u = \dfrac{d}{dx}\,(y \log x)$

$\Rightarrow \quad \dfrac{1}{u}\,\dfrac{du}{dx} = y \dfrac{d}{dx} \log x + \log x\,\dfrac{dy}{dx} \qquad = y \cdot \dfrac{1}{x} + \log x \cdot \dfrac{dy}{dx}$

$\therefore \quad \dfrac{du}{dx} = u\left(\dfrac{y}{x} + \log x \cdot \dfrac{dy}{dx}\right)$

or $\quad \dfrac{du}{dx} = x^y \left(\dfrac{y}{x} + \log x \dfrac{dy}{dx}\right) = x^y \dfrac{y}{x} + x^y \log x \dfrac{dy}{dx}$

or $\quad \dfrac{du}{dx} = x^{y-1}y + x^y \log x \dfrac{dy}{dx} \quad ...(ii) \qquad \left[\because \dfrac{x^y}{x} = \dfrac{x^y}{x^1} = x^{y-1}\right]$

Again $\quad v = y^x$

$\therefore \quad \log v = \log y^x = x \log y \qquad \therefore \dfrac{d}{dx} \log v = \dfrac{d}{dx}(x \log y)$

$\Rightarrow \quad \dfrac{1}{v}\dfrac{dv}{dx} = x\dfrac{d}{dx}(\log y) + \log y \dfrac{d}{dx}x = x \cdot \dfrac{1}{y}\dfrac{dy}{dx} + \log y \cdot 1$

$\Rightarrow \quad \dfrac{dv}{dx} = v\left(\dfrac{x}{y}\dfrac{dy}{dx} + \log y\right)$

$= y^x \left(\dfrac{x}{y}\dfrac{dy}{dx} + \log y\right) = y^x \dfrac{x}{y}\dfrac{dy}{dx} + y^x \log y$

$\Rightarrow \quad \dfrac{dv}{dx} = y^{x-1}x \dfrac{dy}{dx} + y^x \log y \qquad ...(iii)$

Putting values of $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ from $(ii)$ and $(iii)$ in $(i)$, we have

$$x^{y-1}y + x^y \log x \dfrac{dy}{dx} + y^{x-1}x \dfrac{dy}{dx} + y^x \log y = 0$$

or $\quad \dfrac{dy}{dx}(x^y \log x + y^{x-1}x) = -x^{y-1}y - y^x \log y$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{(x^{y-1}y + y^x \log y)}{x^y \log x + y^{x-1}x}.$

**13. $y^x = x^y$.**

**Sol. Given:** $y^x = x^y \Rightarrow x^y = y^x$.

| Form on both sides is $(f(x))^{g(x)}$

Taking logarithms, $\log x^y = \log y^x \Rightarrow y \log x = x \log y$

Differentiating w.r.t. $x$, we have

$y \cdot \dfrac{1}{x} + \log x \cdot \dfrac{dy}{dx} = x \cdot \dfrac{1}{y} \cdot \dfrac{dy}{dx} + \log y \cdot 1$

$\Rightarrow \left(\log x - \dfrac{x}{y}\right)\dfrac{dy}{dx} = \log y - \dfrac{y}{x}$

$\Rightarrow \dfrac{y \log x - x}{y} \cdot \dfrac{dy}{dx} = \dfrac{x \log y - y}{x} \quad \therefore \dfrac{dy}{dx} = \dfrac{y(x \log y - y)}{x(y \log x - x)}.$

**14. $(\cos x)^y = (\cos y)^x$.**

**Sol. Given:** $(\cos x)^y = (\cos y)^x$ [Form on both sides is $(f(x))^{g(x)}$]

$\therefore$ Taking logs on both sides, we have

$\log (\cos x)^y = \log (\cos y)^x$

$\Rightarrow \quad y \log \cos x = x \log \cos y \quad [\because \log m^n = n \log m]$

Differentiating both sides w.r.t. $x$, we have

$$\frac{d}{dx}(y \log \cos x) = \frac{d}{dx}(x \log \cos y)$$

Applying Product Rule on both sides,

$$\Rightarrow \quad y\frac{d}{dx}\log \cos x + \log \cos x \frac{dy}{dx} = x \frac{d}{dx}\log \cos y + \log \cos y \frac{d}{dx}x$$

$$\Rightarrow \quad y \cdot \frac{1}{\cos x}\frac{d}{dx}\cos x + \log \cos x\frac{dy}{dx} = x \cdot \frac{1}{\cos y}\frac{d}{dx}\cos y + \log \cos y$$

$$\Rightarrow \quad y\frac{1}{\cos x}(-\sin x) + \log \cos x\frac{dy}{dx} = x\frac{1}{\cos y}\left(-\sin y\frac{dy}{dx}\right) + \log \cos y$$

$$\Rightarrow \quad -y \tan x + \log \cos x \cdot \frac{dy}{dx} = -x \tan y \frac{dy}{dx} + \log \cos y$$

$$\Rightarrow \quad x \tan y \frac{dy}{dx} + \log \cos x \cdot \frac{dy}{dx} = y \tan x + \log \cos y$$

$$\Rightarrow \quad \frac{dy}{dx}(x \tan y + \log \cos x) = y \tan x + \log \cos y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}.$$

**15.** $\boldsymbol{xy = e^{x-y}}$**.**

**Sol. Given:** $xy = e^{x-y}$

Taking logs on both sides, we have

$$\log(xy) = \log e^{x-y}$$

$$\Rightarrow \quad \log x + \log y = (x - y)\log e$$

$$\Rightarrow \quad \log x + \log y = x - y \qquad (\because \log e = 1)$$

Differentiating both sides w.r.t. $x$, we have

$$\frac{d}{dx}\log x + \frac{d}{dx}\log y = \frac{d}{dx}x - \frac{d}{dx}y$$

$$\Rightarrow \quad \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} + \frac{dy}{dx} = 1 - \frac{1}{x} \Rightarrow \frac{dy}{dx}\left(\frac{1}{y}+1\right) = \frac{x-1}{x}$$

$$\Rightarrow \quad \left(\frac{1+y}{y}\right)\frac{dy}{dx} = \frac{x-1}{x}$$

Cross-multiplying $x(1 + y)\dfrac{dy}{dx} = y(x - 1)$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y(x-1)}{x(1+y)}.$$

**16. Find the derivative of the function given by $f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$ and hence find $f'(1)$.**

**Sol. Given:** $f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$ ...(*i*)

Taking logs on both sides, we have

$$\log f(x) = \log (1 + x) + \log (1 + x^2) + \log (1 + x^4) + \log (1 + x^8)$$

Differentiating both sides w.r.t. $x$, we have

$$\frac{1}{f(x)} \frac{d}{dx} f(x) = \frac{1}{1+x} \frac{d}{dx}(1 + x) + \frac{1}{1+x^2} \frac{d}{dx} (1 + x^2) + \frac{1}{1+x^4} \frac{d}{dx} (1 + x^4) + \frac{1}{1+x^8} \frac{d}{dx} (1 + x^8)$$

$$\Rightarrow \quad \frac{1}{f(x)} f'(x) = \frac{1}{1+x} . 1 + \frac{1}{1+x^2} . 2x + \frac{1}{1+x^4} . 4x^3 + \frac{1}{1+x^8} 8x^7$$

$$\therefore \quad f'(x) = f(x) \left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

Putting the value of $f(x)$ from (*i*),

$$f'(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8) \left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

Putting $x = 1$,

$$f'(1) = (1 + 1)(1 + 1)(1 + 1)(1 + 1) \left[\frac{1}{1+1} + \frac{2}{1+1} + \frac{4}{1+1} + \frac{8}{1+1}\right]$$

$$= 2.2.2.2 \left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right] = 16\left[\frac{15}{2}\right] = 8 \times 15 = 120.$$

**17. Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below:**

**(*i*) by using product rule.**

**(*ii*) by expanding the product to obtain a single polynomial.**

**(*iii*) by logarithmic differentiation.**

**Do they all give the same answer?**

**Sol. Given:** Let $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$ ...(1)

(*i*) **To find $\dfrac{dy}{dx}$ by using Product Rule**

$$\frac{dy}{dx} = (x^2 - 5x + 8) \frac{d}{dx} (x^3 + 7x + 9)$$

$$+ (x^3 + 7x + 9)\frac{d}{dx}(x^2 - 5x + 8)$$

$$= (x^2 - 5x + 8)(3x^2 + 7) + (x^3 + 7x + 9)(2x - 5)$$

$$= 3x^4 + 7x^2 - 15x^3 - 35x + 24x^2 + 56$$

$$+ 2x^4 - 5x^3 + 14x^2 - 35x + 18x - 45$$

$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11 \quad ...(2)$$

(*ii*) **To find $\frac{dy}{dx}$ by expanding the product to obtain a single polynomial.**

From (*i*), $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

$$= x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$$

or $$y = x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$$

$$\therefore \quad \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11 \quad ...(3)$$

(*iii*) **To find $\frac{dy}{dx}$ by logarithmic differentiation**

Taking logs on both sides of (*i*), we have

$$\log y = \log(x^2 - 5x + 8) + \log(x^3 + 7x + 9)$$

$$\therefore \frac{d}{dx}\log y = \frac{d}{dx}\log(x^2 - 5x + 8) + \frac{d}{dx}\log(x^3 + 7x + 9)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 - 5x + 8}\frac{d}{dx}(x^2 - 5x + 8) + \frac{1}{x^3 + 7x + 9}\cdot\frac{d}{dx}(x^3 + 7x + 9)$$

$$= \frac{1}{x^2 - 5x + 8}(2x - 5) + \frac{1}{x^3 + 7x + 9}(3x^2 + 7)$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{(2x - 5)}{x^2 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9}\right]$$

$$= y\left[\frac{(2x - 5)(x^3 + 7x + 9) + (3x^2 + 7)(x^2 - 5x + 8)}{(x^2 - 5x + 8)(x^3 + 7x + 9)}\right]$$

$$= y\,\frac{[2x^4 + 14x^2 + 18x - 5x^3 - 35x - 45 + 3x^4 - 15x^3 + 24x^2 + 7x^2 - 35x + 56]}{(x^2 - 5x + 8)(x^3 + 7x + 9)}$$

or $$\frac{dy}{dx} = y\,\frac{(5x^4 - 20x^3 + 45x^2 - 52x + 11)}{(x^2 - 5x + 8)(x^3 + 7x + 9)}$$

Putting the value of $y$ from (*i*),

$$\frac{dy}{dx} = (x^2 - 5x + 8)(x^3 + 7x + 9)\,\frac{(5x^4 - 20x^3 + 45x^2 - 52x + 11)}{(x^2 - 5x + 8)(x^3 + 7x + 9)}$$

$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11 \quad ...(4)$$

From (2), (3) and (4), we can say that value of $\frac{dy}{dx}$ is same obtained by three different methods used in (*i*), (*ii*) and (*iii*).

**18. If *u*, *v* and *w* are functions of *x*, then show that**

$$\frac{d}{dx}(u\,.\,v\,.\,w) = \frac{du}{dx}\,v\,.\,w + u\,.\,\frac{dv}{dx}\,.\,w + u\,.\,v\,\frac{dw}{dx}$$

**in two ways-first by repeated application of product rule, second by logarithmic differentiation.**

**Sol. Given:** $u$, $v$ and $w$ are functions of $x$.

**To prove:** $\frac{d}{dx}(u\,.\,v\,.\,w) = \frac{du}{dx}\,.\,v\,.\,w + u\,.\,\frac{dv}{dx}\,.\,w + u\,.\,v\,.\,\frac{dw}{dx}$ ...(*i*)

(*i*) **To prove eqn. (*i*): By repeated application of product rule**

L.H.S. $= \frac{d}{dx}(u\,.\,v\,.\,w)$

Let us treat the product $uv$ as a single function

$$= \frac{d}{dx}[(uv)w] \qquad = uv\frac{d}{dx}(w) + w\frac{d}{dx}(uv)$$

Again Applying Product Rule on $\frac{d}{dx}(uv)$

$$\text{L.H.S.} = \frac{d}{dx}(uvw) = uv\frac{dw}{dx} + w\left[u\frac{d}{dx}v + v\frac{d}{dx}u\right]$$

$$= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

Rearranging terms

or $\frac{d}{dx}(uvw) = \frac{du}{dx}\,.\,v\,.\,w + u\,.\,\frac{dv}{dx}\,.\,w + u\,.\,v\,.\,\frac{dw}{dx}$

which proves eqn. (*i*)

(*ii*) **To prove eqn. (*i*): By Logarithmic differentiation**

Let $y = uvw$

Taking logs on both sides

$$\log y = \log(u\,.\,v\,.\,w) = \log u + \log v + \log w$$

$$\therefore \frac{d}{dx}\log y = \frac{d}{dx}\log u + \frac{d}{dx}\log v + \frac{d}{dx}\log w$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right]$$

Putting $y = uvw$, $\frac{d}{dx}(uvw) = uvw\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$

$= \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$ which proves eqn. ($i$).

**Remark.** The result of eqn. ($i$) can be used as a formula for derivative of product of three functions.

It can be used as a formula for doing Q. No. 1 and Q. No. 5 of this Exercise 5.5.

# Exercise 5.6

**If $x$ and $y$ are connected parametrically by the equations given in Exercises 1 to 5, without eliminating the parameter, find $\frac{dy}{dx}$.**

**1. $x = 2at^2$, $y = at^4$.**

**Sol. Given:** $x = 2at^2$ and $y = at^4$

Differentiating both eqns. w.r.t. $t$, we have

$$\frac{dx}{dt} = \frac{d}{dt}(2at^2) = 2a\frac{d}{dt}t^2 = 2a.2t = 4at \quad \text{and} \quad \frac{dy}{dt} = \frac{d}{dt}(at^4) = a\frac{d}{dt}t^4 = a.4t^3 = 4at^3$$

We know that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4at^3}{4at} = t^2$.

**2. $x = a \cos \theta$, $y = b \cos \theta$.**

**Sol. Given:** $x = a \cos \theta$ and $y = b \cos \theta$

Differentiating both eqns. w.r.t. $\theta$, we have

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a \cos \theta) = a\frac{d}{d\theta}\cos \theta = -a \sin \theta \quad \text{and} \quad \frac{dy}{d\theta} = \frac{d}{d\theta}(b \cos \theta) = b\frac{d}{d\theta}\cos \theta = -b \sin \theta$$

We know that $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$.

**3. $x = \sin t$, $y = \cos 2t$.**

**Sol. Given:** $x = \sin t$ and $y = \cos 2t$

Differentiating both eqns. w.r.t. $t$, we have

$$\frac{dx}{dt} = \cos t \quad \text{and} \quad \frac{dy}{dt} = -\sin 2t\frac{d}{dt}(2t) = -(\sin 2t)\,2 = -2\sin 2t$$

We know that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{-2\sin 2t}{\cos t}$

$$= -2 . \frac{2\sin t\cos t}{\cos t} = -4 \sin t.$$

**4. $x = 4t$, $y = \frac{4}{t}$.**

**Sol. Given:** $x = 4t$ and $y = \frac{4}{t}$

$$\therefore \quad \frac{dx}{dt} = \frac{d}{dt}(4t) \quad \text{and} \quad \frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right)$$

$$= 4\frac{d}{dt}t \qquad\qquad = \frac{t\frac{d}{dt}(4) - 4\frac{d}{dt}t}{t^2}$$

$$= 4(1) = 4 \qquad\qquad = \frac{t(0) - 4(1)}{t^2} = -\frac{4}{t^2}$$

We know that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(-\frac{4}{t^2}\right)}{4} = \frac{-1}{(t^2)}$.

**5. $x = \cos\theta - \cos 2\theta$, $y = \sin\theta - \sin 2\theta$.**

**Sol. Given:** $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$

$\therefore \quad \frac{dx}{d\theta} = \frac{d}{d\theta}(\cos\theta) - \frac{d}{d\theta}\cos 2\theta$ and $\frac{dy}{d\theta} = \cos\theta - \frac{d}{d\theta}\sin 2\theta$

$= -\sin\theta - (-\sin 2\theta)\frac{d}{d\theta}2\theta \qquad = \cos\theta - \cos 2\theta\frac{d}{d\theta}2\theta$

$= -\sin\theta + (\sin 2\theta)2 \qquad = \cos\theta - \cos 2\theta(2)$

$= 2\sin 2\theta - \sin\theta \qquad = \cos\theta - 2\cos 2\theta.$

We know that $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$.

**If $x$ and $y$ are connected parametrically by the equations given in Exercises 6 to 10, without eliminating the parameter, find $\frac{dy}{dx}$.**

**6. $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$.**

**Sol.** $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

Differentiating both eqns. w.r.t. $\theta$, we have

$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\theta - \sin\theta)$ and $\frac{dy}{d\theta} = a\frac{d}{d\theta}(1 + \cos\theta)$

$= a\left[\frac{d}{d\theta}\theta - \frac{d}{d\theta}\sin\theta\right]$ and $\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}\cos\theta\right]$

$\Rightarrow \frac{dx}{d\theta} = a(1 - \cos\theta)$ and $\frac{dy}{d\theta} = a(0 - \sin\theta) = -a\sin\theta$

We know that $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{a(1 - \cos\theta)}$

$$= -\frac{\sin\theta}{1 - \cos\theta} = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}.$$

**7. $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$.**

**Sol.** We have $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$

$$\therefore \quad \frac{dx}{dt} = \frac{\sqrt{\cos 2t} \,.\, \frac{d}{dt}(\sin^3 t) - \sin^3 t \,.\, \frac{d}{dt}(\sqrt{\cos 2t})}{(\sqrt{\cos 2t})^2}$$

[By Quotient Rule]

$$= \frac{\sqrt{\cos 2t} \,.\, 3 \sin^2 t \, \frac{d}{dt} . (\sin t) - \sin^3 t \,.\, \frac{1}{2} (\cos 2t)^{-1/2} \,.\, \frac{d}{dt} (\cos 2t)}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t} \,.\, 3 \sin^2 t \cos t - \frac{\sin^3 t}{2\sqrt{\cos 2t}} \,.\, (-2 \sin 2t)}{\cos 2t}$$

$$= \frac{3 \sin^2 t \cos t \cos 2t + \sin^3 t \sin 2t}{(\cos 2t)^{3/2}}$$

$$= \frac{3 \sin^2 t \cos t \cos 2t + \sin^3 t \,.\, 2 \sin t \cos t}{(\cos 2t)^{3/2}}$$

$$= \frac{\sin^2 t \cos t \,(3 \cos 2t + 2 \sin^2 t)}{(\cos 2t)^{3/2}}$$

$$\text{and} \quad \frac{dy}{dt} = \frac{\sqrt{\cos 2t} \,.\, \frac{d}{dt}(\cos^3 t) - \cos^3 t \,.\, \frac{d}{dt}(\sqrt{\cos 2t})}{(\sqrt{\cos 2t})^2}$$

[By Quotient Rule]

$$= \frac{\sqrt{\cos 2t} \,.\, 3 \cos^2 t \,.\, \frac{d}{dt} (\cos t) - \cos^3 t \,.\, \frac{1}{2} (\cos 2t)^{-1/2} \,.\, \frac{d}{dt} (\cos 2t)}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t} \,.\, 3 \cos^2 t \,(-\sin t) - \frac{\cos^3 t}{2\sqrt{\cos 2t}} (-2 \sin 2t)}{\cos 2t}$$

$$= \frac{-3 \cos^2 t \sin t \cos 2t + \cos^3 t \,.\, \sin 2t}{(\cos 2t)^{3/2}}$$

$$= \frac{-3 \cos^2 t \sin t \cos 2t + \cos^3 t \,.\, 2 \sin t \cos t}{(\cos 2t)^{3/2}}$$

$$= \frac{\sin t \cos^2 t \,(2 \cos^2 t - 3 \cos 2t)}{(\cos 2t)^{3/2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{\sin t \cos^2 t\,(2\cos^2 t - 3\cos 2t)}{(\cos 2t)^{3/2}} \cdot \frac{(\cos 2t)^{3/2}}{\sin^2 t \cos t\,(3\cos 2t + 2\sin^2 t)}$$

$$= \frac{\cos t\,[2\cos^2 t - 3(2\cos^2 t - 1)]}{\sin t\,[3(1 - 2\sin^2 t) + 2\sin^2 t]} = \frac{\cos t\,(3 - 4\cos^2 t)}{\sin t\,(3 - 4\sin^2 t)}$$

$$= \frac{-(4\cos^3 t - 3\cos t)}{3\sin t - 4\sin^3 t} = \frac{-\cos 3t}{\sin 3t} = -\cot 3t$$

Hence $\frac{dy}{dx} = -\cot 3t$.

**8.** $\boldsymbol{x = a\left(\cos t + \log \tan \frac{t}{2}\right), y = a \sin t.}$

**Sol.** $x = a\left[\cos t + \log\left(\tan \frac{t}{2}\right)\right]$

$$\Rightarrow \frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \frac{d}{dt}\left(\tan \frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{1}{\tan \frac{t}{2}} . \sec^2 \frac{t}{2} \cdot \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \cdot \frac{1}{\cos^2 \frac{t}{2}} \cdot \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin \frac{t}{2}\cos \frac{t}{2}}\right]$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = a\left[\frac{1}{\sin t} - \sin t\right]$$

$$= a\left(\frac{1 - \sin^2 t}{\sin t}\right) = \frac{a\cos^2 t}{\sin t}$$

$$y = a \sin t \Rightarrow \frac{dy}{dt} = a \cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{\left(\frac{a\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t.$$

**9.** $\boldsymbol{x = a \sec \theta, y = b \tan \theta.}$

**Sol.** $x = a \sec \theta$ and $y = b \tan \theta$

Differentiating both eqns. w.r.t. $\theta$, we have

$$\frac{dx}{d\theta} = a \sec \theta \tan \theta \quad \text{and} \quad \frac{dy}{d\theta} = b \sec^2 \theta$$

We know that $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{b\sec^2\theta}{a\sec\theta\tan\theta} = \dfrac{b\sec\theta}{a\tan\theta}$

$$= \frac{b\,.\,\frac{1}{\cos\theta}}{a\,.\,\frac{\sin\theta}{\cos\theta}} = \frac{b}{\cos\theta}\,.\,\frac{\cos\theta}{a\sin\theta} = \frac{b}{a\sin\theta} = \frac{b}{a}\ \text{cosec}\ \theta.$$

**10. $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$.**

**Sol.** We have $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

$\therefore \quad \dfrac{dx}{d\theta} = a(-\sin\theta + \theta\cos\theta + \sin\theta\,.\,1) = a\theta\cos\theta$

and $\quad \dfrac{dy}{d\theta} = a[\cos\theta - (\theta(-\sin\theta) + \cos\theta\,.\,1)]$

$= a\,[\cos\theta + \theta\sin\theta - \cos\theta] = a\theta\sin\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta.$$

**11. If $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$, show that $\dfrac{dy}{dx} = -\dfrac{y}{x}$.**

**Sol. Given:** $x = \sqrt{a^{\sin^{-1}t}} = (a^{\sin^{-1}t})^{1/2} = a^{1/2\sin^{-1}t}$ ...(*i*)

$\therefore \quad \dfrac{dx}{dt} = a^{1/2\sin^{-1}t}\log a\ \dfrac{d}{dt}\left(\dfrac{1}{2}\sin^{-1}t\right)$

$$\left[\because\ \frac{d}{dx}a^x = a^x\log a \text{ and } \frac{d}{dx}a^{f(x)} = a^{f(x)}\log a\,\frac{d}{dx}f(x)\right]$$

$\Rightarrow \quad \dfrac{dx}{dt} = a^{1/2\sin^{-1}t}\log a\,.\,\dfrac{1}{2}\,\dfrac{1}{\sqrt{1-t^2}}$ ...(*ii*)

Again **given:** $y = \sqrt{a^{\cos^{-1}t}} = (a^{\cos^{-1}t})^{1/2} = a^{1/2\cos^{-1}t}$ ...(*iii*)

$\therefore \quad \dfrac{dy}{dt} = a^{1/2\cos^{-1}t}\log a\ \dfrac{d}{dt}\left(\dfrac{1}{2}\cos^{-1}t\right)$

$= a^{1/2\cos^{-1}t}\log a\,.\,\dfrac{1}{2}\left(\dfrac{-1}{\sqrt{1-t^2}}\right)$ ...(*iv*)

We know that $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

Putting values from (*iv*) and (*ii*),

$$\frac{dy}{dx} = \frac{a^{1/2\cos^{-1}t}\log a\,\frac{1}{2}\left(\frac{-1}{\sqrt{1-t^2}}\right)}{a^{1/2\sin^{-1}t}\log a\,.\,\frac{1}{2}\,\frac{1}{\sqrt{1-t^2}}} = \frac{-a^{1/2\cos^{-1}t}}{a^{1/2\sin^{-1}t}} = -\frac{y}{x}$$

(By (*iii*) and (*i*))

# Exercise 5.7

**Find the second order derivatives of the functions given in Exercises 1 to 5.**

**1. $x^2 + 3x + 2$.**

**Sol.** Let $y = x^2 + 3x + 2$

$\therefore \quad \dfrac{dy}{dx} = 2x + 3.1 + 0 = 2x + 3$

Again differentiating w.r.t. $x$, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = 2(1) + 0 = 2.$

**2. $x^{20}$.**

**Sol.** Let $y = x^{20}$

$\therefore \quad \dfrac{dy}{dx} = 20x^{19}$

Again differentiating w.r.t. $x$, $\dfrac{d^2y}{dx^2} = 20.19x^{18} = 380x^{18}.$

**3. $x \cos x$.**

**Sol.** Let $y = x \cos x$

$\therefore \quad \dfrac{dy}{dx} = x\dfrac{d}{dx}\cos x + \cos x \dfrac{d}{dx}x$ [By Product Rule]

$= -x \sin x + \cos x$

Again differentiating w.r.t. $x$,

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}(x \sin x) + \frac{d}{dx}\cos x$$

$$= -\left[x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}(x)\right] - \sin x$$

$$= -(x \cos x + \sin x) - \sin x = -x \cos x - \sin x - \sin x$$

$$= -x \cos x - 2 \sin x = -(x \cos x + 2 \sin x).$$

**4. $\log x$.**

**Sol.** Let $y = \log x$ $\quad \therefore \quad \dfrac{dy}{dx} = \dfrac{1}{x}$

Again differentiating w.r.t. $x$, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{d}{dx}x^{-1}$

$= (-1)\, x^{-2} = \dfrac{-1}{x^2}.$

**5. $x^3 \log x$.**

**Sol.** Let $y = x^3 \log x$

$\therefore \quad \dfrac{dy}{dx} = x^3 \dfrac{d}{dx}\log x + \log x \dfrac{d}{dx}x^3$ [By Product Rule]

$= x^3 . \dfrac{1}{x} + (\log x)\, 3x^2$

$= x^2 + 3x^2 \log x$

Again differentiating w.r.t. $x$,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}x^2 + 3\frac{d}{dx}(x^2 \log x)$$

$$= 2x + 3\left[x^2 \frac{d}{dx}\log x + \log x \frac{d}{dx}x^2\right]$$

$$= 2x + 3\left(x^2 \cdot \frac{1}{x} + (\log x)\, 2x\right)$$

$$= 2x + 3(x + 2x \log x)$$

$$= 2x + 3x + 6x \log x = 5x + 6x \log x$$

$$= x(5 + 6 \log x).$$

**Find the second order derivatives of the functions given in exercises 6 to 10.**

**6. $e^x$ sin $5x$.**

**Sol.** Let $y = e^x \sin 5x$

$$\therefore \quad \frac{dy}{dx} = e^x \frac{d}{dx}\sin 5x + \sin 5x \frac{d}{dx}e^x \quad \text{[By Product Rule]}$$

$$= e^x \cos 5x \frac{d}{dx}5x + \sin 5x \cdot e^x = e^x \cos 5x \cdot 5 + e^x \sin 5x$$

or $$\frac{dy}{dx} = e^x (5 \cos 5x + \sin 5x)$$

Again applying Product Rule of derivatives

$$\frac{d^2y}{dx^2} = e^x \frac{d}{dx}(5 \cos 5x + \sin 5x) + (5 \cos 5x + \sin 5x)\frac{d}{dx}e^x$$

$$= e^x (5(-\sin 5x) \cdot 5 + (\cos 5x) \cdot 5) + (5 \cos 5x + \sin 5x)\, e^x$$

$$= e^x (-25 \sin 5x + 5 \cos 5x + 5 \cos 5x + \sin 5x)$$

$$= e^x (10 \cos 5x - 24 \sin 5x)$$

$$= 2e^x (5 \cos 5x - 12 \sin 5x).$$

**7. $e^{6x}$ cos $3x$.**

**Sol.** Let $y = e^{6x} \cos 3x$

$$\therefore \quad \frac{dy}{dx} = e^{6x} \frac{d}{dx}\cos 3x + \cos 3x \frac{d}{dx}e^{6x}$$

$$= e^{6x}(-\sin 3x)\frac{d}{dx}(3x) + \cos 3x \cdot e^{6x}\frac{d}{dx}6x$$

$$= -e^{6x} \sin 3x \cdot 3 + \cos 3x \cdot e^{6x} \cdot 6$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{6x}(-3 \sin 3x + 6 \cos 3x)$$

Again applying Product Rule of derivatives,

$$\frac{d^2y}{dx^2} = e^{6x}\frac{d}{dx}(-3 \sin 3x + 6 \cos 3x)$$

$$+ (-3 \sin 3x + 6 \cos 3x)\frac{d}{dx}e^{6x}$$

$= e^{6x}\,[-3\,.\cos 3x\,.\,3 - 6\sin 3x\,.\,3]$
$\qquad\qquad\qquad\qquad + (-3\sin 3x + 6\cos 3x)\,e^{6x}\,.\,6$
$= e^{6x}\,(-9\cos 3x - 18\sin 3x - 18\sin 3x + 36\cos 3x)$
$= e^{6x}\,(27\cos 3x - 36\sin 3x)$
$= 9e^{6x}\,(3\cos 3x - 4\sin 3x).$

**8. $\tan^{-1} x$.**

**Sol.** Let $\quad y = \tan^{-1} x$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+x^2}$$

Again differentiating w.r.t. $x$,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{(1+x^2)\frac{d}{dx}(1) - 1\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)0-(2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}.$$

**9. $\log(\log x)$.**

**Sol.** Let $\quad y = \log(\log x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\log x}\,\frac{d}{dx}\log x \qquad \left[\because \frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right]$$

$$= \frac{1}{\log x}\,\frac{1}{x} = \frac{1}{x\log x}$$

Again differentiating w.r.t. $x$,

$$\frac{d^2y}{dx^2} = \frac{(x\log x)\frac{d}{dx}(1) - 1\frac{d}{dx}(x\log x)}{(x\log x)^2}$$

$$= \frac{(x\log x)\,0 - \left[x\frac{d}{dx}\log x + \log x\frac{d}{dx}(x)\right]}{(x\log x)^2}$$

$$= -\frac{\left[x\,.\,\frac{1}{x} + \log x\,.\,1\right]}{(x\log x)^2} = -\frac{(1+\log x)}{(x\log x)^2}.$$

**10. $\sin(\log x)$.**

**Sol.** Let $\quad y = \sin(\log x)$

$$\therefore \quad \frac{dy}{dx} = \cos(\log x)\,\frac{d}{dx}(\log x) = \cos(\log x)\,.\,\frac{1}{x}$$

$$= \frac{\cos(\log x)}{x}$$

Again differentiating w.r.t. $x$,

$$\frac{d^2y}{dx^2} = \frac{x\frac{d}{dx}\cos(\log x) - \cos(\log x)\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x[-\sin(\log x)]\frac{d}{dx}\log x - \cos(\log x)}{x^2}$$

$$= \frac{-x\sin(\log x)\,.\,\frac{1}{x} - \cos(\log x)}{x^2} \quad = \frac{-[\sin(\log x) + \cos(\log x)]}{x^2}.$$

**11. If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$.**

**Sol. Given:** $y = 5\cos x - 3\sin x$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = -5\sin x - 3\cos x$

Again differentiating w.r.t. $x$, $\frac{d^2y}{dx^2} = -5\cos x + 3\sin x$

$= -(5\cos x - 3\sin x) \;\; - y$ (By (*i*))

or $\quad \frac{d^2y}{dx^2} = -y \quad \therefore \quad \frac{d^2y}{dx^2} + y = 0.$

**12. If $y = \cos^{-1} x$. Find $\frac{d^2y}{dx^2}$ in terms of $y$ alone.**

**Sol. Given:** $y = \cos^{-1} x \quad \Rightarrow \quad x = \cos y$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} = \frac{-1}{\sqrt{1-\cos^2 y}}$ (By (*i*))

$= \frac{-1}{\sqrt{\sin^2 y}} = \frac{-1}{\sin y} = -\operatorname{cosec} y$

or $\quad \frac{dy}{dx} = -\operatorname{cosec} y$ ...(*ii*)

Again differentiating both sides w.r.t. $x$,

$\frac{d^2y}{dx^2} = -\frac{d}{dx}(\operatorname{cosec} y) = -\left[-\operatorname{cosec} y \cot y \frac{dy}{dx}\right]$

$= \operatorname{cosec} y \cot y\,(-\operatorname{cosec} y)$ (By (*ii*))

$= -\operatorname{cosec}^2 y \cot y.$

**13. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$.**

**Sol. Given:** $y = 3\cos(\log x) + 4\sin(\log x)$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = (y_1) = -3\sin(\log x)\frac{d}{dx}\log x + 4\cos(\log x)\frac{d}{dx}\log x$

or $\quad y_1 = -3\sin(\log x)\,.\,\frac{1}{x} + 4\cos(\log x)\,.\,\frac{1}{x}$

Multiplying both sides by L.C.M. $= x$,

$xy_1 = -3\sin(\log x) + 4\cos(\log x)$

Again differentiating both sides w.r.t. $x$,

$$\frac{d}{dx}(xy_1) = -3\cos(\log x)\frac{d}{dx}\log x - 4\sin(\log x)\frac{d}{dx}\log x$$

$$\Rightarrow \quad x\frac{d}{dx}y_1 + y_1\frac{d}{dx}x = -3\cos(\log x)\cdot\frac{1}{x} - 4\sin(\log x)\cdot\frac{1}{x}$$

(By Product Rule)

$$\Rightarrow \quad xy_2 + y_1 = -\frac{[3\cos(\log x) + 4\sin(\log x)]}{x}$$

Cross-multiplying

$$x(xy_2 + y_1) = -[3\cos(\log x) + 4\sin(\log x)]$$

$$\Rightarrow \quad x^2y_2 + xy_1 = -y \qquad \text{(By } (i))$$

$$\Rightarrow \quad x^2y_2 + xy_1 + y = 0.$$

**14. If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$.**

**Sol. Given:** $y = Ae^{mx} + Be^{nx}$ ...($i$)

$$\therefore \quad \frac{dy}{dx} = Ae^{mx}\frac{d}{dx}(mx) + Be^{nx}\frac{d}{dx}(nx)\left[\because \frac{d}{dx}e^{f(x)} = e^{f(x)}\frac{d}{dx}f(x)\right]$$

or $\frac{dy}{dx} = Am\,e^{mx} + Bn\,e^{nx}$ ...($ii$)

$$\therefore \quad \frac{d^2y}{dx^2} = Am.e^{mx}.m + Bne^{nx}.n$$

$= Am^2\,e^{mx} + Bn^2\,e^{nx}$ ...($iii$)

Putting values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from ($i$), ($ii$) and ($iii$) in

L.H.S. $= \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$

$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)(Am\,e^{mx} + Bn\,e^{nx}) + mn(Ae^{mx} + Be^{nx})$

$= Am^2e^{mx} + Bn^2e^{nx} - Am^2\,e^{mx} - Bmn\,e^{nx} - Amn\,e^{mx} - Bn^2\,e^{nx} + Amn\,e^{mx} + Bmn\,e^{nx} = 0 =$ R.H.S.

**15. If $y = 500\,e^{7x} + 600\,e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$.**

**Sol. Given:** $y = 500\,e^{7x} + 600\,e^{-7x}$ ...($i$)

$$\therefore \quad \frac{dy}{dx} = 500\,e^{7x}(7) + 600\,e^{-7x}(-7) = 500(7)\,e^{7x} - 600(7)\,e^{-7x}$$

$$\therefore \quad \frac{d^2y}{dx^2} = 500(7)\,e^{7x}(7) - 600(7)e^{-7x}(-7)$$

$$= 500(49)\,e^{7x} + 600(49)\,e^{-7x}$$

or $\frac{d^2y}{dx^2} = 49[500\,e^{7x} + 600\,e^{-7x}] = 49y$ (By ($i$))

or $\frac{d^2y}{dx^2} = 49y.$

**16. If $e^y (x + 1) = 1$, show that $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$.**

**Sol. Given:** $e^y (x + 1) = 1$

$$\Rightarrow \qquad e^y = \frac{1}{x+1}$$

Taking logs of both sides, $\log e^y = \log \dfrac{1}{x+1}$

or $y \log e = \log 1 - \log (x + 1)$

or $y = - \log (x + 1)$ $[\because \log e = 1 \text{ and } \log 1 = 0]$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{x+1}\frac{d}{dx}(x+1) = \frac{-1}{x+1} = -(x+1)^{-1}$$

$$\therefore \quad \frac{d^2y}{dx^2} = -(-1)(x+1)^{-2}\frac{d}{dx}(x+1)$$

$$\left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$$

$$= \frac{1}{(x+1)^2} \qquad \left[\because \frac{d}{dx}(x+1) = 1+0 = 1\right]$$

$$\text{L.H.S.} = \frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$$

$$\text{R.H.S.} = \left(\frac{dy}{dx}\right)^2 = \left(\frac{-1}{x+1}\right)^2 = \frac{1}{(x+1)^2}$$

$\therefore$ L.H.S. = R.H.S. *i.e.,* $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$.

**17. If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)^2 y_2 + 2x(x^2 + 1)y_1 = 2$.**

**Sol. Given:** $y = (\tan^{-1} x)^2$ ...(*i*)

$$\therefore \quad y_1 = 2(\tan^{-1} x)\frac{d}{dx}\tan^{-1} x \left[\because \frac{d}{dx}(f(x))^n = n\,(f(x))^{n-1}\frac{d}{dx}f(x)\right]$$

$$\Rightarrow \quad y_1 = 2(\tan^{-1} x)\frac{1}{1+x^2} \qquad \Rightarrow \quad y_1 = \frac{2\tan^{-1} x}{1+x^2}$$

Cross-multiplying, $(1 + x^2)\, y_1 = 2 \tan^{-1} x$

Again differentiating both sides w.r.t. $x$,

$$(1+x^2)\frac{d}{dx}y_1 + y_1\frac{d}{dx}(1+x^2) = 2\,.\,\frac{1}{1+x^2}$$

$$\Rightarrow \qquad (1+x^2)\,y_2 + y_1\,.\,2x = \frac{2}{1+x^2}$$

Multiplying both sides by $(1 + x^2)$,

$$(x^2+1)^2\,y_2 + 2xy_1(1+x^2) = 2.$$

# Exercise 5.8

$\dfrac{f(x)}{g(x)}$ $(g(x) \neq 0)$, $\sin x$, $\cos x$, $e^x$, $e^{-x}$, $\log x$ $(x > 0)$ are continuous and derivable for all real $x$.

**Note 2:** Sum, difference, product of two continuous (derivable) functions is continuous (derivable).

**1. Verify Rolle's theorem for $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$.**

**Sol.** **Given:** $f(x) = x^2 + 2x - 8$; $x \in [-4, 2]$ ...(*i*)

Here $f(x)$ is a polynomial function of $x$ (of degree 2).

$\therefore$ $f(x)$ is continuous and derivable everywhere *i.e.,* on $(-\infty, \infty)$.

Hence $f(x)$ is continuous in the closed interval $[-4, 2]$ and derivable in open interval $(-4, 2)$.

Putting $x = -4$ in (*i*), $f(-4) = 16 - 8 - 8 = 0$

Putting $x = 2$ in (*i*), $f(2) = 4 + 4 - 8 = 0$

$\therefore$ $f(-4) = f(2)$ $(= 0)$

$\therefore$ All three conditions of Rolle's Theorem are satisfied.

From (*i*), $f'(x) = 2x + 2$.

Putting $x = c$, $f'(c) = 2c + 2 = 0 \Rightarrow 2c = -2$

$$\Rightarrow \qquad c = -\frac{2}{2} = -1 \in \text{ open interval } (-4, 2).$$

$\therefore$ Conclusion of Rolle's theorem is true.

$\therefore$ Rolle's theorem is verified.

**2. Examine if Rolle's theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's theorem from these examples?**

**(*i*) $f(x) = [x]$ for $x \in [5, 9]$ (*ii*) $f(x) = [x]$ for $x \in [-2, 2]$**

**(*iii*) $f(x) = x^2 - 1$ for $x \in [1, 2]$.**

**Sol.** (*i*) **Given:** $f(x) = [x]$ for $x \in [5, 9]$ ...(*i*)

(of course $[x]$ denotes the greatest integer $\leq x$)

We know that bracket function $[x]$ is discontinuous at all the integers (See Ex. 15, page 155, NCERT, Part I). Hence $f(x) = [x]$ is discontinuous at all integers between 5 and 9 *i.e.,* discontinuous at $x = 6$, $x = 7$ and $x = 8$ and hence discontinuous in the closed interval $[5, 9]$ and hence not derivable in the open interval $(5, 9)$. ...(*ii*) ($\because$ discontinuity $\Rightarrow$ Non-derivability)

Again from (*i*), $f(5) = [5] = 5$ and $f(9) = [9] = 9$

$\therefore$ $f(5) \neq f(9)$

$\therefore$ Conditions of Rolle's Theorem are not satisfied.

$\therefore$ Rolle's Theorem is not applicable to $f(x) = [x]$ in the closed interval $[5, 9]$.

But converse (conclusion) of Rolle's theorem is true for this function $f(x) = [x]$.

*i.e.,* $f'(c) = 0$ for every real $c$ belonging to open interval

(5, 9) other than integers. (*i.e.,* for every real $c \neq 6, 7, 8$) (even though conditions are not satisfied).

Let us prove it.

$$\text{Left Hand derivative} = Lf'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{x \to c^-} \frac{[x] - [c]}{x - c} \qquad \text{(By } (i)\text{)}$$

$$\text{Put } x = c - h,\ h \to 0^+,\ = \lim_{h \to 0^+} \frac{[c - h] - [c]}{c - h - c}$$

$$= \lim_{h \to 0^+} \frac{[c] - [c]}{-h}$$

[$\because$ We know that for $c \in R - Z$, as $h \to 0^+$, $[c - h] = [c]$]

$$= \lim_{h \to 0^+} \frac{0}{-h} = \lim_{h \to 0^+} 0$$

($\because$ $h \to 0^+ \Rightarrow h > 0$ and hence $h \neq 0$)

$$= 0 \qquad ...(iii)$$

$$\text{Right Hand derivative} = Rf'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{x \to c^+} \frac{[x] - [c]}{x - c} \qquad \text{(By } (i)\text{)}$$

$$\text{Put } x = c + h,\ h \to 0^+,\ = \lim_{h \to 0^+} \frac{[c + h] - [c]}{c + h - c} = \lim_{h \to 0^+} \frac{[c] - [c]}{h}$$

[$\because$ We know that for $c \in R - Z$, as $h \to 0^+$, $[c + h] = [c]$]

$$= \lim_{h \to 0^+} \frac{0}{h} = \lim_{h \to 0^+} 0$$

($\because$ $h \to 0^+ \Rightarrow h > 0$ and hence $h \neq 0$)

$$= 0 \qquad ...(iv)$$

From (*iii*) and (*iv*) $Lf'(c) = Rf'(c) = 0$

$\therefore$ $f'(c) = 0$ $\forall$ real $c \in$ open interval (5, 9) other than integers $c = 6, 7, 8$.

(*ii*) **Given:** $f(x) = [x]$ for $x \in [-2, 2]$.

*Reproduce the solution of (i) part* replacing closed interval [5, 9] by [– 2, 2] and integers 6, 7, 8 by – 1, 0 and 1 lying between – 2 and 2.

(*iii*) **Given:** $f(x) = x^2 - 1$ for $x \in [1, 2]$ ...(*i*)

Here $f(x)$ is a polynomial function of $x$ (of degree 2).

$\therefore$ $f(x)$ is continuous and derivable everywhere *i.e.,* on $(-\infty, \infty)$.

Hence $f(x)$ is continuous in the closed interval [1, 2] and derivable in the open interval (1, 2).

Again from (*i*), $f(1) = 1 - 1 = 0$

and $f(2) = 2^2 - 1 = 4 - 1 = 3$

$\therefore$ $f(1) \neq f(2)$.

$\therefore$ Conditions of Rolle's Theorem are not satisfied.

$\therefore$ Rolle's theorem is not applicable to $f(x) = x^2 - 1$ in [1, 2].

Let us examine if converse (*i.e.,* conclusion) is true for this function given by (*i*).

From (*i*), $f'(x) = 2x$

Put $x = c$, $f'(c) = 2c = 0 \Rightarrow c = 0$ does not belong to open interval (1, 2).

$\therefore$ Converse (conclusion) of Rolle's Theorem is also not true for this function.

**3. If $f : [-5, 5] \to \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.**

**Sol. Given:** $f : [-5, 5] \to \mathbf{R}$ is a differentiable function *i.e.,* $f$ is differentiable on its domain closed interval $[-5, 5]$ (and in particular in open interval $(-5, 5)$ also) and hence is continuous also on closed interval $[-5, 5]$ ...(*i*)

**To prove:** $f(-5) \neq f(5)$.

If possible, let $f(-5) = f(5)$ ...(*ii*)

From (*i*) and (*ii*) all the three conditions of Rolle's Theorem are satisfied.

$\therefore$ There exists at least one point $c$ in the open interval $(-5, 5)$ such that $f'(c) = 0$.

*i.e.,* $f'(x) = 0$ *i.e.,* $f'(x)$ vanishes (vanishes $\Rightarrow$ zero) for at least one value of $x$ in the open interval $(-5, 5)$. But this is contrary to given that $f'(x)$ does not vanish anywhere.

$\therefore$ Our supposition in (*ii*) *i.e.,* $f(-5) = f(5)$ is wrong.

$\therefore$ $f(-5) \neq f(5)$.

**4. Verify Mean Value Theorem if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$ where $a = 1$ and $b = 4$.**

**Sol. Given:** $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$ where $a = 1$ and $b = 4$ *i.e.,* in the interval [1, 4] ...(*i*)

Here $f(x)$ is a polynomial function of $x$ and hence is continuous and derivable everywhere.

$\therefore$ $f(x)$ is continuous in the closed interval [1, 4] and derivable in the open interval (1, 4) also.

$\therefore$ Both conditions of L.M.V.T. are satisfied.

From (*i*), $f'(x) = 2x - 4$

Put $x = c$, $f'(c) = 2c - 4$

from (*i*) $f(a) = f(1) = 1 - 4 - 3 = -6$

and $\quad f(b) = f(4) = 16 - 16 - 3 = -3$

Putting these values in $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, we have

$$2c - 4 = \frac{-3 - (-6)}{4 - 1} \quad \Rightarrow \quad 2c - 4 = \frac{-3 + 6}{3}$$

$$\Rightarrow \quad 2c - 4 = \frac{3}{3} = 1 \qquad \Rightarrow \quad 2c = 5$$

$$\Rightarrow \quad c = \frac{5}{2} \in \text{ open interval } (1, 4).$$

$\therefore$ L.M.V.T. is verified.

**5. Verify Mean Value Theorem if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$ where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.**

**Sol. Given:** $\quad f(x) = x^3 - 5x^2 - 3x \qquad ...(i)$

In the interval $[a, b]$ where $a = 1$ and $b = 3$ *i.e.,* in the interval $[1, 3]$.

Here $f(x)$ is a polynomial function of $x$ (of degree 3). Therefore, $f(x)$ is continuous and derivable everywhere *i.e.,* on the real line $(-\infty, \infty)$.

Hence $f(x)$ is continuous in the closed interval $[1, 3]$ and derivable in open interval $(1, 3)$.

$\therefore$ Both conditions of Mean Value Theorem are satisfied.

From (*i*), $\quad f'(x) = 3x^2 - 10x - 3$

Put $x = c$, $\quad f'(c) = 3c^2 - 10c - 3 \qquad ...(ii)$

From (*i*), $\quad f(a) = f(1) = 1 - 5 - 3 = 1 - 8 = -7$

and $f(b) = f(3) = 3^3 - 5 \,.\, 3^2 - 3.3 = 27 - 45 - 9 = 27 - 54 = -27$

Putting these values in the conclusion of Mean Value Theorem *i.e.,*

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \text{ we have}$$

$$3c^2 - 10c - 3 = \frac{-27 - (-7)}{3 - 1} = \frac{-27 + 7}{2} = -\frac{20}{2} = -10$$

$\Rightarrow \quad 3c^2 - 10c - 3 + 10 = 0 \quad \Rightarrow \quad 3c^2 - 10c + 7 = 0$

$\Rightarrow \quad 3c^2 - 3c - 7c + 7 = 0 \quad \Rightarrow \quad 3c(c - 1) - 7(c - 1) = 0$

$\Rightarrow \quad (c - 1)(3c - 7) = 0$

$\therefore$ Either $c - 1 = 0$ or $3c - 7 = 0$

*i.e.,* $c = 1 \notin$ open interval $(1, 3)$ or $3c = 7$ *i.e.,* $c = \dfrac{7}{3}$

which belongs to open interval $(1, 3)$.

Hence mean value theorem is verified.

**Now we are to find all $c \in (1, 3)$ for which $f'(c) = 0$.**

$\therefore$ From (*ii*), $\quad 3c^2 - 10c - 3 = 0$

Solving for $c$, $c = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{10 \pm \sqrt{100 + 36}}{6}$

$= \dfrac{10 \pm \sqrt{136}}{6} = \dfrac{10 \pm \sqrt{4 \times 34}}{6} = \dfrac{10 \pm 2\sqrt{34}}{6} = 2\left(\dfrac{5 \pm \sqrt{34}}{6}\right) = \dfrac{5 \pm \sqrt{34}}{3}$

Taking positive sign, $c = \dfrac{5 + \sqrt{34}}{3} > 3$ and hence $\notin (1, 3)$

Taking negative sign, $c = \dfrac{5 - \sqrt{34}}{3}$ is negative and hence $\notin (1, 3)$.

**6. Examine the applicability of Mean Value Theorem for all the three functions being given below:**

**(*i*) $f(x) = [x]$ for $x \in [5, 9]$ (*ii*) $f(x) = [x]$ for $x \in [-2, 2]$**

**(*iii*) $f(x) = x^2 - 1$ for $x \in [1, 2]$.**

**Sol.** (*i*) *Reproduce solution of Q. No. 2(i) upto eqn. (ii)*

$\therefore$ Both conditions of L.M.V.T. are not satisfied.

$\therefore$ L.M.V.T. is not applicable to $f(x) = [x]$ for $x \in [5, 9]$.

(*ii*) *Reproduce solution of Q. No. 2(i) upto eqn. (ii)* replacing [5, 9] by [– 2, 2] and integers 6, 7, 8 by – 1, 0 and 1 lying between – 2 and 2.

$\therefore$ Both conditions of L.M.V.T. are not satisfied.

$\therefore$ L.M.V.T. is not applicable to $f(x) = [x]$ for $x \in [-2, 2]$.

(*iii*) **Given:** $f(x) = x^2 - 1$ for $x \in [1, 2]$ ...(*i*)

Here $f(x)$ is a polynomial function (of degree 2).

Therefore $f(x)$ is continuous and derivable everywhere *i.e.,* on the real line $(-\infty, \infty)$.

Hence $f(x)$ is continuous in the closed interval [1, 2] and derivable in open interval (1, 2).

$\therefore$ Both conditions of Mean Value Theorem are satisfied.

From (*i*), $f'(x) = 2x$

Put $x = c$, $f'(c) = 2c$

From (*i*), $f(a) = f(1) = 1^2 - 1 = 1 - 1 = 0$

$f(b) = f(2) = 2^2 - 1 = 4 - 1 = 3$

Putting these values in the conclusion of Mean Value Theorem *i.e.,* in $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, we have

$$2c = \frac{3 - 0}{2 - 1} \Rightarrow 2c = 3$$

$\Rightarrow \quad c = \dfrac{3}{2} \in (1, 2)$

$\therefore$ Mean Value Theorem is verified.

# MISCELLANEOUS EXERCISE

**1. $(3x^2 - 9x + 5)^9$.**

**Sol.** Let $y = (3x^2 - 9x + 5)^9$

$$\therefore \quad \frac{dy}{dx} = 9(3x^2 - 9x + 5)^8 \frac{d}{dx}(3x^2 - 9x + 5)$$

$$\left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$$

$$= 9(3x^2 - 9x + 5)^8 [3(2x) - 9.1 + 0]$$

$$= 9(3x^2 - 9x + 5)^8 (6x - 9) = 27(3x^2 - 9x + 5)^8 (2x - 3).$$

**2. $\sin^3 x + \cos^6 x$.**

**Sol.** Let $y = \sin^3 x + \cos^6 x = (\sin x)^3 + (\cos x)^6$

$$\therefore \quad \frac{dy}{dx} = 3(\sin x)^2 \frac{d}{dx}\sin x + 6(\cos x)^5 \frac{d}{dx}\cos x$$

$$\left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$$

$$= 3\sin^2 x \cos x - 6\cos^5 x \sin x$$

$$= 3\sin x \cos x(\sin x - 2\cos^4 x).$$

**3. $(5x)^{3\cos 2x}$.**

**Sol.** Let $y = (5x)^{3\cos 2x}$ ...(*i*) [Form $(f(x))^{g(x)}$]

Taking logs of both sides of (*i*) we have

$$\log y = \log (5x)^{3\cos 2x} = 3\cos 2x \log(5x)$$

Differentiating both sides w.r.t. $x$, we have

$$\frac{d}{dx}(\log y) = 3\frac{d}{dx}(\cos 2x \log(5x))$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = 3\left[\cos 2x \frac{d}{dx}\log(5x) + \log(5x)\frac{d}{dx}\cos 2x\right]$$

$$= 3\left[\cos 2x \cdot \frac{1}{5x}\frac{d}{dx}5x + \log(5x)(-\sin 2x)\frac{d}{dx}2x\right]$$

or $$\frac{1}{y}\frac{dy}{dx} = 3\left[\cos 2x \cdot \frac{1}{5x} \cdot 5 - 2\sin 2x \log 5x\right]$$

Cross-multiplying, $$\frac{dy}{dx} = 3y\left(\frac{\cos 2x}{x} - 2\sin 2x \log 5x\right)$$

Putting the value of $y$ from (*i*),

$$\frac{dy}{dx} = 3(5x)^{3\cos 2x}\left(\frac{\cos 2x}{x} - 2\sin 2x \log 5x\right)$$

**4. $\sin^{-1}(x\sqrt{x})$, $0 \le x \le 1$.**

**Sol.** Let $y = \sin^{-1}(x\sqrt{x}) = \sin^{-1}(x^{3/2})$

$$\left[\because x\sqrt{x} = x^1 \cdot x^{1/2} = x^{1+1/2} = x^{3/2}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-(x^{3/2})^2}} \frac{d}{dx} x^{3/2} \left[\because \frac{d}{dx} \sin^{-1} f(x) = \frac{1}{\sqrt{1-(f(x))^2}} \frac{d}{dx} f(x)\right]$$

$$= \frac{1}{\sqrt{1-x^3}} \frac{3}{2} x^{1/2} = \frac{3\sqrt{x}}{2\sqrt{1-x^3}} = \frac{3}{2} \sqrt{\frac{x}{1-x^3}}.$$

**5.** $\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$**, $-2 < x < 2$.**

**Sol.** Let $y = \dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

Applying Quotient Rule,

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\,\frac{d}{dx}\cos^{-1}\frac{x}{2} - \cos^{-1}\frac{x}{2}\,\frac{d}{dx}\sqrt{2x+7}}{(\sqrt{2x+7})^2}$$

$$= \frac{\sqrt{2x+7}\left(\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\right)\frac{d}{dx}\frac{x}{2} - \left(\cos^{-1}\frac{x}{2}\right)\frac{1}{2}\,(2x+7)^{-1/2}\,\frac{d}{dx}(2x+7)}{2x+7}$$

$$\left[\because \frac{d}{dx}\cos^{-1} f(x) = \frac{-1}{\sqrt{1-(f(x))^2}} \frac{d}{dx} f(x) \text{ and } \frac{d}{dx}(f(x))^n = n(f(x))^{n-1} \frac{d}{dx} f(x)\right]$$

or $$\frac{dy}{dx} = \frac{-\sqrt{2x+7}\,.\,\frac{2}{\sqrt{4-x^2}}\,\frac{1}{2} - \frac{1}{2}\,\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}\,2}{2x+7}$$

$$\left[\because \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} = \frac{1}{\sqrt{1-\frac{x^2}{4}}} = \frac{1}{\sqrt{\frac{4-x^2}{4}}} = \frac{2}{\sqrt{4-x^2}}\right]$$

$$= \frac{-\left[\frac{\sqrt{2x+7}}{\sqrt{4-x^2}} + \frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}\right]}{2x+7} = -\left[\frac{2x+7+\sqrt{4-x^2}\,\cos^{-1}\frac{x}{2}}{\sqrt{4-x^2}\,\sqrt{2x+7}\,(2x+7)}\right]$$

$$= -\left[\frac{2x+7+\sqrt{4-x^2}\,\cos^{-1}\frac{x}{2}}{\sqrt{4-x^2}\,(2x+7)^{3/2}}\right].$$

**Differentiate w.r.t. $x$, the following functions in Exercises 6 to 11.**

**6.** $\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$, $0 < x < \dfrac{\pi}{2}$.

**Sol.** Let $y = \cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$ ...($i$), $0 < x < \dfrac{\pi}{2}$

Let us simplify the given inverse T-function

Now $\sqrt{1+\sin x} = \sqrt{\cos^2\dfrac{x}{2}+\sin^2\dfrac{x}{2}+2\sin\dfrac{x}{2}\cos\dfrac{x}{2}}$

$$= \sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} = \cos\frac{x}{2}+\sin\frac{x}{2} \quad ...(ii)$$

Again $\sqrt{1-\sin x} = \sqrt{\cos^2\dfrac{x}{2}+\sin^2\dfrac{x}{2}-2\sin\dfrac{x}{2}\cos\dfrac{x}{2}}$

$$= \sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2} = \cos\frac{x}{2}-\sin\frac{x}{2}$$

...($iii$)

(**Given:** $0 < x < \dfrac{\pi}{2}$. Dividing by 2, $0 < \dfrac{x}{2} < \dfrac{\pi}{4}$ and therefore

$\cos\dfrac{x}{2} > \sin\dfrac{x}{2} \Rightarrow \cos\dfrac{x}{2} - \sin\dfrac{x}{2} > 0$)

Putting values from ($ii$) and ($iii$) in ($i$), we have

$$y = \cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right) = \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)$$

$$= \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} \qquad \therefore \quad \frac{dy}{dx} = \frac{1}{2}(1) = \frac{1}{2}.$$

**7.** $(\log x)^{\log x}$, $x > 1$.

**Sol.** Let $y = (\log x)^{\log x}$, $x > 1$ ...($i$) [Form $(f(x))^{g(x)}$]

Taking logs of both sides of ($i$), we have

$\log y = \log (\log x)^{\log x} = \log x \log (\log x)$ [$\because$ $\log m^n = n \log m$]

Differentiating both sides w.r.t. $x$, we have

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(\log x \log (\log x))$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \log x \frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}\log x$$

(By Product Rule)

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \log x \,.\, \frac{1}{\log x}\frac{d}{dx}\log x + \log(\log x)\,.\,\frac{1}{x}$$

$$\left[\because \frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right]$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{\log(\log x)}{x} = \frac{1+\log(\log x)}{x}$$

$$\therefore \quad \frac{dy}{dx} = y\left(\frac{1+\log(\log x)}{x}\right)$$

Putting the value of $y$ from $(i)$, $\dfrac{dy}{dx} = (\log x)^{\log x}\left(\dfrac{1+\log(\log x)}{x}\right)$.

**8. cos ($a$ cos $x$ + $b$ sin $x$) for some constants $a$ and $b$.**

**Sol.** Let $y = \cos(a\cos x + b\sin x)$ for some constants $a$ and $b$.

$$\therefore \quad \frac{dy}{dx} = -\sin(a\cos x + b\sin x)\frac{d}{dx}(a\cos x + b\sin x)$$

$$\left[\because \frac{d}{dx}\cos f(x) = -\sin f(x)\frac{d}{dx}f(x)\right]$$

$$= -\sin(a\cos x + b\sin x)[-a\sin x + b\cos x]$$

$$= -(-a\sin x + b\cos x)\sin(a\cos x + b\sin x)$$

$$= (a\sin x - b\cos x)\sin(a\cos x + b\sin x).$$

**9. $(\sin x - \cos x)^{\sin x - \cos x}$, $\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$.**

**Sol.** Let $y = (\sin x - \cos x)^{\sin x - \cos x}$ ...$(i)$ [Form $(f(x))^{g(x)}$]

Taking logs of both sides of $(i)$, we have

$$\log y = \log(\sin x - \cos x)^{(\sin x - \cos x)}$$

$$= (\sin x - \cos x)\log(\sin x - \cos x) \qquad [\therefore \ \log m^n = n\log m]$$

Differentiating both sides w.r.t. $x$, we have

$$\frac{d}{dx}\log y = (\sin x - \cos x)\frac{d}{dx}\log(\sin x - \cos x)$$

$$+ \log(\sin x - \cos x)\,.\,\frac{d}{dx}(\sin x - \cos x)$$

(By Applying Product Rule on R.H. Side)

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x)\frac{1}{(\sin x - \cos x)}\frac{d}{dx}(\sin x - \cos x)$$

$$+ \log(\sin x - \cos x)\,.\,(\cos x + \sin x)$$

$$\left[\because \frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right]$$

$$= (\cos x + \sin x) + (\cos x + \sin x)\log(\sin x - \cos x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (\cos x + \sin x)[1 + \log(\sin x - \cos x)]$$

$\Rightarrow \quad \dfrac{dy}{dx} = y\ (\cos x + \sin x)\ [1 + \log (\sin x - \cos x)]$

Putting the value of $y$ from $(i)$,

$\dfrac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}\ (\cos x + \sin x)\ [1 + \log (\sin x - \cos x)]$

**10. $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$.**

**Sol.** Let $\quad y = x^x + x^a + a^x + a^a$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{d}{dx} x^x + \dfrac{d}{dx} x^a + \dfrac{d}{dx} a^x + \dfrac{d}{dx} a^a$

$= \dfrac{d}{dx} x^x + a\ x^{a-1} + a^x \log a + 0 \quad ...(i)$

[$\because \quad a^a$ is constant as $3^3 = 27$ is constant]

To find $\dfrac{d}{dx}(x^x)$: Let $u = x^x \quad ...(ii)\ (f(x))^{g(x)}]$

$\therefore$ Taking logs on both sides of eqn. $(ii)$, we have

$\log u = \log x^x = x \log x$

$\therefore \quad \dfrac{d}{dx} \log u = \dfrac{d}{dx}(x \log x)$

$\Rightarrow \quad \dfrac{1}{u}\dfrac{du}{dx} = x \dfrac{d}{dx}(\log x) + \log x \dfrac{d}{dx} x \quad$ (Product Rule)

$= x \cdot \dfrac{1}{x} + \log x \cdot 1 = 1 + \log x$

$\Rightarrow \quad \dfrac{du}{dx} = u\ (1 + \log x)$

Putting the value of $u$ from $(ii)$, $\dfrac{d}{dx} x^x = x^x (1 + \log x)$

Putting this value in eqn. $(i)$,

$$\dfrac{dy}{dx} = x^x (1 + \log x) + a\ x^{a-1} + a^x \log a.$$

**11. $x^{x^2-3} + (x-3)^{x^2}$ for $x > 3$.**

**Sol.** Let $y = x^{x^2-3} + (x-3)^{x^2}$ for $x > 3$

(**Caution.** For types $(f(x))^{g(x)} \pm (l(x))^{m(x)}$ or $(f(x))^{g(x)} \pm l(x)$
or $(f(x))^{g(x)} \pm k$ where $k$ is a constant,
Never begin by taking logs of both sides as
$\log (m \pm n) \neq \log m \pm \log n$)

Put $\quad u = x^{x^2-3}$ and $v = (x-3)^{x^2} \quad \therefore \quad y = u + v$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx} \quad ...(i)$

Now $\quad u = x^{(x^2-3)} \quad$ [Type $(f(x))^{g(x)}$]

$\therefore$ Taking logs of both sides, we have

$\log u = \log x^{(x^2-3)} = (x^2-3)\log x \quad [\because \log m^n = n \log m]$

Differentiating both sides w.r.t. $x$, we have

$$\frac{1}{u}\frac{du}{dx} = (x^2-3)\frac{d}{dx}\log x + \log x \frac{d}{dx}(x^2-3)$$

$$= (x^2-3)\frac{1}{x} + \log x \,.\, (2x-0)$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = \frac{x^2-3}{x} + 2x\log x \quad \therefore \quad \frac{du}{dx} = u\left[\frac{x^2-3}{x} + 2x\log x\right]$$

Putting $u = x^{(x^2-3)}$, $\dfrac{du}{dx} = x^{(x^2-3)}\left(\dfrac{x^2-3}{x} + 2x\log x\right)$ ...(*ii*)

Again $v = (x-3)^{x^2}$ $\quad [(f(x))^{g(x)}]$

$\therefore$ Taking logs of both sides, we have

$\log v = \log (x-3)^{x^2} = x^2 \log (x-3) \quad [\because \log m^n = n \log m]$

$$\therefore \quad \frac{d}{dx}\log v = \frac{d}{dx}(x^2 \log (x-3))$$

$$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = x^2\frac{d}{dx}\log (x-3) + \log (x-3)\frac{d}{dx}x^2$$

$$= x^2\frac{1}{x-3}\frac{d}{dx}(x-3) + \log (x-3)\,.\,2x$$

$$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = \frac{x^2}{x-3} + 2x\log (x-3)$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left[\frac{x^2}{x-3} + 2x\log (x-3)\right]$$

Putting $v = (x-3)^{x^2}$,

$$\frac{dv}{dx} = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log (x-3)\right] \quad ...(iii)$$

Putting values of $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*), we have

$$\frac{dy}{dx} = x^{(x^2-3)}\left[\frac{x^2-3}{x} + 2x\log x\right] + (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log (x-3)\right].$$

**12. Find $\dfrac{dy}{dx}$ if $y = 12(1 - \cos t)$ and $x = 10(t - \sin t)$,**

**$-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$.**

**Sol. Given:** $y = 12(1-\cos t)$ and $x = 10(t - \sin t)$

Differentiating both equations w.r.t. $t$, we have

$$\frac{dy}{dt} = 12\frac{d}{dt}(1 - \cos t) \qquad \text{and} \quad \frac{dx}{dt} = 10\frac{d}{dt}(t - \sin t)$$

$$= 12(0 + \sin t) = 12 \sin t \quad \text{and} \quad \frac{dx}{dt} = 10(1 - \cos t)$$

We know that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12 \sin t}{10(1 - \cos t)}$

$$= \frac{6}{5} \cdot \frac{2 \sin\frac{t}{2} \cos\frac{t}{2}}{2 \sin^2 \frac{t}{2}} = \frac{6}{5} \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} = \frac{6}{5} \cot \frac{t}{2}.$$

**13. Find $\frac{dy}{dx}$ if $y = \sin^{-1} x + \sin^{-1} \sqrt{1 - x^2}$, $-1 \le x \le 1$.**

**Sol. Given:** $y = \sin^{-1} x + \sin^{-1} \sqrt{1 - x^2}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - (\sqrt{1 - x^2})^2}} \frac{d}{dx} \sqrt{1 - x^2}$$

$$\left[\because \frac{d}{dx} \sin^{-1} f(x) = \frac{1}{\sqrt{1 - (f(x))^2}} \frac{d}{dx} f(x)\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - (1 - x^2)}} \frac{1}{2} (1 - x^2)^{-1/2} \frac{d}{dx}(1 - x^2)$$

$$= \frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - 1 + x^2}} \frac{1}{2\sqrt{1 - x^2}}(-2x)$$

$$= \frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{x^2}} \left(\frac{-x}{\sqrt{1 - x^2}}\right) = \frac{1}{\sqrt{1 - x^2}} - \frac{x}{x\sqrt{1 - x^2}}$$

or $\quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 - x^2}} = 0.$

**14. If $x\sqrt{1 + y} + y\sqrt{1 + x} = 0$, for $-1 < x < 1$, prove that**

$$\frac{dy}{dx} = \frac{-1}{(1 + x)^2}.$$

**Sol.** $x\sqrt{1 + y} + y\sqrt{1 + x} = 0.$ ...(*i*) (given)

We shall first find $y$ in terms of $x$ because $y$ is not required in the value of $\frac{dy}{dx} = \frac{-1}{(1 + x)^2}$ to be proved.

From eqn. (*i*), $\quad x\sqrt{1 + y} = -y\sqrt{1 + x}$

Squaring both sides, $\quad x^2 (1 + y) = y^2 (1 + x)$

or $\quad x^2 + x^2y = y^2 + y^2x$ or $x^2 - y^2 = -x^2y + y^2x$

or $(x - y)(x + y) = -xy(x - y)$

Dividing both sides by $(x - y) \neq 0$ $(\because x \neq y)$

$$x + y = -xy \quad \text{or} \quad y + xy = -x$$

$$\Rightarrow \quad y(1 + x) = -x \quad \therefore \quad y = -\frac{x}{1 + x}$$

Differentiating both sides w.r.t. $x$, we have

$$\frac{dy}{dx} = -\frac{(1+x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= -\frac{(1+x).1 - x.1}{(1+x)^2} = -\frac{1}{(1+x)^2}.$$

**15. If $(x - a)^2 + (y - b)^2 = c^2$, for some $c > 0$, prove that**

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

**is a constant independent of $a$ and $b$.**

**Sol.** The given equation is $(x - a)^2 + (y - b)^2 = c^2$ ...(*i*)

Differentiating both sides of eqn. (*i*) w.r.t. $x$,

$$2(x - a) + 2(y - b)\frac{dy}{dx} = 0$$

or $$2(y - b)\frac{dy}{dx} = -2(x - a) \quad \therefore \quad \frac{dy}{dx} = -\left(\frac{x-a}{y-b}\right) \quad ...(ii)$$

Again differentiating both sides of (*ii*) w.r.t. $x$,

$$\frac{d^2y}{dx^2} = \frac{-\left[(y-b).1 - (x-a)\frac{dy}{dx}\right]}{(y-b)^2}$$

Putting the value of $\frac{dy}{dx}$ from (*i*),

$$\frac{d^2y}{dx^2} = \frac{-\left[(y-b) - (x-a)\left(\frac{-(x-a)}{y-b}\right)\right]}{(y-b)^2} = \frac{-\left[(y-b) + \frac{(x-a)^2}{y-b}\right]}{(y-b)^2}$$

$$= \frac{-[(y-b)^2 + (x-a)^2]}{(y-b)^3} = \frac{-c^2}{(y-b)^3} \quad \text{[By } (i)] \quad ...(iii)$$

Putting values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (*ii*) and (*iii*) in the given expression

$$\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}, \text{ it is } = \frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{3/2}}{\frac{-c^2}{(y-b)^3}}$$

$$= \frac{[(y-b)^2+(x-a)^2]^{3/2}}{(y-b)^3} \times \frac{(y-b)^3}{-c^2} \qquad [\because \ ((y-b)^2)^{3/2} = (y-b)^3]$$

Putting $(x - a)^2 + (y - b)^2 = c^2$ from $(i)$

$$= \frac{(c^2)^{3/2}}{-c^2} = \frac{-c^3}{c^2} = -c$$

which is a constant and is independent of $a$ and $b$.

**16. If $\cos y = x \cos (a + y)$ with $\cos a \neq \pm 1$, prove that**

$$\frac{dy}{dx} = \frac{\cos^2 (a+y)}{\sin a}.$$

**Sol. Given:** $\cos y = x \cos (a + y)$

$$\therefore \qquad x = \frac{\cos y}{\cos (a+y)} \qquad ...(i)$$

(We have found the value of $x$ because $x$ is not present in the required value of $\frac{dy}{dx}$ )

Differentiating both sides of $(i)$ w.r.t. $y$, $\frac{dx}{dy} = \frac{d}{dy}\left(\frac{\cos y}{\cos (a+y)}\right)$

Applying Quotient Rule,

$$\frac{dx}{dy} = \frac{\cos (a+y)\frac{d}{dy}\cos y - \cos y \frac{d}{dy}\cos (a+y)}{\cos^2 (a+y)}$$

or
$$\frac{dx}{dy} = \frac{\cos (a+y)(-\sin y) - \cos y(-\sin (a+y))}{\cos^2 (a+y)}$$

$$\left[\because \frac{d}{dy}\cos (a+y) = -\sin (a+y)\frac{d}{dy}(a+y) = -\sin (a+y)(0+1) = -\sin (a+y)\right]$$

or
$$\frac{dx}{dy} = \frac{-\cos (a+y)\sin y + \sin (a+y)\cos y}{\cos^2 (a+y)}$$

$$= \frac{\sin (a+y)\cos y - \cos (a+y)\sin y}{\cos^2 (a+y)}$$

$$= \frac{\sin (a+y-y)}{\cos^2 (a+y)} = \frac{\sin a}{\cos^2 (a+y)}$$

$$[\because \ \sin A \cos B - \cos A \sin B = \sin (A - B)]$$

Taking reciprocals $\frac{dy}{dx} = \frac{\cos^2 (a + y)}{\sin a}$.

**17. If $x = a$ (cos $t$ + $t$ sin $t$) and $y = a$ (sin $t$ – $t$ cos $t$), find $\frac{d^2y}{dx^2}$.**

**Sol. Given:** $x = a$ (cos $t$ + $t$ sin $t$) and $y = a$ (sin $t$ – $t$ cos $t$)

Differentiating both eqns. w.r.t. $t$, we have

$$\frac{dx}{dt} = a\left(-\sin t + \frac{d}{dt} t \sin t\right) \text{ and } \frac{dy}{dt} = a\left(\cos t - \frac{d}{dt}(t \cos t)\right)$$

$$= a\left(-\sin t + t\frac{d}{dt}\sin t + \sin t \frac{d}{dt} t\right)$$

and $$\frac{dy}{dt} = a\left(\cos t - \left(t\frac{d}{dt}(\cos t) + \cos t \frac{d}{dt}(t)\right)\right)$$

$\Rightarrow$ $$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t)$$

and $$\frac{dy}{dt} = a(\cos t - (-t\sin t + \cos t))$$

$\Rightarrow$ $$\frac{dx}{dt} = at\cos t \quad ...(i)$$

and $$\frac{dy}{dt} = a(\cos t + t\sin t - \cos t) = at\sin t$$

We know that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at \sin t}{at \cos t} = \frac{\sin t}{\cos t} = \tan t$

Now differentiating both sides w.r.t. $x$, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan t) = \sec^2 t \frac{d}{dx}(t) \quad \rightarrow \textbf{Note}$$

$$= \sec^2 t \frac{dt}{dx} = \sec^2 t \frac{1}{at\cos t} \quad \text{(By }(i))$$

$$= \sec^2 t \cdot \frac{\sec t}{at} = \frac{\sec^3 t}{at}.$$

**18. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real $x$ and find it.**

**Sol. Given:** $f(x) = |x|^3 = x^3$ if $x \geq 0$ ...$(i)$ [$\because |x| = x$ if $x \geq 0$]

and $f(x) = |x|^3 = (-x)^3 = -x^3$ if $x < 0$ ...$(ii)$

[$\because |x| = -x$ if $x < 0$]

Differentiating both eqns. $(i)$ and $(ii)$ w.r.t. $x$,

$f'(x) = 3x^2$ if $x > 0$ and $f'(x) = -3x^2$ if $x < 0$ ...$(iii)$

(At $x = 0$, we can't write the value of $f'(x)$ by usual rule of derivatives because $x = 0$ is a partitioning point of values of $f(x)$ given by $(i)$ and $(ii)$

$\therefore$ $f''(x) = 6x$ if $x > 0$ and $= -6x$ if $x < 0$ ...(*iv*)

$\therefore$ From (*iv*), $f''(x)$ exists for all $x > 0$ and for all $x < 0$

*i.e.,* for all $x \in$ R except at $x = 0$ ...(*v*)

**Let us discuss derivability of $f(x)$ at $x = 0$**

$$\mathrm{L}f'(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x^3 - 0}{x} \quad \text{[By (ii) and (i)]}$$

$$= \lim_{x \to 0^-} -x^2 = 0 \quad \text{(On putting } x = 0)$$

$$\mathrm{R}f'(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^3 - 0}{x - 0} \quad \text{[By (i)]}$$

$$= \lim_{x \to 0^+} x^2 = 0 \quad \text{(On putting } x = 0)$$

$\therefore$ $\mathrm{L}f'(0) = \mathrm{R}f'(0) = 0$

$\therefore$ $f(x)$ is derivable at $x = 0$ and $f'(0) = 0$ ...(*vi*)

**Let us discuss derivability of $f'(x)$ at $x = 0$**

$$\mathrm{L}f''(0) = \lim_{x \to 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^-} \frac{-3x^2 - 0}{x}$$

(By (*iii*) and (*vi*))

$$= \lim_{x \to 0^-} (-3x) = -3(0) = 0 \quad \text{(On putting } x = 0)$$

$$\mathrm{R}f''(0) = \lim_{x \to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^+} \frac{3x^2 - 0}{x} \quad \text{(By (iii) and (vi))}$$

$$= \lim_{x \to 0^+} 3x = 3(0) = 0 \quad \text{(On putting } x = 0)$$

$\therefore$ $\mathrm{L}f''(0) = \mathrm{R}f''(0) = 0$

$\therefore$ $f'(x)$ is derivable at $x = 0$ and $f''(0) = 0$ ...(*vii*)

From (*iv*) and (*vii*), $f''(x)$ exists for all real $x$ and $f''(x) = 6x$ if $x > 0$ and $= -6x$ if $x < 0$ and $f''(0) = 0$.

**19. Using mathematical induction, prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers $n$.**

**Sol.** Let P($n$): $\frac{d}{dx}(x^n) = nx^{n-1}$

then P(1): $\frac{d}{dx}(x^1) = 1x^0$ or $\frac{d}{dx}(x) = 1$ which is true.

$\Rightarrow$ P(1) is true.

**Assume P($k$) is true.** *i.e.,* let $\frac{d}{dx}(x^k) = kx^{k-1}$ ...(*i*)

Now $\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x^k \cdot x) = \frac{d}{dx}(x^k) \cdot x + x^k \cdot \frac{d}{dx}(x)$

$$\left[\because \frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}\right]$$

$= kx^{k-1} \cdot x + x^k \cdot 1$ [Using (*i*)]

$$= kx^k + x^k = (k + 1)\, x^k \qquad \Rightarrow \text{P}(k + 1) \text{ is true.}$$

Hence by P.M.I., the statement is true for all positive integers $n$.

**20. Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.**

**Sol. Given.** sin (A + B) = sin A cos B + cos A sin B

Assuming A and B are functions of $x$ and differentiating both sides w.r.t. $x$, we have

$$\cos(\text{A} + \text{B}) \cdot \frac{d}{dx}(\text{A} + \text{B}) = \left[\frac{d}{dx}(\sin \text{A}) \cdot \cos \text{B} + \sin \text{A} \cdot \frac{d}{dx}(\cos \text{B})\right] + \left[\frac{d}{dx}(\cos \text{A}) \cdot \sin \text{B} + \cos \text{A} \cdot \frac{d}{dx}(\sin \text{B})\right]$$

$$\Rightarrow \cos(\text{A} + \text{B})\left(\frac{d\text{A}}{dx} + \frac{d\text{B}}{dx}\right) = \cos \text{A} \cdot \frac{d\text{A}}{dx} \cdot \cos \text{B} + \sin \text{A}\,(-\sin \text{B})\, \frac{d\text{B}}{dx} - \sin \text{A}\, \frac{d\text{A}}{dx} \cdot \sin \text{B} + \cos \text{A} \cdot \cos \text{B}\, \frac{d\text{B}}{dx}$$

$$= (\cos \text{A} \cos \text{B} - \sin \text{A} \sin \text{B})\, \frac{d\text{A}}{dx} + (\cos \text{A} \cos \text{B} - \sin \text{A} \sin \text{B})\, \frac{d\text{B}}{dx}$$

$$\text{or } \cos(\text{A} + \text{B})\left(\frac{d\text{A}}{dx} + \frac{d\text{B}}{dx}\right) = (\cos \text{A} \cos \text{B} - \sin \text{A} \sin \text{B})\left(\frac{d\text{A}}{dx} + \frac{d\text{B}}{dx}\right)$$

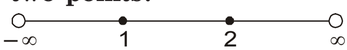
Dividing both sides by $\frac{d\text{A}}{dx} + \frac{d\text{B}}{dx}$, we have

$$\cos(\text{A} + \text{B}) = \cos \text{A} \cos \text{B} - \sin \text{A} \sin \text{B}$$

which is the sum formula for cosines.

**21. Does there exist a function which is continuous everywhere but not differentiable at exactly two points?**

**Sol.** Yes, there exist such function(s).




For example, let us take $f(x) = |x - 1| + |x - 2|$ ...(*i*)

Let us put each expression within modulus equal to 0 *i.e.,* $x - 1 = 0$ and $x - 2 = 0$ *i.e.,* $x = 1$ and $x = 2$.

These two real numbers $x = 1$ and $x = 2$ divide the whole real line $(-\infty, \infty)$ into three sub-intervals $(-\infty, 1]$, $[1, 2]$ and $[2, \infty)$.

In $(-\infty, 1]$ *i.e.,* **For $x \le 1$,** $x - 1 \le 0$ and $x - 2 \le 0$ and therefore $|x - 1| = -(x - 1)$ and $|x - 2| = -(x - 2)$

$\therefore$ From (*i*), $f(x) = -(x - 1) - (x - 2)$

$= -x + 1 - x + 2 = 3 - 2x$ for $x \le 1$ ...(*ii*)

In $[1, 2]$ *i.e.,* for $1 \le x \le 2$, $x - 1 \ge 0$ and $x - 2 \le 0$ and therefore $| x - 1 | = x - 1$ and $| x - 2 | = -(x - 2)$.

From (*i*), $f(x) = x - 1 - (x - 2) = x - 1 - x + 2 = 1$ for $1 \le x \le 2$ ...(*iii*)

Again in $[2, \infty)$ *i.e.*, for $x \ge 2$, $x - 1 \ge 0$ and $x - 2 \ge 0$ and therefore

$| x - 1 | = x - 1$ and $| x - 2 | = x - 2$.

$\therefore$ From (*i*) $f(x) = x - 1 + x - 2 = 2x - 3$ for $x \ge 2$ ...(*iv*)

Hence function (*i*) given in modulus form can be expressed as piece-wise function given by (*ii*), (*iii*) and (*iv*)

*i.e.*, $f(x) = 3 - 2x$ for $x \le 1$ ...(*ii*)

$= 1$ for $1 \le x \le 2$ ...(*iii*)

$= 2x - 3$ for $x \ge 2$ ...(*iv*)

Now the three values of $f(x)$ given by (*ii*), (*iii*) and (*iv*) are polynomial functions and constant function and hence are continuous and derivable for all real values of $x$ except possibly at the partitioning points $x = 1$ and $x = 2$. ...(*v*)

**To examine continuity at $x = 1$**

Left Hand limit $= \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (3 - 2x)$ [By (*ii*)]

Put $x = 1$; $= 3 - 2 = 1$

Right Hand Limit $= \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} 1$ [By (*iii*)]

Put $x = 1$; $= 1$

$\therefore \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)\ (= 1)$

$\therefore \lim\limits_{x \to 1} f(x)$ exists and $= 1 = f(1)$ ($\because$ From (*iii*) $f(1) = 1$]

$\therefore f(x)$ is continuous at $x = 1$ ...(*vi*)

**To examine derivability at $x = 1$**

Left Hand derivative $= \mathrm{L}f'(1) = \lim\limits_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1}$

$= \lim\limits_{x \to 1^-} \dfrac{3 - 2x - 1}{x - 1}$ [By (*ii*) and $f(1) = 1$ (proved above)]

$= \lim\limits_{x \to 1^-} \dfrac{-2x + 2}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{-2(x - 1)}{x - 1}$

$= \lim\limits_{x \to 1^-} (-2) = -2$

Right Hand derivative $= \mathrm{R}f'(1) = \lim\limits_{x \to 1^+} \dfrac{f(x) - f(1)}{x - 1}$

$= \lim\limits_{x \to 1^+} \dfrac{1 - 1}{x - 1}$ (By (*iii*))

$= \lim\limits_{x \to 1^+} \dfrac{0}{x - 1} = \lim\limits_{x \to 1^+} \dfrac{0}{\text{Non-zero}}$

$[x \to 1^+ \Rightarrow x > 1 \Rightarrow x - 1 > 0 \Rightarrow x - 1 \ne 0]$

$= \lim\limits_{x \to 1^+} 0 = 0$

$\therefore$ $Lf'(1) \neq Rf'(1$

$\therefore$ $f(x)$ is not differentiable at $x = 1$ ...(*vii*)

**To examine continuity at $x = 2$**

Left hand limit $= \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} 1$ (By (*ii*)) = 1

Right Hand Limit $= \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (2x - 3)$ [By (*iv*)]

Putting $x = 2$, $= 4 - 3 = 1$

$\therefore$ $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x)$ (= 1)

$\therefore$ $\lim\limits_{x \to 2} f(x)$ exists and $= 1 = f(2)$ [$\because$ From (*iii*), $f(2) = 1$]

$\therefore$ $f(x)$ is continuous at $x = 2$ ...(*viii*)

**To examine derivability at $x = 2$**

$$Lf'(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{1 - 1}{x - 2} \qquad \text{(By (iii))}$$

$$= \lim_{x \to 2^-} \frac{0}{\text{Non-zero}}$$

[$\because$ $x \to 2^- \Rightarrow x < 2 \Rightarrow x - 2 < 0 \Rightarrow x - 2 \neq 0$]

$$= \lim_{x \to 2^-} 0 = 0$$

$$Rf'(2) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{2x - 3 - 1}{x - 2} \qquad \text{[By (iv)]}$$

$$= \lim_{x \to 2^+} \frac{2x - 4}{x - 2} = \lim_{x \to 2^+} \frac{2(x - 2)}{x - 2} = \lim_{x \to 2^+} 2 = 2$$

$\therefore$ $Lf'(2) \neq Rf'(2)$

$\therefore$ $f(x)$ is not differentiable at $x = 2$ ...(*ix*)

From (*v*), (*vi*) and (*viii*), we can say that $f(x)$ is continuous for all real values of $x$ *i.e.,* continuous everywhere.

From (*v*), (*vii*) and (*ix*), we can say that $f(x)$ is not differentiable at exactly two points $x = 1$ and $x = 2$ on the real line.

**22. If** $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$**, prove that**

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}.$$

**Sol. Given:** $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Expanding the determinant along first row,

$$y = f(x)\,(mc - nb) - g(x)\,(lc - na) + h(x)\,(lb - ma)$$

$$\therefore \quad \frac{dy}{dx} = (mc - nb)\frac{d}{dx} f(x) - (lc - na)\frac{d}{dx} g(x) + (lb - ma)\frac{d}{dx} h(x)$$

$$= (mc - nb)\,f'(x) - (lc - na)\,g'(x) + (lb - ma)\,h'(x) \qquad ...(i)$$

$$\text{R.H.S.} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Expanding along first row,

$$= f'(x)\,(mc - nb) - g'(x)\,(lc - na) + h'(x)\,(lb - ma)$$

$$= (mc - nb)\,f'(x) - (lc - na)\,g'(x) + (lb - ma)\,h'(x) \qquad ...(ii)$$

From $(i)$ and $(ii)$, we have L.H.S. = R.H.S.

**23. If $y = e^{a\cos^{-1} x}$, $-1 \le x \le 1$, show that**

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.$$

**Sol. Given:** $y = e^{a\cos^{-1} x}$ ...$(i)$

$$\therefore \quad \frac{dy}{dx} = e^{a\cos^{-1} x}\frac{d}{dx}(a\cos^{-1} x) \qquad \left[\because \frac{d}{dx} e^{f(x)} = e^{f(x)}\frac{d}{dx} f(x)\right]$$

$$\text{or} \quad \frac{dy}{dx} = e^{a\cos^{-1} x}\,.\,a\left(\frac{-1}{\sqrt{1 - x^2}}\right) = \frac{-ae^{a\cos^{-1} x}}{\sqrt{1 - x^2}}$$

Cross-multiplying, $\sqrt{1 - x^2}\,\dfrac{dy}{dx} = -a\,e^{a\cos^{-1} x} = -ay$

(By $(i)$) ...$(ii)$

Again differentiating both sides w.r.t. $x$,

$$\sqrt{1 - x^2}\,\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(1 - x^2)^{1/2} = -a\frac{dy}{dx}$$

$$\Rightarrow \sqrt{1 - x^2}\,\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{1}{2}(1 - x^2)^{-1/2}\frac{d}{dx}(1 - x^2) = -a\frac{dy}{dx}$$

$$\Rightarrow \quad \sqrt{1 - x^2}\,\frac{d^2y}{dx^2} + \frac{1}{2}\frac{dy}{dx}\frac{1}{\sqrt{1 - x^2}}(-2x) = -a\frac{dy}{dx}$$

Multiplying by L.C.M. = $\sqrt{1 - x^2}$,

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -a\sqrt{1 - x^2}\,\frac{dy}{dx}$$

$$= -a(-ay) \qquad \text{[By } (ii)]$$

$$= a^2y$$

$$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.$$