Exercise 11.1

1. If a line makes angles 90°, 135°, 45° with the *x*, *y* and *z*-axes respectively, find its direction cosines.

Sol. We know that direction cosines of a line making angles α, β, γ with the x, y and z-axes respectively are **$\cos \alpha$, $\cos \beta$, $\cos \gamma$.**

Here $\alpha = 90°$, $\beta = 135°$ and $\gamma = 45°$.

Therefore, direction cosines of the required line are cos 90°, cos 135° and cos 45° $= 0, \frac{-1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$.

$$\left[\cos 135° = \underset{\text{(II)}}{\cos (180° - 45°)} = -\cos 45° = \frac{-1}{\sqrt{2}}\right]$$

Result. $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$

2. Find the direction cosines of a line which makes equal angles with the co-ordinate axes.

Sol. Let a line make equal angles α, α, α with the co-ordinate axes.

$\therefore$ Direction cosines of the line are $\cos \alpha$, $\cos \alpha$, $\cos \alpha$ ...(i)

$\therefore$ $\cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha = 1$ $[\because \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1]$

$\Rightarrow 3 \cos^2 \alpha = 1 \quad \Rightarrow \quad \cos^2 \alpha = \frac{1}{3} \quad \Rightarrow \quad \cos \alpha = \pm \sqrt{\frac{1}{3}} = \pm \frac{1}{\sqrt{3}}$

Putting $\cos \alpha = \pm \frac{1}{\sqrt{3}}$ in (i), direction cosines of the required line making equal angles with the co-ordinate axes are

$$\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}.$$

Very Important Remark. Therefore, direction cosines of a line making equal angles with the co-ordinate axes in the positive (*i.e.,* first) octant are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

3. If a line has direction ratios – 18, 12, – 4, then what are its direction cosines?

Sol. We know that if a, b, c are direction ratios of a line, then direction cosines of the line are

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}} \quad ...(i)$$

Here, direction ratios of the line are

$$-18, 12, -4 = a, b, c$$

$\therefore \sqrt{a^2 + b^2 + c^2} = \sqrt{(-18)^2 + (12)^2 + (-4)^2} = \sqrt{324 + 144 + 16}$

$= \sqrt{484} = 22$

Putting these values in (i), direction cosines of the required line are

$$\frac{-18}{22}, \frac{12}{22}, \frac{-4}{22} = \frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}.$$

4. Show that the points (2, 3, 4), (– 1, – 2, 1), (5, 8, 7) are collinear.

Sol. The given points are A(2, 3, 4), B(– 1, – 2, 1) and C(5, 8, 7).

$\therefore$ Direction ratios of the line joining A and B are

$$-1-2,\ -2-3,\ 1-4 \qquad \mid x_2-x_1,\ y_2-y_1,\ z_2-z_1$$

i.e., $-3,\ -5,\ -3$ ...(*i*)

$= a_1, b_1, c_1$ (say)

Again direction ratios of the line joining B and C are

$$5-(-1),\ 8-(-2),\ 7-1 = 6,\ 10,\ 6 \qquad ...(ii)$$

$= a_2, b_2, c_2$ (say)

From (*i*) and (*ii*) direction ratios of AB and BC are proportional

i.e., $$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \qquad \left[\because \frac{-3}{6} = \frac{-5}{10}, \frac{-3}{6} \text{ (each } = \frac{-1}{2})\right]$$

Therefore, AB is parallel to BC. But point B is common to both AB and BC. Hence, points A, B, C are collinear.

5. Find the direction cosines of the sides of the triangle whose vertices are (3, 5, – 4), (– 1, 1, 2) and (– 5, – 5, – 2).

Sol. Direction ratios of line AB are $-1-3,\ 1-5,\ 2-(-4)$

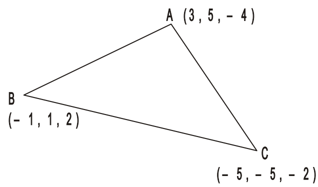
i.e., $-4,\ -4,\ 6$ $\qquad \mid x_2-x_1,\ y_2-y_1,\ z_2-z_1$

Dividing each by $\sqrt{a^2+b^2+c^2} = \sqrt{(-4)^2+(-4)^2+6^2}$

$$= \sqrt{16+16+36} = \sqrt{68} = \sqrt{4\times 17} = 2\sqrt{17}.$$

direction cosines of line AB are

$$\frac{-4}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}, \frac{6}{2\sqrt{17}}.$$



i.e., $$\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}$$

Direction ratios of line BC are

$$-5-(-1),\ -5-1,\ -2-2 = -4,\ -6,\ -4$$

Dividing each by $\sqrt{(-4)^2+(-6)^2+(-4)^2} = \sqrt{16+36+16}$

$$= \sqrt{68} = \sqrt{4\times 17} = 2\sqrt{17}$$

Direction cosines of line BC are $\dfrac{-4}{2\sqrt{17}}, \dfrac{-6}{2\sqrt{17}}, \dfrac{-4}{2\sqrt{17}}$

i.e., $$\frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}$$

Direction ratios of line CA are

$$3-(-5),\ 5-(-5),\ -4-(-2) = 8,\ 10,\ -2$$

Dividing each by $\sqrt{(8)^2+(10)^2+(-2)^2} = \sqrt{64+100+4}$

$$= \sqrt{168} = \sqrt{4\times 42} = 2\sqrt{42}.$$

Direction ratios of line CA are

$$\frac{8}{2\sqrt{42}}, \frac{10}{2\sqrt{42}}, \frac{-2}{2\sqrt{42}} = \frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{42}}$$

Note. If l, m, n are direction cosines of a line, then $-l$, $-m$, $-n$ are also direction cosines of the same line.

Exercise 11.2

1. Show that the three lines with direction cosines

$$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$$

are mutually perpendicular.

Sol. Given: Direction cosines of three lines are

$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; = l_1, m_1, n_1,$ $\quad \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; = l_2, m_2, n_2$

and $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13} = l_3, m_3, n_3$

For first two lines;

$$l_1l_2 + m_1m_2 + n_1n_2 = \frac{12}{13}\left(\frac{4}{13}\right) + \left(\frac{-3}{13}\right)\left(\frac{12}{13}\right) + \left(\frac{-4}{13}\right)\left(\frac{3}{13}\right)$$

$$= \frac{48}{169} - \frac{36}{169} - \frac{12}{169} = \frac{48-36-12}{169} = \frac{0}{169} = 0$$

$\therefore$ The first two lines are perpendicular to each other.

For second and third line,

$$l_2l_3 + m_2m_3 + n_2n_3$$

$$= \frac{4}{13}\left(\frac{3}{13}\right) + \frac{12}{13}\left(\frac{-4}{13}\right) + \frac{3}{13}\left(\frac{12}{13}\right) = \frac{12}{169} - \frac{48}{169} + \frac{36}{169}$$

$$= \frac{12-48+36}{169} = \frac{0}{169} = 0$$

$\therefore$ Second and third lines are perpendicular to each other.

For first and third line,

$$l_1l_3 + m_1m_3 + n_1n_3$$

$$= \frac{12}{13}\left(\frac{3}{13}\right) + \left(\frac{-3}{13}\right)\left(\frac{-4}{13}\right) + \left(\frac{-4}{13}\right)\left(\frac{12}{13}\right) = \frac{36}{169} + \frac{12}{169} - \frac{48}{169}$$

$$= \frac{36+12-48}{169} = \frac{0}{169} = 0$$

$\therefore$ First and third line are also perpendicular to each other.

$\therefore$ The three given lines are mutually perpendicular.

2. Show that the line through the points (1, – 1, 2), (3, 4, – 2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

Sol. We know that direction ratios of the line joining the points A(1, – 1, 2) and B(3, 4, – 2) are $x_2 - x_1, y_2 - y_1, z_2 - z_1$

i.e., $3 - 1, 4 - (-1), -2 - 2 = 2, 5, -4 = a_1, b_1, c_1$ (say)

Again, direction ratios of the line joining the points C(0, 3, 2) and D(3, 5, 6) are $x_2 - x_1, y_2 - y_1, z_2 - z_1$

i.e., $3 - 0, 5 - 3, 6 - 2 = 3, 2, 4 = a_2, b_2, c_2$ (say)

For these lines AB and CD,

$$a_1a_2 + b_1b_2 + c_1c_2 = 2(3) + 5(2) + (-4)(4)$$
$$= 6 + 10 - 16 = 0$$

∴ Given line AB is perpendicular to given line CD.

3. Show that the line through the points (4, 7, 8), (2, 3, 4) is parallel to the line through the points (– 1, – 2, 1), (1, 2, 5).

Sol. We know that direction ratios of the line joining the points A(4, 7, 8) and B(2, 3, 4) are $x_2 - x_1, y_2 - y_1, z_2 - z_1$

i.e., $2 - 4, 3 - 7, 4 - 8$ *i.e.,* $-2, -4, -4 = a_1, b_1, c_1$ (say)

Again, direction ratios of the line joining the points C(– 1, – 2, 1) and D(1, 2, 5) are $1 - (-1), 2 - (-2), 5 - 1 = 2, 4, 4 = a_2, b_2, c_2$ (say)

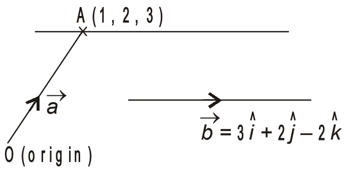
For these lines AB and CD,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \qquad \left(\text{as } \frac{-2}{2} = \frac{-4}{4} = \frac{-4}{4} \ (= -1 \text{ each})\right)$$

∴ Given line AB is parallel to given line CD.

4. Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector

$$\mathbf{3\hat{i} + 2\hat{j} - 2\hat{k}.}$$



Sol. A point on the required line is

$$A(1, 2, 3) = (x_1, y_1, z_1)$$

i.e., Position vector of a point on the required line is

$$\vec{a} = \overrightarrow{OA} = (1, 2, 3) = \hat{i} + 2\hat{j} + 3\hat{k}.$$

The required line is parallel to the vector $\vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}$ (and hence direction ratios of the required line are coefficient of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{b}$ *i.e.,* $3, 2, -2 = a, b, c$)

∴ Vector equation of required line is

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{a}} + \lambda \vec{\boldsymbol{b}} \quad i.e., \ \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$$

where λ is a real number.

Remark. Also cartesian equation of the required line in this Q. No. 4 is

$$\frac{\boldsymbol{x - x_1}}{\boldsymbol{a}} = \frac{\boldsymbol{y - y_1}}{\boldsymbol{b}} = \frac{\boldsymbol{z - z_1}}{\boldsymbol{c}} \quad i.e., \quad \frac{x-1}{3} = \frac{y-2}{2} = \frac{z-3}{-2}$$

5. Find the equation of the line in vector and in cartesian form that passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction $\hat{i} + 2\hat{j} - \hat{k}$.

Sol. Position vector of a point on the required line is

$$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k} = (2, -1, 4) = (x_1, y_1, z_1)$$

The required line is in the direction of the vector

$$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$$

($\Rightarrow$ direction ratios of required line are coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ in $\vec{b}$ *i.e.,* 1, 2, – 1 = a, b, c)

$\therefore$ Equation of the required line in vector form is $\vec{r} = \vec{a} + \lambda \vec{b}$

i.e., $$\vec{r} = (2\hat{i} - \hat{j} + 4\hat{k}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$$

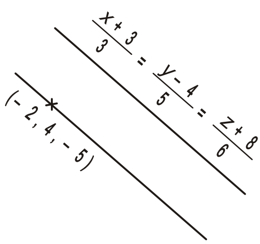
where λ is a real number and equation of line in cartesian form is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad i.e., \quad \frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}.$$

6. Find the cartesian equation of the line which passes through the point (– 2, 4, – 5) and parallel to the line given by

$$\frac{\boldsymbol{x + 3}}{\boldsymbol{3}} = \frac{\boldsymbol{y - 4}}{\boldsymbol{5}} = \frac{\boldsymbol{z + 8}}{\boldsymbol{6}}.$$

Sol. Given: A point on the required line is $(-2, 4, -5) = (x_1, y_1, z_1)$.



Equations of the given line in cartesian form are

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$$

(It is standard form because coefficients of x, y, z are unity each)

$\therefore$ Direction ratios (D.R.'s) of the given line are its denominators 3, 5, 6 and hence d.r.'s of the required parallel line are also 3, 5, 6 = a, b, c.

$\therefore$ Eqations of the required line are

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \quad i.e., \quad \frac{x-(-2)}{3}=\frac{y-4}{5}=\frac{z-(-5)}{6}$$

i.e., $\dfrac{x+2}{3}=\dfrac{y-4}{5}=\dfrac{z+5}{6}$.

7. The cartesian equation of a line is $\dfrac{x-5}{3}=\dfrac{y+4}{7}=\dfrac{z-6}{2}$. Write its vector form.

Sol. Given: The cartesian equation of a line is

$$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$$

i.e $\dfrac{x-5}{3}=\dfrac{y-(-4)}{7}=\dfrac{z-6}{2}$

compairing the given equation with the standard form

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

we have $x_1 = 5$, $y_1 = -4$, $z_1 = 6$; $a = 3$, $b = 7$, $c = 2$

Hence the given line passes through the point

$\vec{a} = (x_1, y_1, z_1) = (5, -4, 6) = 5\hat{i} - 4\hat{j} + 6\hat{k}$

and is parallel (or collinear) with the vector

$\vec{b} = a\hat{i} + b\hat{j} + c\hat{k} = 3\hat{i} + 7\hat{j} + 2\hat{k}$

$\therefore$ Vector equation of the given line is $\vec{r} = \vec{a} + \lambda\vec{b}$

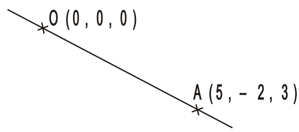
i.e $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

8. Find the vector and cartesian equations of the line that passes through the origin and (5, – 2, 3).

Sol. Vector equation of the line

$\vec{a}$ = Position vector of a point here O (say) on the line

$= (0, 0, 0) = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$



$\vec{b}$ = A vector along the line

$= \overrightarrow{OA}$ = Position vector of point A – Position vector of point O

$= (5, -2, 3) - (0, 0, 0) = (5, -2, 3) = 5\hat{i} - 2\hat{j} + 3\hat{k}$

$\therefore$ Vector equation of the line is $\vec{r} = \vec{a} + \lambda \vec{b}$

i.e., $\vec{r} = \vec{0} + \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$ i.e. $\vec{r} = \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$.

Cartesian equation of the line

Direction ratios of line OA are $5 - 0, -2 - 0, 3 - 0$

i.e., $5, -2, 3$ | $x_2 - x_1, y_2 - y_1, z_2 - z_1 = a, b, c$

A point on the line O is $(0, 0, 0) = (x_1, y_1, z_1)$.

$\therefore$ Cartesian equation of the line is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad i.e., \quad \frac{x - 0}{5} = \frac{y - 0}{-2} = \frac{z - 0}{3}$$

i.e., $\frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$.

Remark. In the solution of the above question we can also take:

$\vec{a}$ = Position vector of point A = $(5, -2, 3) = 5\hat{i} - 2\hat{j} + 3\hat{k}$ for vector form and point A as $(x_1, y_1, z_1) = (5, -2, 3)$ for Cartesian form.

The equation of line in vector form is $\vec{r} = \vec{a} + \lambda \vec{b}$

i.e., $\vec{r} = 5\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$

and equation of line in cartesian form is $\frac{x - 5}{5} = \frac{y + 2}{-2} = \frac{z - 3}{3}$.

9. Find vector and cartesian equations of the line that passes through the points (3, – 2, – 5) and (3, – 2, 6).

Sol. Vector Equation

Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A(3, – 2, – 5) and B(3, – 2, 6).

$\therefore$ $\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$\therefore$ A vector along the line = $\overrightarrow{AB}$ = position vector of point B – position vector of point A

$= \vec{b} - \vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k} - 3\hat{i} + 2\hat{j} + 5\hat{k} = 11\hat{k}$

Vector equation of the line is

$\boldsymbol{\vec{r} = \vec{a} + \lambda \overrightarrow{AB}}$ *i.e.,* $\vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + \lambda(11\hat{k})$

i.e., $\vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + 11\lambda\hat{k}$.

Note. Another vector equation for the same line is

$\vec{r} = \vec{b} + \lambda \overrightarrow{AB}$ *i.e.,* $\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k} + 11\lambda\hat{k}$.

Cartesian Equation

Direction ratios of line AB are $3 - 3, - 2 + 2, 6 + 5$

i.e., 0, 0, 11 | $x_2 - x_1, y_2 - y_1, z_2 - z_1$

$\therefore$ Equations of the line are $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$

i.e., $\dfrac{x-3}{0} = \dfrac{y+2}{0} = \dfrac{z+5}{11}$.

10. Find the angle between the following pairs of lines:

(*i*) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ **and**

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

(*ii*) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ **and**

$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$.

Sol. (*i*) **Given:** Equation of one line is

$\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$

Comparing with $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$,

$\vec{a_1} = 2\hat{i} - 5\hat{j} + \hat{k}$ and a vector along the line is

$\vec{b_1} = 3\hat{i} + 2\hat{j} + 6\hat{k}$ ...(*i*)

(It may be noted that vector $\vec{a_1}$ is the position vector of a point on the line and not a vector along the line).

Given: Equation of second line is

$$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$$

Comparing with $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ we have

$\vec{a_2} = 7\hat{i} - 6\hat{k}$ and a vector along the second line is

$\vec{b_2} = \hat{i} + 2\hat{j} + 2\hat{k}$ ...(*ii*)

Let θ be the angle between the two lines.

We know that $\cos\theta = \dfrac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$

$$= \frac{3(1) + 2(2) + 6(2)}{\sqrt{9+4+36}\sqrt{1+4+4}} = \frac{3+4+12}{\sqrt{49}\sqrt{9}}$$

$$\cos\theta = \frac{19}{7(3)} = \frac{19}{21} \qquad \therefore \qquad \theta = \cos^{-1}\frac{19}{21}.$$

(*ii*) Comparing the equations of the two given lines with

$\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ we have

$\vec{b_1} = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b_2} = 3\hat{i} - 5\hat{j} - 4\hat{k}$.

Let θ be the angle between the two lines

$$\therefore \quad \cos\theta = \frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}||\vec{b_2}|} = \frac{(1)(3)+(-1)(-5)+(-2)(-4)}{\sqrt{1+1+4}\sqrt{9+25+16}}$$

$$= \frac{3+5+8}{\sqrt{6}\sqrt{50}}$$

$$= \frac{16}{\sqrt{300}} = \frac{16}{\sqrt{3\times100}} = \frac{16}{10\sqrt{3}}$$

or $\cos\theta = \dfrac{8}{5\sqrt{3}} \quad \therefore \quad \theta = \cos^{-1}\dfrac{8}{5\sqrt{3}}$.

11. Find the angle between the following pairs of lines:

(*i*) $\dfrac{x-2}{2} = \dfrac{y-1}{5} = \dfrac{z+3}{-3}$ **and** $\dfrac{x+2}{-1} = \dfrac{y-4}{8} = \dfrac{z-5}{4}$

(*ii*) $\dfrac{x}{2} = \dfrac{y}{2} = \dfrac{z}{1}$ **and** $\dfrac{x-5}{4} = \dfrac{y-2}{1} = \dfrac{z-3}{8}$.

Sol. (*i*) **Given:** Equation of one line is $\dfrac{x-2}{2} = \dfrac{y-1}{5} = \dfrac{z+3}{-3}$

(It is standard form because coefficients of x, y, z are unity each)

$\therefore$ Denominators 2, 5, – 3 are direction ratios of this line *i.e.,* a vector along the line is

$$\vec{b_1} = (2, 5, -3) = 2\hat{i} + 5\hat{j} - 3\hat{k} \qquad ...(i)$$

Given: Equation of second line is $\dfrac{x+2}{-1} = \dfrac{y-4}{8} = \dfrac{z-5}{4}$

(It is also standard form)

$\therefore$ Denominators – 1, 8, 4 are direction ratios of this line *i.e.,* a vector along the line is

$$\vec{b_2} = (-1, 8, 4) = -\hat{i} + 8\hat{j} + 4\hat{k} \qquad ...(ii)$$

Let θ be the angle between the two given lines.

We know that $\cos\theta = \dfrac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$

$$= \frac{2(-1)+5(8)+(-3)(4)}{\sqrt{4+25+9}\sqrt{1+64+16}} \qquad \text{(From } (i) \text{ and } (ii))$$

$$\Rightarrow \cos\theta = \frac{-2+40-12}{\sqrt{38}\sqrt{81}} = \frac{26}{9\sqrt{38}} \Rightarrow \theta = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right).$$

(*ii*) **Given:** Equation of one line is

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1} \qquad \text{(Standard Form)}$$

$\therefore$ Denominators 2, 2, 1 are direction ratios of this line *i.e.,* a vector along this line is

$$\vec{b_1} = (2, 2, 1) = 2\hat{i} + 2\hat{j} + \hat{k} \qquad ...(i)$$

Given: Equation of second line is

$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8} \qquad \text{(Standard Form)}$$

$\therefore$ Denominators 4, 1, 8 are direction ratios of this line *i.e.,* a vector along this line is

$$\vec{b_2} = (4, 1, 8) = 4\hat{i} + \hat{j} + 8\hat{k} \qquad ...(ii)$$

Let θ be the angle between the two lines.

We know that $\cos\theta = \dfrac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$

$$= \frac{2(4)+2(1)+1(8)}{\sqrt{4+4+1}\sqrt{16+1+64}} = \frac{8+2+8}{\sqrt{9}\sqrt{81}} = \frac{18}{3\times 9} = \frac{2}{3}$$

$$\therefore \qquad \theta = \cos^{-1}\frac{2}{3}.$$

12. Find the values of p so that the lines $\dfrac{1-x}{3} = \dfrac{7y-14}{2p} = \dfrac{z-3}{2}$ and $\dfrac{7-7x}{3p} = \dfrac{y-5}{1} = \dfrac{6-z}{5}$ are at right angles.

Sol. Let us put the equations of these lines in standard form (*i.e.,* making coeff. of x, y, z unity in each of them)

The first line can be written as

$$-\frac{(x-1)}{3} = \frac{7(y-2)}{2p} = \frac{z-3}{2} \quad \text{or} \quad \frac{x-1}{-3} = \frac{y-2}{\left(\frac{2p}{7}\right)} = \frac{z-3}{2}$$

$\therefore$ direction ratios of this line are $-3, \dfrac{2p}{7}, 2 = a_1, b_1, c_1$.

And the equation of 2nd line can be written as

$$\frac{-7(x-1)}{3p} = \frac{y-5}{1} = \frac{-(z-6)}{5} \quad \text{or} \quad \frac{x-1}{-\frac{3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$$

$\therefore$ The direction ratios of 2nd line are $\frac{-3p}{7}$, 1, – 5 = a_2, b_2, c_2.

$\because$ The two lines are perpendicular, therefore

$$\boldsymbol{a_1a_2 + b_1b_2 + c_1c_2 = 0}$$

$$\Rightarrow -3\left(\frac{-3p}{7}\right) + \left(\frac{2p}{7}\right)(1) + 2 \times (-5) = 0$$

$$\Rightarrow \frac{9p}{7} + \frac{2p}{7} - 10 = 0 \quad \Rightarrow \frac{11p}{7} = 10 \quad \Rightarrow p = \frac{70}{11}.$$

13. Show that the lines $\boldsymbol{\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}}$ and $\boldsymbol{\frac{x}{1} = \frac{y}{2} = \frac{z}{3}}$ are perpendicular to each other.

Sol. Given: Equation of one line is

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1} \qquad \text{(Standard form)}$$

Direction ratios of this line are its denominators 7, – 5, 1

$$= a_1, b_1, c_1 \quad (\Rightarrow \vec{b_1} = 7\hat{i} - 5\hat{j} + \hat{k})$$

Given: Equation of second line is $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ (Standard form)

Direction ratios of this line are its denominators 1, 2, 3

$$= a_2, b_2, c_2 \quad (\Rightarrow \vec{b_2} = \hat{i} + 2\hat{j} + 3\hat{k})$$

$$\vec{b_1} \cdot \vec{b_2} = a_1a_2 + b_1b_2 + c_1c_2 = 7(1) + (-5)(2) + 1(3)$$
$$= 7 - 10 + 3 = 0$$

$\therefore$ The two given lines are perpendicular to each other.

14. Find the shortest distance between the lines

$$\boldsymbol{\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})} \textbf{ and}$$
$$\boldsymbol{\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k}).}$$

Sol. Comparing the equations of the given lines with

$$\vec{r} = \vec{a_1} + \lambda\vec{b_1} \text{ and } \vec{r} = \vec{a_2} + \mu\vec{b_2}, \text{ we have}$$

$\vec{a_1} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b_1} = \hat{i} - \hat{j} + \hat{k}$

and $\vec{a_2} = 2\hat{i} - \hat{j} - \hat{k}$, $\vec{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$

We know **that the S.D. between the two skew lines is given by**

$$d = \frac{|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|} \qquad ...(i)$$

Now $\vec{a_2} - \vec{a_1} = (2\hat{i} - \hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - 3\hat{j} - 2\hat{k}$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = (-2-1)\hat{i} - (2-2)\hat{j} + (1+2)\hat{k}$$

$$= -3\hat{i} - 0\hat{j} + 3\hat{k}$$

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(-3)^2 + 0^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$

Again $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (\hat{i} - 3\hat{j} - 2\hat{k}) \cdot (-3\hat{i} + 3\hat{k})$

$$= (1)(-3) + (-3)(0) + (-2)(3) = -9$$

Putting these values in eqn. (i),

$$\text{S.D. } (d) = \frac{|-9|}{3\sqrt{2}} = \frac{9}{3\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

15. Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}.$$

Sol. Equation of one line is $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$

Comparing with $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$, we have

$$x_1 = -1, y_1 = -1, z_1 = -1; a_1 = 7, b_1 = -6, c_1 = 1$$

$\therefore$ vector form of this line is $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$

where $\vec{a_1} = (x_1, y_1, z_1) = (-1, -1, -1) = -\hat{i} - \hat{j} - \hat{k}$

and $\vec{b_1} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k} = 7\hat{i} - 6\hat{j} + \hat{k}$

Equation of second line is $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

Comparing with $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$; we have

$$x_2 = 3,\ y_2 = 5,\ z_2 = 7;\ a_2 = 1,\ b_2 = -2,\ c_2 = 1$$

$\therefore$ vector form of this second line is $\vec{r} = \vec{a_2} + \mu \vec{b_2}$

where $\vec{a_2} = (x_2, y_2, z_2) = (3, 5, 7) = 3\hat{i} + 5\hat{j} + 7\hat{k}$

and $\vec{b_2} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k} = \hat{i} - 2\hat{j} + \hat{k}$

we know that S.D. between two skew lines is given by

$$d = \frac{|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|} \quad \ldots (i)$$

Now $\vec{a_2} - \vec{a_1} = 3\hat{i} + 5\hat{j} + 7\hat{k} - (-\hat{i} - \hat{j} - \hat{k})$

$$= 4\hat{i} + 6\hat{j} + 8\hat{k}$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$= (-6 + 2)\hat{i} - (7 - 1)\hat{j} + (-14 + 6)\hat{k}$

$= -4\hat{i} - 6\hat{j} - 8\hat{k}$

$\therefore |\vec{b_1} \times \vec{b_2}| = \sqrt{(-4)^2 + (-6)^2 + (-8)^2}$

$$= \sqrt{16 + 36 + 64} = \sqrt{116}$$

again $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 4(-4) + 6(-6) + 8(-8)$

$$= -16 - 36 - 64 = -116$$

Putting these values in eqn. (i),

S.D. $(d) = \dfrac{|-116|}{\sqrt{116}} = \dfrac{116}{\sqrt{116}} = \sqrt{116}$

$$= \sqrt{4 \times 29} = 2\sqrt{29}$$

16. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \text{ and}$$

$$\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k}).$$

Sol. Equation of the first line is

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) = \vec{a_1} + \lambda\vec{b_1}$$

Comparing, $\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b_1} = \hat{i} - 3\hat{j} + 2\hat{k}$

Equation of second line is

$$\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k}) = \vec{a_2} + \mu\vec{b_2}$$

Comparing $\vec{a_2} = 4\hat{i} + 5\hat{j} + 6\hat{k}$ and $\vec{b_2} = 2\hat{i} + 3\hat{j} + \hat{k}$

We know that length of S.D. between two (skew) lines is

$$\frac{|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|} \qquad ...(i)$$

Now $\vec{a_2} - \vec{a_1} = 4\hat{i} + 5\hat{j} + 6\hat{k} - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= 4\hat{i} + 5\hat{j} + 6\hat{k} - \hat{i} - 2\hat{j} - 3\hat{k} = 3\hat{i} + 3\hat{j} + 3\hat{k}$

Again $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

Expanding along first row,

$\vec{b_1} \times \vec{b_2} = \hat{i}(-3-6) - \hat{j}(1-4) + \hat{k}(3+6) = -9\hat{i} + 3\hat{j} + 9\hat{k}$

$\therefore \quad (\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 3(-9) + 3(3) + 3(9)$

$= -27 + 9 + 27 = 9$

and $\quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(-9)^2 + (3)^2 + (9)^2} = \sqrt{81 + 9 + 81}$

$= \sqrt{171} = \sqrt{9 \times 19} = 3\sqrt{19}$

Putting these values in (*i*),

length of shortest distance $= \dfrac{|9|}{3\sqrt{19}} = \dfrac{9}{3\sqrt{19}} = \dfrac{3}{\sqrt{19}}$.

17. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - 2t)\hat{k} \text{ and}$$

$$\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}.$$

Sol. The first line is $\vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - 2t)\hat{k}$

$= \hat{i} - t\hat{i} + t\hat{j} - 2\hat{j} + 3\hat{k} - 2t\hat{k}$

$= (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}) = \vec{a}_1 + t\vec{b}_1$

Comparing $\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b}_1 = -\hat{i} + \hat{j} - 2\hat{k}$

The second line is $\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}$

$= s\hat{i} + \hat{i} + 2s\hat{j} - \hat{j} - 2s\hat{k} - \hat{k}$

$= (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}) = \vec{a}_2 + s\vec{b}_2$

Comparing $\vec{a}_2 = \hat{i} - \hat{j} - \hat{k}$, $\vec{b}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$

We know that the S.D. between the two (skew) lines is given by

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) . (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|} \quad ...(i)$$

Now $\vec{a}_2 - \vec{a}_1 = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \hat{j} - 4\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$= (-2 + 4)\hat{i} - (2 + 2)\hat{j} + (-2 - 1)\hat{k} = 2\hat{i} - 4\hat{j} - 3\hat{k}$

$\therefore |\vec{b}_1 \times \vec{b}_2| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{29}$

Again $(\vec{a}_2 - \vec{a}_1) . (\vec{b}_1 \times \vec{b}_2) = (\hat{j} - 4\hat{k}) . (2\hat{i} - 4\hat{j} - 3\hat{k})$

$= (0)(2) + (1)(-4) + (-4)(-3) = 8$

Putting these values in eqn. (*i*),

$$\text{S.D. } (d) = \frac{|8|}{\sqrt{29}} = \frac{8}{\sqrt{29}}.$$

Exercise 11.3

Note: Formula for question numbers 1 and 2.
If p is the length of perpendicular from the origin to a plane and $\hat{n}$ is a unit normal vector to the plane, then equation of the plane is $\vec{r} \cdot \hat{n} = p$ (where of course p being length is > 0).

1. In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a) $z = 2$ **(b) $x + y + z = 1$**

(c) $2x + 3y - z = 5$ **(d) $5y + 8 = 0$**

Sol. (a) **Given:** Equation of the plane is $z = 2$

Let us first reduce it to vector form $\vec{r} \cdot \vec{n} = d$ where $d > 0$

or $0x + 0y + 1z = 2$ (Here $d = 2 > 0$)

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (0\hat{i} + 0\hat{j} + \hat{k}) = 2$

$(\because a_1a_2 + b_1b_2 + c_1c_2 = (a_1\hat{i} + b_1\hat{j} + c_1\hat{k}) \cdot (a_2\hat{i} + b_2\hat{j} + c_2\hat{k}))$

$\Rightarrow \vec{r} \cdot \vec{n} = 2$ where we know that

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ = (Position vector of point P(x, y, z))

and here $\vec{n} = 0\hat{i} + 0\hat{j} + \hat{k}$

Now let us reduce $\vec{r} \cdot \vec{n} = d$ to $\vec{r} \cdot \hat{n} = p$

Dividing both sides by $|\vec{n}|$, $\dfrac{\vec{r} \cdot \vec{n}}{|\vec{n}|} = 2$

i.e., $\vec{r} \cdot \hat{n} = 2 = p$ where $\hat{n} = \dfrac{\vec{n}}{|\vec{n}|} = \dfrac{0\hat{i} + 0\hat{j} + \hat{k}}{\sqrt{0+0+1} = 1}$

i.e., $\hat{n} = 0\hat{i} + 0\hat{j} + \hat{k}$ and $p = 2$

$\therefore$ By definition, direction cosines of normal to the plane are coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{n}$ *i.e.,* 0, 0, 1 and length of perpendicular from the origin to the plane is $p = 2$.

(*b*) **Given:** Equation of the plane is $x + y + z = 1$

$\Rightarrow$ $1x + 1y + 1z = 1$ (Here $d = 1 > 0$)

$\Rightarrow$ $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$

i.e., $\vec{r} \cdot \vec{n} = 1$ where $\vec{n} = \hat{i} + \hat{j} + \hat{k}$

Dividing both sides by $|\vec{n}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$, we have

$$\vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{1}{|\vec{n}|}$$

i.e., $\vec{r} \cdot \hat{n} = \frac{1}{\sqrt{3}} = p$ where $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\hat{i} + \hat{j} + \hat{k}}{|\vec{n}| = \sqrt{3}}$

i.e., $\hat{n} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$ and $p = \frac{1}{\sqrt{3}}$

$\therefore$ By definition, direction cosines of the normal to the plane are the coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{n}$ *i.e.,* $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and length of perpendicular from the origin to the plane is $p = \frac{1}{\sqrt{3}}$.

(*c*) **Given:** Equation of the plane is $2x + 3y - z = 5$

$\Rightarrow$ $2x + 3y + (-1)z = 5$ (Here $d = 5 > 0$)

$\Rightarrow$ $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 5$

i.e., $\vec{r} \cdot \vec{n} = 5$ where $\vec{n} = 2\hat{i} + 3\hat{j} - \hat{k}$

Dividing both sides by $|\vec{n}| = \sqrt{4 + 9 + 1} = \sqrt{14}$,

we have $\vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{5}{|\vec{n}|}$

i.e., $\vec{r} \cdot \hat{n} = \frac{5}{\sqrt{14}} = p$ where $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} + 3\hat{j} - \hat{k}}{\sqrt{4 + 9 + 1} = \sqrt{14}}$

i.e., $\hat{n} = \frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} - \frac{1}{\sqrt{14}}\hat{k}$

$\therefore$ By definition, direction cosines of the normal to the plane are coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ in $\hat{n}$ *i.e.*, $\frac{2}{\sqrt{14}}$, $\frac{3}{\sqrt{14}}$, $\frac{-1}{\sqrt{14}}$ and length of perpendicular from the origin to the plane is $\frac{5}{\sqrt{14}}$.

(*d*) **Given:** Equation of the plane is

$$5y + 8 = 0 \quad \text{or} \quad 5y = -8$$

Dividing both sides by -1 to make R.H.S. ($= d$) as positive,

$-5y = 8$ or $0x + (-5)y + 0z = 8$ | Now $d = 8 > 0$

$\Rightarrow$ $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (0\hat{i} - 5\hat{j} + 0\hat{k}) = 8$

i.e., $\vec{r} \cdot \vec{n} = 8$ where $\vec{n} = 0\hat{i} - 5\hat{j} + 0\hat{k}$

Dividing both sides by $|\vec{n}| = \sqrt{0^2 + (-5)^2 + 0^2}$

i.e., $|\vec{n}| = \sqrt{25} = 5$

we have $\vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{8}{5}$ *i.e.*, $\vec{r} \cdot \hat{n} = \frac{8}{5} = p$

where $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{0\hat{i} - 5\hat{j} + 0\hat{k}}{5}$

$= \frac{0}{5}\hat{i} - \frac{5}{5}\hat{j} + \frac{0}{5}\hat{k} = 0\hat{i} - \hat{j} + 0\hat{k}$ and $p = \frac{8}{5}$.

$\therefore$ By definition, direction cosines of the normal to the plane are coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ in $\hat{n}$ *i.e.*, 0, -1, 0 and length of perpendicular from the origin to the plane is $\frac{8}{5}$.

2. Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector

$$\mathbf{3\hat{i} + 5\hat{j} - 6\hat{k}}.$$

Sol. Here $\vec{n} = 3\hat{i} + 5\hat{j} - 6\hat{k}$

$\therefore$ The unit vector perpendicular to plane is

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{(3)^2 + (5)^2 + (-6)^2}} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}$$

Also $\quad p = 7$ (given)

Hence, the equation of the required plane is $\vec{r} \cdot \hat{n} = p$

i.e., $$\vec{r} \cdot \frac{(3\hat{i} + 5\hat{j} - 6\hat{k})}{\sqrt{70}} = 7$$

or $$\vec{r} \cdot (3\hat{i} + 5\hat{j} - 6\hat{k}) = 7\sqrt{70}.$$

3. Find the Cartesian equation of the following planes:

(*a*) $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$ (*b*) $\vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$

(*c*) $\vec{r} \cdot [(s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}] = 15.$

Sol. (*a*) Vector equation of the plane is

$$\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2 \quad ...(i)$$

Putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (*i*) (we know that in 3-D, $\vec{r}$ is the position vector of any point, P(x, y, z)),

Cartesian equation of the plane is

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$$

$\Rightarrow \quad x(1) + y(1) + z(-1) = 2 \quad \Rightarrow \quad x + y - z = 2.$

(*b*) We know that $\vec{r}$ is the position vector of any arbitrary point P(x, y, z) on the plane.

$\therefore \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$

$\therefore \quad \vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$ (given)

$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$

$\Rightarrow \quad 2x + 3y - 4z = 1$

which is the required Cartesian equation of the plane.

(*c*) Vector equation of the plane is

$$\vec{r} \cdot [(s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}] = 15 \quad ...(i)$$

We know that $\vec{r}$ is the position vector of any point P(x, y, z) on plane (*i*).

$\therefore \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (*i*), Cartesian equation of the required plane is

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot [(s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}] = 15$$

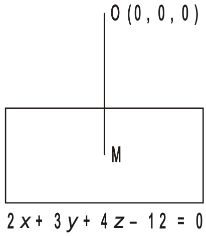
i.e., $$x(s - 2t) + y(3 - t) + z(2s + t) = 15.$$

4. In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.

(*a*) $2x + 3y + 4z - 12 = 0$ **(*b*) $3y + 4z - 6 = 0$**

(*c*) $x + y + z = 1$

(*d*) $5y + 8 = 0$.



Sol. (*a*) **Given:** Equation of the plane is

$2x + 3y + 4z - 12 = 0$ ...(*i*)

Given point is O(0, 0, 0)

Let M be the foot of perpendicular drawn from the origin (0, 0, 0) to plane (*i*).

$\therefore$ By definition, direction ratios of perpendicular OM to plane (*i*) are coefficients of x, y, z in (*i*) *i.e.*, 2, 3, 4 = a, b, c.

$\therefore$ Equations of perpendicular OM are

$$\frac{x-0}{2} = \frac{y-0}{3} = \frac{z-0}{4} = \lambda \text{(say)} \quad \left| \; \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \right.$$

$$\Rightarrow \frac{x}{2} = \frac{y}{3} = \frac{z}{4} = \lambda \qquad \Rightarrow \frac{x}{2} = \lambda, \frac{y}{3} = \lambda \text{ and } \frac{z}{4} = \lambda$$

$\Rightarrow$ $x = 2\lambda, y = 3\lambda, z = 4\lambda$

$\therefore$ Point M of this line OM is M$(2\lambda, 3\lambda, 4\lambda)$ ...(*ii*)

for some real λ.

But point M lies on plane (*i*)

Putting $x = 2\lambda, y = 3\lambda, z = 4\lambda$ in (*i*), we have

$$2(2\lambda) + 3(3\lambda) + 4(4\lambda) - 12 = 0$$

$\Rightarrow$ $4\lambda + 9\lambda + 16\lambda = 12 \Rightarrow 29\lambda = 12$

$\Rightarrow$ $\lambda = \frac{12}{29}$

Putting $\lambda = \frac{12}{29}$ in (*i*), foot of perpendicular M$\left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$.

(*b*) For figure, see figure of part (*a*).

Given: Equation of the plane is $3y + 4z - 6 = 0$ ...(*i*)

Given point is O(0, 0, 0)

Let M be the foot of perpendicular drawn from the origin to plane (*i*).

$\therefore$ By definition direction ratios of perpendicular OM to plane (*i*) are coefficients of x, y, z in (*i*) *i.e.*, 0, 3, 4 = a, b, c.

$\therefore$ Equations of perpendicular OM are

$$\frac{x-0}{0} = \frac{y-0}{3} = \frac{z-0}{4} = \lambda \text{(say)} \quad \left| \; \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \right.$$

$$\Rightarrow \frac{x}{0} = \frac{y}{3} = \frac{z}{4} = \lambda \text{(say)} \qquad \Rightarrow \frac{x}{0} = \lambda, \frac{y}{3} = \lambda \text{ and } \frac{z}{4} = \lambda$$

$\Rightarrow \qquad x = 0, y = 3\lambda, z = 4\lambda$

$\therefore$ Point M of this line OM is M(0, 3λ, 4λ) ...(*ii*)

for some real λ.

But point M lies on plane (*i*)

Putting $x = 0$, $y = 3\lambda$, $z = 4\lambda$ in (*i*), we have

$$3(3\lambda) + 4(4\lambda) - 6 = 0 \quad \text{or} \quad 9\lambda + 16\lambda = 6$$

$$\Rightarrow \qquad 25\lambda = 6 \quad \Rightarrow \quad \lambda = \frac{6}{25}$$

Putting $\lambda = \frac{6}{25}$ in (*ii*), the required foot M of perpendicular is $\left(0, \frac{18}{25}, \frac{24}{25}\right)$.

(*c*) For figure, see figure of part (*a*).

Given: Equation of the plane is

$$x + y + z = 1 \qquad ...(i)$$

Given point is O(0, 0, 0)

Let M be the foot of perpendicular drawn from the origin (0, 0, 0) to plane (*i*).

$\therefore$ By definition direction ratios of perpendicular OM to plane (*i*) are coefficients of x, y, z in (*i*) *i.e.,* $1, 1, 1 = a, b, c$.

$\therefore$ Equations of perpendicular OM are

$$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{1} \qquad \left| \; \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \right.$$

i.e., $\qquad \frac{x}{1} = \frac{y}{1} = \frac{z}{1} = \lambda(\text{say}) \quad \therefore \quad \frac{x}{1} = \lambda, \; \frac{y}{1} = \lambda$ and $\frac{z}{1} = \lambda$

$\Rightarrow \qquad x = \lambda, y = \lambda, z = \lambda$

$\therefore$ Point M of line OM is M($\lambda, \lambda, \lambda$) ...(*ii*)

for some real λ.

But point M lies on plane (*i*)

Putting $x = \lambda$, $y = \lambda$, $z = \lambda$ in (*i*), we have

$$\lambda + \lambda + \lambda = 1 \quad \Rightarrow \quad 3\lambda = 1 \quad \Rightarrow \quad \lambda = \frac{1}{3}$$

Putting $\lambda = \frac{1}{3}$ in (*ii*), required foot M of perpendicular is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

(*d*) For figure, see figure of part (*a*).

Given: Equation of the plane is

$$5y + 8 = 0 \qquad ...(i)$$

Given point is O(0, 0, 0)

Let M be the foot of perpendicular drawn from the origin (0, 0, 0) to plane (*i*).

∴ By definition, direction ratios of perpendicular OM to plane (*i*) are coefficients of x, y, z in (*i*) *i.e.,* $0, 5, 0 = a, b, c$.

∴ Equations of perpendicular OM are

$$\frac{x-0}{0} = \frac{y-0}{5} = \frac{z-0}{0} \qquad \left| \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \right.$$

i.e., $\frac{x}{0} = \frac{y}{5} = \frac{z}{0} = \lambda$(say) ∴ $\frac{x}{0} = \lambda, \frac{y}{5} = \lambda$ and $\frac{z}{0} = \lambda$

$\Rightarrow \qquad x = 0, y = 5\lambda, z = 0$

∴ Point M of line OM is M(0, 5λ, 0) ...(*ii*)

for some real λ.

But point M lies on plane (*i*)

Putting $x = 0$, $y = 5\lambda$ and $z = 0$ in (*i*), we have

$$5(5\lambda) + 8 = 0 \quad \text{or} \quad 25\lambda = -8$$

$$\Rightarrow \qquad \lambda = -\frac{8}{25}$$

Putting $\lambda = -\frac{8}{25}$ in (*i*), required foot M of perpendicular is

$$\left(0, \frac{-40}{25}, 0\right) = \left(0, \frac{-8}{5}, 0\right).$$

5. Find the vector and cartesian equations of the planes

(*a*) that passes through the point (1, 0, – 2) and the normal to the plane is $\hat{i} + \hat{j} - \hat{k}$.

(*b*) that passes through the point (1, 4, 6) and the normal vector to the plane is $\hat{i} - 2\hat{j} + \hat{k}$.

Sol. (*a*) *Vector form of equation of the plane*

The given point on the plane is (1, 0, – 2)

∴ The position vector of the given point is

$$\vec{a} = (1, 0, -2) = \hat{i} + 0\hat{j} - 2\hat{k} = \hat{i} - 2\hat{k}$$

Also Given: Normal vector to the plane is

$$\vec{n} = \hat{i} + \hat{j} - \hat{k}$$

∴ Vector equation of the required plane is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0 \quad i.e., \quad \vec{r} \cdot \vec{n} - \vec{a} \cdot \vec{n} = 0$$

i.e., $$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

Putting values of $\vec{a}$ and $\vec{n}$,

$$\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = (\hat{i} - 2\hat{k}) \cdot (\hat{i} + \hat{j} - \hat{k})$$

i.e., $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 1(1) + 0(1) + (-2)(-1) = 1 + 2 = 3$

i.e., $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 3$

Cartesian form of equation of the plane

The plane passes through the point $(1, 0, -2) = (x_1, y_1, z_1)$

Normal vector to the plane is $\vec{n} = \hat{i} + \hat{j} - \hat{k}$

$\therefore$ Direction ratios of normal to the plane are coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{n}$ *i.e.,* $1, 1, -1$.

$\therefore$ Cartesian equation of the required plane is

$$\boldsymbol{a(x - x_1) + b(y - y_1) + c(z - z_1) = 0}$$

or $\quad 1(x - 1) + 1(y - 0) - (z + 2) = 0$

i.e., $\quad x - 1 + y - z - 2 = 0$

i.e., $\quad x + y - z = 3.$

(*b*) **Vector form of the equation of the plane**

The given point on the plane is (1, 4, 6).

$\therefore$ The position vector of the given point is

$$\vec{a} = (1, 4, 6) = \hat{i} + 4\hat{j} + 6\hat{k}$$

Also **Given:** normal vector to the plane is $\vec{n} = \hat{i} - 2\hat{j} + \hat{k}$.

$\therefore$ Equation of the plane is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

or $\quad \vec{r} \cdot \vec{n} - \vec{a} \cdot \vec{n} = 0 \quad$ *i.e.,* $\quad \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

Putting values of $\vec{a}$ and $\vec{n}$

$$\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = (\hat{i} + 4\hat{j} + 6\hat{k}) \cdot (\hat{i} - 2\hat{j} + \hat{k})$$

$$= 1 - 8 + 6 = -1 \qquad \text{...}(i)$$

Cartesian Form

The plane passes through the point $(1, 4, 6) = (x_1, y_1, z_1)$.

Normal vector to the plane is $\vec{n} = \hat{i} - 2\hat{j} + \hat{k}$.

$\therefore$ D.R.'s of the normal to the plane are coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ in $\vec{n}$

i.e., $\quad 1, -2, 1 = a, b, c$

$\therefore$ Equation of the required plane is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

or $\quad 1(x - 1) - 2(y - 4) + 1(z - 6) = 0$

or $\quad x - 1 - 2y + 8 + z - 6 = 0$

or $\quad x - 2y + z + 1 = 0$

Alternatively for Cartesian form

From eqn. (*i*), $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} - 2\hat{j} + \hat{k}) = -1$

or $\quad x - 2y + z = -1$ or $x - 2y + z + 1 = 0.$

6. Find the equations of the planes that passes through three points:

(*a*) (1, 1, – 1), (6, 4, – 5), (– 4, – 2, 3)

(*b*) (1, 1, 0), (1, 2, 1), (– 2, 2, – 1)

Sol. We know that through three collinear points A, B, C *i.e.,* through a straight line, we can pass an infinite number of planes.

(*a*) The three given points are

$$A(1, 1, -1),\ B(6, 4, -5),\ C(-4, -2, 3)$$

Let us examine whether these points are collinear.

Direction ratios of line AB are

$6 - 1, 4 - 1, -5 + 1 \qquad | \; x_2 - x_1, y_2 - y_1, z_2 - z_1$

$= 5, 3, -4 = a_1, b_1, c_1$

Again direction ratios of line BC are

$-4 - 6, -2 - 4, 3 - (-5) = -10, -6, 8 = a_2, b_2, c_2$

Here $\quad \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \Rightarrow \dfrac{5}{-10} = \dfrac{3}{-6} = -\dfrac{4}{8}$

$\Rightarrow \quad -\dfrac{1}{2} = -\dfrac{1}{2} = -\dfrac{1}{2}$ which is true.

$\therefore$ Lines AB and BC are parallel.

But B is their common point.

$\therefore$ Points A, B and C are collinear and hence an infinite number of planes can be drawn through the three given collinear points.

(*b*) The three given points are

$$A(1, 1, 0) = (x_1, y_1, z_1),\ B(1, 2, 1) = (x_2, y_2, z_2)$$

and $C(-2, 2, -1) = (x_3, y_3, z_3)$

Let us examine whether these points are collinear.

Direction ratios of line AB are

$1 - 1, 2 - 1, 1 - 0 \qquad | \; x_2 - x_1, y_2 - y_1, z_2 - z_1$

i.e., $\quad 0, 1, 1 = a_1, b_1, c_1$

Direction ratios of line BC are

$-2 - 1, 2 - 2, -1 - 1 = -3, 0, -2 = a_2, b_2, c_2$

Here $\quad \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \Rightarrow \dfrac{0}{-3} = \dfrac{1}{0} = \dfrac{1}{-2}$

which is not true.

$\therefore$ Points A, B, C are not collinear.

$\therefore$ Equation of the unique plane passing through these three points A, B, C is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z-0 \\ 1-1 & 2-1 & 1-0 \\ -2-1 & 2-1 & -1-0 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

Expanding along first row,

$$(x - 1)(- 1 - 1) - (y - 1)(0 + 3) + z(0 + 3) = 0$$

$\Rightarrow$ $- 2(x - 1) - 3(y - 1) + 3z = 0$

$\Rightarrow$ $- 2x + 2 - 3y + 3 + 3z = 0$

$\Rightarrow$ $- 2x - 3y + 3z + 5 = 0$

$\Rightarrow$ $2x + 3y - 3z - 5 = 0$

or $2x + 3y - 3z = 5$

which is the equation of required plane.

7. Find the intercepts cut off by the plane $2x + y - z = 5$.

Sol. Equation of the plane is $2x + y - z = 5$

Dividing every term by 5, (to make R.H.S. 1)

$$\frac{2x}{5} + \frac{y}{5} - \frac{z}{5} = 1 \text{ or } \frac{x}{\left(\frac{5}{2}\right)} + \frac{y}{5} + \frac{z}{-5} = 1$$

Comparing with intercept form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, we have $a = \frac{5}{2}$, $b = 5$, $c = -5$ which are the intercepts cut off by the plane on x-axis, y-axis and z-axis respectively.

8. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOX plane.

Sol. We know that equation of ZOX plane is $y = 0$.

$\therefore$ Equation of any plane parallel to ZOX plane is $y = k$ ...(*i*)

($\because$ Equation of any plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$ *i.e.,* change only the constant term)

To find k. Plane (*i*) makes an intercept 3 on the y-axis ($\Rightarrow$ $x = 0$ and $z = 0$) *i.e.,* plane (*i*) passes through (0, 3, 0).

Putting $x = 0$, $y = 3$ and $z = 0$ in (*i*), $3 = k$.

Putting $k = 3$ in (*i*), equation of required plane is $y = 3$.

9. Find the equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point (2, 2, 1).

Sol. Equations of the given planes are

$$3x - y + 2z - 4 = 0 \quad \text{and} \quad x + y + z - 2 = 0$$

(Here R.H.S. of each equation is already zero)

We know that equation of any plane through the intersection of these two planes is

L.H.S. of plane I + λ(L.H.S. of plane II) = 0

i.e., $3x - y + 2z - 4 + \lambda(x + y + z - 2) = 0$ ...(*i*)

To find λ. Given: Required plane (*i*) passes through the point (2, 2, 1).

Putting $x = 2$, $y = 2$ and $z = 1$ in (*i*),

$$6 - 2 + 2 - 4 + \lambda(2 + 2 + 1 - 2) = 0$$

or $2 + 3\lambda = 0 \Rightarrow 3\lambda = -2 \Rightarrow \lambda = -\frac{2}{3}$

Putting $\lambda = -\frac{2}{3}$ in (*i*), equation of required plane is

$$3x - y + 2z - 4 - \frac{2}{3}(x + y + z - 2) = 0$$

$\Rightarrow$ $9x - 3y + 6z - 12 - 2x - 2y - 2z + 4 = 0$

$\Rightarrow$ $7x - 5y + 4z - 8 = 0.$

10. Find the vector equation of the plane passing through the intersection of the planes $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$, $\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and through the point (2, 1, 3).

Sol. Vector equation of first plane is

$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$ i.e $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$

i.e. $2x + 2y - 3z - 7 = 0$ (making R.H.S. zero) ...(*i*)

Vector equation of second plane is

$\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ i.e $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$

i.e. $2x + 5y + 3z - 9 = 0$ (making R.H.S. zero) ...(*ii*)

We know that equation of any plane passing through the line of intersection of planes (*i*) and (*ii*) is

L.H.S of (*i*) + λ L.H.S of (*ii*) = 0

i.e. $2x + 2y - 3z - 7 + \lambda\ (2x + 5y + 3z - 9) = 0$

i.e. $2x + 2y - 3z - 7 + 2\lambda x + 5\lambda y + 3\lambda z - 9\lambda = 0$

i.e. $(2 + 2\lambda)\ x + (2 + 5\lambda)\ y + (-3 + 3\lambda)\ z = 7 + 9\lambda$ ...(*iii*)

To find λ : Given plane (*iii*) passes through the point (2,1,3) putting $x = 2$, $y = 1$, $z = 3$ in (*iii*),

$(2 + 2\lambda)\ 2 + (2 + 5\lambda)\ 1 + (-3 + 3\lambda)\ 3 = 7 + 9\lambda$

or $4 + 4\lambda + 2 + 5\lambda - 9 + 9\lambda = 7 + 9\lambda$

$9\lambda - 3 = 7 \Rightarrow 9\lambda = 10 \Rightarrow \lambda = \frac{10}{9}$

Putting $\lambda = \frac{10}{9}$ in (*iii*), equation of required plane is

$$\left(2 + \frac{20}{9}\right)x + \left(2 + \frac{50}{9}\right)y + \left(-3 + \frac{30}{9}\right)z = 7 + 10$$

or $\frac{38}{9}x + \frac{68}{9}y + \frac{3}{9}z = 17$

Multiplying by L.C.M. = 9, $38x + 68y + 3z = 153$

$$\text{or } x\,(38) + y\,(68) + z\,(3) = 153$$

$$\text{or } (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (38\hat{i} + 68\hat{j} + 3\hat{k}) = 153$$

$$\text{i.e. } \vec{r} \cdot (38\hat{i} + 68\hat{j} + 3\hat{k}) = 153$$

which is the required vector equation of the plane.

11. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$.

Sol. Equations of the given planes are

$$x + y + z = 1 \quad \text{and} \quad 2x + 3y + 4z = 5$$

Making R.H.S. zero, equations of the planes are

$$x + y + z - 1 = 0 \quad \text{and} \quad 2x + 3y + 4z - 5 = 0.$$

We know that equation of any plane through the intersection of the two planes is

(L.H.S. of I) + λ(L.H.S. of II) = 0

i.e., $x + y + z - 1 + \lambda(2x + 3y + 4z - 5) = 0$ ...(*i*)

i.e., $x + y + z - 1 + 2\lambda x + 3\lambda y + 4\lambda z - 5\lambda = 0$

i.e., $(1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z - 1 - 5\lambda = 0$

Given: This plane is perpendicular to the plane

$$x - y + z = 0$$

$\therefore$ $\boldsymbol{a_1a_2 + b_1b_2 + c_1c_2 = 0}$

i.e., Product of coefficients of x + ... = 0

$\therefore$ $(1 + 2\lambda) - (1 + 3\lambda) + 1 + 4\lambda = 0$

$\Rightarrow$ $1 + 2\lambda - 1 - 3\lambda + 1 + 4\lambda = 0 \Rightarrow 3\lambda + 1 = 0 \Rightarrow 3\lambda = -1$

$\Rightarrow$ $\lambda = \frac{-1}{3}$

Putting $\lambda = \frac{-1}{3}$ in (*i*), equation of required plane is

$$x + y + z - 1 - \frac{1}{3}(2x + 3y + 4z - 5) = 0$$

Multiplying by L.C.M. = 3,

$3x + 3y + 3z - 3 - 2x - 3y - 4z + 5 = 0 \quad \Rightarrow x - z + 2 = 0.$

12. Find the angle between the planes whose vector equations are

$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5$ and $\vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3$.

Sol. Equation of one plane is

$$\vec{r} \, . \, (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5 \qquad ...(i)$$

Comparing (*i*) with $\vec{r} \, . \, \vec{n_1} = d_1$, we have

normal vector to plane (*i*) is $\vec{n_1} = 2\hat{i} + 2\hat{j} - 3\hat{k}$

Equation of second plane is $\vec{r} \, . \, (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3 \qquad ...(ii)$

Comparing (*ii*) with $\vec{r} \, . \, \vec{n_2} = d_2$, we have

normal vector to plane (*ii*) is $\vec{n_2} = 3\hat{i} - 3\hat{j} + 5\hat{k}$

Let θ be the **acute** angle between planes (*i*) and (*ii*).

$\therefore$ By definition, angle between normals $\vec{n_1}$ and $\vec{n_2}$ to planes (*i*) and (*ii*) is also θ.

$$\therefore \quad \cos\theta = \frac{|\vec{n_1} . \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|} = \frac{|2(3) + 2(-3) + (-3)5|}{\sqrt{4+4+9}\,\sqrt{9+9+25}}$$

$$= \frac{|6-6-15|}{\sqrt{17}\,\sqrt{43}} = \frac{|-15|}{\sqrt{17 \times 43}} = \frac{15}{\sqrt{731}} \quad \therefore \theta = \cos^{-1}\frac{15}{\sqrt{731}}.$$

13. In the following cases, determine whether the given planes are parallel or perpendicular and in case they are neither, find the angle between them.

(*a*) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

(*b*) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

(*c*) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

(*d*) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

(*e*) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$.

Sol. (*a*) Equations of the given planes are

$$7x + 5y + 6z + 30 = 0$$

$(a_1x + b_1y + c_1z + d_1 = 0)$

and $\quad 3x - y - 10z + 4 = 0 \quad (a_2x + b_2y + c_2z + d_2 = 0)$

Here $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ becomes $\frac{7}{3} = \frac{5}{-1} = \frac{6}{-10}$ which is not true.

$\therefore$ The two planes are not parallel.

Again $a_1a_2 + b_1b_2 + c_1c_2 = 21 - 5 - 60 = 21 - 65 = -44 \neq 0$

$\therefore$ Planes are not perpendicular.

Now let θ be the angle between the two planes.

$$\therefore \cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{|7(3) + 5(-1) + 6(-10)|}{\sqrt{(7)^2 + (5)^2 + (6)^2}\,\sqrt{(3)^2 + (-1)^2 + (-10)^2}}$$

$$= \frac{|21 - 5 - 60|}{\sqrt{49 + 25 + 36}\ \sqrt{9 + 1 + 100}} = \frac{|-44|}{\sqrt{110}\ \sqrt{110}}$$

$$= \frac{|-44|}{110} = \frac{44}{110} = \frac{2}{5} \quad \therefore \quad \theta = \cos^{-1}\left(\frac{2}{5}\right).$$

(b) Equations of the given planes are

$$2x + y + 3z - 2 = 0 \quad (a_1x + b_1y + c_1z + d_1 = 0)$$

and $x - 2y + 5 = 0$ *i.e.,* $x - 2y + 0.z + 5 = 0$

$$(a_2x + b_2y + c_2z + d_2 = 0)$$

Are these planes parallel?

Here $\underset{\downarrow}{\dfrac{a_1}{a_2}} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \Rightarrow \dfrac{2}{1} = \dfrac{1}{-2} = \dfrac{3}{0}$ which is not true.

(Ratio of coefficients of x in equations of two planes)

$\therefore$ The given planes are not parallel.

Are these planes perpendicular?

$\underset{\downarrow}{a_1a_2} + b_1b_2 + c_1c_2 = 2(1) + 1(-2) + 3(0) = 2 - 2 + 0 = 0$

(Product of coefficients of x)

$\therefore$ Given planes are perpendicular.

(c) Equations of the given planes are

$$2x - 2y + 4z + 5 = 0 \quad (a_1x + b_1y + c_1z + d_1 = 0)$$

and $3x - 3y + 6z - 1 = 0 \quad (a_2x + b_2y + c_2z + d_2 = 0)$

Are these planes parallel?

Here $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \Rightarrow \dfrac{2}{3} = \dfrac{-2}{-3} = \dfrac{4}{6} \Rightarrow \dfrac{2}{3} = \dfrac{2}{3} = \dfrac{2}{3}$

which is true.

$\therefore$ The given planes are parallel.

(d) Equations of the given planes are

$$2x - y + 3z - 1 = 0 \quad (a_1x + b_1y + c_1z + d_1 = 0)$$

and $2x - y + 3z + 3 = 0 \quad (a_2x + b_2y + c_2z + d_2 = 0)$

Are these planes parallel?

Here $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \Rightarrow \dfrac{2}{2} = \dfrac{-1}{-1} = \dfrac{3}{3} \Rightarrow 1 = 1 = 1$

which is true.

$\therefore$ The given planes are parallel.

(e) Equations of the given planes are

$$4x + 8y + z - 8 = 0 \quad (a_1x + b_1y + c_1z + d_1 = 0)$$

and $y + z - 4 = 0$ *i.e.,* $0x + y + z - 4 = 0$

$$(a_2x + b_2y + c_2z + d_2 = 0)$$

Are these planes parallel?

Here $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \Rightarrow \dfrac{4}{0} = \dfrac{8}{1} = \dfrac{1}{1}$ which is not true.

$\therefore$ The given planes are not parallel.

Are these planes perpendicular?

Here $a_1a_2 + b_1b_2 + c_1c_2 = 4(0) + 8(1) + 1(1)$

$= 0 + 8 + 1 = 9 \neq 0$ ...(*i*)

$\therefore$ The given planes are not perpendicular.

To find the **(acute)** angle θ between the given planes.

$$\therefore \cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{|\vec{n_1} . \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}$$

$$= \frac{|4(0) + 8(1) + 1(1)|}{\sqrt{16 + 64 + 1}\sqrt{0^2 + 1^2 + 1^2}} = \frac{|8 + 1|}{\sqrt{81}\sqrt{2}}$$

$$= \frac{9}{9\sqrt{2}} = \frac{1}{\sqrt{2}} = \cos 45° \qquad \therefore \theta = 45°.$$

14. In the following cases find the distances of each of the given points from the corresponding given plane.

	Point	**Plane**
(*a*)	**(0, 0, 0)**	$3x - 4y + 12z = 3$
(*b*)	**(3, – 2, 1)**	$2x - y + 2z + 3 = 0$
(*c*)	**(2, 3, – 5)**	$x + 2y - 2z = 9$
(*d*)	**(– 6, 0, 0)**	$2x - 3y + 6z - 2 = 0.$

Sol. (*a*) Distance (of course perpendicular) of the point (0, 0, 0) from the plane $3x - 4y + 12z = 3$ or $3x - 4y + 12z - 3 = 0$ (Making R.H.S. zero) is

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|3(0) - 4(0) + 12(0) - 3|}{\sqrt{(3)^2 + (-4)^2 + (12)^2}}$$

$$= \frac{|-3|}{\sqrt{9 + 16 + 144}} = \frac{3}{\sqrt{169}} = \frac{3}{13}.$$

(*b*) Length of perpendicular from the point (3, – 2, 1) on the plane $2x - y + 2z + 3 = 0$

(Substitute the point for x, y, z in L.H.S. of Eqn. of plane and divide by $\sqrt{a^2 + b^2 + c^2}$)

$$= \frac{|2(3) - (-2) + 2(1) + 3|}{\sqrt{(2)^2 + (-1)^2 + (2)^2}} = \frac{|6 + 2 + 2 + 3|}{\sqrt{4 + 1 + 4} = \sqrt{9}} = \frac{13}{3}$$

(*c*) Length of perpendicular from the point (2, 3, – 5) on the plane

$x + 2y - 2z = 9$ or $x + 2y - 2z - 9 = 0$ (Making R.H.S. zero)

$$= \frac{|2 + 2(3) - 2(-5) - 9|}{\sqrt{(1)^2 + (2)^2 + (-2)^2}} = \frac{|2 + 6 + 10 - 9|}{\sqrt{1 + 4 + 4}} = \frac{9}{\sqrt{9}} = \frac{9}{3} = 3.$$

(*d*) Distance of the point (– 6, 0, 0) from the plane

$$2x - 3y + 6z - 2 = 0$$

(Here R.H.S. is already zero)

$$= \frac{|\boldsymbol{a x_1 + b y_1 + c z_1 + d}|}{\sqrt{\boldsymbol{a^2 + b^2 + c^2}}} = \frac{|2(-6) - 3(0) + 6(0) - 2|}{\sqrt{(2)^2 + (-3)^2 + (6)^2}}$$

$$= \frac{|-12 - 2|}{\sqrt{4 + 9 + 36}} = \frac{|-14|}{\sqrt{49}} = \frac{14}{7} = 2.$$

MISCELLANEOUS EXERCISE

1. Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line determined by the points (3, 5, – 1), (4, 3, – 1).

Sol. We know that direction ratios of the line joining the origin (0, 0, 0) to the point (2, 1, 1) are

$$x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 = 2 - 0,\ 1 - 0,\ 1 - 0$$
$$= 2,\ 1,\ 1 = a_1,\ b_1,\ c_1.$$

Similarly, direction ratios of the line joining the points (3, 5, – 1) and (4, 3, – 1) are

$$4 - 3,\ 3 - 5,\ -1 - (-1) = 1,\ -2,\ 0 = a_2,\ b_2,\ c_2.$$

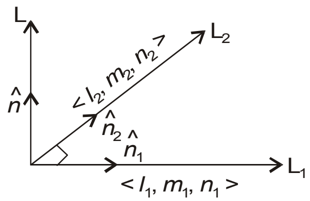
For these two lines $a_1a_2 + b_1b_2 + c_1c_2$

$$= 2(1) + 1(-2) + 1(0) = 2 - 2 + 0 = 0$$

Therefore, the two given lines are perpendicular to each other.

2. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both of these are $m_1n_2 - m_2n_1$, $n_1l_2 - n_2l_1$, $l_1m_2 - l_2m_1$.

Sol.



l_1, m_1, n_1; and l_2, m_2, n_2 are d.c.'s of two mutually perpendicular given lines L_1 and L_2 (say).

Let $\hat{n}_1$ and $\hat{n}_2$ be the unit vectors along these lines L_1 and L_2.

$$\therefore \quad \vec{n}_1 = l_1\hat{i} + m_1\hat{j} + n_1\hat{k} \quad \text{and} \quad \hat{n}_2 = l_2\hat{i} + m_2\hat{j} + n_2\hat{k}$$

Let L be the line $\perp$ to both the lines L_1 and L_2. Let $\hat{n}$ be a unit vector along line L $\perp$ to both lines L_1 and L_2.

$$\therefore \quad \hat{n} = \frac{\hat{n}_1 \times \hat{n}_2}{|\hat{n}_1 \times \hat{n}_2|} = \frac{\hat{n}_1 \times \hat{n}_2}{|\hat{n}_1|\ |\hat{n}_2| \sin 90^\circ}$$

$$\Rightarrow \quad \hat{n} = \hat{n}_1 \times \hat{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

or $\hat{n} = (m_1n_2 - m_2n_1)\hat{i} - (l_1n_2 - l_2n_1)\hat{j} + (l_1m_2 - l_2m_1)\hat{k}$

Now because $\hat{n}$ is a unit vector, therefore its components are its direction cosines.

Thus d.c.'s of $\hat{n}$ are $m_1n_2 - m_2n_1$, $l_2n_1 - l_1n_2$, $l_1m_2 - l_2m_1$
i.e., d.c.'s of line L are $m_1n_2 - m_2n_1$, $l_2n_1 - l_1n_2$, $l_1m_2 - l_2m_1$.

3. Find the angle between the lines whose direction ratios are *a*, *b*, *c* and *b* – *c*, *c* – *a*, *a* – *b*.

Sol. Given: Direction ratios of one line are a, b, c

$\Rightarrow$ A vector along this line is $\vec{b_1} = a\hat{i} + b\hat{j} + c\hat{k}$

Given: Direction ratios of second line are $b - c$, $c - a$, $a - b$

$\Rightarrow$ A vector along the second line is

$$\vec{b_2} = (b - c)\hat{i} + (c - a)\hat{j} + (a - b)\hat{k}$$

Let θ be the angle between the two lines.

We know that $\cos\theta = \dfrac{|\vec{b_1} . \vec{b_2}|}{|\vec{b_1}||\vec{b_2}|}$

$$= \frac{a(b-c) + b(c-a) + c(a-b)}{\sqrt{a^2+b^2+c^2}\ \sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}$$

or $\cos\theta = \dfrac{ab - ac + bc - ab + ac - bc}{\sqrt{a^2+b^2+c^2}\ \sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}$

$$= \frac{0}{\sqrt{a^2+b^2+c^2}\ \sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}$$

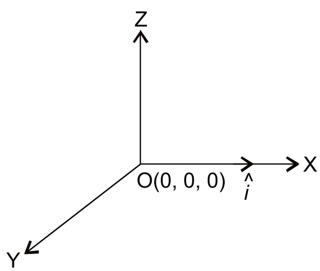
$= 0 = \cos 90^\circ \quad \therefore \theta = 90^\circ$.

4. Find the equation of a line parallel to *x*-axis and passing through the origin.

Sol. We know that a unit vector along x-axis is $\hat{i} = \hat{i} + 0\hat{j} + 0\hat{k}$

$\therefore$ By definition, direction cosines of x-axis are coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ in the unit vector *i.e.,* 1, 0, 0 = l, m, n.

$\therefore$ Equation of the required line



passing through the origin (0, 0, 0) and parallel to x-axis. (In fact this required line is x-axis itself)

is $\dfrac{x-0}{1} = \dfrac{y-0}{0} = \dfrac{z-0}{0}$ *i.e.,* $\dfrac{x}{1} = \dfrac{y}{0} = \dfrac{z}{0}$

Remark. Whenever it is not mentioned "find vector equation of the line (or plane)", we should find cartesian equation only.

However vector equation of the required line in the above question

is $\vec{r} = \vec{a} + \lambda \vec{b}$

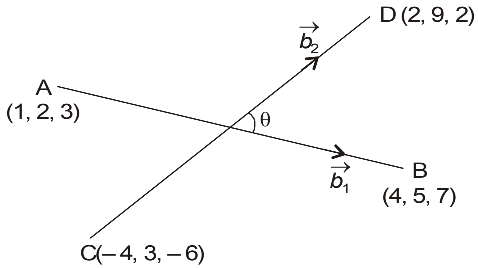
(Here $\vec{a} = \vec{0}$ and $\vec{b} = \hat{i}$)

i.e., $\vec{r} = \vec{0} + \lambda \hat{i} \Rightarrow \boldsymbol{\vec{r} = \lambda \hat{i}}$ **.**

5. If the coordinates of the points A, B, C, D be (1, 2, 3), (4, 5, 7), (– 4, 3, – 6) and (2, 9, 2) respectively, then find the angle between the lines AB and CD.

Sol. Given: Points A(1, 2, 3), B(4, 5, 7), C(– 4, 3, – 6) and D(2, 9, 2).

$\therefore$ Direction ratios of line AB are $x_2 - x_1, y_2 - y_1, z_2 - z_1$



i.e., $4 - 1, 5 - 2, 7 - 3 = 3, 3, 4 = a_1, b_1, c_1$

$\therefore$ A vector along the line AB is $\vec{b_1} = 3\hat{i} + 3\hat{j} + 4\hat{k}$

Similarly direction ratios of line CD are

$2 - (-4), 9 - 3, 2 - (-6) = 6, 6, 8 = a_2, b_2, c_2.$

$\therefore$ A vector along the line CD is $\vec{b_2} = 6\hat{i} + 6\hat{j} + 8\hat{k}$

Let θ be the angle between the lines AB and CD.

We know that $\cos\theta = \dfrac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

$$= \frac{|\vec{b_1} \cdot \vec{b_2}|}{|\vec{b_1}||\vec{b_2}|} = \frac{|3(6) + 3(6) + 4(8)|}{\sqrt{9+9+16}\sqrt{36+36+64}}$$

$$= \frac{|18+18+32|}{\sqrt{34}\sqrt{136}} = \frac{68}{\sqrt{34 \times 136}} = \frac{68}{\sqrt{34 \times 34 \times 4}}. = \frac{68}{34 \times 2}$$

$= 1 = \cos 0°$ $\qquad \therefore\ \theta = 0$

$\therefore$ Lines AB and CD are parallel.

6. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.

Sol. Given: Equation of one line is $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$

(It is standard form because coefficient of x, y, z each is unity)

Direction ratios of this line are its denominators

$$-3,\ 2k,\ 2 = a_1,\ b_1,\ c_1$$

($\Rightarrow$ a vector along this line is $\vec{b_1} = -3\hat{i} + 2k\hat{j} + 2\hat{k}$)

Equation of second line is $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ (Standard form)

Direction ratios of this line are its denominators

$$3k,\ 1,\ -5 = a_2,\ b_2,\ c_2$$

($\Rightarrow$ a vector along this line is $\vec{b_2} = 3k\hat{i} + \hat{j} - 5\hat{k}$)

Because the lines are given to be perpendicular, therefore

$$\vec{b_1} \cdot \vec{b_2}\ (= a_1a_2 + b_1b_2 + c_1c_2) = 0$$

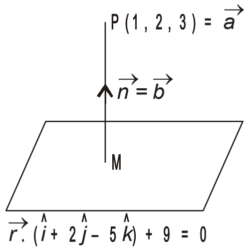
$\Rightarrow \quad (-3)(3k) + (2k)(1) + 2(-5) = 0$

$\Rightarrow \quad -9k + 2k - 10 = 0$

$\Rightarrow \quad -7k = 10 \Rightarrow k = \frac{-10}{7}$

7. Find the vector equation of the line passing through (1, 2, 3) and perpendicular to the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$.

Sol. The required line passes through the point P(1, 2, 3).



$\therefore$ Position vector $\vec{a}$ (say) of point P is (1, 2, 3)

$\Rightarrow \quad \vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

Equation of the given plane is

$$\vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$$

$\Rightarrow \quad \vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) = -9$

Comparing with $\vec{r} \cdot \vec{n} = d$, we have normal vector $\vec{n}$ to the given plane is $\vec{n} = \hat{i} + 2\hat{j} - 5\hat{k}$.

Because required line is perpendicular to the given plane, therefore vector $\vec{b}$ along the required line PM is $\vec{b} = \vec{n} = \hat{i} + 2\hat{j} - 5\hat{k}$.

$\therefore$ Equation of required line is $\vec{r} = \vec{a} + \lambda \vec{b}$

i.e., $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} + 2\hat{j} - 5\hat{k})$.

8. Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$.

Sol. Equation of the given plane is $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$

$[\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$

$\Rightarrow x + y + z = 2$

$\therefore$ Equation of any plane parallel to this plane is

$x + y + z = \lambda$ ... (i) (changing constant term only)

To find λ: Plane (1) passes through the point (a,b,c) (given). Putting $x = a, y = b, z = c$ in (i) $a + b + c = \lambda$

Putting $\lambda = a + b + c$ in (i), equation of required plane is $x + y + z = a + b + c$

9. Find the shortest distance between the lines

$$\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$$

and $$\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k}).$$

Sol. Given: vector equation of one line is

$$\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$$

Comparing with $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ we have

$$\vec{a_1} = 6\hat{i} + 2\hat{j} + 2\hat{k} \text{ and } \vec{b_1} = \hat{i} - 2\hat{j} + 2\hat{k}$$

Given: Vector equation of second line is

$$\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$$

Comparing with $\vec{r} = \vec{a_2} + \mu \vec{b_2}$, we have

$$\vec{a_2} = -4\hat{i} - \hat{k} \quad \text{and} \quad \vec{b_2} = 3\hat{i} - 2\hat{j} - 2\hat{k}$$

We know that length of shortest distance between two (skew) lines is

$$\frac{|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|} \qquad ...(i)$$

Now $\vec{a_2} - \vec{a_1} = -4\hat{i} - \hat{k} - (6\hat{i} + 2\hat{j} + 2\hat{k})$

$$= -4\hat{i} - \hat{k} - 6\hat{i} - 2\hat{j} - 2\hat{k} = -10\hat{i} - 2\hat{j} - 3\hat{k}$$

Again $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix}$

Expanding along first row

$$= \hat{i}(4 + 4) - \hat{j}(-2 - 6) + \hat{k}(-2 + 6) = 8\hat{i} + 8\hat{j} + 4\hat{k}$$

$\therefore \quad (\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (-10)8 + (-2)8 + (-3)4$

$$= -80 - 16 - 12 = -108$$

and $\quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(8)^2 + (8)^2 + (4)^2}$

$$= \sqrt{64 + 64 + 16} = \sqrt{144} = 12$$

Putting these values in (i), length of shortest distance

$$= \frac{|-108|}{12} = \frac{108}{12} = 9.$$

10. Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the YZ-plane.

Sol. Given: A line through the points A(5, 1, 6) and B(3, 4, 1).

$\therefore$ Direction ratios of this line AB are

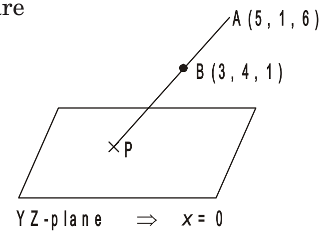
$$x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1$$

i.e., $3 - 5,\ 4 - 1,\ 1 - 6$

i.e., $-2,\ 3,\ -5 = a,\ b,\ c$

$\therefore$ Equation of line AB is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$



i.e., $\dfrac{x - 5}{-2} = \dfrac{y - 1}{3} = \dfrac{z - 6}{-5} \qquad ...(i)$

Let us find the coordinates of the point where this line AB crosses ($\Rightarrow$ cuts or meets) the YZ-plane ($\Rightarrow$ $x = 0$) ...(*ii*)

To find this point P, let us solve (*i*) and (*ii*) for x, y, z.

Putting $x = 0$ from (*ii*) in (*i*), $\frac{-5}{-2} = \frac{y-1}{3} = \frac{z-6}{-5}$

$$\Rightarrow \quad \frac{5}{2} = \frac{y-1}{3} = \frac{z-6}{5}$$

$$\Rightarrow \quad \frac{y-1}{3} = \frac{5}{2} \quad \text{and} \quad \frac{z-6}{-5} = \frac{5}{2}$$

$$\Rightarrow \quad 2y - 2 = 15 \quad \text{and} \quad 2z - 12 = -25$$

$$\Rightarrow \quad 2y = 17 \quad \text{and} \quad 2z = -13$$

$$\Rightarrow \quad y = \frac{17}{2} \quad \text{and} \quad z = \frac{-13}{2}$$

$\therefore$ Required point is $P\left(0, \frac{17}{2}, \frac{-13}{2}\right)$.

11. Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the ZX-plane.

Sol. Given: A line through the points A(5, 1, 6) and B(3, 4, 1).

For figure, see the figure for Q. No. 10

$\therefore$ Direction ratios of this line AB are

$$x_2 - x_1, y_2 - y_1, z_2 - z_1 \quad i.e., \quad 3 - 5, 4 - 1, 1 - 6$$

i.e., $-2, 3, -5 = a, b, c$

$\therefore$ Equation of line AB is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

i.e., $$\frac{x-5}{-2} = \frac{y-1}{3} = \frac{z-6}{-5} \qquad ...(i)$$

Let us find the coordinates of the point where this line AB crosses ($\Rightarrow$ cuts or meets) the ZX-plane ($\Rightarrow$ $y = 0$) ...(*ii*)

To find this point P, let us solve (*i*) and (*ii*) for x, y, z.

Putting $y = 0$ from (*ii*) in (*i*), we have

$$\frac{x-5}{-2} = \frac{-1}{3} = \frac{z-6}{-5} \quad \Rightarrow \quad \frac{x-5}{-2} = \frac{-1}{3} \text{ and } \frac{z-6}{-5} = \frac{-1}{3}$$

$$\Rightarrow \quad 3x - 15 = 2 \quad \text{and} \quad 3z - 18 = 5$$

$$\Rightarrow \quad 3x = 17 \quad \text{and} \quad 3z = 23$$

$$\Rightarrow \quad x = \frac{17}{3} \quad \text{and} \quad z = \frac{23}{3}$$

$\therefore$ Required point is $P\left(\frac{17}{3}, 0, \frac{23}{3}\right)$.

12. Find the coordinates of the point where the line through (3, – 4, – 5) and (2, – 3, 1) crosses the plane $2x + y + z = 7$.

Sol. Direction ratios of the line joining the points

A(3, – 4, – 5) and B(2, – 3, 1) are

$$2 - 3, -3 - (-4), 1 - (-5) \; i.e., \; -1, 1, 6$$

$\therefore$ Equations of the line AB are $\dfrac{x-3}{-1} = \dfrac{y+4}{1} = \dfrac{z+5}{6}$ ...(*i*)

$$\left| \dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c} \right.$$

Equation of the plane is $2x + y + z = 7$ ...(*ii*)

Let us find the point where line (*i*) crosses (*i.e.,* cuts *i.e.,* meets) plane (*ii*).

$\therefore$ For this, let us solve (*i*) and (*ii*) for x, y, z

From (*i*), $\dfrac{x-3}{-1} = \dfrac{y+4}{1} = \dfrac{z+5}{6} = \lambda$ (say)

$\therefore$ $x - 3 = -\lambda,\ y + 4 = \lambda,\ z + 5 = 6\lambda$

$\therefore$ $x = 3 - \lambda,\ y = -4 + \lambda,\ z = -5 + 6\lambda$ ...(*iii*)

Putting these values of x, y, z in Eqn. (*ii*), we have

$$2(3 - \lambda) + (-4 + \lambda) + (-5 + 6\lambda) = 7$$

or $6 - 2\lambda - 4 + \lambda - 5 + 6\lambda = 7$ or $5\lambda = 10$ or $\lambda = 2$.

Putting $\lambda = 2$ in (*iii*), point of intersection of line (*i*) and plane (*ii*) is

$$x = 3 - 2 = 1,\ y = -4 + 2 = -2,\ z = -5 + 12 = 7.$$

$\therefore$ Required point of intersection is $(1, -2, 7)$.

13. Find the equation of the plane passing through the point (– 1, 3, 2) and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

Sol. We know that equation of any plane through the point $(-1, 3, 2)$ is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

i.e., $a(x + 1) + b(y - 3) + c(z - 2) = 0$ ...(*i*)

i.e., $ax + a + by - 3b + cz - 2c = 0$

or $ax + by + cz = -a + 3b + 2c$

Given: This required plane is perpendicular to the plane

$$x + 2y + 3z = 5 \quad \therefore \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$$

i.e., Product of coefficients of $x + ... = 0$

$\therefore$ $a(1) + b(2) + c(3) = 0$ ...(*ii*)

Given: Again the required plane is perpendicular to the plane

$$3x + 3y + z = 0$$

$\therefore$ $a_1a_2 + b_1b_2 + c_1c_2 = 0$

i.e., $a(3) + b(3) + c(1) = 0$ ...(*iii*)

Solving (*ii*) and (*iii*) for a, b, c

$$\frac{a}{2-9} = \frac{-b}{1-9} = \frac{c}{3-6} \qquad i.e., \quad \frac{a}{-7} = \frac{b}{8} = \frac{c}{-3}$$

Putting these value of a, b, c in (*i*), equation of required plane is

$$-7(x + 1) + 8(y - 3) - 3(z - 2) = 0$$

i.e., $-7x - 7 + 8y - 24 - 3z + 6 = 0$

i.e., $-7x + 8y - 3z - 25 = 0$

i.e., $7x - 8y + 3z + 25 = 0.$

14. If the points (1, 1, p) and (– 3, 0, 1) be equidistant from the plane $\vec{r}\,.\,(3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$, then find the value of p.

Sol. Equation of the given plane is $\vec{r}\,.\,(3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$

i.e., $(x\hat{i} + y\hat{j} + z\hat{k})\,.\,(3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$

[$\because$ $\vec{r}$ = Position vector of any point (x, y, z) on the plane $= x\hat{i} + y\hat{j} + z\hat{k}$]

$\Rightarrow \quad 3x + 4y - 12z + 13 = 0 \qquad ...(i)$

Given: The points (1, 1, p) and (– 3, 0, 1) are equidistant from plane (i).

$\Rightarrow$ (Perpendicular) distance of point (1, 1, p) from plane (i)

= Distance of point (– 3, 0, 1) from plane (i)

$$\Rightarrow \quad \frac{|3(1) + 4(1) - 12(p) + 13|}{\sqrt{9+16+144}} = \frac{|3(-3) + 4(0) - 12(1) + 13|}{\sqrt{9+16+144}} \qquad \left| \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}} \right.$$

$$\Rightarrow \quad \frac{|3 + 4 - 12p + 13|}{13} = \frac{|-9 - 12 + 13|}{13}$$

$\Rightarrow \quad |20 - 12p| = |-8| = 8$

$\Rightarrow \quad 20 - 12p = \pm 8$ [$\because$ If $|x| = a$, $a \geq 0$, then $x = \pm a$]

Taking positive sign $20 - 12p = 8 \Rightarrow -12p = -12$

$\Rightarrow p = 1$

Taking negative sign $20 - 12p = -8$

$\Rightarrow -12p = -28 \Rightarrow p = \dfrac{-28}{-12} = \dfrac{7}{3}$ Hence, $p = 1$ or $p = \dfrac{7}{3}$

15. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}\,.\,(\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r}\,.\,(2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and parallel to x-axis.

Sol. Given: Equation of first plane is

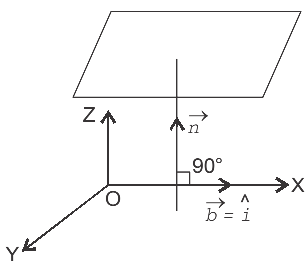
$\vec{r}\,.\,(\hat{i} + \hat{j} + \hat{k}) = 1$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k})\,.\,(\hat{i} + \hat{j} + \hat{k}) = 1$

$\Rightarrow x + y + z = 1$

Making R.H.S. zero

$\Rightarrow x + y + z - 1 = 0 \qquad ...(i)$



Equation of second plane is

$\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$

$\Rightarrow \ 2x + 3y - z + 4 = 0$ ...(*ii*)

(Here R.H.S. is already zero)

We know that equation of any plane passing through the line of intersection of these two planes is L.H.S. of (*i*) + λ [L.H.S. of (*ii*)] = 0

i.e. $x + y + z - 1 + \lambda\,(2x + 3y - z + 4) = 0$

$\Rightarrow x + y + z - 1 + 2\lambda x + 3\lambda y - \lambda z + 4\lambda = 0$

$\Rightarrow (1 + 2\lambda)\,x + (1 + 3\lambda)y + (1 - \lambda)z - 1 + 4\lambda = 0$ ...(*iii*)

$\Rightarrow ax + by + cz + d = 0$

Given : Required plane (*iii*) is parallel to x-axis $\Rightarrow$ $\boldsymbol{al + bm + cn = 0}$ ...(*iv*)

we know that a vector $\vec{b}$ along x-axis is $\hat{i} + 0\hat{j} + 0\hat{k}$.

$\therefore$ D.R's of x-axis are 1,0,0 | coeff of $\hat{i}, \hat{j}, \hat{k} = l, m, n$

Putting values of a,b,c, l,m,n, in (*iv*), we have

$(1 + 2\lambda)\,1 + (1 + 3\lambda)\,0 + (1 - \lambda)\,0 = 0$

$\Rightarrow 1 + 2\lambda = 0 \ \Rightarrow \ 2\lambda = -1 \Rightarrow \lambda = -\dfrac{1}{2}$

Putting $\lambda = -\dfrac{1}{2}$ in (*iii*), Equation of required plane is

$(1 - 1)\,x + \left(1 - \dfrac{3}{2}\right)y + \left(1 + \dfrac{1}{2}\right)z - 1 - 2 = 0$

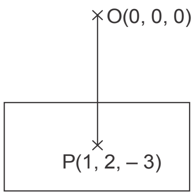
or $-\dfrac{y}{2} + -\dfrac{3z}{2} - 3 = 0$

Multiplying by –2, y – 3z + 6 = 0

Note: Condition $al + bm + cn = 0$ is cartesian equivalent of

$\vec{b} \cdot \vec{n} = 0$

16. If O be the origin and the coordinates of P be (1, 2, – 3), then find the equation of the plane passing through P and perpendicular to OP.



Sol. Given: Origin O(0, 0, 0) and point P(1, 2, – 3).

We are to find the equation of the plane passing through P(1, 2, – 3) = (x_1, y_1, z_1) and perpendicular to OP.

$\therefore$ Direction ratios of normal OP to the plane are

$1 - 0,\ 2 - 0,\ -3 - 0$ | $x_2 - x_1, y_2 - y_1, z_2 - z_1$

i.e., $1, 2, -3 = a, b, c.$

$\therefore$ Equation of the required plane is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

i.e., $1(x - 1) + 2(y - 2) - 3(z + 3) = 0$

i.e., $x - 1 + 2y - 4 - 3z - 9 = 0$

i.e., $x + 2y - 3z - 14 = 0.$

17. Find the equation of the plane which contains the line of intersection of the planes $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0$, $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$ and which is perpendicular to the plane $\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$.

Sol. Equation of first plane is $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0$

i.e. $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0$

i.e. $x + 2y + 3z - 4 = 0$ ... (*i*) (R.H.S. already zero)

Equation of second plane is $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$

i.e. $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$

i.e. $2x + y - z + s = 0$ ... (*ii*) (R.H.S. already zero)

We know that equation of any plane passing through the line of intersection of planes (*i*) and (*ii*) is

L.H.S. of (*i*) + λ L.H.S. of (*ii*) = 0

i.e. $x + 2y + 3z - 4 + \lambda(2x + y - z + 5) = 0$

i.e. $x + 2y + 3z - 4 + 2\lambda x + \lambda y - \lambda z + 5\lambda = 0$

i.e. $(1 + 2\lambda)x + (2 + \lambda)y + (3 - \lambda)z - 4 + 5\lambda = 0$ ... (*iii*)

Given : Required plane (*iii*) is perpendicular to the plane

$\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$

i.e. $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$

i.e. $5x + 3y - 6z + 8 = 0$ ... (*iv*)

$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 0$

i.e. product of coeffient of x in (*iii*) and (*iv*) ......+.......... = 0

$\therefore (1 + 2\lambda)5 + (2 + \lambda)3 + (3 - \lambda)(-6) = 0$

$\Rightarrow 5 + 10\lambda + 6 + 3\lambda - 18 + 6\lambda = 0$

$$\Rightarrow 19\lambda - 7 = 0 \Rightarrow 19\lambda = 7 \Rightarrow \lambda = \frac{7}{19}$$

Putting $\lambda = \frac{7}{19}$ in (*iii*), equation of required plane is

$$\left(1 + \frac{14}{19}\right)x + \left(2 + \frac{7}{19}\right)y + \left(3 - \frac{7}{19}\right)z - 4 + \frac{35}{19} = 0$$

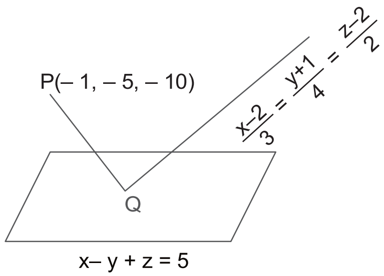
$$\Rightarrow \frac{33}{19}x + \frac{45}{19}y + \frac{50}{19}z - \frac{41}{19} = 0$$

Multiplying by 19, equation of required plane is

$$33x + 45y + 50z = 41$$

18. Find the distance of the point (– 1, – 5, – 10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$.

Sol. Given: Point P (say) (– 1, – 5, – 10).



Given: Vector Equation of the line is

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) = \vec{a} + \lambda\vec{b}$$

This line passes through the point $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ $= (2, -1, 2) = (x_1, y_1, z_1)$ and is parallel to the vector $3\hat{i} + 4\hat{j} + 2\hat{k}$ i.e. has d.r's 3,4,2.

∴ Equation of the given line in cartesian form is

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = \lambda \text{ (say)} \qquad \text{... }(i)$$

$$\left(\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}\right)$$

Vector equation of the given plane is $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$

i.e. $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$

$$\Rightarrow \quad x - y + z = 5 \qquad \text{... }(ii)$$

Let us solve (i) and (ii) for x, y, z *to* find point of intersection (say Q) of line (i) and plane (ii).

From (i) $x - 2 = 3\lambda$, $y + 1 = 4\lambda$, $z - 2 = 2\lambda$

i.e. $x = 2 + 3\lambda$, $y = -1 + 4\lambda$, $z = 2 + 2\lambda$

∴ Point Q is $(2 + 3\lambda\ -1 + 4\lambda, 2 + 2\lambda)$ for some real λ ... (iii)

Putting these values of x, y, z in (ii)

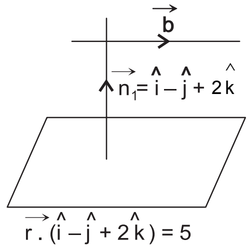
$2 + 3\lambda + 1 - 4\lambda + 2 + 2\lambda = 5 \quad \Rightarrow \lambda = 0$

Putting $\lambda = 0$ in (iii), point Q is (2, –1, 2)

∴ Distance of the given point P(– 1, – 5, – 10) from the point Q(2, – 1, 2)

$$= PQ = \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$$
$$\left(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\right)$$
$$= \sqrt{9 + 16 + 144} = \sqrt{169} = 13.$$

19. Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r} \cdot (\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\vec{r} \cdot (3\hat{i} + \hat{j} + \hat{k}) = 6$.



Sol. Given: The required line passes through the point A(1, 2, 3) (= $\vec{a}$)

$\Rightarrow$ $\vec{a}$ = Position vector of point A

$= \hat{i} + 2\hat{j} + 3\hat{k}$

Let $\vec{b}$ be any vector along the required line.

$\therefore$ Vector equation of required line is $\vec{r} = \vec{a} + \lambda\vec{b}$

$\Rightarrow$ $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda\vec{b}$ ...(*i*)

Because the required line is parallel to the plane

$\vec{r} \cdot (\hat{i} - \hat{j} + 2\hat{k}) = 5$

(Form $\vec{r} \cdot \vec{n_1} = d_1$ where $\vec{n_1} = \hat{i} - \hat{j} + 2\hat{k}$) $\therefore$ $\boldsymbol{\vec{b} \cdot \vec{n_1} = 0}$

Similarly, $\vec{b} \cdot \vec{n_2} = 0$

[$\because$ The required line is also parallel to the plane $\vec{r} \cdot (3\hat{i} + \hat{j} + \hat{k}) = 6$ *i.e.*, $\vec{r} \cdot \vec{n_2} = d_2$ where $\vec{n_2} = 3\hat{i} + \hat{j} + \hat{k}$]

Now $\vec{b} \cdot \vec{n_1} = 0$ and $\vec{b} \cdot \vec{n_1} = 0$

$\Rightarrow$ $\vec{b}$ is perpendicular to both $\vec{n_1}$ and $\vec{n_2}$

$\Rightarrow$ $\vec{b} = \vec{n_1} \times \vec{n_2}$ (By definition of cross-product)

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{vmatrix}$$

Expanding along first row

$\vec{b} = \hat{i}(-1 - 2) - \hat{j}(1 - 6) + \hat{k}(1 + 3) = -3\hat{i} + 5\hat{j} + 4\hat{k}$

Putting this value of $\vec{b}$ in (*i*), vector equation of required line is

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-3\hat{i} + 5\hat{j} + 4\hat{k}).$$

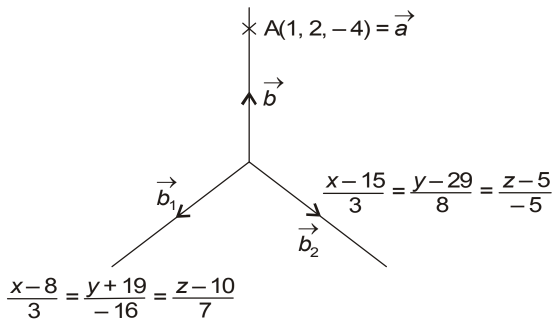
20. Find the vector equation of the line passing through the point (1, 2, – 4) and perpendicular to the two lines:

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

Sol. Given: A point on the required line is A(1, 2, – 4).

$\therefore$ Position vector of point A is $\vec{a} = (1, 2, -4)$

$$= \hat{i} + 2\hat{j} - 4\hat{k}$$



Equations of the two given lines are

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \qquad \text{(Standard Form)}$$

and $$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \qquad \text{(Standard Form)}$$

$\therefore$ Direction ratios of first given line are its denominators 3, – 16, 7

i.e., a vector along this line is $\vec{b_1} = 3\hat{i} - 16\hat{j} + 7\hat{k}$ and direction ratios of the second given line are also its denominators 3, 8, – 5

i.e., a vector along the second line is $\vec{b_2} = 3\hat{i} + 8\hat{j} - 5\hat{k}$

Let $\vec{b}$ be the vector along the required line perpendicular to the two given lines.

$\therefore$ By definition of cross-product $\vec{b} = \vec{b_1} \times \vec{b_2}$

i.e., $$\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$$

Expanding along first row

$$= \hat{i}(80 - 56) - \hat{j}(-15 - 21) + \hat{k}(24 + 48)$$

$$= 24\hat{i} + 36\hat{j} + 72\hat{k}, \ \vec{b} = 12(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$\therefore$ Equation of the required line is $\vec{r} = \vec{a} + \lambda\vec{b}$

Putting values of $\vec{a}$ and $\vec{b}$,

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(12)(2\hat{i} + 3\hat{j} + 6\hat{k})$$

Replacing 12λ by λ,

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}).$$

21. Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}.$$

Sol. We know that equation of a plane making intercepts a, b, c (on the axes) is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

or $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$ ...(i)

Perpendicular distance of the origin (0, 0, 0) from plane (i) = p (given)

$$\therefore \quad \frac{\left|\frac{0}{a} + \frac{0}{b} + \frac{0}{c} - 1\right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = p \qquad \left| \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \right.$$

Squaring both sides, $\dfrac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} = p^2$

Cross-multiplying, $p^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = 1$

Dividing by p^2, $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$.

Choose the correct answer in Exercises Q. 22 and 23.

22. Distance between the two planes:

$$\mathbf{2x + 3y + 4z = 4 \text{ and } 4x + 6y + 8z = 12 \text{ is}}$$

(A) 2 units **(B) 4 units**

(C) 8 units **(D) $\frac{2}{\sqrt{29}}$ units.**

Sol. Given: Equation of one plane is

$$2x + 3y + 4z = 4$$

or $2x + 3y + 4z - 4 = 0$ ...(i) (Making R.H.S. zero)

$(ax + by + cz + d_1 = 0)$

Equation of second plane is

$4x + 6y + 8z = 12$ or $4x + 6y + 8z - 12 = 0$

The two planes are parallel as $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ is satisfied.

$$\left(\because \frac{2}{4} = \frac{3}{6} = \frac{4}{8}\right)$$

Dividing every term of second equation by 2 to make coefficients of x, y, z equal in the equations of the two planes

i.e., $2x + 3y + 4z - 6 = 0$ ...(*ii*)

$(ax + by + cz + d_2 = 0)$

We know that distance between parallel planes (*i*) and (*ii*)

$$= \frac{|\boldsymbol{d_1 - d_2}|}{\sqrt{\boldsymbol{a^2 + b^2 + c^2}}} = \frac{|-4-(-6)|}{\sqrt{(2)^2+(3)^2+(4)^2}} = \frac{|-4+6|}{\sqrt{4+9+16}} = \frac{2}{\sqrt{29}}$$

$\therefore$ Option (D) is the correct answer.

23. The planes: $2x - y + 4z = 5$ and $5x - 2.5y + 10z = 6$ are

(A) Perpendicular **(B) Parallel**

(C) intersect y-axis **(D) passes through $\left(0, 0, \frac{5}{4}\right)$.**

Sol. Equations of the given planes are

$2x - y + 4z = 5$

$(a_1x + b_1y + c_1z + d_1 = 0)$

$5x - 2.5y + 10z = 6$

$(a_2x + b_2y + c_2z + d_2 = 0)$

Let us test option (A). Are these planes perpendicular?

$a_1a_2 + b_1b_2 + c_1c_2 = 2(5) + (-1)(-2.5) + 4(10)$

$= 10 + 2.5 + 40 = 52.5 \neq 0$

$\therefore$ Planes are not perpendicular.

$\therefore$ Option (A) is not correct answer.

Let us test option (B). Are these planes parallel?

Here, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$\Rightarrow \frac{2}{5} = \frac{-1}{-2.5} = \frac{4}{10}$

$\Rightarrow \frac{2}{5} = \frac{1}{\left(\frac{25}{10}\right)} = \frac{2}{5} \Rightarrow \frac{2}{5} = \frac{10}{25} = \frac{2}{5}$

$\Rightarrow \frac{2}{5} = \frac{2}{5} = \frac{2}{5}$ which is true. $\therefore$ Planes are parallel.

$\therefore$ Option (B) is correct answer.