Exercise 4.1

1. $\begin{vmatrix} \mathbf{2} & \mathbf{4} \\ \mathbf{-5} & \mathbf{-1} \end{vmatrix}$

Sol. Determinant $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 2(-1) - 4(-5)$

$= -2 + 20 = 18.$

2. (*i*) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (*ii*) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Sol. (*i*) Determinant $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos\theta\,(\cos\theta) - (-\sin\theta)\,(\sin\theta)$

$= \cos^2\theta + \sin^2\theta = 1.$

(*ii*) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = (x^2 - x + 1)(x + 1) - (x + 1)(x - 1)$

$= (x^3 + 1) - (x^2 - 1) = x^3 - x^2 + 2.$

3. If A = $\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that | 2A | = 4 | A |.

Sol. Given: Matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$

$\therefore \quad 2A = 2\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2\times1 & 2\times2 \\ 2\times4 & 2\times2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$

$\therefore \quad \text{L.H.S.} = |2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 2(4) - 4(8) = 8 - 32 = -24$ ...(*i*)

$\text{R.H.S.} = 4\,|A| = 4\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 4(2 - 8) = 4(-6) = -24$ ...(*ii*)

From (*i*) and (*ii*), we have L.H.S. = R.H.S.

4. If A = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that | 3A | = 27 | A |.

Sol. $3A = 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$

$\therefore$ L.H.S. $= |3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix}$

Expanding along first column $= 3[36 - 0] = 3 \times 36 = 108$.

Also R.H.S. $= 27 \,|A|\, = 27 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$

Expanding along first column

($\because$ There are two zeros in it)

$= 27 [1(4 - 0)] = 27 \times 4$

$= 108 =$ L.H.S. $\quad \therefore \quad |3A| = 27 \,|A|$.

5. Evaluate the determinants:

(*i*) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ **(*ii*)** $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(*iii*) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ **(*iv*)** $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Sol. (*i*) Given determinant is $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ and is of order 3.

Expanding along first row

$= 3\begin{vmatrix} 0 & -1 \\ -5 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 0 & 0 \\ 3 & -5 \end{vmatrix}$

$= 3(0 - 5) + 1(0 - (-3)) - 2(0 - 0)$

$= -15 + 3 - 0 \quad = -12.$

(*ii*) Given determinant is $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$ and is of order 3.

Expanding along first row

$$= 3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$= 3(1 + 6) + 4(1 - (-4)) + 5(3 - 2)$

$= 3(7) + 4(5) + 5(1) = 21 + 20 + 5 = 46.$

(*iii*) Given determinant is $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ and is of order 3.

Expanding along first row

$$= 0\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & -3 \\ -2 & 0 \end{vmatrix} + 2\begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix}$$

$= 0(0 + 9) - (0 - 6) + 2(-3 - 0) = 0 + 6 - 6 = 0.$

(*iv*) Given determinant is $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ and is of order 3.

Expanding along first row

$$= 2\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix}$$

$= 2(0 - 5) + (0 + 3) - 2(0 - 6) \quad = -10 + 3 + 12 = 5.$

6. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find | A |.

Sol. Matrix $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$ $\therefore$ Det A *i.e.,* $|A| = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$

Expanding along first row

$$= 1 \begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1 \begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$$

$= (-9 - (-12)) - (-18 - (-15)) - 2(8 - 5)$

$= -9 + 12 - (-18 + 15) - 2(3) = 3 - (-3) - 6$

$= 3 + 3 - 6 = 0$

Note. Such a matrix A for which $|A| = 0$ is called a **singular matrix.**

7. Find values of x, if

(*i*) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ **(*ii*)** $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Sol. (*i*) **Given:** $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$\Rightarrow \quad 2 - 20 = 2x^2 - 24 \quad \Rightarrow \quad -18 = 2x^2 - 24$

$\Rightarrow \quad -2x^2 = -24 + 18 = -6$

Dividing by -2, $x^2 = 3$

Taking square roots, $x = \pm\sqrt{3}$.

(*ii*) **Given:** $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

$\Rightarrow \quad 10 - 12 = 5x - 6x \quad \Rightarrow \quad -2 = -x$

Dividing by -1, $2 = x$ *i.e.,* $x = 2$.

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$**, then x is equal to**

(A) 6 (B) ± 6 (C) – 6 (D) 0.

Sol. Given: $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

$\Rightarrow \quad x^2 - 36 = 36 - 36 \quad \Rightarrow \quad x^2 - 36 = 0 \quad \Rightarrow \quad x^2 = 36$

Taking square roots, $x = \pm 6$. $\therefore$ Option (B) is the correct answer.

Exercise 4.2

Using the properties of determinants and without expanding in Exercises 1 to 5, prove that:

1. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0.$

Sol. On $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$, operate $C_1 \to C_1 + C_2$

$$= \begin{vmatrix} x+a & a & x+a \\ y+b & b & y+b \\ z+c & c & z+c \end{vmatrix} = 0. \quad (\because \ C_1 \text{ and } C_3 \text{ are identical})$$

2. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0.$

Sol. On $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$, operate $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix} = \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0.$$

($\because$ All entries of one column here first are zero)

Note: The reader can do the above problem by operating $R_1 \to R_1 + R_2 + R_3$ also.

3. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0.$

Sol. On $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$, operate $C_3 \to C_3 - C_1$ $= \begin{vmatrix} 2 & 7 & 63 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix}$

Taking 9 common from third column $= 9 \begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix} = 9(0) = 0.$

[Because two columns (one and three) are identical]

4. $\begin{vmatrix} \mathbf{1} & \boldsymbol{bc} & \boldsymbol{a(b+c)} \\ \mathbf{1} & \boldsymbol{ca} & \boldsymbol{b(c+a)} \\ \mathbf{1} & \boldsymbol{ab} & \boldsymbol{c(a+b)} \end{vmatrix} = \mathbf{0}$

Sol. The given determinant is $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = \begin{vmatrix} 1 & bc & ab+ac \\ 1 & ca & bc+ba \\ 1 & ab & ac+bc \end{vmatrix}$

Operate $C_3 \to C_3 + C_2$, $= \begin{vmatrix} 1 & bc & ab+bc+ac \\ 1 & ca & ab+bc+ac \\ 1 & ab & ab+bc+ac \end{vmatrix}$

Taking $(ab + bc + ac)$ common from C_3,

$= (ab + bc + ac) \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$

$= (ab + bc + ac)\ 0 = 0.$ $\quad (\because \ C_1$ and C_3 are identical)

5. $\begin{vmatrix} \boldsymbol{b+c} & \boldsymbol{q+r} & \boldsymbol{y+z} \\ \boldsymbol{c+a} & \boldsymbol{r+p} & \boldsymbol{z+x} \\ \boldsymbol{a+b} & \boldsymbol{p+q} & \boldsymbol{x+y} \end{vmatrix} = \mathbf{2} \begin{vmatrix} \boldsymbol{a} & \boldsymbol{p} & \boldsymbol{x} \\ \boldsymbol{b} & \boldsymbol{q} & \boldsymbol{y} \\ \boldsymbol{c} & \boldsymbol{r} & \boldsymbol{z} \end{vmatrix}.$

Sol. L.H.S. $= \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$

Operate $R_1 \to R_1 + R_2 + R_3$

Taking 2 common from R_1

$= 2 \begin{vmatrix} a+b+c & p+q+r & x+y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$

Operate $R_1 \to R_1 - R_2$ (to get single letter entries as required in the determinant on R.H.S.)

$$= 2 \begin{vmatrix} b & q & y \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Now operate $R_3 \to R_3 - R_1$ (to get single letter entries as required in the determinant on R.H.S.)

$$= 2 \begin{vmatrix} b & q & y \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix}$$

Now operate $R_2 \to R_2 - R_3$ (objective being same as in the above two operations)

$$= 2 \begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$$

Interchanging R_2 and R_3, $= -2 \begin{vmatrix} b & q & y \\ a & p & x \\ c & r & z \end{vmatrix}$

Interchanging R_1 and R_2, Operate $R_1 \to R_1 - R_2$ (to get single letter entries as required inthe determinant on R.H.S.)

$$= -(-2) \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \text{R.H.S.}$$

By using properties of determinants, in Exercise 6 to 14, show that:

6. $$\begin{vmatrix} \mathbf{0} & \boldsymbol{a} & -\boldsymbol{b} \\ -\boldsymbol{a} & \mathbf{0} & -\boldsymbol{c} \\ \boldsymbol{b} & \boldsymbol{c} & \mathbf{0} \end{vmatrix} = \mathbf{0}.$$

Sol. Let $\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$ ...(*i*)

Taking (–1) common from each row, we have

$$\Delta = (-1)^3 \begin{vmatrix} 0 & -a & b \\ a & 0 & c \\ -b & -c & 0 \end{vmatrix}$$

Interchanging rows and columns in the determinant on R.H.S.,

$$\Delta = -\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} \qquad (\because \ (-1)^3 = -1)$$

$\Rightarrow \quad \Delta = -\Delta$ (By (i))

Shifting $-\Delta$ from R.H.S. to L.H.S., $\Delta + \Delta = 0$ or $2\Delta = 0$

$\therefore \quad \Delta = \frac{0}{2} = 0.$

Note. 1. We can also do this question by taking (– 1) common from each column.

2. When you are asked to prove that a determinant is equal to zero or two determinants are equal, then it is to be proved so only without expanding.

3. It may be remarked that the determinant of Q. No. 6 above is determinant of a skew symmetric matrix of order 3.

7. $$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2.$$

Sol. L.H.S. $= \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$

Taking a, b, c common from R_1, R_2, R_3 respectively,

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Operate $R_1 \to R_1 + R_2$ (to create two zeros in a line (here first row))

$$= abc \begin{vmatrix} 0 & 0 & 2c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Expanding along first row ($\because$ There are two zeros in it)

$$= abc \cdot 2c \begin{vmatrix} a & -b \\ a & b \end{vmatrix} = abc \cdot 2c\,(ab + ab)$$

$= abc \cdot 2c \cdot 2ab = 4a^2b^2c^2 = \text{R.H.S.}$

Note. Whenever we are asked to find the value of a determinant by using "Properties of Determinants", we must create two zeros in a line (Row or Column).

8. **(*i*)** $\begin{vmatrix} \mathbf{1} & \boldsymbol{a} & \boldsymbol{a^2} \\ \mathbf{1} & \boldsymbol{b} & \boldsymbol{b^2} \\ \mathbf{1} & \boldsymbol{c} & \boldsymbol{c^2} \end{vmatrix} = \boldsymbol{(a - b)(b - c)(c - a)}$

(*ii*) $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{a^3} & \boldsymbol{b^3} & \boldsymbol{c^3} \end{vmatrix} = \boldsymbol{(a - b)(b - c)(c - a)(a + b + c)}.$

Sol. (*i*) L.H.S. $= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

Expanding along first column

$$= 1\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} (b-a) & (b-a)(b+a) \\ (c-a) & (c-a)(c+a) \end{vmatrix}$$

Taking out $(b - a)$ common from first row and $(c - a)$ common from second row

$$= (b - a)(c - a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$= (b - a)(c - a)(c + a - b - a) = (b - a)(c - a)(c - b)$
$= -(a - b)(c - a)(-(b - c)) = (a - b)(b - c)(c - a).$

Remark. For expanding a determinant of order 3 we should make all entries except one entry of a row or column as zeros (*i.e.,* we should make two entries as zeros) and then expand the determinant along this row or column. For doing so, the ideal situation is that all entries of a row or column are 1 each.

If each entry of a **column** is 1, then, to create two zeros, subtract first **row** from each of the remaining two **rows.**
If each entry of a **row** is 1, then to create two zeros, subtract first **column** from each of the remaining two columns.

(ii) L.H.S. = $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

Here all entries of a row are 1 each.
So operate $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ (to creat two zeros in a line (here first row))

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$$

Expanding along first row, $= 1 \begin{vmatrix} b-a & c-a \\ b^3-a^3 & c^3-a^3 \end{vmatrix}$

(Forming factors)

$$= \begin{vmatrix} (b-a) & (c-a) \\ (b-a)(b^2+a^2+ab) & (c-a)(c^2+a^2+ac) \end{vmatrix}$$

Taking $(b-a)$ common from C_1 and $(c-a)$ common from C_2,

$$= (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b^2+a^2+ab & c^2+a^2+ac \end{vmatrix}$$

$= (b-a)(c-a)\ (c^2+a^2+ac-b^2-a^2-ab)$
$= (b-a)(c-a)(c^2-b^2+ac-ab)$
$= (b-a)(c-a)\ [(c-b)(c+b)+a(c-b)]$
$= (b-a)(c-a)(c-b)(c+b+a)$
$= -\ (a-b(c-a)\ [-\ (b-c)]\ (a+b+c)$
$= (a-b)(b-c)(c-a)(a+b+c) =$ R.H.S.

9. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$ **$= (x-y)(y-z)(z-x)(xy+yz+zx)$.**

Sol. L.H.S. = $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$

Multiplying R_1, R_2, R_3 by x, y, z respectively (to make each entry of third column same here (xyz))

$$= \frac{1}{xyz}\begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix}$$

Taking xyz common from C_3, $= \dfrac{xyz}{xyz}\begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix} = \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$

Now all entries of a **column** are same. So operate $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ to create two zeros in a column.

$$= \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 - x^2 & y^3 - x^3 & 0 \\ z^2 - x^2 & z^3 - x^3 & 0 \end{vmatrix}$$

Expanding along third column $= 1\begin{vmatrix} y^2 - x^2 & y^3 - x^3 \\ z^2 - x^2 & z^3 - x^3 \end{vmatrix}$

(Forming factors) $= \begin{vmatrix} (y-x)(y+x) & (y-x)(y^2+x^2+xy) \\ (z-x)(z+x) & (z-x)(z^2+x^2+zx) \end{vmatrix}$

Taking $(y - x)$ common from R_1 and $(z - x)$ common from R_2

$$= (y-x)(z-x)\begin{vmatrix} y+x & y^2+x^2+xy \\ z+x & z^2+x^2+zx \end{vmatrix}$$

$= (y-x)(z-x)\,[(y+x)(z^2+x^2+zx) - (z+x)(y^2+x^2+xy)]$

$= (y-x)(z-x)\,[yz^2 + yx^2 + xyz + xz^2 + x^3 + x^2z$
$\qquad - zy^2 - zx^2 - xyz - xy^2 - x^3 - x^2y]$

$= (y-x)(z-x)\,[yz^2 - zy^2 + xz^2 - xy^2]$

$= (y-x)(z-x)\,\mathbf{[yz(z-y) + x(z^2-y^2)]}$

$= (y-x)(z-x)\,[yz(z-y) + x(z-y)(z+y)]$

$= (y-x)(z-x)(z-y)\,[yz + x(z+y)]$

$= -(x-y)(z-x)\,[-(y-z)]\,(yz+xz+xy)$

$= (x-y)(y-z)(z-x)(xy+yz+zx) =$ R.H.S.

10. (*i*) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x + 4)(4 - x)^2.$

(*ii*) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y + k).$

Sol. (*i*) L.H.S. $= \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

Here sum of entries of each **column** is same $(= 5x + 4)$, so let us operate $\mathbf{R_1 \to R_1 + R_2 + R_3}$ to make all entries of first row equal $(= 5x + 4)$.

$$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Taking $(5x + 4)$ common from R_1,

$$= (5x + 4) \begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Now each entry of one (here first) row is 1, so let us operate $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ to create two zeros in a zero.

$$= (5x + 4) \begin{vmatrix} 1 & 0 & 0 \\ 2x & 4-x & 0 \\ 2x & 0 & 4-x \end{vmatrix}$$

Expanding along first row

$$= (5x + 4) \,.\, 1 \begin{vmatrix} 4-x & 0 \\ 0 & 4-x \end{vmatrix}$$

$$= (5x + 4)(4 - x)^2 = \text{R.H.S.}$$

Remark. We could also start here by operating
$C_1 \to C_1 + C_2 + C_3$.

(*ii*) L.H.S. $= \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$

Here sum of entries of each row is same $(= 3y + k)$, so let us operate $\mathbf{C_1 \to C_1 + C_2 + C_3}$ to make all entries of first column equal $(= 3y + k)$

$$= \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix}$$

Taking $(3y + k)$ common from C_1,

$$= (3y + k) \begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix}$$

Now each entry of one (here first) column is 1, so let us operate $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ to create two zeros in a column,

$$= (3y + k) \begin{vmatrix} 1 & y & y \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix}$$

Expanding along first column, $= (3y + k) \cdot 1 \begin{vmatrix} k & 0 \\ 0 & k \end{vmatrix}$

$= (3y + k)k^2 = k^2(3y + k) =$ R.H.S.

Remark. We could also start here by operating

$R_1 \to R_1 + R_2 + R_3$.

11. **(*i*)** $\begin{vmatrix} \boldsymbol{a-b-c} & \boldsymbol{2a} & \boldsymbol{2a} \\ \boldsymbol{2b} & \boldsymbol{b-c-a} & \boldsymbol{2b} \\ \boldsymbol{2c} & \boldsymbol{2c} & \boldsymbol{c-a-b} \end{vmatrix} = \boldsymbol{(a + b + c)^3}$.

(*ii*) $\begin{vmatrix} \boldsymbol{x+y+2z} & \boldsymbol{x} & \boldsymbol{y} \\ \boldsymbol{z} & \boldsymbol{y+z+2x} & \boldsymbol{y} \\ \boldsymbol{z} & \boldsymbol{x} & \boldsymbol{z+x+2y} \end{vmatrix} = \boldsymbol{2(x + y + z)^3}$.

Sol. (*i*) L.H.S. $= \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Here sum of entries of each **column** is same $(= a + b + c)$, so let us operate $\mathbf{R_1 \to R_1 + R_2 + R_3}$ to make all entries of first row equal $(= a + b + c)$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking $(a + b + c)$ common from R_1,

$$= (a + b + c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Now each entry of one (here first) row is 1, so let us operate $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ to create two zeros in a row,

$$= (a + b + c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

Expanding along first row

$$= (a + b + c) \cdot 1 \cdot \begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix}$$

$$= (a + b + c) [(-b - c - a)(-c - a - b)]$$
$$= (a + b + c) (-) (b + c + a) (-) (c + a + b)$$
$$= (a + b + c)^3 = \text{R.H.S.}$$

Remark. Here we can't operate $C_1 \to C_1 + C_2 + C_3$ because sum of entries of each row is not same.

(*ii*) $$\text{L.H.S.} = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

Here sum of entries of each **row** is same $(= 2x + 2y + 2z = 2(x + y + z))$, so let us operate $\mathbf{C_1 \to C_1 + C_2 + C_3}$ to make all entries of first column equal $(= 2(x + y + z))$

$$= \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$

Taking $2(x + y + z)$ common from C_1,

$$= 2(x + y + z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

Now each entry of one (here first) column is 1, so let us operate $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ to create two zeros in a column

$$= 2(x + y + z) \begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

Expanding along first column

$$= 2(x + y + z) \cdot 1 \cdot \begin{vmatrix} x+y+z & 0 \\ 0 & x+y+z \end{vmatrix}$$

$$= 2(x + y + z) [(x + y + z)^2 - 0]$$
$$= 2(x + y + z)^3 = \text{R.H.S.}$$

12. $$\begin{vmatrix} \mathbf{1} & \boldsymbol{x} & \boldsymbol{x^2} \\ \boldsymbol{x^2} & \mathbf{1} & \boldsymbol{x} \\ \boldsymbol{x} & \boldsymbol{x^2} & \mathbf{1} \end{vmatrix} = \mathbf{(1 - x^3)^2}.$$

Sol. L.H.S. = $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$

Here sum of entries of each **column** is same $(= 1 + x + x^2)$, so let us operate $\mathbf{R_1 \to R_1 + R_2 + R_3}$ to make all entries of first **row** equal $(= 1 + x + x^2)$

$$= \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Taking $(1 + x + x^2)$ common from R_1,

$$= (1 + x + x^2) \begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Now each entry of one (here first) row is 1, so let us operate $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ to create two zeros in a row.

$$= (1 + x + x^2) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix}$$

Expanding along first row

$$= (1 + x + x^2) \,.\, 1 \begin{vmatrix} 1-x^2 & x-x^2 \\ x^2-x & 1-x \end{vmatrix}$$

$$= (1 + x + x^2) \begin{vmatrix} (1-x)(1+x) & x(1-x) \\ -x(1-x) & (1-x) \end{vmatrix}$$

$= (1 + x + x^2) [(1 - x)^2 (1 + x) + x^2(1 - x)^2]$

$= (1 + x + x^2) (1 - x)^2 (1 + x + x^2) = (1 + x + x^2)^2 (1 - x)^2$

$= [(1 + x + x^2) (1 - x)]^2$ $\quad (\because \;\; A^2B^2 = (AB)^2)$

$= (1 - x + x - x^2 + x^2 - x^3)^2 = (1 - x^3)^2 =$ R.H.S.

Remark. For the above question, we could also operate $C_1 \to C_1 + C_2 + C_3$ because sum of entries of each row is also same and $(= 1 + x + x^2)$.

13. $\begin{vmatrix} \mathbf{1+a^2-b^2} & \mathbf{2ab} & \mathbf{-2b} \\ \mathbf{2ab} & \mathbf{1-a^2+b^2} & \mathbf{2a} \\ \mathbf{2b} & \mathbf{-2a} & \mathbf{1-a^2-b^2} \end{vmatrix} = \mathbf{(1 + a^2 + b^2)^3}.$

Sol. Operating $C_1 \to C_1 - b\, C_3, C_2 \to C_2 + a\, C_3$ in L.H.S. as suggested by the factor $(1 + a^2 + b^2)$ in R.H.S.

$$\text{L.H.S.} = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

[$\because$ $2b - b(1 - a^2 - b^2) = 2b - b + a^2b + b^3$
$= b + a^2b + b^3 = b(1 + a^2 + b^2)$]

Taking out $(1 + a^2 + b^2)$ common from C_1 and C_2

$$= (1 + a^2 + b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

Operating $R_3 \to R_3 - bR_1$ (to create another zero in first column)

$$= (1 + a^2 + b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1-a^2+b^2 \end{vmatrix}$$

Expanding along C_1

$$= (1 + a^2 + b^2)^2 \,.\, 1 \begin{vmatrix} 1 & 2a \\ -a & 1-a^2+b^2 \end{vmatrix}$$

$$= (1 + a^2 + b^2)^2 (1 - a^2 + b^2 + 2a^2) = (1 + a^2 + b^2)^3.$$

14. $$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$$

Sol. Multiplying C_1, C_2, C_3 by a, b, c respectively and in return dividing the determinant by abc,

$$\Delta = \frac{1}{abc} \begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}$$

Taking out a, b, c common from R_1, R_2, R_3 respectively,

$$= \frac{abc}{abc} \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$ Operating $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

Taking out $(1 + a^2 + b^2 + c^2)$ common from C_1,

$$= (1 + a^2 + b^2 + c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$,

$$= (1 + a^2 + b^2 + c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along first column,

$$= (1 + a^2 + b^2 + c^2) \times 1\,(1 - 0) = 1 + a^2 + b^2 + c^2.$$

Choose the correct answer in Exercises 15 and 16:

15. Let A be a square matrix of order 3 × 3, then | kA | is equal to

(A) k | A | **(B) k^2 | A |**

(C) k^3 | A | **(D) $3k$ | A |.**

Sol. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3×3. ...(*i*)

$\therefore$ By definition of scalar multiplication of a matrix,

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix} \quad \therefore \ |kA| = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix}$$

Taking k common from each row,

$$= k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= k^3\,|A| \qquad \text{[By } (i)]$$

Remark. In general, if A is a square matrix of order $n \times n$; then we can prove that **| kA | = k^n | A |.**

$\therefore$ Option (C) is the correct answer.

16. Which of the following is correct:

(A) Determinant is a square matrix.

(B) Determinant is a number associated to a matrix.

(C) Determinant is a number associated to a square matrix.

(D) None of these.

Sol. Option (C) is the correct answer.

i.e., Determinant is a number associated to a square matrix.

Exercise 4.3

1. Find the area of the triangle with vertices at the points given in each of the following:

(*i*) (1, 0), (6, 0), (4, 3) **(*ii*) (2, 7), (1, 1), (10, 8)**

(*iii*) (– 2, – 3), (3, 2), (– 1, – 8).

Sol. (*i*) Area of the triangle having vertices at (1, 0), (6, 0), (4, 3)

$$= \text{Modulus of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

Expanding along first row,

$$= \frac{1}{2}[1(0 - 3) - 0(6 - 4) + 1(18 - 0)]$$

i.e., Area of triangle = modulus of $\frac{1}{2}(- 3 + 18)$

$$= \left|\frac{15}{2}\right| = \frac{15}{2} \text{ sq. units}$$

($\because$ Modulus of a positive number is number itself)

(*ii*) Area of the triangle having vertices at (2, 7), (1, 1), (10, 8).

$$= \text{Modulus of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$$

Expanding along first row

$$= \frac{1}{2}[2(1 - 8) - 7(1 - 10) + 1(8 - 10)]$$

$$= \frac{1}{2}[2(- 7) - 7(- 9) - 2] = \frac{1}{2}(- 14 + 63 - 2)$$

$$= \frac{1}{2}(63 - 16)$$

i.e., Area of triangle = $\left|\frac{47}{2}\right| = \frac{47}{2}$ sq. units.

($\because$ Modulus of a positive real number is number itself)

(*iii*) Area of the triangle having vertices at

(– 2, – 3), (3, 2), (– 1, – 8) is

$$\text{Modulus of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[- 2(2 + 8) - (- 3)(3 + 1) + 1(- 24 + 2)]$$

$$= \frac{1}{2}[- 2(10) + 3(4) - 22] = \frac{1}{2}(- 20 + 12 - 22)$$

$= \frac{1}{2}(-42 + 12) = \frac{1}{2}(-30) = -15$

$\therefore$ Area of triangle = Modulus of -15 *i.e.,* $= |-15|$
$= 15$ sq. units

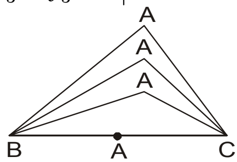
($\because$ Modulus of a negative real number is negative of itself)

2. Show that the points A(a, $b + c$), B(b, $c + a$), C(c, $a + b$) are collinear.

Sol. The given points are A(a, $b + c$), B(b, $c + a$), C(c, $a + b$).

$\therefore$ Area of triangle ABC is modulus of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$= \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$



Expanding along first row,

$= \frac{1}{2}[a(c + a - a - b) - (b + c)(b - c) + 1(b(a + b) - c(c + a))]$

$= \frac{1}{2}[a(c - b) - (b^2 - c^2) + (ab + b^2 - c^2 - ac)]$

$= \frac{1}{2}(ac - ab - b^2 + c^2 + ab + b^2 - c^2 - ac) = \frac{1}{2}(0) = 0$

i.e., Area of ΔABC = 0

$\therefore$ Points A, B, C are collinear (See above figure).

3. Find values of k if area of triangle is 4 sq. units and vertices are:

(*i*) (k, 0), (4, 0), (0, 2)   (*ii*) (– 2, 0), (0, 4), (0, k).

Sol. (*i*) **Given:** Area of the triangle whose vertices are (k, 0), (4, 0), (0, 2) is 4 sq. units.

$\Rightarrow$ Modulus of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 4$

$\Rightarrow$ Modulus of $\frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 4$

Expanding along first row, $\left| \frac{1}{2}\{k(0 - 2) - 0 + 1(8 - 0)\} \right| = 4$

$\Rightarrow \left| \frac{1}{2}(-2k + 8) \right| = 4 \Rightarrow |-k + 4| = 4$

$\Rightarrow -k + 4 = \pm 4$

[$\because$ If $x \in$ R and $|x| = a$ where $a \geq 0$, then $x = \pm a$]

Taking positive sign, $-k + 4 = 4$

$\Rightarrow -k = 0 \qquad \Rightarrow k = 0$

Taking negative sign, $-k + 4 = -4$

$\Rightarrow \quad -k = -8 \qquad \Rightarrow \quad k = 8 \qquad$ Hence $\qquad k = 0, k = 8.$

(*ii*) **Given:** Area of the triangle whose vertices are $(-2, 0)$, $(0, 4)$, $(0, k)$ is 4 sq. units.

$$\Rightarrow \text{ Modulus of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 4$$

$$\Rightarrow \text{ Modulus of } \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix} = 4$$

Expanding along first row, $\left| \frac{1}{2}\{-2(4-k) - 0 + 1(0-0)\} \right| = 4$

$\Rightarrow \quad \left| \frac{1}{2}(-8 + 2k) \right| = 4 \quad \Rightarrow \quad |-4 + k| = 4$

$\Rightarrow \qquad -4 + k = \pm 4$

($\because$ If $|x| = a$ where $a \geq 0$, then $x = \pm a$)

Taking positive sign, $-4 + k = 4 \quad \Rightarrow \quad k = 4 + 4 = 8$

Taking negative sign, $-4 + k = -4 \quad \Rightarrow \quad k = 0$

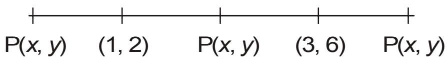
Hence, $\qquad k = 0, k = 8.$

4. (*i*) Find the equation of the line joining (1, 2) and (3, 6) using determinants.

(*ii*) Find the equation of the line joining (3, 1) and (9, 3) using determinants.

Sol. (*i*) Let P(x, y) be **any** point on the line joining the points (1, 2) and (3, 6).

$\therefore$ Three points are collinear.



$\therefore$ Area of triangle that could be formed by them is zero.

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0 \qquad \left[\left| \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right| \right]$$

Multiplying both sides by 2, and expanding the determinant on left hand side along first row,

$$x(2 - 6) - y(1 - 3) + 1(6 - 6) = 0$$

$\Rightarrow \quad -4x + 2y = 0.$ Dividing by -2, $2x - y = 0$

or $\quad -y = -2x \quad$ *i.e.,* $\quad y = 2x$ which is the required equation of the line.

(*ii*) Let P(x, y) be any point on the line joining the points (3, 1) and (9, 3).

$\therefore$ Three points are collinear.

P(x, y) (3, 1) P(x, y) (9, 3) P(x, y)

$\therefore$ Area of triangle that could be formed by them is zero.

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 0$$

Multiplying both sides by 2 and expanding the determinant on left hand side along first row,

$$x(1 - 3) - y(3 - 9) + 1(9 - 9) = 0$$

$$\Rightarrow \quad -2x + 6y = 0$$

Dividing by -2, $x - 3y = 0$ which is the required equation of the line.

5. If area of triangle is 35 sq. units with vertices (2, – 6), (5, 4) and (*k*, 4). Then *k* is

(A) 12 (B) – 2 (C) – 12, – 2 (D) 12, – 2.

Sol. Given: Area of triangle having vertices $(2, -6)$, $(5, 4)$ and $(k, 4)$ is 35 sq. units.

$$\therefore \quad \text{Modulus of } \left(\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix}\right) = 35 \quad \text{(Given)}$$

Expanding along first row,

$$\left|\frac{1}{2}\{2(4-4) - (-6)(5-k) + 1(20-4k)\}\right| = 35$$

$$\Rightarrow \quad \left|\frac{1}{2}\{0 + 30 - 6k + 20 - 4k\}\right| = 35$$

$$\Rightarrow \quad \left|\frac{1}{2}(50 - 10k)\right| = 35 \quad \Rightarrow \quad |\,25 - 5k\,| = 35$$

$$\Rightarrow \quad 25 - 5k = \pm 35$$

[$\because$ If $|\,x\,| = a$ where $a \geq 0$, then $x = \pm a$]

Taking positive sign, $25 - 5k = 35 \quad \Rightarrow \quad -5k = 10$

$$\Rightarrow \quad k = \frac{-10}{5} = -2$$

Taking negative sign, $25 - 5k = -35$

$\Rightarrow \quad -5k = -60 \quad \Rightarrow \quad k = 12$

Thus, $k = 12, -2$ $\therefore$ Option (D) is the correct answer.

Exercise 4.4

Note. Minor (M_{ij}) and Cofactor (A_{ij}) of an element a_{ij} of a determinant Δ are defined **not for the value** of the element but for **(i, j)th position** of the element.

Def. 1. Minor M_{ij} of an element a_{ij} of a determinant Δ is the determinant obtained by omitting its ith row and jth column in which element a_{ij} lies.

Def. 2. Cofactor A_{ij} of an element a_{ij} of Δ is defined as $A_{ij} = (-1)^{i+j} M_{ij}$ where M_{ij} is the minor of a_{ij}.

1. Write minors and cofactors of the elements of the following determinants:

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ **(ii)** $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Sol. (i) Let $\Delta = \begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

M_{11} = Minor of a_{11} $= | 3 | = 3$;

$A_{11} = (-1)^{1+1} M_{11} = (-1)^{1+1} (3) = (-1)^2\, 3 = 3$

(Omit first row and first column of Δ)

M_{12} = Minor of a_{12} $= | 0 | = 0$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^{1+2} (0) = (-1)^3 . 0 = 0$

M_{21} = Minor of a_{21} $= | -4 | = -4$,

$A_{21} = (-1)^{2+1} M_{21} = (-1)^{2+1} (-4) = (-1)^3(-4) = 4$

M_{22} = Minor of a_{22} $= | 2 | = 2$,

$A_{22} = (-1)^{2+2} M_{22} = (-1)^{2+2}\, 2 = (-1)^4\, 2 = 2$

(ii) Let $\Delta = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

M_{11} = Minor of a_{11} $= | d | = d$,

$A_{11} = (-1)^{1+1} d = (-1)^2 d = d$

M_{12} = Minor of a_{12} $= | b | = b$,

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 b = -b$

M_{21} = Minor of a_{21} $= | c | = c$,

$A_{21} = (-1)^{2+1} c = (-1)^3 c = -c$

M_{22} = Minor of a_{22} $= | a | = a$,

$A_{22} = (-1)^{2+2} a = (-1)^4 a = a$.

following determinants:

2. Write Minors and Cofactors of the elements of the

(i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ **(ii)** $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Sol. (*i*) Let $\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$\therefore \quad M_{11} = \text{Minor of } a_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 \, 1 = 1$

$M_{12} = \text{Minor of } a_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0$

(Omitting first row and second column of Δ)

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 \, 0 = 0$

$M_{13} = \text{Minor of } a_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0,$

$A_{13} = (-1)^{1+3} M_{13} = (-1)^4 \, 0 = 0$

$M_{21} = \text{Minor of } a_{21} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0,$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 \, 0 = 0$

$M_{22} = \text{Minor of } a_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1,$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 \, 1 = 1$

$M_{23} = \text{Minor of } a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0,$

$A_{23} = (-1)^{2+3} M_{23} = (-1)^5 \, 0 = 0$

$M_{31} = \text{Minor of } a_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0,$

$A_{31} = (-1)^{3+1} M_{31} = (-1)^4 \, 0 = 0$

$M_{32} = \text{Minor of } a_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0,$

$A_{32} = (-1)^{3+2} M_{32} = (-1)^5 \, 0 = 0$

$M_{33} = \text{Minor of } a_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1,$

$A_{33} = (-1)^{3+3} M_{33} = (-1)^6 \, 1 = 1.$

(*ii*) Let $\Delta = \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

M_{11} = Minor of $a_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 - (-1) = 10 + 1 = 11,$

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2\ 11 = 11$

M_{12} = Minor of $a_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6,$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3\ 6 = -6$

M_{13} = Minor of $a_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3,$

$A_{13} = (-1)^{1+3} M_{13} = (-1)^4\ 3 = 3$

M_{21} = Minor of $a_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4,$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3\ (-4) = 4$

M_{22} = Minor of $a_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2,$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4\ 2 = 2$

M_{23} = Minor of $a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1,$

$A_{23} = (-1)^{2+3} M_{23} = (-1)^5\ 1 = -1$

M_{31} = Minor of $a_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20,$

$A_{31} = (-1)^{3+1} M_{31} = (-1)^4\ (-20) = -20$

M_{32} = Minor of $a_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13,$

$A_{32} = (-1)^{3+2} M_{32} = (-1)^5\ (-13) = 13$

M_{33} = Minor of $a_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5,$

$A_{33} = (-1)^{3+3} M_{33} = (-1)^6\ 5 = 5.$

Note. Two Most Important Results

1. Sum of the products of the elements of any row or column of a determinant Δ with their corresponding factors is = Δ.
i.e., $\boldsymbol{\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}}$ **etc.**

2. Sum of the products of the elements of any row or column of a determinant Δ with the cofactors of any other row or column of Δ is zero.

For example, $\boldsymbol{a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0}$.

3. Using Cofactors of elements of second row, evaluate

$$\boldsymbol{\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}}.$$

Sol. $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Elements of second row of Δ are $a_{21} = 2$, $a_{22} = 0$, $a_{23} = 1$

A_{21} = Cofactor of $a_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix}$ $(\because \; A_{ij} = (-1)^{i+j} M_{ij}]$

$\downarrow \; \downarrow$

(determinant obtained by omitting second row and first column of Δ)

$= (-1)^3 (9 - 16) = -(-7) = 7$

A_{22} = Cofactor of $a_{22} = (-1)^{2+2} \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = (-1)^4 (15 - 8) = 7$

A_{23} = Cofactor $a_{23} = (-1)^{2+3} \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = (-1)^5 (10 - 3) = -7$

Now by Result I of Note after the solution of Q. No. 2,

$\boldsymbol{\Delta = a_{21} A_{21} + a_{22}A_{22} + a_{23}A_{23}}$

$= 2(7) + 0(7) + 1(-7) = 14 - 7 = 7.$

Remark. The above method of finding the value of Δ is equivalent to expanding Δ along second row.

4. Using Cofactors of elements of third column, evaluate

$$\boldsymbol{\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}}.$$

Sol. $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

Here elements of third column of Δ are

$a_{13} = yz,\ a_{23} = zx,\ a_{33} = xy$

$A_{13} = \text{Cofactor of } a_{13} = (-1)^{1+3} \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix}$

$= (-1)^4$

$(z - y) = z - y$

$\downarrow$

(determinant obtained by omitting first row and third column of Δ)

$A_{23} = \text{Cofactor of } a_{23} = (-1)^{2+3} \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = (-1)^5 (z - x) = -(z - x)$

$A_{33} = \text{Cofactor of } a_{33} = (-1)^{3+3} \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = (-1)^6 (y - x) = y - x$

Now by Result I of Note after the solution of Q. NO. 2,

$\boldsymbol{\Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}}$

$= yz(z - y) + zx[-(z - x)] + xy(y - x)$

$= yz^2 - y^2z - z^2x + zx^2 + xy^2 - x^2y$

$= (yz^2 - y^2z) + (xy^2 - xz^2) + (zx^2 - x^2y)$

$= yz(z - y) + x(y^2 - z^2) - x^2(y - z)$

$= -yz(y - z) + x(y + z)(y - z) - x^2(y - z)$

$= (y - z)[-yz + xy + xz - x^2]$

$= (y - z)[-y(z - x) + x(z - x)]$

$= (y - z)(z - x)(-y + x) = (x - y)(y - z)(z - x)$

Remark. The above method of finding the value of Δ is equivalent to expanding Δ along third column.

5. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactor of a_{ij}, then value of Δ is given by

(A) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$

(B) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$

(C) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$

(D) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$.

Sol. Option (D) is correct answer as given in Result I of Note after solution of Q. No. 2 and used in the solution of Q. No. 3 and 4 above.

Remark. The values of expressions given in options (A) and (C) are each equal to zero as given in Result II of Note after solution of Q. No. 2.

Exercise 4.5

Find adjoint of each of the matrices in Exercises 1 and 2.

1. $\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}$.

Sol. Here $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$| A | = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$\therefore \quad A_{11}$ = Cofactor of $a_{11} = (-1)^2\, 4 = 4$,
A_{12} = Cofactor of $a_{12} = (-1)^3\, 3 = -3$
A_{21} = Cofactor of $a_{21} = (-1)^3\, 2 = -2$,
A_{22} = Cofactor of $a_{22} = (-1)^4\, 1 = 1$

$$\therefore \quad \text{adj. } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}' = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}' = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}.$$

Remark. For writing the Cofactors of the elements of a determinant of order 2, assign a **positive** sign to the Cofactors of **diagonal** elements and a **negative sign** to the Cofactors of **non-diagonal** elements.

2. $\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} & \mathbf{5} \\ \mathbf{-2} & \mathbf{0} & \mathbf{1} \end{bmatrix}$.

Sol. Here $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} | A | = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{vmatrix}$

$\therefore \quad A_{11} = + \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3,\ A_{12} = - \begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix}$

$= -(2 + 10) = -12,$ (See Note 2, below)

$A_{13} = + \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 6,\ A_{21} = - \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1) = 1,$

$A_{22} = + \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 + 4 = 5,\ A_{23} = - \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix}$

$= -(-2) = 2,$

$$A_{31} = + \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5 - 6 = -11,$$

$$A_{32} = - \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5-4) = -1, \quad A_{33} = + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 3 + 2 = 5$$

$$\therefore \text{ adj. A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}' = \begin{bmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{bmatrix}'$$

$$= \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}.$$

Note. 1. Adjoint of matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **is** $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

i.e., To write adjoint of a 2 × 2 matrix, interchange the diagonal elements and change the signs of non-diagonal elements.

The above result can be used as a formula.

2. For writing the Cofactors of the elements of a determinant of order 3 × 3, using the rule $(-1)^{i+j} M_{ij}$, the signs to be assigned to 9 cofactors are alternately + and – beginning with +.

Verify A(adj. A) = (adj. A)A = | A | I in Exercises 3 and 4:

3. $\begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{-4} & \mathbf{-6} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

$\therefore$ By Note 1, above, adj. A $= \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$

$$\left(\because \text{ adj.} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is } \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$$

$$\therefore \text{ A. (adj. A)} = \begin{bmatrix} \boxed{2 \quad 3} \\ \boxed{-4 \quad -6} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} -6 \\ 4 \end{matrix}} & \boxed{\begin{matrix} -3 \\ 2 \end{matrix}} \end{bmatrix}$$

$$= \begin{bmatrix} -12+12 & -6+6 \\ 24-24 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad ...(i)$$

Again (adj. A). $A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

$$= \begin{bmatrix} -12+12 & -18+18 \\ 8-8 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad ...(ii)$$

Now $|A| = \begin{vmatrix} 2 & 3 \\ -4 & -6 \end{vmatrix} = 2(-6) - 3(-4) = -12 + 12 = 0$

Again $|A|\,I = |A|\,I_2$ (I is I_2 because A is of order 2×2)

$$= 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad ...(iii)$$

From (i), (ii) and (iii), A (adj. A) = (adj. A) A = | A | I.

4. $\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} \\ \mathbf{3} & \mathbf{0} & \mathbf{-2} \\ \mathbf{1} & \mathbf{0} & \mathbf{3} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$ $\therefore\ |A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix}$

Let A_{ij} denote Cofactor of a_{ij}

(For rule of signs to be assigned see Note 2 at the end of solution of Q. No. 2).

$\therefore\ A_{11} = + \begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} = +(0 + 0) = 0,$

$A_{12} = - \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -(9 + 2) = -11,$

$A_{13} = + \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = +(0 - 0) = 0,$

$A_{21} = - \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = -(-3 - 0) = 3,$

$A_{22} = + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = +(3 - 2) = 1,$

$A_{23} = - \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -(0 + 1) = -1,$

$A_{31} = + \begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} = +(2 - 0) = 2,$

$A_{32} = - \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -(-2 - 6) = 8,$

$$A_{33} = + \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = +(0+3) = 3$$

$$\therefore \text{ adj. A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}' = \begin{bmatrix} 0 & -11 & 0 \\ 3 & 1 & -1 \\ 2 & 8 & 3 \end{bmatrix}' = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\therefore \text{ A (adj. A)} = \begin{bmatrix} \boxed{1 \quad -1 \quad 2} \\ \boxed{3 \quad 0 \quad -2} \\ \boxed{1 \quad 0 \quad 3} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 0 \\ -11 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 3 \\ 1 \\ -1 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 8 \\ 3 \end{matrix}} \end{bmatrix}$$

$$= \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0-0-0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \qquad ...(i)$$

$$\text{Now (adj. A) A} = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+9+2 & 0+0+0 & 0-6+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0-0+0 & 0+2+9 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \qquad ...(ii)$$

$$\text{Now } |A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix}$$

Expanding along first row

$= 1(0 - 0) - (-1)(9 + 2) + 2(0 - 0) = 0 + 11 + 0 = 11$

Again $|A|\,I = |A|\,I_3$ $\quad$ ($\because$ A is 3×3, therefore I must be I_3)

$$= 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \qquad ...(iii)$$

From (*i*), (*ii*) and (*iii*)

A. (adj. A) = (adj. A) A = | A | I.

Find the inverse of the matrix (if it exists) given in Exercises 5 to 11.

5. $\begin{bmatrix} \mathbf{2} & \mathbf{-2} \\ \mathbf{4} & \mathbf{3} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6 - (-8) = 6 + 8 = 14 \neq 0$

$\therefore$ Matrix A is non-singular and hence A^{-1} exists.

We know that adj. $A = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix} \left(\because \text{ Adj.} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{is} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$

We know that $A^{-1} = \dfrac{1}{|A|}$ adj. $A = \dfrac{1}{14} \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$.

6. $\begin{bmatrix} \mathbf{-1} & \mathbf{5} \\ \mathbf{-3} & \mathbf{2} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} -1 & 5 \\ -3 & 2 \end{vmatrix} = -2 - (-15) = -2 + 15 = 13 \neq 0$

$\therefore \quad A^{-1}$ exists.

We know that adj. $A = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix} \quad \left(\because \text{ Adj.} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$

$\therefore \quad A^{-1} = \dfrac{1}{|A|}$ (adj. A) $= \dfrac{1}{13} \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$.

7. $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{2} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{5} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ $\qquad \therefore \quad |A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{vmatrix}$

Expanding along first row

$= 1(10 - 0) - 2(0 - 0) + 3(0 - 0) = 10 \neq 0$

$\therefore \quad A^{-1}$ exists.

$A_{11} = + \begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} = + (10 - 0) = 10,$

$A_{12} = - \begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} = - (0 - 0) = 0,$

$A_{13} = + \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = (0 - 0) = 0,$

$$A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = -(10-0) = -10,$$

$$A_{22} = +\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = (5-0) = 5,$$

$$A_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = -(0-0) = 0,$$

$$A_{31} = +\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = (8-6) = 2,$$

$$A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -(4-0) = -4,$$

$$A_{33} = +\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = +(2-0) = 2$$

$$\therefore \quad \text{adj. A} = \begin{bmatrix} 10 & 0 & 0 \\ -10 & 5 & 0 \\ 2 & -4 & 2 \end{bmatrix}' = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{ adj. A} = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}.$$

8. $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{3} & \mathbf{0} \\ \mathbf{5} & \mathbf{2} & \mathbf{-1} \end{bmatrix}$**.**

Sol. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ $\therefore \quad |A| = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{vmatrix}$

Expanding along first row $|A| = 1(-3-0) - 0 + 0 = -3 \neq 0$

$$A_{11} = +\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = (-3-0) = -3,$$

$$A_{12} = -\begin{vmatrix} 3 & 0 \\ 5 & -1 \end{vmatrix} = -(-3-0) = 3,$$

$$A_{13} = +\begin{vmatrix} 3 & 3 \\ 5 & 2 \end{vmatrix} = +(6-15) = -9,$$

$$A_{21} = -\begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} = -(0-0) = 0,$$

$$A_{22} = +\begin{vmatrix} 1 & 0 \\ 5 & -1 \end{vmatrix} = +(-1-0) = -1,$$

$$A_{23} = -\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = -(2-0) = -2,$$

$$A_{31} = +\begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} = (0-0) = 0,$$

$$A_{32} = -\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = -(0-0) = 0,$$

$$A_{33} = +\begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} = +(3-0) = 3$$

$$\therefore \quad \text{adj. } A = \begin{bmatrix} -3 & 3 & -9 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}' = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{ adj. } A = \frac{-1}{3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}.$$

9. $\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{4} & \mathbf{-1} & \mathbf{0} \\ \mathbf{-7} & \mathbf{2} & \mathbf{1} \end{bmatrix}$.

Sol. Let $|A| = \begin{vmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{vmatrix}$

Expanding by first row,

$$= 2(-1) - 1(4) + 3(8-7) = -2 - 4 + 3 = -3 \neq 0$$

$\Rightarrow$ A is non-singular $\quad \therefore \; A^{-1}$ exists.

$$A_{11} = +\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1, \qquad A_{12} = -\begin{vmatrix} 4 & 0 \\ -7 & 1 \end{vmatrix} = -4,$$

$$A_{13} = +\begin{vmatrix} 4 & -1 \\ -7 & 2 \end{vmatrix} = 8 - 7 = 1, \qquad A_{21} = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5,$$

$$A_{22} = +\begin{vmatrix} 2 & 3 \\ -7 & 1 \end{vmatrix} = 23, \qquad A_{23} = -\begin{vmatrix} 2 & 1 \\ -7 & 2 \end{vmatrix} = -11,$$

$$A_{31} = \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} = 3, \qquad A_{32} = -\begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = 12,$$

$$A_{33} = \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = -6$$

$$\therefore \quad \text{adj. A} = \begin{bmatrix} -1 & -4 & 1 \\ 5 & 23 & -11 \\ 3 & 12 & -6 \end{bmatrix}' = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{ adj. A} = -\frac{1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}.$$

10. $\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & \mathbf{-3} \\ \mathbf{3} & \mathbf{-2} & \mathbf{4} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ $\therefore \quad |A| = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{vmatrix}$

Expanding along first row,

$$= 1(8 - 6) - (-1)(0 + 9) + 2(0 - 6)$$
$$= 2 + 9 - 12 = -1 \neq 0$$

$\therefore \quad A^{-1}$ exists.

$$A_{11} = + \begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} = (8 - 6) = 2,$$

$$A_{12} = - \begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} = -(0 + 9) = -9$$

$$A_{13} = + \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = +(0 - 6) = -6,$$

$$A_{21} = - \begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix} = -(-4 + 4) = 0$$

$$A_{22} = + \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (4 - 6) = -2,$$

$$A_{23} = - \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -(-2 + 3) = -1$$

$$A_{31} = + \begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} = 3 - 4 = -1,$$

$$A_{32} = - \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = -(-3 - 0) = 3$$

$$A_{33} = + \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = (2 - 0) = 2$$

$$\therefore \quad \text{adj. A} = \begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}' = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{ adj. } A = \frac{1}{-1}\begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} \qquad \left(\because \frac{1}{-1} = \frac{-1}{1} = -1\right)$$

11. $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{cos}\,\alpha & \mathbf{sin}\,\alpha \\ \mathbf{0} & \mathbf{sin}\,\alpha & -\mathbf{cos}\,\alpha \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix} \quad \therefore |A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{vmatrix}$

Expanding along first row

$$= 1(-\cos^2\alpha - \sin^2\alpha) - 0 + 0 = -(\cos^2\alpha + \sin^2\alpha)$$

or $|A| = -1 \neq 0$

$\therefore$ A^{-1} exists.

$$A_{11} = +\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix} = (-\cos^2\alpha - \sin^2\alpha)$$

$$= -(\cos^2\alpha + \sin^2\alpha) = -1$$

$$A_{12} = -\begin{vmatrix} 0 & \sin\alpha \\ 0 & -\cos\alpha \end{vmatrix} = -(0-0) = 0,$$

$$A_{13} = +\begin{vmatrix} 0 & \cos\alpha \\ 0 & \sin\alpha \end{vmatrix} = 0$$

$$A_{21} = -\begin{vmatrix} 0 & 0 \\ \sin\alpha & -\cos\alpha \end{vmatrix} = -(0-0) = 0,$$

$$A_{22} = +\begin{vmatrix} 1 & 0 \\ 0 & -\cos\alpha \end{vmatrix} = (-\cos\alpha - 0) = -\cos\alpha$$

$$A_{23} = -\begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = -(\sin\alpha - 0) = -\sin\alpha,$$

$$A_{31} = +\begin{vmatrix} 0 & 0 \\ \cos\alpha & \sin\alpha \end{vmatrix} = 0 - 0 = 0$$

$$A_{32} = -\begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = -\sin\alpha,$$

$$A_{33} = + \begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} = \cos\alpha.$$

$$\therefore \text{adj. } A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{ adj. } A = - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

($\because$ $|A| = -1$, obtained above)

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}.$$

12. Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Sol. Given: Matrix $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$.

Therefore $|A| = \begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix} = 15 - 14 = 1 \neq 0$

$$\therefore A^{-1} = \frac{1}{|A|} \text{ adj. } A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} \left(\because \text{ adj.} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$$

Given: Matrix $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$

$$\therefore \quad |B| = \begin{vmatrix} 6 & 8 \\ 7 & 9 \end{vmatrix} = 54 - 56 = -2 \neq 0$$

$$\therefore \quad B^{-1} = \frac{1}{|B|} \text{ adj. } B = \frac{1}{-2} \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$$

$$\text{Now} AB = \begin{bmatrix} \boxed{3 \quad 7} \\ \boxed{2 \quad 5} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 6 \\ 7 \end{matrix}} & \boxed{\begin{matrix} 8 \\ 9 \end{matrix}} \end{bmatrix} = \begin{bmatrix} 18 + 49 & 24 + 63 \\ 12 + 35 & 16 + 45 \end{bmatrix} = \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$$

$$\therefore \quad |AB| = \begin{vmatrix} 67 & 87 \\ 47 & 61 \end{vmatrix} = 67(61) - 87(47) = 4087 - 4089$$

$$= -2 \neq 0$$

$$\therefore \text{ L.H.S.} = (AB)^{-1} = \frac{1}{|AB|} \text{ adj. } (AB)$$

$$= \frac{1}{-2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} \quad ...(i)$$

$$\text{R.H.S.} = B^{-1}A^{-1} = \frac{-1}{2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} 45+16 & -63-24 \\ -35-12 & 49+18 \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} \quad ...(ii)$$

From (*i*) and (*ii*) we have L.H.S. = R.H.S. *i.e.,* $(AB)^{-1} = B^{-1}A^{-1}$.

13. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.

Sol. Given: $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$\therefore A^2 = A\,.\,A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

L.H.S. = $A^2 - 5A + 7I = A^2 - 5A + 7I_2$

(I is I_2 here because A is 2×2)

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 8-15 & 5-5 \\ -5+5 & 3-10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -7+7 & 0+0 \\ 0+0 & -7+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O = \text{R.H.S.}$$

$$\Rightarrow \quad A^2 - 5A + 7I_2 = O \quad ...(i)$$

Hence to find A^{-1}. Multiplying both sides of eqn. (*i*) by A^{-1},

$$A^2A^{-1} - 5AA^{-1} + 7I_2A^{-1} = O.A^{-1}$$

$$\Rightarrow \quad A - 5I_2 + 7A^{-1} = O$$

[$\because A^2A^{-1} = A.A.A^{-1} = AI_2 = A$ and $AA^{-1} = I_2$ and $IB = B$]

$$\Rightarrow \quad 7A^{-1} = -A + 5I_2$$

$$= -\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\Rightarrow \quad 7A^{-1} = \begin{bmatrix} -3+5 & -1+0 \\ 1+0 & -2+5 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}.$$

Caution. Because we were to find; **Hence** A^{-1} *i.e.,* A^{-1} from $A^2 - 5A + 7I = O$, so don't use

$$A^{-1} = \frac{\text{adj. A}}{|A|} \text{ to find } A^{-1} \text{ here.}$$

14. For the matrix A = $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find numbers *a* and *b* such that $A^2 + aA + bI = O$.

Sol. Given: Matrix A = $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$$\therefore \quad A^2 = A.A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Putting values of A^2 and A in $A^2 + aA + bI_2 = O$,
(Here I is I_2 because A is 2×2), we have

$$\begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O$$

$$\Rightarrow \quad \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2a \\ a & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 11+3a+b & 8+2a+0 \\ 4+a+0 & 3+a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating corresponding entries, we have

$11 + 3a + b = 0$ ...(*i*)

$8 + 2a = 0 \quad (\Rightarrow \quad 2a = -8 \quad \Rightarrow \quad a = -4)$

$4 + a = 0 \quad (\Rightarrow \quad a = -4), \; 3 + a + b = 0$ ...(*ii*)

Value of $a = -4$ is same from both equations.
Therefore, $a = -4$ is correct.
Putting $a = -4$ in (*i*), $11 - 12 + b = 0$ or $b - 1 = 0$ *i.e.,* $b = 1$
Again putting $a = -4$ in (*ii*), $3 - 4 + b = 0$
i.e., $-1 + b = 0$ or $b = 1$
The two values of $b = 1$ are same from both equations.
$\therefore$ $A^2 + aA + bI = 0$ holds true when $a = -4$ and $b = 1$.

15. For the matrix A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, show that $A^3 - 6A^2 + 5A + 11I = O$. Hence find A^{-1}.

Sol. $A^2 = A \,.\, A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

Performing row by column multiplication,

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \quad ...(i)$$

$$\therefore \quad A^3 = A^2 \,.\, A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

Now, putting values of A^3, A^2, A and I_3 in $A^3 - 6A^2 + 5A + 11I_3$
(Here I is I_3 because matrix A is of order 3×3)

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6 \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24+5 & 7-12+5 & 1-6+5 \\ -23+18+5 & 27-48+10 & -69+84-15 \\ 32-42+10 & -13+18-5 & 58-84+15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_{3 \times 3}.$$

$\therefore \quad A^3 - 6A^2 + 5A + 11I_3 = O_{3 \times 3}$ **(proved above)**

Hence find A^{-1}.

(See caution at the end of solution of Q. No. 13)
Now multiplying both sides by A^{-1}.

$(A^{-1}A)\, A^2 - 6(A^{-1}A)\, A + 5(A^{-1}A) + 11A^{-1}I_3 = A^{-1} \,.\, O_{3 \times 3}$

$\Rightarrow \quad A^2 - 6IA + 5I + 11A^{-1} = 0 \qquad (\because \; A^{-1}A = I \text{ and } A^{-1}0 = 0)$

$\Rightarrow \quad A^2 - 6A + 5I + 11A^{-1} = 0$

$\Rightarrow \quad 11A^{-1} = 6A - 5I - A^2$

or $11A^{-1} = 6\begin{bmatrix}1 & 1 & 1\\1 & 2 & -3\\2 & -1 & 3\end{bmatrix} - 5\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

$- \begin{bmatrix}4 & 2 & 1\\-3 & 8 & -14\\7 & -3 & 14\end{bmatrix}$ (From (i))

or $11A^{-1} = \begin{bmatrix}6-5-4 & 6-2 & 6-1\\6+3 & 12-5-8 & -18+14\\12-7 & -6+3 & 18-5-14\end{bmatrix}$

or $11A^{-1} = \begin{bmatrix}-3 & 4 & 5\\9 & -1 & -4\\5 & -3 & -1\end{bmatrix}$ or $A^{-1} = \dfrac{1}{11}\begin{bmatrix}-3 & 4 & 5\\9 & -1 & -4\\5 & -3 & -1\end{bmatrix}$.

16. If $A = \begin{bmatrix}2 & -1 & 1\\-1 & 2 & -1\\1 & -1 & 2\end{bmatrix}$, verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}.

Sol. Given: $A = \begin{bmatrix}2 & -1 & 1\\-1 & 2 & -1\\1 & -1 & 2\end{bmatrix}$

$\therefore \quad A^2 = A.A = \begin{bmatrix}2 & -1 & 1\\-1 & 2 & -1\\1 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\-1 & 2 & -1\\1 & -1 & 2\end{bmatrix}$

$\Rightarrow \quad A^2 = \begin{bmatrix}4+1+1 & -2-2-1 & 2+1+2\\-2-2-1 & 1+4+1 & -1-2-2\\2+1+2 & -1-2-2 & 1+1+4\end{bmatrix}$

$= \begin{bmatrix}6 & -5 & 5\\-5 & 6 & -5\\5 & -5 & 6\end{bmatrix}$

$\therefore \quad A^3 = A^2 \,.\, A = \begin{bmatrix}6 & -5 & 5\\-5 & 6 & -5\\5 & -5 & 6\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\-1 & 2 & -1\\1 & -1 & 2\end{bmatrix}$

$= \begin{bmatrix}12+5+5 & -6-10-5 & 6+5+10\\-10-6-5 & 5+12+5 & -5-6-10\\10+5+6 & -5-10-6 & 5+5+12\end{bmatrix} = \begin{bmatrix}22 & -21 & 21\\-21 & 22 & -21\\21 & -21 & 22\end{bmatrix}$

L.H.S. $= A^3 - 6A^2 + 9A - 4I = A^3 - 6A^2 + 9A - 4I_3$

(Here I is I_3 because A is 3×3)

Putting values

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$+ 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix}$$

$$+ \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36 & -21+30 & 21-30 \\ -21+30 & 22-36 & -21+30 \\ 21-30 & -21+30 & 22-36 \end{bmatrix} + \begin{bmatrix} 18-4 & -9-0 & 9-0 \\ -9-0 & 18-4 & -9-0 \\ 9-0 & -9-0 & 18-4 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 9 & -9 \\ 9 & -14 & 9 \\ -9 & 9 & -14 \end{bmatrix} + \begin{bmatrix} 14 & -9 & 9 \\ -9 & 14 & -9 \\ 9 & -9 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} -14+14 & 9-9 & -9+9 \\ 9-9 & -14+14 & 9-9 \\ -9+9 & 9-9 & -14+14 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O = \text{R.H.S.}$$

$\therefore \quad A^3 - 6A^2 + 9A - 4I_3 = O$ ...(*i*)

Hence to find A^{-1}

(See **caution** at the end of solution of Q. No. 13)

Multiplying both sides of (*i*) by A^{-1},

$$A^3A^{-1} - 6A^2\,A^{-1} + 9A\,A^{-1} - 4I_3A^{-1} = O.A^{-1}$$

or $$A^2 - 6A + 9I_3 - 4A^{-1} = 0$$

$[\because \quad A^3A^{-1} = A^2AA^{-1} = A^2I = A^2$ etc. and also $IB = B]$

$\Rightarrow \quad -4A^{-1} = -A^2 + 6A - 9I_3$

Dividing by -4, $A^{-1} = \frac{1}{4}A^2 - \frac{6}{4}A + \frac{9}{4}I_3 \qquad = \frac{1}{4}[A^2 - 6A + 9I_3]$

$$\Rightarrow A^{-1} = \frac{1}{4}\left[\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$\Rightarrow A^{-1} = \frac{1}{4}\left[\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right]$$

$$= \frac{1}{4}\begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

17. Let A be a non-singular matrix of order 3 × 3. Then | adj. A | is equal to

(A) | A | (B) $| A |^2$ (C) $| A |^3$ (D) 3 | A |.

Sol. If A is a non-singular matrix of order $n \times n$,

then $|$ **adj. A** $| = |$ **A** $|^{n-1}$.

Putting $n = 3$, $|\text{ adj. A }| = |\text{ A }|^2$

$\therefore$ Option (B) is the correct answer.

18. If A is an invertible matrix of order 2, then det (A^{-1}) is equal to

(A) det A (B) $\frac{1}{\det A}$ (C) 1 (D) 0.

Sol. We know that $AA^{-1} = I$ for every invertible matrix A.

Taking determinants on both sides, we have

$| AA^{-1} | = | I | \Rightarrow | A | | A^{-1} | = 1$

Dividing by $| A |$, $| A^{-1} | = \frac{1}{| A |}$ *i.e.,* $\det (A^{-1}) = \frac{1}{\det A}$

$\therefore$ Option (B) is the correct answer.

Exercise 4.6

Examine the consistency of the system of equations in Exercises 1 to 3.

1. $\boldsymbol{x + 2y = 2}$
$\boldsymbol{2x + 3y = 3.}$

Sol. Given linear equations are

$$x + 2y = 2$$
$$2x + 3y = 3$$

Their matrix form is $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $(\Rightarrow \quad AX = B)$

Comparing A = $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$| A | = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = 3 - 4 = -1 \neq 0$$

$\therefore$ (Unique) solution and hence equations are consistent.

2. $\boldsymbol{2x - y = 5}$
$\boldsymbol{x + y = 4.}$

Sol. Given linear equations are

$$2x - y = 5$$
$$x + y = 4$$

Their matrix form is $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ $(\Rightarrow \quad AX = B)$

Comparing A = $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$$| A | = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = 2 - (-1) = 3 \neq 0$$

$\therefore$ (Unique) solution and hence equations are consistent.

3. $\boldsymbol{x + 3y = 5}$
$\boldsymbol{2x + 6y = 8.}$

Sol. Given linear equations are

$$x + 3y = 5$$
$$2x + 6y = 8$$

Their matrix form is $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ $(\Rightarrow \quad AX = B)$

Comparing A = $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} 5 \\ 8 \end{bmatrix}$ $| A | = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = 6 - 6 = 0$

So we are to find (adj. A) B

$$\text{adj. A} = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} \qquad \left(\because \text{ adj.} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$$

$$\therefore \text{(adj. A) B} = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 30 - 24 \\ -10 + 8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq \text{O}$$

$$\left(\because \text{ The matrix} \begin{bmatrix} 6 \\ -2 \end{bmatrix} \text{has non-zero entries}\right)$$

$\therefore$ Given Equations are Inconsistent *i.e.,* have no common solution.

Examine the consistency of the system of equations in Exercises 4 to 6.

4. $\boldsymbol{x + y + z = 1}$

$\boldsymbol{2x + 3y + 2z = 2}$

$\boldsymbol{ax + ay + 2az = 4}$

Sol. The given equations are

$$x + y + z = 1 \qquad \text{...}(i)$$
$$2x + 3y + 2z = 2 \qquad \text{...}(ii)$$
$$ax + ay + 2az = 4 \qquad \text{...}(iii)$$

Their matrix form is $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ ($\Rightarrow$ AX = B)

$$\therefore \text{ Matrix } \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix} \qquad \therefore \ |A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{vmatrix}$$

Expanding along first row,

$$|A| = 1(6a - 2a) - 1(4a - 2a) + 1(2a - 3a)$$
$$= 4a - 2a - a = a$$

Case I. $\boldsymbol{a \neq 0}$ $\therefore$ $|A| = a \neq 0$

$\therefore$ (Unique) solution and hence equations are consistent.

Case II. $\boldsymbol{a = 0}$ $\therefore$ $|A| = a = 0.$

Putting $a = 0$ in given equation (*iii*), we have $0 = 4$ which is impossible.

$\therefore$ Given equations are inconsistent if $a = 0$.

5. $\boldsymbol{3x - y - 2z = 2}$

$\boldsymbol{2y - z = -1}$

$\boldsymbol{3x - 5y = 3}$

Sol. The given equations are

$$3x - y - 2z = 2$$
$$2y - z = -1 \quad i.e., \quad 0x + 2y - z = -1$$
and $$3x - 5y = 3 \quad i.e., \quad 3x - 5y + 0z = 3$$

Their matrix form is $\begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ $(\Rightarrow \quad AX = B)$

Comparing $\quad A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Expanding along first row,

$$|A| = 3(0-5) - (-1)(0+3) + (-2)(0-6)$$
$$= 3(-5) + 3 + 12 = -15 + 15 = 0$$

So now we are to find (adj. A) B

To find adj. A for $|A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

$$A_{11} = + \begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} = (0-5) = -5,$$

$$A_{12} = - \begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} = -(0+3) = -3,$$

$$A_{13} = + \begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix} = (0-6) = -6,$$

$$A_{21} = - \begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} = -(0-10) = 10,$$

$$A_{22} = + \begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} = (0+6) = 6,$$

$$A_{23} = - \begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix} = -(-15+3) = 12,$$

$$A_{31} = + \begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} = (1+4) = 5,$$

$$A_{32} = - \begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} = -(-3-0) = 3,$$

$$A_{33} = + \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = +(6-0) = 6.$$

$$\therefore \quad \text{adj. } A = \begin{bmatrix} -5 & -3 & -6 \\ 10 & 6 & 12 \\ 5 & 3 & 6 \end{bmatrix}' = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$\therefore \quad (\text{adj. } A)\, B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -10-10+15 \\ -6-6+9 \\ -12-12+18 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq O \qquad \left[\because \text{ The matrix } \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \text{ has non-zero entries} \right]$$

$\therefore$ Given equations are inconsistent.

6. $\mathbf{5x - y + 4z = 5}$
$\mathbf{2x + 3y + 5z = 2}$
$\mathbf{5x - 2y + 6z = -1}$

Sol. The given equations are

$5x - y + 4z = 5$
$2x + 3y + 5z = 2$
$5x - 2y + 6z = -1$

Their matrix form is $\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$ $(\Rightarrow AX = B)$

$$\therefore A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} \qquad |A| = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix}$$

Expanding along first row

$$= 5(18 + 10) - (-1)(12 - 25) + 4(-4 - 15)$$
$$= 5(28) + (-13) + 4(-19)$$
$$= 140 - 13 - 76 = 140 - 89 = 51 \neq 0$$

$\therefore$ Given system of equations has a (unique) solution and hence equations are consistent.

Solve the system of linear equations, using matrix method, in Exercises 7 to 10.

7. $\mathbf{5x + 2y = 4}$
$\mathbf{7x + 3y = 5.}$

Sol. The given equations are

$5x + 2y = 4$
$7x + 3y = 5$

Their matrix form is $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ ($\Rightarrow$ AX = B)

Comparing A = $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \end{bmatrix}$ and B = $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix} = 15 - 14 = 1 \neq 0$$

$\therefore$ Solution is unique and $\mathbf{X = A^{-1}B}$

$$\Rightarrow \quad X = \frac{1}{|A|} (\text{adj. } A) \,.\, B$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1} \begin{bmatrix} \boxed{3 \quad -2} \\ \boxed{-7 \quad 5} \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} \left(\because \text{ adj.} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 - 10 \\ -28 + 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Equating corresponding entries, we have $x = 2$ and $y = -3$.

8. $\mathbf{2x - y = -2}$
$\mathbf{3x + 4y = 3.}$

Sol. The given equations are

$2x - y = -2$
$3x + 4y = 3$

Their matrix form is $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ ($\Rightarrow$ AX = B)

Comparing A = $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \end{bmatrix}$ and B = $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 8 - (-3) = 8 + 3 = 11 \neq 0$$

$\therefore$ Solution is unique and $\mathbf{X = A^{-1}B}$

$$\Rightarrow X = \frac{1}{|A|} (\text{adj. } A) \,.\, B \quad \Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11} \begin{bmatrix} \boxed{4 \quad 1} \\ \boxed{-3 \quad 2} \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad = \frac{1}{11} \begin{bmatrix} -8 + 3 \\ 6 + 6 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -5 \\ 12 \end{bmatrix} = \begin{bmatrix} -\frac{5}{11} \\ \frac{12}{11} \end{bmatrix}$$

Equating corresponding entries, we have $x = -\frac{5}{11}$ and $y = \frac{12}{11}$.

9. $\boldsymbol{4x - 3y = 3}$
$\boldsymbol{3x - 5y = 7.}$

Sol. The given equations are
$4x - 3y = 3$
$3x - 5y = 7$

Their matrix form is $\begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ $(\Rightarrow \quad AX = B)$

Comparing $A = \begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & -3 \\ 3 & -5 \end{vmatrix} = -20 - (-9) = -20 + 9 = -11 \neq 0$$

$\therefore$ Solution is unique and $\mathbf{X = A^{-1}B}$

$$\Rightarrow \quad X = \frac{1}{|A|} \text{ (adj. A) B}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} -15+21 \\ -9+28 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} 6 \\ 19 \end{bmatrix} = \begin{bmatrix} -\frac{6}{11} \\ -\frac{19}{11} \end{bmatrix}$$

Equating corresponding entries, we have $x = -\frac{6}{11}$ and $y = -\frac{19}{11}$.

10. $\boldsymbol{5x + 2y = 3}$
$\boldsymbol{3x + 2y = 5.}$

Sol. The given equations are
$5x + 2y = 3$
$3x + 2y = 5$

Their matrix form is $\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ $(\Rightarrow \quad AX = B)$

Comparing $A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 5 & 2 \\ 3 & 2 \end{vmatrix} = 10 - 6 = 4 \neq 0$$

$\therefore$ Solution is unique and $\mathbf{X = A^{-1}B}$

$$\Rightarrow \quad X = \frac{1}{|A|} \text{ (adj. A) B}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}\left(\because \text{adj.}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ 16 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Equating corresponding entries, we have $x = -1$ and $y = 4$.

Solve the system of linear equations, using matrix method, in Exercises 11 to 14.

11. $2x + y + z = 1$

$x - 2y - z = \frac{3}{2}$

$3y - 5z = 9.$

Sol. The given equations are

$2x + y + z = 1$

$x - 2y - z = \frac{3}{2}$

$3y - 5z = 9$ or $0.x + 3y - 5z = 9$

Their matrix form is $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$

$(\Rightarrow AX = B)$

Comparing $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{vmatrix}$$

Expanding along first row, $= 2(10 + 3) - 1(-5 - 0) + 1(3 - 0)$

or $|A| = 2(13) + 5 + 3 = 26 + 5 + 3 = 34 \neq 0$

$\therefore$ Solution is unique and $\mathbf{X = A^{-1}B} = \frac{1}{|A|}$ (adj. A) B ...(*i*)

Let us find adj. A

$$A_{11} = + \begin{vmatrix} -2 & -1 \\ 3 & -5 \end{vmatrix} = 10 + 3 = 13,$$

$$A_{12} = -\begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix} = -(-5-0) = 5,$$

$$A_{13} = +\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix} = (3-0) = 3,$$

$$A_{21} = -\begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix} = -(-5-3) = 8,$$

$$A_{22} = +\begin{vmatrix} 2 & 1 \\ 0 & -5 \end{vmatrix} = (-10-0) = -10,$$

$$A_{23} = -\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = -(6-0) = -6,$$

$$A_{31} = +\begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} = (-1+2) = 1,$$

$$A_{32} = -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -(-2-1) = 3,$$

$$A_{33} = +\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -4-1 = -5.$$

$$\therefore \quad \text{Adj. A} = \begin{bmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{bmatrix}' = \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

Putting values in eqn. (i), $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{1}{34}\begin{bmatrix} \boxed{13 \quad 8 \quad 1} \\ \boxed{5 \quad -10 \quad 3} \\ \boxed{3 \quad -6 \quad -5} \end{bmatrix}\begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$

$$= \frac{1}{34}\begin{bmatrix} 13+12+9 \\ 5-15+27 \\ 3-9-45 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{-3}{2} \end{bmatrix}$$

Equating corresponding entries, we have $x = 1$,

$$y = \frac{1}{2},\ z = -\frac{3}{2}.$$

12. $\boldsymbol{x - y + z = 4}$

$\boldsymbol{2x + y - 3z = 0}$

$\boldsymbol{x + y + z = 2.}$

Sol. The given equations are

$$x - y + z = 4$$
$$2x + y - 3z = 0$$
$$x + y + z = 2$$

Their matrix form is $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ $(\Rightarrow AX = B)$

Comparing $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

Expanding along first row,

$$= 1(1 + 3) - (-1)(2 + 3) + 1(2 - 1)$$

or $|A| = 4 + 5 + 1 = 10 \neq 0$

$\therefore$ Solution is unique and $\mathbf{X = A^{-1}B} = \dfrac{1}{|A|}$ (adj. A) B ...(*i*)

To find adj. A

$$A_{11} = + \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = (1 + 3) = 4,$$

$$A_{12} = - \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -(2 + 3) = -5,$$

$$A_{13} = + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (2 - 1) = 1,$$

$$A_{21} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-1 - 1) = 2,$$

$$A_{22} = + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = (1 - 1) = 0,$$

$$A_{23} = - \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -(1 + 1) = -2,$$

$$A_{31} = + \begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = (3 - 1) = 2,$$

$$A_{32} = - \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -(-3 - 2) = 5,$$

$$A_{33} = + \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1 + 2 = 3.$$

$$\therefore \quad \text{adj. A} = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}' = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

Putting these values in eqn. (i), we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4-0+6 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Equating corresponding entries, we have

$x = 2, \; y = -1, z = 1.$

13. $\boldsymbol{2x + 3y + 3z = 5}$

$\boldsymbol{x - 2y + z = -4}$

$\boldsymbol{3x - y - 2z = 3.}$

Sol. The given equations are

$$2x + 3y + 3z = 5$$
$$x - 2y + z = -4$$
$$3y - y - 2z = 3$$

Their matrix form is $\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$ $(\Rightarrow \; AX = B)$

Comparing $A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix}$$

Expanding along first row, $|A| = 2(4 + 1) - 3(-2 - 3) + 3(-1 + 6)$

$$= 2(5) - 3(-5) + 3(5) = 10 + 15 + 15 = 40 \neq 0$$

$\therefore$ Solution is unique and $\mathbf{X = A^{-1}B} = \dfrac{1}{|A|}$ (adj. A) B ...(i)

Let us find adj. A

$$A_{11} = +\begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix} = 4 + 1 = 5,$$

$$A_{12} = -\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -(-2 - 3) = 5,$$

$$A_{13} = +\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = -1 + 6 = 5,$$

$$A_{21} = -\begin{vmatrix} 3 & 3 \\ -1 & -2 \end{vmatrix} = -(-6 + 3) = 3,$$

$$A_{22} = + \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -4 - 9 = -13,$$

$$A_{23} = - \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -(-2 - 9) = 11,$$

$$A_{31} = + \begin{vmatrix} 3 & 3 \\ -2 & 1 \end{vmatrix} = 3 + 6 = 9,$$

$$A_{32} = - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -(2 - 3) = 1,$$

$$A_{33} = + \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -4 - 3 = -7.$$

$$\therefore \quad \text{adj. A} = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}' = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

Putting these values in eqn. (i), $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$= \frac{1}{40} \begin{bmatrix} \boxed{5 \quad 3 \quad 9} \\ \boxed{5 \quad -13 \quad 1} \\ \boxed{5 \quad 11 \quad -7} \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Equating corresponding entries, we have $x = 1$, $y = 2$, $z = -1$.

14. $\boldsymbol{x - y + 2z = 7}$

$\boldsymbol{3x + 4y - 5z = -5}$

$\boldsymbol{2x - y + 3z = 12.}$

Sol. The given equations are

$$x - y + 2z = 7$$
$$3x + 4y - 5z = -5$$
$$2x - y + 3z = 12$$

Their matrix form is $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$ ($\Rightarrow$ AX = B)

Comparing, $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix}$$

Expanding along first row,

$$|A| = 1(12 - 5) - (-1)(9 + 10) + 2(-3 - 8)$$
$$= 7 + 19 - 22 = 4 \neq 0$$

$\therefore$ Solution is unique and $\mathbf{X = A^{-1}B} = \dfrac{1}{|A|}$ (adj. A) B ...(*i*)

Let us find adj. A

$$A_{11} = + \begin{vmatrix} 4 & -5 \\ -1 & 3 \end{vmatrix} = 12 - 5 = 7,$$

$$A_{12} = - \begin{vmatrix} 3 & -5 \\ 2 & 3 \end{vmatrix} = -(9 + 10) = -19,$$

$$A_{13} = + \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = -3 - 8 = -11,$$

$$A_{21} = - \begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix} = -(-3 + 2) = 1,$$

$$A_{22} = + \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1,$$

$$A_{23} = - \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -(-1 + 2) = -1,$$

$$A_{31} = + \begin{vmatrix} -1 & 2 \\ 4 & -5 \end{vmatrix} = 5 - 8 = -3,$$

$$A_{32} = - \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} = -(-5 - 6) = 11,$$

$$A_{33} = + \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = 4 + 3 = 7.$$

$$\therefore \quad \text{adj. A} = \begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}' = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

Putting values in eqn. (*i*),

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 49 - 5 - 36 \\ -133 + 5 + 132 \\ -77 + 5 + 84 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Equating corresponding entries, we have $x = 2$, $y = 1$, $z = 3$.

15. If A = $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Using A^{-1}, solve the system of equations

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3.$$

Sol. Given: Matrix A = $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

To find A^{-1} $|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$

Expanding along first row,

$$|A| = 2(-4 + 4) - (-3)(-6 + 4) + 5(3 - 2)$$
$$= 0 + 3(-2) + 5 = -6 + 5 = -1 \neq 0$$

$\therefore$ A^{-1} exists and $A^{-1} = \dfrac{1}{|A|}$ (adj. A) ...(*i*)

To find adj. A from $|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$

$$A_{11} = + \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = (-4 + 4) = 0,$$

$$A_{12} = - \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = -(-6 + 4) = 2,$$

$$A_{13} = + \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1,$$

$$A_{21} = - \begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = -(6 - 5) = -1,$$

$$A_{22} = + \begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -4 - 5 = -9,$$

$$A_{23} = - \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -(2 + 3) = -5,$$

$$A_{31} = + \begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = (12 - 10) = 2,$$

$$A_{32} = - \begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = -(-8 - 15) = 23,$$

$$A_{33} = + \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = (4 + 9) = 13.$$

$$\therefore \quad \text{adj. A} = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}' = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

Putting this value of adj. A in (*i*),

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \ldots(ii) \left(\because \frac{1}{-1} = -1\right)$$

Now using (this) A^{-1}, we are to solve the equations

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

Their matrix form is $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$ $(\Rightarrow AX = B)$

Comparing $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

Solution is unique and $X = A^{-1}B$ $(\because\ A^{-1}$ exists by (*ii*))

Putting values, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Equating corresponding entries, we have $x = 1, y = 2, z = 3$.

16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹ 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹ 90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is ₹ 70. Find cost of each item per kg by matrix method.

Sol. Let ₹ x, ₹ y, ₹ z per kg be the prices of onion, wheat and rice respectively.

$\therefore$ According to the given data, we have the following three equations

$$4x + 3y + 2z = 60,$$
$$2x + 4y + 6z = 90,$$
and $$6x + 2y + 3z = 70.$$

We know that these equations can be expressed in the matrix form as

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

or $$AX = B,$$

where $A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$ $|A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix}$

Expanding along first row,

$$|A| = 4(12 - 12) - 3(6 - 36) + 2(4 - 24)$$
$$= 0 - 3(-30) + 2(-20) = 90 - 40 = 50 \neq 0$$

Hence A is non-singular

$\therefore$ A^{-1} exists.

$\therefore$ Unique solution is $X = A^{-1} B$ ...(*i*)

$A_{11} = +(12 - 12) = 0,$ $A_{12} = -(6 - 36) = 30,$

$A_{13} = +(4 - 24) = -20$

$A_{21} = -(9 - 4) = -5,$ $A_{22} = +(12 - 12) = 0,$

$A_{23} = -(8 - 18) = 10$ $A_{31} = +(18 - 8) = 10,$

$A_{32} = -(24 - 4) = -20,$ $A_{33} = +(16 - 6) = 10$

$$\therefore \text{ adj. } A = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}' = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

Putting values of X, A^{-1} and B in (*i*), we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix} \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$= \frac{1}{50} \begin{bmatrix} -450 + 700 \\ 1800 - 1400 \\ -1200 + 900 + 700 \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix}$$

or $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$\Rightarrow$ $x = 5, y = 8, z = 8.$

$\therefore$ The cost of onion, wheat and rice are respectively ₹ 5, ₹ 8 and ₹ 8 per kg.

MISCELLANEOUS EXERCISE

1. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ

Sol. Let $\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

Expanding along first row

$$\Delta = x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta \begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta \begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix}$$

$$= x(-x^2 - 1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta)$$

$$= -x^3 - x + x\sin^2\theta + \boldsymbol{\sin\theta\cos\theta - \sin\theta\cos\theta} + x\cos^2\theta$$

$$= -x^3 - x + x(\sin^2\theta + \cos^2\theta) = -x^3 - x + x$$

$= -x^3$ which is free from θ *i.e.,* independent of θ.

2. Without expanding the determinants, prove that

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

Sol. L.H.S. $= \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

Multiplying R_1 by a, R_2 by b and R_3 by c (by looking at the partial products in third column)

$$= \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$$

Taking abc common from C_3, $= \frac{abc}{abc}\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$

Interchanging C_1 and C_3, $= -\begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix}$

Interchanging C_2 and C_3, $= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ $(\because (-1)(-1) = 1)$

$=$ R.H.S.

3. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$**.**

Sol. Let $\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Expanding along R_1,

$= \cos\alpha\cos\beta\begin{vmatrix} \cos\beta & 0 \\ \sin\alpha\sin\beta & \cos\alpha \end{vmatrix} - \cos\alpha\sin\beta$

$\begin{vmatrix} -\sin\beta & 0 \\ \sin\alpha\cos\beta & \cos\alpha \end{vmatrix} - \sin\alpha\begin{vmatrix} -\sin\beta & \cos\beta \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta \end{vmatrix}$

$= \cos\alpha\cos\beta(\cos\alpha\cos\beta - 0) - \cos\alpha\sin\beta(-\cos\alpha\sin\beta$

$- 0) - \sin\alpha(-\sin\alpha\sin^2\beta - \sin\alpha\cos^2\beta)$

$= \cos^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha(\sin^2\beta + \cos^2\beta)$

$= \cos^2\alpha(\cos^2\beta + \sin^2\beta) + \sin^2\alpha(\sin^2\beta + \cos^2\beta)$

$= \cos^2\alpha + \sin^2\alpha = 1.$

4. If a, b and c are real numbers and

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0.$$

Show that either $a + b + c = 0$ or $a = b = c$.

Sol. Given: $(\Delta =)\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

Operate $R_1 \to R_1 + R_2 + R_3$

($\because$ Sum of entries of each **column** is same and $= 2a + 2b + 2c = 2(a + b + c)$)

$$\Rightarrow \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Taking out $2(a + b + c)$ common from R_1.

$$\Rightarrow \quad 2(a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

$\therefore$ Either $2(a + b + c) = 0$ *i.e.,* $a + b + c = \frac{0}{2} = 0$ ...(*i*)

or
$$\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Now each entry of first row is 1.

So operate $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ (to create two zeros in first row)

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ c+a & a+b-c-a & b+c-c-a \\ a+b & b+c-a-b & c+a-a-b \end{vmatrix} = 0$$

Expanding along R_1, $\Rightarrow \begin{vmatrix} b-c & b-a \\ c-a & c-b \end{vmatrix} = 0$

$\Rightarrow \quad (b - c)(c - b) - (b - a)(c - a) = 0$

$\Rightarrow \quad bc - b^2 - c^2 + bc - bc + ab + ac - a^2 = 0$

$\Rightarrow \quad -a^2 - b^2 - c^2 + ab + bc + ac = 0$

Multiplying by -2, $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac = 0$

or $\quad a^2 + a^2 + b^2 + b^2 + c^2 + c^2 - 2ab - 2bc - 2ac = 0$

or $\quad (a^2 + b^2 - 2ab) + (b^2 + c^2 - 2bc) + (a^2 + c^2 - 2ac) = 0$

$\Rightarrow \quad (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$\Rightarrow \quad a - b = 0$ and $b - c = 0$ and $c - a = 0$

[$\because$ $x^2 + y^2 + z^2 = 0$ and $x, y, z \in R$ $\Rightarrow$ $x = 0, y = 0$ and $z = 0$]

$\Rightarrow \quad a = b$ and $b = c$ and $c = a$ $\Rightarrow$ $a = b = c$ ...(*ii*)

From (*i*) and (*ii*) either $a + b + c = 0$ or $a = b = c$.

Note. We can also start doing this question by operating $C_1 \to C_1 + C_2 + C_3$.

5. Solve the equation $\begin{vmatrix} \boldsymbol{x+a} & \boldsymbol{x} & \boldsymbol{x} \\ \boldsymbol{x} & \boldsymbol{x+a} & \boldsymbol{x} \\ \boldsymbol{x} & \boldsymbol{x} & \boldsymbol{x+a} \end{vmatrix}$ **= 0,** $\boldsymbol{a \neq 0}$**.**

Sol. Given: The equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$

Sum of entries of each **column** is same and = $(3x + a)$, so let us operate $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Taking out $(3x + a)$ common from R_1

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

$\therefore$ Either $3x + a = 0$ *i.e.,* $3x = -a$ *i.e.,* $x = -\dfrac{a}{3}$ ...(*i*)

or $\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$

Now each entry of first row is 1,
so operate $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ (to create two zeros in first row)

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0$$

Expanding along first row $1(a^2 - 0) = 0$ *i.e.,* $a^2 = 0$
$\Rightarrow$ $a = 0$. But this is contrary to given that $a \neq 0$.

$\therefore$ From (*i*) $x = -\dfrac{a}{3}$ is the only solution (root).

Note. We can also start doing this question by operating $C_1 \to C_1 + C_2 + C_3$.

6. Prove that $\begin{vmatrix} \boldsymbol{a^2} & \boldsymbol{bc} & \boldsymbol{ac+c^2} \\ \boldsymbol{a^2+ab} & \boldsymbol{b^2} & \boldsymbol{ac} \\ \boldsymbol{ab} & \boldsymbol{b^2+bc} & \boldsymbol{c^2} \end{vmatrix}$ $\boldsymbol{= 4a^2b^2c^2}$**.**

Sol. L.H.S. = $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2 & bc & c(a+c) \\ a(a+b) & b^2 & ac \\ ab & b(b+c) & c^2 \end{vmatrix}$$

Taking a, b, c common from C_1, C_2, C_3 respectively

$$= abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Operate $R_1 \to R_1 - R_2 - R_3$ (to create one zero in R_1)

$$= abc \begin{vmatrix} a-a-b-b & c-b-b-c & a+c-a-c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$$= abc \begin{vmatrix} -2b & -2b & 0 \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Operate $C_2 \to C_2 - C_1$ (to create another zero in R_1)

$$= abc \begin{vmatrix} -2b & 0 & 0 \\ a+b & -a & a \\ b & c & c \end{vmatrix}$$

Expanding along R_1 $= abc(-2b) \begin{vmatrix} -a & a \\ c & c \end{vmatrix}$

$= abc(-2b)(-ac-ac)$

$= abc(-2b)(-2ac) = 4a^2b^2c^2$ = R.H.S.

7. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$.

Sol. Given: $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

We know that $\mathbf{(AB)^{-1} = B^{-1} A^{-1}}$ (Reversal Law) ...(i)

Now A^{-1} is given, so let us find B^{-1}.

$$|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

Expanding along first row,

$|B| = 1(3-0) - 2(-1-0) + (-2)(2-0)$

$= 3 + 2 - 4 = 1 \neq 0$

$\therefore$ B^{-1} exists.

To find adj. B

$$\mathbf{B_{11} = + \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} = (3 - 0) = 3,}$$

$$B_{12} = - \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -(-1 - 0) = 1$$

$$B_{13} = + \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2,$$

$$B_{21} = - \begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = -(2 - 4) = 2,$$

$$B_{22} = + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1,$$

$$B_{23} = - \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = -(-2 - 0) = 2,$$

$$B_{31} = + \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = (0 + 6) = 6,$$

$$B_{32} = - \begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} = -(0 - 2) = 2,$$

$$B_{33} = + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = (3 + 2) = 5.$$

$$\therefore \quad \text{adj. } B = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}' = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$\therefore \quad B^{-1} = \frac{1}{|B|} (\text{adj. } B) = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \qquad (\because \ |B| = 1)$$

Putting values of B^{-1} and A^{-1} in eqn. (i), we have

$$(AB)^{-1} = \begin{bmatrix} \boxed{3 \quad 2 \quad 6} \\ \boxed{1 \quad 1 \quad 2} \\ \boxed{2 \quad 2 \quad 5} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 3 \\ -15 \\ 5 \end{matrix}} & \boxed{\begin{matrix} -1 \\ 6 \\ -2 \end{matrix}} & \boxed{\begin{matrix} 1 \\ -5 \\ 2 \end{matrix}} \end{bmatrix}$$

$$= \begin{bmatrix} 9 - 30 + 30 & -3 + 12 - 12 & 3 - 10 + 12 \\ 3 - 15 + 10 & -1 + 6 - 4 & 1 - 5 + 4 \\ 6 - 30 + 25 & -2 + 12 - 10 & 2 - 10 + 10 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

8. Let A = $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, verify that

(*i*) (adj. A)$^{-1}$ = adj. (A^{-1}) (*ii*) (A^{-1})$^{-1}$ = A.

Sol. Given: Matrix A = $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ $\therefore$ $|A| = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{vmatrix}$

$= 1(15 - 1) - (-2)(-10 - 1) + 1(-2 - 3)$

$= 14 - 22 - 5 = -13 \neq 0$

To find adj. A

$$A_{11} = + \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} = 15 - 1 = 14,$$

$$A_{12} = - \begin{vmatrix} -2 & 1 \\ 1 & 5 \end{vmatrix} = -(-10 - 1) = 11$$

$$A_{13} = + \begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} = (-2 - 3) = -5,$$

$$A_{21} = - \begin{vmatrix} -2 & 1 \\ 1 & 5 \end{vmatrix} = -(-10 - 1) = 11$$

$$A_{22} = + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = (5 - 1) = 4,$$

$$A_{23} = - \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = -(1 + 2) = -3$$

$$A_{31} = + \begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = -2 - 3 = -5,$$

$$A_{32} = - \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = -(1 + 2) = -3$$

$$A_{33} = + \begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix} = 3 - 4 = -1.$$

$$\therefore \quad \text{adj. A } (= \text{B (say)}) = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}'$$

$$= \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{ adj. A} = \frac{-1}{13} \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} \quad ...(i)$$

Now let us find (adj. A)$^{-1}$ = B^{-1}

$$|B| = \begin{vmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{vmatrix}$$

Expanding along first row,

$$|B| = 14(-4-9) - 11(-11-15) - 5(-33+20)$$
$$= 14(-13) - 11(-26) - 5(-13)$$
$$= -182 + 286 + 65 = -182 + 351 = 169 \neq 0$$

To find adj. B

$$B_{11} = + \begin{vmatrix} 4 & -3 \\ -3 & -1 \end{vmatrix} = (-4-9) = -13,$$

$$B_{12} = - \begin{vmatrix} 11 & -3 \\ -5 & -1 \end{vmatrix} = -(-11-15) = 26,$$

$$B_{13} = + \begin{vmatrix} 11 & 4 \\ -5 & -3 \end{vmatrix} = +(-33+20) = -13,$$

$$B_{21} = - \begin{vmatrix} 11 & -5 \\ -3 & -1 \end{vmatrix} = -(-11-15) = 26,$$

$$B_{22} = + \begin{vmatrix} 14 & -5 \\ -5 & -1 \end{vmatrix} = (-14-25) = -39,$$

$$B_{23} = - \begin{vmatrix} 14 & 11 \\ -5 & -3 \end{vmatrix} = -(-42+55) = -13,$$

$$B_{31} = + \begin{vmatrix} 11 & -5 \\ 4 & -3 \end{vmatrix} = -33+20 = -13,$$

$$B_{32} = - \begin{vmatrix} 14 & -5 \\ 11 & -3 \end{vmatrix} = -(-42+55) = -13,$$

$$B_{33} = + \begin{vmatrix} 14 & 11 \\ 11 & 4 \end{vmatrix} = (56-121) = -65.$$

$$\therefore \quad \text{adj. } B = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}' = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$$

Taking – 13 common from this **matrix** adj. B

$$= -13 \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$\therefore \quad B^{-1} \textit{ i.e.,} \text{ (adj. A)}^{-1} = \frac{1}{|B|} \text{ adj. } B$$

$$= \frac{1}{169}\ (-\ 13) \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$= \frac{-1}{13} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} \qquad ...(ii)$$

Now let us find $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{C}^{-1}$ (say) where

$$C = A^{-1} = \frac{-1}{13} \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} \qquad [\text{By } (i)]$$

or
$$C = A^{-1} = \begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}$$

$$\therefore \quad |\,C\,| = |\,A^{-1}\,| = -\frac{14}{13}\left(-\frac{4}{169} - \frac{9}{169}\right)$$

$$-\left(-\frac{11}{13}\right)\left(-\frac{11}{169} - \frac{15}{169}\right) + \frac{5}{13}\left(\frac{-33}{169} + \frac{20}{169}\right)$$

$$= -\frac{14}{13}\left(-\frac{13}{169}\right) + \frac{11}{13}\left(-\frac{26}{169}\right) + \frac{5}{13}\left(-\frac{13}{169}\right)$$

$$= \frac{14}{169} - \frac{22}{169} - \frac{5}{169} = \frac{14 - 22 - 5}{169} = -\frac{13}{169} = \frac{-1}{13} \neq 0$$

Let us now find adj. C *i.e.*, adj. ($\mathbf{A}^{-1}$)

$$C_{11} = + \begin{vmatrix} -\frac{4}{13} & \frac{3}{13} \\ \frac{3}{13} & \frac{1}{13} \end{vmatrix} = \left(-\frac{4}{169} - \frac{9}{169}\right) = \frac{-13}{169} = \frac{-1}{13}$$

$$C_{12} = - \begin{vmatrix} -\frac{11}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{1}{13} \end{vmatrix} = -\left(-\frac{11}{169} - \frac{15}{169}\right) = \frac{26}{169} = \frac{2}{13}$$

$$C_{13} = + \begin{vmatrix} -\frac{11}{13} & -\frac{4}{13} \\ \frac{5}{13} & \frac{3}{13} \end{vmatrix} = +\left(-\frac{33}{169} + \frac{20}{169}\right) = \frac{-13}{169} = \frac{-1}{13}$$

$$C_{21} = -\begin{vmatrix} -\frac{11}{13} & \frac{5}{13} \\ \frac{3}{13} & \frac{1}{13} \end{vmatrix} = -\left(-\frac{11}{169} - \frac{15}{169}\right) = \frac{26}{169} = \frac{2}{13}$$

$$C_{22} = +\begin{vmatrix} -\frac{14}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{1}{13} \end{vmatrix} = -\frac{14}{169} - \frac{25}{169} = -\frac{39}{169} = -\frac{3}{13}$$

$$C_{23} = -\begin{vmatrix} -\frac{14}{13} & -\frac{11}{13} \\ \frac{5}{13} & \frac{3}{13} \end{vmatrix} = -\left(-\frac{42}{169} + \frac{55}{169}\right) = -\frac{13}{169} = -\frac{1}{13}$$

$$C_{31} = +\begin{vmatrix} -\frac{11}{13} & \frac{5}{13} \\ \frac{-4}{13} & \frac{3}{13} \end{vmatrix} = -\frac{33}{169} + \frac{20}{169} = -\frac{13}{169} = -\frac{1}{13}$$

$$C_{32} = -\begin{vmatrix} -\frac{14}{13} & \frac{5}{13} \\ -\frac{11}{13} & \frac{3}{13} \end{vmatrix} = -\left(-\frac{42}{169} + \frac{55}{169}\right) = -\frac{13}{169} = -\frac{1}{13}$$

$$C_{33} = +\begin{vmatrix} -\frac{14}{13} & -\frac{11}{13} \\ -\frac{11}{13} & -\frac{4}{13} \end{vmatrix} = \frac{56}{169} - \frac{121}{169} = -\frac{65}{169} = -\frac{5}{13}$$

$$\therefore \quad \text{adj. C} = \text{adj. }(A^{-1}) = \begin{bmatrix} -\frac{1}{13} & \frac{2}{13} & -\frac{1}{13} \\ \frac{2}{13} & -\frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & -\frac{1}{13} & -\frac{5}{13} \end{bmatrix}'$$

$$= \begin{bmatrix} -\frac{1}{13} & \frac{2}{13} & -\frac{1}{13} \\ \frac{2}{13} & -\frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & -\frac{1}{13} & -\frac{5}{13} \end{bmatrix}$$

Taking $-\frac{1}{13}$ common,

$$\Rightarrow \quad \text{adj. C} = \text{adj. }(A^{-1}) = -\frac{1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} \qquad ...(iii)$$

From (*ii*) and (*iii*), we can say that (adj. A)$^{-1}$ = adj. (A^{-1})

($\because$ R.H.Sides of eqns. (*ii*) and (*iii*) same)

Hence first part is verified.

Again $(A^{-1})^{-1} = C^{-1}$ $\qquad (\because \ C = A^{-1})$

$$= \frac{1}{|C|}\text{ adj. C} = \frac{1}{\left(-\frac{1}{13}\right)}\left(-\frac{1}{13}\right)\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} \text{(By }(iii))$$

$$\Rightarrow \quad (A^{-1})^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = A \text{ (given)}$$

Hence second part is verified.

9. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$.

Sol. Let $\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Operate $R_1 \to R_1 + R_2 + R_3$ (because sum of entries of each **column** is same and $= 2x + 2y = 2(x + y)$)

$$= \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Taking out $2(x + y)$ common from R_1,

$$= 2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Now each entry of first row is 1, so let us operate $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ (to create two zeros in first row)

$$= 2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x+y-y & x-y \\ x+y & x-x-y & y-x-y \end{vmatrix}$$

$$= 2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix}$$

Expanding along R_1, $\Delta = 2(x + y) \,.\, 1 \begin{vmatrix} x & x-y \\ -y & -x \end{vmatrix}$

$= 2(x + y)\,(-x^2 + y(x - y)) = 2(x + y)\,(-x^2 + xy - y^2)$
$= -2(x + y)\,(x^2 + y^2 - xy) = -2(x^3 + y^3)$
$[\because\ x^3 + y^3 = (x + y)\,(x^2 + y^2 - xy)]$

Remark. This question can also be done by operating $C_1 \to C_1 + C_2 + C_3$.

10. Evaluate $\begin{vmatrix} \mathbf{1} & \boldsymbol{x} & \boldsymbol{y} \\ \mathbf{1} & \boldsymbol{x+y} & \boldsymbol{y} \\ \mathbf{1} & \boldsymbol{x} & \boldsymbol{x+y} \end{vmatrix}$**.**

Sol. Let $\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Each entry of one column (here first) is 1.
Sol let us operate $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ (to create two zeros in first column)

$$= \begin{vmatrix} 1 & x & y \\ 0 & x+y-x & 0 \\ 0 & 0 & x+y-y \end{vmatrix} = \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}$$

Expanding along first column, $\Delta = 1 \begin{vmatrix} y & 0 \\ 0 & x \end{vmatrix} = xy.$

Using properties of determinants in Exercises 11 to 15, prove that:

11. $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$ $\boldsymbol{= (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)(\alpha + \beta + \gamma).}$

Sol. L.H.S. $= \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$

Operate $C_3 \to C_3 + C_1$ to make all entries of third column equal

$$= \begin{vmatrix} \alpha & \alpha^2 & \alpha+\beta+\gamma \\ \beta & \beta^2 & \alpha+\beta+\gamma \\ \gamma & \gamma^2 & \alpha+\beta+\gamma \end{vmatrix}$$

Taking out $(\alpha + \beta + \gamma)$ common from C_3,

$$= (\alpha + \beta + \gamma) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta & \beta^2 & 1 \\ \gamma & \gamma^2 & 1 \end{vmatrix}$$

Now each entry of one column (here third is 1), so let us operate $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ to create two zeros in third column

$$= (\alpha + \beta + \gamma) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta - \alpha & \beta^2 - \alpha^2 & 0 \\ \gamma - \alpha & \gamma^2 - \alpha^2 & 0 \end{vmatrix}$$

Expanding along third column,

$$= (\alpha + \beta + \gamma) \begin{vmatrix} \beta - \alpha & \beta^2 - \alpha^2 \\ \gamma - \alpha & \gamma^2 - \alpha^2 \end{vmatrix}$$

$$= (\alpha + \beta + \gamma) \begin{vmatrix} (\beta - \alpha) & (\beta - \alpha)(\beta + \alpha) \\ (\gamma - \alpha) & (\gamma - \alpha)(\gamma + \alpha) \end{vmatrix}$$

Taking $(\beta - \alpha)$ common from R_1 and $(\gamma - \alpha)$ common from R_2

$$= (\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha) \begin{vmatrix} 1 & \beta + \alpha \\ 1 & \gamma + \alpha \end{vmatrix}$$

$$= (\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha)(\gamma + \alpha - \beta - \alpha)$$

$$= (\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha)(\gamma - \beta)$$

$$= (\alpha + \beta + \gamma)\,[-(\alpha - \beta)]\,(\gamma - \alpha)\,[-(\beta - \gamma)]$$

$$= (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha + \beta + \gamma)$$

$$= \text{R.H.S.}$$

12. $\begin{vmatrix} x & x^2 & 1 + px^3 \\ y & y^2 & 1 + py^3 \\ z & z^2 & 1 + pz^3 \end{vmatrix} = (1 + pxyz)(x - y)(y - z)(z - x).$

Sol. L.H.S. $= \begin{vmatrix} x & x^2 & 1 + px^3 \\ y & y^2 & 1 + py^3 \\ z & z^2 & 1 + pz^3 \end{vmatrix}$

(Here each entry of third column is the sum of two entries, so we can write this determinant as sum of two determinants, the first two columns being same in both determinants)

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$$

$$\uparrow \qquad \uparrow$$

$$= \Delta_1 \quad + \quad \Delta_2 \qquad \text{...}(i)$$

Now $\Delta_2 = \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$

In an effort to make Δ_2 similar to Δ_1,
taking x, y, z common from R_1, R_2, R_3 respectively and p common from C_3,

$$\Delta_2 = pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Operate $C_1 \leftrightarrow C_3$, $= -\, pxyz \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}$

Operate $C_1 \leftrightarrow C_2$, $\Delta_2 = pxyz \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} = pxyz\Delta_1$

Putting this value of Δ_2 in (i), L.H.S. $= \Delta_1 + pxyz\Delta_1$

$$= (1 + pxyz)\Delta_1 \qquad \text{...}(ii)$$

Now $\Delta_1 = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$

Operate $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ (to create two zeros in third column)

$$= \begin{vmatrix} x & x^2 & 1 \\ y - x & y^2 - x^2 & 0 \\ z - x & z^2 - x^2 & 0 \end{vmatrix}$$

Expanding along third column

$$\Delta_1 = \begin{vmatrix} y-x & y^2-x^2 \\ z-x & z^2-x^2 \end{vmatrix} = \begin{vmatrix} (y-x) & (y-x)(y+x) \\ (z-x) & (z-x)(z+x) \end{vmatrix}$$

Taking $(y-x)$ common from R_1 and $(z-x)$ common from R_2,

$$\Delta_1 = (y-x)(z-x)\begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix}$$

$$= (y-x)(z-x)(z+x-y-x)$$

$$= (y-x)(z-x)(z-y) = -(x-y)(z-x)(-(y-z))$$

or $\Delta_1 = (x-y)(y-z)(z-x)$

Putting this value of Δ_1 in (ii),

L.H.S. $= (1+pxyz)(x-y)(y-z)(z-x)$ = R.H.S.

13. $$\begin{vmatrix} \boldsymbol{3a} & \boldsymbol{-a+b} & \boldsymbol{-a+c} \\ \boldsymbol{-b+a} & \boldsymbol{3b} & \boldsymbol{-b+c} \\ \boldsymbol{-c+a} & \boldsymbol{-c+b} & \boldsymbol{3c} \end{vmatrix} = \boldsymbol{3(a+b+c)(ab+bc+ca).}$$

Sol. L.H.S. $= \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$

Here the sum of entries of each **row** is same and $= a+b+c$. So let us operate $C_1 \to C_1 + C_2 + C_3$

$\therefore$ L.H.S. $= \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$

Taking out $(a+b+c)$ common from C_1,

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Now each entry of first column is 1. So let us operate $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ to create two zeros in first column,

$$\text{L.H.S.} = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 3b+a-b & -b+c+a-c \\ 0 & -c+b+a-b & 3c+a-c \end{vmatrix}$$

Expanding along first column,

$$= (a+b+c)\,.\,1\begin{vmatrix} 2b+a & a-b \\ a-c & 2c+a \end{vmatrix}.$$

$\therefore$ L.H.S. $= (a+b+c)\,[(2b+a)(2c+a)-(a-b)(a-c)]$

$= (a+b+c)\,[4bc+2ab+2ac+a^2-a^2+ac+ab-bc]$

$= (a+b+c)(3ab+3bc+3ac)$

$= 3(a + b + c)(ab + bc + ac)$
$= \text{R.H.S.}$

14. $\begin{vmatrix} \mathbf{1} & \mathbf{1+p} & \mathbf{1+p+q} \\ \mathbf{2} & \mathbf{3+2p} & \mathbf{4+3p+2q} \\ \mathbf{3} & \mathbf{6+3p} & \mathbf{10+6p+3q} \end{vmatrix} = \mathbf{1.}$

Sol. L.H.S. $= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$

On looking at $a_{11} = 1$,

Operate $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$ (to make entries a_{21} and a_{31} of first column as zeros)

$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$

$$\begin{array}{lcccc} R_2: & 2 & 3+2p & 4+3p+2q \\ 2R_1: & 2 & 2+2p & 2+2p+2q \\ - & - & - & - \\ \hline R_2 - 2R_1: & 0 & 1 & 2+p \end{array}$$

$$\begin{array}{lcccc} R_3: & 3 & 6+3p & 10+6p+3q \\ 3R_1: & 3 & 3+3p & 3+3p+3q \\ - & - & - & - \\ \hline R_3 - 3R_1: & 0 & 3 & 7+3p \end{array}$$

Expanding along first column,

L.H.S. $= 1\begin{vmatrix} 1 & 2+p \\ 3 & 7+3p \end{vmatrix} - 0 + 0$

$= 7 + 3p - 3(2 + p) = 7 + 3p - 6 - 3p = 1 = \text{R.H.S.}$

15. $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = \mathbf{0}$

Sol. L.H.S. $= \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$

$= \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$

Operate $C_3 \to C_3 + (\sin\delta)\, C_1$

$= \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta + \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta - \sin\beta\sin\delta + \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta + \sin\gamma\sin\delta \end{vmatrix}$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta \end{vmatrix}$$

Taking $\cos\delta$ common from C_3,

$$= \cos\delta \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha \\ \sin\beta & \cos\beta & \cos\beta \\ \sin\gamma & \cos\gamma & \cos\gamma \end{vmatrix}$$

$= \cos\delta\,(0)$ [$\because$ C_2 and C_3 have become identical]

$= 0$

= R.H.S.

16. Solve the system of the following equations: (Using matrices)

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4,\ \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1,\ \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2.$$

Sol. Put $\frac{1}{x} = u,\ \frac{1}{y} = v,\ \frac{1}{z} = w.$

$\therefore$ The given equations become

$2u + 3v + 10w = 4,\ 4u - 6v + 5w = 1,\ 6u + 9v - 20w = 2$

The matrix form of these equations is

$$\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

It is of the form AX = B

where $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix},\ X = \begin{bmatrix} u \\ v \\ w \end{bmatrix},\ B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$$

Expanding by first row

$= 2(120 - 45) - 3(-80 - 30) + 10(36 + 36)$

$= 2(75) - 3(-110) + 10(72) = 150 + 330 + 720 = 1200 \neq 0$

$\therefore$ Matrix A is non-singular

$\therefore$ A^{-1} exists. $\therefore$ Unique solution is $X = A^{-1}B$ ...(*i*)

Now let us find adj. A.

Now $A_{11} = +(120 - 45) = 75,\ A_{12} = -(-80 - 30) = 110,$

$A_{13} = +(36 + 36) = 72 \quad A_{21} = -(-60 - 90) = 150,$

$A_{22} = +(-40 - 60) = -100,\ A_{23} = -(18 - 18) = 0$

$A_{31} = +(15 + 60) = 75 \quad A_{32} = -(10 - 40) = 30,$

$A_{33} = +(-12 - 12) = -24$

$$\therefore \quad \text{adj. } A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}' = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj. A}}{|A|} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Putting these values of X, A^{-1} and B in (*i*), we have

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

Equating corresponding entries, $u = \frac{1}{2}, v = \frac{1}{3}, w = \frac{1}{5}$

$$\therefore \quad \frac{1}{u} = 2, \ \frac{1}{v} = 3, \ \frac{1}{w} = 5$$

i.e., $x = \frac{1}{u} = 2, \ y = \frac{1}{v} = 3, \ z = \frac{1}{w} = 5.$

Choose the correct answer in Exercises 17 to 19.

17. If *a*, *b*, *c*, are in A.P., then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix} \textbf{ is}$$

(A) 0 **(B) 1** **(C) x** **(D) $2x$.**

Sol. $\because$ a, b and c are in A.P. $\therefore$ $b - a = c - b$ ...(*i*)

$$\text{Let} \quad \Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, [By looking at (*i*)], we have

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ 1 & 1 & 2(b-a) \\ 1 & 1 & 2(c-b) \end{vmatrix}$$

Putting $c - b = b - a$ from (i), $\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ 1 & 1 & 2(b-a) \\ 1 & 1 & 2(b-a) \end{vmatrix}$

$= 0$ $\quad (\because\ R_2$ and R_3 are identical$)$

Alternative Method

Operate $R_1 \to R_1 + R_3 - 2R_2$.

18. If x, y, z are non zero real numbers, then the inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

(A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ **(B)** $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(C) $\dfrac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ **(D)** $\dfrac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Sol. Given: Matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ $\quad \therefore\ |A| = \begin{vmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{vmatrix}$

Expanding along first row,

$$|A| = x(yz - 0) - 0 + 0 = xyz \neq 0$$

$(\because$ It is given that x, y, z are non-zero real numbers$)$

$\therefore\ A^{-1}$ exists and $A^{-1} = \dfrac{1}{|A|}$ adj. A ...(i)

To find adj. A

$A_{11} = +\begin{vmatrix} y & 0 \\ 0 & z \end{vmatrix} = yz$, $\quad A_{12} = -\begin{vmatrix} 0 & 0 \\ 0 & z \end{vmatrix} = -(0-0) = 0$,

$A_{13} = +\begin{vmatrix} 0 & y \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$, $\quad A_{21} = -\begin{vmatrix} 0 & 0 \\ 0 & z \end{vmatrix} = -(0-0) = 0$,

$A_{22} = +\begin{vmatrix} x & 0 \\ 0 & z \end{vmatrix} = xz$, $\quad A_{23} = -\begin{vmatrix} x & 0 \\ 0 & 0 \end{vmatrix} = -(0-0) = 0$,

$A_{31} = +\begin{vmatrix} 0 & 0 \\ y & 0 \end{vmatrix} = (0-0) = 0$, $\quad A_{32} = -\begin{vmatrix} x & 0 \\ 0 & 0 \end{vmatrix} = -(0-0) = 0$,

$A_{33} = +\begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix} = xy$.

$$\therefore \quad \text{adj. A} = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}' = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

Putting values in eqn. (*i*), $A^{-1} = \dfrac{1}{xyz}\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$

$$= \begin{bmatrix} \dfrac{yz}{xyz} & 0 & 0 \\ 0 & \dfrac{xz}{xyz} & 0 \\ 0 & 0 & \dfrac{xy}{xyz} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{x} & 0 & 0 \\ 0 & \dfrac{1}{y} & 0 \\ 0 & 0 & \dfrac{1}{z} \end{bmatrix}$$

$$= \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

$\therefore$ Option (A) is the correct answer.

Remark. The answer of this Q. No. 18 should be used as a formula for one mark questions and Entrance Examinations.

For example, inverse matrix of diagonal matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is diagonal matrix $\begin{bmatrix} \dfrac{1}{2} & 0 & 0 \\ 0 & \dfrac{1}{3} & 0 \\ 0 & 0 & \dfrac{1}{5} \end{bmatrix}$.

19. Let A = $\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$. Then

(A) Det (A) = 0 **(B) Det (A) $\in$ (2, ∞)**

(C) Det (A) $\in$ (2, 4) **(D) Det (A) $\in$ [2, 4].**

Sol. Given: Matrix $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$

$$\therefore \qquad |A| = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$$

Expanding along first row

det A *i.e.,* $|A| = 1(1 + \sin^2\theta) - \sin\theta$

$(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1)$

$= 1 + \sin^2\theta + 1 + \sin^2\theta = 2 + 2\sin^2\theta$ ...(*i*)

We know that $-1 \le \sin\theta \le 1$

$\therefore \quad 0 \le \sin^2\theta \le 1$

($\because$ $\sin^2\theta$ can never be negative)

Multiplying by 2, $0 \le 2\sin^2\theta \le 2$

Adding 2 to all sides $2 \le 2 + 2\sin^2\theta \le 4$

i.e., $2 \le$ det. $A \le 4$ (By (*i*))

$\therefore$ (Value of) det $A \in$ closed interval [2, 4].

Therefore option (D) is correct answer.