Exercise 7.1

Find an antiderivative (or integral) of the following functions by the method of inspection in Exercises 1 to 5.

1. sin $2x$

Sol. To find an anti derivative of sin $2x$ by Inspection Method.

We know that $\frac{d}{dx}$ (cos $2x$) = – 2 sin $2x$

Dividing by – 2, $\frac{-1}{2}\frac{d}{dx}$ (cos $2x$) = sin $2x$

or $\frac{d}{dx}\left(\frac{-1}{2}\cos 2x\right) = \sin 2x$

∴ By definition; **an** integral or **an** antiderivative of sin $2x$ is $\frac{-1}{2}$ cos $2x$.

Note. In fact anti derivative or integral of sin $2x$ is $\frac{-1}{2}\cos 2x + c$. For different values of c, we get different antiderivatives. So we omitted c for writing **an** anti derivative.

2. cos $3x$

Sol. To find an anti derivative of cos $3x$ by Inspection Method.

We know that $\frac{d}{dx}$ (sin $3x$) = 3 cos $3x$

Dividing by 3, $\frac{1}{3}\frac{d}{dx}$ (sin $3x$) = cos $3x$ or $\frac{d}{dx}\left(\frac{1}{3}\sin 3x\right) = \cos 3x$

∴ By definition, **an** integral or **an** antiderivative of cos $3x$ is $\frac{1}{3}$ sin $3x$.

(See note after solution of Q.No.1 for not adding c to the answer.)

3. e^{2x}.

Sol. To find an antiderivative of e^{2x} by Inspection Method.

We know that $\frac{d}{dx} e^{2x} = e^{2x}\frac{d}{dx}(2x) = 2e^{2x}$

Dividing by 2, $\frac{1}{2}\frac{d}{dx} e^{2x} = e^{2x}$ or $\frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = e^{2x}$

∴ An antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$.

4. $(ax + b)^2$.

Sol. To find an anti derivative of $(ax + b)^2$.

We know that $\frac{d}{dx}(ax + b)^3 = 3(ax + b)^2\frac{d}{dx}(ax + b) = 3(ax + b)^2 a$.

Dividing by $3a$, $\frac{1}{3a}\frac{d}{dx}(ax + b)^3 = (ax + b)^2$

or $$\frac{d}{dx}\left[\frac{1}{3a}(ax+b)^3\right] = (ax+b)^2$$

$\therefore$ An anti derivative of $(ax+b)^2$ is $\frac{1}{3a}(ax+b)^3$.

5. $\sin 2x - 4e^{3x}$.

Sol. To find an anti derivative of $\sin 2x - 4e^{3x}$ by Inspection Method.

We know that $$\frac{d}{dx}(\cos 2x) = -2\sin 2x$$

Dividing by -2, $$\frac{d}{dx}\left(\frac{-1}{2}\cos 2x\right) = \sin 2x \qquad ...(i)$$

Again $\frac{d}{dx}e^{3x} = 3e^{3x}$ $\qquad \therefore \frac{d}{dx}\left(\frac{1}{3}e^{3x}\right) = e^{3x}$

Multiplying by -4, $$\frac{d}{dx}\left(\frac{-4}{3}e^{3x}\right) = -4e^{3x} \qquad ...(ii)$$

Adding eqns. (i) and (ii)

$$\frac{d}{dx}\left(\frac{-1}{2}\cos 2x\right) + \frac{d}{dx}\left(\frac{-4}{3}e^{3x}\right) = \sin 2x - 4e^{3x}$$

or $$\frac{d}{dx}\left(\frac{-1}{2}\cos 2x - \frac{4}{3}e^{3x}\right) = \sin 2x - 4e^{3x}$$

$\therefore$ An anti derivative of $\sin 2x - 4e^{3x}$ is $\frac{-1}{2}\cos 2x - \frac{4}{3}e^{3x}$.

Evaluate the following integrals in Exercises 6 to 11.

6. $\int (4e^{3x}+1)\,dx$.

Sol. $\int (4e^{3x}+1)\,dx = \int 4e^{3x}\,dx + \int 1\,dx$

$$= 4\int e^{3x}\,dx + x = 4\frac{e^{3x}}{3} + x + c. \left[\because \int e^{ax}\,dx = \frac{e^{ax}}{a} \text{ and } \int 1\,dx = x\right]$$

7. $\int x^2\left(1-\frac{1}{x^2}\right)dx$.

Sol. $\int x^2\left(1-\frac{1}{x^2}\right)dx = \int \left(x^2 - \frac{x^2}{x^2}\right)dx = \int (x^2-1)\,dx$

$$= \int x^2\,dx - \int 1\,dx = \frac{x^3}{3} - x + c. \left[\because \int x^n\,dx = \frac{x^{n+1}}{n+1} \text{ if } n \neq -1\right]$$

8. $\int (ax^2+bx+c)\,dx$.

Sol. $\int (ax^2+bx+c)\,dx = \int ax^2\,dx + \int bx\,dx + \int c\,dx$

$= a\int x^2\ dx + b\int x^1\,dx + c\int 1\ dx \qquad = a\,\dfrac{x^3}{3} + b\,\dfrac{x^2}{2} + cx + c_1$

where c_1 is the constant of integration.

9. $\int (2x^2 + e^x)\ dx.$

Sol. $\int (2x^2 + e^x)\ dx = \int 2x^2\,dx + \int e^x\ dx$

$= 2\int x^2\ dx + \int e^x\ dx \qquad = 2\,\dfrac{x^{2+1}}{2+1} + e^x + c = 2\,\dfrac{x^3}{3} + e^x + c.$

10. $\int \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^2\ dx.$

Sol. $\int \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^2\ dx$

Opening the square $= \int \left((\sqrt{x})^2 + \left(\dfrac{1}{\sqrt{x}}\right)^2 - 2\sqrt{x}\,\dfrac{1}{\sqrt{x}}\right)\ dx$

$= \int \left(x + \dfrac{1}{x} - 2\right)\ dx \ = \int x\ dx + \int \dfrac{1}{x}\ dx - \int 2\ dx$

$= \dfrac{x^2}{2} + \log|x| - 2x + c. \qquad \left[\because \int 2\,dx = 2\int 1\,dx = 2x\right]$

11. $\int \dfrac{x^3 + 5x^2 - 4}{x^2}\ dx.$

Sol. $\int \dfrac{x^3 + 5x^2 - 4}{x^2}\ dx = \int \left(\dfrac{x^3}{x^2} + \dfrac{5x^2}{x^2} - \dfrac{4}{x^2}\right)\ dx$

$\left[\text{Using } \dfrac{a+b-c}{d} = \dfrac{a}{d} + \dfrac{b}{d} - \dfrac{c}{d}\right]$

$= \int (x + 5 - 4x^{-2})\ dx = \int x^1\ dx + \int 5\ dx - \int 4x^{-2}\ dx$

$= \dfrac{x^2}{2} + 5\int 1\ dx - 4\int x^{-2}\ dx = \dfrac{x^2}{2} + 5x - 4\,\dfrac{x^{-2+1}}{-2+1} + c$

$= \dfrac{x^2}{2} + 5x + \dfrac{4}{x} + c.$

Evaluate the following integrals in Exercises 12 to 16.

12. $\int \dfrac{x^3 + 3x + 4}{\sqrt{x}}\ dx.$

Sol. $\int \dfrac{x^3 + 3x + 4}{\sqrt{x}}\ dx = \int \left(\dfrac{x^3}{x^{1/2}} + \dfrac{3x}{x^{1/2}} + \dfrac{4}{x^{1/2}}\right)\ dx$

$= \int (x^{3-1/2} + 3x^{1-1/2} + 4x^{-1/2})\,dx = \int (x^{5/2} + 3x^{1/2} + 4x^{-1/2})\ dx$

$= \int x^{5/2}\ dx + 3\int x^{1/2}\ dx + 4\int x^{-1/2}\ dx$

$$= \frac{x^{5/2+1}}{\frac{5}{2}+1} + 3\frac{x^{1/2+1}}{\frac{1}{2}+1} + 4\frac{x^{-1/2+1}}{\frac{-1}{2}+1} + c = \frac{x^{7/2}}{\frac{7}{2}} + 3\frac{x^{3/2}}{\frac{3}{2}} + 4\frac{x^{1/2}}{\frac{1}{2}} + c$$

$$= \frac{2}{7}x^{7/2} + 2x^{3/2} + 8x^{1/2} + c.$$

13. $\displaystyle\int \frac{x^3 - x^2 + x - 1}{x - 1}\, dx.$

Sol. $\displaystyle\int \frac{x^3 - x^2 + x - 1}{x-1}\, dx = \int \frac{x^2(x-1)+(x-1)}{x-1}\, dx$

$$= \int \frac{(x-1)(x^2+1)}{(x-1)}\, dx = \int (x^2+1)\, dx$$

$$= \int x^2\, dx + \int 1\, dx = \frac{x^{2+1}}{2+1} + x + c = \frac{x^3}{3} + x + c.$$

14. $\displaystyle\int (1-x)\sqrt{x}\, dx.$

Sol. $\displaystyle\int (1-x)\sqrt{x}\, dx = \int (\sqrt{x} - x\sqrt{x})\, dx$

$$= \int (x^{1/2} - x^1 x^{1/2})\, dx = \int (x^{1/2} - x^{1+1/2})\, dx$$

$$= \int (x^{1/2} - x^{3/2})\, dx = \frac{x^{1/2+1}}{\frac{1}{2}+1} - \frac{x^{3/2+1}}{\frac{3}{2}+1} + c$$

$$= \frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{5/2}}{\frac{5}{2}} + c = \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + c.$$

15. $\displaystyle\int \sqrt{x}\,(3x^2 + 2x + 3)\, dx.$

Sol. $\displaystyle\int \sqrt{x}\,(3x^2+2x+3)\, dx = \int x^{1/2}(3x^2+2x+3)\, dx$

$$= \int (3x^2 x^{1/2} + 2x\, x^{1/2} + 3x^{1/2})\, dx = \int (3x^{5/2} + 2x^{3/2} + 3x^{1/2})\, dx$$

$$\left(\because\ 2 + \frac{1}{2} = \frac{4+1}{2} = \frac{5}{2}, 1 + \frac{1}{2} = \frac{2+1}{2} = \frac{3}{2}\right)$$

$$= 3\int x^{5/2}\, dx + 2\int x^{3/2}\, dx + 3\int x^{1/2}\, dx$$

$$= 3\frac{x^{5/2+1}}{\frac{5}{2}+1} + 2\frac{x^{3/2+1}}{\frac{3}{2}+1} + 3\frac{x^{1/2+1}}{\frac{1}{2}+1} + c = 3\frac{x^{7/2}}{\frac{7}{2}} + 2\frac{x^{5/2}}{\frac{5}{2}} + 3\frac{x^{3/2}}{\frac{3}{2}} + c$$

$$= \frac{6}{7}x^{7/2} + \frac{4}{5}x^{5/2} + 2x^{3/2} + c.$$

16. $\int (2x - 3\cos x + e^x)\ dx.$

Sol. $\int (2x - 3\cos x + e^x)\ dx = \int 2x\ dx - \int 3\cos x\ dx + \int e^x\, dx$

$= 2\int x^1\ dx - 3\int \cos x\ dx + \int e^x\, dx = 2\dfrac{x^2}{2} - 3\sin x + e^x + c$

$= x^2 - 3\sin x + e^x + c.$

Evaluate the following integrals in Exercises 17 to 20.

17. $\int (2x^2 - 3\sin x + 5\sqrt{x})\ dx.$

Sol. $\int (2x^2 - 3\sin x + 5\sqrt{x})\ dx$

$= 2\int x^2\ dx - 3\int \sin x\ dx + 5\int x^{1/2}\ dx$

$= 2\dfrac{x^{2+1}}{2+1} - 3(-\cos x) + 5\dfrac{x^{1/2+1}}{\frac{1}{2}+1} + c = 2\dfrac{x^3}{3} + 3\cos x + 5\dfrac{x^{3/2}}{\frac{3}{2}} + c$

$= 2\dfrac{x^3}{3} + 3\cos x + \dfrac{10}{3}x^{3/2} + c.$

18. $\int \sec x(\sec x + \tan x)\ dx.$

Sol. $\int \sec x(\sec x + \tan x)\ dx = \int (\sec^2 x + \sec x\tan x)\, dx$

$= \int \sec^2 x\ dx + \int \sec x\tan x\ dx \qquad = \tan x + \sec x + c.$

19. $\int \dfrac{\sec^2 x}{\text{cosec}^2 x}\ dx.$

Sol. $\int \dfrac{\sec^2 x}{\text{cosec}^2 x}\ dx = \int \dfrac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}}\ dx = \int \dfrac{\sin^2 x}{\cos^2 x}\ dx$

$= \int \tan^2 x\ dx = \int (\sec^2 x - 1)\ dx$

$(\because \sec^2 x - \tan^2 x = 1 \Rightarrow \sec^2 x - 1 = \tan^2 x)$

$= \int \sec^2 x\ dx - \int 1\ dx = \tan x - x + c.$

Note. Similarly $\int \cot^2 x\ dx = \int (\text{cosec}^2 x - 1)\ dx$

$= \int \text{cosec}^2 x\, dx - \int 1\ dx = -\cot x - x + c.$

20. $\int \dfrac{2 - 3\sin x}{\cos^2 x}\ dx.$

Sol. $\int \dfrac{2 - 3\sin x}{\cos^2 x}\ dx = \int \left(\dfrac{2}{\cos^2 x} - \dfrac{3\sin x}{\cos^2 x}\right) dx$

$= \int \left(2\sec^2 x - \dfrac{3\sin x}{\cos x\cos x}\right) dx = \int (2\sec^2 x - 3\tan x\sec x)\ dx$

$= 2\int \sec^2 x\ dx - 3\int \sec x\tan x\ dx = 2\tan x - 3\sec x + c.$

21. Choose the correct answer:

The anti derivative of $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$ equals

(A) $\frac{1}{3}x^{1/3} + 2x^{1/2} + C$ **(B)** $\frac{2}{3}x^{2/3} + \frac{1}{2}x^2 + C$

(C) $\frac{2}{3}x^{3/2} + 2x^{1/2} + C$ **(D)** $\frac{3}{2}x^{3/2} + \frac{1}{2}x^{1/2} + C.$

Sol. The anti derivative of the $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$

$$= \int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right) dx = \int (x^{1/2} + x^{-1/2})\ dx$$

$$= \int x^{1/2}\ dx + \int x^{-1/2}\ dx = \frac{x^{1/2+1}}{\frac{1}{2}+1} + \frac{x^{1/2+1}}{\frac{1}{2}+1} + C$$

$$= \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{1/2}}{\frac{1}{2}} + C = \frac{2}{3}x^{3/2} + 2x^{1/2} + C$$

$\therefore$ Option (C) is the correct answer.

22. Choose the correct answer:

If $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$. Then $f(x)$ is

(A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$ **(B)** $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$ **(D)** $x^3 + \frac{1}{x^4} - \frac{129}{8}.$

Sol. Given: $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$ and $f(2) = 0$

$\therefore$ By definition of anti derivative (*i.e.,* Integral),

$$f(x) = \int \left(4x^3 - \frac{3}{x^4}\right) dx = 4\int x^3\ dx - 3\int \frac{1}{x^4}\ dx$$

$$= 4 \cdot \frac{x^4}{4} - 3\int x^{-4}\ dx = x^4 - 3\ \frac{x^{-3}}{-3} + c$$

or $f(x) = x^4 + \frac{1}{(x^3)} + c$ ...(*i*)

To find *c*. Let us make use of $f(2) = 0$ (given)

Putting $x = 2$ on both sides of (*i*),

$$f(2) = 16 + \frac{1}{8} + c \qquad \text{or} \qquad 0 = \frac{128+1}{8} + c$$

$(\because \ f(2) = 0 \text{ (given)})$

$$\text{or} \quad c + \frac{129}{8} = 0 \qquad \text{or} \quad c = \frac{-129}{8}$$

Putting $c = \dfrac{-129}{8}$ in (i), $f(x) = x^4 + \dfrac{1}{(x^3)} - \dfrac{129}{8}$

$\therefore$ Option (A) is the correct answer.

Exercise 7.2

Integrate the functions in Exercises 1 to 8:

1. $\dfrac{2x}{1+x^2}$

Sol. To evaluate $\int \dfrac{2x}{1+x^2}\, dx$

Put $1 + x^2 = t$. Therefore $2x = \dfrac{dt}{dx}$ or $2x\, dx = dt$

$\therefore \quad \int \dfrac{2x}{1+x^2}\, dx = \int \dfrac{dt}{t} = \int \dfrac{1}{t}\, dt = \log |t| + c$

Putting $t = 1 + x^2$, $\quad = \log |1 + x^2| + c = \log (1 + x^2) + c.$

$(\because \; 1 + x^2 > 0.$ Therefore $|1 + x^2| = 1 + x^2)$

2. $\dfrac{(\log x)^2}{x}$.

Sol. To evaluate $\int \dfrac{(\log x)^2}{x}\, dx$

Put $\log x = t$. Therefore $\dfrac{1}{x} = \dfrac{dt}{dx} \quad \Rightarrow \quad \dfrac{dx}{x} = dt$

$\therefore \quad \int \dfrac{(\log x)^2}{x}\, dx = \int t^2\, dt = \dfrac{t^3}{3} + c$

Putting $t = \log x$, $= \dfrac{1}{3} (\log x)^3 + c.$

3. $\dfrac{1}{x + x \log x}$

Sol. To evaluate $\int \dfrac{1}{x + x\log x}\, dx = \int \dfrac{1}{x(1+\log x)}\, dx$

Put $1 + \log x = t$. Therefore $\dfrac{1}{x} = \dfrac{dt}{dx} \quad \Rightarrow \quad \dfrac{dx}{x} = dt$

$\therefore \int \dfrac{1}{x + x\log x}\, dx = \int \dfrac{1}{1+\log x} \dfrac{dx}{x} = \int \dfrac{1}{t}\, dt = \log |t| + c$

Putting $t = 1 + \log x$, $\log |1 + \log x| + c.$

4. $\sin x \sin (\cos x)$

Sol. To evaluate $\int \sin x \sin (\cos x)\, dx = -\int \sin (\cos x) (-\sin x)\, dx$

Put $\cos x = t$. Therefore $-\sin x = \dfrac{dt}{dx}$

$\therefore \qquad -\sin x\, dx = dt$

$\therefore \quad \int \sin x \sin (\cos x)\, dx = -\int \sin (\cos x)(-\sin x\, dx)$

$= -\int \sin t\, dt = -(-\cos t) + c$

$= \cos t + c$

Putting $t = \cos x$, $= \cos(\cos x) + c.$

5. sin (*ax* + *b*) cos (*ax* + *b*)

Sol. To evaluate $\int \sin(ax+b)\cos(ax+b)\ dx$

$$= \frac{1}{2}\int 2\sin(ax+b)\cos(ax+b)\ dx \quad = \frac{1}{2}\int \sin 2(ax+b)\ dx$$

$(\because\ 2\sin\theta\cos\theta = \sin 2\theta)$

$$= \frac{1}{2}\int \sin(2ax+2b)\ dx = \frac{1}{2}\ \frac{[-\cos(2ax+2b)]}{2a \to \text{Coeff. of } x} + c$$

$$= \frac{-1}{4a}\cos 2(ax+b) + c.$$

6. $\sqrt{ax+b}$

Sol. To evaluate $\int \sqrt{ax+b}\ dx = \int (ax+b)^{1/2}\ dx$

$$= \frac{(ax+b)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right) a \to \text{Coeff. of } x} + c = \frac{(ax+b)^{\frac{3}{2}}}{\frac{3}{2}a} + c$$

$$\left[\because\ \int (ax+b)^n\ dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \text{ if } n \neq -1\right]$$

$$= \frac{2}{3a}(ax+b)^{3/2} + c.$$

7. $x\sqrt{x+2}$

Sol. To evaluate $\int x\sqrt{x+2}\ dx$

$$= \int x\sqrt{x+2}\ dx = \int ((x+2)-2)\sqrt{x+2}\ dx$$

$$= \int \left((x+2)(x+2)^{\frac{1}{2}} - 2(x+2)^{\frac{1}{2}}\right) dx \quad = \int \left((x+2)^{\frac{3}{2}} - 2(x+2)^{\frac{1}{2}}\right) dx$$

$$= \int (x+2)^{\frac{3}{2}}\ dx - 2\int (x+2)^{\frac{1}{2}}\ dx$$

$$= \frac{(x+2)^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right) 1 \to \text{Coeff. of } x} - 2\frac{(x+2)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right).1} + c = \frac{(x+2)^{\frac{5}{2}}}{\frac{5}{2}} - 2\frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + c.$$

OR

To evaluate $\int x\sqrt{x+2}\ dx$

Put $\sqrt{\textbf{Linear}}$ **= *t*,** *i.e.,* $\sqrt{x+2} = t.$
Squaring $x + 2 = t^2$ $(\Rightarrow \ x = t^2 - 2)$

$\therefore \quad \frac{dx}{dt} = 2t, \quad$ *i.e.,* $\frac{dx}{dt} = 2t \quad$ or $\quad dx = 2t\ dt$

$\therefore \int x\sqrt{x+2}\ dx = \int (t^2-2)\,t\,.\,2t\ dt = \int 2t^2(t^2-2)\ dt$

$= \int 2t^2(t^2-2)\ dt = 2\int t^4\ dt - 4\int t^2\ dt = 2\frac{t^5}{5} - 4\frac{t^3}{3} + c$

Putting $t = \sqrt{x+2}$, $\quad = \frac{2}{5}(\sqrt{x+2})^5 - \frac{4}{3}(\sqrt{x+2})^3 + c$

$= \frac{2}{5}((x+2)^{1/2})^5 - \frac{4}{3}((x+2)^{1/2})^3 + c = \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + c.$

8. $\boldsymbol{x\sqrt{1+2x^2}}$

Sol. To evaluate $\int x\sqrt{1+2x^2}\ dx$

Let $I = \int x\sqrt{1+2x^2}\ dx = \frac{1}{4}\int \sqrt{1+2x^2}\,(4x\,dx)$ ...(*i*)

$\left[\because \frac{d}{dx}(1+2x^2) = 0 + 2\,.\,2x = 4x\right]$

Put $1 + 2x^2 = t$. Therefore $4x = \frac{dt}{dx}$ or $4x\ dx = dt$

$\therefore$ From (*i*), $I = \frac{1}{4}\int \sqrt{t}\ dt = \frac{1}{4}\int t^{1/2}\ dt$

$$= \frac{1}{4}\frac{t^{3/2}}{\frac{3}{2}} + c = \frac{1}{4}\,.\,\frac{2}{3}\,t^{3/2} + c$$

Putting $t = 1 + 2x^2$, $= \frac{1}{6}(1+2x^2)^{3/2} + c.$

Integrate the functions in Exercises 9 to 17:

9. $\boldsymbol{(4x+2)\sqrt{x^2+x+1}}$ **.**

Sol. Let $I = \int (4x+2)\sqrt{x^2+x+1}\ dx = \int 2(2x+1)\sqrt{x^2+x+1}\ dx$

$= \int 2\sqrt{x^2+x+1}\ (2x+1)\ dx$ ...(*i*)

Put $\boldsymbol{x^2 + x + 1 = t}$**.** Therefore $(2x+1) = \frac{dt}{dx}$

$\therefore \quad (2x+1)\ dx = dt$

$\therefore \quad$ From (*i*), $I = \int 2\sqrt{t}\ dt = 2\int t^{1/2}\ dt$

$= 2\frac{t^{3/2}}{\frac{3}{2}} + c = \frac{4}{3}t^{3/2} + c$

Putting $t = x^2 + x + 1$, $I = \frac{4}{3} (x^2 + x + 1)^{3/2} + c$.

10. $\dfrac{\mathbf{1}}{\boldsymbol{x - \sqrt{x}}}$

Sol. Let $I = \int \frac{1}{x - \sqrt{x}} \, dx$ ...(*i*)

Put $\sqrt{\textbf{Linear}}$ **= *t*,** *i.e.,* $\sqrt{x} = t$

Squaring $x = t^2$. Therefore $\frac{dx}{dt} = 2t$ or $dx = 2t\, dt$

$\therefore$ From (*i*), $I = \int \frac{1}{t^2 - t} 2t \, dt = 2 \int \frac{t}{t(t-1)} \, dt$

$= 2\int \frac{1}{t-1} \, dt = 2 \log | t - 1 | + c \left(\because \int \frac{1}{ax+b} dx = \frac{1}{a} \log | ax + b | \right)$

Putting $t = \sqrt{x}$, $I = 2 \log | \sqrt{x} - 1 | + c$.

11. $\dfrac{\boldsymbol{x}}{\boldsymbol{\sqrt{x+4}}}$**,** $\boldsymbol{x > 0}$

Sol. Let $I = \int \frac{x}{\sqrt{x+4}} \, dx$ ...(*i*)

$$= \int \frac{x + 4 - 4}{\sqrt{x+4}} \, dx = \int \left(\frac{x+4}{\sqrt{x+4}} - \frac{4}{\sqrt{x+4}} \right) dx$$

$$= \int \sqrt{x+4} \, dx - 4 \int \frac{1}{\sqrt{x+4}} \, dx \left[\because \frac{t}{\sqrt{t}} = \frac{t\sqrt{t}}{\sqrt{t}\sqrt{t}} = \frac{t\sqrt{t}}{t} = \sqrt{t} \right]$$

$$= \int (x+4)^{1/2} \, dx - 4\int (x+4)^{-1/2} \, dx$$

$$= \frac{(x+4)^{3/2}}{\frac{3}{2}(1)} - \frac{4(x+4)^{1/2}}{\frac{1}{2}(1)} + c = \frac{2}{3} (x+4)^{3/2} - 8(x+4)^{1/2} + c$$

$$= \frac{2}{3} (x+4) \sqrt{x+4} - 8\sqrt{x+4} + c$$

$$\left[\because t^{3/2} = t^{\frac{2}{2} + \frac{1}{2}} = t^{1 + \frac{1}{2}} = t^1 \,.\, t^{1/2} = t\sqrt{t} \right]$$

$$= 2\sqrt{x+4} \left(\frac{x+4}{3} - 4 \right) + c = 2\sqrt{x+4} \left(\frac{x+4-12}{3} \right) + c$$

$$= \frac{2}{3} \sqrt{x+4} \, (x - 8) + c.$$

OR

Put $\sqrt{\textbf{Linear}}$ = $\boldsymbol{t}$, *i.e.,* $\sqrt{x+4} = t$.

Squaring $x + 4 = t^2 \quad \Rightarrow \quad x = t^2 - 4$.

Therefore $\frac{dx}{dt} = 2t$ or $dx = 2t\ dt$

$$\therefore \quad I = \int \frac{x}{\sqrt{x+4}}\ dx = \int \frac{t^2-4}{t}\ .\ 2t\ dt$$

$$= 2\int (t^2-4)\ dt = 2\left[\int t^2\ dt - 4\int 1\ dt\right]$$

$$= 2\left[\frac{t^3}{3} - 4t\right] + c = \frac{2t}{3}\ (t^2 - 12) + c.$$

Putting $t = \sqrt{x+4}$, $= \frac{2}{3}\ \sqrt{x+4}\ (x + 4 - 12) + c$

$$= \frac{2}{3}\ \sqrt{x+4}\ (x - 8) + c.$$

12. $\boldsymbol{(x^3 - 1)^{1/3}\ x^5}$

Sol. Let $I = \int (x^3-1)^{1/3}\ x^5\ dx = \int (x^3-1)^{1/3}\ x^3\ x^2\ dx$

$$= \frac{1}{3}\int (x^3-1)^{1/3}\ x^3\ (3x^2\ dx) \quad ...(i) \quad \left[\because \frac{d}{dx}(x^3-1) = 3x^2\right]$$

Put $x^3 - 1 = t \quad \Rightarrow \quad x^3 = t + 1$

$\therefore \quad 3x^2 = \frac{dt}{dx} \quad \Rightarrow \quad 3x^2\ dx = dt$

$\therefore$ From (i), $I = \frac{1}{3}\int t^{1/3}\ (t+1)\ dt$

$$= \frac{1}{3}\int (t^{4/3} + t^{1/3})\, dt \qquad \left[\because \frac{1}{3} + 1 = \frac{1+3}{3} = \frac{4}{3}\right]$$

$$= \frac{1}{3}\left(\int t^{4/3}\ dt + \int t^{1/3}\ dt\right)$$

$$= \frac{1}{3}\left(\frac{t^{7/3}}{\frac{7}{3}} + \frac{t^{4/3}}{\frac{4}{3}}\right) + c = \frac{1}{3}\left(\frac{3}{7}t^{7/3} + \frac{3}{4}t^{4/3}\right) + c = \frac{1}{7}t^{7/3} + \frac{1}{4}t^{4/3} + c$$

Putting $t = x^3 - 1$, $= \frac{1}{7}\ (x^3-1)^{7/3} + \frac{1}{4}\ (x^3-1)^{4/3} + c.$

13. $\boldsymbol{\dfrac{x^2}{(2+3x^3)^3}}$

Sol. Let $I = \int \frac{x^2}{(2+3x^3)^3}\ dx$

$$= \frac{1}{9}\int \frac{9x^2}{(2+3x^3)^3}\ dx \quad ...(i) \quad \left[\because \frac{d}{dx}(2+3x^3) = 9x^2\right]$$

Put $2 + 3x^3 = t$. Therefore $9x^2 = \frac{dt}{dx} \quad \Rightarrow \quad 9x^2\ dx = dt$

$\therefore$ From (i), $\text{I} = \frac{1}{9} \int t^{-3}\ dt \qquad = \frac{1}{9} \frac{t^{-2}}{-2} + c = \frac{-1}{18t^2} + c$

Putting $t = 2 + 3x^3$; $= \frac{-1}{18(2+3x^3)^2} + c.$

14. $\boldsymbol{\frac{1}{x (\log x)^m}, x > 0}$ ***(Important)***

Sol. Let $\text{I} = \int \frac{1}{x(\log x)^m}\ dx\ (x > 0) \quad \Rightarrow \quad \text{I} = \int \frac{\frac{1}{x} dx}{(\log x)^m}$ ...(i)

Put $\log x = t$. Therefore $\frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{dx}{x} = dt$

$\therefore$ From (i), $\text{I} = \int \frac{dt}{t^m} = \int t^{-m}\ dt = \frac{t^{-m+1}}{-m+1} + c$

(Assuming $m \neq 1$)

Putting $t = \log x$, $= \frac{(\log x)^{1-m}}{1-m} + c.$

15. $\boldsymbol{\frac{x}{9 - 4x^2}}$

Sol. Let $\text{I} = \int \frac{x}{9-4x^2}\ dx = \frac{-1}{8} \int \frac{-8x}{9-4x^2}\ dx$ ...(i)

$\left[\because \frac{d}{dx}(9-4x^2) = -8x\right]$

Put $9 - 4x^2 = t$. Therefore $-8x = \frac{dt}{dx} \quad \Rightarrow \quad -8x\ dx = dt$

$\therefore$ From (i), $\text{I} = \frac{-1}{8} \int \frac{dt}{t} = \frac{-1}{8} \int \frac{1}{t}\ dt \qquad = \frac{-1}{8} \log |t| + c$

Putting $t = 9 - 4x^2$, $= \frac{-1}{8} \log |9 - 4x^2| + c.$

16. $\boldsymbol{e^{2x+3}}$

Sol. $\int e^{2x+3}\ dx = \frac{e^{2x+3}}{2 \to \text{Coeff. of } x} + c \qquad \left[\because \int e^{ax+b}\ dx = \frac{e^{ax+b}}{a}\right]$

$= \frac{1}{2} e^{2x+3} + c.$

17. $\boldsymbol{\frac{x}{e^{x^2}}}$

Sol. Let $\text{I} = \int \frac{x}{(e^{x^2})}\ dx = \frac{1}{2} \int \frac{2x}{(e^{x^2})}\ dx$ ...(i)

Put $x^2 = t$. Therefore $2x = \frac{dt}{dx} \quad \Rightarrow \quad 2x\ dx = dt.$

$\therefore$ From (i), $\quad I = \dfrac{1}{2}\displaystyle\int \dfrac{dt}{(e^t)} = \dfrac{1}{2}\int e^{-t}\, dt$

$$= \frac{1}{2}\,\frac{e^{-t}}{-1 \to \text{Coeff. of } t} + c = \frac{-1}{2(e^t)} + c$$

Putting $t = x^2$, $I = \dfrac{-1}{2(e^{x^2})} + c.$

Integrate the functions in Exercises 18 to 26:

18. $\dfrac{e^{\tan^{-x} x}}{1 + x^2}$

Sol. Let $I = \displaystyle\int \dfrac{e^{\tan^{-1} x}}{1 + x^2}\, dx$ ...(i)

Put $\tan^{-1} x = t.$

$\therefore \quad \dfrac{1}{1 + x^2} = \dfrac{dt}{dx} \quad \Rightarrow \quad \dfrac{dx}{1 + x^2} = dt$

$\therefore$ From (i), $\quad I = \displaystyle\int e^t\, dt = e^t + c = e^{\tan^{-1} x} + c.$

19. $\dfrac{e^{2x} - 1}{e^{2x} + 1}$

Sol. Let $I = \displaystyle\int \dfrac{e^{2x} - 1}{e^{2x} + 1}\, dx$

Multiplying every term in integrand by e^{-x},

$$I = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx \qquad ...(i) \quad [\because \ e^{2x} \,.\, e^{-x} = e^{2x - x} = e^x]$$

Put denominator $e^x + e^{-x} = t$

$\therefore \quad e^x + e^{-x}\dfrac{d}{dx}(-x) = \dfrac{dt}{dx} \quad \Rightarrow \quad (e^x - e^{-x})\, dx = dt$

$\therefore$ From (i), $I = \displaystyle\int \dfrac{dt}{t} = \int \dfrac{1}{t}\, dt = \log|t| + c$

Putting $t = e^x + e^{-x}$, $I = \log|e^x + e^{-x}| + c$ or $I = \log(e^x + e^{-x}) + c$

$$\left[\because \ e^x + e^{-x} = e^x + \frac{1}{(e^x)} > 0 \text{ for all real } x \text{ and hence } |e^x + e^{-x}| = e^x + e^{-x}\right]$$

20. $\dfrac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$

Sol. Let $I = \displaystyle\int \dfrac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\, dx = \dfrac{1}{2}\int \dfrac{2(e^{2x} - e^{-2x})}{e^{2x} + e^{-2x}}\, dx$ ...(i)

Put denominator $\quad e^{2x} + e^{-2x} = t$

$\therefore \quad e^{2x}\dfrac{d}{dx}\,2x + e^{-2x}\dfrac{d}{dx}(-2x) = \dfrac{dt}{dx}$

$\Rightarrow \quad e^{2x} \,.\, 2 - 2e^{-2x} = \dfrac{dt}{dx} \quad \Rightarrow \quad 2(e^{2x} - e^{-2x})\, dx = dt$

$\therefore$ From (i), $\quad I = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \log|t| + c$

Putting $t = e^{2x} + e^{-2x}$, $= \frac{1}{2}\log|e^{2x} + e^{-2x}| + c = \frac{1}{2}\log(e^{2x} + e^{-2x}) + c$

$[\because \; e^{2x} + e^{-2x} > 0 \quad \Rightarrow \quad |e^{2x} + e^{-2x}| = e^{2x} + e^{-2x}]$

21. $\tan^2 (2x - 3)$

Sol. $\int \tan^2(2x-3)\, dx = \int (\sec^2(2x-3) - 1)\, dx \; (\because \tan^2\theta = \sec^2\theta - 1)$

$= \int \sec^2(2x-3)\, dx - \int 1\, dx$

$= \frac{\tan(2x-3)}{2 \to \text{Coeff. of } x} - x + c = \frac{1}{2}\tan(2x-3) - x + c$

$\left[\because \int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + c\right]$

22. $\sec^2 (7 - 4x)$

Sol. $\int \sec^2(7-4x)\, dx = \frac{\tan(7-4x)}{-4 \to \text{Coeff. of } x} + c$

$\left[\because \int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + c\right]$

$= \frac{-1}{4}\tan(7-4x) + c.$

23. $\dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$

Sol. Let $I = \int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$ ...(i)

Put $\sin^{-1} x = t \quad \therefore \quad \frac{1}{\sqrt{1-x^2}} = \frac{dt}{dx} \quad \Rightarrow \quad \frac{dx}{\sqrt{1-x^2}} = dt$

$\therefore$ From (i), $I = \int t\, dt = \frac{t^2}{2} + c$

Putting $t = \sin^{-1} x$, $I = \frac{1}{2}(\sin^{-1} x)^2 + c.$

24. $\dfrac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$

Sol. Let $I = \int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}\, dx = \int \frac{2\cos x - 3\sin x}{2(2\sin x + 3\cos x)}\, dx$

$= \frac{1}{2}\int \frac{2\cos x - 3\sin x}{2\sin x + 3\cos x}\, dx$ ...(i)

Put DENOMINATOR $2\sin x + 3\cos x = t$

$\therefore \quad 2\cos x - 3\sin x = \frac{dt}{dx} \Rightarrow (2\cos x - 3\sin x)\, dx = dt$

$\therefore$ From (i), $\quad I = \dfrac{1}{2} \displaystyle\int \dfrac{dt}{t} = \dfrac{1}{2} \log |t| + c.$

Putting $t = 2 \sin x + 3 \cos x$, $= \dfrac{1}{2} \log | 2 \sin x + 3 \cos x | + c.$

25. $\mathbf{\dfrac{1}{\cos^2 x (1 - \tan x)^2}}$

Sol. Let $I = \displaystyle\int \dfrac{1}{\cos^2 x (1 - \tan x)^2} dx = \int \dfrac{\sec^2 x}{(1 - \tan x)^2} dx$

$$= - \int \frac{- \sec^2 x}{(1 - \tan x)^2} dx \qquad ...(i)$$

Put $1 - \tan x = t$.

$\therefore \quad - \sec^2 x = \dfrac{dt}{dx} \qquad \Rightarrow \quad - \sec^2 x \, dx = dt$

$\therefore$ From (i), $I = - \displaystyle\int \dfrac{dt}{t^2} = -\int t^{-2} \, dt$

$$= - \frac{t^{-1}}{-1} + c = \frac{1}{t} + c = \frac{1}{1 - \tan x} + c.$$

26. $\mathbf{\dfrac{\cos \sqrt{x}}{\sqrt{x}}}$

Sol. Let $I = \displaystyle\int \dfrac{\cos \sqrt{x}}{\sqrt{x}} dx \qquad ...(i)$

Put $\sqrt{\textbf{Linear}}$ **= t,** *i.e.,* $\sqrt{x} = t$

Squaring, $x = t^2$. Therefore $\dfrac{dx}{dt} = 2t \qquad \therefore \quad dx = 2t \, dt$

$\therefore$ From (i), $\quad I = \displaystyle\int \dfrac{\cos t}{t} 2t \, dt = 2\int \cos t \, dt = 2 \sin t + c$

Putting $t = \sqrt{x}$, $I = 2 \sin \sqrt{x} + c.$

Integrate the functions in Exercises 27 to 37:

27. $\mathbf{\sqrt{\sin 2x} \cos 2x}$

Sol. Let $I = \displaystyle\int \sqrt{\sin 2x} \cos 2x \, dx = \dfrac{1}{2} \int \sqrt{\sin 2x} \, (2 \cos 2x \, dx) \qquad ...(i)$

Put $\sin 2x = t$

$\therefore \quad \cos 2x \dfrac{d}{dx}(2x) = \dfrac{dt}{dx} \qquad \Rightarrow \quad 2 \cos 2x \, dx = dt$

$\therefore$ From (i), $\quad I = \dfrac{1}{2} \displaystyle\int \sqrt{t} \, dt = \dfrac{1}{2} \int t^{1/2} \, dt$

$$= \frac{1}{2} \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{1}{2} \frac{t^{3/2}}{\frac{3}{2}} + c = \frac{1}{3} (\sin 2x)^{3/2} + c.$$

28. $\dfrac{\cos x}{\sqrt{1+\sin x}}$

Sol. Let $I = \int \dfrac{\cos x}{\sqrt{1+\sin x}}\, dx$ ...(i)

Put $1 + \sin x = t$

$\therefore \quad \cos x = \dfrac{dt}{dx}$ or $\cos x\, dx = dt$

$\therefore$ From (i), $I = \int \dfrac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \dfrac{t^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + c$

$= \dfrac{t^{1/2}}{\frac{1}{2}} + c = 2\sqrt{t} + c = 2\sqrt{1+\sin x} + c.$

29. cot x log sin x

Sol. Let $I = \int \cot x \log \sin x\, dx$ ...(i)

Put $\log \sin x = t$

$\therefore \quad \dfrac{1}{\sin x}\dfrac{d}{dx}(\sin x) = \dfrac{dt}{dx}$ or $\dfrac{1}{\sin x}\cos x = \dfrac{dt}{dx}$

or $\cot x\, dx = dt$

$\therefore$ From (i), $I = \int t\, dt = \dfrac{t^2}{2} + c \quad = \dfrac{1}{2}(\log \sin x)^2 + c.$

30. $\dfrac{\sin x}{1+\cos x}$

Sol. Let $I = \int \dfrac{\sin x}{1+\cos x}\, dx \quad = -\int \dfrac{-\sin x}{1+\cos x}\, dx$ ...(i)

Put $1 + \cos x = t$. Therefore $-\sin x = \dfrac{dt}{dx}$

$\therefore \quad -\sin x\, dx = dt$

$\therefore$ From (i), $I = -\int \dfrac{dt}{t} \quad = -\log|t| + c$

Putting $t = 1 + \cos x$, $= -\log|1 + \cos x| + c.$

31. $\dfrac{\sin x}{(1+\cos x)^2}$

Sol. Let $I = \int \dfrac{\sin x}{(1+\cos x)^2}\, dx = -\int \dfrac{-\sin x\, dx}{(1+\cos x)^2}$ ...(i)

Put $1 + \cos x = t$. Therefore $-\sin x = \dfrac{dt}{dx}$

$\Rightarrow \quad -\sin x \, dx = dt$

$\therefore$ From (*i*), $\quad I = -\int \frac{dt}{t^2} = -\int t^{-2} \, dt = \frac{-t^{-1}}{-1} + c$

$$= \frac{1}{t} + c = \frac{1}{1+\cos x} + c.$$

32. $\mathbf{\dfrac{1}{1+\cot x}}$

Sol. Let $I = \int \frac{1}{1+\cot x} dx = \int \frac{1}{1+\frac{\cos x}{\sin x}} dx = \int \frac{1}{\left(\frac{\sin x + \cos x}{\sin x}\right)} dx$

$= \int \frac{\sin x}{\sin x + \cos x} dx = \frac{1}{2} \int \frac{2 \sin x}{\sin x + \cos x} dx = \frac{1}{2} \int \frac{\sin x + \sin x}{\sin x + \cos x} dx$

Adding and subtracting $\cos x$ in the numerator of integrand,

$$I = \frac{1}{2} \int \frac{\sin x + \cos x - \cos x + \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2} \int \frac{(\sin x + \cos x) - (\cos x - \sin x)}{\sin x + \cos x} dx$$

$$= \frac{1}{2} \int \left(\frac{\sin x + \cos x}{\sin x + \cos x} - \frac{(\cos x - \sin x)}{\sin x + \cos x} \right) dx \quad \left[\because \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}\right]$$

$$= \frac{1}{2} \int \left(1 - \frac{(\cos x - \sin x)}{\sin x + \cos x} \right) dx$$

$$= \frac{1}{2} \left[\int 1 \, dx - \int \frac{\cos x - \sin x}{\sin x + \cos x} dx \right] = \frac{1}{2} [x - I_1] \qquad ...(i)$$

where $I_1 = \int \frac{\cos x - \sin x}{\sin x + \cos x} dx$

Put DENOMINATOR $\sin x + \cos x = t$

$\therefore \quad \cos x - \sin x = \frac{dt}{dx} \qquad \Rightarrow \quad (\cos x - \sin x) \, dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{t} = \log |t| = \log |\sin x + \cos x|.$

Note. Alternative solution for finding I_1

$$I_1 = \int \frac{\cos x - \sin x}{\sin x + \cos x} dx = \log |\sin x + \cos x|$$

$$\left[\because \int \frac{f'(x)}{f(x)} dx = \log |f(x)|\right]$$

Putting this value of I_1 in (*i*), required integral

$$= \frac{1}{2} [x - \log |\sin x + \cos x|] + c.$$

33. $\dfrac{\mathbf{1}}{\mathbf{1-\tan x}}$

Sol. Let I $= \displaystyle\int \frac{1}{1-\tan x}\, dx = \int \frac{1}{1-\dfrac{\sin x}{\cos x}}\, dx = \int \frac{1}{\left(\dfrac{\cos x - \sin x}{\cos x}\right)}\, dx$

$= \displaystyle\int \frac{\cos x}{\cos x - \sin x} dx = \frac{1}{2}\int \frac{2\cos x}{\cos x - \sin x} dx = \frac{1}{2}\int \frac{\cos x + \cos x}{\cos x - \sin x} dx$

Subtracting and adding $\sin x$ in the Numerator,

$= \displaystyle\frac{1}{2}\int \frac{\cos x - \sin x + \sin x + \cos x}{\cos x - \sin x}\, dx$

$= \displaystyle\frac{1}{2}\int \left(\frac{\cos x - \sin x}{\cos x - \sin x} + \frac{\sin x + \cos x}{\cos x - \sin x}\right) dx = \frac{1}{2}\int \left(1 + \frac{\sin x + \cos x}{\cos x - \sin x}\right) dx$

$= \displaystyle\frac{1}{2}\left[\int 1\, dx - \int \frac{-\sin x - \cos x}{\cos x - \sin x}\, dx\right]$

$= \displaystyle\frac{1}{2}\,[x - \log|\cos x - \sin x|] + c \qquad \left[\because \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)|\right]$

Note. Alternative solution for evaluating $\displaystyle\int \frac{-\sin x - \cos x}{\cos x - \sin x}\, dx$, put denominator $\cos x - \sin x = t$.

34. $\dfrac{\sqrt{\tan x}}{\sin x \cos x}$

Sol. Let I $= \displaystyle\int \frac{\sqrt{\tan x}}{\sin x \cos x}\, dx = \int \frac{\sqrt{\tan x}}{\dfrac{\sin x}{\cos x}\cos x \cos x}\, dx$

$= \displaystyle\int \frac{\sqrt{\tan x}}{\tan x \cos^2 x}\, dx = \int \frac{\sec^2 x}{\sqrt{\tan x}}\, dx \qquad ...(i) \qquad \left[\because \frac{\sqrt{t}}{t} = \frac{1}{\sqrt{t}}\right]$

Put $\tan x = t$.

$\therefore \quad \sec^2 x = \dfrac{dt}{dx} \qquad \Rightarrow \quad \sec^2 x\, dx = dt$

$\therefore$ From (i),

$\text{I} = \displaystyle\int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{1/2}}{\dfrac{1}{2}} + c = 2\sqrt{t} + c = 2\sqrt{\tan x} + c.$

35. $\dfrac{\mathbf{(1+\log x)^2}}{\boldsymbol{x}}$

Sol. Let I $= \displaystyle\int \frac{(1+\log x)^2}{x}\, dx \qquad ...(i)$

Put $1 + \log x = t$

$\therefore \quad \dfrac{1}{x} = \dfrac{dt}{dx} \qquad \Rightarrow \quad \dfrac{dx}{x} = dt$

$\therefore$ From (i), I $= \displaystyle\int t^2\, dt = \frac{t^3}{3} + c = \frac{1}{3}(1+\log x)^3 + c.$

36. $\dfrac{\boldsymbol{(x+1)(x+\log x)^2}}{\boldsymbol{x}}$

Sol. Let I = $\displaystyle\int \frac{(x+1)(x+\log x)^2}{x}\, dx$ ...(*i*)

Put x + log x = t

$\therefore \quad 1 + \dfrac{1}{x} = \dfrac{dt}{dx} \quad \Rightarrow \quad \dfrac{x+1}{x} = \dfrac{dt}{dx} \quad \Rightarrow \quad \left(\dfrac{x+1}{x}\right) dx = dt$

$\therefore$ From (*i*), I = $\displaystyle\int t^2\, dt = \frac{t^3}{3} + c$

Putting $t = x + \log x$, $\dfrac{1}{3}(x + \log x)^3 + c$.

37. $\dfrac{\boldsymbol{x^3 \sin(\tan^{-1} x^4)}}{\boldsymbol{1+x^8}}$

Sol. Let I = $\displaystyle\int \frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}\, dx = \frac{1}{4}\int \sin(\tan^{-1} x^4) \cdot \frac{4x^3}{1+x^8}\, dx$ ...(*i*)

Put (tan^{-1} x^4) = t

[Rule for $\int \sin(f(x))\, f'(x)\, dx$; put $f(x) = t$]

$\therefore \quad \dfrac{1}{1+(x^4)^2}\dfrac{d}{dx}x^4 = \dfrac{dt}{dx} \quad \left[\because \dfrac{d}{dx}\tan^{-1} f(x) = \dfrac{1}{1+(f(x))^2}\dfrac{d}{dx}f(x)\right]$

$\Rightarrow \quad \dfrac{4x^3}{1+x^8}\, dx = dt$

$\therefore$ From (*i*),

$\text{I} = \dfrac{1}{4}\displaystyle\int \sin t\ dt = -\frac{1}{4}\cos t + c \quad = \frac{-1}{4}\cos(\tan^{-1} x^4) + c.$

Choose the correct answer in Exercises 38 and 39:

38. $\displaystyle\int \frac{\boldsymbol{10x^9 + 10^x \log_e 10\, dx}}{\boldsymbol{x^{10} + 10^x}}$ **equals**

(A) $\boldsymbol{10^x - x^{10} + C}$ **(B)** $\boldsymbol{10^x + x^{10} + C}$

(C) $\boldsymbol{(10^x - x^{10})^{-1} + C}$ **(D)** $\boldsymbol{\log(10^x + x^{10}) + C.}$

Sol. Let I = $\displaystyle\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\, dx$ ...(*i*)

Put x^{10} + 10^x = t

$\therefore \quad (10x^9 + 10^x \log_e 10)\, dx = dt \qquad \left[\because \dfrac{d}{dx}(a^x) = a^x \log_e a\right]$

$\therefore$ From (*i*), I = $\displaystyle\int \frac{dt}{t} = \log|t| + c$

Putting $t = x^{10} + 10^x$, I = $\log|x^{10} + 10^x| + c$

or I = $\log(10^x + x^{10}) + c$.

$\therefore$ Option (D) is the correct answer.

OR

$$\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\, dx = \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c$$

$$= \log |x^{10} + 10^x| + c$$

$\therefore$ Option (D) is the correct answer.

39. $\displaystyle\int \frac{dx}{\sin^2 x \cos^2 x}$ **equals**

(A) $\tan x + \cot x + C$ **(B)** $\tan x - \cot x + C$

(C) $\tan x \cot x + C$ **(D)** $\tan x - \cot 2x + C.$

Sol. $\displaystyle\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\, dx \qquad [\because \ 1 = \sin^2 x + \cos^2 x]$

$$= \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx \qquad \left[\because \ \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}\right]$$

$$= \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right) dx = \int (\sec^2 x + \text{cosec}^2 x)\ dx$$

$$= \int \sec^2 x\ dx + \int \text{cosec}^2 x\ dx = \tan x - \cot x + c$$

$\therefore$ Option (B) is the correct answer.

Exercise 7.3

Find the integrals of the following functions in Exercises 1 to 9:

1. $\sin^2 (2x + 5)$

Sol. $\int \sin^2(2x+5)\ dx = \int \frac{1}{2}(1 - \cos 2(2x+5))\ dx$

$$\left[\because \quad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)\,;\ \text{put } \theta = 2x + 5\right]$$

$= \frac{1}{2} \int (1 - \cos(4x + 10))\ dx \qquad = \frac{1}{2}\left[\int 1\,dx - \int \cos(4x+10)\,dx\right]$

$= \frac{1}{2}\left[x - \frac{\sin(4x+10)}{4 \to \text{Coeff. of } x}\right] + c \ = \frac{1}{2}\ x - \frac{1}{8}\ \sin\ (4x + 10) + c.$

2. $\sin 3x \cos 4x$

Sol. $\int \sin 3x \cos 4x\ dx = \frac{1}{2}\int 2\sin 3x \cos 4x\ dx$

$= \frac{1}{2}\int (\sin(3x + 4x) + \sin(3x - 4x)\ dx$

$$[\because \quad 2 \sin A \cos B = \sin(A + B) + \sin(A - B)]$$

$= \frac{1}{2}\int (\sin 7x + \sin(-x))\ dx \quad = \frac{1}{2}\int (\sin 7x - \sin x)\ dx$

$= \frac{1}{2}\left[\int \sin 7x\,dx - \int \sin x\,dx\right] = \frac{1}{2}\left[\frac{-\cos 7x}{7} - (-\cos x)\right] + c$

$= \frac{-1}{14}\ \cos\ 7x + \frac{1}{2}\ \cos\ x + c.$

3. $\cos 2x \cos 4x \cos 6x$

Sol. $\cos 2x \cos 4x \cos 6x = \frac{1}{2}\ (2 \cos 6x \cos 4x) \cos 2x$

$= \frac{1}{2}\ [\cos(6x + 4x) + \cos(6x - 4x)] \cos 2x$

$$[\because \quad 2 \cos x \, . \, \cos y = \cos(x + y) + \cos(x - y)]$$

$= \frac{1}{2}\ (\cos 10x + \cos 2x) \cos 2x = \frac{1}{4}\ (2 \cos 10x \cos 2x + 2 \cos^2 2x)$

$= \frac{1}{4}\ [\cos(10x + 2x) + \cos(10x - 2x) + 1 + \cos 4x]$

$= \frac{1}{4}\ (\cos 12x + \cos 8x + \cos 4x + 1)$

$\therefore \int \cos 2x \cos 4x \cos 6x\ dx = \frac{1}{4}\int (\cos 12x + \cos 8x + \cos 4x + 1)\,dx$

$= \frac{1}{4}\left[\int \cos 12x\,dx + \int \cos 8x\,dx + \int \cos 4x\,dx + \int 1\,dx\right]$

$= \frac{1}{4}\left(\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4} + x\right) + c.$

Exercise 7.3

Find the integrals of the following functions in Exercises 1 to 9:

1. $\sin^2 (2x + 5)$

Sol. $\int \sin^2(2x+5)\ dx = \int \frac{1}{2}(1 - \cos 2(2x+5))\ dx$

$$\left[\because \ \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)\ ;\ \text{put}\ \theta = 2x+5\right]$$

$= \frac{1}{2} \int (1 - \cos(4x+10))\ dx \quad = \frac{1}{2}\left[\int 1\, dx - \int \cos(4x+10)\, dx\right]$

$= \frac{1}{2}\left[x - \frac{\sin(4x+10)}{4 \to \text{Coeff. of } x}\right] + c \ = \frac{1}{2}x - \frac{1}{8}\sin(4x+10) + c.$

2. $\sin 3x \cos 4x$

Sol. $\int \sin 3x \cos 4x\ dx = \frac{1}{2}\int 2\sin 3x \cos 4x\ dx$

$= \frac{1}{2}\int (\sin(3x+4x) + \sin(3x-4x))\ dx$

$$[\because \ 2 \sin A \cos B = \sin (A + B) + \sin (A - B)]$$

$= \frac{1}{2}\int (\sin 7x + \sin(-x))\ dx \quad = \frac{1}{2}\int (\sin 7x - \sin x)\ dx$

$= \frac{1}{2}\left[\int \sin 7x\, dx - \int \sin x\, dx\right] = \frac{1}{2}\left[\frac{-\cos 7x}{7} - (-\cos x)\right] + c$

$= \frac{-1}{14}\cos 7x + \frac{1}{2}\cos x + c.$

3. $\cos 2x \cos 4x \cos 6x$

Sol. $\cos 2x \cos 4x \cos 6x = \frac{1}{2}(2\cos 6x \cos 4x)\cos 2x$

$= \frac{1}{2}[\cos(6x+4x) + \cos(6x-4x)]\cos 2x$

$$[\because \ 2 \cos x \,.\, \cos y = \cos(x+y) + \cos(x-y)]$$

$= \frac{1}{2}(\cos 10x + \cos 2x)\cos 2x = \frac{1}{4}(2\cos 10x \cos 2x + 2\cos^2 2x)$

$= \frac{1}{4}[\cos(10x+2x) + \cos(10x-2x) + 1 + \cos 4x]$

$= \frac{1}{4}(\cos 12x + \cos 8x + \cos 4x + 1)$

$\therefore \int \cos 2x \cos 4x \cos 6x\ dx = \frac{1}{4}\int (\cos 12x + \cos 8x + \cos 4x + 1)\, dx$

$= \frac{1}{4}\left[\int \cos 12x\, dx + \int \cos 8x\, dx + \int \cos 4x\, dx + \int 1\, dx\right]$

$= \frac{1}{4}\left(\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4} + x\right) + c.$

Note. We know that $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$

$\therefore \quad 4 \sin^3 \theta = 3 \sin \theta - \sin 3\theta$

Dividing by 4, $\sin^3 \theta = \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta$ ...(*i*)

Similarly, $\cos^3 \theta = \frac{3}{4} \cos \theta + \frac{1}{4} \cos 3\theta$ ...(*ii*)

$[\because \quad \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta]$

4. $\sin^3 (2x + 1)$

Sol. To evaluate $\int \sin^3 (2x+1) \ dx$

We know by Eqn. (*i*) of above note that $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4} \sin 3\theta$

Putting $\theta = 2x + 1$, we have

$\sin^3 (2x + 1) = \frac{3}{4} \sin (2x + 1) - \frac{1}{4} \sin 3 (2x + 1)$

$= \frac{3}{4} \sin (2x + 1) - \frac{1}{4} \sin (6x + 3)$

$\therefore \int \sin^3 (2x+1) \ dx = \frac{3}{4} \int \sin (2x+1) \ dx - \frac{1}{4} \int \sin (6x+3) \ dx$

$$= \frac{3}{4} \left(\frac{-\cos (2x+1)}{2} \right) - \frac{1}{4} \left(\frac{-\cos (6x+3)}{6 \to \text{Coeff. of } x} \right) + c$$

$$= \frac{-3}{8} \cos (2x + 1) + \frac{1}{24} \cos (6x + 3) + c.$$

OR

To integrate $\sin^n x$ where n is odd, put $\cos x = t$.

$\therefore \quad \int \sin^3 (2x+1) \ dx = \int \sin^2 (2x+1) \ \sin (2x + 1) \ dx$

$= \frac{-1}{2} \int [1 - \cos^2 (2x+1)] \ (-2 \sin (2x + 1)) \ dx$ ...(*i*)

Put $\cos (2x + 1) = t$

$\therefore \quad -\sin (2x + 1) \frac{d}{dx} (2x + 1) = \frac{dt}{dx} \quad \therefore -2 \sin (2x + 1) \ dx = dt$

$\therefore \quad$ From (*i*), the given integral $= \frac{-1}{2} \int (1 - t^2) \ dt$

$$= \frac{-1}{2} \left(t - \frac{t^3}{3} \right) + c = \frac{-1}{2} t + \frac{1}{6} t^3 + c$$

$$= \frac{-1}{2} \cos (2x + 1) + \frac{1}{6} \cos^3 (2x + 1) + c.$$

5. $\sin^3 x \cos^3 x$

Sol. $\int \sin^3 x \cos^3 x \ dx = \int (\sin x \cos x)^3 \ dx$

$$= \int \left(\frac{1}{2} 2 \sin x \cos x \right)^3 dx = \int \left(\frac{1}{2} \sin 2x \right)^3 dx$$

$$= \frac{1}{8} \int \sin^3 2x \ dx = \frac{1}{8} \int \left(\frac{3}{4} \sin 2x - \frac{1}{4} \sin 6x \right) dx$$

$$\left(\text{Putting } \theta = 2x \text{ in } \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right)$$

$$= \frac{3}{32}\int \sin 2x \; dx - \frac{1}{32}\int \sin 6x \; dx$$

$$= \frac{-3}{32}\frac{\cos 2x}{2} - \frac{1}{32}\left(\frac{-\cos 6x}{6}\right) + c = \frac{-3}{64}\cos 2x + \frac{1}{192}\cos 6x + c.$$

OR

To evaluate $\int \sin^3 x \cos^3 x \, dx$, Put either $\sin x = t$ or $\cos x = t$. (The form of answer given in N.C.E.R.T. book II can be obtained by putting $\cos x = t$)

6. $\sin x \sin 2x \sin 3x$

Sol. $\sin x \sin 2x \sin 3x = \frac{1}{2}(2 \sin 3x \sin 2x) \sin x$

$$= \frac{1}{2}[\cos(3x - 2x) - \cos(3x + 2x)] \sin x$$

$$[\because \; 2 \sin x \sin y = \cos(x - y) - \cos(x + y)]$$

$$= \frac{1}{2}(\cos x - \cos 5x) \sin x = \frac{1}{4}(2 \cos x \sin x - 2 \cos 5x \sin x)$$

$$= \frac{1}{4}[\sin 2x - \{\sin(5x + x) - \sin(5x - x)\}]$$

$$[\because \; 2 \cos x \sin y = \sin(x + y) - \sin(x - y)]$$

$$= \frac{1}{4}(\sin 2x - \sin 6x + \sin 4x)$$

$$\therefore \int \sin x \sin 2x \sin 3x \; dx = \frac{1}{4}\int (\sin 2x + \sin 4x - \sin 6x) \; dx$$

$$= \frac{1}{4}\left[\int \sin 2x \, dx + \int \sin 4x \, dx - \int \sin 6x \, dx\right]$$

$$= \frac{1}{4}\left(-\frac{\cos 2x}{2} - \frac{\cos 4x}{4} + \frac{\cos 6x}{6}\right) + c.$$

7. $\sin 4x \sin 8x$

Sol. $\int \sin 4x \sin 8x \; dx = \frac{1}{2}\int 2 \sin 4x \sin 8x \; dx$

$$= \frac{1}{2}\int [\cos(4x - 8x) - \cos(4x + 8x)] \; dx$$

$$[\because \; 2 \sin A \sin B = \cos(A - B) - \cos(A + B)]$$

$$= \frac{1}{2}\int (\cos(-4x) - \cos 12x) \; dx = \frac{1}{2}\int (\cos 4x - \cos 12x) \; dx$$

$$[\because \; \cos(-\theta) = \cos\theta]$$

$$= \frac{1}{2}\left[\int \cos 4x \, dx - \int \cos 12x \, dx\right] = \frac{1}{2}\left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12}\right] + c.$$

8. $\dfrac{1 - \cos x}{1 + \cos x}$

Sol. $\int \frac{1-\cos x}{1+\cos x}\, dx = \int \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}\, dx = \int \tan^2\frac{x}{2}\, dx$

$\left(\because \quad 1-\cos\theta = 2\sin^2\frac{\theta}{2} \text{ and } 1+\cos\theta = 2\cos^2\frac{\theta}{2}\right)$

$= \int \left(\sec^2\frac{x}{2} - 1\right) dx \qquad (\because \quad \tan^2\theta = \sec^2\theta - 1)$

$= \int \sec^2\frac{x}{2}\, dx - \int 1\, dx = \frac{\tan\frac{x}{2}}{\frac{1}{2} \to \text{Coeff. of } x} - x + c = 2\tan\frac{x}{2} - x + c.$

9. $\mathbf{\frac{\cos x}{1+\cos x}}$

Sol. $\int \frac{\cos x}{1+\cos x}\, dx$

Adding and subtracting 1 in the numerator of integrand,

$= \int \frac{1+\cos x - 1}{1+\cos x} dx = \int \left(\frac{1+\cos x}{1+\cos x} - \frac{1}{1+\cos x}\right) dx \quad \left(\because \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}\right)$

$= \int \left(1 - \frac{1}{2\cos^2\frac{x}{2}}\right) dx = \int 1\, dx - \frac{1}{2}\int \sec^2\frac{x}{2}\, dx$

$= x - \frac{1}{2}\frac{\tan\frac{x}{2}}{\frac{1}{2}} + c = x - \tan\frac{x}{2} + c.$

Find the integrals of the functions in Exercises 10 to 18:

10. $\mathbf{\sin^4 x}$

Sol. $\int \sin^4 x\, dx = \int (\sin^2 x)^2\, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$

$= \int \frac{(1-\cos 2x)^2}{4}\, dx = \frac{1}{4}\int (1+\cos^2 2x - 2\cos 2x)\, dx$

$= \frac{1}{4}\int \left(1 + \left(\frac{1+\cos 4x}{2}\right) - 2\cos 2x\right) dx \quad \left[\because \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$

$= \frac{1}{4}\int \left(\frac{2+1+\cos 4x - 4\cos 2x}{2}\right) dx = \frac{1}{8}\int (3+\cos 4x - 4\cos 2x)\, dx$

$= \frac{1}{8}\left[3\int 1\, dx + \int \cos 4x\, dx - 4\int \cos 2x\, dx\right]$

$$= \frac{1}{8}\left[3x + \frac{\sin 4x}{4} - \frac{4\sin 2x}{2}\right] + c = \frac{3}{8}x + \frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + c$$

11. $\boldsymbol{\cos^4 2x}$

Sol. $\int \cos^4 2x \; dx = \int (\cos^2 2x)^2 \; dx$

$$= \int \left(\frac{1+\cos 4x}{2}\right)^2 dx = \int \frac{1}{4}(1+\cos 4x)^2 \; dx$$

$$= \frac{1}{4}\int (1 + \cos^2 4x + 2\cos 4x)\, dx$$

$$= \frac{1}{4}\int \left(1 + \frac{1+\cos 8x}{2} + 2\cos 4x\right) dx \qquad \left[\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$= \frac{1}{4}\int \left(\frac{2+1+\cos 8x + 4\cos 4x}{2}\right) dx = \frac{1}{8}\int (3 + \cos 8x + 4\cos 4x)\, dx$$

$$= \frac{1}{8}\left[3\int 1\, dx + \int \cos 8x\, dx + 4\int \cos 4x\, dx\right]$$

$$= \frac{1}{8}\left[3x + \frac{\sin 8x}{8} + \frac{4\sin 4x}{4}\right] + c = \frac{3}{8}x + \frac{1}{64}\sin 8x + \frac{1}{8}\sin 4x + c$$

12. $\boldsymbol{\dfrac{\sin^2 x}{1+\cos x}}$

Sol. $\int \frac{\sin^2 x}{1+\cos x}\; dx = \int \frac{1-\cos^2 x}{1+\cos x}\; dx = \int \frac{(1-\cos x)(1+\cos x)}{1+\cos x}\; dx$

$$= \int (1-\cos x)\; dx = \int 1\; dx - \int \cos x\; dx = x - \sin x + c.$$

Note. It may be noted that letters a, b, c, d, ..., q of English Alphabet and letters α, β, γ, δ of Greek Alphabet are generally treated as constants.

13. $\boldsymbol{\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}}$

Sol. $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\; dx = \int \frac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{\cos x - \cos\alpha}\; dx$

$$= \int \frac{2\cos^2 x - 1 - 2\cos^2\alpha + 1}{\cos x - \cos\alpha}\; dx \qquad = \int \frac{2\cos^2 x - 2\cos^2\alpha}{\cos x - \cos\alpha}\; dx$$

$$= 2\int \frac{\cos^2 x - \cos^2\alpha}{\cos x - \cos\alpha}\; dx \qquad = 2\int \frac{(\cos x - \cos\alpha)(\cos x + \cos\alpha)}{(\cos x - \cos\alpha)}\; dx$$

$$= 2\int (\cos x + \cos\alpha)\; dx \qquad = 2\left[\int \cos x\, dx + \int \cos\alpha\, dx\right]$$

$$= 2\,[\sin x + \boldsymbol{\cos\alpha} \int \boldsymbol{1} \;\boldsymbol{dx}] = 2\,[\sin x + (\cos\alpha)\, x] + c$$

$$= 2\sin x + 2x\cos\alpha + c.$$

Remark. $\int \sin a \; dx = \sin a \int 1 \; dx = x \sin a.$

Please note that $\int \sin a\, dx \neq -\cos a.$

14. $\mathbf{\dfrac{\cos x - \sin x}{1 + \sin 2x}}$

Sol. Let $I = \int \dfrac{\cos x - \sin x}{1 + \sin 2x} dx = \int \dfrac{\cos x - \sin x}{\cos^2 x + \sin^2 x + 2 \sin x \cos x} \; dx$

$$= \int \frac{\cos x - \sin x}{(\cos x + \sin x)^2} \; dx \qquad ...(i)$$

Put $\cos x + \sin x = t$.

$\therefore - \sin x + \cos x = \dfrac{dt}{dx}$. Therefore $(\cos x - \sin x) \; dx = dt$.

$\therefore$ From (i), $I = \int \dfrac{dt}{t^2} = \int t^{-2} \; dt = \dfrac{t^{-1}}{-1} + c$

$\Rightarrow I = \dfrac{-1}{t} + c = \dfrac{-1}{\cos x + \sin x} + c.$

15. $\mathbf{\tan^3 2x \sec 2x}$

Sol. Let $I = \int \tan^3 2x \; \sec 2x \; dx = \int \tan^2 2x \tan 2x \sec 2x \; dx$

$= \int (\sec^2 2x - 1) \sec 2x \tan 2x \; dx \qquad [\because \; \tan^2 \theta = \sec^2 \theta - 1]$

$= \dfrac{1}{2} \int (\sec^2 2x - 1)(2 \sec 2x \tan 2x) \; dx \qquad ...(i)$

Put $\sec 2x = t$. Therefore $\sec 2x \tan 2x \dfrac{d}{dx} (2x) = \dfrac{dt}{dx}$

$\therefore \; 2 \sec 2x \tan 2x \; dx = dt$

$\therefore$ From (i), $I = \dfrac{1}{2} \int (t^2 - 1) \; dt = \dfrac{1}{2} \left(\int t^2 \, dt - \int 1 \, dt \right)$

$$= \frac{1}{2} \left(\frac{t^3}{3} - t \right) + c = \frac{1}{6} t^3 - \frac{1}{2} t + c$$

Putting $t = \sec 2x$, $= \dfrac{1}{6} \sec^3 2x - \dfrac{1}{2} \sec 2x + c.$

16. $\mathbf{\tan^4 x}$

Sol. $\int \tan^4 x \; dx = \int \tan^2 x \tan^2 x \; dx = \int \tan^2 x (\sec^2 x - 1) \; dx$

$= \int (\tan^2 x \sec^2 x - \tan^2 x) \; dx = \int \tan^2 x \sec^2 x \; dx - \int \tan^2 x \; dx$

$= \int \tan^2 x \sec^2 x \; dx - \int (\sec^2 x - 1) \; dx$

$= \underbrace{\int \tan^2 x \sec^2 x \; dx}_{\downarrow} - \int \sec^2 x \; dx + \int 1 \; dx$

For this integral, put **$\tan x = t$.**

$\therefore \quad \sec^2 x = \dfrac{dt}{dx} \quad$ or $\quad \sec^2 x \; dx = dt$

$$= \int t^2 \, dt - \tan x + x + c = \frac{t^3}{3} - \tan x + x + c$$

Put $t = \tan x$, $= \frac{1}{3} \tan^3 x - \tan x + x + c.$

17. $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$

Sol. $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} \, dx = \int \left(\frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x} \right) dx$

$$\left(\because \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \right)$$

$$= \int \left(\frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x} \right) dx = \int \left(\frac{\sin x}{\cos x \cos x} + \frac{\cos x}{\sin x \sin x} \right) dx$$

$$= \int (\tan x \sec x + \cot x \operatorname{cosec} x) \, dx$$

$$= \int \sec x \tan x \, dx + \int \operatorname{cosec} x \cot x \, dx = \sec x - \operatorname{cosec} x + c.$$

18. $\dfrac{\cos 2x + 2 \sin^2 x}{\cos^2 x}$

Sol. $\int \frac{\cos 2x + 2 \sin^2 x}{\cos^2 x} \, dx = \int \frac{(1 - 2 \sin^2 x) + 2 \sin^2 x}{\cos^2 x} \, dx$

$$= \int \frac{1}{\cos^2 x} \, dx = \int \sec^2 x \, dx = \tan x + c.$$

Integrate the functions in Exercises 19 to 22:

Note. Method to evaluate $\int \frac{1}{\sin^p x \cos^q x} \, dx$ if $(p + q)$ is a negative even integer $(= -n$ (say)); then multiply Numerator and Denominator of integrand by $\sec^n x$.

19. $\dfrac{1}{\sin x \cos^3 x}$

Sol. Let $I = \int \frac{1}{\sin x \cos^3 x} \, dx$ ...(*i*)

Here $p + q = -1 - 3 = -4$ is a negative even integer.
So multiplying both Numerator and Denominator of integrand of (*i*) by $\sec^4 x$,

$$I = \int \frac{\sec^4 x}{\sin x \cos^3 x \sec^4 x} \, dx = \int \frac{\sec^4 x}{\tan x} \, dx$$

$$\left(\because \sin x \cos^3 x \sec^4 x = \sin x \cos^3 x \, . \frac{1}{\cos^4 x} = \frac{\sin x}{\cos x} = \tan x \right)$$

or $I = \int \frac{\sec^2 x \sec^2 x}{\tan x} \, dx = \int \frac{(1 + \tan^2 x) \sec^2 x}{\tan x} \, dx$ ...(*ii*)

Put $\tan x = t$

$\therefore \quad \sec^2 x = \dfrac{dt}{dx} \quad \Rightarrow \quad \sec^2 x\, dx = dt$

$\therefore \quad$ From (ii), $\text{I} = \displaystyle\int \frac{(1+t^2)}{t}\, dt = \int \left(\frac{1}{t} + \frac{t^2}{t}\right) dt$

$= \displaystyle\int \left(\frac{1}{t} + t\right) dt = \int \frac{1}{t}\, dt + \int t\, dt = \log|t| + \frac{t^2}{2} + c$

Putting $t = \tan x$, $= \log|\tan x| + \dfrac{1}{2}\tan^2 x + c$.

20. $\dfrac{\cos 2x}{(\cos x + \sin x)^2}$

Sol. Let $\text{I} = \displaystyle\int \frac{\cos 2x}{(\cos x + \sin x)^2}\, dx = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2}\, dx$

$= \displaystyle\int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)(\cos x + \sin x)}\, dx = \int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx \quad ...(i)$

Put DENOMINATOR $\cos x + \sin x = t$

$\therefore \quad -\sin x + \cos x = \dfrac{dt}{dx} \quad \Rightarrow \quad (\cos x - \sin x)\, dx = dt$

$\therefore \quad$ From (i), $\text{I} = \displaystyle\int \frac{dt}{t} = \log|t| + c = \log|\cos x + \sin x| + c$

Note. Another method to evaluate integral (i) is, apply

$$\int \frac{f'(x)}{f(x)}\, dx = \log|f(x)|.$$

21. $\sin^{-1}(\cos x)$

Sol. $\displaystyle\int \sin^{-1}(\cos x)\, dx = \int \sin^{-1} \sin\left(\frac{\pi}{2} - x\right) dx$

$= \displaystyle\int \left(\frac{\pi}{2} - x\right) dx = \int \frac{\pi}{2}\, dx - \int x\, dx$

$= \dfrac{\pi}{2}\displaystyle\int 1\, dx - \int x^1\, dx = \frac{\pi}{2}\, x - \frac{x^2}{2} + c.$

22. $\dfrac{1}{\cos(x-a)\cos(x-b)}$

Sol. Let $\text{I} = \displaystyle\int \frac{1}{\cos(x-a)\cos(x-b)}\, dx \quad ...(i)$

Here $(x - a) - (x - b) = x - a - x + b = b - a \quad ...(ii)$

By looking at Eqn. (ii), dividing and multiplying the integrand in (i) by $\sin(b - a)$,

$\text{I} = \dfrac{1}{\sin(b-a)} \displaystyle\int \frac{\sin(b-a)}{\cos(x-a)\cos(x-b)}\, dx$

$= \dfrac{1}{\sin(b-a)} \displaystyle\int \frac{\sin[(x-a)-(x-b)]}{\cos(x-a)\cos(x-b)}\, dx \quad$ [By (ii)]

$= \dfrac{1}{\sin(b-a)} \displaystyle\int \frac{\sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b)}{\cos(x-a)\cos(x-b)}\, dx$

[$\because \sin(\text{A} - \text{B}) = \sin \text{A} \cos \text{B} - \cos \text{A} \sin \text{B}$]

$$= \frac{1}{\sin(b-a)} \int \left[\frac{\sin(x-a)\cos(x-b)}{\cos(x-a)\cos(x-b)} - \frac{\cos(x-a)\sin(x-b)}{\cos(x-a)\cos(x-b)} \right] dx$$

$$\left(\because \ \frac{A-B}{C} = \frac{A}{C} - \frac{B}{C} \right)$$

$$= \frac{1}{\sin(b-a)} \int [\tan(x-a) - \tan(x-b)] \, dx$$

$$= \frac{1}{\sin(b-a)} [-\log|\cos(x-a)| + \log|\cos(x-b)|] + c$$

$$(\because \ \int \tan x \, dx = -\log|\cos x|)$$

$$= \frac{1}{\sin(b-a)} \log \left| \frac{\cos(x-b)}{\cos(x-a)} \right| + c. \quad \left(\because \ \log m - \log n = \log \frac{m}{n} \right)$$

Choose the correct answer in Exercises 23 and 24:

23. $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} \, dx$ **is equal to**

(A) $\tan x + \cot x + C$ **(B) $\tan x + \text{cosec } x + C$**

(C) $-\tan x + \cot x + C$ **(D) $\tan x + \sec x + C$**

Sol. $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} \, dx$

$$= \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} - \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx \qquad \left[\because \ \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c} \right]$$

$$= \int \left(\frac{1}{\cos^2 x} - \frac{1}{\sin^2 x} \right) dx = \int (\sec^2 x - \text{cosec}^2 x) \, dx$$

$$= \int \sec^2 x \, dx - \int \text{cosec}^2 x \, dx = \tan x - (-\cot x) + C$$

$= \tan x + \cot x + C$ $\quad\therefore$ Option (A) is the correct answer.

24. $\displaystyle\int \frac{e^x (1+x)}{\cos^2 (e^x x)} \, dx$ **equals**

(A) $-\cot (ex^x) + C$ **(B) $\tan (xe^x) + C$**

(C) $\tan (e^x) + C$ **(D) $\cot (e^x) + C$**

Sol. Let $I = \displaystyle\int \frac{e^x (1+x)}{\cos^2 (e^x x)} \, dx$ ...(*i*)

Put $e^x \cdot x = t$

[To evaluate $\int$ (T-function or Inverse T-function $f(x)$) $f'(x) \, dx$, put $f(x) = t$]

Applying Product Rule, $e^x \cdot 1 + xe^x = \dfrac{dt}{dx}$

or $\quad e^x (1 + x) \, dx = dt$

$\therefore$ From (*i*), $\quad I = \displaystyle\int \frac{dt}{\cos^2 t} = \int \sec^2 t \, dt$

$= \tan t + C = \tan (x e^x) + C$ $\therefore$ Option (B) is the correct answer.

Exercise 7.4

Integrate the following functions in Exercises 1 to 9:

1. $\dfrac{3x^2}{x^6+1}$

Sol. Let I $= \int \dfrac{3x^2}{x^6+1}\,dx = \int \dfrac{3x^2}{(x^3)^2+1^2}\,dx$ ...(*i*)

Put $x^3 = t$

$\therefore \quad 3x^2 = \dfrac{dt}{dx} \quad \Rightarrow \quad 3x^2\,dx = dt$

$\therefore \quad$ From (*i*), I $= \int \dfrac{dt}{t^2+1^2} = \dfrac{1}{1}\tan^{-1}\dfrac{t}{1} + C$

$$\left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

Putting $t = x^3$; $= \tan^{-1}(x^3) + C$.

Note. $ax^2 + b$ $(a \neq 0)$ is called a **pure quadratic.**

2. $\dfrac{1}{\sqrt{1+4x^2}}$

Sol. Let I $= \int \dfrac{1}{\sqrt{1+4x^2}}\,dx = \int \dfrac{1}{\sqrt{(2x)^2+1^2}}\,dx$

Using $\int \dfrac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right|$,

I $= \dfrac{\log\left|(2x)+\sqrt{(2x)^2+1^2}\right|}{2 \to \text{Coeff. of } x} + C \quad = \dfrac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right| + C.$

3. $\dfrac{1}{\sqrt{(2-x)^2+1}}$

Sol. Let I $= \int \dfrac{1}{\sqrt{(2-x)^2+1}}\,dx = \int \dfrac{1}{\sqrt{(2-x)^2+1}}\,dx$

Using $\int \dfrac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right|$,

$= \dfrac{\log\left|(2-x)+\sqrt{(2-x)^2+1^2}\right|}{-1 \to \text{Coeff. of } x} + C$

$= -\log\left|2-x+\sqrt{4+x^2-4x+1}\right| + C$

$= \log\left|\dfrac{1}{2-x+\sqrt{x^2-4x+5}}\right| + C.$

$$\left[\because -\log\frac{m}{n} = -(\log m - \log n) = \log n - \log m = \log\frac{n}{m}\right]$$

4. $\dfrac{\mathbf{1}}{\sqrt{\mathbf{9-25}x^{\mathbf{2}}}}$

Sol. Let $\mathrm{I} = \int \frac{1}{\sqrt{9-25x^2}}\,dx = \int \frac{1}{\sqrt{3^2-(5x)^2}}\,dx$

$$= \frac{\sin^{-1}\frac{5x}{3}}{5 \to \text{Coeff. of } x} + \mathrm{C} \qquad \left[\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + \mathrm{C}.$$

5. $\dfrac{\mathbf{3}x}{\mathbf{1+2}x^{\mathbf{4}}}$

Sol. Let $\mathrm{I} = \int \frac{3x}{1+2x^4}\,dx = \frac{3}{2}\int \frac{2x}{1+2(x^2)^2}\,dx$ ...(*i*)

Put $x^2 = t$. $\therefore\ 2x = \frac{dt}{dx} \Rightarrow 2x\,dx = dt$

$\therefore$ From (*i*), $\mathrm{I} = \frac{3}{2}\int \frac{dt}{1+2t^2} = \frac{3}{2}\int \frac{1}{(\sqrt{2}t)^2+1^2}\,dt$

$$= \frac{3}{2}\,\frac{\frac{1}{1}\tan^{-1}\frac{\sqrt{2}t}{1}}{\sqrt{2} \to \text{Coeff. of } t} + \mathrm{C} \quad = \frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\,t) + \mathrm{C}$$

Putting $t = x^2$, $= \frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\,x^2) + \mathrm{C}$.

6. $\dfrac{x^{\mathbf{2}}}{\mathbf{1}-x^{\mathbf{6}}}$

Sol. Let $\mathrm{I} = \int \frac{x^2}{1-x^6}\,dx = \int \frac{x^2}{1-(x^3)^2}\,dx = \frac{1}{3}\int \frac{3x^2}{1-(x^3)^2}\,dx$

Put $x^3 = t$. Therefore $3x^2 = \frac{dt}{dx} \Rightarrow 3x^2\,dx = dt$.

$\therefore\ \mathrm{I} = \frac{1}{3}\int \frac{dt}{1-t^2} = \frac{1}{3}\int \frac{1}{1^2-t^2}\,dt = \frac{1}{3}\,\frac{1}{2\times 1}\log\left|\frac{1+t}{1-t}\right| + \mathrm{C}$

$$\left[\because \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|\right]$$

Putting $t = x^3$, $= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + \mathrm{C}$.

7. $\dfrac{x-\mathbf{1}}{\sqrt{x^{\mathbf{2}}-\mathbf{1}}}$

Sol. Let $\mathrm{I} = \int \frac{x-1}{\sqrt{x^2-1}}\,dx = \int \left(\frac{x}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}}\right)dx$

$$= \int \frac{x}{\sqrt{x^2-1}}\,dx - \int \frac{1}{\sqrt{x^2-1^2}}\,dx$$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}\,dx - \log\left| x+\sqrt{x^2-1^2}\right| \quad ...(i)$$

$$\left(\because \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log|x+\sqrt{x^2-a^2}|\right)$$

Let $I_1 = \int \frac{2x}{\sqrt{x^2-1}}\,dx$

Put $x^2 - 1 = t$. Therefore $2x = \frac{dt}{dx} \Rightarrow 2x\,dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\,dt = \frac{t^{1/2}}{\frac{1}{2}} = 2\sqrt{t} \qquad = 2\sqrt{x^2-1} + C$

Putting this value of $I_1 = \int \frac{2x}{\sqrt{x^2-1}}\,dx$ in (i),

$$I = \frac{1}{2}\,(2\sqrt{x^2-1} + C) - \log|x+\sqrt{x^2-1}|$$

$$= \sqrt{x^2-1} + \frac{C}{2} - \log|x+\sqrt{x^2-1}|$$

$$= \sqrt{x^2-1} - \log|x+\sqrt{x^2-1}| + C_1 \text{ where } C_1 = \frac{C}{2}.$$

8. $\boldsymbol{\frac{x^2}{\sqrt{x^6+a^6}}}$

Sol. Let $I = \int \frac{x^2}{\sqrt{x^6+a^6}}\,dx = \frac{1}{3}\int \frac{3x^2}{\sqrt{(x^3)^2+a^6}}\,dx \quad ...(i)$

Put $x^3 = t$. Therefore $3x^2 = \frac{dt}{dx} \Rightarrow 3x^2\,dx = dt.$

$\therefore$ From (i), $I = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+a^6}} = \frac{1}{3}\int \frac{1}{\sqrt{t^2+(a^3)^2}}\,dt$

$$= \frac{1}{3}\log\left|t+\sqrt{t^2+(a^3)^2}\right| + C\left[\because \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

Putting $t = x^3$, $= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C.$

9. $\boldsymbol{\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}}$

Sol. Let $I = \int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}\,dx \quad ...(i)$

Put tan x = t. $\therefore$ $\sec^2 x = \dfrac{dt}{dx}$ $\Rightarrow$ $\sec^2 x\, dx = dt$

$\therefore$ From (i), $I = \int \dfrac{dt}{\sqrt{t^2+4}} = \int \dfrac{1}{\sqrt{t^2+2^2}}\, dt$

$= \log\left| t + \sqrt{t^2+2^2} \right| + C \quad \left[\because \int \dfrac{1}{\sqrt{x^2+a^2}}\,dx = \log\left| x + \sqrt{x^2+a^2} \right|\right]$

Putting $t = \tan x$, $I = \log\left| \tan x + \sqrt{\tan^2 x + 4} \right| + C.$

Integrate the following functions in Exercises 10 to 18:

Note. Rule to evaluate

$$\int \frac{1}{\textbf{Quadratic}}\,dx \textbf{ or } \int \frac{1}{\sqrt{\textbf{Quadratic}}}\,dx \textbf{ or } \int \sqrt{\textbf{Quadratic}}\,dx$$

Write Quadratic. Take coefficient of x^2 common to make it unity. Then complete squares by adding and subtracting $\left(\dfrac{1}{2}\text{ coefficient of } x\right)^2$

10. $\dfrac{1}{\sqrt{x^2+2x+2}}$

Sol. $\int \dfrac{1}{\sqrt{x^2+2x+2}}\,dx = \int \dfrac{1}{\sqrt{x^2+2x+1+1}}\,dx = \int \dfrac{1}{\sqrt{(x+1)^2+1^2}}\,dx$

Using $\int \dfrac{1}{\sqrt{x^2+a^2}}\,dx = \log|x + \sqrt{x^2+a^2}|$

$= \log|x + 1 + \sqrt{(x+1)^2+1^2}| + c = \log|x + 1 + \sqrt{x^2+2x+2}| + c.$

11. $\dfrac{1}{9x^2+6x+5}$

Sol. Let $I = \int \dfrac{1}{9x^2+6x+5}\,dx$ ...(i)

$\int \dfrac{1}{\text{Quadratic}}\,dx$

Here Quadratic expression $= 9x^2 + 6x + 5$

Making coefficient of x^2 unity, $= 9\left(x^2 + \dfrac{6x}{9} + \dfrac{5}{9}\right)$

$= 9\left(x^2 + \dfrac{2x}{3} + \dfrac{5}{9}\right)$

To complete squares, adding and subtracting $\left(\dfrac{1}{2}\text{ Coefficient of } x\right)^2$

$= \left(\left(\dfrac{1}{2}\times\dfrac{2}{3}\right)^2 = \left(\dfrac{1}{3}\right)^2 = \dfrac{1}{9}\right) = 9\left(x^2 + \dfrac{2x}{3} + \left(\dfrac{1}{3}\right)^2 - \dfrac{1}{9} + \dfrac{5}{9}\right)$

$$= 9\left(\left(x+\frac{1}{3}\right)^2+\frac{4}{9}\right) \Rightarrow 9x^2 + 6x + 5 = 9\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2\right]$$

Putting this value in (i), $\quad \mathrm{I} = \int \frac{1}{9\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2\right]}\,dx$

$$= \frac{1}{9}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2}\,dx$$

$$= \frac{1}{9}\,\frac{1}{\left(\frac{2}{3}\right)}\tan^{-1}\frac{x+\frac{1}{3}}{\frac{2}{3}} + c \qquad \left(\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right)$$

$$= \frac{1}{9}\cdot\frac{3}{2}\tan^{-1}\left(\frac{\frac{3x+1}{3}}{\frac{2}{3}}\right) + c = \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right) + c.$$

12. $\dfrac{\mathbf{1}}{\sqrt{\mathbf{7-6}x-x^2}}$

Sol. Let $\mathrm{I} = \int \frac{1}{\sqrt{7-6x-x^2}}\,dx \qquad ...(i) \qquad \Big|\ \text{Type} \int \frac{1}{\text{Quadratic}}\,dx$

Here Quadratic expression is $7 - 6x - x^2 = -x^2 - 6x + 7$.
Making coefficient of x^2 unity, $= -(x^2 + 6x - 7)$.

To complete squares, adding and subtracting $\left(\frac{1}{2}\text{coefficient of } x\right)^2$

$$= \left(\frac{1}{2}\times 6\right)^2 = 9$$

$$= -[x^2 + 6x + 9 - 9 - 7] = -[(x+3)^2 - 16] \qquad ...(ii)$$

$$= -(x+3)^2 + 16 = 4^2 - (x+3)^2 \qquad ...(iii)$$

(**Note.** Must adjust negative sign outside Eqn. (ii) in the bracket as shown above because otherwise we shall get $\sqrt{-1} = i$ on taking square roots.]

Putting the value of quadratic expression from (iii) in (i),

$$\mathrm{I} = \int \frac{1}{\sqrt{4^2-(x+3)^2}}\,dx = \sin^{-1}\left(\frac{x+3}{4}\right) + c$$

$$\left[\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}\right]$$

13. $\dfrac{\mathbf{1}}{\sqrt{(x-\mathbf{1})(x-\mathbf{2})}}$

Sol. Let $\mathrm{I} = \int \frac{1}{\sqrt{(x-1)(x-2)}}\,dx = \int \frac{1}{\sqrt{x^2-2x-x+2}}\,dx$

$$= \int \frac{1}{\sqrt{x^2 - 3x + 2}} \qquad ...(i)$$

Here quadratic expression is $x^2 - 3x + 2$. Coefficient of x^2 is already unity. To complete squares, adding and subtracting

$$\left(\frac{1}{2}\text{ coefficient of } x\right)^2, \textit{ i.e., } \left(-\frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$$

$$x^2 - 3x + 2 = x^2 - 3x + \left(\frac{3}{2}\right)^2 - \frac{9}{4} + 2$$

$$= \left(x - \frac{3}{2}\right)^2 - \frac{1}{4} \qquad \left[\because \frac{-9}{4} + 2 = \frac{-9+8}{4} = \frac{-1}{4}\right]$$

$$= \left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \qquad ...(ii)$$

Putting this value in (i), $\mathrm{I} = \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\, dx$

$$= \log\left| x - \frac{3}{2} + \sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right| + c$$

$$\left[\because \int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log\left| x + \sqrt{x^2 - a^2} \right|\right]$$

$$= \log\left| x - \frac{3}{2} + \sqrt{x^2 - 3x + 2} \right| + c. \qquad [\text{By } (ii)]$$

14. $\boldsymbol{\frac{1}{\sqrt{8 + 3x - x^2}}}$

Sol. Let $\mathrm{I} = \int \frac{1}{\sqrt{8 + 3x - x^2}}\, dx \qquad ...(i)$

Here quadratic expression is $8 + 3x - x^2 = -x^2 + 3x + 8$.
Making coefficient of x^2 unity, $= -(x^2 - 3x - 8)$.
To complete squares, adding and subtracting

$$\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{3}{2}\right)^2$$

$$8 + 3x - x^2 = -\left(x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 8\right)$$

$$= -\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 8\right] = -\left[\left(x - \frac{3}{2}\right)^2 - \frac{41}{4}\right] = \frac{41}{4} - \left(x - \frac{3}{2}\right)^2$$

(See **Note** given in the solution of Q.N. 12)

$$= \left(\frac{\sqrt{41}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2 \qquad ...(ii)$$

Putting this value in (i), $\text{I} = \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}} \, dx$

$$= \sin^{-1} \frac{x - \frac{3}{2}}{\frac{\sqrt{41}}{2}} + c \qquad \left[\because \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}\right]$$

$$= \sin^{-1} \left(\frac{2x - 3}{\sqrt{41}}\right) + c.$$

15. $\frac{1}{\sqrt{(x-a)(x-b)}}$

Sol. Let $\text{I} = \int \frac{1}{\sqrt{(x-a)(x-b)}} \, dx = \int \frac{1}{\sqrt{x^2 - bx - ax + ab}} \, dx$

$$= \int \frac{1}{\sqrt{x^2 - x(a+b) + ab}} \qquad ...(i)$$

Here Quadratic expression $= x^2 - x(a + b) + ab$

Adding and subtracting $\left(\frac{1}{2}\text{coefficient of } x\right)^2 = \left(\frac{a+b}{2}\right)^2$

$$= x^2 - x(a+b) + \left(\frac{a+b}{2}\right)^2 - \left(\frac{a+b}{2}\right)^2 + ab$$

$$= \left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{(a+b)^2}{4} - ab\right)$$

$$= \left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{(a+b)^2 - 4ab}{4}\right) = \left(x - \left(\frac{a+b}{2}\right)\right)^2 - \frac{(a-b)^2}{4}$$

$$= \left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{a-b}{2}\right)^2 \qquad ...(ii)$$

$(\because (a+b)^2 - 4ab = a^2 + b^2 + 2ab - 4ab$
$= a^2 + b^2 - 2ab = (a-b)^2)$

Putting this value in (i),

$$\text{I} = \int \frac{1}{\sqrt{\left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{a-b}{2}\right)^2}} \, dx$$

$$= \log \left| x - \left(\frac{a+b}{2}\right) + \sqrt{\left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{a-b}{2}\right)^2} \right| + c$$

$$\left[\because \int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \log |x + \sqrt{x^2 - a^2}|\right]$$

$$= \log \left| x - \left(\frac{a+b}{2}\right) + \sqrt{x^2 - x(a+b) + ab} \right| + c \quad \text{[By } (ii)\text{]}$$

Note. Method to evaluate $\displaystyle\int \frac{\textbf{Linear}}{\textbf{Quadratic}}\, dx$ **or** $\displaystyle\int \frac{\textbf{Linear}}{\sqrt{\textbf{Quadratic}}}\, dx$ **or** $\displaystyle\int \textbf{Linear}\sqrt{\textbf{Quadratic}}\, dx.$

Write linear = A $\dfrac{d}{dx}$ (Quadratic) + B.

Find values of A and B by comparing coefficients of x and constant terms on both sides.

16. $\dfrac{\mathbf{4x+1}}{\sqrt{\mathbf{2x^2+x-3}}}$

Sol. Let $I = \displaystyle\int \frac{4x+1}{\sqrt{2x^2+x-3}}\, dx$ ...(i)

Here $\dfrac{d}{dx}$ (Quadratic $2x^2 + x - 3$) is $(4x + 1)$, the numerator.

So put $2x^2 + x - 3 = t$.

$\therefore \quad (4x + 1) = \dfrac{dt}{dx} \quad \Rightarrow \quad (4x + 1)\, dx = dt$

$\therefore \quad$ From (i), $I = \displaystyle\int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{1/2}}{\frac{1}{2}} + c$

$$= 2\sqrt{t} + c = 2\sqrt{2x^2 + x - 3} + c.$$

17. $\dfrac{\mathbf{x+2}}{\sqrt{\mathbf{x^2-1}}}$

Sol. Let $I = \displaystyle\int \frac{x+2}{\sqrt{x^2-1}}\, dx = \int \left(\frac{x}{\sqrt{x^2-1}} + \frac{2}{\sqrt{x^2-1}}\right) dx$

$$= \int \frac{x}{\sqrt{x^2-1}}\, dx + 2\int \frac{1}{\sqrt{x^2-1^2}}\, dx$$

$$= \int \frac{x}{\sqrt{x^2-1}}\, dx + 2 \log | x + \sqrt{x^2-1^2} | + c \qquad ...(i)$$

Let $I_1 = \displaystyle\int \frac{x}{\sqrt{x^2-1}}\, dx = \frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}}\, dx$

Put $x^2 - 1 = t$. Therefore $2x = \dfrac{dt}{dx}$ or $2x\, dx = dt$

$\therefore \quad I_1 = \dfrac{1}{2} \displaystyle\int \frac{dt}{\sqrt{t}} = \frac{1}{2} \int t^{-1/2}\, dt = \frac{1}{2} \frac{t^{1/2}}{\frac{1}{2}} = \sqrt{t} = \sqrt{x^2 - 1}$

Putting this value of (I_1 =) $\int \frac{x}{\sqrt{x^2-1}}\, dx = \sqrt{x^2-1}$ in (*i*)

$$I = \sqrt{x^2-1} + 2 \log | x + \sqrt{x^2-1} | + c.$$

18. $\dfrac{\mathbf{5x - 2}}{\mathbf{1 + 2x + 3x^2}}$

Sol. Let $I = \int \frac{5x-2}{1+2x+3x^2}\, dx$ ...(*i*) $\left| \int \frac{\text{Linear}}{\text{Quadratic}}\, dx \right.$

Let Linear = A $\frac{d}{dx}$ (Quadratic) + B

i.e., $5x - 2 = A \frac{d}{dx}(1 + 2x + 3x^2) + B$

or $5x - 2 = A(2 + 6x) + B$ ...(*ii*)

i.e., $5x - 2 = 2A + 6Ax + B$

Comparing coefficients of x, $6A = 5 \Rightarrow A = \frac{5}{6}$

Comparing constants, $2A + B = -2$

Putting $A = \frac{5}{6}$, $\frac{10}{6} + B = -2$

$\Rightarrow \quad B = -2 - \frac{10}{6} = \frac{-22}{6}$ or $B = \frac{-11}{3}$

Putting values of A and B in (*ii*), $5x - 2 = \frac{5}{6}(2 + 6x) - \frac{11}{3}$

Putting this value of $5x - 2$ in (*i*),

$$I = \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2}\, dx$$

$$\Rightarrow \quad I = \frac{5}{6} \int \frac{2+6x}{1+2x+3x^2}\, dx - \frac{11}{3} \int \frac{1}{1+2x+3x^2}\, dx$$

$$= \frac{5}{6} I_1 - \frac{11}{3} I_2 \qquad ...(iii)$$

Here $I_1 = \int \frac{2+6x}{1+2x+3x^2}\, dx$

Put Denominator $1 + 2x + 3x^2 = t$.

$\therefore\ 2 + 6x = \frac{dt}{dx} \quad \Rightarrow \quad (2 + 6x)\, dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{t} = \int \frac{1}{t}\, dt = \log | t | = \log | 1 + 2x + 3x^2 |$ ...(*iv*)

Again $I_2 = \int \frac{1}{1+2x+3x^2}\, dx = \int \frac{1}{3x^2+2x+1}\, dx \left| \int \frac{1}{\text{Quadratic}}\, dx \right.$

Now Quadratic Expression $= 3x^2 + 2x + 1$.

Making coefficient of x^2 unity $= 3\left(x^2 + \frac{2}{3}x + \frac{1}{3}\right)$

Completing squares $= 3\left[x^2 + \frac{2}{3}x + \left(\frac{1}{3}\right)^2 + \frac{1}{3} - \frac{1}{9}\right]$

$= 3\left[\left(x + \frac{1}{3}\right)^2 + \frac{2}{9}\right]$ $\quad \Big| \because \frac{1}{3} - \frac{1}{9} = \frac{3-1}{9} = \frac{2}{9}$

$\Rightarrow \quad I_2 = \int \frac{1}{3\left[\left(x+\frac{1}{3}\right)^2 + \frac{2}{9}\right]} dx = \frac{1}{3}\int \frac{1}{\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2} dx$

$= \frac{1}{3} \frac{1}{\left(\frac{\sqrt{2}}{3}\right)} \tan^{-1} \frac{x + \frac{1}{3}}{\frac{\sqrt{2}}{3}}$ $\quad \left[\because \int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

$\Rightarrow \quad I_2 = \frac{1}{3} \cdot \frac{3}{\sqrt{2}} \tan^{-1} \frac{3x+1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)$ ...(*v*)

Putting values of I_1 and I_2 from (*iv*) and (*v*) in (*iii*), we have

$I = \frac{5}{6} \log |1 + 2x + 3x^2| - \frac{11}{3} \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + c.$

Integrate the functions in Exercises 19 to 23:

19. $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$

Sol. Let $I = \int \frac{6x+7}{\sqrt{(x-5)(x-4)}} dx = \int \frac{6x+7}{\sqrt{x^2 - 4x - 5x + 20}} dx$

i.e., $I = \int \frac{6x+7}{\sqrt{x^2 - 9x + 20}} dx$ ...(*i*) $\quad \Big| \int \frac{\text{Linear}}{\sqrt{\text{Quadratic}}} dx$

Let Linear = A $\frac{d}{dx}$ (Quadratic) + B

i.e., $6x + 7 = A(2x - 9) + B$ ...(*ii*)

$= 2Ax - 9A + B$

Comparing coefficients of x, $2A = 6 \Rightarrow A = 3$

Comparing constants, $-9A + B = 7$.

Putting $A = 3$, $-27 + B = 7 \Rightarrow B = 34$

Putting values of A and B in (*ii*),

$6x + 7 = 3(2x - 9) + 34$

Putting this value of $6x + 7$ in (*i*),

$I = \int \frac{3(2x-9)+34}{\sqrt{x^2 - 9x + 20}} dx$

$= 3\int \frac{2x-9}{\sqrt{x^2 - 9x + 20}} dx + 34 \int \frac{1}{\sqrt{x^2 - 9x + 20}} dx$

$= 3 I_1 + 34 I_2$ ...(*iii*)

$I_1 = \int \frac{2x-9}{\sqrt{x^2 - 9x + 20}} dx$

Put $x^2 - 9x + 20 = t$. $\therefore$ $2x - 9 = \frac{dt}{dx}$

$\Rightarrow \quad (2x - 9)\, dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{1/2}}{1/2} = 2\sqrt{t}$

$= 2\sqrt{x^2 - 9x + 20}$ ...(*iv*)

$$I_2 = \int \frac{1}{\sqrt{x^2 - 9x + 20}}\, dx = \int \frac{1}{\sqrt{x^2 - 9x + \left(\frac{9}{2}\right)^2 + 20 - \frac{81}{4}}}\, dx$$

$$= \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}}\, dx = \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\, dx$$

$$= \log \left| x - \frac{9}{2} + \sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right|$$

$$\left[\because \int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log | x + \sqrt{x^2 - a^2} | \right]$$

$I_2 = \log \left| x - \frac{9}{2} + \sqrt{x^2 - 9x + 20} \right|$ ...(*v*)

$$\left(\because \left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = x^2 + \frac{81}{4} - 9x - \frac{1}{4} = x^2 - 9x + 20\right)$$

Putting values of I_1 and I_2 from (*iv*) and (*v*) in (*iii*),

$$I = 6\sqrt{x^2 - 9x + 20} + 34 \log \left| x - \frac{9}{2} + \sqrt{x^2 - 9x + 20} \right| + c.$$

20. $\boldsymbol{\frac{x + 2}{\sqrt{4x - x^2}}}$

Sol. Let $I = \int \frac{x + 2}{\sqrt{4x - x^2}}\, dx$ ...(*i*) $\quad \left| \int \frac{\text{Linear}}{\sqrt{\text{Quadratic}}}\, dx \right.$

Let Linear $= A \frac{d}{dx}$ (Quadratic) + B

i.e., $x + 2 = A(4 - 2x) + B$ ...(*ii*)

$= 4A - 2Ax + B$

Comparing coefficients of x: $-2A = 1 \Rightarrow A = \frac{-1}{2}$

Comparing constants: $4A + B = 2$

Putting $A = \frac{-1}{2}$, $-2 + B = 2 \Rightarrow B = 4$

Putting values of A and B in (*ii*), $x + 2 = \frac{-1}{2}(4 - 2x) + 4$

Putting this value of $x + 2$ in (*i*),

$$\text{I} = \int \frac{\frac{-1}{2}(4-2x)+4}{\sqrt{4x-x^2}}\,dx = \frac{-1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}\,dx + 4\int \frac{1}{\sqrt{4x-x^2}}\,dx$$

$$= \frac{-1}{2}\,\text{I}_1 + 4\,\text{I}_2 \quad ...(iii) \qquad \text{I}_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}\,dx$$

Put $4x - x^2 = t \quad \therefore \quad 4 - 2x = \frac{dt}{dx} \quad \Rightarrow \quad (4 - 2x)\,dx = dt$

$$\therefore \quad \text{I}_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\,dt = \frac{t^{1/2}}{1/2} = 2\sqrt{t} = 2\sqrt{4x-x^2} \qquad ...(iv)$$

$$\text{I}_2 = \int \frac{1}{\sqrt{4x-x^2}}\,dx$$

Quadratic Expression is $4x - x^2 = -x^2 + 4x$
$= -(x^2 - 4x) = -(x^2 - 4x + 4 - 4) = -((x-2)^2 - 2^2) = 2^2 - (x-2)^2$

$$\therefore \quad \text{I}_2 = \int \frac{1}{\sqrt{2^2-(x-2)^2}}\,dx = \sin^{-1}\frac{x-2}{2} \qquad ...(v)$$

$$\left(\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}\right)$$

Putting values of I_1 and I_2 from (iv) and (v) in (iii),

$$\text{I} = -\sqrt{4x-x^2} + 4\sin^{-1}\frac{x-2}{2} + c.$$

21. $\dfrac{x+2}{\sqrt{x^2+2x+3}}$

Sol. Let $\text{I} = \displaystyle\int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx \qquad ...(i)$

Let Linear = A $\dfrac{d}{dx}$ (Quadratic) + B

i.e., $x + 2 = \text{A}(2x + 2) + \text{B} \qquad ...(ii)$
$= 2\text{A}x + 2\text{A} + \text{B}$

Comparing coefficients of x, $2\text{A} = 1 \quad \Rightarrow \quad \text{A} = \frac{1}{2}$

Comparing constants, $2\text{A} + \text{B} = 2$

Putting $\text{A} = \frac{1}{2}$, $1 + \text{B} = 2 \quad \Rightarrow \quad \text{B} = 1$

Putting values of A and B in (ii), $x + 2 = \frac{1}{2}(2x + 2) + 1$

Putting this value of $(x + 2)$ in (i),

$$\text{I} = \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}\,dx = \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx + \int \frac{dx}{\sqrt{x^2+2x+3}}$$

$\Rightarrow \quad I = \frac{1}{2} I_1 + I_2 \quad ...(iii) \qquad I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}\, dx$

Put $x^2 + 2x + 3 = t \quad \therefore \quad (2x + 2) = \frac{dt}{dx} \Rightarrow (2x + 2)\, dx = dt$

$I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{1/2}}{\frac{1}{2}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \quad ...(iv)$

$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}\, dx = \int \frac{1}{\sqrt{x^2+2x+1+2}}\, dx$

$= \int \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}\, dx = \log | x + 1 + \sqrt{(x+1)^2+(\sqrt{2})^2} |$

$\left[\because \int \frac{1}{\sqrt{x^2+a^2}}\, dx = \log | x + \sqrt{x^2+a^2} |\right]$

$= \log | x + 1 + \sqrt{x^2+2x+3} | \quad ...(v)$

Putting values from (iv) and (v) in (iii),

$I = \sqrt{x^2+2x+3} + \log | x + 1 + \sqrt{x^2+2x+3} | + c.$

22. $\frac{x+3}{x^2-2x-5}$

Sol. Let $I = \int \frac{x+3}{x^2-2x-5}\, dx \quad ...(i)$

Let $x + 3 = A \frac{d}{dx}(x^2 - 2x - 5) + B$

or $x + 3 = A(2x - 2) + B \quad ...(ii)$
$= 2Ax - 2A + B$

Comparing coefficients of x on both sides, $2A = 1 \Rightarrow A = \frac{1}{2}$

Comparing constants, $-2A + B = 3$

Putting $A = \frac{1}{2}$, $-1 + B = 3 \quad \Rightarrow \quad B = 4$

Putting values of A and B in (ii), $x + 3 = \frac{1}{2}(2x - 2) + 4$

Putting this value in (i),

$I = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}\, dx = \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}\, dx + 4\int \frac{1}{x^2-2x-5}\, dx$

$= \frac{1}{2} I_1 + 4 I_2 \quad ...(iii)$

$I_1 = \int \frac{2x-2}{x^2-2x-5}\, dx$

Put $x^2 - 2x - 5 = t$. Therefore $(2x - 2) = \frac{dt}{dx} \Rightarrow (2x - 2)\, dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{t} = \log |t| = \log |x^2 - 2x - 5|$ ...(*iv*)

Again $I_2 = \int \frac{1}{x^2 - 2x - 5} dx$

$$= \int \frac{1}{x^2 - 2x + 1 - 1 - 5} dx = \int \frac{1}{(x-1)^2 - 6} dx$$

$$= \int \frac{1}{(x-1)^2 - 6} dx = \frac{1}{2\sqrt{6}} \log \left| \frac{x - 1 - \sqrt{6}}{x - 1 + \sqrt{6}} \right| \quad ...(v)$$

$$\left[\because \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| \right]$$

Putting values of I_1 and I_2 from (*iv*) and (*v*) in (*iii*),

$$I = \frac{1}{2} \log |x^2 - 2x - 5| + \frac{2}{\sqrt{6}} \log \left| \frac{x - 1 - \sqrt{6}}{x - 1 + \sqrt{6}} \right| + c.$$

23. $\dfrac{5x + 3}{\sqrt{x^2 + 4x + 10}}$

Sol. Let $I = \int \frac{5x + 3}{\sqrt{x^2 + 4x + 10}} dx$ ...(*i*)

Let Linear = A $\frac{d}{dx}$ (Quadratic) + B

i.e., $5x + 3 = A(2x + 4) + B$ ...(*ii*)

$= 2Ax + 4A + B$

Comparing coefficients of x on both sides, $2A = 5 \Rightarrow A = \frac{5}{2}$

Comparing constants, $4A + B = 3$

Putting $A = \frac{5}{2}$, $10 + B = 3 \quad \Rightarrow \quad B = -7$

Putting values of A and B in (*ii*), $5x + 3 = \frac{5}{2}(2x + 4) - 7$

Putting this value in (*i*), $I = \int \frac{\frac{5}{2}(2x + 4) - 7}{\sqrt{x^2 + 4x + 10}} dx$

$$= \frac{5}{2} \int \frac{2x + 4}{\sqrt{x^2 + 4x + 10}} dx - 7 \int \frac{1}{\sqrt{x^2 + 4x + 10}} dx$$

or $\quad I = \frac{5}{2} I_1 - 7 I_2$ ...(*iii*)

$$I_1 = \int \frac{2x + 4}{\sqrt{x^2 + 4x + 10}} dx$$

Put $x^2 + 4x + 10 = t$. Therefore $2x + 4 = \frac{dt}{dx} \Rightarrow (2x + 4) dx = dt$

$$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{1/2}}{\frac{1}{2}} = 2\sqrt{t}$$

$$= 2\sqrt{x^2+4x+10} \qquad ...(iv)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}\, dx = \int \frac{1}{\sqrt{x^2+4x+4+6}}\, dx$$

$$= \int \frac{1}{\sqrt{(x+2)^2+(\sqrt{6})^2}}\, dx = \log \left| x + 2 + \sqrt{(x+2)^2+(\sqrt{6})^2} \right|$$

$$\left[\because \int \frac{1}{\sqrt{x^2+a^2}}\, dx = \log \left| x + \sqrt{x^2+a^2} \right| \right]$$

$$= \log \left| x + 2 + \sqrt{x^2+4x+10} \right| \qquad ...(v)$$

Putting values of I_1 and I_2 from (iv) and (v) in (iii),

$I = 5\sqrt{x^2+4x+10} - 7 \log \left| x + 2 + \sqrt{x^2+4x+10} \right| + c.$

Choose the correct answer in Exercises 24 and 25.

24. $\int \frac{dx}{x^2+2x+2}$ **equals**

(A) $x \tan^{-1}(x+1) + C$ **(B)** $\tan^{-1}(x+1) + C$
(C) $(x+1)\tan^{-1} x + C$ **(D)** $\tan^{-1} x + C.$

Sol. $\int \frac{dx}{x^2+2x+2} = \int \frac{1}{x^2+2x+1+1}\, dx = \int \frac{1}{(x+1)^2+1^2}\, dx$

$$= \frac{1}{1}\tan^{-1}\frac{(x+1)}{1} + C \quad \left[\because \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \tan^{-1}(x+1) + C$$

$\therefore$ Option (B) is the correct answer.

25. $\int \frac{dx}{\sqrt{9x-4x^2}}$ **equals**

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right) + C$ **(B)** $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right) + C$

(C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right) + C$ **(D)** $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{8}\right) + C.$

Sol. Let $I = \int \frac{dx}{\sqrt{9x-4x^2}} = \int \frac{dx}{\sqrt{-4x^2+9x}} \qquad ...(i)$

Here Quadratic expression is $-4x^2 + 9x = -4\left(x^2 - \frac{9}{4}x\right)$

$$= -4\left[x^2 - \frac{9}{4}x + \left(\frac{9}{8}\right)^2 - \left(\frac{9}{8}\right)^2\right] = -4\left[\left(x-\frac{9}{8}\right)^2 - \left(\frac{9}{8}\right)^2\right]$$

$$= 4\left[-\left(x-\frac{9}{8}\right)^2 + \left(\frac{9}{8}\right)^2\right] = 4\left[\left(\frac{9}{8}\right)^2 - \left(x-\frac{9}{8}\right)^2\right]$$

Putting this value in (*i*),

$$I = \int \frac{1}{\sqrt{4\left[\left(\frac{9}{8}\right)^2 - \left(x - \frac{9}{8}\right)^2\right]}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{\left[\left(\frac{9}{8}\right)^2 - \left(x - \frac{9}{8}\right)^2\right]}}\, dx$$

$$= \frac{1}{2}\sin^{-1}\frac{x - \frac{9}{8}}{\frac{9}{8}} + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{8x - 9}{9}\right) + C \qquad \left[\because \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a}\right]$$

$\therefore$ Option (B) is the correct answer.

Exercise 7.5

Integrate the (rational) functions in Exercises 1 to 6:

1. $\dfrac{x}{(x+1)(x+2)}$

Sol. To integrate the (rational) function $\dfrac{x}{(x+1)(x+2)}$.

Let integrand $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$ ...(*i*)

(Partial Fractions)

Multiplying by L.C.M. = $(x + 1)(x + 2)$,

$x = A(x + 2) + B(x + 1) \quad = Ax + 2A + Bx + B$

Comparing coefficients of x on both sides, $A + B = 1$ ...(*ii*)

Comparing constants, $2A + B = 0$ ...(*iii*)

Let us solve Eqns. (*ii*) and (*iii*) for A and B.

Eqn. (*iii*) – Eqn. (*ii*) gives, $A = -1$

Putting $A = -1$ in (*ii*), $-1 + B = 1 \Rightarrow B = 2$

Putting values of A and B in (*i*), $\dfrac{x}{(x+1)(x+2)} = \dfrac{-1}{x+1} + \dfrac{2}{x+2}$

$$\therefore \int \frac{x}{(x+1)(x+2)}\,dx = -\int \frac{1}{x+1}\,dx + 2\int \frac{1}{x+2}\,dx$$

$$= -\log|x+1| + 2\log|x+2| + c$$

$$= \log|x+2|^2 - \log|x+1| + c = \log\frac{(x+2)^2}{|x+1|} + c.$$

$(\because \ |t|^2 = t^2)$

2. $\dfrac{1}{x^2-9}$

Sol. To integrate the (rational) function $\dfrac{1}{x^2-9}$

$$\int \frac{1}{x^2-9}\,dx = \int \frac{1}{x^2-3^2}\,dx$$

$$= \frac{1}{2\times 3}\log\left|\frac{x-3}{x+3}\right| + c \quad \left[\because \int \frac{1}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

$$= \frac{1}{6}\log\left|\frac{x-3}{x+3}\right| + c.$$

OR

Integrand $\dfrac{1}{x^2-9} = \dfrac{1}{(x-3)(x+3)} = \dfrac{A}{x-3} + \dfrac{B}{x+3}$

Now proceed as in the solution of Q.No.1.

3. $\dfrac{\mathbf{3x - 1}}{\mathbf{(x - 1)(x - 2)(x - 3)}}$.

Sol. To integrate the (rational) function $\dfrac{3x - 1}{(x - 1)(x - 2)(x - 3)}$

Let integrand $\dfrac{3x - 1}{(x - 1)(x - 2)(x - 3)} = \dfrac{A}{x - 1} + \dfrac{B}{x - 2} + \dfrac{C}{x - 3}$ ...(*i*)

Multiplying by L.C.M. = $(x - 1)(x - 2)(x - 3)$, we have

$3x - 1 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)$

$= A(x^2 - 5x + 6) + B(x^2 - 4x + 3) + C(x^2 - 3x + 2)$

$= Ax^2 - 5Ax + 6A + Bx^2 - 4Bx + 3B + Cx^2 - 3Cx + 2C$

Comparing coefficients of x^2, x and constant terms on both sides, we have

Coefficients of x^2: $A + B + C = 0$ ...(*ii*)

Coefficient of x: $-5A - 4B - 3C = 3$ or $5A + 4B + 3C = -3$ ...(*iii*)

Constants: $6A + 3B + 2C = -1$ ...(*iv*)

Let us solve (*ii*), (*iii*) and (*iv*) for A, B, C.

Let us first form two Eqns. in two unknowns say A and B.

Eqn. (*iii*) – 3 Eqn. (*i*) gives (to eliminate C),

$5A + 4B + 3C - 3A - 3B - 3C = -3$

or $2A + B = -3$ ...(*v*)

Eqn. (*iv*) – 2 Eqn. (*i*) gives (to eliminate C),

$6A + 3B + 2C - 2A - 2B - 2C = -1$

or $4A + B = -1$ ...(*vi*)

Eqn. (*vi*) – Eqn. (*v*) gives (to eliminate B),

$2A = -1 + 3 = 2 \Rightarrow A = \dfrac{2}{2} = 1.$

Putting A = 1 in (*v*), $2 + B = -3 \Rightarrow B = -5$

Putting A = 1 and B = – 5 in (*ii*), $1 - 5 + C = 0$

or $C - 4 = 0$ or $C = 4$

Putting values of A, B, C in (*i*),

$$\frac{3x - 1}{(x - 1)(x - 2)(x - 3)} = \frac{1}{x - 1} - \frac{5}{x - 2} + \frac{4}{x - 3}$$

$$\therefore \int \frac{3x - 1}{(x - 1)(x - 2)(x - 3)}$$

$$= \int \frac{1}{x - 1}\, dx - 5\int \frac{1}{x - 2}\, dx + 4\int \frac{1}{x - 3}\, dx$$

$$= \log |x - 1| - 5 \log |x - 2| + 4 \log |x - 3| + c.$$

4. $\dfrac{\boldsymbol{x}}{\boldsymbol{(x-1)(x-2)(x-3)}}$

Sol. To integrate the (rational) function $\dfrac{x}{(x-1)(x-2)(x-3)}$.

Let integrand $\dfrac{x}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$ ...(*i*)

(Partial fractions)

Multiplying by L.C.M. = $(x-1)(x-2)(x-3)$,

$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

$= A(x^2 - 5x + 6) + B(x^2 - 4x + 3) + C(x^2 - 3x + 2)$

$= Ax^2 - 5Ax + 6A + Bx^2 - 4Bx + 3B + Cx^2 - 3Cx + 2C$

Comparing coefficients of x^2, x and constant terms on both sides, we have

x^2: $A + B + C = 0$ ...(*ii*)

x: $-5A - 4B - 3C = 1$ or $5A + 4B + 3C = -1$ ...(*iii*)

Constants: $6A + 3B + 2C = 0$ ...(*iv*)

Let us solve Eqns. (*ii*), (*iii*) and (*iv*) for A, B, C.

Let us first form two Eqns. in two unknowns say A and B.

Eqn. (*iii*) – 3 × Eqn. (*ii*) gives | To eliminate C

$5A + 4B + 3C - 3A - 3B - 3C = -1$ or $2A + B = -1$ ...(*v*)

Eqn. (*iv*) – 2 × Eqn. (*ii*) gives | To eliminate C

$4A + B = 0$ ...(*vi*)

Eqn. (*vi*) – Eqn. (*v*) gives (To eliminate B)

$2A = 1 \quad \therefore \quad A = \dfrac{1}{2}$

Putting $A = \dfrac{1}{2}$ in (*v*), $1 + B = -1 \Rightarrow B = -2$

Putting $A = \dfrac{1}{2}$ and $B = -2$ in (*ii*),

$\dfrac{1}{2} - 2 + C = 0 \Rightarrow C = \dfrac{-1}{2} + 2 = \dfrac{-1+4}{2} = \dfrac{3}{2}$

Putting these values of A, B, C in (*i*), we have

$$\frac{x}{(x-1)(x-2)(x-3)} = \frac{\frac{1}{2}}{x-1} - \frac{2}{x-2} + \frac{\frac{3}{2}}{x-3}$$

$$\therefore \int \frac{x}{(x-1)(x-2)(x-3)}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}\,dx - 2\int \frac{1}{x-2}\,dx + \frac{3}{2}\int \frac{1}{x-3}\,dx$$

$$= \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + c.$$

5. $\dfrac{\mathbf{2x}}{\mathbf{x^2+3x+2}}$

Sol. To integrate the (rational) function $\dfrac{2x}{x^2+3x+2}$.

Now $x^2 + 3x + 2 = x^2 + 2x + x + 2 = x(x + 2) + 1(x + 2)$
$= (x + 1)(x + 2)$

$\therefore$ Integrand $\dfrac{2x}{x^2+3x+2} = \dfrac{2x}{(x+1)(x+2)}$

$= \dfrac{A}{x+1} + \dfrac{B}{x+2}$ ...(*i*)

(Partial Fractions)

Multiplying both sides by L.C.M. $= (x + 1)(x + 2)$,

$2x = A(x + 2) + B(x + 1) \qquad = Ax + 2A + Bx + B$

Comparing coefficients of x and constant terms on both sides, we have

Coefficients of x: $A + B = 2$ ...(*ii*)

Constant terms: $2A + B = 0$ ...(*iii*)

Let us solve (*ii*) and (*iii*) for A and B.

(*iii*) – (*ii*) gives $A = -2$.

Putting $A = -2$ in (*ii*), $-2 + B = 2$. $\quad \therefore \quad B = 4$

Putting values of A and B in (*i*), $\dfrac{2x}{x^2+3x+2} = \dfrac{-2}{x+1} + \dfrac{4}{x+2}$

$$\therefore \int \frac{2x}{x^2+3x+2}\,dx = -2\int \frac{1}{x+1}\,dx + 4\int \frac{1}{x+2}\,dx$$

$$= -2 \log |x + 1| + 4 \log |x + 2| + c$$

$$= 4 \log |x + 2| - 2 \log |x + 1| + c$$

Remark: Alternative method to evaluate $\displaystyle\int \frac{2x}{x^2+3x+2}\,dx$

is $\displaystyle\int \frac{\text{Linear}}{\text{Quadratic}}\,dx$ as explained in solutions in Exercise 7.4 (Exercise 18 and Exercise 22.

6. $\dfrac{\mathbf{1-x^2}}{\mathbf{x(1-2x)}}$

Sol. To integrate (rational) function $\dfrac{1-x^2}{x(1-2x)} = \dfrac{1-x^2}{x-2x^2} = \dfrac{-x^2+1}{-2x^2+x}$

[Here Degree of numerator = Degree of Denominator = 2

$\therefore$ We must divide numerator by denominator to make the degree of numerator smaller than degree of denominator so that we can form partial fractions.]

$$-2x^2 + x \overline{\big)\, -x^2 + 1 \big(}\ \frac{1}{2}$$
$$-x^2 + \frac{x}{2}$$
$$+ \quad -$$
$$-\frac{x}{2} + 1$$

$$\therefore \quad \frac{1-x^2}{x(1-2x)} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}} = \frac{1}{2} + \frac{\left(-\frac{x}{2}+1\right)}{x(1-2x)}$$

$$\therefore \quad \int \frac{1-x^2}{x(1-2x)}\,dx = \int \left(\frac{1}{2} + \frac{\left(-\frac{x}{2}+1\right)}{x(1-2x)}\right) dx$$

$$= \frac{1}{2}\int 1\,dx + \int \frac{-\frac{x}{2}+1}{x(1-2x)}\,dx \qquad ...(i)$$

Let integrand $\dfrac{-\frac{x}{2}+1}{x(1-2x)} = \dfrac{A}{x} + \dfrac{B}{1-2x}$ ...(*ii*)

Multiplying by L.C.M. = $x(1 - 2x)$,

$$-\frac{x}{2} + 1 = A(1-2x) + Bx \qquad = A - 2Ax + Bx$$

Comparing coefficients of x, $-2A + B = \dfrac{-1}{2}$ ...(*iii*)

Comparing constants, $A = 1$ ...(*iv*)

Putting A = 1 from (*iv*) in (*iii*),

$$-2 + B = \frac{-1}{2} \Rightarrow B = \frac{-1}{2} + 2 = \frac{-1+4}{2} \quad \text{or} \quad B = \frac{3}{2}$$

Putting values of A and B in (*ii*),

$$\frac{-\frac{x}{2}+1}{x(1-2x)} = \frac{1}{x} + \frac{\frac{3}{2}}{1-2x}$$

$$\therefore \quad \int \frac{-\frac{x}{2}+1}{x(1-2x)}\,dx = \int \frac{1}{x}\,dx + \frac{3}{2}\int \frac{1}{1-2x}\,dx$$

$$= \log|x| + \frac{3}{2}\log\frac{|1-2x|}{-2 \to \text{Coefficient of } x} + c$$

$$= \log|x| - \frac{3}{4}\log|1-2x| + c$$

Putting this value in (*i*),

$$\int \frac{1-x^2}{x(1-2x)}\,dx = \frac{1}{2}x + \log|x| - \frac{3}{4}\log|1-2x| + c.$$

Integrate the following functions in Exercises 7 to 12:

7. $\dfrac{x}{(x^2+1)(x-1)}$

Sol. To integrate the (rational) function $\dfrac{x}{(x^2+1)(x-1)}$.

Let integrand $\dfrac{x}{(x^2+1)(x-1)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-1}$ ...(*i*)

(Partial Fractions)

Multiplying by L.C.M. = $(x^2 + 1)(x - 1)$ on both sides,

$$x = (Ax + B)(x - 1) + C(x^2 + 1)$$

$$\Rightarrow \quad x = Ax^2 - Ax + Bx - B + Cx^2 + C,$$

Comparing coefficients of x^2, x and constant terms on both sides, we have

x^2 $\quad A + C = 0$ ...(*ii*)

x $\quad - A + B = 1$ ...(*iii*)

Constants $\quad - B + C = 0$ ...(*iv*)

Let us solve Eqns. (*ii*), (*iii*) and (*iv*) for A, B, C

Adding (*ii*) and (*iii*) to eliminate A, B + C = 1 ...(*v*)

Adding (*iv*) and (*v*), $2C = 1 \Rightarrow C = \dfrac{1}{2}$

From (*iv*), $- B = - C \Rightarrow B = C = \dfrac{1}{2}$

From (*ii*), $A = - C = \dfrac{-1}{2}$

Putting these values of A, B, C in (*i*),

$$\frac{x}{(x^2+1)(x-1)} = \frac{\frac{-1}{2}x + \frac{1}{2}}{x^2+1} + \frac{\frac{1}{2}}{x-1}$$

$$= \frac{-1}{2}\frac{x}{x^2+1} + \frac{1}{2} \cdot \frac{1}{x^2+1} + \frac{1}{2}\frac{1}{x-1}$$

$$= \frac{-1}{4}\frac{2x}{x^2+1} + \frac{1}{2}\frac{1}{x^2+1} + \frac{1}{2}\frac{1}{x-1}$$

$$\therefore \int \frac{x}{(x^2+1)(x-1)}\,dx$$

$$= \frac{-1}{4}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$

$$\Rightarrow \int \frac{x}{(x^2+1)(x-1)}\,dx = \frac{-1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x$$

$$+ \frac{1}{2}\log|x-1| + c \qquad \left(\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right)$$

$$= \frac{-1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + c$$

$[\because \; x^2 + 1 > 0 \Rightarrow |x^2 + 1| = x^2 + 1]$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + c.$$

8. $\dfrac{\boldsymbol{x}}{(\boldsymbol{x}-\mathbf{1})^2(\boldsymbol{x}+\mathbf{2})}$

Sol. To integrate the (rational) function $\dfrac{x}{(x-1)^2(x+2)}$.

Let integrand $\dfrac{x}{(x-1)^2(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2}$ ...(*i*)

(Partial fractions)

Multiplying both sides of (*i*) by L.C.M. = $(x-1)^2(x+2)$

$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$

or $x = A(x^2 + \mathbf{2x} - \boldsymbol{x} - 2) + B(x+2) + C(x^2 + 1 - 2x)$

or $x = Ax^2 + Ax - 2A + Bx + 2B + Cx^2 + C - 2Cx$

Comparing coefficients of x^2, x and constant terms on both sides

$\boldsymbol{x^2}$ $A + C = 0$ ...(*ii*)

$\boldsymbol{x}$ $A + B - 2C = 1$ ...(*iii*)

Constants $-2A + 2B + C = 0$ ...(*iv*)

Let us solve (*ii*), (*iii*) and (*iv*) for A, B, C

From (*ii*), $A = -C$

Putting $A = -C$ in (*iv*), $2C + 2B + C = 0$

$\Rightarrow 2B = -3C \Rightarrow B = \dfrac{-3C}{2}$

Putting values of A and B in (*iii*),

$$-C - \frac{-3C}{2} - 2C = 1 \Rightarrow -2C - 3C - 4C = 2$$

$\Rightarrow -9C = 2 \Rightarrow C = \dfrac{-2}{9}$

Putting $C = \dfrac{-2}{9}$, $B = \dfrac{-3C}{2} = \dfrac{-3}{2}\left(\dfrac{-2}{9}\right) = \dfrac{1}{3}$ $\therefore A = -C = \dfrac{2}{9}$

Putting these values of A, B, C in (*i*),

$$\frac{x}{(x-1)^2(x+2)} = \frac{\frac{2}{9}}{x-1} + \frac{\frac{1}{3}}{(x-1)^2} - \frac{\frac{2}{9}}{x+2}$$

$$\therefore \int \frac{x}{(x-1)^2(x+2)}\, dx$$

$$= \frac{2}{9}\int \frac{1}{x-1}\, dx + \frac{1}{3}\int (x-1)^{-2}\, dx - \frac{2}{9}\int \frac{1}{x+2}\, dx$$

$$= \frac{2}{9} \log|x-1| + \frac{1}{3}\frac{(x-1)^{-1}}{(-1)(1)} - \frac{2}{9}\log|x+2| + c$$

$$= \frac{2}{9}(\log|x-1| - \log|x+2|) - \frac{1}{3(x-1)} + c$$

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + c.$$

9. $\dfrac{\mathbf{3}\boldsymbol{x}+\mathbf{5}}{\boldsymbol{x}^3 - \boldsymbol{x}^2 - \boldsymbol{x} + \mathbf{1}}$

Sol. To integrate the (rational) function $\dfrac{3x+5}{x^3 - x^2 - x + 1}$.

Now denominator = $x^3 - x^2 - x + 1$

$= x^2 (x - 1) - 1(x - 1) = (x - 1)(x^2 - 1)$

$= (x - 1)(x - 1)(x + 1) = (x - 1)^2 (x + 1)$

$\therefore$ Integrand $\dfrac{3x+5}{x^3 - x^2 - x + 1} = \dfrac{3x+5}{(x-1)^2 (x+1)}$

$= \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1}$ ...(*i*) (Partial fractions)

Multiplying by L.C.M. = $(x - 1)^2 (x + 1)$,

$3x + 5 = A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^2$

$= A(x^2 - 1) + B(x + 1) + C(x^2 + 1 - 2x)$

$= Ax^2 - A + Bx + B + Cx^2 + C - 2Cx$

Comparing coefficients of x^2, x and constant terms on both sides,

x^2 $A + C = 0$ ...(*ii*)

x $B - 2C = 3$ ...(*iii*)

Constants $- A + B + C = 5$ ...(*iv*)

Let us solve Eqns. (*ii*), (*iii*) and (*iv*) for A, B, C.

From (*ii*), A = – C and from (*iii*), B = 2C + 3

Putting these values of A and B in (*iv*),

$$C + 2C + 3 + C = 5 \quad \Rightarrow \quad 4C = 2 \quad \Rightarrow \quad C = \frac{2}{4} = \frac{1}{2}$$

$\therefore \quad A = - C = - \dfrac{1}{2}$

and $\quad B = 2C + 3 = 2\left(\dfrac{1}{2}\right) + 3 = 1 + 3 = 4.$

Putting these values of A, B, C in (*i*)

$$\frac{3x+5}{x^3 - x^2 - x + 1} = \frac{\frac{-1}{2}}{x-1} + \frac{4}{(x-1)^2} + \frac{\frac{1}{2}}{x+1}$$

$\therefore \displaystyle\int \frac{3x+5}{x^3 - x^2 - x + 1}\, dx$

$$= \frac{-1}{2} \int \frac{1}{x-1}\, dx + 4 \int (x-1)^{-2}\, dx + \frac{1}{2} \int \frac{1}{x+1}\, dx$$

$$= \frac{-1}{2} \log | x - 1 | + 4 \frac{(x-1)^{-1}}{(-1)\underset{\substack{\downarrow \\ \text{Coeff. of } x}}{(1)}} + \frac{1}{2} \log | x + 1 | + c$$

$$= \frac{1}{2} (\log | x + 1 | - \log | x - 1 |) - \frac{4}{x-1} + c$$

$$= \frac{1}{2} \log \left| \frac{x+1}{x-1} \right| - \frac{4}{x-1} + c.$$

10. $\dfrac{\mathbf{2x - 3}}{\mathbf{(x^2 - 1)(2x + 3)}}$

Sol. To integrate the rational function $\dfrac{2x-3}{(x^2-1)(2x+3)}$.

Let integrand $\dfrac{2x-3}{(x^2-1)(2x+3)} = \dfrac{2x-3}{(x-1)(x+1)(2x+3)}$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3} \quad ...(i)$$

Multiplying both sides by L.C.M. = $(x - 1)(x + 1)(2x + 3)$,

$2x - 3 = A(x + 1)(2x + 3) + B(x - 1)(2x + 3) + C(x - 1)(x + 1)$

or $2x - 3 = A(2x^2 + \mathbf{3x + 2x} + 3) + B(2x^2 + \mathbf{3x - 2x} - 3) + C(x^2 - 1)$

Comparing coefficients of x^2, x and constant terms on both sides,

$\boldsymbol{x^2}$ $\quad 2A + 2B + C = 0 \quad ...(ii)$

$\boldsymbol{x}$ $\quad 5A + B = 2 \quad ...(iii)$

Constants $\quad 3A - 3B - C = -3 \quad ...(iv)$

Let us solve Eqns. (*ii*), (*iii*) and (*iv*) for A, B, C.

Eqn. (*ii*) + Eqn. (*iv*) gives (to eliminate C)

$$5A - B = -3 \quad ...(v)$$

Adding Eqns. (*iii*) and (*v*), $10A = -1 \Rightarrow A = \dfrac{-1}{10}$

Putting $A = \dfrac{-1}{10}$ in (*iii*), $\dfrac{-5}{10} + B = 2 \Rightarrow B = 2 + \dfrac{1}{2} = \dfrac{5}{2}$

Putting values of A and B in (*ii*),

$$\frac{-1}{5} + 5 + C = 0 \qquad \therefore \quad C = \frac{1}{5} - 5 = \frac{1-25}{25} = \frac{-24}{5}$$

Putting values of A, B, C in (*i*),

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{\frac{-1}{10}}{x-1} + \frac{\frac{5}{2}}{x+1} - \frac{\frac{24}{5}}{2x+3}$$

$$\therefore \int \frac{2x-3}{(x^2-1)(2x+3)}\,dx$$

$$= \frac{-1}{10}\int \frac{1}{x-1}\,dx + \frac{5}{2}\int \frac{1}{x+1}\,dx - \frac{24}{5}\int \frac{1}{2x+3}\,dx$$

$$= \frac{-1}{10}\,\frac{\log|x-1|}{1 \to \text{Coeff. of } x} + \frac{5}{2}\,\frac{\log|x+1|}{1} - \frac{24}{5}\,\frac{\log|2x+3|}{2 \to \text{Coeff. of } x} + c$$

$$= \frac{-1}{10}\log|x-1| + \frac{5}{2}\log|x+1| - \frac{12}{5}\log|2x+3| + c$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + c.$$

11. $\boldsymbol{\dfrac{5x}{(x+1)(x^2-4)}}$

Sol. To integrate the rational function $\dfrac{5x}{(x+1)(x^2-4)}$.

Let integrand $\dfrac{5x}{(x+1)(x^2-4)} = \dfrac{5x}{(x+1)(x+2)(x-2)}$

$$= \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-2} \quad ...(i) \text{ (Partial fractions)}$$

Multiplying both sides of (i) by L.C.M.

$= (x + 1)(x + 2)(x - 2),$

$5x = A(x + 2)(x - 2) + B(x + 1)(x - 2) + C(x + 1)(x + 2)$

$= A(x^2 - 4) + B(x^2 - x - 2) + C(x^2 + 3x + 2)$

$= Ax^2 - 4A + Bx^2 - Bx - 2B + Cx^2 + 3Cx + 2C.$

Comparing coefficients of x^2, x and constant terms on both sides,

$\boldsymbol{x^2}$ $\quad A + B + C = 0$ ...(ii)

$\boldsymbol{x}$ $\quad - B + 3C = 5$ ...(iii)

Constants $\quad - 4A - 2B + 2C = 0$

Dividing by $- 2$, $2A + B - C = 0$ ...(iv)

Let us solve (ii), (iii) and (iv) for A, B, C

Eqn. (ii) × 2 – Eqn. (iv) gives (To eliminate A) because Eqn. (iii) does not involve A.

$2A + 2B + 2C - (2A + B - C) = 0,$

i.e., $\quad 2A + 2B + 2C - 2A - B + C = 0$

$\Rightarrow \quad B + 3C = 0$ ...(v)

Adding Eqns. (iii) and (v),

$$6C = 5 \qquad \Rightarrow \qquad C = \frac{5}{6}$$

Putting $C = \frac{5}{6}$ in (iii), $- B + \frac{15}{6} = 5 \quad \Rightarrow \quad - B = 5 - \frac{15}{6}$

$$\Rightarrow \quad - B = \frac{30 - 15}{6} = \frac{15}{6} = \frac{5}{2} \quad \Rightarrow \quad B = \frac{-5}{2}$$

Putting $B = \frac{-5}{2}$ and $C = \frac{5}{6}$ in (ii), $A - \frac{5}{2} + \frac{5}{6} = 0$

$$\Rightarrow \quad A = \frac{5}{2} - \frac{5}{6} = \frac{15 - 5}{6} = \frac{10}{6} = \frac{5}{3}$$

Putting values of A, B, C in (i),

$$\frac{5x}{(x + 1)(x^2 - 4)} = \frac{\frac{5}{3}}{x + 1} - \frac{\frac{5}{2}}{x + 2} + \frac{\frac{5}{6}}{x - 2}$$

$$\therefore \quad \int \frac{5x}{(x + 1)(x^2 - 4} \, dx = \frac{5}{3} \int \frac{1}{x + 1} \, dx - \frac{5}{2} \int \frac{1}{x + 2} \, dx + \frac{5}{6} \int \frac{1}{x - 2} \, dx$$

$$= \frac{5}{3} \log | x + 1 | - \frac{5}{2} \log | x + 2 | + \frac{5}{6} \log | x - 2 | + c.$$

12. $\boldsymbol{\dfrac{x^3 + x + 1}{x^2 - 1}}$

Sol. Here degree of numerator is greater than degree of denominator. Therefore, dividing the numerator by the denominator,

$$\begin{array}{r|l} x^2 - 1 & x^3 + x + 1 \;(\; x \\ & x^3 - x \\ & \underline{- \quad +} \\ & \quad 2x + 1 \end{array}$$

$$\therefore \quad \frac{x^3 + x + 1}{x^2 - 1} = x + \frac{2x + 1}{x^2 - 1} \qquad ...(i)$$

$$\left[\textbf{Rational function = Quotient + }\frac{\textbf{Remainder}}{\textbf{Divisor}}\right]$$

Let $\frac{2x+1}{x^2-1} = \frac{2x+1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$ ...(*ii*)

Multiplying by L.C.M. = $(x + 1)(x - 1)$, we have

$2x + 1 = A(x - 1) + B(x + 1)$

or $2x + 1 = Ax - A + Bx + B$

By equating the coefficients of x and constant terms, we get

$A + B = 2$ ...(*iii*)

and $-A + B = 1$ ...(*iv*)

(*iii*) + (*iv*) gives $2B = 3 \Rightarrow B = \frac{3}{2}$

Putting $B = \frac{3}{2}$ in (*iii*), we get $A + \frac{3}{2} = 2$ or $A = \frac{1}{2}$

Putting values of A and B in eqn. (*ii*), we have

$$\frac{2x+1}{x^2-1} = \frac{\frac{1}{2}}{x+1} + \frac{\frac{3}{2}}{x-1}$$

Putting this value of $\frac{2x+1}{x^2-1}$ in (*i*),

$$\frac{x^3+x+1}{x^2-1} = x + \frac{\frac{1}{2}}{x+1} + \frac{\frac{3}{2}}{x-1}$$

$$\therefore \int \frac{x^3+x+1}{x^2-1}\,dx = \int x\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{3}{2}\int \frac{1}{x-1}\,dx$$

$$= \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + c.$$

Integrate the following functions in Exercises 13 to 17:

13. $\frac{2}{(1-x)(1+x^2)}$

Sol. To find integral of the Rational function $\frac{2}{(1-x)(1+x^2)}$.

Let integrand $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}$ ...(*i*)

(Partial Fractions)

Multiplying by L.C.M. = $(1 - x)(1 + x^2)$

$2 = A(1 + x^2) + (Bx + C)(1 - x)$

or $2 = A + Ax^2 + Bx - Bx^2 + C - Cx$

Comparing coefficients of x^2, x and constant terms, we have

$\boldsymbol{x^2}$ $A - B = 0$ ...(*ii*)

$\boldsymbol{x}$ $B - C = 0$ ...(*iii*)

Constant terms $A + C = 2$ ...(*iv*)

Let us solve (*ii*), (*iii*), (*iv*) for A, B, C

From (*ii*), $A = B$ and from (*iii*), $B = C$

$\therefore \quad A = B = C$

Putting A = C in (*iv*), C + C = 2 or 2C = 2 or C = 1

$\therefore \quad A = C = 1 \qquad \therefore \quad B = A = 1$

Putting these values of A, B, C in (*i*),

$$\frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2} = \frac{1}{1-x} + \frac{x}{1+x^2} + \frac{1}{1+x^2}$$

$$= \frac{1}{1-x} + \frac{1}{2}\frac{2x}{1+x^2} + \frac{1}{1+x^2}$$

$$\therefore \int \frac{2}{(1-x)(1+x^2)} dx = \int \frac{1}{1-x} dx + \frac{1}{2}\int \frac{2x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= \frac{\log|1-x|}{-1 \rightarrow \text{Coefficient of } x} + \frac{1}{2} \log|1+x^2| + \tan^{-1} x + c$$

$$\left[\because \int \frac{2x}{1+x^2} dx = \int \frac{f'(x)}{f(x)} dx = \log|f(x)|\right]$$

$$= -\log|1-x| + \frac{1}{2}\log(1+x^2) + \tan^{-1} x + c$$

$(\because \ 1 + x^2 > 0$, therefore $|1 + x^2| = 1 + x^2)$

Note. $\log|1-x| = \log|-(x-1)|$

$= \log|x-1|$ because $|-t| = |t|$.

14. $\boldsymbol{\dfrac{3x-1}{(x+2)^2}}$

Sol. To find integral of rational function $\dfrac{3x-1}{(x+2)^2}$.

Let $I = \int \dfrac{3x-1}{(x+2)^2} dx$ ...(*i*)

Form $\int \dfrac{\text{Polynomial function}}{(\text{Linear})^k} dx$ where k is a positive integer,

put Linear = t.

Here put $x + 2 = t \quad \Rightarrow \quad x = t - 2$

$\therefore \quad \dfrac{dx}{dt} = 1 \quad \Rightarrow \quad dx = dt$

Putting these values in (*i*),

$$I = \int \frac{3(t-2)-1}{t^2} dt = \int \frac{3t-6-1}{t^2} dt = \int \frac{3t-7}{t^2} dt$$

$$= \int \left(\frac{3t}{t^2} - \frac{7}{t^2}\right) dt = \int \left(\frac{3}{t} - \frac{7}{t^2}\right) dt$$

$$= 3\int \frac{1}{t} dt - 7\int t^{-2} dt = 3\log|t| - 7\frac{t^{-1}}{-1} + c$$

$$= 3\log|t| + \frac{7}{t} + c$$

Putting $t = x + 2$, $= 3 \log | x + 2 | + \dfrac{7}{x+2} + c$.

Remark. Alternative solution is Let $\dfrac{3x-1}{(x+2)^2} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2}$.

15. $\dfrac{\mathbf{1}}{\boldsymbol{x^4 - 1}}$

Sol. To find integral of $\dfrac{1}{x^4-1}$.

Let integrand $\dfrac{1}{x^4-1} = \dfrac{1}{(x^2-1)(x^2+1)}$.

Put $x^2 = y$ **only** to form partial fractions.

$$= \frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1} \qquad ...(i)$$

Multiplying by L.C.M. $= (y - 1)(y + 1)$

$1 = A(y + 1) + B(y - 1)$ or $1 = Ay + A + By - B$

Comparing coeffs. of y and constant terms, we have

Coefficients of y: $A + B = 0$ ...(*ii*)

Constant terms $A - B = 1$ ...(*iii*)

Adding (*ii*) and (*iii*), $2A = 1$ $\Rightarrow$ $A = \dfrac{1}{2}$

Putting $A = \dfrac{1}{2}$ in (*ii*), $\dfrac{1}{2} + B = 0$ $\Rightarrow$ $B = \dfrac{-1}{2}$

Putting values of A, B **and y** in (*i*),

$$\frac{1}{x^4-1} = \frac{\frac{1}{2}}{x^2-1} - \frac{\frac{1}{2}}{x^2+1}$$

$$\therefore \int \frac{1}{x^4-1}\,dx = \frac{1}{2}\int \frac{1}{x^2-1^2}\,dx - \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\,\frac{1}{2.1}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1} x + c$$

$$\left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

Note. Must put $y = x^2$ in (*i*) along with values of A and B before writing values of integrals.

Remark. Alternative solution is:

$$\frac{1}{x^4-1} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x-1)(x+1)(x^2+1)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

But the above given solution is better.

16. $\mathbf{\dfrac{1}{x(x^n+1)}}$

Sol. Let I = $\int \dfrac{1}{x(x^n+1)}\,dx$

Multiplying both numerator and denominator of integrand by nx^{n-1}.

$$\left[\because \frac{d}{dx}(x^n+1) = nx^{n-1}\right]$$

$$I = \int \frac{nx^{n-1}}{n\,x^{n-1}\,x(x^n+1)}\,dx = \frac{1}{n}\int \frac{n\,x^{n-1}}{x^n\,(x^n+1)}\,dx \quad ...(i)$$

$$(\because \ n-1+1 = n)$$

Put $x^n = t$. Therefore $n\,x^{n-1} = \dfrac{dt}{dx}$. $\quad \therefore \ n\,x^{n-1}\,dx = dt$.

$\therefore$ From (i), I = $\dfrac{1}{n}\displaystyle\int \frac{dt}{t(t+1)} = \frac{1}{n}\int \frac{1}{t(t+1)}\,dt$

Adding and subtracting t in the numerator of integrand,

$$= \frac{1}{n}\int \frac{t+1-t}{t(t+1)}\,dt = \frac{1}{n}\int \left(\frac{t+1}{t(t+1)} - \frac{t}{t(t+1)}\right)dt \quad \left[\because \ \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}\right]$$

$$= \frac{1}{n}\left[\int \frac{1}{t}\,dt - \int \frac{1}{t+1}\,dt\right] = \frac{1}{n}\,[\log|t| - \log|t+1| + c]$$

$$= \frac{1}{n}\log\left|\frac{t}{t+1}\right| + c$$

Putting $t = x^n$, $= \dfrac{1}{n}\log\left|\dfrac{x^n}{x^n+1}\right| + c$

Remark: Alternative solution for $\displaystyle\int \frac{1}{t(t+1)}\,dt$ is:

Let $\dfrac{1}{t(t+1)} = \dfrac{A}{t} + \dfrac{B}{t+1}$.

But the above given solution is better.

17. $\mathbf{\dfrac{\cos x}{(1-\sin x)(2-\sin x)}}$

Sol. Let I = $\displaystyle\int \frac{\cos x}{(1-\sin x)(2-\sin x)}\,dx$ ...(i)

Put $\sin x = t$. Therefore $\cos x = \dfrac{dt}{dx} \ \Rightarrow \ \cos x\,dx = dt$,

$\therefore$ From (i), $\displaystyle\int \frac{1}{(1-t)(2-t)}\,dt = \int \frac{(2-t)-(1-t)}{(1-t)(2-t)}\,dt$

[$\because$ Difference of two factors in the denominator namely $1-t$ and $2-t$ is $(2-t)-(1-t) = 2-t-1+t = 1$]

$$= \int\left(\frac{2-t}{(1-t)(2-t)} - \frac{(1-t)}{(1-t)(2-t)}\right)dt \quad \left[\because \ \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}\right]$$

$$= \int \left(\frac{1}{1-t} - \frac{1}{2-t} \right) dt = \int \frac{1}{1-t}\, dt - \int \frac{1}{2-t}\, dt$$

$$= \frac{\log|1-t|}{-1 \to \text{Coefficient of } t} - \frac{\log|2-t|}{-1} + c$$

$$= -\log|1-t| + \log|2-t| + c$$

$$= \log|2-t| - \log|1-t| + c \qquad = \log\left|\frac{2-t}{1-t}\right| + c$$

Putting $t = \sin x$, $= \log\left|\dfrac{2-\sin x}{1-\sin x}\right| + c$

Remark: Alternative solution for $\int \dfrac{1}{(1-t)(2-t)}\, dt$ is

Let $\dfrac{1}{(1-t)(2-t)} = \dfrac{A}{1-t} + \dfrac{B}{2-t}$

Integrate the following functions for Exercises 18 to 21:

18. $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Sol. To integrate the rational function $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$. ...(*i*)

Put $x^2 = y$ in the integrand to get

$$= \frac{(y+1)(y+2)}{(y+3)(y+4)} = \frac{y^2+3y+2}{y^2+7y+12} \qquad ...(ii)$$

Here degree of numerator = degree of denominator (= 2)
So have to perform long division to make the degree of numerator smaller than degree of denominator so that the concept of forming partial fractions becomes valid.

$$\begin{array}{r|l} y^2 + 7y + 12 & y^2 + 3y + 2 \;(\, 1 \\ & \underline{\begin{array}{l} y^2 + 7y + 12 \\ - \quad - \quad - \end{array}} \\ & \quad -4y - 10 \end{array}$$

$\therefore$ From (*i*) and (*ii*),

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{(y+1)(y+2)}{(y+3)(y+4)} = 1 + \frac{(-4y-10)}{(y+3)(y+4)} \qquad ...(iii)$$

Let us form partial fractions of $\dfrac{(-4y-10)}{(y+3)(y+4)}$.

Let $\dfrac{-4y-10}{(y+3)(y+4)} = \dfrac{A}{y+3} + \dfrac{B}{y+4}$ ...(*iv*)

Multiplying by L.C.M. = $(y+3)(y+4)$

$$-4y - 10 = A(y+4) + B(y+3) = Ay + 4A + By + 3B$$

Comparing coefficients of y, $A + B = -4$ ...(*v*)
Comparing constants, $4A + 3B = -10$ ...(*vi*)
Let us solve Eqns. (*v*) and (*vi*) for A and B.
Eqn. (*v*) $\times$ 4 gives, $4A + 4B = -16$ ...(*vii*)

Eqn. (vi) – Eqn. (vii) gives, $-B = 6$ or $B = -6$.
Putting $B = -6$ in (v), $A - 6 = -4 \Rightarrow A = -4 + 6 = 2$
Putting these values of A and B in (iv),

$$\frac{-4y-10}{(y+3)(y+4)} = \frac{2}{y+3} - \frac{6}{y+4}$$

Putting this value in (iii),

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 + \frac{2}{y+3} - \frac{6}{y+4}$$

In R.H.S., Putting $y = x^2$ (before integration)

$$= 1 + \frac{2}{x^2+3} - \frac{6}{x^2+4}$$

$$\therefore \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\, dx$$

$$= \int 1\, dx + 2\int \frac{1}{x^2+(\sqrt{3})^2}\, dx - 6\int \frac{1}{x^2+2^2}\, dx$$

$$= x + 2 \cdot \frac{1}{\sqrt{3}} \tan^{-1}\frac{x}{\sqrt{3}} - 6 \cdot \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + c.$$

19. $\mathbf{\dfrac{2x}{(x^2+1)(x^2+3)}}$

Sol. Let $I = \int \frac{2x}{(x^2+1)(x^2+3)}\, dx$

Put $x^2 = t$. Differentiating both sides $2x\, dx = dt$

$$\therefore \quad I = \int \frac{dt}{(t+1)(t+3)}$$

Dividing and multiplying by 2,
$(\because \ (t+3) - (t+1) = t + 3 - t - 1 = 2)$

$$= \frac{1}{2}\int \frac{2}{(t+1)(t+3)}\, dt = \frac{1}{2}\int \frac{(t+3)-(t+1)}{(t+1)(t+3)}\, dt$$

$$= \frac{1}{2}\int \left(\frac{1}{t+1} - \frac{1}{t+3}\right) dt = \frac{1}{2}\,[\log|t+1| - \log|t+3|] + c$$

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + c = \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + c = \frac{1}{2}\log\left(\frac{x^2+1}{x^2+3}\right) + c.$$

20. $\mathbf{\dfrac{1}{x(x^4-1)}}$

Sol. Let $I = \int \frac{1}{x(x^4-1)}\, dx$

Multiplying both numerator and denominator of integrand by $4x^3$.

$$\left(\because \ \frac{d}{dx}(x^4-1) = 4x^3\right)$$

$$I = \int \frac{4x^3}{4x^4\,(x^4-1)}\,dx = \frac{1}{4}\int \frac{4x^3}{x^4\,(x^4-1)}\,dx \qquad ...(i)$$

Put $x^4 = t$. Therefore $4x^3 = \frac{dt}{dx} \Rightarrow 4x^3\,dx = dt.$

$$\therefore \text{ From } (i),\ I = \frac{1}{4}\int \frac{dt}{t(t-1)} = \frac{1}{4}\int \frac{t-(t-1)}{t(t-1)}\,dt$$

$[\because\ t-(t-1) = t-t+1 = 1]$

$$= \frac{1}{4}\int \left(\frac{t}{t(t-1)} - \frac{(t-1)}{t(t-1)}\right) dt = \frac{1}{4}\int \left(\frac{1}{t-1} - \frac{1}{t}\right) dt$$

$$= \frac{1}{4}\left[\int \frac{1}{t-1}\,dt - \int \frac{1}{t}\,dt\right] = \frac{1}{4}\,[\log|t-1| - \log|t|] + c$$

$$= \frac{1}{4}\log\left|\frac{t-1}{t}\right| + c$$

Putting $t = x^4$, $= \frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + c.$

Remark: Alternative solution is:

$$\frac{1}{x(x^4-1)} = \frac{1}{x(x^2-1)(x^2+1)} = \frac{1}{x(x-1)(x+1)(x^2+1)}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{Dx+E}{x^2+1}$$

But the solution given above is much better.

21. $\boldsymbol{\dfrac{1}{(e^x-1)}}$

Sol. Let $I = \int \frac{1}{e^x-1}\,dx \qquad ...(i)$

Put $e^x = t$. Therefore $e^x = \frac{dt}{dx} \Rightarrow e^x\,dx = dt \Rightarrow dx = \frac{dt}{e^x}$

$$\left(\textbf{Rule to evaluate } \int f(e^x)\,dx, \textbf{ put } e^x = t\right)$$

$$\therefore \text{ From } (i),\ I = \int \frac{1}{t-1}\,\frac{dt}{e^x} = \int \frac{1}{t-1}\,\frac{dt}{t} = \int \frac{1}{t(t-1)}\,dt$$

$$= \int \frac{t-(t-1)}{t(t-1)}\,dt = \int \left(\frac{t}{t(t-1)} - \frac{(t-1)}{t(t-1)}\right) dt = \int \frac{1}{t-1}\,dt - \int \frac{1}{t}\,dt$$

$$= \log|t-1| - \log|t| + c = \log\left|\frac{t-1}{t}\right| + c.$$

Putting $t = e^x$, $= \log\left|\frac{e^x-1}{e^x}\right| + c.$

Choose the correct answer in each of the Exercises 22 and 23:

22. $\int \frac{x\,dx}{(x-1)(x-2)}$ **equals**

(A) $\log \left| \frac{(x-1)^2}{x-2} \right| + C$ **(B)** $\log \left| \frac{(x-2)^2}{x-1} \right| + C$

(C) $\log \left| \left(\frac{x-1}{x-2}\right)^2 \right| + C$ **(D)** $\log | (x-1)(x-2) | + C.$

Sol. Let integrand $\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$ ...(*i*)

(Partial fractions)

Multiplying by L.C.M. = $(x - 1)(x - 2)$,

$$x = A(x - 2) + B(x - 1)$$
$$= Ax - 2A + Bx - B$$

Comparing coefficients of x and constant terms on both sides,

Coefficients of x: $A + B = 1$ ...(*ii*)

Constant terms: $-2A - B = 0$ ...(*iii*)

Let us solve (*ii*) and (*iii*) for A and B

Adding (*ii*) and (*iii*), $-A = 1$ or $A = -1$

Putting $A = -1$ in (*ii*) $-1 + B = 1$ or $B = 2$

Putting values of A and B in (*i*),

$$\frac{x}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{2}{x-2}$$

$$\therefore \int \frac{x}{(x-1)(x-2)}\,dx = -\int \frac{1}{x-1}\,dx + 2\int \frac{1}{x-2}\,dx$$

$$= -\log | x - 1 | + 2 \log | x - 2 | + c$$
$$= \log | (x - 2)^2 | - \log | x - 1 | + c$$

$(\because\ n \log m = \log m^n)$

$$= \log \left| \frac{(x-2)^2}{x-1} \right| + c$$

$\therefore$ Option (B) is the correct answer.

23. $\int \frac{dx}{x(x^2+1)}$ **equals**

(A) $\log | x | - \frac{1}{2} \log (x^2 + 1) + C$

(B) $\log | x | + \frac{1}{2} \log (x^2 + 1) + C$

(C) $-\log | x | + \frac{1}{2} \log (x^2 + 1) + C$

(D) $\frac{1}{2} \log | x | + \log (x^2 + 1) + C.$

Sol. Let $I = \int \frac{1}{x(x^2+1)}\,dx$

Multiplying both numerator and denominator of integrand by $2x$.

$$\left(\because\ \frac{d}{dx}(x^2+1) = 2x\right)$$

$\Rightarrow \quad \mathrm{I} = \int \dfrac{2x}{2x^2(x^2+1)}\, dx$ ...(*i*)

Put $x^2 = t$. $\quad \therefore \quad 2x = \dfrac{dt}{dx} \quad \Rightarrow \quad 2x\, dx = dt$

$\therefore$ From (*i*), $\mathrm{I} = \int \dfrac{dt}{2t(t+1)} = \dfrac{1}{2}\int \dfrac{1}{t(t+1)}\, dt$

Adding and subtracting t in the numerator of integrand,

$$= \frac{1}{2}\int \frac{(t+1)-t}{t(t+1)}\, dt = \frac{1}{2}\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt$$

$$= \frac{1}{2}\left(\log|t| - \log|t+1|\right) + c$$

Putting $t = x^2$, $\mathrm{I} = \dfrac{1}{2}\left(\log|x^2| - \log|x^2+1|\right) + c$

$$= \frac{1}{2}\left(2\log|x| - \log(x^2+1)\right) + c$$

$(\because \; x^2 + 1 \geq 1 > 0$ and hence $|x^2+1| = x^2+1)$

$$= \log|x| - \frac{1}{2}\log(x^2+1) + c$$

$\therefore$ Option (A) is the correct answer.

Exercise 7.6

Integrate the functions in Exercises 1 to 8:

1. $\boldsymbol{x \sin x}$

Sol. $\int \underset{\text{I}}{x} \underset{\text{II}}{\sin x}\ dx$

Applying Product Rule I $\int \text{II}\, dx - \int \left(\frac{d}{dx}(\text{I}) \int \text{II}\, dx\right) dx$

$$= x \int \sin x\ dx - \int \left(\frac{d}{dx}(x) \int \sin x\, dx\right) dx$$

$= x\,(-\cos x) - \int 1\,(-\cos x)\ dx = -x \cos x - \int -\cos x\ dx$

$= -x \cos x + \int \cos x\ dx = -x \cos x + \sin x + c$

Note. $\int \sin x\ dx = -\cos x.$

2. $\boldsymbol{x \sin 3x}$

Sol. $\int \underset{\text{I}}{x} \underset{\text{II}}{\sin 3x}\ dx$

Applying Product Rule I $\int \text{II}\, dx - \int \left(\frac{d}{dx}(\text{I}) \int \text{II}\, dx\right) dx$

$= x \int \sin 3x\ dx - \int \left(\frac{d}{dx}(x) \int \sin 3x\, dx\right) dx$

$= x \left(\frac{-\cos 3x}{3}\right) - \int \left[1\left(\frac{-\cos 3x}{3}\right)\right] dx + c$

$= \frac{-1}{3} x \cos 3x + \frac{1}{3} \int \cos 3x\ dx + c$

$= \frac{-1}{3} x \cos 3x + \frac{1}{3} \frac{\sin 3x}{3} + c = \frac{-1}{3} x \cos 3x + \frac{1}{9} \sin 3x + c.$

3. $\boldsymbol{x^2 e^x}$

Sol. $\int \underset{\text{I}}{x^2} \underset{\text{II}}{e^x}\ dx$

Applying Product Rule I $\int \text{II}\, dx - \int \left(\frac{d}{dx}(\text{I}) \int \text{II}\, dx\right) dx$

$= x^2 \int e^x\, dx - \int \left[\left(\frac{d}{dx} x^2\right) \int e^x\, dx\right] dx = x^2 e^x - \int 2x\, e^x\ dx$

$= x^2 e^x - 2 \int \underset{\text{I}}{x} \underset{\text{II}}{e^x}\ dx$

Again Applying Product Rule

$$= x^2 e^x - 2\left[x\int e^x\,dx - \int\left[\frac{d}{dx}(x)\int e^x\,dx\right]dx\right]$$

$$= x^2 e^x - 2\left(x e^x - \int 1.e^x\,dx\right) = x^2 e^x - 2\left(x e^x - \int e^x\,dx\right)$$

$$= x^2 e^x - 2x e^x + 2\int e^x\,dx + c \qquad = x^2 e^x - 2x e^x + 2e^x + c$$

$$= e^x (x^2 - 2x + 2) + c.$$

4. $x \log x$

Sol. $\int x \log x \; dx = \int \underset{\text{I}}{(\log x)} . \underset{\text{II}}{x}\; dx$

Applying Product Rule $\text{I}\int \text{II}\;dx - \int\left[\frac{d}{dx}(\text{I})\int \text{II}\,dx\right]dx$

$$= (\log x)\int x\;dx - \int\left[\frac{d}{dx}(\log x)\int x\,dx\right]dx$$

$$= (\log x)\frac{x^2}{2} - \int\frac{1}{x}\frac{x^2}{2}\,dx = \frac{1}{2}x^2\log x - \frac{1}{2}\int x\;dx$$

$$\left(\because \frac{x^2}{x} = \frac{x\,.\,x}{x} = x\right)$$

$$= \frac{1}{2}x^2\log x - \frac{1}{2}\frac{x^2}{2} + c = \frac{x^2}{2}\log x - \frac{x^2}{4} + c.$$

5. $x \log 2x$

Sol. $\int x \log 2x \; dx = \int \underset{\text{I}}{(\log 2x)} . \underset{\text{II}}{x}\; dx$

Applying Product Rule $\text{I}\int \text{II}\,dx - \int\left(\frac{d}{dx}(\text{I})\int \text{II}\,dx\right)dx$

$$= (\log 2x)\int x\;dx - \int\left(\frac{d}{dx}(\log 2x)\int x\,dx\right)dx$$

$$= (\log 2x)\frac{x^2}{2} - \int\frac{1}{2x}\,.\,2\,.\,\frac{x^2}{2}\,dx$$

$$= \frac{1}{2}x^2\log 2x - \frac{1}{2}\int x\;dx \qquad \left[\because \frac{x^2}{x} = \frac{x\,.\,x}{x} = x\right]$$

$$= \frac{1}{2}x^2\log 2x - \frac{1}{2}\frac{x^2}{2} + c = \frac{x^2}{2}\log 2x - \frac{x^2}{4} + c.$$

6. $x^2 \log x$

Sol. $\int x^2 \log x \; dx = \int \underset{\text{I}}{(\log x)}\, \underset{\text{II}}{x^2}\; dx$

Applying Product Rule: $\text{I}\int \text{II}\,dx - \int\left(\frac{d}{dx}(\text{I})\int \text{II}\,dx\right)dx$

$$= \log x\int x^2\;dx - \int\left(\frac{d}{dx}(\log x)\int x^2\,dx\right)dx$$

$$= (\log x)\frac{x^3}{3} - \int \frac{1}{x}\ \frac{x^3}{3}\ dx = \frac{x^3}{3}\ \log x - \frac{1}{3}\int x^2\ dx \left[\because\ \frac{x^3}{x} = x^2\right]$$

$$= \frac{x^3}{3}\ \log x - \frac{1}{3}\ \frac{x^3}{3} + c = \frac{x^3}{3}\ \log x - \frac{x^3}{9} + c.$$

7. $\boldsymbol{x \sin^{-1} x}$

Sol. Let $I = \int x \sin^{-1} x\ dx.$

Put $\boldsymbol{x = \sin\theta}$. Differentiating both sides $dx = \cos\theta\ d\theta$

$$\therefore\quad I = \int \sin\theta\ .\ \theta\ .\ \cos\theta\ d\theta = \frac{1}{2}\int \theta\ .\ 2\sin\theta\cos\theta\ d\theta$$

$$= \frac{1}{2}\int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sin 2\theta}\ d\theta$$

Integrating by parts

$$= \frac{1}{2}\left[\theta\left(-\frac{\cos 2\theta}{2}\right) - \int 1.\left(-\frac{\cos 2\theta}{2}\right)d\theta\right]$$

$$= \frac{1}{4}\left[-\theta\cos 2\theta + \int \cos 2\theta\ d\theta\right] = \frac{1}{4}\left[-\theta\cos 2\theta + \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{1}{4}\ [-\theta\ (1 - 2\sin^2\theta) + \sin\theta\cos\theta] + c$$

$$(\because\ \sin 2\theta = 2\sin\theta\cos\theta)$$

$$= \frac{1}{4}\ [-\sin^{-1} x\ .\ (1 - 2x^2) + x\sqrt{1 - x^2}\ ] + c$$

$$\left[\because\ \cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - x^2}\right]$$

$$= \frac{1}{4}\ (2x^2 - 1)\sin^{-1} x + \frac{x\sqrt{1 - x^2}}{4} + c.$$

8. $\boldsymbol{x \tan^{-1} x}$

Sol. Let $I = \int x \tan^{-1} x\ dx = \int (\tan^{-1} x)\ .\ x\ dx$

$$= (\tan^{-1} x)\ .\ \frac{x^2}{2} - \int \frac{1}{1 + x^2}\ .\ \frac{x^2}{2}\ dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1 + x^2}\ dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1 + x^2}\right)dx$$

$$\left[\because\ \frac{x^2}{1 + x^2} = \frac{1 + x^2 - 1}{1 + x^2} = 1 - \frac{1}{1 + x^2}\right]$$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} (x - \tan^{-1} x) + c$

$= \frac{1}{2} [x^2 \tan^{-1} x - x + \tan^{-1} x] + c \quad = \frac{1}{2} [(x^2 + 1) \tan^{-1} x - x] + c.$

Integrate the functions in Exercises 9 to 15:

9. $\boldsymbol{x \cos^{-1} x}$

Sol. Let $\quad I = \int x \cos^{-1} x \; dx \quad$ ...(*i*)

Put $\boldsymbol{\cos^{-1} x = \theta}$. Therefore $x = \cos \theta$.

$\therefore \qquad \frac{dx}{d\theta} = -\sin \theta \Rightarrow dx = -\sin \theta \, d\theta$

$\therefore$ From(*i*), $I = \int (\cos \theta)\, \theta\, (-\sin \theta\, d\theta) = \frac{-1}{2} \int \theta\, (2 \sin \theta \cos \theta)\, d\theta$

$= \frac{-1}{2} \int \underset{\text{I}}{\theta} \underset{\text{II}}{\sin 2\theta}\, d\theta$

Applying Product Rule: $\text{I} \int \text{II} \; d\theta - \int \left[\frac{d}{d\theta} (\text{I}) \int \text{II}\, d\theta \right] d\theta$

$= \frac{-1}{2} \left[\theta \left(\frac{-\cos 2\theta}{2} \right) - \int 1 \left(\frac{-\cos 2\theta}{2} \right) d\theta \right]$

$= \frac{-1}{2} \left[\frac{-1}{2} \theta \cos 2\theta + \frac{1}{2} \int \cos 2\theta\, d\theta \right] = \frac{1}{4} \theta \cos 2\theta - \frac{1}{4} \left(\frac{\sin 2\theta}{2} \right) + c$

$= \frac{1}{4} \theta \cos 2\theta - \frac{1}{8} (2 \sin \theta \cos \theta) + c$

$= \frac{1}{4} \theta\, (2 \cos^2 \theta - 1) - \frac{1}{4} \sqrt{1 - \cos^2 \theta} \, . \cos \theta + c$

Putting $\cos \theta = x$ and $\theta = \cos^{-1} x$;

$= \frac{1}{4} (\cos^{-1} x)(2x^2 - 1) - \frac{1}{4} \sqrt{1 - x^2} \, . \, x + c$

$= (2x^2 - 1) \frac{\cos^{-1} x}{4} - \frac{x}{4} \sqrt{1 - x^2} + c.$

10. $\boldsymbol{(\sin^{-1} x)^2}$

Sol. **Put $\boldsymbol{x = \sin \theta}$.** Differentiating both sides, $dx = \cos \theta \, d\theta$

$\therefore \int (\sin^{-1} x)^2\, dx = \int \underset{\text{I}}{\theta^2} \underset{\text{II}}{\cos \theta} \; d\theta = \theta^2 \sin \theta - \int 2\theta \sin \theta \; d\theta$

$= \theta^2 \sin \theta - 2 \int \underset{\text{I}}{\theta} \underset{\text{II}}{\sin \theta} \; d\theta$

$= \theta^2 \sin \theta - 2 \left[\theta\, (-\cos \theta) - \int 1\, .\, (-\cos \theta)\, d\theta \right]$

$= \theta^2 \sin \theta + 2\theta \cos \theta - 2 \int \cos \theta\, d\theta \; = \theta^2 \sin \theta + 2\theta \cos \theta - 2 \sin \theta + c$

$= x (\sin^{-1} x)^2 + 2 \sqrt{1 - x^2} \sin^{-1} x - 2x + c.$

$\left(\because \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}\right)$

11. $\dfrac{\boldsymbol{x \cos^{-1} x}}{\boldsymbol{\sqrt{1-x^2}}}$

Sol. Let $I = \int \dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}\, dx$ ...(*i*)

Put $\boldsymbol{\cos^{-1} x = \theta}$**.** $\Rightarrow$ $x = \cos\theta$

Therefore $\dfrac{dx}{d\theta} = -\sin\theta \Rightarrow dx = -\sin\theta\, d\theta$

$\therefore$ From (*i*), $I = \int \dfrac{(\cos\theta)\,\theta}{\sqrt{1-\cos^2\theta}}\,(-\sin\theta\, d\theta)$

$= -\int \dfrac{\theta\cos\theta\sin\theta}{\sin\theta}\, d\theta \quad (\because \sqrt{1-\cos^2\theta} = \sqrt{\sin^2\theta} = \sin\theta)$

$= -\int \underset{\text{I}}{\theta}\, \underset{\text{II}}{\cos\theta}\, d\theta$

Applying Product Rule: $\text{I}\int \text{II}\, d\theta - \int\left[\dfrac{d}{d\theta}(\text{I})\int \text{II}\, d\theta\right] d\theta$

$= -\left[\theta\,.\sin\theta - \int 1\,.\sin\theta\, d\theta\right] = -\theta\sin\theta + \int \sin\theta\, d\theta$

$= -\theta\sin\theta - \cos\theta + c = -\theta\sqrt{1-\cos^2\theta} - \cos\theta + c$

Putting $\theta = \cos^{-1}x$ and $\cos\theta = x$,

$= -(\cos^{-1}x)\sqrt{1-x^2} - x + c = -[\sqrt{1-x^2}\cos^{-1}x + x] + c.$

12. $\boldsymbol{x \sec^2 x}$

Sol. $\int \underset{\text{I}}{x}\, \underset{\text{II}}{\sec^2 x}\, dx$

Applying Product Rule: $\text{I}\int \text{II}\, dx - \int\left[\dfrac{d}{dx}(\text{I})\int \text{II}\, dx\right] dx$

$= x\int \sec^2 x\, dx - \int\left[\dfrac{d}{dx}(x)\int \sec^2 x\, dx\right] dx$

$= x\tan x - \int 1\,.\tan x\, dx = x\tan x - \int \tan x\, dx$

$= x\tan x - (-\log|\cos x|) + c = x\tan x + \log|\cos x| + c.$

13. $\boldsymbol{\tan^{-1} x}$

Sol. Let $I = \int \tan^{-1}x\, dx = \int (\tan^{-1}x)\,.\,1\, dx$

$= \tan^{-1}x\,.\,x - \int \dfrac{1}{1+x^2}\,.\,x\, dx = x\tan^{-1}x - \dfrac{1}{2}\int \dfrac{2x}{1+x^2}\, dx$

$= x\tan^{-1}x - \dfrac{1}{2}\log|(1+x^2)| + c.\left[\because \int \dfrac{f'(x)}{f(x)}\, dx = \log|f(x)|\right]$

$$= x \tan^{-1} x - \frac{1}{2} \log (1 + x^2) + c$$

$[\because \quad 1 + x^2 \geq 1 > 0 \text{ and hence } | 1 + x^2 | = 1 + x^2]$

14. $x (\log x)^2$

Sol. $\int x (\log x)^2 \; dx = \int \underset{\text{I}}{(\log x)^2} . \underset{\text{II}}{x} \; dx$

Applying Product Rule: $\text{I} \int \text{II} \; dx - \int \left[\frac{d}{dx} (\text{I}) \int \text{II} \, dx \right] dx$

$$= (\log x)^2 \int x \; dx - \int \left[\frac{d}{dx} (\log x)^2 \int x \, dx \right] dx$$

$$= (\log x)^2 \frac{x^2}{2} - \int \frac{2 (\log x)}{x} \frac{x^2}{2} \; dx$$

$$\left[\because \frac{d}{dx} (\log x)^2 = 2(\log x)^1 \frac{d}{dx} (\log x) = 2 \log x . \frac{1}{x} = \frac{2 \log x}{x} \right]$$

$$= \frac{x^2}{2} (\log x)^2 - \int \underset{\text{I}}{(\log x)} \underset{\text{II}}{x} \; dx \qquad \left[\because \frac{x^2}{x} = \frac{x . x}{x} = x \right]$$

Again applying Product Rule: $\text{I} \int \text{II} \; dx - \int \left[\frac{d}{dx} (\text{I}) \int \text{II} \, dx \right] dx$

$$= \frac{x^2}{2} (\log x)^2 - \left[(\log x) \frac{x^2}{2} - \int \left(\frac{1}{x} \frac{x^2}{2} \right) dx \right] + c$$

$$= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{1}{2} \int x \; dx + c$$

$$= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{x^2}{4} + c.$$

15. $(x^2 + 1) \log x$

Sol. $\int (x^2 + 1) \; \log x \, dx = \int \underset{\text{I}}{(\log x)} \underset{\text{II}}{(x^2 + 1)} \; dx$

Applying Product Rule: $\text{I} \int \text{II} \; dx - \int \left[\frac{d}{dx} (\text{I}) \int \text{II} \, dx \right] dx$

$$= \log x \left(\frac{x^3}{3} + x \right) - \int \frac{1}{x} \left(\frac{x^3}{3} + x \right) dx$$

$$= \left(\frac{x^3}{3} + x \right) \log x - \int \left(\frac{x^2}{3} + 1 \right) dx$$

$$= \left(\frac{x^3}{3} + x \right) \log x - \frac{1}{3} \int x^2 \; dx - \int 1 \; dx$$

$$= \left(\frac{x^3}{3} + x \right) \log x - \frac{1}{3} \frac{x^3}{3} - x + c = \left(\frac{x^3}{3} + x \right) \log x - \frac{x^3}{9} - x + c.$$

Integrate the functions in Exercises 16 to 22:

16. $e^x (\sin x + \cos x)$

Sol. Here $I = \int e^x \ (\sin x + \cos x)\ dx$

It is of the form $\int e^x\ [f(x) + f'(x)]\ dx$

Let us take $f(x) = \sin x$ so that $f'(x) = \cos x$

$$I = e^x f(x) + c = e^x \sin x + c.$$

$$\left[\because \int e^x (f(x) + f'(x))\, dx = e^x f(x) + c\right]$$

17. $\dfrac{x e^x}{(1+x)^2}$

Sol. Here $I = \int \dfrac{xe^x}{(x+1)^2}\ dx = \int \dfrac{(x+1)-1}{(x+1)^2}\ e^x\ dx$

$$= \int e^x \left[\frac{x+1}{(x+1)^2} - \frac{1}{(x+1)^2}\right] dx = \int e^x \left[\frac{1}{x+1} + \frac{-1}{(x+1)^2}\right] dx$$

It is of the form $\int e^x\ [f(x) + f'(x)]\ dx$

Let us take $f(x) = \dfrac{1}{x+1}$ so that $f'(x) = \dfrac{d}{dx}\ [(x+1)^{-1}]$

$$= -(x+1)^{-2} = \frac{-1}{(x+1)^2}$$

$\therefore\ I = e^x f(x) + c = \dfrac{e^x}{x+1} + c.$ $\left[\because \int e^x (f(x) + f'(x))\, dx = e^x f(x) + c\right]$

18. $e^x \left(\dfrac{1+\sin x}{1+\cos x}\right)$

Sol. Here $I = \int e^x \cdot \dfrac{1+\sin x}{1+\cos x}\ dx = \int e^x \cdot \dfrac{1 + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\ dx$

$$= \int e^x \cdot \left[\frac{1}{2\cos^2\frac{x}{2}} + \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right] dx = \int e^x \left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right) dx$$

$$= \int e^x \left(\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\right) dx$$

It is of the form $\int e^x\ [f(x) + f'(x)]\ dx$

Let us take $f(x) = \tan\dfrac{x}{2}$ so that $f'(x) = \dfrac{1}{2}\ \sec^2\dfrac{x}{2}$

$\therefore\ \ I = e^x f(x) + c = e^x \tan\dfrac{x}{2} + c.$

$$\left[\because \int e^x (f(x) + f'(x))\, dx = e^x f(x) + c\right]$$

19. $e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$

Sol. Let $I = \int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)\,dx$

It is of the form $\int e^x\,(f(x)+f'(x))\,dx$

Here $f(x) = \frac{1}{x} = x^{-1}$ and so $f'(x) = (-1)\,x^{-2} = \frac{-1}{x^2}$

$\therefore\quad I = e^x\,f(x) + c \qquad [\because \int e^x\,(f(x) + f'(x)]\,dx = e^x\,f(x) + c$

$= e^x\,\frac{1}{x} + c = \frac{e^x}{x} + c.$

20. $\dfrac{(x-3)\,e^x}{(x-1)^3}$

Sol. Here $I = \int \frac{(x-3)\,e^x}{(x-1)^3}\,dx = \int \frac{(x-1)-2}{(x-1)^3}\,e^x\,dx$

$= \int e^x\left[\frac{x-1}{(x-1)^3}-\frac{2}{(x-1)^3}\right]dx = \int e^x\left[\frac{1}{(x-1)^2}+\frac{-2}{(x-1)^3}\right]dx$

It is of the form $\int e^x\,[f(x)+f'(x)]\,dx$

Let us take $f(x) = \frac{1}{(x-1)^2}$ so that $f'(x) = \frac{d}{dx}\,[(x-1)^{-2}]$

$$= -2(x-1)^{-3} = \frac{-2}{(x-1)^3}$$

$\therefore\quad I = e^x\,f(x) + c = \frac{e^x}{(x-1)^2} + c.$

$$\left[\because \int e^x\,(f(x)+f'(x))\,dx = e^x\,f(x)\right]$$

Note. Rule to evaluate $\int e^{ax}\sin bx\,dx$ or $\int e^{ax}\cos bx\,dx$

Let $I = \int \underset{\text{I}}{e^{ax}}\,\underset{\text{II}}{\sin bx}\,dx$ or $\int \underset{\text{I}}{e^{ax}}\,\underset{\text{II}}{\cos bx}\,dx$

Integrate twice by product Rule and transpose term containing I from R.H.S. to L.H.S.

21. $e^{2x}\sin x$

Sol. Let $I = \int \underset{\text{I}}{e^{2x}}\,\underset{\text{II}}{\sin x}\,dx$ ...(*i*)

Applying Product Rule: $\text{I}\int \text{II}\,dx - \int\left[\frac{d}{dx}(\text{I})\int \text{II}\,dx\right]dx$

$\Rightarrow\quad I = e^{2x}\,(-\cos x) - \int e^{2x}\,.\,2\,.\,(-\cos x)\,dx$

$$\left[\because \frac{d}{dx}e^{2x} = e^{2x}\frac{d}{dx}(2x) = 2e^{2x}\right]$$

$\Rightarrow \quad \mathrm{I} = -\, e^{2x} \cos x + 2 \int \underset{\mathrm{I}}{e^{2x}} \underset{\mathrm{II}}{\cos x}\ dx$

Again Applying Product Rule:

$$\mathrm{I} = -\, e^{2x} \cos x + 2 \left[e^{2x} \sin x - \int 2\, e^{2x} \sin x\, dx \right]$$

$\Rightarrow \quad \mathrm{I} = -\, e^{2x} \cos x + 2\, e^{2x} \sin x - 4 \int e^{2x} \sin x\, dx$

$\Rightarrow \quad \mathrm{I} = e^{2x} (-\cos x + 2 \sin x) - 4\mathrm{I}$ [By (i)]

Transposing $-\,4\mathrm{I}$ to L.H.S.; $5\mathrm{I} = e^{2x} (2 \sin x - \cos x)$

$\therefore \quad \mathrm{I} \left(= \int e^{2x} \sin x\, dx \right) = \dfrac{e^{2x}}{5} (2 \sin x - \cos x) + c$

Remark: The above question can also be done as:
Applying Product Rule: taking $\sin x$ as first function and e^{2x} as second function.

22. $\sin^{-1} \left(\dfrac{2x}{1 + x^2} \right)$

Sol. Put $x = \tan \theta$. Differentiating both sides $dx = \sec^2 \theta\, d\theta$.

$$\therefore \quad \int \sin^{-1} \left(\frac{2x}{1 + x^2} \right) dx = \int \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) . \sec^2 \theta\, d\theta$$

$$= \int \sin^{-1} (\sin 2\theta) . \sec^2 \theta\, d\theta = \int 2\theta \sec^2 \theta\ d\theta$$

$$= 2 \int \underset{\mathrm{I}}{\theta}\ \underset{\mathrm{II}}{\sec^2 \theta}\, d\theta$$

Applying product rule

$$= 2 [\theta . \tan \theta - \int 1 . \tan \theta\, d\theta] = 2 [\theta \tan \theta - \int \tan \theta\, d\theta]$$

$$= 2 [\theta \tan \theta - \log \sec \theta] + c$$

$$= 2 [\tan^{-1} x . x - \log \sqrt{1 + x^2}\,] + c$$

$$\left[\because \quad \sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + x^2}\,\right]$$

$$= 2 \left[x \tan^{-1} x - \frac{1}{2} \log (1 + x^2) \right] + c$$

$$= 2x \tan^{-1} x - \log (1 + x^2) + c.$$

Choose the correct answer in Exercises 23 and 24.

23. $\int x^2 e^{x^3}\ dx$ **equals**

(A) $\dfrac{1}{3} e^{x^3} + \mathrm{C}$ **(B)** $\dfrac{1}{3} e^{x^2} + \mathrm{C}$

(C) $\dfrac{1}{2} e^{x^3} + \mathrm{C}$ **(D)** $\dfrac{1}{2} e^{x^2} + \mathrm{C}$

Sol. Let $\mathrm{I} = \int x^2 e^{x^3}\ dx = \dfrac{1}{3} \int e^{(x^3)} (3x^2)\ dx \quad \left[\because \ \dfrac{d}{dx} x^3 = 3x^2 \right]$ ...(i)

Put $x^3 = t$. Therefore $3x^2 = \dfrac{dt}{dx}$. Therefore $3x^2\, dx = dt$

$\therefore$ From (i), $\quad I = \frac{1}{3} \int e^t \, dt = \frac{1}{3} e^t + C$

Putting $t = x^3$, $\quad = \frac{1}{3} e^{x^3} + C$

$\therefore$ Option (B) is the correct answer.

24. $\int e^x \sec x \,(1 + \tan x)\, dx$ **equals**

(A) $e^x \cos x + C$ **(B)** $e^x \sec x + C$

(C) $e^x \sin x + C$ **(D)** $e^x \tan x + C$

Sol. Let $I = \int e^x \sec x \,(1 + \tan x)\, dx = \int e^x \,(\sec x + \sec x \tan x)\, dx$

It is of the form $\int e^x \,(f(x) + f'(x))\, dx$

Here $f(x) = \sec x$ and so $f'(x) = \sec x \tan x$

$\therefore$ $I = e^x f(x) + C$ $\quad \left[\because \int e^x (f(x) + f'(x))\, dx = e^x f(x) + C\right]$

$= e^x \sec x + C$

$\therefore$ Option (B) is the correct answer.

Exercise 7.7

I. Rule to evaluate $\int \sqrt{\textbf{Pure Quadratic}}\ dx$, *i.e.*,

$$\int \sqrt{ax^2 + b}\ dx.$$

Apply directly one of these formulae according to form of integrand:

1. $\int \sqrt{a^2 - x^2}\ dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$.
2. $\int \sqrt{x^2 + a^2}\ dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right|$.
3. $\int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right|$.

II. Rule to evaluate $\int \sqrt{\textbf{Quadratic}}\ dx$, *i.e.*, $\int \sqrt{ax^2 + bx + c}\ dx$

Step I. Make coefficient of x^2 unity by taking $|\ a\ |$ common.
Now complete the squares by adding and subtracting $\left(\frac{1}{2}\text{ Coefficient of } x\right)^2$.

Now applying one of the above three formulae (according to the form of the integrand) will give value of required integral.

1. Integrate the functions in Exercises 1 to 9:

$\sqrt{4 - x^2}$

Sol. $\int \sqrt{4 - x^2}\ dx = \int \sqrt{2^2 - x^2}\ dx$

$$= \frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2} + c$$

$$\left[\because \int \sqrt{a^2 - x^2}\ dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2} + c$$

2. $\sqrt{1-4x^2}$

Sol. $\int \sqrt{1-4x^2}\;dx = \int \sqrt{1^2-(2x)^2}\;dx$

$$= \frac{\frac{(2x)}{2}\sqrt{1^2-(2x)^2}+\frac{1^2}{2}\sin^{-1}\left(\frac{2x}{1}\right)}{2 \to \text{Coefficient of } x \text{ in } 2x} + c$$

$$\left[\because \int \sqrt{a^2-x^2}\;dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{2}\left[x\sqrt{1-4x^2}+\frac{1}{2}\sin^{-1}\frac{2x}{1}\right]+c = \frac{x}{2}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+c.$$

3. $\sqrt{x^2+4x+6}$

Sol. $\int \sqrt{x^2+4x+6}\;dx$

Coefficient of x^2 is unity. So let us complete squares by adding and subtracting $\left(\frac{1}{2}\text{Coefficient of } x\right)^2 = 2^2$

$$= \int \sqrt{x^2+4x+4+6-4}\;dx = \int \sqrt{(x+2)^2+2}\;dx$$

$$= \int \sqrt{(x+2)^2+(\sqrt{2})^2}\;dx = \left(\frac{x+2}{2}\right)\sqrt{(x+2)^2+(\sqrt{2})^2}$$

$$+\frac{(\sqrt{2})^2}{2}\log\left|x+2+\sqrt{(x+2)^2+(\sqrt{2})^2}\right|+c$$

$$\left[\because \int \sqrt{x^2+a^2}\;dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|\right]$$

$$= \frac{(x+2)}{2}\sqrt{x^2+4+4x+2}$$

$$+\frac{2}{2}\log|x+2+\sqrt{x^2+4+4x+2}|+c$$

$$= \frac{(x+2)}{2}\sqrt{x^2+4x+6}+\log|x+2+\sqrt{x^2+4x+6}|+c.$$

4. $\sqrt{x^2+4x+1}$

Sol. $\int \sqrt{x^2+4x+1}\;dx = \int \sqrt{x^2+4x+2^2+1-4}\;dx$

$$\left(\text{We have added and subtracted}\left(\frac{1}{2}\text{coefficient of } x\right)^2 = 2^2\right)$$

$$= \int \sqrt{(x+2)^2 - 3}\ dx = \int \sqrt{(x+2)^2 - (\sqrt{3})^2}\ dx$$

$$= \left(\frac{x+2}{2}\right) \sqrt{(x+2)^2 - (\sqrt{3})^2} - \frac{(\sqrt{3})^2}{2} \log \left| x + 2 + \sqrt{(x+2)^2 - (\sqrt{3})^2} \right| + c$$

$$\left[\because \int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log | x + \sqrt{x^2 - a^2} |\right]$$

$$= \left(\frac{x+2}{2}\right) \sqrt{x^2 + 4x + 1} - \frac{3}{2} \log \left| x + 2 + \sqrt{x^2 + 4x + 1} \right| + c$$

$[\because\ (x + 2)^2 - (\sqrt{3})^2 = x^2 + 4x + 4 - 3 = x^2 + 4x + 1]$

5. $\int \sqrt{\mathbf{1 - 4x - x^2}}$

Sol. $\int \sqrt{1 - 4x - x^2}\ dx = \int \sqrt{-x^2 - 4x + 1}\ dx$

Making coefficient of x^2 unity

$$= \int \sqrt{-(x^2 + 4x - 1)}\ dx$$

(**Note.** You can't take this (–) sign out of this bracket because square root of – 1 is imaginary)

$$= \int \sqrt{-(x^2 + 4x + 2^2 - 4 - 1)}\ dx = \int \sqrt{-[(x+2)^2 - 5]}\ dx$$

$$= \int \sqrt{5 - (x+2)^2}\ dx = \int \sqrt{(\sqrt{5})^2 - (x+2)^2}\ dx$$

$$= \frac{x+2}{2} \sqrt{(\sqrt{5})^2 - (x+2)^2} + \frac{(\sqrt{5})^2}{2} \sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + c$$

$$\left[\because \int \sqrt{a^2 - x^2}\ dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}\right]$$

$$= \frac{x+2}{2} \sqrt{1 - 4x - x^2} + \frac{5}{2} \sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + c$$

$[\because\ (\sqrt{5})^2 - (x + 2)^2 = 5 - (x^2 + 4 + 4x)$
$= 5 - x^2 - 4 - 4x = 1 - 4x - x^2]$

6. $\sqrt{\mathbf{x^2 + 4x - 5}}$

Sol. $\int \sqrt{x^2 + 4x - 5}\ dx = \int \sqrt{x^2 + 4x + 2^2 - 4 - 5}\ dx$

$$= \int \sqrt{(x+2)^2 - 9}\ dx = \int \sqrt{(x+2)^2 - 3^2}\ dx$$

$$= \left(\frac{x+2}{2}\right) \sqrt{(x+2)^2 - 3^2} - \frac{3^2}{2} \log \left| x + 2 + \sqrt{(x+2)^2 - 3^2} \right| + c$$

$$\left[\because \int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log | x + \sqrt{x^2 - a^2} |\right]$$

$$= \left(\frac{x+2}{2}\right)\sqrt{x^2+4x-5} - \frac{9}{2}\log\left|x+2+\sqrt{x^2+4x-5}\right| + c$$

$[\because \ (x+2)^2 - 3^2 = x^2+4x+4-9 = x^2+4x-5]$

7. $\sqrt{1+3x-x^2}$

Sol. $\int \sqrt{1+3x-x^2}\, dx = \int \sqrt{-x^2+3x+1}\, dx$

$$= \int \sqrt{-(x^2-3x-1)}\, dx$$

$$= \int \sqrt{-\left[x^2-3x+\left(\frac{3}{2}\right)^2 - \frac{9}{4} - 1\right]}\, dx = \int \sqrt{-\left[\left(x-\frac{3}{2}\right)^2 - \frac{13}{4}\right]}\, dx$$

$$= \int \sqrt{\frac{13}{4} - \left(x-\frac{3}{2}\right)^2}\, dx = \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2}\, dx$$

$$= \left(\frac{x-\frac{3}{2}}{2}\right)\sqrt{\left(\frac{\sqrt{13}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2} + \frac{\left(\frac{\sqrt{13}}{2}\right)^2}{2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right) + c$$

$$\left[\because \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \left(\frac{2x-3}{4}\right)\sqrt{1+3x-x^2} + \frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + c$$

$$\left[\because \left(\frac{\sqrt{13}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2 = \frac{13}{4} - \left(x^2+\frac{9}{4}-3x\right)\right.$$

$$\left. = \frac{13}{4} - x^2 - \frac{9}{4} + 3x = 1+3x-x^2\right]$$

8. $\sqrt{x^2+3x}$

Sol. $\int \sqrt{x^2+3x}\, dx = \int \sqrt{x^2+3x+\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\, dx = \int \sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\, dx$

$$= \frac{x+\frac{3}{2}}{2}\sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2} - \frac{\left(\frac{3}{2}\right)^2}{2}\log\left|x+\frac{3}{2}+\sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\right| + c$$

$$\left[\because \int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|\right]$$

$$= \frac{2x+3}{4}\sqrt{x^2+3x} - \frac{9}{8}\log\left|x+\frac{3}{2}+\sqrt{x^2+3x}\right| + c$$

$$\left[\because \left(x+\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 = x^2+3x+\frac{9}{4}-\frac{9}{4} = x^2+3x\right]$$

9. $\sqrt{1+\frac{x^2}{9}}$

Sol. $\int \sqrt{1+\frac{x^2}{9}}\,dx = \int \sqrt{\frac{9+x^2}{9}}\,dx = \int \frac{\sqrt{x^2+3^2}}{3}\,dx = \frac{1}{3}\int \sqrt{x^2+3^2}\,dx$

$$= \frac{1}{3}\left[\frac{x}{2}\sqrt{x^2+3^2} + \frac{3^2}{2}\log\left|x+\sqrt{x^2+3^2}\right|\right] + c$$

$$\left[\because \int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|\right]$$

$$= \frac{x}{6}\sqrt{x^2+9} + \frac{3}{2}\log|x+\sqrt{x^2+9}| + c.$$

Choose the correct answer in Exercises 10 to 11:

10. $\int \sqrt{1+x^2}\,dx$ **is equal to**

(A) $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|\left(x+\sqrt{1+x^2}\right)\right| + C$

(B) $\frac{2}{3}(1+x^2)^{3/2} + C$

(C) $\frac{2}{3}x(1+x^2)^{3/2} + C$

(D) $\frac{x^2}{2}\sqrt{1+x^2} + \frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right| + C.$

Sol. $\int \sqrt{1+x^2}\,dx = \int \sqrt{x^2+1^2}\,dx$

$$= \frac{x}{2}\sqrt{x^2+1^2} + \frac{1^2}{2}\log|x+\sqrt{x^2+1^2}| + C$$

$$\left[\because \int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|\right]$$

$$= \frac{x}{2}\sqrt{x^2+1} + \frac{1}{2}\log|x+\sqrt{x^2+1}| + C.$$

11. $\int \sqrt{x^2-8x+7}\,dx$ **is equal to**

(A) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} + 9\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

(B) $\frac{1}{2}(x+4)\sqrt{x^2-8x+7} + 9\log\left|x+4+\sqrt{x^2-8x+7}\right| + C$

(C) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - 3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

(D) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - \frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + C.$

Sol. $\int \sqrt{x^2 - 8x + 7}\ dx = \int \sqrt{x^2 - 8x + 4^2 - 16 + 7}\ dx$

$= \int \sqrt{(x-4)^2 - 9}\ dx = \int \sqrt{(x-4)^2 - 3^2}\ dx$

$= \left(\frac{x-4}{2}\right) \sqrt{(x-4)^2 - 3^2} - \frac{3^2}{2} \log \left| x - 4 + \sqrt{(x-4)^2 - 3^2} \right| + C$

$\left[\because \int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right|\right]$

$= \left(\frac{x-4}{2}\right) \sqrt{x^2 - 8x + 7} - \frac{9}{2} \log \left| x - 4 + \sqrt{x^2 - 8x + 7} \right| + C.$

$[\because (x-4)^2 - 3^2 = x^2 - 8x + 16 - 9 = x^2 - 8x + 7]$

Exercise 7.8

Definition of definite integral as the limit of a sum:

$$\int_a^b f(x)\ dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+(n-1)h)]$$

where $nh = b - a$

Note. The series within brackets represents the sum of n terms.

Evaluate the following definite integrals as limit of sums:

1. $\int_a^b x\, dx$

Sol. Step I. Comparing $\int_a^b x\ dx$ with $\int_a^b f(x)\ dx$ we have

$$a = a,\ b = b \text{ and } f(x) = x \quad \ldots(i)$$

$\therefore \quad nh = b - a = b - a$

Step II. Putting $x = a, a + h, a + 2h, \ldots\ldots, a + (n-1)h$ in (i), we have $f(a) = a, f(a+h) = a + h,$
$f(a+2h) = a + 2h, \ldots.., f(a+(n-1)h) = a + (n-1)h$

Step III. Putting these values in

$$\int_a^b f(x)\, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h\ [f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+(n-1)h)]$$

where $nh = b - a$, we have

$$\int_a^b x\, dx = \underset{h \to 0}{\text{Lt}}\ h\ [a + (a+h) + (a+2h) + \ldots\ldots + (a+(n-1)h)]$$

where $nh = b - a$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[na + h(1 + 2 + 3 + \ldots\ldots + (n-1)]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[anh + hh\frac{n(n-1)}{2}\right] \left[\because\ \ 1 + 2 + 3 + \ldots\ldots + (n-1) = \frac{n(n-1)}{2}\right]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[anh + \frac{nh(nh-h)}{2}\right].$$

Step IV. Putting $nh = b - a$,

$$= \underset{h \to 0}{\text{Lt}} \left[a(b-a) + \frac{(b-a)(b-a-h)}{2} \right].$$

Step V. Taking Limits as $h \to 0$ (*i.e.,* putting $h = 0$ here)

$$= a(b-a) + \frac{(b-a)(b-a)}{2}$$

$$= (b-a)\left[a + \frac{b-a}{2}\right] = (b-a)\left[\frac{2a+b-a}{2}\right]$$

$$= \frac{(b-a)(b+a)}{2} = \frac{b^2-a^2}{2}.$$

2. $\int_0^5 (x+1)\,dx$

Sol. Step I. Comparing $\int_0^5 (x+1)\,dx$ with $\int_a^b f(x)\,dx$, we have

$$a = 0,\ b = 5 \text{ and } f(x) = x + 1 \qquad ...(i)$$

$\therefore \quad nh = b - a = 5 - 0 = 5.$

Step II. Putting $x = a, a + h, a + 2h, ..., a + (n-1)h$ in (i), we have

$f(a) = f(0) = 0 + 1 = 1,\ f(a+h) = f(h) = h + 1,$

$f(a+2h) = f(2h) = 2h + 1, ...,$

$f(a + (n-1)h) = f((n-1)h) = (n-1)h + 1.$

Step III. Putting these values in

$$\int_a^b f(x)\,dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)],$$ we have

$$\int_0^5 (x+1)\,dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[1 + (h+1) + (2h+1) + ... + [(n-1)h + 1)]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[n + h(1 + 2 + ... + (n-1)] = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[nh + hh\frac{n(n-1)}{2}\right]$$

$$\left[\because \ 1 + 2 + 3 + ... + (n-1) = \frac{n(n-1)}{2}\right]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[nh + \frac{(nh)(nh-h)}{2}\right].$$

Step IV. Putting $nh = 5$, $= \underset{h \to 0}{\text{Lt}} \left[5 + \frac{5(5-h)}{2}\right]$.

Step V. Taking limits as $h \to 0$ (*i.e.,* putting $h = 0$ here)

$$= 5 + \frac{5(5-0)}{2} = 5 + \frac{25}{2} = \frac{10+25}{2} = \frac{35}{2}.$$

3. $\boldsymbol{\int_2^3 x^2\, dx}$

Sol. **Step I.** Comparing $\int_2^3 x^2 \ dx$ with $\int_a^b f(x)$, we have

$a = 2$, $b = 3$ and $f(x) = x^2$ ...(*i*)

$\therefore \quad nh = b - a = 3 - 2 = 1.$

Step II. Putting $x = a, a + h, a + 2h, \ldots\ldots, a + (n-1)h$ in (*i*), we have

$f(a) = f(2) = 2^2 = 4$

$f(a + h) = f(2 + h) = (2 + h)^2 = 4 + 4h + h^2$

$f(a + 2h) = f(2 + 2h) = (2 + 2h)^2 = 4 + 8h + 2^2h^2$

|

$f(a + (n-1)h) = f(2 + (n-1)h) = (2 + (n-1)h)^2$

$= 4 + 4(n-1)h + (n-1)^2h^2.$

Step III. Putting these values in

$$\boldsymbol{\int_a^b f(x)\, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[f(a) + f(a+h) + f(a+2h) + \ldots.. + f(a + (n-1)h)]}$$

where $nh = 1$, we have

$$\int_2^3 x^2 \ dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[4 + (4 + 4h + h^2) + (4 + 8h + 2^2h^2) + \ldots\ldots + (4 + 4(n-1)h + (n-1)^2h^2)]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[4n + 4h(1 + 2 + \ldots\ldots + (n-1)) + h^2\,(1^2 + 2^2) + \ldots.. + (n-1)^2)]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[4nh + 4hh\frac{n(n-1)}{2} + hhh\frac{n(n-1)(2n-1)}{6}\right]$$

$$\left[\because \quad 1 + 2 + \ldots. + (n-1) = \frac{n(n-1)}{2} \text{ and } 1^2 + 2^2 + \ldots. + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\right.$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[4nh + 4nh\frac{(nh-h)}{2} + \frac{nh(nh-h)(2nh-h)}{6}\right].$$

Step IV. Putting $nh = 1$;

$$= \underset{h \to 0}{\text{Lt}} \left[4 + 2(1-h) + 1\frac{(1-h)(2-h)}{6}\right].$$

Step V. Taking limits as $h \to 0$ (*i.e.,* putting $h = 0$ here)

$= 4 + 2(1 - 0) + \dfrac{1(2)}{6} = 6 + \dfrac{1}{3} = \dfrac{19}{3}$.

4. $\int_1^4 (x^2 - x)\, dx$

Sol. **Step I.** Comparing $\int_1^4 (x^2 - x)\, dx$ with $\int_a^b f(x)\, dx$,

we have

$$a = 1,\ b = 4,\ f(x) = x^2 - x \quad \text{...}(i)$$

$$\therefore \quad nh = b - a = 4 - 1 = 3.$$

Step II. Putting $x = a, a + h, a + 2h, ... a + (n - 1)h$ in (i),

$$f(a) = f(1) = 1^2 - 1 = 1 - 1 = 0$$

$$f(a + h) = f(1 + h) = (1 + h)^2 - (1 + h)$$
$$= 1 + h^2 + 2h - 1 - h = h + h^2$$

$$f(a + 2h) = f(1 + 2h) = (1 + 2h)^2 - (1 + 2h)$$
$$= 1 + 4h^2 + 4h - 1 - 2h$$
$$= 2h + 4h^2$$

$$\vdots$$

$$f(a + (n - 1)h) = (1 + (n - 1)h)^2 - (1 + (n - 1)h)$$
$$= 1 + (n - 1)^2 h^2 + 2(n - 1)h - 1 - (n - 1)h$$
$$= (n - 1)h + (n - 1)^2 h^2.$$

Step III. Putting these values in

$$\int_a^b f(x)\, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[f(a) + f(a + h) + f(a + 2h) + ... + f(a + (n - 1)h)]$$

we have

$$\int_1^4 (x^2 - x)\, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[0 + h + h^2 + 2h + 4h^2 + ... + (n - 1)h + (n - 1)^2 h^2)$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[h(1 + 2 + ... + (n - 1)) + h^2(1^2 + 2^2 + ... + (n - 1)^2)]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[h \cdot h \cdot \frac{n(n-1)}{2} + h \cdot h \cdot h \frac{n(n-1)(2n-1)}{6}\right]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[nh \frac{(nh - h)}{2} + \frac{(nh)(nh - h)(2nh - h)}{6}\right].$$

Step IV. Putting $nh = 3$

$$= \underset{h \to 0}{\text{Lt}} \left[\frac{3(3 - h)}{2} + \frac{3(3 - h)(6 - h)}{6}\right].$$

Step V. Taking limits as $h \to 0$ (Putting $h = 0$ here)

$$= \frac{3(3 - 0)}{2} + \frac{3(3 - 0)(6 - 0)}{6} = \frac{9}{2} + 9 = \frac{27}{2}.$$

5. $\int_{-1}^{1} e^x \, dx$

Sol. Step I. Comparing $\int_{-1}^{1} e^x \, dx$ with $\int_a^b f(x) \, dx$, we have

$$a = -1,\ b = 1 \text{ and } f(x) = e^x \qquad ...(i)$$

$\therefore \quad nh = b - a = 1 - (-1) = 2.$

Step II. Putting $x = a, a + h, a + 2h, ..., a + (n - 1)h$ in (i), we have

$$f(a) = f(-1) = e^{-1}$$
$$f(a+h) = f(-1+h) = e^{-1+h} = e^{-1} \cdot e^h$$
$$f(a+2h) = f(-1+2h) = e^{-1+2h} = e^{-1} \cdot e^{2h}$$
$$\vdots$$
$$f(a+(n-1)h) = f(-1+(n-1)h) = e^{-1+(n-1)h} = e^{-1} e^{(n-1)h}.$$

Step III. Putting these values in

$$\int_a^b f(x)\, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)],$$

we have

$$\int_{-1}^{1} e^x \, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[e^{-1} + e^{-1} e^h + e^{-1} e^{2h} + ... + e^{-1} e^{(n-1)h}]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h\, e^{-1} \frac{\left[(e^h)^n - 1\right]}{e^h - 1} \left[\because \text{ The series within brackets}\right.$$

is a G.P. series with First term $A = e^{-1}$ and common ratio $R = e^h$,

$$\left.\text{Number of terms is } n \text{ and } S_n \text{ of G.P.} = A\, \frac{(R^n - 1)}{R - 1}\right].$$

$$= \int_{-1}^{1} e^x \, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h\, e^{-1} \frac{(e^{nh} - 1)}{e^h - 1}.$$

Step IV. Putting $nh = 2$, $= \underset{h \to 0}{\text{Lt}}\ h\, e^{-1} \dfrac{(e^2 - 1)}{e^h - 1}$

$$= e^{-1}(e^2 - 1) \underset{h \to 0}{\text{Lt}} \frac{h}{e^h - 1} = e^{-1}(e^2 - 1) \times 1 \left[\because \underset{x \to 0}{\text{Lt}} \frac{x}{e^x - 1} = 1\right]$$
$$= e^{-1+2} - e^{-1} = e^1 - e^{-1} = e - e^{-1}.$$

6. $\int_0^4 (x + e^{2x})\, dx$

Sol. Step I. Comparing $\int_0^4 (x + e^{2x})\, dx$ with $\int_a^b f(x)\, dx$, we have

$$a = 0,\ b = 4 \text{ and } f(x) = x + e^{2x} \qquad ...(i)$$

$\therefore \quad nh = b - a = 4 - 0 = 4.$

Step II. Putting $x = a, a + h, a + 2h, ..., a + (n - 1)h$ in (i), we have

$$f(a) = f(0) = 0 + e^0 = 1$$

$f(a + h) = f(h) = h + e^{2h}$

$f(a + 2h) = f(2h) = 2h + e^{4h}$

.

.

$f(a + (n - 1)h) = f((n - 1)h) = (n - 1)h + e^{2(n-1)h}$.

Step III. Putting these values in

$$\int_a^b f(x)\,dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + (n - 1)h)],$$

we have

$$\int_0^4 (x + e^{2x})\,dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[1 + (h + e^{2h}) + (2h + e^{4h}) + \ldots\ldots ((n - 1)h + e^{2(n-1)h})]$$

(G.P. series : A = 1, R = e^{2h}, $n = n$)

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[(h + 2h + .. + (n - 1)h) + (1 + e^{2h} + e^{4h} + \ldots + e^{2(n-1)h})]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h\left[h(1 + 2 + \ldots\ldots + (n - 1)) + \text{A}\left(\frac{\text{R}^n - 1}{\text{R} - 1}\right)\right]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h\left[h\,\frac{n(n - 1)}{2} + \frac{1((e^{2h})^n - 1)}{e^{2h} - 1}\right]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ \left[\frac{nh(nh - h)}{2} + \frac{h(e^{2nh} - 1)}{e^{2h} - 1}\right].$$

Step IV. Putting $nh = 4$, $= \underset{h \to 0}{\text{Lt}}\ \left[\frac{4(4 - h)}{2} + \frac{h(e^8 - 1)}{e^{2h} - 1}\right].$

Step V. Taking limits as $h \to 0$

$$= \frac{4(4 - 0)}{2} + (e^8 - 1)\ \underset{h \to 0}{\text{Lt}}\ \frac{h}{e^{2h} - 1} = 8 + (e^8 - 1)\ \frac{1}{2}\ \underset{h \to 0}{\text{Lt}}\ \frac{2h}{e^{2h} - 1}$$

$$= 8 + \frac{(e^8 - 1)}{2}. \quad \left[\because\ \underset{h \to 0}{\text{Lt}}\ \frac{e^{2h}}{e^{2h} - 1}\left(\Rightarrow\ \underset{x \to 0}{\text{Lt}}\ \frac{x}{e^x - 1}\right) = 1\right]$$

Exercise 7.9

Evaluate the definite integrals in Exercises 1 to 11:

Result. If $\int f(x)\ dx = \phi(x)$, then $\int_a^b f(x)\ dx = \phi(b) - \phi(a)$ ...(*i*)

(This is known as **Second Fundamental Theorem**).

1. $\int_{-1}^{1} (x+1)\ dx$

Sol. $\int_{-1}^{1} (x+1)\ dx = \left(\frac{x^2}{2} + x\right)_{-1}^{1} = \phi(b) - \phi(a)$

(By Second Fundamental Theorem given in Eqn. (*i*) page 496)

$$= \left(\frac{1^2}{2} + 1\right) - \left(\frac{(-1)^2}{2} - 1\right) = \frac{1}{2} + 1 - \left(\frac{1}{2} - 1\right)$$

$$= \frac{1}{2} + 1 - \frac{1}{2} + 1 = 2.$$

Remark. [Constant c will never occur in the value of a definite integral because c in the value of $\phi(b)$ gets cancelled with c in $\phi(a)$ when we subtract them to get $\phi(b) - \phi(a)$].

2. $\int_2^3 \frac{1}{x}\ dx$

Sol. $\int_2^3 \frac{1}{x}\ dx = (\log |x|)_2^3 = \phi(b) - \phi(a) = \log |3| - \log |2|$

$$= \log 3 - \log 2 = \log \frac{3}{2}. \qquad [\because\ |x| = x \text{ if } x \geq 0]$$

3. $\int_1^2 (4x^3 - 5x^2 + 6x + 9)\ dx$

Sol. $\int_1^2 (4x^3 - 5x^2 + 6x + 9)\ dx = \left(4\frac{x^4}{4} - 5\frac{x^3}{3} + 6\frac{x^2}{2} + 9x\right)_1^2$

$$= \left(x^4 - \frac{5}{3}x^3 + 3x^2 + 9x\right)_1^2$$

$$= \left[2^4 - \frac{5}{3}(2)^3 + 3(2)^2 + 9(2)\right] - \left[1 - \frac{5}{3} + 3 + 9\right]$$

$$= \left(16 - \frac{40}{3} + 12 + 18\right) - \left(13 - \frac{5}{3}\right)$$

$$= 46 - \frac{40}{3} - \left(13 - \frac{5}{3}\right) = 46 - \frac{40}{3} - 13 + \frac{5}{3}$$

$$= 33 - \frac{40}{3} + \frac{5}{3} = \frac{99 - 40 + 5}{3} = \frac{104 - 40}{3} = \frac{64}{3}.$$

4. $\int_0^{\frac{\pi}{4}} \sin 2x \, dx$

Sol. $\int_0^{\frac{\pi}{4}} \sin 2x \, dx = \left(\frac{-\cos 2x}{2}\right)_0^{\frac{\pi}{4}} = \frac{-\cos\frac{\pi}{2}}{2} - \left(\frac{-\cos 0}{2}\right)$

$= \frac{-0}{2} - \left(\frac{-1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}.$

5. $\int_0^{\frac{\pi}{2}} \cos 2x \, dx$

Sol. $\int_0^{\frac{\pi}{2}} \cos 2x \, dx = \left(\frac{\sin 2x}{2}\right)_0^{\frac{\pi}{2}} = \frac{\sin \pi}{2} - \frac{\sin 0}{2}$

$= \frac{0}{2} - \frac{0}{2} = 0$

$[\because \sin \pi = \sin 180° = \sin (180° - 0°) = \sin 0 = 0]$

6. $\int_4^5 e^x \, dx$

Sol. $\int_4^5 e^x \, dx = \left(e^x\right)_4^5 = e^5 - e^4 = e^4 (e - 1).$

7. $\int_0^{\frac{\pi}{4}} \tan x \, dx$

Sol. $\int_0^{\frac{\pi}{4}} \tan x \, dx = \left(\log|\sec x|\right)_0^{\frac{\pi}{4}}$

$= \log\left|\sec\frac{\pi}{4}\right| - \log|\sec 0| = \log|\sqrt{2}| - \log|1|$

$= \log\sqrt{2} - \log 1 = \log 2^{1/2} - 0 = \frac{1}{2}\log 2.$

8. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$

Sol. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx = \left(\log|\operatorname{cosec} x - \cot x|\right)_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right|$

$= \log|\sqrt{2} - 1| - \log|2 - \sqrt{3}|$

$= \log(\sqrt{2} - 1) - \log(2 - \sqrt{3})$ $\quad [\because |x| = x \text{ if } x \geq 0]$

$= \log\left(\frac{\sqrt{2} - 1}{2 - \sqrt{3}}\right).$

9. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Sol. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \left(\sin^{-1} x\right)_0^1$ $\left[\because \int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\frac{x}{a}\right]$

$= \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$. $\left[\because \sin\frac{\pi}{2} = 1 \text{ and } \sin 0 = 0\right]$

10. $\int_0^1 \frac{dx}{1+x^2}$

Sol. $\int_0^1 \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_0^1$ $\left[\because \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

$= \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$.

$\left[\because \tan\frac{\pi}{4} = 1 \text{ and } \tan 0 = 0\right]$

11. $\int_2^3 \frac{dx}{x^2-1}$

Sol. $\int_2^3 \frac{1}{x^2-1}\, dx = \int_2^3 \frac{1}{x^2-1^2}\, dx$

$= \left(\frac{1}{2(1)}\log\left|\frac{x-1}{x+1}\right|\right)_2^3$ $\left[\because \int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$

$= \frac{1}{2}\log\left|\frac{3-1}{3+1}\right| - \frac{1}{2}\log\left|\frac{2-1}{2+1}\right| = \frac{1}{2}\log\left|\frac{1}{2}\right| - \frac{1}{2}\log\left|\frac{1}{3}\right|$

$= \frac{1}{2}\left(\log\frac{1}{2} - \log\frac{1}{3}\right)$ $[\because\ |x| = x \text{ if } x \geq 0]$

$= \frac{1}{2}\left[\log\left(\frac{\frac{1}{2}}{\frac{1}{3}}\right)\right] = \frac{1}{2}\log\frac{3}{2}$.

Evaluate the definite integrals in Exercises 12 to 20:

12. $\int_0^{\frac{\pi}{2}} \cos^2 x \; dx$

Sol. $\int_0^{\frac{\pi}{2}} \cos^2 x \; dx = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2} \; dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2x) \; dx$

$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1+\cos 2x) \; dx = \frac{1}{2} \left(x + \frac{\sin 2x}{2}\right)_0^{\frac{\pi}{2}}$

$= \frac{1}{2} \left[\frac{\pi}{2} + \frac{1}{2}\sin \pi - \left(0 + \frac{1}{2}\sin 0\right)\right] = \frac{1}{2} \left[\frac{\pi}{2} + 0 - 0\right]$

$= \frac{\pi}{4}$. [$\because$ $\sin \pi = \sin 180° = \sin (180° - 0°) = \sin 0 = 0$]

13. $\int_2^3 \frac{x \, dx}{x^2+1}$

Sol. $\int_2^3 \frac{x}{x^2+1} \; dx = \frac{1}{2} \int_2^3 \frac{2x}{x^2+1} \; dx$

$= \frac{1}{2} \left(\log |x^2+1|\right)_2^3$. $\left[\because \int \frac{f'(x)}{f(x)} dx = \log |f(x)|\right]$

(Here $f(x) = x^2 + 1$ and $f'(x) = 2x$)

$= \frac{1}{2}$ (log | 10 | – log | 5 |) $= \frac{1}{2}$ (log 10 – log 5)

$= \frac{1}{2} \log \frac{10}{5} = \frac{1}{2} \log 2.$

14. $\int_0^1 \frac{2x+3}{5x^2+1} \; dx$

Sol. $\int_0^1 \frac{2x+3}{5x^2+1} \; dx = \int_0^1 \left(\frac{2x}{5x^2+1} + \frac{3}{5x^2+1}\right) dx$

$= \int_0^1 \frac{2x}{5x^2+1} \; dx + 3 \int_0^1 \frac{dx}{5x^2+1}$

$= \frac{1}{5} \int_0^1 \frac{10x}{5x^2+1} \; dx + 3\int_0^1 \frac{dx}{(\sqrt{5}x)^2 + 1^2}$

$= \frac{1}{5} \left(\log |5x^2+1|\right)_0^1 + 3 \cdot \frac{1}{1} \; \frac{\left(\tan^{-1}\left(\frac{\sqrt{5}x}{1}\right)\right)_0^1}{\sqrt{5} \rightarrow \text{Coefficient of } x}$

$\left[\because \int \frac{f'(x)}{f(x)} dx = \log |f(x)| \text{ and } \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}\right]$

$$= \frac{1}{5} (\log 6 - \log 1) + \frac{3}{\sqrt{5}} (\tan^{-1} \sqrt{5} - \tan^{-1} 0)$$

$$= \frac{1}{5} \log 6 + \frac{3}{\sqrt{5}} \tan^{-1} \sqrt{5}.$$

15. $\int_0^1 x\, e^{x^2} \, dx$

Sol. To evaluate $\int_0^1 x\, e^{x^2} \, dx$

Let us first evaluate $\int x\, e^{x^2} \, dx$

$$= \frac{1}{2} \int e^{x^2} (2x\, dx) \quad ...(i)$$

Put $x^2 = t$. Therefore $2x = \frac{dt}{dx}$ $\therefore$ $2x\, dx = dt$

$\therefore$ From (i), $\int x\, e^{x^2} \, dx = \frac{1}{2} \int e^t \, dt = \frac{1}{2} e^t$

Putting $t = x^2$, $= \frac{1}{2} e^{x^2}$ ...(ii)

$\therefore$ The given integral $\int_0^1 x\, e^{x^2} \, dx = \frac{1}{2} \left(e^{x^2}\right)_0^1$ [By (ii)]

$$= \frac{1}{2} (e^1 - e^0) = \frac{1}{2} (e - 1).$$

Note. Please note that limits 0 and 1 specified in the given integral are limits for x.
Therefore after substituting $x^2 = t$ and evaluating the integral, we must put back $t = x^2$ and only then use $\int_a^b f(x) \, dx = \phi(b) - \phi(a)$.
Remark. In the next Exercise 7.10 we shall also learn to change the limits of integration from values of x to values of t and then we may use our discretion even here also.

16. $\int_1^2 \frac{5x^2}{x^2 + 4x + 3} \, dx$

Sol. $\int_1^2 \frac{5x^2}{x^2 + 4x + 3} \, dx = \int_1^2 \frac{5x^2}{(x+1)(x+3)}$ ...(i)

[$\because$ $x^2 + 4x + 3 = x^2 + 3x + x + 3$
$= x(x + 3) + 1(x + 3) = (x + 1)(x + 3)$]

The integrand $\frac{5x^2}{(x+1)(x+3)}$ is a rational function and degree of numerator = degree of denominator.
So let us apply long division.

$$(x+1)(x+3) = x^2 + 4x + 3 \;\overline{)\; 5x^2 \qquad\qquad (}5$$
$$\underline{\begin{array}{l} 5x^2 + 20x + 15 \\ - \quad\; - \qquad - \end{array}}$$
$$-20x - 15$$

$$\therefore \quad \frac{5x^2}{(x+1)(x+3)} = 5 + \frac{(-20x-15)}{(x+1)(x+3)}$$

Putting this value in (*i*),

$$\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx = \int_1^2 \left(5 + \frac{(-20x-15)}{(x+1)(x+3)}\right) dx$$

$$= \int_1^2 5\,dx + \int_1^2 \frac{-20x-15}{(x+1)(x+3)}\,dx = 5\,(x)_1^2 + \mathbf{I}$$

$= 5(2-1) + \mathbf{I} = 5 + \mathbf{I}$ ...(*ii*)

where $\mathrm{I} = \int_1^2 \frac{-20x-15}{(x+1)(x+3)}\,dx$

Let integrand of I $= \frac{-20x-15}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$ ...(*iii*)

(Partial Fractions)

Multiplying both sides by L.C.M. $= (x+1)(x+3)$,

$$-20x - 15 = A(x+3) + B(x+1)$$
$$= Ax + 3A + Bx + B$$

Comparing coefficients of x and constant terms on both sides, we have

Coefficients of x: $A + B = -20$ ...(*iv*)

Constant terms: $3A + B = -15$ ...(*v*)

Subtracting (*iv*) and (*v*), $-2A = -5$. Therefore $A = \frac{5}{2}$.

Putting $A = \frac{5}{2}$ in (*iv*), $\frac{5}{2} + B = -20 \Rightarrow B = -20 - \frac{5}{2}$

or $\quad B = \frac{-40-5}{2} = \frac{-45}{2}$

Putting these values of A and B in (*iii*),

$$\frac{-20x-15}{(x+1)(x+3)} = \frac{\frac{5}{2}}{x+1} - \frac{\frac{45}{2}}{x+3}$$

$$\therefore \quad \mathrm{I} = \int_1^2 \frac{-20x-15}{(x+1)(x+3)}\,dx = \frac{5}{2}\int_1^2 \frac{1}{x+1}\,dx - \frac{45}{2}\int_1^2 \frac{1}{x+3}\,dx$$

$$= \frac{5}{2}\left(\log|x+1|\right)_1^2 - \frac{45}{2}\left(\log|x+3|\right)_1^2$$

$$= \frac{5}{2}\left(\log|3| - \log|2|\right) - \frac{45}{2}\left(\log|5| - \log|4|\right)$$

$$= \frac{5}{2} \log \frac{3}{2} - \frac{45}{2} \log \frac{5}{4} \qquad [\because \ |x| = x \text{ if } x \geq 0]$$

$$= \frac{5}{2} \left(\log \frac{3}{2} - 9 \log \frac{5}{4} \right)$$

Putting this value of I in (*ii*),

$$\int_1^2 \frac{5x^2}{x^2 + 4x + 3} \, dx = 5 + \frac{5}{2} \left(\log \frac{3}{2} - 9 \log \frac{5}{4} \right) = 5 - \frac{5}{2} \left(9 \log \frac{5}{4} - \log \frac{3}{2} \right)$$

17. $\int_0^{\frac{\pi}{4}} (2 \sec^2 x + x^3 + 2) \, dx$

Sol. $\int_0^{\frac{\pi}{4}} (2 \sec^2 x + x^3 + 2) \, dx = 2 \int_0^{\frac{\pi}{4}} \sec^2 x \, dx + \int_0^{\frac{\pi}{4}} x^3 \, dx + 2 \int_0^{\frac{\pi}{4}} 1 \, dx$

$$= 2 \, (\tan x)_0^{\frac{\pi}{4}} + \left(\frac{x^4}{4} \right)_0^{\frac{\pi}{4}} + 2 \, (x)_0^{\frac{\pi}{4}}$$

$$= 2\left(\tan \frac{\pi}{4} - \tan 0 \right) + \frac{\left(\frac{\pi}{4} \right)^4}{4} - 0 + 2 \left(\frac{\pi}{4} - 0 \right)$$

$$= 2(1 - 0) + \frac{\left(\frac{\pi^4}{256} \right)}{4} + \frac{2\pi}{4} \qquad = 2 + \frac{\pi^4}{1024} + \frac{\pi}{2} \, .$$

18. $\int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$

Sol. $\int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx = \int_0^{\pi} \left[\left(\frac{1 - \cos x}{2} \right) - \left(\frac{1 + \cos x}{2} \right) \right] dx$

$$\left(\because \ \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \text{ and } \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \right)$$

$$= \int_0^{\pi} \left(\frac{1 - \cos x - 1 - \cos x}{2} \right) dx = \int_0^{\pi} \frac{-2 \cos x}{2} \, dx$$

$$= - \int_0^{\pi} \cos x \, dx = - \left(\sin x \right)_0^{\pi} = -(\sin \pi - \sin 0) = -(0 - 0) = 0.$$

$$[\because \ \sin \pi = \sin 180° = \sin (180° - 0) = \sin 0 = 0]$$

19. $\int_0^2 \frac{6x + 3}{x^2 + 4} \, dx$

Sol. $\int_0^2 \frac{6x + 3}{x^2 + 4} \, dx = \int_0^2 \frac{6x}{x^2 + 4} \, dx + 3 \int_0^2 \frac{1}{x^2 + 4} \, dx$

$$= 3 \int_0^2 \frac{2x}{x^2 + 4} \, dx + 3 \, \frac{1}{2} \left(\tan^{-1} \frac{x}{2} \right)_0^2$$

$$= 3 \left(\log |x^2 + 4| \right)_0^2 + \frac{3}{2} \, (\tan^{-1} 1 - \tan^{-1} 0)$$

$$\left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| \text{ and } \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= 3\ (\log 8 - \log 4) + \frac{3}{2}\left(\frac{\pi}{4} - 0\right) \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$= 3\log\frac{8}{4} + \frac{3\pi}{8} = 3\log 2 + \frac{3\pi}{8}.$$

20. $\int_0^1 \left(x\,e^x + \sin\frac{\pi x}{4}\right) dx$

Sol. $\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right) dx = \int_0^1 \underset{\text{I II}}{x\,e^x}\ dx + \int_0^1 \sin\frac{\pi x}{4}\ dx$

Applying Product Rule on first definite integral,

$$\left(\text{I}\int \text{II}\,dx\right)_0^1 - \int_0^1\left(\frac{d}{dx}(\text{I})\int \text{II}\,dx\right)dx$$

$$= \left(x\,e^x\right)_0^1 - \int_0^1 1\,.\,e^x\ dx - \frac{\left(\cos\frac{\pi x}{4}\right)_0^1}{\frac{\pi}{4} \to \text{Coefficient of } x \text{ in } \frac{\pi x}{4}}$$

$$= e^1 - 0 - \int_0^1 e^x\ dx - \frac{4}{\pi}\left[\cos\frac{\pi}{4} - \cos 0\right] = e - \left(e^x\right)_0^1 - \frac{4}{\pi}\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= e - (e - e^0) - \frac{4}{\pi\sqrt{2}} + \frac{4}{\pi}$$

$$= e - e + 1 - \frac{2\,.\,2}{\pi\sqrt{2}} + \frac{4}{\pi} = 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}.$$

Choose the correct answer in Exercises 21 and 22:

21. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ **equals**

(A) $\frac{\pi}{3}$ **(B)** $\frac{2\pi}{3}$ **(C)** $\frac{\pi}{6}$ **(D)** $\frac{\pi}{12}$

Sol. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \left(\tan^{-1}x\right)_1^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1} 1$

$$= \frac{\pi}{3} - \frac{\pi}{4} \qquad \left[\because \tan\frac{\pi}{3} = \sqrt{3} \text{ and } \tan\frac{\pi}{4} = 1\right]$$

$$= \frac{4\pi - 3\pi}{12} = \frac{\pi}{12}$$

$\therefore$ Option (D) is the correct answer.

22. $\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$ **equals**

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{12}$ **(C)** $\frac{\pi}{24}$ **(D)** $\frac{\pi}{4}$

Sol. $\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = \int_0^{\frac{2}{3}} \frac{dx}{(3x)^2+2^2} = \left[\frac{1}{2}\,\frac{\tan^{-1}\frac{3x}{2}}{3 \to \text{Coefficient of } x \text{ in } 3x}\right]$

$= \frac{1}{6}\left[\tan^{-1}\frac{3x}{2}\right]_0^{\frac{2}{3}} = \frac{1}{6}\left[\tan^{-1}\left(\frac{3}{2}\times\frac{2}{3}\right) - \tan^{-1} 0\right]$

$= \frac{1}{6}\left(\tan^{-1} 1 - \tan^{-1} 0\right) = \frac{1}{6}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{24}$

$\left[\because \tan\frac{\pi}{4} = 1 \text{ and } \tan 0 = 0\right]$

$\therefore$ Option (C) is the correct answer.

Exercise 7.10

Evaluate the integrals in Exercises 1 to 8 using substitution:

1. $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx$

Sol. Let I = $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx$ ...(*i*)

Put $x^2 + 1 = t$. Therefore $2x = \dfrac{dt}{dx} \Rightarrow 2x\,dx = dt$.

To change the limits of integration from values of *x* to values of *t*.

When $x = 0$, $t = 0 + 1 = 1$

When $x = 1$, $t = 1 + 1 = 2$

$\therefore$ From (*i*), $\displaystyle I = \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{1}{2}\left(\log|t|\right)_1^2 = \frac{1}{2}(\log|2| - \log|1|)$

$$= \frac{1}{2}(\log 2 - \log 1) = \frac{1}{2}(\log 2 - 0) = \frac{1}{2}\log 2.$$

2. $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^5\phi\;d\phi$

Sol. Let I = $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^5\phi\,d\phi$ ...(*i*)

Put $\sin\phi = t$.

($\because$ one factor of integrand is $\cos^5\phi$ where $n = 5$ is odd.)

$\therefore$ $\cos\phi = \dfrac{dt}{d\phi}$ *i.e.,* $\cos\phi\,d\phi = dt$.

To change the limits of integration from ϕ to t

When $\phi = 0$, $t = \sin\phi = \sin 0 = 0$

When $\phi = \dfrac{\pi}{2}$, $t = \sin\phi = \sin\dfrac{\pi}{2} = 1$

Now Integrand $\sqrt{\sin\phi}\,\cos^5\phi = \sqrt{\sin\phi}\,\cos^4\phi\cos\phi$

$$= \sqrt{\sin\phi}\,(\cos^2\phi)^2\cos\phi = \sqrt{\sin\phi}\,(1 - \sin^2\phi)^2\cos\phi$$

$\therefore$ From (*i*), $\displaystyle I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,(1-\sin^2\phi)^2\cos\phi\,d\phi$

$$= \int_0^1 \sqrt{t}\,(1-t^2)^2\,dt = \int_0^1 t^{1/2}(1+t^4-2t^2)\,dt$$

$$= \int_0^1 \left(t^{\frac{1}{2}} + t^{\frac{1}{2}+4} - 2t^{\frac{1}{2}+2}\right)dt = \int_0^1 (t^{1/2} + t^{9/2} - 2t^{5/2})\,dt$$

$$= \int_0^1 t^{1/2}\ dt + \int_0^1 t^{9/2}\ dt - 2\int_0^1 t^{5/2}\ dt$$

$$= \frac{\left(t^{3/2}\right)_0^1}{\frac{3}{2}} + \frac{\left(t^{11/2}\right)_0^1}{\frac{11}{2}} - 2\frac{\left(t^{7/2}\right)_0^1}{\frac{7}{2}}$$

$$= \frac{2}{3}\ (1-0) + \frac{2}{11}\ (1-0) - \frac{4}{7}\ (1-0)$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7} = \frac{2(77)+2(21)-4(33)}{3(11)(7)}$$

$$= \frac{154+42-132}{231} = \frac{196-132}{231} = \frac{64}{231}.$$

3. $\displaystyle\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)\ dx$

Sol. Let $\displaystyle I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)\ dx$ ...(*i*)

Put $x = \tan\theta$. $\quad \therefore \dfrac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta\ d\theta$

To change the limits of integration

When $x = 0$, $\tan\theta = 0 = \tan 0 \Rightarrow \theta = 0$

When $x = 1$, $\tan\theta = 1 = \tan\dfrac{\pi}{4} \Rightarrow \theta = \dfrac{\pi}{4}$

$\therefore$ From (*i*), $\displaystyle I = \int_0^{\frac{\pi}{4}} \left(\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\right)\sec^2\theta\ d\theta$

$$= \int_0^{\frac{\pi}{4}} (\sin^{-1}(\sin 2\theta))\sec^2\theta\ d\theta = \int_0^{\frac{\pi}{4}} 2\theta\sec^2\theta\ d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sec^2\theta}\ d\theta$$

Applying Product Rule of Integration

$$\left(\int_a^b \text{I}\,.\,\text{II}\,dx = \left(\text{I}\int \text{II}\right)_a^b - \int_a^b\left(\frac{d}{dx}(\text{I})\int \text{II}\,dx\right)dx\right)$$

$$= 2\left[(\theta\,.\,\tan\theta)_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 1\,.\tan\theta\,d\theta\right]$$

$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} - 0 - \int_0^{\frac{\pi}{4}}\tan\theta\,d\theta\right] = 2\left[\frac{\pi}{4} - (\log\sec\theta)_0^{\frac{\pi}{4}}\right]$$

$$= 2\left[\frac{\pi}{4} - \left(\log\sec\frac{\pi}{4} - \log\sec 0\right)\right] = 2\left[\frac{\pi}{4} - (\log\sqrt{2} - \log 1)\right]$$

$$= \frac{\pi}{2} - 2\log 2^{1/2}\quad (\because\ \log 1 = 0)$$

$= \frac{\pi}{2} - 2 \cdot \frac{1}{2} \log 2 = \frac{\pi}{2} - \log 2.$

4. $\int_0^2 x\sqrt{x+2}\, dx$

Sol. Let $I = \int_0^2 x\sqrt{x+2}\, dx$ ...(*i*)

Put $\sqrt{\text{Linear}} = t$, *i.e.*, $\sqrt{x+2} = t$. Therefore $x + 2 = t^2$.

$\therefore \quad \frac{dx}{dt} = 2t \quad \Rightarrow \quad dx = 2t\, dt$

To change the limits of Integration

When $x = 0$, $t = \sqrt{x+2} = \sqrt{2}$

When $x = 2$, $t = \sqrt{x+2} = \sqrt{2+2} = \sqrt{4} = 2.$

$\therefore$ From (*i*), $I = \int_{\sqrt{2}}^2 (t^2 - 2)\ t \cdot 2t\, dt$

$[\because \quad x + 2 = t^2 \quad \Rightarrow \quad x = t^2 - 2]$

$= 2 \int_{\sqrt{2}}^2 t^2(t^2 - 2)\, dt = 2 \int_{\sqrt{2}}^2 (t^4 - 2t^2)\, dt$

$= 2 \left[\left(\frac{t^5}{5}\right)_{\sqrt{2}}^2 - 2\left(\frac{t^3}{3}\right)_{\sqrt{2}}^2 \right] \quad = 2 \left[\frac{1}{5}(2^5 - (\sqrt{2})^5) - \frac{2}{3}(2^3 - (\sqrt{2})^3) \right]$

$= 2 \left[\frac{1}{5}(32 - 4\sqrt{2}) - \frac{2}{3}(8 - 2\sqrt{2}) \right] [\because \quad (\sqrt{2})^3 = \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} = 2\sqrt{2},$

and $(\sqrt{2})^5 = \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} = 4\sqrt{2}\,]$

$= 2 \left[\frac{32}{5} - \frac{4\sqrt{2}}{5} - \frac{16}{3} + \frac{4\sqrt{2}}{3} \right] = 2 \left[\frac{96 - 12\sqrt{2} - 80 + 20\sqrt{2}}{15} \right]$

$= \frac{2}{15} (16 + 8\sqrt{2}) = \frac{16}{15} (2 + \sqrt{2}) = \frac{16}{15} (\sqrt{2} \cdot \sqrt{2} + \sqrt{2})$

$= \frac{16\sqrt{2}}{15} (\sqrt{2} + 1).$

5. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x}\, dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x}\, dx = - \int_0^{\frac{\pi}{2}} \frac{-\sin x}{1 + \cos^2 x}\, dx$ ...(*i*)

Put $\cos x = t$. Therefore $-\sin x = \frac{dt}{dx} \Rightarrow -\sin x\, dx = dt$.

To change the limits of Integration.

When $x = 0$, $t = \cos 0 = 1$, When $x = \frac{\pi}{2}$, $t = \cos \frac{\pi}{2} = 0$

$$\therefore \text{ From } (i),\ \mathrm{I} = -\int_1^0 \frac{dt}{1+t^2} = -\int_1^0 \frac{1}{t^2+1}\, dt$$

$$= -\left(\tan^{-1} t\right)_1^0 = -(\tan^{-1} 0 - \tan^{-1} 1) = -\left(0 - \frac{\pi}{4}\right)$$

$$\left[\because \tan 0 = 0 \Rightarrow \tan^{-1} 0 = 0 \text{ and } \tan\frac{\pi}{4} = 1 \Rightarrow \tan^{-1} 1 = \frac{\pi}{4}\right] = \frac{\pi}{4}.$$

6. $\displaystyle\int_0^2 \frac{dx}{x+4-x^2}$

Sol. $\displaystyle\int_0^2 \frac{dx}{4+x-x^2} = \int_0^2 \frac{dx}{-x^2+x+4} = \int_0^2 \frac{dx}{-(x^2-x-4)}$

(Making coeff. of x^2 numerically unity)

Completing squares by adding and subtracting

$$\left(\frac{1}{2}\text{coeff. of } x\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}. = \int_0^2 \frac{dx}{-\left[x^2 - x + \frac{1}{4} - \frac{1}{4} - 4\right]}$$

$$= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2 - \frac{17}{4}\right]} = \int_0^2 \frac{dx}{\frac{17}{4} - \left(x-\frac{1}{2}\right)^2} = \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\times\frac{\sqrt{17}}{2}}\left[\log\left|\frac{\frac{\sqrt{17}}{2}+\left(x-\frac{1}{2}\right)}{\frac{\sqrt{17}}{2}-\left(x-\frac{1}{2}\right)}\right|\right]_0^2 \quad \left(\because \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|\right)$$

$$= \frac{1}{\sqrt{17}}\left[\log\left|\frac{\sqrt{17}+2x-1}{\sqrt{17}-2x+1}\right|\right]_0^2$$

$$= \frac{1}{\sqrt{17}}\left[\log\left|\frac{\sqrt{17}+3}{\sqrt{17}-3}\right| - \log\left|\frac{\sqrt{17}-1}{\sqrt{17}+1}\right|\right]$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3}\times\frac{\sqrt{17}+1}{\sqrt{17}-1}\right) = \frac{1}{\sqrt{17}}\log\frac{20+4\sqrt{17}}{20-4\sqrt{17}}$$

($\because$ $(\sqrt{17}+3)(\sqrt{17}+1) = 17 + \sqrt{17} + 3\sqrt{17} + 3 = 20 + 4\sqrt{17}$.
Similarly $(\sqrt{17}-3)(\sqrt{17}-1) = 20 - 4\sqrt{17}$)

$$= \frac{1}{\sqrt{17}}\log\frac{4(5+\sqrt{17})}{4(5-\sqrt{17})} = \frac{1}{\sqrt{17}}\log\frac{5+\sqrt{17}}{5-\sqrt{17}}$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{5+\sqrt{17}}{5-\sqrt{17}} \times \frac{5+\sqrt{17}}{5+\sqrt{17}} \right) = \frac{1}{\sqrt{17}} \log \frac{(5+\sqrt{17})^2}{25-17}$$

$$= \frac{1}{\sqrt{17}} \log \frac{42+10\sqrt{17}}{8} = \frac{1}{\sqrt{17}} \log \frac{21+5\sqrt{17}}{4}.$$

7. $\displaystyle\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

Sol. Let I $= \displaystyle\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \int_{-1}^{1} \frac{dx}{x^2+2x+1+4}$ (To complete squares)

$$= \int_{-1}^{1} \frac{1}{(x+1)^2+2^2}\, dx \qquad ...(i)$$

Put $x + 1 = t$. $\quad \therefore \ \dfrac{dx}{dt} = 1 \quad \Rightarrow \quad dx = dt$

To change the limits of Integration

When $x = -1$, $t = -1 + 1 = 0$

When $x = 1$, $t = 1 + 1 = 2$

$\therefore$ From (i), I $= \displaystyle\int_{0}^{2} \frac{1}{t^2+2^2}\, dt = \frac{1}{2}\left(\tan^{-1}\frac{t}{2}\right)_0^2$

$$\left[\because \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{2}{2} - \tan^{-1}\frac{0}{2}\right] = \frac{1}{2}\,(\tan^{-1} 1 - \tan^{-1} 0)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}. \qquad \left[\because \tan\frac{\pi}{4} = 1 \text{ and } \tan 0 = 0\right]$$

8. $\displaystyle\int_{1}^{2} \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x}\, dx$

Sol. Let I $= \displaystyle\int_{1}^{2} \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x}\, dx \qquad ...(i)$

[Type $\int (f(x) + g(x))\ e^{ax}\, dx$. Put $ax = t$ and it will become $\int (f(t) + f'(t))\ e^t\, dt = e^t f(t)$]

Put $2x = t$ $\quad \therefore \ 2 = \dfrac{dt}{dx} \quad \Rightarrow \quad 2dx = dt \quad \Rightarrow \quad dx = \dfrac{dt}{2}$

To change the limits of Integration

When $x = 1$, $t = 2x = 2$, When $x = 2$, $t = 2x = 4$

$\therefore$ From (i), I $= \displaystyle\int_{2}^{4} \left(\frac{1}{\frac{t}{2}} - \frac{1}{2\left(\frac{t}{2}\right)^2}\right) e^t\, \frac{dt}{2} \quad \left[\because \ 2x = t \ \Rightarrow \ x = \frac{t}{2}\right]$

$$\therefore \quad I = \int_2^4 \left(\frac{2}{t} - \frac{2}{t^2}\right) e^t \frac{dt}{2} = \int_2^4 \frac{1}{2} \cdot 2\left(\frac{1}{t} - \frac{1}{t^2}\right) e^t \, dt$$

$$= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t \, dt = \int_2^4 (f(t) + f'(t)) \, e^t \, dt$$

$$\left(\text{Here } f(t) = \frac{1}{t} = t^{-1} \text{ and therefore } f'(t) = (-1)t^{-2} = \frac{-1}{t^2}\right)$$

$$= \left(e^t f(t)\right)_2^4 = \left(\frac{e^t}{t}\right)_2^4 = \frac{e^4}{4} - \frac{e^2}{2} = \frac{e^4 - 2e^2}{4} = \frac{e^2(e^2 - 2)}{4}.$$

Choose the correct answer in Exercises 9 and 10.

9. The value of the integral $\int_{\frac{1}{3}}^{1} \frac{(x - x^3)^{1/3}}{x^4} dx$ is

(A) 6 **(B) 0** **(C) 3** **(D) 4**

Sol. Let $I = \int_{\frac{1}{3}}^{1} \frac{(x - x^3)^{1/3}}{x^4} \, dx$

$$= \int_{\frac{1}{3}}^{1} \frac{\left[x^3\left(\frac{x}{x^3} - 1\right)\right]^{1/3}}{x^4} \, dx = \int_{\frac{1}{3}}^{1} \frac{(x^3)^{1/3}\left(\frac{1}{x^2} - 1\right)^{1/3}}{x^4}$$

$$= \int_{\frac{1}{3}}^{1} \frac{x\,(x^{-2} - 1)^{1/3}}{x^4} \, dx = \int_{\frac{1}{3}}^{1} (x^{-2} - 1)^{1/3} \, x^{-3} \, dx$$

$$I = \frac{-1}{2} \int_{\frac{1}{3}}^{1} (x^{-2} - 1)^{1/3} (-2x^{-3}) \, dx \qquad \text{...}(i)$$

Put $x^{-2} - 1 = t$

Therefore $-2x^{-3} = \frac{dt}{dx} \quad \Rightarrow \quad -2x^{-3} \, dx = dt$

To change the limits of Integration

When $x = \frac{1}{3}$, $t = x^{-2} - 1 = \left(\frac{1}{3}\right)^{-2} - 1$

$= (3^{-1})^{-2} - 1 = 3^2 - 1 = 9 - 1 = 8$

When $x = 1$, $t = 1^{-2} - 1 = 1 - 1 = 0$

$$\therefore \quad \text{From } (i),\ I = \frac{-1}{2} \int_8^0 t^{1/3} \, dt = \frac{-1}{2} \left(\frac{t^{4/3}}{\frac{4}{3}}\right)_8^0$$

$$= \frac{-1}{2} \cdot \frac{3}{4} [0 - 8^{4/3}] = \frac{-3}{8} [-(2^3)^{4/3}] = \frac{-3}{8}(-2^4) = \frac{3}{8} \times 16 = 6$$

$\therefore$ Option (A) is the correct answer.

10. If $f(x) = \int_0^x t \sin t \ dt$, then $f'(x)$ is

(A) $\cos x + x \sin x$ **(B) $x \sin x$**

(C) $x \cos x$ **(D) $\sin x + x \cos x$**

Sol. $f(x) = \int_0^x \underset{\text{I}}{t} \underset{\text{II}}{\sin t} \ dt$

Applying Product Rule of Integration

$$\left[\int_a^b \text{I}\,.\,\text{II}\,dx = \left(\text{I} \int \text{II}\,dx\right)_a^b - \int_a^b \frac{d}{dx}(\text{I}) \int \text{II}\,dx\,dx\right]$$

$\Rightarrow \quad f(x) = \left(t\,(-\cos t)\right)_0^x - \int_0^x 1\,(-\cos t)\,dt$

$= -x \cos x - 0 + \int_0^x \cos t \ dt = -x \cos x + (\sin t)_0^x$

$= -x \cos x + \sin x - \sin 0 = -x \cos x + \sin x$

$\therefore \quad f'(x) = -(x(-\sin x) + (\cos x)1) + \cos x$

$= x \sin x - \cos x + \cos x = x \sin x$

$\therefore$ Option (B) is the correct answer.

OR

$f(x) = \int_0^x \sin t\,dt \qquad \therefore \quad f'(x) = (t \sin t)_0^x$

[$\because$ Derivative operator and integral operator cancel with each other]

$= x \sin x - 0 = x \sin x.$

Exercise 7.11

By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 6:

1. $\int_0^{\frac{\pi}{2}} \cos^2 x \; dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x \; dx$ ...(*i*)

$\therefore \quad I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right) dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

or $\quad I = \int_0^{\frac{\pi}{2}} \sin^2 x \; dx$ ...(*ii*)

Adding Eqns. (*i*) and (*ii*),

$$2I = \int_0^{\frac{\pi}{2}} (\cos^2 x + \sin^2 x)\; dx = \int_0^{\frac{\pi}{2}} 1 \; dx = (x)_0^{\frac{\pi}{2}}$$

$\Rightarrow \quad 2I = \frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi}{4}.$

2. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \; dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \; dx$ ...(*i*)

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} \; dx$$

$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

or $\quad I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \; dx$ ...(*ii*)

Adding Eqns. (*i*) and (*ii*), we have

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\right) dx = \int_0^{\frac{\pi}{2}} 1 \; dx$$

$\Rightarrow 2I = (x)_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \qquad \Rightarrow \quad I = \frac{\pi}{4}.$

3. $\int_0^{\frac{\pi}{2}} \frac{\sin^{3/2} x \; dx}{\sin^{3/2} x + \cos^{3/2} x}$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} \, dx$ ...(*i*)

Changing x to $\frac{\pi}{2} - x$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^{3/2}\left(\frac{\pi}{2} - x\right)}{\sin^{3/2}\left(\frac{\pi}{2} - x\right) + \cos^{3/2}\left(\frac{\pi}{2} - x\right)} \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x} \quad ...(ii)$$

Adding Eqns. (*i*) and (*ii*),

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{3/2} x + \cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} \, dx = \int_0^{\frac{\pi}{2}} 1 \, dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4}.$$

4. $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x \, dx}{\sin^5 x + \cos^5 x}$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} \, dx$ ...(*i*)

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{\cos^5\left(\frac{\pi}{2} - x\right)}{\sin^5\left(\frac{\pi}{2} - x\right) + \cos^5\left(\frac{\pi}{2} - x\right)} \, dx$$

$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

or $I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} \, dx$ ...(*ii*)

Adding Eqns. (*i*) and (*ii*), we have

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{\cos^5 x}{\sin^5 x + \cos^5 x} + \frac{\sin^5 x}{\cos^5 x + \sin^5 x} \right) dx$$

$$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x + \sin^5 x}{\sin^5 x + \cos^5 x} \, dx = \int_0^{\frac{\pi}{2}} 1 \, dx = (x)_0^{\frac{\pi}{2}}$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} \qquad \Rightarrow \quad I = \frac{\pi}{4}.$$

5. $\int_{-5}^{5} |x+2| \, dx$

Sol. Let $I = \int_{-5}^{5} |x+2| \, dx$ ...(*i*)

We can evaluate this integral only if we can get rid of the modulus.

Putting expression within modulus equal to 0, we have

$x + 2 = 0$, *i.e.*, $x = -2 \in (-5, 5)$

$\therefore$ From (*i*), $I = \int_{-5}^{5} |x+2| dx$

$$= \int_{-5}^{-2} |x+2| \, dx + \int_{-2}^{5} |x+2| \, dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \text{ where } a < c < b\right]$$

$$= \int_{-5}^{-2} -(x+2)\ dx + \int_{-2}^{5} (x+2)\ dx$$

[$\because$ On $(-5, -2)$, $x < -2 \Rightarrow x + 2 < 0$

$\Rightarrow |x + 2| = -(x + 2)$ and on $(-2, 5)$; $x > -2$

$\Rightarrow x + 2 > 0 \Rightarrow |x + 2| = x + 2$, by definition of modulus function]

$$= -\left(\frac{x^2}{2} + 2x\right)_{-5}^{-2} + \left(\frac{x^2}{2} + 2x\right)_{-2}^{5}$$

$$= -\left[\left(\frac{4}{2} - 4\right) - \left(\frac{25}{2} - 10\right)\right] + \left[\left(\frac{25}{2} + 10\right) - \left(\frac{4}{2} - 4\right)\right]$$

$$= -\left[-2 - \frac{5}{2}\right] + \left[\frac{45}{2} + 2\right] = 2 + \frac{5}{2} + \frac{45}{2} + 2$$

$$= 4 + \frac{50}{2} = 4 + 25 = 29.$$

6. $\int_2^8 |x - 5|\ dx$

Sol. We know by definition of modulus function, that

$$|x - 5| = \begin{cases} x - 5 & \text{if } \ x - 5 \geq 0, i.e., x \geq 5 \quad ...(i) \\ -(x-5) = 5 - x, & \text{if } \ x < 5 \quad ...(ii) \end{cases}$$

$$\therefore \int_2^8 |x-5|\ dx = \int_2^5 |x-5|\ dx + \int_5^8 |x-5|\ dx$$

$$= \underset{\text{[By (ii)]}}{\int_2^5 (5-x)\ dx} + \underset{\text{[By (i)]}}{\int_5^8 (x-5)\ dx} = \left(5x - \frac{x^2}{2}\right)_2^5 + \left(\frac{x^2}{2} - 5x\right)_5^8$$

$$= \left(25 - \frac{25}{2}\right) - (10 - 2) + (32 - 40) - \left(\frac{25}{2} - 25\right)$$

$$= 25 - \frac{25}{2} - 8 - 8 - \frac{25}{2} + 25 \qquad = 34 - \frac{50}{2} = 34 - 25 = 9$$

By using the properties of definite integrals, evaluate the integrals in Exercises 7 to 11:

7. $\int_0^1 x(1-x)^n\ dx$

Sol. Let $\mathrm{I} = \int_0^1 x(1-x)^n\ dx$

$$\therefore \quad \mathrm{I} = \int_0^1 (1-x)\ (1 - (1 - x))^n\ dx \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

or $\quad \mathrm{I} = \int_0^1 (1-x)\ (1 - 1 + x)^n\ dx$

or $\quad I = \int_0^1 (1-x)\ x^n\ dx = \int_0^1 (x^n - x^{n+1})\ dx$

$$= \left(\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right)_0^1 = \frac{1}{n+1} - \frac{1}{n+2} - (0-0)$$

$$= \frac{n+2-n-1}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}.$$

8. $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\ dx$

Sol. Let $I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)\ dx$ ...(*i*)

Changing x to $\frac{\pi}{4} - x$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$I = \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right] dx = \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan x}{1+\tan x}\right] dx$$

$$\left[\because \tan\left(\frac{\pi}{4}-x\right) = \frac{\tan\frac{\pi}{4} - \tan x}{1+\tan\frac{\pi}{4}\tan x} = \frac{1-\tan x}{1+\tan x}\right]$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan x}\right) dx \quad ...(ii)$$

Adding Eqns. (*i*) and (*ii*), we have

$$2I = \int_0^{\frac{\pi}{4}} \left[\log(1+\tan x) + \log\left(\frac{2}{1+\tan x}\right)\right] dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left[(1+\tan x)\frac{2}{(1+\tan x)}\right] dx = \int_0^{\frac{\pi}{4}} \log 2\ dx$$

or $\quad 2I = (\log 2)\ [x]_0^{\frac{\pi}{4}} = \frac{\pi}{4}\log 2$ Dividing by 2, $I = \frac{\pi}{8}\log 2.$

9. $\int_0^2 x\sqrt{2-x}\ dx$

Sol. Let $I = \int_0^2 x\sqrt{2-x}\ dx$

Changing x to $2 - x$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$I = \int_0^2 (2-x)\sqrt{2-(2-x)}\ dx$$

$$= \int_0^2 (2-x)\sqrt{x}\,dx = \int_0^2 (2x^{1/2} - x^{3/2})\ dx$$

$$= \left[2 \cdot \frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2}\right]_0^2 = \left(\frac{4}{3} \cdot 2^{3/2} - \frac{2}{5} \cdot 2^{5/2}\right) - (0-0)$$

$$= \frac{4}{3} \times 2\sqrt{2} - \frac{2}{5} \times 4\sqrt{2} = \left(\frac{8}{3} - \frac{8}{5}\right)\sqrt{2}$$

$(\because\ 2^{3/2} = (2^{1/2})^3 = (\sqrt{2})^3 = \sqrt{2}\ \sqrt{2}\ \sqrt{2} = 2\sqrt{2}$

and $2^{5/2} = (2^{1/2})^5 = (\sqrt{2})^5 = \sqrt{2}\ \sqrt{2}\ \sqrt{2}\ \sqrt{2}\ \sqrt{2} = 2.2.\sqrt{2}$

$$= 4\sqrt{2}\ ) = \frac{16\sqrt{2}}{15}.$$

10. $\displaystyle\int_0^{\frac{\pi}{2}} (2 \log \sin x - \log \sin 2x)\ dx$

Sol. Let $\displaystyle I = \int_0^{\pi/2} (2 \log \sin x - \log \sin 2x)\, dx$

$$= \int_0^{\pi/2} (\log \sin^2 x - \log \sin 2x)\, dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin^2 x}{\sin 2x}\right) dx = \int_0^{\pi/2} \log\left(\frac{\sin^2 x}{2 \sin x \cos x}\right) dx$$

or $\displaystyle I = \int_0^{\pi/2} \log\left(\frac{1}{2} \tan x\right) dx$ ...(*i*)

$\therefore$ $\displaystyle I = \int_0^{\pi/2} \log\left(\frac{1}{2} \tan\left(\frac{\pi}{2} - x\right) dx\right) \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$

or $\displaystyle I = \int_0^{\pi/2} \log\left(\frac{1}{2} \cot x\right) dx$ ...(*ii*)

Adding Eqns. (*i*) and (*ii*),

$$2I = \int_0^{\pi/2} \left[\log\left(\frac{1}{2} \tan x\right) + \log\left(\frac{1}{2} \cot x\right)\right] dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log\left(\frac{1}{2} \tan x \frac{1}{2} \cot x\right) dx = \int_0^{\pi/2} \log \frac{1}{4}\, dx = \log \frac{1}{4}\, (x)_0^{\pi/2}$$

$$= (\log 1 - \log 4)\ \frac{\pi}{2} = -\frac{\pi}{2} \log 4 \qquad (\because\ \log 1 = 0)$$

$$\therefore \quad I = -\frac{\pi}{4} \log 4 = -\frac{\pi}{4} \log 2^2 = -\frac{2\pi}{4} \log 2 = -\frac{\pi}{2} \log 2.$$

11. $\displaystyle\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\ dx$

Sol. Let $\displaystyle I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\ dx$ or $\displaystyle I = 2 \int_0^{\frac{\pi}{2}} \sin^2 x\ dx$ ...(*i*)

$[\because$ For $f(x) = \sin^2 x$, $f(-x) = \sin^2(-x) = (-\sin x)^2 = \sin^2 x = f(x)$

$\therefore$ $f(x)$ is an even function of x and hence

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx]$$

$$\therefore \quad I = 2\int_0^{\frac{\pi}{2}} \sin^2\left(\frac{\pi}{2} - x\right) dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\text{or} \quad I = 2\int_0^{\frac{\pi}{2}} \cos^2 x \; dx \qquad ...(ii)$$

Adding Eqns. (i) and (ii), we have

$$2I = 2\int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x)\; dx$$

$$\text{or} \quad 2I = 2\int_0^{\frac{\pi}{2}} 1 \; dx = 2\; (x)_0^{\frac{\pi}{2}} = 2 \,.\, \frac{\pi}{2} = \pi \;\therefore\; I = \frac{\pi}{2}.$$

Using properties of definite integrals, evaluate the following integrals in Exercises 12 to 18:

12. $\displaystyle\int_0^{\pi} \frac{x\,dx}{1 + \sin x}$

Sol. Let $\displaystyle I = \int_0^{\pi} \frac{x}{1 + \sin x}\; dx \qquad ...(i)$

Changing x to $\pi - x$, $\displaystyle I = \int_0^{\pi} \frac{\pi - x}{1 + \sin(\pi - x)}\; dx$

$$\text{or} \quad I = \int_0^{\pi} \frac{\pi - x}{1 + \sin x} dx \qquad ...(ii) \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

Adding Eqns. (i) and (ii), we have

$$2I = \int_0^{\pi} \left(\frac{x}{1+\sin x} + \frac{\pi - x}{1 + \sin x}\right) dx = \int_0^{\pi} \frac{x + \pi - x}{1 + \sin x}\; dx$$

$$= \int_0^{\pi} \frac{\pi}{1 + \sin x}\; dx = \pi \int_0^{\pi} \frac{1}{1 + \sin x}\; dx$$

$$\text{or} \quad 2I = 2\pi \int_0^{\pi/2} \frac{dx}{1 + \sin x}$$

$$\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(2a - x) = f(x)\right]$$

$$= 2\pi \int_0^{\pi/2} \frac{dx}{1 + \sin\left(\frac{\pi}{2} - x\right)} \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= 2\pi \int_0^{\pi/2} \frac{dx}{1 + \cos x}$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{dx}{2\cos^2 \frac{x}{2}} = \frac{\pi}{2} \int_0^{\pi/2} \sec^2 \frac{x}{2}\; dx = \frac{\pi}{2} \left[\frac{\tan \frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2}$$

$$= \pi\left(\tan\frac{\pi}{4} - \tan 0\right) = \pi(1 - 0) = \pi.$$

13. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \; dx$

Sol. Let I = $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \; dx$

Here Integrand $f(x) = \sin^7 x$

$\therefore \quad f(-x) = \sin^7(-x) = (-\sin x)^7 = -\sin^7 x = -f(x)$

$\therefore \quad f(x)$ is an odd function of x.

$\therefore \quad I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \; dx = 0.$

$\left[\because \text{ If } f(x) \text{ is an odd function of } x, \text{ then } \int_{-a}^{a} f(x)\,dx = 0\right]$

14. $\int_0^{2\pi} \cos^5 x \; dx$

Sol. $\int_0^{2\pi} \cos^5 x \; dx = 2\int_0^{\pi} \cos^5 x \; dx$

$\left[\because \int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx, \text{ if } f(2a - x) = f(x)\right]$

Here $f(x) = \cos^5 x \quad \therefore \quad f(2\pi - x) = \cos^5(2\pi - x) = \cos^5 x$

$= f(x) = 2(0) = 0$

$\left[\because \int_0^{2a} f(x)\; dx = 0, \text{ if } f(2a - x) = -f(x). \text{ Here } f(x) = \cos^5 x\right.$

$\therefore \quad f(\pi - x) = \cos^5(\pi - x) = (-\cos x)^5 = -\cos^5 x = -f(x)]$

Alternatively. To evaluate $\int_0^{2\pi} \cos^5 x \; dx$, put $\sin x = t$.

Remark. In fact $\int_0^{2\pi} \cos^n x \; dx$ or $\int_0^{\pi} \cos^n x \; dx$ for all positive **odd integers** n is equal to zero.
This is a very important result for I.I.T. Entrance Examination.

15. $\int_0^{\frac{\pi}{2}} \dfrac{\sin x - \cos x}{1 + \sin x \cos x} \; dx$

Sol. Let I = $\int_0^{\frac{\pi}{2}} \dfrac{\sin x - \cos x}{1 + \sin x \cos x} \; dx$ ...(*i*)

Changing x to $\frac{\pi}{2} - x$ in integrand of (*i*),

$\left[\because \int_0^{a} f(x)\,dx = \int_0^{a} f(a - x)\,dx\right]$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)} \; dx = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \cos x \sin x} \; dx$$

$$= -\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx \qquad ...(ii)$$

Adding equations (i) and (ii), we have $2I = 0$ or $I = 0$.

16. $\int_0^{\pi} \log(1 + \cos x)\, dx$

Sol. Let $I = \int_0^{\pi} \log(1 + \cos x)\, dx \qquad ...(i)$

$\therefore \quad I = \int_0^{\pi} \log(1 + \cos(\pi - x))\, dx \quad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$

or $\quad I = \int_0^{\pi} \log(1 - \cos x)\, dx \qquad ...(ii)$

Adding Eqns. (i) and (ii), we have

$$2I = \int_0^{\pi} [\log(1 + \cos x) + \log(1 - \cos x)]\, dx$$

$$= \int_0^{\pi} \log((1 + \cos x)(1 - \cos x))\, dx = \int_0^{\pi} \log(1 - \cos^2 x)\, dx$$

$\Rightarrow 2I = \int_0^{\pi} \log \sin^2 x\, dx = 2 \int_0^{\pi} \log \sin x\, dx \quad (\because \log m^n = n \log m)$

Dividing by 2, $I = \int_0^{\pi} \log \sin x\, dx = 2 \int_0^{\frac{\pi}{2}} \log \sin x\, dx \qquad ...(iii)$

$\Big[\because$ For $f(x) = \log \sin x$, $f(\pi - x) = \log \sin(\pi - x) = \log \sin x =$ $f(x)$ and if $f(2a - x) = f(x)$; then $\int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx\Big]$

$\therefore \quad I = 2 \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx \quad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$

or $\quad I = 2 \int_0^{\frac{\pi}{2}} \log \cos x\, dx \qquad ...(iv)$

Adding Eqns. (iii) and (iv), we have

$$2I = 2 \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x)\, dx$$

Dividing by 2, $I = \int_0^{\frac{\pi}{2}} (\log \sin x \cos x)\, dx$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{2 \sin x \cos x}{2}\right) dx = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right) dx$$

or $\quad I = \int_0^{\frac{\pi}{2}} (\log \sin 2x - \log 2)\, dx$

or $\quad I = \int_0^{\frac{\pi}{2}} \log \sin 2x\, dx - \int_0^{\frac{\pi}{2}} \log 2\, dx$

or $\quad I = \int_0^{\frac{\pi}{2}} \log \sin 2x\, dx - \log 2\, (x)_0^{\frac{\pi}{2}}$

or $\quad I = \int_0^{\frac{\pi}{2}} \log \sin 2x \; dx - \frac{\pi}{2} \log 2$

or $\quad I = I_1 - \frac{\pi}{2} \log 2$ ...(*v*)

where $\quad I_1 = \int_0^{\frac{\pi}{2}} \log \sin 2x \; dx$ ...(*vi*)

Put $2x = t$ to make I_1 look as I given by (*iii*)

$\therefore \quad 2 = \frac{dt}{dx} \quad$ or $\quad 2\,dx = dt \quad$ or $\quad dx = \frac{dt}{2}$

To change the limits: When $x = 0$, $\quad t = 2x = 0$

When $x = \frac{\pi}{2}$, $\; t = 2x = \pi$

$\therefore$ From (*vi*), $I_1 = \int_0^{\pi} \log \sin t \; \frac{dt}{2} = \frac{1}{2} \int_0^{\pi} \log \sin t \; dt$

or $\quad I_1 = \frac{1}{2} \times 2 \int_0^{\frac{\pi}{2}} \log \sin t \, dt$

(For reason see Explanation within brackets below Eqn. (*iii*))

or $\quad I_1 = \int_0^{\frac{\pi}{2}} \log \sin t \; dt = \int_0^{\frac{\pi}{2}} \log \sin x \, dx \left[\because \int_a^b f(t)\,dt = \int_a^b f(x)\,dx\right]$

or $\quad I_1 = \frac{I}{2}$ [By Eqn. (*iii*)]

Putting this value of I_1 in Eqn. (*v*), $I = \frac{I}{2} - \frac{\pi}{2} \log 2$

Multiplying by L.C.M. = 2, $2I = I - \pi \log 2$

or $\quad 2I - I = -\pi \log 2 \quad$ or $\quad I = -\pi \log 2$.

17. $\displaystyle \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} \, dx$

Sol. Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} \, dx$ ...(*i*)

$\therefore \quad I = \int_0^a \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{a - (a - x)}} \, dx = \int_0^a \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} \, dx$ ...(*ii*)

Adding Eqns. (*i*) and (*ii*), we have

$$2I = \int_0^a \left(\frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} + \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} \right) dx = \int_0^a \left(\frac{\sqrt{x} + \sqrt{a - x}}{\sqrt{x} + \sqrt{a - x}} \right) dx$$

or $\quad 2I = \int_0^a 1 \; dx = (x)_0^a = a \quad \therefore \quad I = \frac{a}{2}$.

18. $\int_0^4 |x-1|\, dx$

Sol. Let I = $\int_0^4 |x-1|\, dx$ ...(*i*)

Putting the expression $(x-1)$ within modulus equal to zero, we have

$x = 1 \in (0, 4)$

$\therefore$ From (*i*), $I = \int_0^4 |x-1|\, dx = \int_0^1 |x-1|\, dx + \int_1^4 |x-1|\, dx$

$= -\int_0^1 (x-1)\, dx + \int_1^4 (x-1)\, dx$

[$\because$ On (0, 1); $x < 1 \Rightarrow x - 1 < 0$ and hence $|x-1| = -(x-1)$ and on (1, 4), $x > 1 \Rightarrow x - 1 > 0$ and hence $|x-1| = (x-1)$ by definition of modulus function]

$= -\left(\frac{x^2}{2} - x\right)_0^1 + \left(\frac{x^2}{2} - x\right)_1^4 = -\left(\left(\frac{1}{2} - 1\right) - 0\right) + \left(\frac{16}{2} - 4 - \left(\frac{1}{2} - 1\right)\right)$

$= \frac{-1}{2} + 1 + 8 - 4 - \frac{1}{2} + 1 \quad = 6 - \frac{2}{2} = 6 - 1 = 5.$

19. Show that $\int_0^a f(x)\, g(x)\, dx = 2\int_0^a f(x)\, dx$, if f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$.

Sol. Given: $f(x) = f(a-x)$ ...(*i*)

and $g(x) + g(a-x) = 4$ ...(*ii*)

Let $I = \int_0^a f(x)\, g(x)\, dx$ ...(*iii*)

$\therefore \quad I = \int_0^a f(a-x)\, g(a-x)\, dx \quad \left[\because \int_0^a F(x)\, dx = \int_0^a F(a-x)\, dx\right]$

Putting $f(a-x) = f(x)$ from (*i*),

$I = \int_0^a f(x)\, g(a-x)\, dx$ ...(*iv*)

Adding Eqns. (*iii*) and (*iv*), we have

$2I = \int_0^a (f(x)\, g(x) + f(x)\, g(a-x))\, dx = \int_0^a f(x)\,(g(x) + g(a-x))\, dx$

or $2I = \int_0^a f(x)\,(4)\, dx$ [By (*ii*)] $= 4\int_0^a f(x)\, dx$

Dividing by 2, $I = 2\int_0^a f(x)\, dx$ = R.H.S.

Choose the correct answer in Exercises 20 and 21:

20. The value of $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)\, dx$ is

(A) 0 **(B) 2** **(C) π** **(D) 1**

Sol. Let $I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)\, dx$

$$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x^3 \; dx + \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x \cos x \; dx + \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \tan^5 x \; dx + \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 1 \, dx$$

$$= 0 + 0 + 0 + \left(x \right)_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2} - \left(\frac{-\pi}{2} \right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

[$\because$ Each of the three functions x^3, $x \cos x$ and $\tan^5 x$ is an odd function of x as $f(-x) = -f(x)$ for each of them and $\int_{-a}^{a} f(x)\,dx = 0$ for each odd function $f(x)$]

$\therefore$ Option (C) is the correct option.

21. The value of $\int_0^{\frac{\pi}{2}} \log\left(\frac{4 + 3 \sin x}{4 + 3 \cos x} \right) dx$ is

(A) 2 **(B) $\frac{3}{4}$** **(C) 0** **(D) – 2**

Sol. Let $I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4 + 3 \sin x}{4 + 3 \cos x} \right) dx$ ...(*i*)

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4 + 3 \sin\left(\frac{\pi}{2} - x\right)}{4 + 3 \cos\left(\frac{\pi}{2} - x\right)} \right) dx$$

or $I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4 + 3 \cos x}{4 + 3 \sin x} \right) dx$ ...(*ii*)

Adding Eqns. (*i*) and (*ii*), we get

$$2I = \int_0^{\frac{\pi}{2}} \left[\log\left(\frac{4 + 3 \sin x}{4 + 3 \cos x} \right) + \log\left(\frac{4 + 3 \cos x}{4 + 3 \sin x} \right) \right] dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left[\frac{4 + 3 \sin x}{4 + 3 \cos x} \cdot \frac{4 + 3 \cos x}{4 + 3 \sin x} \right] dx = \int_0^{\frac{\pi}{2}} \log 1 \; dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0 \; dx = 0 \quad \Rightarrow \quad I = \frac{0}{2} = 0.$$

MISCELLANEOUS EXERCISE

Integrate the functions in Exercises 1 to 11:

1. $\dfrac{1}{x - x^3}$

Sol. The integrand $\dfrac{1}{x - x^3}$ is a rational function of x and the denominator $x - x^3 = x(1 - x^2) = x(1 - x)(1 + x)$ is the product of more than one factor. So, will form partial fractions.

$$\frac{1}{x - x^3} = \frac{1}{x\,(1 - x^2)} = \frac{1}{x(1 - x)(1 + x)}$$

$$= \frac{A}{x} + \frac{B}{1 - x} + \frac{C}{1 + x} \quad ...(i)$$

Multiplying every term of Eqn. (i) by L.C.M. $= x(1 - x)(1 + x)$,

$1 = A(1 - x)(1 + x) + Bx(1 + x) + Cx(1 - x)$

or $1 = A(1 - x^2) + B(x + x^2) + C(x - x^2)$

$\Rightarrow$ $1 = A - Ax^2 + Bx + Bx^2 + Cx - Cx^2$

Comparing coefficients of like powers on both sides,

$\boldsymbol{x^2}$**:** $-A + B - C = 0$ ...(ii)

$\boldsymbol{x}$**:** $B + C = 0$ ....(iii)

Constants: $A = 1$

Putting $A = 1$ in (ii), $-1 + B - C = 0$ or $B - C = 1$ ...(iv)

Adding Eqns. (iii) and (iv), $2B = 1$ $\Rightarrow$ $B = \dfrac{1}{2}$

From (iii), $C = -B = \dfrac{-1}{2}$

Putting these values of A, B, C in (i),

$$\frac{1}{x - x^3} = \frac{1}{x} + \frac{\frac{1}{2}}{1 - x} - \frac{\frac{1}{2}}{1 + x}$$

$$\therefore \int \frac{1}{x - x^3}\,dx = \int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{1 - x}\,dx - \frac{1}{2}\int \frac{1}{1 + x}\,dx$$

$$= \log|x| + \frac{1}{2}\,\frac{\log|1 - x|}{-1} - \frac{1}{2}\log|1 + x|$$

$$= \frac{1}{2}\,[\,2\log|x| - \log|1 - x| - \log|1 + x|] + C$$

$$= \frac{1}{2}\,[\,\log|x|^2 - (\log|1 - x| + \log|1 + x|] + C$$

$$= \frac{1}{2} [\log | x |^2 - \log | 1 - x | | 1 + x |] + C$$

$$= \frac{1}{2} [\log | x |^2 - \log | 1 - x^2 |] + C \qquad = \frac{1}{2} \log \left| \frac{x^2}{1 - x^2} \right| + C.$$

2. $\dfrac{\mathbf{1}}{\sqrt{\boldsymbol{x + a}} + \sqrt{\boldsymbol{x + b}}}$

Sol. $\displaystyle\int \frac{1}{\sqrt{x + a} + \sqrt{x + b}} \, dx$

Rationalising, $\displaystyle = \int \frac{\sqrt{x+a} - \sqrt{x+b}}{(\sqrt{x+a} + \sqrt{x+b})(\sqrt{x+a} - \sqrt{x+b})} \, dx$

$$= \int \frac{(\sqrt{x+a} - \sqrt{x+b})}{x + a - (x + b)} dx = \int \frac{\sqrt{x+a} - \sqrt{x+b}}{a - b} dx$$

$[\because x + a - (x + b) = x + a - x - b = a - b]$

$$= \frac{1}{a-b} \int (\sqrt{x+a} - \sqrt{x+b}) \, dx$$

$$= \frac{1}{a-b} \left[\int (x+a)^{1/2} dx - \int (x+b)^{1/2} dx \right]$$

$$= \frac{1}{a-b} \left[\frac{(x+a)^{3/2}}{\frac{3}{2}(1)} - \frac{(x+b)^{3/2}}{\frac{3}{2}(1)} \right] + C$$

$$= \frac{1}{a-b} \left[\frac{2}{3} (x+a)^{3/2} - \frac{2}{3} (x+b)^{3/2} \right] + C$$

$$= \frac{2}{3(a-b)} [(x + a)^{3/2} - (x + b)^{3/2}] + C.$$

3. $\dfrac{\mathbf{1}}{\boldsymbol{x}\sqrt{\boldsymbol{ax - x^2}}}$

Sol. $\displaystyle I = \int \frac{dx}{x\sqrt{ax - x^2}}$ $\left[\textbf{Form} \displaystyle\int \frac{\boldsymbol{dx}}{\textbf{Linear} \sqrt{\textbf{Quadratic}}} \right.$

Put Linear $= \dfrac{1}{t}$, *i.e.*, $x = \dfrac{1}{t} = t^{-1}$.

Differentiating both sides $dx = -\dfrac{1}{t^2} \, dt$

$$\therefore \qquad I = \int \frac{-\frac{1}{t^2} dt}{\frac{1}{t}\sqrt{\frac{a}{t} - \frac{1}{t^2}}} = -\int \frac{dt}{\sqrt{at - 1}}$$

$$= -\int (at - 1)^{-1/2} \, dt = -\frac{(at-1)^{1/2}}{\frac{1}{2} \times a} + c$$

$$= -\frac{2}{a}\sqrt{\frac{a}{x}-1} + c = -\frac{2}{a}\sqrt{\frac{a-x}{x}} + c.$$

4. $\dfrac{1}{x^2(x^4+1)^{3/4}}$

Sol. $I = \int \frac{dx}{x^2(x^4+1)^{3/4}} = \int \frac{dx}{x^2\left[x^4\left(1+\frac{1}{x^4}\right)\right]^{3/4}} = \int \frac{dx}{x^2 \,.\, x^3\left(1+\frac{1}{x^4}\right)^{3/4}}$

$$\left[\because (x^4)^{\frac{3}{4}} = x^3\right]$$

$$= \int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-3/4} dx$$

Put $1 + \frac{1}{x^4} = t$ or $1 + x^{-4} = t$.

Differentiating both sides, $-4x^{-5}\, dx = dt$

or $-\frac{4}{x^5}\, dx = dt$ or $\frac{1}{x^5}\, dx = -\frac{1}{4}\, dt$

$\therefore\ I = -\frac{1}{4}\int t^{-3/4}\, dt = -\frac{1}{4} \,.\, \frac{t^{1/4}}{1/4} + c = -\left(1+\frac{1}{x^4}\right)^{1/4} + c.$

5. $\dfrac{1}{x^{1/2}+x^{1/3}}$

Sol. Here the denominators of fractional powers $\frac{1}{2}$ and $\frac{1}{3}$ of x are 2 and 3. L.C.M. of 2 and 3 is 6.

Put $x = t^6$. Differentiating both sides, $dx = 6t^5\, dt$

$\therefore\ I = \int \frac{dx}{x^{1/2}+x^{1/3}} = \int \frac{6t^5}{t^3+t^2}\, dt = 6\int \frac{t^5}{t^2(t+1)}\, dt$

$$= 6\int \frac{t^3}{t+1}\, dt = 6\int \frac{(t^3+1)-1}{t+1}\, dt = 6\int \left[\frac{t^3+1}{t+1} - \frac{1}{t+1}\right] dt$$

$$= 6\int \left[\frac{(t+1)(t^2-t+1)}{t+1} - \frac{1}{t+1}\right] dt = 6\int \left(t^2 - t + 1 - \frac{1}{t+1}\right) dt$$

$[\because a^3 + b^3 = (a+b)(a^2 - ab + b^2)]$

$$= 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1|\right] + c$$

$$= 2t^3 - 3t^2 + 6t - 6\log|t+1| + c$$

Putting $t = x^{1/6}$ $(\because\ x = t^6 \Rightarrow t = x^{1/6})$

$$= 2\sqrt{x} - 3x^{1/3} + 6x^{1/6} - 6\log|x^{1/6}+1| + c.$$

6. $\dfrac{\mathbf{5x}}{\mathbf{(x+1)(x^2+9)}}$

Sol. Let I = $\displaystyle\int \frac{5x}{(x+1)(x^2+9)}\, dx$ ...(*i*)

Let $\dfrac{5x}{(x+1)(x^2+9)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+9}$ ...(*ii*)

L.C.M. = $(x + 1)(x^2 + 9)$

Multiplying every term of (*ii*) by L.C.M.,

$5x = A(x^2 + 9) + (Bx + C)(x + 1)$

or $5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$

Comparing coefficients of x^2, x and constant terms on both sides,

$\boldsymbol{x^2}$**:** $A + B = 0$ ...(*iii*)

$\boldsymbol{x}$**:** $B + C = 5$ ...(*iv*)

Constant terms : $9A + C = 0$ ...(*v*)

Let us solve Eqns. (*iii*), (*iv*) and (*v*) for A, B, C.

(*iii*) – (*iv*) gives, (to eliminate B), $A - C = -5$ ...(*vi*)

Adding (*v*) and (*vi*), $10A = -5$

$\therefore \quad A = \dfrac{-5}{10} = \dfrac{-1}{2}$

Putting $A = \dfrac{-1}{2}$ in (*iii*), $\dfrac{-1}{2} + B = 0 \Rightarrow B = \dfrac{1}{2}$

Putting $B = \dfrac{1}{2}$ in (*iv*), $\dfrac{1}{2} + C = 5 \Rightarrow C = 5 - \dfrac{1}{2} = \dfrac{9}{2}$

Putting these values of A, B, C in (*ii*),

$$\frac{5x}{(x+1)(x^2+9)} = \frac{\frac{-1}{2}}{x+1} + \frac{\frac{1}{2}x + \frac{9}{2}}{x^2+9}$$

$$\therefore \int \frac{5x}{(x+1)(x^2+9)}\, dx$$

$$= \frac{-1}{2}\int \frac{1}{x+1}\, dx + \frac{1}{2}\int \frac{x}{x^2+9}\, dx + \frac{9}{2}\int \frac{1}{x^2+3^2}\, dx$$

$$= \frac{-1}{2}\log|x+1| + \frac{1}{4}\int \frac{2x}{x^2+9}\, dx + \frac{9}{2}\cdot\frac{1}{3}\tan^{-1}\frac{x}{3} + c$$

$$= \frac{-1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{3}{2}\tan^{-1}\frac{x}{3} + c$$

$$\left[\because \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)|\right]$$

$= \frac{-1}{2} \log|x+1| + \frac{1}{4} \log(x^2+9) + \frac{3}{2} \tan^{-1} \frac{x}{3} + c.$

$(\because \ x^2 + 9 \geq 9 > 0$ and hence $|x^2 + 9| = x^2 + 9)$

7. $\dfrac{\sin x}{\sin (x-a)}$

Sol. $\int \frac{\sin x}{\sin (x-a)} dx = \int \frac{\sin (x-a+a)}{\sin (x-a)} dx$

$= \int \frac{\sin (x-a) \cos a + \cos (x-a) \sin a}{\sin (x-a)} dx$

$[\because \ \sin (A+B) = \sin A \cos B + \cos A \sin B]$

$= \int \left[\frac{\sin (x-a) \cos a}{\sin (x-a)} + \frac{\cos (x-a) \sin a}{\sin (x-a)}\right] dx \qquad \left[\because \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}\right]$

$= \int [\cos a + \sin a \cot (x-a)] dx = \int \cos a \, dx + \int \sin a \cot (x-a) dx$

$= \cos a \int 1 \, dx + \sin a \int \cot (x-a) \, dx$

$= (\cos a)x + \sin a \frac{\log|\sin (x-a)|}{1} + c \left[\because \int \cot x \, dx = \log|\sin x|\right]$

$= x \cos a + \sin a \log|\sin (x-a)| + c.$

8. $\dfrac{e^{5 \log x} - e^{4 \log x}}{e^{3 \log x} - e^{2 \log x}}$

Sol. $\int \frac{e^{5 \log x} - e^{4 \log x}}{e^{3 \log x} - e^{2 \log x}} dx = \int \frac{e^{\log x^5} - e^{\log x^4}}{e^{\log x^3} - e^{\log x^2}} dx \quad [\because \ n \log m = \log m^n]$

$= \int \frac{x^5 - x^4}{x^3 - x^2} dx \qquad [\because \ e^{\log f(x)} = f(x)]$

$= \int \frac{x^4 (x-1)}{x^2 (x-1)} dx = \int x^2 \, dx = \frac{x^3}{3} + c.$

9. $\dfrac{\cos x}{\sqrt{4 - \sin^2 x}}$

Sol. Let $I = \int \frac{\cos x}{\sqrt{4 - \sin^2 x}} dx \qquad \ldots(i)$

Put $\sin x = t$. Therefore $\cos x = \frac{dt}{dx} \Rightarrow \cos x \, dx = dt$

$\therefore$ From (i), $I = \int \frac{dt}{\sqrt{4 - t^2}} = \int \frac{dt}{\sqrt{2^2 - t^2}} dt$

$$= \sin^{-1}\left(\frac{t}{2}\right) + c \qquad \left[\because \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a}\right]$$

$$= \sin^{-1}\left[\frac{1}{2}\sin x\right] + c.$$

10. $\dfrac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x}$

Sol. Let $I = \int \dfrac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x}\, dx$ ...(*i*)

Now numerator of integrand $= \sin^8 x - \cos^8 x$

$= (\sin^4 x)^2 - (\cos^4 x)^2$

$= (\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)$ $\quad [\because a^2 - b^2 = (a - b)(a + b)]$

$= [(\sin^2 x)^2 - (\cos^2 x)^2]\,[(\sin^2 x)^2 + (\cos^2 x)^2]$

$= (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)$

$[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x]$

$[\because a^2 + b^2 = a^2 + b^2 + 2ab - 2ab = (a + b)^2 - 2ab]$

$= 1\,[-(\cos^2 x - \sin^2 x)]\,(1 - 2\sin^2 x \cos^2 x)$

$\Rightarrow \sin^8 x - \cos^8 x = -\cos 2x\,(1 - 2\sin^2 x \cos^2 x)$

Putting this value of $\sin^8 x - \cos^8 x$ in numerator of (*i*),

$$I = \int \frac{-\cos 2x\,(1 - 2\sin^2 x \cos^2 x)}{1 - 2\sin^2 x \cos^2 x}\, dx = \int -\cos 2x\, dx = -\frac{\sin 2x}{2} + c.$$

11. $\dfrac{1}{\cos(x + a)\cos(x + b)}$

Sol. Let $I = \int \dfrac{1}{\cos(x + a)\cos(x + b)}\, dx$ ...(*i*)

We know that $(x + a) - (x + b) = x + a - x - b = a - b$ ...(*ii*)

Dividing and multiplying by $\boldsymbol{\sin(a - b)}$ in (*i*),

$$I = \frac{1}{\sin(a - b)}\int \frac{\sin(a - b)}{\cos(x + a)\cos(x + b)}\, dx$$

Replacing $(a - b)$ by $(x + a) - (x + b)$ in $\sin(a - b)$

[Using (*ii*)],

$$= \frac{1}{\sin(a - b)}\int \frac{\sin[(x + a) - (x + b)]}{\cos(x + a)\cos(x + b)}\, dx$$

$$= \frac{1}{\sin(a - b)}\int \frac{\sin(x + a)\cos(x + b) - \cos(x + a)\sin(x + b)}{\cos(x + a)\cos(x + b)}\, dx$$

$[\because \sin(A - B) = \sin A \cos B - \cos A \sin B]$

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x+a)\cos(x+b)}{\cos(x+a)\cos(x+b)} - \frac{\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)} \right] dx$$

$$\left(\because \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c} \right)$$

$$= \frac{1}{\sin(a-b)} \int [\tan(x+a) - \tan(x+b)]\ dx$$

$$= \frac{1}{\sin(a-b)} [-\log|\cos(x+a)| + \log|\cos(x+b)|] + c$$

$$\left[\because \int \tan x\, dx = -\log|\cos x| \right]$$

$$= \frac{1}{\sin(a-b)} \log\left| \frac{\cos(x+b)}{\cos(x+a)} \right| + c. \quad \left[\because \log m - \log n = \log \frac{m}{n} \right].$$

Integrate the functions in Exercises 12 to 22:

12. $\boldsymbol{\dfrac{x^3}{\sqrt{1-x^8}}}$

Sol. Let $I = \int \frac{x^3}{\sqrt{1-x^8}}\ dx = \frac{1}{4} \int \frac{4x^3}{\sqrt{1-(x^4)^2}}\ dx$ ...(*i*)

Put $x^4 = t$. Therefore $4x^3 = \frac{dt}{dx} \Rightarrow 4x^3\ dx = dt$

$\therefore$ From (*i*), $I = \frac{1}{4} \int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{4} \sin^{-1} t + c$

or $I = \frac{1}{4} \sin^{-1}(x^4) + c.$

13. $\boldsymbol{\dfrac{e^x}{(1+e^x)(2+e^x)}}$

Sol. Let $I = \int \frac{e^x}{(1+e^x)(2+e^x)}\ dx$ ...(*i*)

[Rule to evaluate $\int f(e^x)\ dx$, put $e^x = t$]

Put $e^x = t$. Therefore $e^x = \frac{dt}{dx} \Rightarrow e^x\ dx = dt$

$\therefore$ From (*i*), $I = \int \frac{dt}{(1+t)(2+t)} = \int \frac{1}{(t+1)(t+2)}\ dt$ ...(*ii*)

Now $t + 2 - (t + 1) = t + 2 - t - 1 = 1$

Replacing 1 in the numerator of integrand in (*ii*) by (this) $(t + 2) - (t + 1)$,

$$I = \int \frac{(t+2)-(t+1)}{(t+1)(t+2)}\ dt = \int \left(\frac{t+2}{(t+1)(t+2)} - \frac{t+1}{(t+1)(t+2)} \right) dt$$

$$= \int \left(\frac{1}{t+1} - \frac{1}{t+2} \right) dt$$

$$= \log | t + 1 | - \log | t + 2 | + c = \log \left| \frac{t+1}{t+2} \right| + c$$

Putting $t = e^x$, $= \log \left| \frac{e^x+1}{e^x+2} \right| + c \qquad = \log \left(\frac{e^x+1}{e^x+2} \right) + c.$

[$\because$ $e^x + 1 > 0$ and $e^x + 2 > 0$ and $| t | = t$ if $t \geq 0$]

14. $\dfrac{\mathbf{1}}{\mathbf{(x^2+1)(x^2+4)}}$

Sol. Let $I = \int \frac{1}{(x^2+1)(x^2+4)} \, dx$ ...(*i*)

Put x^2 = y only in the integrand.

Now the integrand is $\frac{1}{(y+1)\,(y+4)}$

Let $\frac{1}{(y+1)\,(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$ ...(*ii*)

Multiplying by L.C.M. = $(y + 1)(y + 4)$,

$1 = A(y + 4) + B(y + 1)$

or $1 = Ay + 4A + By + B$

comparing coefficient of y, $A + B = 0$ ...(*iii*)

comparing constants, $4A + B = 1$ ...(*iv*)

Let us solve (*iii*) and (*iv*) for A and B.

(*iv*) – (*iii*) gives $3A = 1 \quad \therefore A = \frac{1}{3}$

From (*iii*) $B = -A = -\frac{1}{3}$

Putting values of A, B and y in (*ii*),

$$\frac{1}{\left(x^2+1\right)\left(x^2+4\right)} = \frac{\frac{1}{3}}{x^2+1} + \frac{-\frac{1}{3}}{x^2+4} = \frac{1}{3}\left(\frac{1}{x^2+1} - \frac{1}{x^2+4} \right)$$

Putting this value in (*i*),

$$I = \frac{1}{3} \int \left(\frac{1}{x^2+1} - \frac{1}{x^2+2^2} \right) dx = \frac{1}{3} \left[\int \frac{1}{x^2+1} dx - \int \frac{1}{x^2+2^2} dx \right]$$

$$= \frac{1}{3} \left[\tan^{-1} x - \frac{1}{2} \tan^{-1} \frac{x}{2} \right] + c.$$

15. $\cos^3 x\, e^{\log \sin x}$

Sol. Let I $= \int \cos^3 x \; e^{\log \sin x}\, dx \qquad = \int \cos^3 x \;\; \sin x\; dx$

$$= - \int \cos^3 x \;\; (-\sin x)\; dx \qquad ...(i)$$

Put $\cos x = t$. $\therefore \quad -\sin x = \dfrac{dt}{dx} \quad \Rightarrow \quad -\sin x\, dx = dt$

$\therefore$ From (i), I $= - \int t^3 \;\; dt = \dfrac{-t^4}{4} + c = \dfrac{-1}{4} \cos^4 x + c.$

16. $e^{3 \log x}\, (x^4 + 1)^{-1}$

Sol. Let I $= \int e^{3 \log x}\, (x^4+1)^{-1}\;\; dx = \int \dfrac{e^{\log x^3}}{x^4+1}\; dx \qquad = \int \dfrac{x^3}{x^4+1}\; dx$

$[\because \quad e^{\log f(x)} = f(x)]$

$$\Rightarrow \quad \text{I} = \frac{1}{4} \int \frac{4x^3}{x^4+1}\; dx \qquad ...(i)$$

Put $x^4 + 1 = t$. Therefore $4x^3 = \dfrac{dt}{dx} \quad \Rightarrow \quad 4x^3\; dx = dt$

$\therefore$ From (i), $\qquad$ I $= \dfrac{1}{4} \int \dfrac{dt}{t} = \dfrac{1}{4} \log |\, t \,| + c$

Putting $t = x^4 + 1$, $= \dfrac{1}{4} \log |\, x^4 + 1 \,| + c \qquad = \dfrac{1}{4} \log (x^4 + 1) + c.$

$[\because \quad x^4 + 1 > 0 \quad \Rightarrow \quad |\, x^4 + 1 \,| = x^4 + 1]$

17. $\int \boldsymbol{f'(ax+b)(f(ax+b))^n\; dx}$

Sol. Let I $= \int f'(ax+b)\,(f(ax+b))^n \;\; dx$

$$= \frac{1}{a} \int (f(ax+b))^n \; af'(ax+b)\;\; dx \qquad ...(i)$$

Put $f(ax + b) = t$. Therefore $f'(ax + b)\, \dfrac{d}{dx}\,(ax + b) = \dfrac{dt}{dx}$

$\Rightarrow \quad af'(ax + b)\; dx = dt$

$\therefore$ From (i), I $= \dfrac{1}{a} \int t^n \;\; dt = \dfrac{1}{a}\; \dfrac{t^{n+1}}{n+1} + c$ if $n \neq -1$

and if $n = -1$, then I $= \dfrac{1}{a} \int t^{-1}\;\; dt = \dfrac{1}{a} \int \dfrac{1}{t}\; dt$

$$= \frac{1}{a} \log |\, t \,| + c.$$

Putting $t = f(ax + b)$, I $= \dfrac{(f(ax+b))^{n+1}}{a(n+1)} + c$ if $n \neq -1$

and $\qquad = \dfrac{1}{a}\; |\log f(ax + b)| + c$ if $n = -1.$

18. $\dfrac{\mathbf{1}}{\sqrt{\sin^3 x \sin (x+\alpha)}}$

Sol. $I = \int \dfrac{dx}{\sqrt{\sin^3 x \sin (x+\alpha)}} \quad = \int \dfrac{dx}{\sqrt{\sin^3 x (\sin x \cos \alpha + \cos x \sin \alpha)}}$

$= \int \dfrac{dx}{\sqrt{\sin^3 x \,.\, \sin x (\cos \alpha + \cot x \sin \alpha)}}$

$= \int \dfrac{dx}{\sin^2 x \sqrt{\cos \alpha + \cot x \sin \alpha}} \quad = \int \dfrac{\operatorname{cosec}^2 x \, dx}{\sqrt{\cos \alpha + \cot x \sin \alpha}}$

Put $\cos \alpha + \cot x \sin \alpha = t$. Differentiating both sides

$- \operatorname{cosec}^2 x \sin \alpha \, dx = dt$

or $\operatorname{cosec}^2 x \, dx = -\dfrac{dt}{\sin \alpha}$

$\therefore \quad I = \int -\dfrac{dt}{\sin \alpha \sqrt{t}} = -\dfrac{1}{\sin \alpha} \int t^{-1/2} \, dt$

$= -\dfrac{1}{\sin \alpha} \cdot \dfrac{t^{1/2}}{1/2} + c = -\dfrac{2}{\sin \alpha} \sqrt{\cos \alpha + \cot x \sin \alpha} + c$

$= -\dfrac{2}{\sin \alpha} \sqrt{\cos \alpha + \dfrac{\cos x}{\sin x} \sin \alpha} + c$

$= -\dfrac{2}{\sin \alpha} \sqrt{\dfrac{\sin x \cos \alpha + \cos x \sin \alpha}{\sin x}} + c$

$= -\dfrac{2}{\sin \alpha} \sqrt{\dfrac{\sin (x+\alpha)}{\sin x}} + c.$

19. $\dfrac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}$, $x \in [0, 1]$

Sol. We know that $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \dfrac{\pi}{2}$

$\therefore \quad \cos^{-1}\sqrt{x} = \dfrac{\pi}{2} - \sin^{-1}\sqrt{x}$

$\therefore I = \int \dfrac{\sin^{-1}\sqrt{x} - \left(\dfrac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\dfrac{\pi}{2}} \, dx$

$= \dfrac{2}{\pi} \int \left(2 \sin^{-1}\sqrt{x} - \dfrac{\pi}{2}\right) dx = \dfrac{4}{\pi} \int \sin^{-1}\sqrt{x} \, dx - \int 1 \, dx$

$= \dfrac{4}{\pi} \int \sin^{-1}\sqrt{x} \, dx - x + c \qquad ...(i)$

Now let us evaluate $\int \sin^{-1}\sqrt{x}\ dx$

Put $\sqrt{x} = \sin\theta$. $\therefore\ x = \sin^2\theta$.
Differentiating both sides, $dx = 2\sin\theta\cos\theta\, d\theta = \sin 2\theta\, d\theta$

$$\therefore \int \sin^{-1}\sqrt{x}\ dx = \int \sin^{-1}(\sin\theta)\,.\,\sin 2\theta\, d\theta = \int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sin 2\theta}\, d\theta$$

Applying Product Rule

$$= \theta\left(\frac{-\cos 2\theta}{2}\right) - \int 1\,.\left(\frac{-\cos 2\theta}{2}\right) d\theta$$

$$= -\frac{1}{2}\theta\cos 2\theta + \frac{1}{2}\int \cos 2\theta\, d\theta \quad = -\frac{1}{2}\theta\cos 2\theta + \frac{1}{2}\ \frac{\sin 2\theta}{2}$$

$$= -\frac{1}{2}\theta(1 - 2\sin^2\theta) + \frac{1}{4}\ 2\sin\theta\cos\theta$$

$$= -\frac{1}{2}\theta(1 - 2\sin^2\theta) + \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta}$$

Putting $\sin\theta = \sqrt{x}$

$$= -\frac{1}{2}(\sin^{-1}\sqrt{x})(1 - 2x) + \frac{1}{2}\sqrt{x}\sqrt{1-x}$$

Putting this value of $\int \sin^{-1}\sqrt{x}\ dx$ in (i),

$$\text{I} = \frac{4}{\pi}\left[-\frac{1}{2}(1-2x)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x}\sqrt{1-x}\right] - x + c$$

$$= -\frac{2}{\pi}(1-2x)\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x}\sqrt{1-x} - x + c.$$

20. $\sqrt{\dfrac{\mathbf{1-\sqrt{x}}}{\mathbf{1+\sqrt{x}}}}$

Sol. Let $\text{I} = \int \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}\ dx$

Put $\sqrt{x} = t$, *i.e.*, $\sqrt{\text{Linear}} = t$. $\therefore\ x = t^2$
Differentiating both sides, $dx = 2t\, dt$

$$\therefore\ \text{I} = \int \sqrt{\frac{1-t}{1+t}}\ 2t\, dt = 2\int t\sqrt{\frac{1-t}{1+t}}\ dt$$

$$= 2\int t\sqrt{\frac{1-t}{1+t}\times\frac{1-t}{1-t}}\ dt \qquad \text{(Rationalising)}$$

$$= 2\int \frac{t(1-t)}{\sqrt{1-t^2}}\ dt = 2\int \frac{t-t^2}{\sqrt{1-t^2}}\ dt \qquad ...(i)$$

$$= 2\int \frac{(1-t^2)+t-1}{\sqrt{1-t^2}}\ dt$$

$$= 2\left[\int \sqrt{1-t^2}\,dt + \int \frac{t}{\sqrt{1-t^2}}\,dt - \int \frac{1}{\sqrt{1-t^2}}\,dt\right]$$

$$= 2\left[\frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1}t + \int \frac{t}{\sqrt{1-t^2}}\,dt - \sin^{-1}t\right] + c$$

$$\left[\because \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

or $\quad I = 2\left[\frac{1}{2}t\sqrt{1-t^2} - \frac{1}{2}\sin^{-1}t + \int \frac{t}{\sqrt{1-t^2}}\,dt\right] + c \qquad ...(ii)$

To evaluate $\int \frac{t}{\sqrt{1-t^2}}\,dt$

Put $1 - t^2 = z$

Differentiating both sides $-2t\,dt = dz$ or $t\,dt = -\frac{1}{2}\,dz.$

$$\therefore \int \frac{t}{\sqrt{1-t^2}}\,dt = \int \frac{-\frac{1}{2}dz}{\sqrt{z}} = -\frac{1}{2}\int z^{-1/2}\,dz$$

$$= -\frac{1}{2}\frac{z^{1/2}}{\frac{1}{2}} = -\sqrt{1-t^2} \qquad ...(iii)$$

Putting the value of $\int \frac{t}{\sqrt{1-t^2}}\,dt = -\sqrt{1-t^2}$ from (iii) in (ii),

We have $\quad I = 2\left[\frac{1}{2}t\sqrt{1-t^2} - \frac{1}{2}\sin^{-1}t - \sqrt{1-t^2}\right] + c$

$$= t\sqrt{1-t^2} - \sin^{-1}t - 2\sqrt{1-t^2} + c$$

$$= (t-2)\sqrt{1-t^2} - \sin^{-1}t + c$$

Putting $t = \sqrt{x}$ $\qquad = (\sqrt{x} - 2)\sqrt{1-x} - \sin^{-1}\sqrt{x} + c.$

Remark. Second method to integrate after arriving at equation (i) namely $I = 2\int \frac{t-t^2}{\sqrt{1-t^2}}\,dt$, is **put $t = \sin\theta$.**

21. $\dfrac{\mathbf{2+\sin 2x}}{\mathbf{1+\cos 2x}}\ e^x$

Sol. Let I $= \int \dfrac{2+\sin 2x}{1+\cos 2x}\ e^x\ dx = \int e^x\ \dfrac{(2+2\sin x\cos x)}{2\cos^2 x}\ dx$

$= \int e^x \left(\dfrac{2}{2\cos^2 x} + \dfrac{2\sin x\cos x}{2\cos^2 x}\right) dx$

$= \int e^x \left(\dfrac{1}{\cos^2 x} + \dfrac{\sin x}{\cos x}\right) dx = \int e^x\ (\sec^2 x + \tan x)\ dx$

$= \int e^x\ (\tan x + \sec^2 x)\ dx = \int e^x\ (f(x) + f'(x))\ dx$

where $f(x) = \tan x$ and $f'(x) = \sec^2 x$

$= e^x f(x) + c = e^x \tan x + c.$ $\left[\because \int e^x\ (f(x)+f'(x))\ dx = e^x f(x) + c\right]$

22. $\dfrac{\mathbf{x^2+x+1}}{\mathbf{(x+1)^2\ (x+2)}}$

Sol. Let I $= \int \dfrac{x^2+x+1}{(x+1)^2\ (x+2)}\ dx$ ...(*i*)

The integrand $\dfrac{x^2+x+1}{(x+1)^2\ (x+2)}$ is a rational function of x and degree of numerator is less than degree of denominator. So we can form partial fractions of integrand.

Let integrand $\dfrac{x^2+x+1}{(x+1)^2\ (x+2)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+2}$ ...(*ii*)

Multiplying both sides of (*ii*) L.C.M. $= (x+1)^2\ (x+2)$, we have

$x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$

or $x^2 + x + 1 = A(x^2 + 3x + 2) + B(x+2) + C(x^2 + 1 + 2x)$

$= Ax^2 + 3Ax + 2A + Bx + 2B + Cx^2 + C + 2Cx$

Comparing coefficients of x^2, x and constant terms on both sides, we have

x^2: $A + C = 1$ ...(*iii*)

x: $3A + B + 2C = 1$ ...(*iv*)

Constant terms: $2A + 2B + C = 1$ ...(*v*)

Let us solve Eqns. (*iii*), (*iv*) and (*v*) for A, B, C.

Eqn. (*iv*) – 2 × Eqn. (*iii*) gives (to eliminate C)

$3A + B + 2C - 2A - 2C = 1 - 2$

or $A + B = -1$ ...(*vi*)

Eqn. (*v*) – Eqn. (*iii*) gives (To eliminate C)

$A + 2B = 0$ ...(*vii*)

Eqn. (*vii*) – Eqn. (*vi*) gives $B = 0 + 1 = 1.$

Putting B = 1 in (*vi*), $A + 1 = -1 \Rightarrow A = -2$

Putting A = – 2 in (*iii*), $-2 + C = 1 \Rightarrow C = 3$

Putting values of A, B, C in (*ii*)

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}$$

$$\therefore \int \frac{x^2+x+1}{(x+1)^2(x+2)}\, dx$$

$$= -2 \int \frac{1}{x+1}\, dx + \int (x+1)^{-2}\, dx + 3 \int \frac{1}{x+2}\, dx$$

$$= -2 \log|x+1| + \frac{(x+1)^{-2+1}}{-2+1} + 3 \log|x+2| + c$$

$$= -2 \log|x+1| - \frac{1}{x+1} + 3 \log|x+2| + c \left(\because \frac{(x+1)^{-1}}{-1} = \frac{-1}{x+1}\right)$$

Evaluate the integrals in Exercises 23 and 24:

23. $\tan^{-1} \sqrt{\dfrac{1-x}{1+x}}$

Sol. Let $I = \int \tan^{-1} \sqrt{\dfrac{1-x}{1+x}}\, dx$ ...(*i*)

Put $x = \cos 2\theta$ $\Rightarrow \dfrac{dx}{d\theta} = -2 \sin 2\theta$

$\Rightarrow dx = -2 \sin 2\theta\, d\theta$

and $\tan^{-1} \sqrt{\dfrac{1-x}{1+x}} = \tan^{-1} \sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}} = \tan^{-1} \sqrt{\dfrac{2\sin^2\theta}{2\cos^2\theta}}$

$= \tan^{-1} \sqrt{\tan^2\theta} = \tan^{-1} \tan\theta = \theta$

$\therefore$ From (*i*), $I = \int \theta(-2 \sin 2\theta\, d\theta) = -2 \int \underset{\text{I}}{\theta}\, \underset{\text{II}}{\sin 2\theta}\, d\theta$

Applying Product Rule of Integration,

$$\left(\int \text{I}.\text{II}\, dx = \text{I} \int \text{II}\, dx - \int \left(\frac{d}{dx}(\text{I}) \int \text{II}\, dx\right) dx\right)$$

$$I = -2 \left[\theta\left(\frac{-\cos 2\theta}{2}\right) - \int 1\left(\frac{-\cos 2\theta}{2}\right) d\theta\right]$$

$$= -2\left[\frac{-1}{2}\theta \cos 2\theta + \frac{1}{2}\int \cos 2\theta\, d\theta\right] = \theta \cos 2\theta - \frac{\sin 2\theta}{2} + c$$

$$= \theta \cos 2\theta - \frac{1}{2}\sqrt{1-\cos^2 2\theta} + c (\because \sin^2\alpha + \cos^2\alpha = 1)$$

$$= \frac{1}{2}(\cos^{-1} x)\, x - \frac{1}{2}\sqrt{1-x^2} + c$$

$$\left[\because \cos 2\theta = x \Rightarrow 2\theta = \cos^{-1} x \Rightarrow \theta = \frac{1}{2}\cos^{-1} x\right]$$

$$= \frac{1}{2} x \cos^{-1} x - \frac{1}{2}\sqrt{1-x^2} + c$$

$$= \frac{1}{2}[x \cos^{-1} x - \sqrt{1-x^2}] + c.$$

24. $\dfrac{\sqrt{x^2+1}\,[\log(x^2+1)-2\log x]}{x^4}$

Sol. $I = \int \dfrac{\sqrt{x^2+1}\,[\log(x^2+1)-2\log x]}{x^4}\,dx$

$$= \int \frac{\sqrt{x^2+1}}{x^4}\,[\log(x^2+1)-\log x^2]\,dx$$

$$= \int \frac{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x^4}\log\left(\frac{x^2+1}{x^2}\right)dx$$

$$= \int \frac{\sqrt{1+\frac{1}{x^2}}}{x^3}\log\left(1+\frac{1}{x^2}\right)dx = \int \sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right).\frac{dx}{x^3}$$

Put $1+\dfrac{1}{x^2}=t$ or $1+x^{-2}=t.$

Differentiating both sides, $-\dfrac{2}{x^3}\,dx = dt$ or $\dfrac{dx}{x^3} = -\dfrac{1}{2}\,dt$

$$\therefore\quad I = -\frac{1}{2}\int \sqrt{t}\,\log t\,dt = -\frac{1}{2}\int \underset{\text{I}}{(\log t)}\,.\,\underset{\text{II}}{t^{1/2}}\,dt$$

Integrating by Product Rule,

$$= -\frac{1}{2}\left[(\log t).\frac{t^{3/2}}{3/2} - \int \frac{1}{t}.\frac{t^{3/2}}{3/2}\,dt\right] = -\frac{1}{3}t^{3/2}\log t + \frac{1}{3}\int t^{1/2}\,dt$$

$$= -\frac{1}{3}t^{3/2}\log t + \frac{1}{3}.\frac{t^{3/2}}{3/2} + c$$

$$= \frac{2}{9}t^{3/2} - \frac{1}{3}t^{3/2}\log t + c = \frac{1}{3}t^{3/2}\left[\frac{2}{3}-\log t\right] + c$$

Putting $t = 1+\dfrac{1}{x^2}$, we have $= \dfrac{1}{3}\left(1+\dfrac{1}{x^2}\right)^{3/2}\left[\dfrac{2}{3}-\log\left(1+\dfrac{1}{x^2}\right)\right]+c.$

Evaluate the definite integrals in Exercises 25 to 33:

25. $\displaystyle\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

Sol. Let $I = \displaystyle\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx = \int_{\frac{\pi}{2}}^{\pi} e^x\left[\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right]dx$

$$=\int_{\frac{\pi}{2}}^{\pi} e^x\left[\frac{1}{2\sin^2\frac{x}{2}} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right]dx = \int_{\frac{\pi}{2}}^{\pi} e^x\left[\frac{1}{2}\text{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right]dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left[-\cot\frac{x}{2} + \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right] dx \qquad = \int_{\frac{\pi}{2}}^{\pi} e^x\,(f(x) + f'(x))\;dx$$

where $f(x) = -\cot\dfrac{x}{2}$. Therefore $f'(x) = \dfrac{1}{2}\operatorname{cosec}^2\dfrac{x}{2}$

$$= \left(e^x f(x)\right)_{\frac{\pi}{2}}^{\pi} = \left(-e^x\cot\frac{x}{2}\right)_{\frac{\pi}{2}}^{\pi} \quad \left[\because \int e^x\,(f(x)+f'(x))\,dx = e^x f(x)\right]$$

$$= -e^{\pi}\cot\frac{\pi}{2} - \left(-e^{\frac{\pi}{2}}\cot\frac{\pi}{4}\right)$$

$$= -e^{\pi}\;(0) + e^{\pi/2}\;(1) \qquad \left[\because \cot\frac{\pi}{2} = \frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}} = \frac{0}{1} = 0\right]$$

$= e^{\pi/2}$.

26. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x\cos x}{\cos^4 x + \sin^4 x}\,dx$

Sol. Let $\mathrm{I} = \displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x\cos x}{\cos^4 x + \sin^4 x}\,dx$

Dividing every term by $\cos^4 x$,

$$\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{\dfrac{\sin x\cos x}{\cos x\,.\,\cos x\,.\,\cos^2 x}}{1 + \dfrac{\sin^4 x}{\cos^4 x}}\,dx = \int_0^{\frac{\pi}{4}} \frac{\tan x\sec^2 x}{1+\tan^4 x}\,dx$$

Dividing and multiplying by 2,

$$\mathrm{I} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{2\tan x\sec^2 x}{1+\tan^4 x}\,dx \qquad ...(i)$$

Put $\tan^2 x = t$.

$\therefore\quad 2\tan x\,\dfrac{d}{dx}(\tan x) = \dfrac{dt}{dx} \quad\Rightarrow\quad 2\tan x\sec^2 x\,dx = dt.$

To change the limits of integration

When $x = 0$, $t = \tan^2 x = \tan^2 0 = 0$

When $x = \dfrac{\pi}{4}$, $t = \tan^2\dfrac{\pi}{4} = 1$

$$\therefore\quad \text{From } (i),\ \mathrm{I} = \frac{1}{2}\int_0^1 \frac{dt}{1+t^2} = \frac{1}{2}\left(\tan^{-1} t\right)_0^1$$

$$= \frac{1}{2}(\tan^{-1} 1 - \tan^{-1} 0) = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}. \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

27. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\, dx}{\cos^2 x + 4\sin^2 x}$

Sol. Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\, dx$

Dividing every term of integrand by $\cos^2 x$,

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{(1+4\tan^2 x)}\, dx \qquad ...(i)$$

Put $\tan x = t$.

$\therefore \quad \sec^2 x = \dfrac{dt}{dx} \quad \Rightarrow \quad \sec^2 x\, dx = dt$

$\Rightarrow \quad dx = \dfrac{dt}{\sec^2 x} = \dfrac{dt}{1+\tan^2 x} = \dfrac{dt}{1+t^2}$

To change the limits:

When $x = 0$, $t = \tan 0 = 0$

When $x = \dfrac{\pi}{2}$, $t = \tan \dfrac{\pi}{2} = \infty$

$\therefore$ From (i), $I = \displaystyle\int_0^{\infty} \frac{1}{1+4t^2} \frac{dt}{1+t^2}$

$$= \int_0^{\infty} \frac{1}{(4t^2+1)(t^2+1)}\, dt \qquad ...(ii)$$

Put $t^2 = y$ only in the integrand of (ii) to form partial fractions.

The new integrand is $\dfrac{1}{(4y+1)(y+1)}$

Let $\dfrac{1}{(4y+1)(y+1)} = \dfrac{A}{4y+1} + \dfrac{B}{y+1} \qquad ...(iii)$

Multiplying by L.C.M. $= (4y + 1)(y + 1)$

$1 = A(y + 1) + B(4y + 1)$

or $1 = Ay + A + 4By + B$

Comparing coefficient of y on both sides, $A + 4B = 0 \qquad ...(iv)$

Comparing constants, $A + B = 1 \qquad ...(v)$

$(iv) - (v)$ gives $3B = -1 \Rightarrow B = -\dfrac{1}{3}$

$\therefore$ From (iv) $A = -4B = -4\left(-\dfrac{1}{3}\right) = \dfrac{4}{3}$

Putting values of A, B and y in (iii), we have

$$\frac{1}{(4t^2+1)(t^2+1)} = \frac{\frac{4}{3}}{4t^2+1} - \frac{\frac{1}{3}}{t^2+1} = \frac{1}{3}\left(\frac{4}{(4t^2+1)} - \frac{1}{(t^2+1)}\right)$$

Putting this value in (ii)

$$I = \frac{1}{3}\left[4\int_0^{\infty}\frac{1}{(4t^2+1)}\,dt - \int_0^{\infty}\frac{1}{t^2+1}\,dt\right]$$

$$= \frac{1}{3}\left[4\int_0^{\infty}\frac{1}{(2t)^2+1^2}\,dt - \left(\tan^{-1}t\right)_0^{\infty}\right]$$

$$= \frac{1}{3}\left[4\,\frac{\left(\frac{1}{1}\tan^{-1}\frac{2t}{1}\right)_0^{\infty}}{2 \to \text{Coeff. of } t} - \left(\tan^{-1}t\right)_0^{\infty}\right]$$

$$= \frac{1}{3}\,[2\,(\tan^{-1}\infty - \tan^{-1}0) - (\tan^{-1}\infty - \tan^{-1}0)]$$

$$= \frac{1}{3}\left[2.\left(\frac{\pi}{2}-0\right)-\left(\frac{\pi}{2}-0\right)\right] = \frac{1}{3}\left(\frac{2\pi}{2}-\frac{\pi}{2}\right) = \frac{1}{3}\times\frac{\pi}{2} = \frac{\pi}{6}.$$

28. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin x+\cos x}{\sqrt{\sin 2x}}\,dx$

Sol. Let $I = \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin x+\cos x}{\sqrt{\sin 2x}}\,dx$ ...(*i*)

Put $\sin x - \cos x = t$. Differentiating both sides w.r.t.x,

$(\cos x + \sin x)\,dx = dt$

Also, squaring $\sin^2 x + \cos^2 x - 2\sin x\cos x = t^2$

$\Rightarrow\ 1 - \sin 2x = t^2 \qquad \Rightarrow\ \sin 2x = 1 - t^2$

To change the limits of Integration

When $x = \frac{\pi}{6}$, $t = \sin\frac{\pi}{6} - \cos\frac{\pi}{6}$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2} = \frac{-(\sqrt{3}-1)}{2} = -\alpha \text{ (say)}$$

where $\alpha = \dfrac{\sqrt{3}-1}{2}$ ...(*ii*)

When $x = \frac{\pi}{3}$, $t = \sin\frac{\pi}{3} - \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2} = \alpha$

$\therefore$ From (*i*), $I = \displaystyle\int_{-\alpha}^{\alpha}\frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1}t\right]_{-\alpha}^{\alpha}$

$= \sin^{-1}\alpha - \sin^{-1}(-\alpha)$

$= \sin^{-1}\alpha + \sin^{-1}\alpha = 2\sin^{-1}\left(\dfrac{\sqrt{3}-1}{2}\right)$. [By (*ii*)]

29. $\displaystyle\int_0^1\frac{dx}{\sqrt{1+x}-\sqrt{x}}$

Sol. Let $I = \displaystyle\int_0^1\frac{1}{\sqrt{1+x}-\sqrt{x}}\,dx$

Rationalising $= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{(\sqrt{1+x}+\sqrt{x})(\sqrt{1+x}-\sqrt{x})}\, dx$

$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x}\, dx \quad = \int_0^1 (\sqrt{1+x}+\sqrt{x})\, dx \quad (\because \; 1 + x - x = 1)$

$= \int_0^1 (1+x)^{1/2}\, dx + \int_0^1 x^{1/2}\, dx \qquad = \frac{\left((1+x)^{\frac{3}{2}}\right)_0^1}{\frac{3}{2}(1)} + \frac{\left(x^{\frac{3}{2}}\right)_0^1}{\frac{3}{2}}$

$= \frac{2}{3}\,[(2)^{3/2} - (1)^{3/2}] + \frac{2}{3}\,[(1)^{3/2} - 0] = \frac{2}{3}\,(2\sqrt{2} - 1) + \frac{2}{3}\,(1 - 0)$

$\left[\because \; x^{\frac{3}{2}} = x^{\frac{2+1}{2}} = x^{1+\frac{1}{2}} = x^1 . x^{\frac{1}{2}} = x\sqrt{x}\right]$

$= \frac{4\sqrt{2}}{3} - \frac{2}{3} + \frac{2}{3} = \frac{4\sqrt{2}}{3}.$

30. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x}\, dx$

Sol. Let I $= \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x}\, dx$

Put $\sin x - \cos x = t$. Differentiating both sides

$(\cos x + \sin x)\, dx = dt$

Also $(\sin x - \cos x)^2 = t^2 \quad \therefore \sin^2 x + \cos^2 x - 2 \sin x \cos x = t^2$

or $\quad 1 - t^2 = \sin 2x$

Let us change the limits of Integration

When $x = 0$, $t = 0 - 1 = -1$

When $x = \frac{\pi}{4}$, $t = \sin \frac{\pi}{4} - \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

$\therefore \quad I = \int_{-1}^0 \frac{dt}{9 + 16(1 - t^2)} = \int_{-1}^0 \frac{dt}{25 - 16t^2}$

$= \int_{-1}^0 \frac{dt}{16\left(\frac{25}{16} - t^2\right)} = \frac{1}{16} \int_{-1}^0 \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$

$= \frac{1}{16} \times \left[\frac{1}{2 \times 5/4} \log\left|\frac{5/4 + t}{5/4 - t}\right|\right]_{-1}^0 \quad \left[\because \int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right|\right]$

$= \frac{1}{40}\left[\log 1 - \log \frac{1/4}{9/4}\right] = \frac{1}{40}\left[0 - \log \frac{1}{9}\right]$

$= \frac{1}{40}\,[-(\log 1 - \log 9)] = \frac{1}{40} \log 9$

$= \frac{1}{40} \log 3^2 = \frac{2}{40} \log 3 = \frac{1}{20} \log 3.$

31. $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1} (\sin x)\, dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1} (\sin x)\, dx = \int_0^{\frac{\pi}{2}} 2 \sin x \cos x \tan^{-1} (\sin x)\, dx$

Put $\sin x = t$. Differentiating both sides $\cos x\, dx = dt$

To change the limits of Integration

When $x = 0,\ t = 0$

When $x = \frac{\pi}{2},\ t = \sin \frac{\pi}{2} = 1 \quad \therefore \quad I = 2 \int_0^1 t \tan^{-1} t\, dt$ ...(*i*)

Now $\int t \tan^{-1} t\, dt = \int \underset{\text{I}}{(\tan^{-1} t)}\ \underset{\text{II}}{t}\, dt$ Integrating by parts

$= \tan^{-1} t \cdot \frac{t^2}{2} - \int \frac{1}{1+t^2} \cdot \frac{t^2}{2}\, dt$

$= \frac{t^2}{2} \tan^{-1} t - \frac{1}{2} \int \frac{(1+t^2)-1}{1+t^2}\, dt$

$= \frac{t^2}{2} \tan^{-1} t - \frac{1}{2} \int \left(1 - \frac{1}{1+t^2}\right) dt = \frac{t^2}{2} \tan^{-1} t - \frac{1}{2} (t - \tan^{-1} t)$

$= \frac{t^2}{2} \tan^{-1} t - \frac{1}{2} t + \frac{1}{2} \tan^{-1} t + c = \frac{1}{2} [(t^2 + 1) \tan^{-1} t - t]$

From (*i*), $I = 2 \left[\frac{1}{2} \{(t^2+1) \tan^{-1} t - t\}\right]_0^1 = (2 \tan^{-1} 1 - 1) - (0 - 0)$

$= 2 \times \frac{\pi}{4} - 1 = \frac{\pi}{2} - 1.$

32. $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$

Sol. Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx = \int_0^{\pi} \frac{x \frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}\, dx$

$= \int_0^{\pi} \frac{x \sin x}{1 + \sin x}\, dx$ ...(*i*)

Using $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x)}{1 + \sin (\pi - x)}\, dx = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \sin x}\, dx$ ...(*ii*)

Adding Eqns. (*i*) and (*ii*), we have

$2I = \int_0^{\pi} \frac{x \sin x + (\pi - x) \sin x}{1 + \sin x} dx = \int_0^{\pi} \frac{x \sin x + \pi \sin x - x \sin x}{1 + \sin x} dx$

$$= \int_0^{\pi} \frac{\pi \sin x}{1+\sin x}\, dx = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x}\, dx$$

or $\quad 2I = \pi \int_0^{\pi} \frac{(1+\sin x)-1}{1+\sin x}\, dx$

$\Rightarrow \quad 2I = \pi \int_0^{\pi} \left(1 - \frac{1}{1+\sin x}\right) dx = \pi \int_0^{\pi} dx - \pi \int_0^{\pi} \frac{dx}{1+\sin x}$

$$= \pi \Big[x\Big]_0^{\pi} - 2\pi \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$$

$$\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x)\right]$$

$$= \pi(\pi) - 2\pi \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin\left(\frac{\pi}{2}-x\right)} = \pi^2 - 2\pi \int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$$

$$= \pi^2 - 2\pi \int_0^{\frac{\pi}{2}} \frac{dx}{2\cos^2 \frac{x}{2}} = \pi^2 - \pi \int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2}\, dx$$

or $\quad 2I = \pi^2 - \pi \left[\dfrac{\tan \frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2} = \pi^2 - 2\pi(1)$

Dividing both sides by 2, $I = \dfrac{\pi^2}{2} - \pi = \pi\left(\dfrac{\pi}{2} - 1\right) = \pi\left(\dfrac{\pi-2}{2}\right)$.

33. $\int_1^4 [\,|x-1| + |x-2| + |x-3|\,]\, dx$

Sol. Let $I = \int_1^4 (|x-1| + |x-2| + |x-3|)\,dx$ ...(*i*)

Putting each expression within modulus equal to 0, we have

$x - 1 = 0,\ x - 2 = 0,\ x - 3 = 0$ *i.e.,* $x = 1,\ x = 2,\ x = 3$

Here 2 and 3 $\in$ (1, 4)

$\therefore$ From (*i*), $I = \int_1^2 (|x-1| + |x-2| + |x-3|)\,dx$

$+ \int_2^3 (|x-1| + |x-2| + |x-3|)\, dx + \int_3^4 (|x-1| + |x-2| + |x-3|)\,dx$ ...(*ii*)

$$\left[\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^b f(x)\,dx \quad \text{where } a < c < d < b\right]$$

Let $I_1 = \int_1^2 (|x-1| + |x-2| + |x-3|)\, dx$

On this interval (1, 2) (for example taking $x = 1.3$; $(x - 1)$ is positive, $(x - 2)$ is negative and $(x - 3)$ is negative and hence $|x - 1| = (x - 1)$, $|x - 2| = -(x - 2)$ and $|x - 3| = -(x - 3)$).

Therefore $I_1 = \int_1^2 ((x-1)-(x-2)-(x-3))\ dx$

$= \int_1^2 (x-1-x+2-x+3)\ dx = \int_1^2 (4-x)\ dx$

$= \left(4x - \frac{x^2}{2}\right)_1^2 = (8 - 2) - \left(4 - \frac{1}{2}\right)$

$= 6 - 4 + \frac{1}{2} = 2 + \frac{1}{2} = \frac{5}{2}$ ...(*iii*)

Let $I_2 = \int_2^3 (|x-1|+|x-2|+|x-3|)\ dx$

On this interval (2, 3) (for example taking x = 2.8; $(x - 1)$ is positive, $(x - 2)$ is positive and $(x - 3)$ is negative and hence $|x - 1| = x - 1$, $|x - 2| = x - 2$ and $|x - 3| = -(x - 3))$

Therefore $I_2 = \int_2^3 ((x-1+x-2-(x-3))\,dx = \int_2^3 (2x-3-x+3)\,dx$

$= \int_2^3 x\ dx = \left(\frac{x^2}{2}\right)_2^3 = \frac{9}{2} - \frac{4}{2} = \frac{5}{2}$ ...(*iv*)

Let $I_3 = \int_3^4 (|x-1|+|x-2|+|x-3|)\ dx$

On this interval (3, 4), (for example taking x = 3.4; $(x - 1)$ is positive, $(x - 2)$ is positive and $(x - 3)$ is positive and hence $|x - 1| = x - 1$, $|x - 2| = x - 2$ and $|x - 3| = x - 3))$

Therefore $I_3 = \int_3^4 (x-1+x-2+x-3)\ dx = \int_3^4 (3x-6)\ dx$

$= \left(\frac{3x^2}{2} - 6x\right)_3^4 = (24 - 24) - \left(\frac{27}{2} - 18\right)$

$= 0 - \left(\frac{27-36}{2}\right) = -\left(-\frac{9}{2}\right) = \frac{9}{2}$ ...(*v*)

Putting values of I_1, I_2, I_3 from (*iii*), (*iv*) and (*v*) in (*ii*),

$I = \frac{5}{2} + \frac{5}{2} + \frac{9}{2} = \frac{19}{2}$.

Prove the following (Exercises 34 to 40):

34. $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$

Sol. Let $I = \int_1^3 \frac{dx}{x^2(x+1)} = \int_1^3 \frac{1}{x^2(x+1)}\ dx$ ...(*i*)

Let integrand $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$ ...(*ii*)

(Partial fractions)

Multiplying by L.C.M. $= x^2(x + 1)$

$1 = Ax(x + 1) + B(x + 1) + Cx^2$

$\Rightarrow \quad 1 = Ax^2 + Ax + Bx + B + Cx^2$

Comparing coefficients of x^2, x and constant terms on both sides, we have

x^2: $\quad A + C = 0 \qquad ...(iii)$

x: $\quad A + B = 0 \qquad ...(iv)$

Constants: $\quad B = 1 \qquad ...(v)$

Let us solve (iii), (iv), (v) for A, B, C.

Putting B = 1 from (v) in (iv), A + 1 = 0 or A = – 1

Putting A = – 1 in (iii), – 1 + C = 0 $\Rightarrow$ C = 1

Putting values of A, B, C in (ii),

$$\frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

$\therefore$ From (i), $I = \int_1^3 \frac{dx}{x^2(x+1)}$

$$= -\int_1^3 \frac{1}{x}\,dx + \int_1^3 \frac{1}{x^2}\,dx + \int_1^3 \frac{1}{x+1}\,dx$$

$$= -\left(\log|x|\right)_1^3 + \int_1^3 x^{-2}\,dx + \left(\log|x+1|\right)_1^3$$

$$= -(\log|3| - \log|1|) + \left(\frac{x^{-1}}{-1}\right)_1^3 + (\log|4| - \log|2|)$$

$$= -\log 3 + 0 - \left(\frac{1}{x}\right)_1^3 + \log 4 - \log 2$$

$$= -\log 3 - \left(\frac{1}{3} - 1\right) + \log 2^2 - \log 2$$

$$= -\log 3 - \left(\frac{1-3}{3}\right) + 2\log 2 - \log 2$$

$$= -\log 3 + \frac{2}{3} + \log 2 = \frac{2}{3} + \log 2 - \log 3$$

$$= \frac{2}{3} + \log\frac{2}{3}.$$

35. $\int_0^1 x\,e^x\,dx = 1$

Sol. $\int_0^1 x\,e^x$

I II

Applying Product Rule of definite Integration

$$\left(\int \mathrm{I}\,.\,\mathrm{II}\,dx = \left(\mathrm{I}\int \mathrm{II}\,dx\right)_a^b - \int_a^b\left(\frac{d}{dx}(\mathrm{I})\int \mathrm{II}\,dx\right)dx\right)$$

$$= \left(x\,e^x\right)_0^1 - \int_0^1 1\,.\,e^x\,dx$$

$$= e - 0 - \int_0^1 e^x\,dx = e - \left(e^x\right)_0^1 = e - (e - e^0) = e - e + e^0 = 1.$$

36. $\int_{-1}^{1} x^{17} \cos^4 x \, dx = 0$

Sol. Let $I = \int_{-1}^{1} x^{17} \cos^4 x \; dx$ ...(*i*)

Here the integrand $f(x) = x^{17} \cos^4 x$

$\therefore \quad f(-x) = (-x)^{17} \cos^4 (-x)$

$\qquad\qquad = -x^{17} \cos^4 x = -f(x)$

$\therefore$ $f(x)$ is an odd function of x.

$\therefore$ From (*i*), $I = \int_{-1}^{1} x^{17} \cos^4 x \; dx = 0$

$\left[\because \text{ If } f(x) \text{ is an odd function of } x, \text{ then } \int_{-a}^{a} f(x) \; dx = 0\right]$

37. $\int_0^{\frac{\pi}{2}} \sin^3 x \, dx = \frac{2}{3}$

Sol. $\int_0^{\frac{\pi}{2}} \sin^3 x \; dx = \int_0^{\frac{\pi}{2}} \frac{1}{4}(3 \sin x - \sin 3x) \; dx$

$$\left[\because \ \sin 3A = 3 \sin A - 4 \sin^3 A \quad \Rightarrow \ \sin^3 A = \frac{1}{4}(3 \sin A - \sin 3A)\right]$$

$$= \frac{1}{4}\left[3(-\cos x) - \left(-\frac{\cos 3x}{3}\right)\right]_0^{\pi/2} = \frac{1}{4}\left(-3 \cos x + \frac{1}{3}\cos 3x\right)_0^{\pi/2}$$

$$= \frac{1}{4}\left[\left(-3 \cos \frac{\pi}{2} + \frac{1}{3}\cos\frac{3\pi}{2}\right) - \left(-3 \cos 0 + \frac{1}{3}\cos 0\right)\right]$$

$$= \frac{1}{4}\left[-3 \times 0 + \frac{1}{3} \times 0 + 3 \times 1 - \frac{1}{3} \times 1\right] = \frac{1}{4}\left(3 - \frac{1}{3}\right)$$

$$= \frac{1}{4} \times \frac{8}{3} = \frac{2}{3}.$$

$$\left[\because \ \cos\frac{3\pi}{2} = \cos 270° = \cos(180° + 90°) = -\cos 90° = 0\right]$$

38. $\int_0^{\frac{\pi}{4}} 2 \tan^3 x \, dx = 1 - \log 2$

Sol. Let $I = \int_0^{\frac{\pi}{4}} 2 \tan^3 x \; dx = 2 \int_0^{\frac{\pi}{4}} \tan x \,.\, \tan^2 x \, dx$

Replacing $\tan^2 x$ by $(\sec^2 x - 1)$ in the integrand,

$$I = 2 \int_0^{\frac{\pi}{4}} \tan x(\sec^2 x - 1) \, dx = 2\left[\int_0^{\frac{\pi}{4}} (\tan x \sec^2 x - \tan x) \, dx\right]$$

$$= 2\left[\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan x \, dx\right] \qquad ...(i)$$

Let $I_1 = \int_0^{\frac{\pi}{4}} \tan x \sec^2 x \; dx$

Put $\tan x = t$. Therefore $\sec^2 x = \frac{dt}{dx}$ $\therefore$ $\sec^2 x\, dx = dt$

To change the limits of Integration

When $x = 0$, $t = \tan x = \tan 0 = 0$

When $x = \frac{\pi}{4}$, $t = \tan \frac{\pi}{4} = 1$

$$\therefore \quad I_1 = \int_0^1 t \; dt = \left(\frac{t^2}{2}\right)_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

Putting this value of I_1 in (i),

$$I = 2\left[\frac{1}{2} - \left(\log|\sec x|\right)_0^{\pi/4}\right] = 1 - 2\left(\log\sec\frac{\pi}{4} - \log\sec 0\right)$$

$$= 1 - 2\,(\log\sqrt{2} - \log 1) = 1 - 2\,(\log 2^{1/2} - 0)$$

$$= 1 - 2\left(\frac{1}{2}\log 2\right) = 1 - \log 2.$$

39. $\int_0^1 \sin^{-1} x\, dx = \frac{\pi}{2} - 1$

Sol. Put $x = \sin\theta$. Differentiating both sides $dx = \cos\theta\, d\theta$

To change the limits of Integration

When $x = 0$, $\theta = 0$,

When $x = 1$, $\sin\theta = 1$ and therefore $\theta = \frac{\pi}{2}$

$$\therefore \quad \int_0^1 \sin^{-1} x \; dx = \int_0^{\frac{\pi}{2}} \underset{\text{I}}{\theta}\, \underset{\text{II}}{\cos\theta}\; d\theta$$

Integrating by parts

$$= \Big[\theta\sin\theta\Big]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} 1\,.\sin\theta\, d\theta = \left(\frac{\pi}{2} - 0\right) + \Big[\cos\theta\Big]_0^{\pi/2}$$

$$= \frac{\pi}{2} + \left(\cos\frac{\pi}{2} - \cos 0\right) = \frac{\pi}{2} + (0 - 1) = \frac{\pi}{2} - 1.$$

40. Evaluate $\int_0^1 e^{2-3x}\, dx$ as a limit of a sum.

Sol. Step I. Comparing $\int_0^1 e^{2-3x} \; dx$ with $\int_a^b f(x) \; dx$, we have

$a = 0,\ b = 1,\ f(x) = e^{2-3x}$

$\therefore\ nh = b - a = 1$

Step II. Putting $x = a,\ a + h,\ a + 2h,\ a + (n - 1)\,h$ in $f(x)$, we have

$$f(a) = f(0) = e^2$$
$$f(a + h) = f(h) = e^{2-3h}$$
$$f(a + 2h) = f(2h) = e^{2-6h}$$
$$f(a + (n - 1)h) = f((n - 1)\,h) = e^{2-3(n-1)\,h}$$

Step III. Putting these values in

$$\int_a^b f(x) = \lim_{\substack{n \to \infty \\ h \to 0}} h[f(a) + f(a+h) + f(a+2h) + \ldots + f\{a + (n-1)h\}],$$

we have $\int_0^1 e^{2-3x}\, dx = \lim\limits_{\substack{n \to \infty \\ h \to 0}} h[e^2 + e^{2-3h} + e^{2-6h} + \ldots + e^{2-3(n-1)h}]$

$$= \lim_{\substack{h \to 0 \\ n \to \infty}} h \,.\, e^2 [1 + e^{-3h} + e^{-6h} + \ldots + e^{-3(n-1)h}]$$

$\left[\textbf{The series within brackets is a G.P. series of } \boldsymbol{n} \textbf{ terms with } \boldsymbol{a = 1, r = e^{-3h}} \textbf{ and using } \mathbf{S}_{\boldsymbol{n}} \textbf{ of G.P.} = \boldsymbol{a \frac{(r^n - 1)}{r - 1}}\right]$

$$= e^2 \lim_{\substack{h \to 0 \\ n \to \infty}} h \,.\, \left[\frac{e^{-3nh} - 1}{e^{-3h} - 1}\right] \qquad \left[\because (e^{-3h})^n = e^{-3nh}\right]$$

Step IV. Putting $nh = 1$

$$= e^2 \lim_{h \to 0} h \,.\, \left[\frac{e^{-3} - 1}{e^{-3h} - 1}\right]$$

Step V. Taking limits as $h \to 0$,

$$= e^2 (e^{-3} - 1) \lim_{h \to 0} \frac{-3h}{e^{-3h} - 1} \times \left(-\frac{1}{3}\right)$$

$$= (e^{-1} - e^2) \times 1 \times \left(-\frac{1}{3}\right) \qquad \left[\because \lim_{x \to 0} \frac{x}{e^x - 1} = 1\right]$$

$$= \frac{1}{3}\left(e^2 - \frac{1}{e}\right)$$

41. Choose the correct answer: $\boldsymbol{\int \frac{dx}{e^x + e^{-x}}}$ **is equal to**

(A) $\boldsymbol{\tan^{-1}(e^x) + c}$ **(B)** $\boldsymbol{\tan^{-1}(e^{-x}) + c}$
(C) $\boldsymbol{\log(e^x - e^{-x}) + c}$ **(D)** $\boldsymbol{\log(e^x + e^{-x}) + c}$

Sol. Let $I = \int \frac{dx}{e^x + e^{-x}} = \int \frac{1}{e^x + \left(\frac{1}{e^x}\right)} dx$

$$= \int \frac{1}{\left(\frac{e^{2x} + 1}{e^x}\right)} dx = \int \frac{e^x}{e^{2x} + 1} dx \qquad \ldots(i)$$

$$[\because e^x \,.\, e^x = e^{x+x} = e^{2x}]$$

Put $e^x = t$. $\left[\because \text{ For } \int f(e^x)\,dx, \text{ put } e^x = t\right]$

Therefore $e^x = \dfrac{dt}{dx}$. Therefore $e^x\,dx = dt$

$\therefore$ From (i), $\text{I} = \int \dfrac{dt}{t^2+1} = \tan^{-1} t + c$

$= \tan^{-1}(e^x) + c$

$\therefore$ Option (A) is the correct answer.

42. Choose the correct answer:

$$\int \frac{\cos 2x}{(\sin x + \cos x)^2}\,dx \text{ is equal to}$$

(A) $\dfrac{-1}{\sin x + \cos x} + c$ **(B)** $\log|\sin x + \cos x| + c$

(C) $\log|\sin x - \cos x| + c$ **(D)** $\dfrac{1}{(\sin x + \cos x)^2}$.

Sol. Let $\text{I} = \int \dfrac{\cos 2x}{(\sin x + \cos x)^2}\,dx = \int \dfrac{\cos^2 x - \sin^2 x}{(\sin x + \cos x)^2}\,dx$

$= \int \dfrac{(\cos x + \sin x)(\cos x - \sin x)}{(\sin x + \cos x)(\sin x + \cos x)}\,dx = \int \dfrac{\cos x - \sin x}{\sin x + \cos x}\,dx$

$= \log|\sin x + \cos x| + c. \left[\because \int \dfrac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$

OR

Put denominator $\sin x + \cos x = t$.

$\therefore$ Option (B) is the correct answer.

43. Choose the correct answer:

If $f(a + b - x) = f(x)$, then $\int_a^b x\,f(x)\,dx$ is equal to

(A) $\dfrac{a+b}{2}\int_a^b f(b-x)\,dx$ **(B)** $\dfrac{a+b}{2}\int_a^b f(b+x)\,dx$

(C) $\dfrac{b-a}{2}\int_a^b f(x)\,dx$ **(D)** $\dfrac{a+b}{2}\int_a^b f(x)\,dx$.

Sol. Given: $f(a + b - x) = f(x)$ ...(i)

Let $\text{I} = \int_a^b x\,f(x)\,dx$ ...(ii)

Changing x to $(a + b - x)$ in the Integrand on Right side (ii).

$\text{I} = \int_a^b (a+b-x)\,f(a+b-x)\,dx$ ...(iii)

$\left[\because \text{ By Property of definite integrals, } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$

Putting $f(a+b-x) = f(x)$ from (i) in integrand of (iii),

$$I = \int_a^b f(a+b-x)\, f(x)\, dx \qquad ...(iv)$$

Adding (ii) and (iv), we have $2I = \int_a^b [x\, f(x) + (a+b-x)\, f(x)]\, dx$

$$2I = \int_a^b (x+a+b-x)\, f(x)\, dx = \int_a^b (a+b)\, f(x)\, dx = (a+b)\int_a^b f(x)\, dx$$

Dividing by 2, $I = \left(\dfrac{a+b}{2}\right) \int_a^b f(x)\, dx$

or $\int_a^b x\, f(x)\, dx = \left(\dfrac{a+b}{2}\right) \int_a^b f(x)\, dx$

$\therefore$ Option (D) is the correct answer.

44. The value of $\int_0^1 \tan^{-1}\left(\dfrac{2x-1}{1+x-x^2}\right) dx$ is

(A) 1 **(B) 0** **(C) – 1** **(D) $\dfrac{\pi}{4}$**

Sol. Let $I = \int_0^1 \tan^{-1}\left(\dfrac{2x-1}{1+x-x^2}\right) dx = \int_0^1 \tan^{-1}\left(\dfrac{x+x-1}{1-x^2+x}\right) dx$

$$= \int_0^1 \tan^{-1}\left(\frac{x+(x-1)}{1-x(x-1)}\right) dx = \int_0^1 (\tan^{-1} x + \tan^{-1}(x-1))\, dx$$

$$\left[\because \tan^{-1}\frac{x+y}{1-xy} = \tan^{-1} x + \tan^{-1} y\right]$$

$$= \int_0^1 \tan^{-1} x\ dx + \int_0^1 \tan^{-1}(x-1)\ dx$$

Changing x to $(1-x)$ in integrand of second integral

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^1 \tan^{-1} x\ dx + \int_0^1 \tan^{-1}(1-x-1)\ dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(-x)\, dx = \int_0^1 \tan^{-1} x\ dx - \int_0^1 \tan^{-1} x\, dx$$

$$[\because \tan^{-1}(-x) = -\tan^{-1} x]$$

$= 0.$

$\therefore$ Option (B) is the correct answer.