Exercise 3.1

1. In the matrix A = $\begin{bmatrix} \mathbf{2} & \mathbf{5} & \mathbf{19} & \mathbf{-7} \\ \mathbf{35} & \mathbf{-2} & \mathbf{5/2} & \mathbf{12} \\ \mathbf{\sqrt{3}} & \mathbf{1} & \mathbf{-5} & \mathbf{17} \end{bmatrix}$**, write**

(*i*) The order of the matrix (*ii*) The number of elements

(*iii*) Write the elements a_{13}, a_{21}, a_{33}, a_{24}, a_{23}.

Sol. (*i*) There are 3 horizontal lines (rows) and 4 vertical lines (columns) in the given matrix A.

$\therefore$ Order of the matrix A is **3 × 4**.

(*ii*) The number of elements in this matrix A is 3 × 4 = 12.

($\because$ The number of elements in a $m \times n$ matrix is $m \,.\, n$)

(*iii*) $a_{13} \Rightarrow$ Element in first row and third column = 19

$a_{21} \Rightarrow$ Element in second row and first column = 35

$a_{33} \Rightarrow$ Element in third row and third column = – 5

$a_{24} \Rightarrow$ Element in second row and fourth column = 12

$a_{23} \Rightarrow$ Element in second row and third column $= \dfrac{5}{2}$.

2. If a matrix has 24 elements, what are the possible orders it can have? What, if it has 13 elements?

Sol. We know that a matrix having mn elements is of order $m \times n$.

(*i*) Now 24 = 1 × 24, 2 × 12, 3 × 8, 4 × 6 and hence

= 24 × 1, 12 × 2, 8 × 3, 6 × 4 also.

$\therefore$ There are 8 possible matrices having 24 elements of orders 1 × 24, 2 × 12, 3 × 8, 4 × 6, 24 × 1, 12 × 2, 8 × 3, 6 × 4.

(*ii*) Again (prime number) 13 = 1 × 13 and 13 × 1 only.

$\therefore$ There are 2 possible matrices of order 1 × 13 (Row matrix) and 13 × 1 (Column matrix)

3. If a matrix has 18 elements, what are the possible orders it can have? What if has 5 elements?

Sol. We know that a matrix having mn elements is of order $m \times n$.

(*i*) Now 18 = 1 × 18, 2 × 9, 3 × 6 and hence 18 × 1, 9 × 2, 6 × 3 also.

$\therefore$ There are 6 possible matrices having 18 elements of orders 1 × 18, 2 × 9, 3 × 6, 18 × 1, 9 × 2 and 6 × 3.

(*ii*) Again (Prime number) 5 = 1 × 5 and 5 × 1 only.

$\therefore$ There are 2 possible matrices of order 1 × 5 and 5 × 1.

4. Construct a 2 × 2 matrix A = [a_{ij}] whose elements are given by:

(*i*) $a_{ij} = \dfrac{(i+j)^2}{2}$ **(*ii*)** $a_{ij} = \dfrac{i}{j}$ **(*iii*)** $a_{ij} = \dfrac{(i+2j)^2}{2}$

Sol. To construct a 2 × 2 matrix A = [a_{ij}]

(*i*) **Given:** $a_{ij} = \dfrac{(i+j)^2}{2}$ ...(*i*)

In (*i*),

Put $i = 1, j = 1,$ $\therefore$ $a_{11} = \dfrac{(1+1)^2}{2} = \dfrac{2^2}{2} = \dfrac{4}{2} = 2$

Put $i = 1, j = 2,$ $\therefore$ $a_{12} = \dfrac{(1+2)^2}{2} = \dfrac{3^2}{2} = \dfrac{9}{2}$

Put $i = 2, j = 1;$ $\therefore$ $a_{21} = \dfrac{(2+1)^2}{2} = \dfrac{9}{2}$

Put $i = 2, j = 2;$ $\therefore$ $a_{22} = \dfrac{(2+2)^2}{2} = \dfrac{4^2}{2} = \dfrac{16}{2} = 8$

$$\therefore \quad A_{2\times 2} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & \dfrac{9}{2} \\ \dfrac{9}{2} & 8 \end{bmatrix}.$$

(*ii*) **Given:** $a_{ij} = \dfrac{i}{j}$ ...(*i*)

In (*i*),

Put $i = 1, j = 1,$ $\therefore$ $a_{11} = \dfrac{1}{1} = 1$

Put $i = 1, j = 2,$ $\therefore$ $a_{12} = \dfrac{1}{2}$

Put $i = 2, j = 1;$ $\therefore$ $a_{21} = \dfrac{2}{1} = 2$

Put $i = 2, j = 2;$ $\therefore$ $a_{22} = \dfrac{2}{2} = 1$

$$\therefore \quad A_{2\times 2} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & \dfrac{1}{2} \\ 2 & 1 \end{bmatrix}.$$

(*iii*) **Given:** $a_{ij} = \dfrac{(i+2j)^2}{2}$ ...(*i*)

In (*i*),

Put $i = 1, j = 1$; $\therefore\ a_{11} = \dfrac{(1+2)^2}{2} = \dfrac{3^2}{2} = \dfrac{9}{2}$

Put $i = 1, j = 2$; $\therefore\ a_{12} = \dfrac{(1+4)^2}{2} = \dfrac{5^2}{2} = \dfrac{25}{2}$

Put $i = 2, j = 1$; $\therefore\ a_{21} = \dfrac{(2+2)^2}{2} = \dfrac{16}{2} = 8$

Put $i = 2, j = 2$; $\therefore\ a_{22} = \dfrac{(2+4)^2}{2} = \dfrac{6^2}{2} = \dfrac{36}{2} = 18$

$$\therefore\ A_{2\times 2} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{9}{2} & \dfrac{25}{2} \\ 8 & 18 \end{bmatrix}.$$

5. Construct a 3 × 4 matrix, whose elements are given by:

(*i*) $a_{ij} = \dfrac{1}{2}\,|-3i+j|$ (*ii*) $a_{ij} = 2i - j$.

Sol. (*i*) To construct a 3 × 4 matrix say A.

Given: $a_{ij} = \dfrac{1}{2}|-3i+j|$ ...(*i*)

In (*i*),

Put $i = 1, j = 1$,

$\therefore\ a_{11} = \dfrac{1}{2}|-3+1| = \dfrac{1}{2}|-2| = \dfrac{1}{2}(2) = 1$

Put $i = 1, j = 2$,

$\therefore\ a_{12} = \dfrac{1}{2}|-3+2| = \dfrac{1}{2}|-1| = \dfrac{1}{2}(1) = \dfrac{1}{2}$

$i = 1, j = 3$,

$\therefore\ a_{13} = \dfrac{1}{2}|-3+3| = \dfrac{1}{2}|0| = \dfrac{1}{2}(0) = 0$

$i = 1, j = 4$,

$\therefore\ a_{14} = \dfrac{1}{2}|-3+4| = \dfrac{1}{2}|1| = \dfrac{1}{2}(1) = \dfrac{1}{2}$

$i = 2, j = 1$,

$\therefore\ a_{21} = \dfrac{1}{2}|-6+1| = \dfrac{1}{2}|-5| = \dfrac{5}{2}$

$i = 2, j = 2$,

$\therefore\ a_{22} = \dfrac{1}{2}|-6+2| = \dfrac{1}{2}|-4| = \dfrac{4}{2} = 2$

$i = 2, j = 3$,

$\therefore\ a_{23} = \dfrac{1}{2}|-6+3| = \dfrac{1}{2}|-3| = \dfrac{3}{2}$

$i = 2, j = 4$,

$\therefore\ a_{24} = \dfrac{1}{2}|-6+4| = \dfrac{1}{2}|-2| = \dfrac{2}{2} = 1$

$i = 3, j = 1,$

$\therefore \quad a_{31} = \frac{1}{2} | -9 + 1 | = \frac{1}{2} | -8 | = \frac{8}{2} = 4$

$i = 3, j = 2,$

$\therefore \quad a_{32} = \frac{1}{2} | -9 + 2 | = \frac{1}{2} | -7 | = \frac{7}{2}$

$i = 3, j = 3,$

$\therefore \quad a_{33} = \frac{1}{2} | -9 + 3 | = \frac{1}{2} | -6 | = \frac{6}{2} = 3$

$i = 3, j = 4,$

$\therefore \quad a_{34} = \frac{1}{2} | -9 + 4 | = \frac{1}{2} | -5 | = \frac{5}{2}$

$$\therefore \quad A_{3 \times 4} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}.$$

(ii) **Given:** $a_{ij} = 2i - j$

$\therefore \quad a_{11} = 2 - 1 = 1, \qquad a_{12} = 2 - 2 = 0$

$a_{13} = 2 - 3 = -1, \qquad a_{14} = 2 - 4 = -2$

$a_{21} = 4 - 1 = 3, \qquad a_{22} = 4 - 2 = 2$

$a_{23} = 4 - 3 = 1, \qquad a_{24} = 4 - 4 = 0$

$a_{31} = 6 - 1 = 5, \qquad a_{32} = 6 - 2 = 4$

$a_{33} = 6 - 3 = 3, \qquad a_{34} = 6 - 4 = 2$

$$\therefore \quad A_{3 \times 4} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 \end{bmatrix}.$$

6. Find the values of x, y and z from the following equations:

(i) $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$ **(ii)** $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$

(iii) $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

Sol. (i) **Given:** $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$

By definition of Equal matrices, equating corresponding entries, we have $4 = y$, $3 = z$, $x = 1$, $5 = 5$

$\therefore \quad x = 1, y = 4, z = 3.$

(ii) **Given:** $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$

Equating corresponding entries, we have

$x + y = 6$ ...(*i*)

$5 + z = 5 \quad i.e., \quad z = 5 - 5 = 0$

and $xy = 8$ ...(*ii*)

Let us solve (*i*) and (*ii*) for x and y.

From (*i*), $y = 6 - x$

Putting this value of y in (*ii*), we have

$x(6 - x) = 8$ or $6x - x^2 = 8$

or $-x^2 + 6x - 8 = 0$ or $x^2 - 6x + 8 = 0$

or $x^2 - 4x - 2x + 8 = 0$ or $x(x - 4) - 2(x - 4) = 0$

or $(x - 4)(x - 2) = 0$

$\therefore$ Either $x - 4 = 0$ or $x - 2 = 0$

i.e., $x = 4$ or $x = 2$.

When $x = 4$, then $y = 6 - x = 6 - 4 = 2$

$\therefore$ $x = 4$, $y = 2$, $z = 0$.

When $x = 2$, then $y = 6 - x = 6 - 2 = 4$

$\therefore$ $x = 2$, $y = 4$, $z = 0$.

(*iii*) **Given:** $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

Equating corresponding entries, we have

$x + y + z = 9$ ...(*i*)

$x + z = 5$ ...(*ii*)

$y + z = 7$ ...(*iii*)

Eqn. (*i*) – eqn. (*ii*) gives $y = 9 - 5 = 4$

Eqn. (*i*) – eqn. (*iii*) gives $x = 9 - 7 = 2$

Putting $x = 2$ and $y = 4$ in (*i*), $2 + 4 + z = 9$

or $6 + z = 9$

$\therefore$ $z = 3$

Hence $x = 2$, $y = 4$, $z = 3$.

7. Find the values of *a*, *b*, *c* and *d* from the equation

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}.$$

Sol. Equating corresponding entries of given equal matrices, we have

$a - b = -1$ ...(*i*)

$2a - b = 0$ ...(*ii*)

$2a + c = 5$ ...(*iii*)

and $3c + d = 13$ ...(*iv*)

Eqn. (*i*) – eqn. (*ii*) gives $-a = -1$ or $a = 1$

Putting $a = 1$ in (*i*), $1 - b = -1$ or $-b = -2$ or $b = 2$

Putting $a = 1$ in (*iii*), $2 + c = 5 \Rightarrow c = 5 - 2 = 3$

Putting $c = 3$ in (*iv*), $9 + d = 13$ or $d = 13 - 9 = 4$

$\therefore$ $a = 1$, $b = 2$, $c = 3$, $d = 4$.

8. A = $[a_{ij}]_{m \times n}$ is a square matrix, if
(A) $m < n$ (B) $m > n$ (C) $m = n$ (D) None of these.

Sol. (C) is the correct option.
($\because$ By definition of square matrix $m = n$)

9. Which of the given values of x and y make the following pair of matrices equal

$$\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix}, \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$$

(A) $x = \frac{-1}{3}, y = 7$ (B) Not possible to find

(C) $y = 7, x = \frac{-2}{3}$ (D) $x = \frac{-1}{3}, y = \frac{-2}{3}$.

Sol. According to given, matrix $\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix}$ = matrix $\begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$

Equating corresponding entries, we have

$3x + 7 = 0 \Rightarrow 3x = -7 \Rightarrow x = -\frac{7}{3}$ ...(*i*)

$5 = y - 2 \Rightarrow 5 + 2 = y \Rightarrow y = 7$

$y + 1 = 8 \Rightarrow y = 8 - 1 = 7$

and $2 - 3x = 4 \Rightarrow -3x = 2 \Rightarrow x = -\frac{2}{3}$ ...(*ii*)

The two values of $x = -\frac{7}{3}$ given by (*i*) and $x = -\frac{2}{3}$ given by (*ii*) are not equal.

$\therefore$ No values of x and y exist to make the two matrices equal.

$\therefore$ Option (B) is the correct answer.

10. The number of all possible matrices of order 3 × 3 with each entry 0 or 1 is:
(A) 27 (B) 18 (C) 81 (D) 512.

Sol. We know that general matrix of order 3 × 3 is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

This matrix has 3 × 3 = 9 elements.

The number of choices for a_{11} is 2 (as 0 or 1 can be used)

Similarly, the number of choices for each other element is 2.

Hence, total possible arrangements (matrices)

$$= \frac{2 \times 2 \times \ldots \times 2}{9 \text{ times}} \quad \text{(By fundamental principle of counting)}$$

$$= 2^9 = 512$$

$\therefore$ Option (D) is the correct answer.

Exercise 3.2

1. Let A = $\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, B = $\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$, C = $\begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$.

Find each of the following:

(*i*) A + B **(*ii*) A – B** **(*iii*) 3A – C**

(*iv*) AB **(*v*) BA.**

Sol. (*i*) $A + B = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2+1 & 4+3 \\ 3-2 & 2+5 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$

(*ii*) $A - B = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2-1 & 4-3 \\ 3+2 & 2-5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$

(*iii*) $3A - C = 3\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} - C = \begin{bmatrix} 3\times 2 & 3\times 4 \\ 3\times 3 & 3\times 2 \end{bmatrix} - C$

$= \begin{bmatrix} 6 & 12 \\ 9 & 6 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6+2 & 12-5 \\ 9-3 & 6-4 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$

(*iv*) $AB = \begin{bmatrix} \boxed{2 \quad 4} \\ \boxed{3 \quad 2} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 1 \\ -2 \end{matrix}} & \boxed{\begin{matrix} 3 \\ 5 \end{matrix}} \end{bmatrix}$

Performing row by column multiplication,

$= \begin{bmatrix} 2(1)+4(-2) & 2(3)+4(5) \\ 3(1)+2(-2) & 3(3)+2(5) \end{bmatrix} = \begin{bmatrix} 2-8 & 6+20 \\ 3-4 & 9+10 \end{bmatrix} = \begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix}$

(*v*) $BA = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$

Performing row by column multiplication,

$= \begin{bmatrix} 1(2)+3(3) & 1(4)+3(2) \\ (-2)2+5(3) & (-2)(4)+5(2) \end{bmatrix} = \begin{bmatrix} 2+9 & 4+6 \\ -4+15 & -8+10 \end{bmatrix} = \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$

Note. From solutions of part (*iv*) and (*v*), we can easily observe that AB need not be equal to BA *i.e.,* matrix multiplication need not be commutative.

2. Compute the following:

(*i*) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

(*ii*) $\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

(*iii*) $\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$

(iv) $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$.

Sol. (i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a+a & b+b \\ -b+b & a+a \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 2a \end{bmatrix}$

(ii) $\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

$= \begin{bmatrix} a^2+b^2+2ab & b^2+c^2+2bc \\ a^2+c^2-2ac & a^2+b^2-2ab \end{bmatrix} = \begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} -1+12 & 4+7 & -6+6 \\ 8+8 & 5+0 & 16+5 \\ 2+3 & 8+2 & 5+4 \end{bmatrix} = \begin{bmatrix} 11 & 11 & 0 \\ 16 & 5 & 21 \\ 5 & 10 & 9 \end{bmatrix}$

(iv) $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$

$= \begin{bmatrix} \cos^2 x + \sin^2 x & \sin^2 x + \cos^2 x \\ \sin^2 x + \cos^2 x & \cos^2 x + \sin^2 x \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

3. Compute the indicated products:

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [2 \quad 3 \quad 4]$

(iii) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$ **(vi)** $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$

Sol. (i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is defined because the pre-matrix has 2 columns which is equal to the number of rows of the post-matrix.

Performing row by column multiplication,

$= \begin{bmatrix} a(a)+b(b) & a(-b)+b(a) \\ (-b)a+a(b) & (-b)(-b)+a(a) \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & b^2+a^2 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1} \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}_{1\times 3}$ is defined because the pre-matrix has one column which is equal to the number of rows of the post-matrix.

Performing row by column multiplication,

$$= \begin{bmatrix} 1(2) & 1(3) & 1(4) \\ 2(2) & 2(3) & 2(4) \\ 3(2) & 3(3) & 3(4) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}_{3\times 3}$$

(iii) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1(1)+(-2)2 & 1(2)+(-2)3 & 1(3)+(-2)1 \\ 2(1)+3(2) & 2(2)+3(3) & 2(3)+3(1) \end{bmatrix}$$

(Row by column multiplication)

$$= \begin{bmatrix} 1-4 & 2-6 & 3-2 \\ 2+6 & 4+9 & 6+3 \end{bmatrix} = \begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$$

(iv) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

Performing row by column multiplication

$$= \begin{bmatrix} 2(1)+3(0)+4(3) & 2(-3)+3(2)+4(0) & 2(5)+3(4)+4(5) \\ 3(1)+4(0)+5(3) & 3(-3)+4(2)+5(0) & 3(5)+4(4)+5(5) \\ 4(1)+5(0)+6(3) & 4(-3)+5(2)+6(0) & 4(5)+5(4)+6(5) \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+12 & -6+6+0 & 10+12+20 \\ 3+0+15 & -9+8+0 & 15+16+25 \\ 4+0+18 & -12+10+0 & 20+20+30 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}$$

(v) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$ is defined because the pre-matrix has 2 columns which is equal to the number of rows of the post-matrix.

Performing row by column multiplication,

$$= \begin{bmatrix} 2(1)+1(-1) & 2(0)+1(2) & 2(1)+1(1) \\ 3(1)+2(-1) & 3(0)+2(2) & 3(1)+2(1) \\ (-1)1+1(-1) & (-1)0+1(2) & (-1)1+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 0+2 & 2+1 \\ 3-2 & 0+4 & 3+2 \\ -1-1 & 0+2 & -1+1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ -2 & 2 & 0 \end{bmatrix}.$$

(*vi*) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6-1+9 & -9-0+3 \\ -2+0+6 & 3+0+2 \end{bmatrix}$

(Row by column multiplication)

$= \begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$.

4. If A = $\begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, B = $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$ and C = $\begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$, then compute (A + B) and (B – C). Also, verify that A + (B – C) = (A + B) – C.

Sol. $A + B = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1+3 & 2-1 & -3+2 \\ 5+4 & 0+2 & 2+5 \\ 1+2 & -1+0 & 1+3 \end{bmatrix}$

$\Rightarrow \quad A + B = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$ ...(*i*)

Again $B - C = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3-4 & -1-1 & 2-2 \\ 4-0 & 2-3 & 5-2 \\ 2-1 & 0+2 & 3-3 \end{bmatrix}$

$\Rightarrow \quad B - C = \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$ ...(*ii*)

Putting the value of (B – C) from (*ii*) in L.H.S.

$= A + (B - C)$

$= \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1-1 & 2-2 & -3+0 \\ 5+4 & 0-1 & 2+3 \\ 1+1 & -1+2 & 1+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$ ...(*iii*)

Putting the value of (A + B) from (*i*) in R.H.S. = (A + B) – C

$= \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 4-4 & 1-1 & -1-2 \\ 9-0 & 2-3 & 7-2 \\ 3-1 & -1+2 & 4-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix} \quad ...(iv)$$

From (*iii*) and (*iv*), we have L.H.S. = R.H.S.

5. If $A = \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix}$ and $B = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix}$, then compute 3A – 5B.

Sol. $3A - 5B = 3 \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix} - 5 \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix}$

Multiplying each entry of first matrix by 3 and each entry of second matrix by 5

$$= \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \\ 7 & 6 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \\ 7 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 2-2 & 3-3 & 5-5 \\ 1-1 & 2-2 & 4-4 \\ 7-7 & 6-6 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Remark. Here answer is a zero matrix.

6. Simplify $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$.

Sol. $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

Multiplying each entry of first matrix by $\cos\theta$ and each entry of second matrix by $\sin\theta$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Remark. The answer matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ of this question is identity (unit) matrix I_2.

7. Find X and Y if

(*i*) $\mathbf{X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}}$ **and** $\mathbf{X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}}$

(*ii*) $\mathbf{2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}}$ **and** $\mathbf{3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}}$

Sol. (*i*) **Given:** $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ ...(*i*)

and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ ...(*ii*)

Adding eqns. (*i*) and (*ii*), we have

$$2X = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$$

$$\therefore \quad X = \frac{1}{2}\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} \frac{10}{2} & \frac{0}{2} \\ \frac{2}{2} & \frac{8}{2} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}.$$

Eqn. (*i*) – eqn. (*ii*) gives

$$2Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 7-3 & 0-0 \\ 2-0 & 5-3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\therefore \quad Y = \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{2} & \frac{0}{2} \\ \frac{2}{2} & \frac{2}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

(*ii*) **Given:** $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ ...(*i*)

and $3X + 2Y = \begin{bmatrix} -2 & -2 \\ -1 & 5 \end{bmatrix}$ ...(*ii*)

Multiplying equation (*i*) by 2, we have

$$4X + 6Y = 2\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 0 \end{bmatrix} \quad ...(iii)$$

Multiplying equation (*ii*) by 3, we have

$$9X + 6Y = 3\begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ -3 & 15 \end{bmatrix} \quad ...(iv)$$

Equation (*iv*) – equation (*iii*) gives

$$5X = \begin{bmatrix} 6 & -6 \\ -3 & 15 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} 6-4 & -6-6 \\ -3-8 & 15-0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -12 \\ -11 & 15 \end{bmatrix}$$

$$\therefore \quad X = \frac{1}{5}\begin{bmatrix} 2 & -12 \\ -11 & 15 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{-12}{5} \\ \frac{-11}{5} & 3 \end{bmatrix}.$$

Now from equation (i),

$$3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - 2X$$

$$= \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - 2\begin{bmatrix} \frac{2}{5} & \frac{-12}{5} \\ \frac{-11}{5} & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} & \frac{-24}{5} \\ \frac{-22}{5} & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - \frac{4}{5} & 3 + \frac{24}{5} \\ 4 + \frac{22}{5} & 0 - 6 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & \frac{39}{5} \\ \frac{42}{5} & -6 \end{bmatrix}$$

$$\Rightarrow Y = \frac{1}{3}\begin{bmatrix} \frac{6}{5} & \frac{39}{5} \\ \frac{42}{5} & -6 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}$$

8. Find X if Y = $\begin{bmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{1} & \mathbf{4} \end{bmatrix}$ and 2X + Y = $\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{3} & \mathbf{2} \end{bmatrix}$.

Sol. $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - Y$

$$\Rightarrow \quad 2X = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1-3 & 0-2 \\ -3-1 & 2-4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}$$

$$\Rightarrow \quad X = \frac{1}{2}\begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}.$$

9. Find x and y, if $2\begin{bmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{0} & \boldsymbol{x} \end{bmatrix} + \begin{bmatrix} \boldsymbol{y} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{5} & \mathbf{6} \\ \mathbf{1} & \mathbf{8} \end{bmatrix}$.

Sol. Given: $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Equating corresponding entries, we have

$2 + y = 5$ and $2x + 2 = 8$

$\Rightarrow \quad y = 5 - 2 = 3$ and $2x = 8 - 2 = 6 \Rightarrow x = 3$

$\therefore \quad x = 3, y = 3.$

10. Solve the equation for x, y, z and t if

$$2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

Sol. Given: $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} 2x & 2z \\ 2y & 2t \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2x+3 & 2z-3 \\ 2y+0 & 2t+6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

Since the two matrices are equal, so the corresponding elements are equal.

Thus, $2x + 3 = 9$

$\Rightarrow \quad 2x = 9 - 3 = 6 \quad \Rightarrow \quad x = 3$

Also $\quad 2z - 3 = 15 \quad \Rightarrow \quad 2z = 18 \quad \Rightarrow \quad z = 9$

Also $\quad 2y = 12 \quad \Rightarrow \quad y = 6$

and $\quad 2t + 6 = 18$ and $2t = 12 \quad \Rightarrow \quad t = 6$

$\therefore\ x = 3,\ y = 6,\ z = 9$ and $t = 6$.

11. If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find the values of x and y.

Sol. Given: $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} 2x \\ 3x \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2x - y \\ 3x + y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Equating corresponding entries, we have

$2x - y = 10$ ...(*i*)

and $\quad 3x + y = 5$ ...(*ii*)

Adding eqns. (*i*) and (*ii*) we have $5x = 15$

or $\quad x = \dfrac{15}{5} = 3$

Putting $x = 3$ in (*ii*), $9 + y = 5 \quad \Rightarrow \quad y = 5 - 9 = -4$

$\therefore \quad x = 3,\ y = -4$.

12. Given: $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$; find the values of x, y, z and w.

Sol. Given: $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$$

Equating corresponding entries, we have

$3x = x + 4 \quad \Rightarrow \quad 2x = 4 \quad \Rightarrow \quad x = 2$ ...(*i*)

and $\quad 3y = 6 + x + y \quad \Rightarrow \quad 2y = 6 + x = 6 + 2$ (By (*i*))

$\Rightarrow \quad 2y = 8 \quad \Rightarrow \quad y = 4$ ...(*ii*) and $\quad 3z = -1 + z + w \Rightarrow \; 2z - w = -1$ ...(*iii*)

and $\quad 3w = 2w + 3 \quad \Rightarrow \quad w = 3.$

Putting $w = 3$ in eqn. (*iii*),

$2z - 3 = -1 \quad \Rightarrow \quad 2z = 2 \Rightarrow z = 1$

$\therefore \quad x = 2, \quad y = 4, \quad z = 1, \quad w = 3.$

13. If $\mathbf{F}(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $\mathbf{F}(x)\,\mathbf{F}(y)$ $= \mathbf{F}(x + y)$.

Sol. Given: $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ...(*i*)

Changing x to y in (*i*), $F(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\text{L.H.S.} = F(x)\,F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing row by column multiplication,

$$= \begin{bmatrix} \cos x \cos y - \sin x \sin y + 0 & -\cos x \sin y - \sin x \cos y + 0 & 0 - 0 + 0 \\ \sin x \cos y + \cos x \sin y + 0 & -\sin x \sin y + \cos x \cos y + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos (x + y) & -\sin (x + y) & 0 \\ \sin (x + y) & \cos (x + y) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [\because \; -\cos x \sin y - \sin x \cos y$$

$= -(\cos x \sin y + \sin x \cos y) = -\sin (x + y)]$

Now, changing x to $x + y$ in (*i*), we get

$$F(x + y) = \begin{bmatrix} \cos (x + y) & -\sin (x + y) & 0 \\ \sin (x + y) & \cos (x + y) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Thus, L.H.S. = R.H.S.}$$

14. Show that:

(*i*) $\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$

(*ii*) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Sol. (*i*) L.H.S. $= \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5(2) + (-1)3 & 5(1) + (-1)4 \\ 6(2) + 7(3) & 6(1) + 7(4) \end{bmatrix}$

$$= \begin{bmatrix} 10-3 & 5-4 \\ 12+21 & 6+28 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 33 & 34 \end{bmatrix} \quad ...(i)$$

$$\text{R.H.S.} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 2(5)+1(6) & 2(-1)+1(7) \\ 3(5)+4(6) & 3(-1)+4(7) \end{bmatrix}$$

$$= \begin{bmatrix} 10+6 & -2+7 \\ 15+24 & -3+28 \end{bmatrix} = \begin{bmatrix} 16 & 5 \\ 39 & 25 \end{bmatrix} \quad ...(ii)$$

From (i) and (ii), we can say that L.H.S. $\neq$ R.H.S.

(Because corresponding entries of matrices $\begin{bmatrix} 7 & 1 \\ 33 & 34 \end{bmatrix}$ and $\begin{bmatrix} 16 & 5 \\ 39 & 25 \end{bmatrix}$ are not same).

(ii) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

Here, matrices A and B are both of order 3×3 respectively, therefore AB and BA are both of same order 3×3.

$$\text{Now, } AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

Performing row by column multiplication,

$$= \begin{bmatrix} 1(-1)+2(0)+3(2) & 1(1)+2(-1)+3(3) & 1(0)+2(1)+3(4) \\ 0(-1)+1(0)+0(2) & 0(1)+1(-1)+0(3) & 0(0)+1(1)+0(4) \\ 1(-1)+1(0)+0(2) & 1(1)+1(-1)+0(3) & 1(0)+1(1)+0(4) \end{bmatrix}$$

$$\text{or } AB = \begin{bmatrix} -1+6 & 1-2+9 & 2+12 \\ 0 & -1 & 1 \\ -1 & 1-1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 14 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \quad ...(i)$$

$$\text{Again, } BA = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Performing row by column multiplication,

$$= \begin{bmatrix} (-1)1+1(0)+0(1) & (-1)2+1(1)+0(1) & (-1)3+1(0)+0(0) \\ 0(1)+(-1)0+1(1) & 0(2)+(-1)1+1(1) & 0(3)+(-1)0+1(0) \\ 2(1)+3(0)+4(1) & 2(2)+3(1)+4(1) & 2(3)+3(0)+4(0) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2+1 & -3 \\ 1 & -1+1 & 0 \\ 2+4 & 4+3+4 & 6 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ 1 & 0 & 0 \\ 6 & 11 & 6 \end{bmatrix} \quad ...(ii)$$

From (i) and (ii), AB $\neq$ BA because corresponding entries of matrices AB and BA are not same.

Remark. From both questions (i), (ii) we can learn that matrix multiplication is not commutative.

15. Find $A^2 - 5A + 6I$ if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$.

Sol. $A^2 = A \cdot A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Performing row by column multiplication,

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1-0 & 1-3+0 \end{bmatrix} \text{ or } A^2 = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$\therefore$ $A^2 - 5A + 6I = A^2 - 5A + 6I_3$ (Here I is I_3 because matrices A and A^2 are of order 3×3)

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10+6 & -1-0+0 & 2-5+0 \\ 9-10+0 & -2-5+6 & 5-15+0 \\ 0-5+0 & -1+5+0 & -2-0+6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

Remark. The above question can also be stated as:

If $f(x) = x^2 - 5x + 6$ and $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$; then find $f(A)$.

16. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, prove that $A^3 - 6A^2 + 7A + 2I = 0$.

Sol. Given: $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ $\therefore$ $A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & 2+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$\therefore \quad A^3 = A^2 \,.\, A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5+0+16 & 0+0+0 & 10+0+24 \\ 2+0+10 & 0+8+0 & 4+4+15 \\ 8+0+26 & 0+0+0 & 16+0+39 \end{bmatrix} \text{or } A^3 = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

L.H.S. $= A^3 - 6A^2 + 7A + 2I$

$= A^3 - 6A^2 + 7A + 2I_3$

[Here I is I_3 because A, A^2, A^3 are matrices of order 3×3]

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 0 & -14 \\ 0 & -16 & -7 \\ -14 & 0 & -23 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 14 \\ 0 & 16 & 7 \\ 14 & 0 & 23 \end{bmatrix}$$

$$= \begin{bmatrix} -9+9 & 0+0 & -14+14 \\ 0+0 & -16+16 & -7+7 \\ -14+14 & 0+0 & -23+23 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

= (zero matrix) O = R.H.S.

17. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find k so that $A^2 = kA - 2I$.

Sol. Given: $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$

$$\therefore \quad A^2 = A \,.\, A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$$

Putting values of A^2, A and I in the given equation $A^2 = kA - 2I$, we have

$$\begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = k\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3k & -2k \\ 4k & -2k \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{bmatrix}$$

Equating corresponding entries, we have

$3k - 2 = 1 \Rightarrow 3k = 3 \Rightarrow k = 1$ and $-2 = -2k \Rightarrow k = 1$

and $4k = 4 \Rightarrow k = 1$ and $-4 = -2k - 2 \Rightarrow 2k = -2 + 4 = 2$

$\Rightarrow \quad k = 1$

Therefore, value of $k = 1$ and is same from all the four equations. Therefore, k exists and $= 1$.

18. If A = $\begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, show that I + A = (I – A) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

Sol. A = $\begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2

i.e., $\quad I = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\text{L.H.S.} = I + A = I_2 + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix} \quad \ldots(i)$$

Again, $\quad I - A = I_2 - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}$$

$$\text{R.H.S.} = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Performing row by column multiplication,

$$= \begin{bmatrix} \cos\alpha + \sin\alpha\tan\frac{\alpha}{2} & -\sin\alpha + \cos\alpha\tan\frac{\alpha}{2} \\ -\cos\alpha\tan\frac{\alpha}{2} + \sin\alpha & \sin\alpha\tan\frac{\alpha}{2} + \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha + \sin\alpha\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & -\sin\alpha + \cos\alpha\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \\ -\cos\alpha\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} + \sin\alpha & \sin\alpha\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} + \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\cos\alpha\cos\frac{\alpha}{2} + \sin\alpha\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \dfrac{-\sin\alpha\cos\frac{\alpha}{2} + \cos\alpha\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \\ \dfrac{-\cos\alpha\sin\frac{\alpha}{2} + \sin\alpha\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \dfrac{\sin\alpha\sin\frac{\alpha}{2} + \cos\alpha\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \end{bmatrix}$$

Numerator of a_{12} is $= -\left(\sin\alpha\cos\frac{\alpha}{2} - \cos\alpha\sin\frac{\alpha}{2}\right)$

$$= \begin{bmatrix} \dfrac{\cos\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} & \dfrac{-\sin\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} \\ \dfrac{\sin\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} & \dfrac{\cos\left(\alpha - \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}} \end{bmatrix} = \begin{bmatrix} \dfrac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \dfrac{-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \\ \dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \dfrac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \end{bmatrix}$$

[$\because$ $\cos A\cos B + \sin A\sin B = \cos(A - B)$
and $\sin A\cos B - \cos A\sin B = \sin(A - B)$]

$$= \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix} \qquad \ldots(ii)$$

From equations (*i*) and (*ii*), we have L.H.S. = R.H.S.

i.e., $\quad I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$

19. A trust fund has ₹ 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹ 30,000 in two types of bonds, if the trust fund must obtain an annual interest of

(*a*) ₹ 1800 **(*b*) ₹ 2000.**

Sol. Let the investment in first bond be ₹ x,

then the investment in second bond = ₹ $(30,000 - x)$

Interest paid by first bond = 5% = $\dfrac{5}{100}$ per rupee

Interest paid by second bond = 7% = $\dfrac{7}{100}$ per rupee

Matrix of investment is $A = [x \quad 30000 - x]_{1 \times 2}$

Matrix of annual interest per rupee is $B = \begin{bmatrix} \dfrac{5}{100} \\ \dfrac{7}{100} \end{bmatrix}_{2\times 1}$

Matrix of total annual interest is

$$AB = [x \qquad 30000 - x]\begin{bmatrix} \dfrac{5}{100} \\ \dfrac{7}{100} \end{bmatrix} = \left[\frac{5x}{100} + \frac{7(30000 - x)}{100}\right]$$

$$= \left[\frac{5x + 210000 - 7x}{100}\right] = \left[\frac{210000 - 2x}{100}\right]$$

$\therefore$ Total annual interest = ₹ $\dfrac{2,10,000 - 2x}{100}$

(a) total annual interest is given to be ₹ 1,800

$$\therefore \qquad \frac{2,10,000 - 2x}{100} = 1,800$$

$\Rightarrow \qquad 2,10,000 - 2x = 1,80,000 \quad \therefore \quad x = 15,000$

Hence, investment in first bond = ₹ 15,000

and investment in second bond = ₹ $(30,000 - x)$

= ₹ (30,000 – 15,000) = ₹ 15,000.

(b) Total annual interest is given to be ₹ 2,000

$$\therefore \qquad \frac{2,10,000 - 2x}{100} = 2,000$$

$\Rightarrow \quad 2,10,000 - 2x = 2,00,000 \qquad \therefore \quad x = 5,000$

Hence, investment in first bond = ₹ 5,000 and investment in second bond = ₹ $(30,000 - x)$ = ₹ (30,000 – 5,000) **= ₹ 25,000.**

20. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are ₹ 80, ₹ 60 and ₹ 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

Sol. Let us represent the number of books as a 1 × 3 row matrix

$$B = \begin{bmatrix} 10 \text{ dozen} & 8 \text{ dozen} & 10 \text{ dozen} \\ 10 \times 12 = 120 & 8 \times 12 = 96 & 10 \times 12 = 120 \end{bmatrix}$$

Let us represent the selling prices of each book as a 3 × 1 column matrix $S = \begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix}$

∴ [Total amount received by selling all books]$_{1 \times 1}$

$$= BS = [120 \quad 96 \quad 120]_{1 \times 3} \begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix}_{3 \times 1}$$

$$= [120(80) + 96(60) + 120(40)]_{1 \times 1}$$
$$= [9600 + 5760 + 4800] = [20160]$$

Equating corresponding entries,

Total amount received by selling all the books = ₹ 20160.

Assume X, Y, Z, W and P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$ respectively. Choose the correct answer in Exercises 21 and 22.

21. The restriction on n, k and p so that PY + WY will be defined are:

(A) $k = 3, p = n$ **(B) k is arbitrary, $p = 2$**
(C) p is arbitrary, $k = 3$ **(D) $k = 2, p = 3$.**

Sol. Given: Matrix PY + WY is defined (⇒ possible).

Matrix P is of order $p \times k$ and matrix Y is of order $3 \times k$ and matrix W is of order $n \times 3$.

Now PY + WY = (P + W) Y ... (*i*)

We know that sum $\underset{p \times k}{\underset{\downarrow}{P}} + \underset{n \times 3}{\underset{\downarrow}{W}}$ is defined if two matrices

P and W are of same order. Therefore $p = n$ and $k = 3$ and order of P + W is $n \times 3$ (or $p \times k$)

Therefore from (1), PY + WY = $\underset{n \times 3}{\underset{\downarrow}{(P + W)}} \; \underset{3 \times k}{\underset{\downarrow}{Y}}$ is defined as

Number of columns in P + W is same as number of rows in Y.

∴ $p = n$ and $k = 3$

∴ Option (A) is the correct answer *i.e.*, $k = 3$ and $p = n$.

22. If $n = p$, then order of the matrix 7X – 5Z is

(A) $p \times 2$ (B) $2 \times n$ (C) $n \times 3$ (D) $p \times n$.

Sol. Since $n = p$ (given), the order of matrices X and Z are equal.

$\therefore$ 7X – 5Z is well defined and the order of 7X – 5Z is same as the order of X and Z.

$\therefore$ The order of 7X – 5Z is either equal to $2 \times n$ or $2 \times p$

($\because$ $n = p$)

$\therefore$ The correct option is (B), *i.e.,* the order of 7X – 5Z is $2 \times n$.

Exercise 3.3

1. Find the transpose of each of the following matrices:

(i) $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ **(iii)** $\begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$.

Sol. (*i*) Let A = $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ (is a column matrix 3×1)

Changing column of A into a row, (row will automatically become column)

Transpose of A (*i.e.*, A′ or A^T) = $\begin{bmatrix} 5 & \frac{1}{2} & -1 \end{bmatrix}$

(which is a row matrix 1×3)

(*ii*) Let A = $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Changing rows of A to columns of A,

(columns will automatically beocme rows),

A' or A^T = $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

(*iii*) Let A = $\begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$

(Making) changing rows of A as columns of the new matrix,

we have A' or A^T = $\begin{bmatrix} -1 & \sqrt{3} & 2 \\ 5 & 5 & 3 \\ 6 & 6 & -1 \end{bmatrix}$.

2. If A = $\begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$ **and B =** $\begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$**, then verify that**

(*i*) (A + B)′ = A′ + B′ **(*ii*) (A – B)′ = A′ – B′.**

Sol. (*i*) To verify (A + B)′ = A′ + B′

$$A + B = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1-4 & 2+1 & 3-5 \\ 5+1 & 7+2 & 9+0 \\ -2+1 & 1+3 & 1+1 \end{bmatrix} = \begin{bmatrix} -5 & 3 & -2 \\ 6 & 9 & 9 \\ -1 & 4 & 2 \end{bmatrix}$$

(Making) changing rows of A + B as columns of the new matrix, we have

$$\text{L.H.S.} = (A + B)' = \begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix} \qquad ...(i)$$

$$\text{R.H.S.} = A' + B' = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}' + \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}'$$

$$= \begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1-4 & 5+1 & -2+1 \\ 2+1 & 7+2 & 1+3 \\ 3-5 & 9+0 & 1+1 \end{bmatrix} = \begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix} \qquad ...(ii)$$

From (*i*) and (*ii*), we have L.H.S. = R.H.S.

i.e., (A + B)′ = A′ + B′

(*ii*) To verify (A – B)′ = A′ – B′

$$A - B = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+4 & 2-1 & 3+5 \\ 5-1 & 7-2 & 9-0 \\ -2-1 & 1-3 & 1-1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 8 \\ 4 & 5 & 9 \\ -3 & -2 & 0 \end{bmatrix}$$

(Making) changing rows of A – B as columns of the new matrix, we have

$$\text{L.H.S.} = (A - B)' = \begin{bmatrix} 3 & 4 & -3 \\ 1 & 5 & -2 \\ 8 & 9 & 0 \end{bmatrix} \quad ...(i)$$

$$\text{R.H.S.} = A' - B' = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}' - \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}'$$

$$= \begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+4 & 5-1 & -2-1 \\ 2-1 & 7-2 & 1-3 \\ 3+5 & 9-0 & 1-1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -3 \\ 1 & 5 & -2 \\ 8 & 9 & 0 \end{bmatrix} \quad ...(ii)$$

From (*i*) and (*ii*), we have L.H.S. = R.H.S.

Note $(A')' = A$.

3. If $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then verify that

(*i*) $(A + B)' = A' + B'$ (*ii*) $(A - B)' = A' - B'$.

Sol. Given: $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

Making rows of A′ as columns of the new matrix (transpose of A′ *i.e.,* $(A')'$) *i.e.,* $A = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$

(*i*) $A + B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 3-1 & -1+2 & 0+1 \\ 4+1 & 2+2 & 1+3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 4 & 4 \end{bmatrix}$$

$$\therefore \quad \text{L.H.S.} = (A + B)' = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 4 & 4 \end{bmatrix}' = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 4 \end{bmatrix} \quad ...(i)$$

$$\text{R.H.S.} = A' + B' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}' \quad \text{(given)}$$

$$= \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3-1 & 4+1 \\ -1+2 & 2+2 \\ 0+1 & 1+3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 4 \end{bmatrix} \quad ...(ii)$$

From (i) and (ii), we have L.H.S. = R.H.S.

(ii) $A - B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 3+1 & -1-2 & 0-1 \\ 4-1 & 2-2 & 1-3 \end{bmatrix} = \begin{bmatrix} 4 & -3 & -1 \\ 3 & 0 & -2 \end{bmatrix}$$

$$\therefore \text{ L.H.S.} = (A - B)' = \begin{bmatrix} 4 & -3 & -1 \\ 3 & 0 & -2 \end{bmatrix}' = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix} \quad ...(i)$$

$$\text{R.H.S.} = A' - B' = \underset{\text{(given)}}{\begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}} - \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}'$$

$$= \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3+1 & 4-1 \\ -1-2 & 2-2 \\ 0-1 & 1-3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix} \quad ...(ii)$$

From (i) and (ii), we have L.H.S. = R.H.S.

4. If $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, then find $(A + 2B)'$.

Sol. **Given:** $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$

Making rows of A′ as columns of the new matrix (transpose of A′ *i.e.,* (A′)′) *i.e.,* $A = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$

$$\therefore \quad A + 2B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} + 2\begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2-2 & 1+0 \\ 3+2 & 2+4 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 5 & 6 \end{bmatrix}$$

Making rows of this matrix as columns of new matrix, we have

$$(A + 2B)' = \begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}.$$

5. For the matrices A and B, verify that (AB)′ = B′A′, where

(i) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $B = [-1 \ \ 2 \ \ 1]$ $\qquad$ (ii) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $B = [1 \ \ 5 \ \ 7]$.

Sol. (i) **Given:** $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$ and $B = [-1 \ \ 2 \ \ 1]$

$\therefore \quad AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}_{3\times 1} [\,-1 \quad 2 \quad 1]_{1\times 3}$ is a matrix of order

3×3 and $= \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$

(Using row by column multiplication rule)

$$\text{L.H.S.} = (AB)' = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} \quad ...(i)$$

$$\text{R.H.S.} = B'A' = [-1 \quad 2 \quad 1]' \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}'$$

$$= \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} [\,1 - \ 4 \quad 3] = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} \quad ...(ii)$$

From (*i*) and (*ii*), we have L.H.S. = R.H.S. *i.e.,* $(AB)' = B'A'$.

(*ii*) **Given:** $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $B = [1 \quad 5 \quad 7]$

$$\therefore \quad AB = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}_{3\times 1} [\,1 \quad 5 \quad 7\,]_{1\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 5 & 7 \\ 2 & 10 & 14 \end{bmatrix}$$

$$\text{L.H.S.} = (AB)' = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 5 & 7 \\ 2 & 10 & 14 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix} \quad ...(i)$$

$$\text{R.H.S.} = B'A' = [1 \quad 5 \quad 7]' \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}_{3\times 1} [\,0 \quad 1 \quad 2]_{1\times 3}$$

$$= \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix} \quad ...(ii)$$

From (*i*) and (*ii*) we have L.H.S. = R.H.S.

i.e., $(AB)' = B'A'$.

Remark. Result to remember form this Q.No. 5:

$(AB)' = B'A'$ | Reversal Law

6. (*i*) If A = $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then verify that A′A = I

(*ii*) If A = $\begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$, then verify that A′A = I.

Sol. (*i*) **Given:** $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$$\therefore \quad \text{L.H.S.} = A'A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}' \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

(Row by Column Multiplication)

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \ (= I) = \text{R.H.S.}$$

(*ii*) **Given:** $A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$

$$\therefore \quad \text{L.H.S.} = A'A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}' A$$

$$= \begin{bmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix} \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2\alpha + \cos^2\alpha & \sin\alpha\cos\alpha - \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha - \sin\alpha\cos\alpha & \cos^2\alpha + \sin^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \ (= I) = \text{R.H.S.}$$

7. (*i*) Show that the matrix A = $\begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$ is a symmetric matrix.

(*ii*) Show that the matrix A = $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Sol. (*i*) **Given:** $A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$ ...(*i*)

(Making) changing rows of matrix A as the columns of the new matrix $A' = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix} = A$ [By (*i*)]

$\therefore$ $A' = A$

$\therefore$ By definition of symmetric matrix, A is a symmetric matrix.

(*ii*) **Given:** Matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ ...(*i*)

$\therefore$ $A' = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}' = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

Taking (– 1) common from R.H.S. of A′, we have

$A' = -\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = -A$ [By (*i*)]

$\therefore$ By definition, matrix A is a skew-symmetric matrix.

8. For the matrix $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, verify that

(*i*) (A + A′) is a symmetric matrix.

(*ii*) (A – A′) is a skew symmetric matrix.

Sol. (*i*) **Given:** $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$

Let $B = A + A' = A + \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}' = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix}$

$= \begin{bmatrix} 1+1 & 5+6 \\ 6+5 & 7+7 \end{bmatrix} = \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix}$ ...(*i*)

$\therefore$ $B' = \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix}' = \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix} = B$ [By (*i*)]

$\therefore$ B *i.e.,* (A + A′) is a symmetric matrix.

(*ii*) **Given:** $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$

Let $B = A - A' = A - \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}'$

$= \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1-1 & 5-6 \\ 6-5 & 7-7 \end{bmatrix}$

or $\quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ...(*i*)

$\therefore \quad B' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Taking (– 1) common from R.H.S. of B′,

$$B' = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -B \quad \text{[By } (i)]$$

$\therefore$ Matrix B *i.e.*, A – A′ is a skew symmetric matrix.

9. Find $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$ when $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

Sol. Given: $\quad A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

$$\therefore \quad A' = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}' = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$$

$$\therefore \quad A + A' = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} + \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & a-a & b-b \\ -a+a & 0+0 & c-c \\ -b+b & -c+c & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad \frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Again $\quad A - A' = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0-0 & a+a & b+b \\ -a-a & 0-0 & c+c \\ -b-b & -c-c & 0-0 \end{bmatrix} = \begin{bmatrix} 0 & 2a & 2b \\ -2a & 0 & 2c \\ -2b & -2c & 0 \end{bmatrix}$$

$$\therefore \quad \frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & 2a & 2b \\ -2a & 0 & 2c \\ -2b & -2c & 0 \end{bmatrix}$$

Multiplying each entry by $\frac{1}{2}$, $= \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

10. Express the following matrices as the sum of a symmetric and skew symmetric matrix:

(*i*) $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ **(*ii*)** $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ **(*iv*)** $\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$

Note Formula. Every square matrix A can be expressed as the sum of a symmetric matrix $\frac{1}{2}(A + A')$ and skew symmetric matrix $\frac{1}{2}(A - A')$.

Sol. (*i*) **Given:** Matrix (say) $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$

therefore, $A' = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}' = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$

By Formula above, symmetric matrix part of A

$$= \frac{1}{2}(A + A') = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix} 6 & 6 \\ 6 & -2 \end{pmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} \qquad ...(i)$$

and skew symmetric matrix part of A.

$$= \frac{1}{2}(A - A') = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 3-3 & 5-1 \\ 1-5 & -1+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(0) & \frac{1}{2}(4) \\ \frac{1}{2}(-4) & \frac{1}{2}(0) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \qquad ...(ii)$$

$\therefore$ Given matrix A is sum of matrices (*i*) and (*ii*)

= symmetric matrix $\begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix}$ + skew symmetric matrix $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

(*ii*) **Given:** matrix say $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$$\therefore \quad A' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$\therefore$ Symmetric part of A = $\mathbf{\frac{1}{2}(A + A')}$

$$= \frac{1}{2}\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 12 & -4 & 4 \\ -4 & 6 & -2 \\ 4 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \qquad ...(i)$$

and skew symmetric part of A = $\mathbf{\frac{1}{2}(A - A')}$

$$= \frac{1}{2}\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 6-6 & -2+2 & 2-2 \\ -2+2 & 3-3 & -1+1 \\ 2-2 & -1+1 & 3-3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad ...(ii)$$

$\therefore$ Given matrix A = sum of matrices (*i*) and (*ii*)

$$= \text{symmetric matrix } \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$+ \text{ skew symmetric matrix } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(*iii*) **Given:** matrix say $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

$$\therefore \quad A' = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

$\therefore$ Symmetric part of A = $\mathbf{\frac{1}{2}}$ **(A + A′)**

$$= \frac{1}{2}\left(\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix} \quad ...(i)$$

and skew symmetric part of A = $\mathbf{\frac{1}{2}}$ **(A – A′)**

$$= \frac{1}{2}\left(\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 3-3 & 3+2 & -1+4 \\ -2-3 & -2+2 & 1+5 \\ -4+1 & -5-1 & 2-2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3-3 & 3+2 & -1+4 \\ -2-3 & -2+2 & 1+5 \\ -4+1 & -5-1 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ -\frac{5}{2} & 0 & 3 \\ -\frac{3}{2} & -3 & 0 \end{bmatrix} \quad ...(ii)$$

$\therefore$ Given matrix A = sum of matrices (i) and (ii)

$$= \text{symmetric matrix} \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix}$$

$$+ \text{skew symmetric matrix} \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ -\frac{5}{2} & 0 & 3 \\ -\frac{3}{2} & -3 & 0 \end{bmatrix}$$

(*iv*) **Given:** matrix say $A = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix} \therefore A' = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}' = \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}$

$\therefore$ Symmetric part of $A = \mathbf{\frac{1}{2}\,(A + A')}$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \quad ...(i)$$

and skew symmetric part of $A = \mathbf{\frac{1}{2}\,(A - A')}$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}\right) = \frac{1}{2}\left(\begin{bmatrix} 1-1 & 5+1 \\ -1-5 & 2-2 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \quad ...(ii)$$

$\therefore$ Given matrix = Sum of matrices (*i*) and (*ii*)

= Symmetric matrix $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

+ skew-symmetric matrix $\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$.

Choose the correct answer in Exercises 11 and 12

11. If A and B are symmetric matrices of same order, AB – BA is a

(A) Skew-symmetric matrix **(B) Symmetric Matrix**
(C) Zero matrix **(D) Identity matrix.**

Sol. Given: A and B are symmetric matrices

$\Rightarrow \quad A' = A$ and $B' = B$ ...(*i*)

Now $(AB - BA)' = (AB)' - (BA)'$ $[\because (P - Q)' = P' - Q']$

$= B'A' - A'B'$ [Reversal Law]

$= BA - AB$ [Using (*i*)]

$= -(AB - BA)$

$\therefore$ (AB – BA) is a skew symmetric.

Thus, option (A) is the correct answer.

12. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, then A + A′ = I, if the value of α is

(A) $\frac{\pi}{6}$ **(B) $\frac{\pi}{3}$** **(C) π** **(D) $\frac{3\pi}{2}$.**

Sol. Given: $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Also given $A + A' = I$

$$\Rightarrow \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}' = I = I_2$$

$$\Rightarrow \quad \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating corresponding entries, we have

$2\cos\alpha = 1$

$$\Rightarrow \quad \cos\alpha = \frac{1}{2} = \cos\frac{\pi}{3} \qquad \therefore \qquad \alpha = \frac{\pi}{3}.$$

Thus, option (B) is the correct answer.

Exercise 3.4

Using elementary transformations, find the inverse of each of the matrices, if it exists in Exercises 1 to 6.

1. $\begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$.

Sol. Let A = $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

We shall find A^{-1}, if it exists; by elementary **(Row)** transformations (only)

So we must write **A = IA only and not A = AI**

$$\therefore \quad \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

(Here I is I_2 because A is 2 × 2)

We shall reduce the matrix on left side to I_2.

Here $a_{11} = 1$

Operate $R_2 \to R_2 - 2R_1$ to make $a_{21} = 0$

$$\begin{array}{rrr} R_2 \to & 2 & 3 \\ 2R_1 \to & 2 & -2 \\ - & - & + \\ \hline \therefore\ R_2 - 2R_1 = & 0 & 5 \\ R_2 \to & 0 & 1 \\ 2R_1 \to & 2 & 0 \\ - & - & - \\ \hline \therefore\ R_2 - 2R_1 = & -2 & 1 \end{array}$$

$$\therefore \quad \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

Operate $R_2 \to \frac{1}{5} R_2$ to make $a_{22} = 1$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-2}{5} & \frac{1}{5} \end{bmatrix} A$$

Now operate $R_1 \to R_1 + R_2$ to make $a_{12} = 0$

$$\Rightarrow \quad \begin{bmatrix} 1+0 & -1+1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{2}{5} & 0 + \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} A$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{bmatrix} A$$

$\therefore$ By definition of inverse of a matrix, $A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{bmatrix}$

Note. Any row operation done on left hand side matrix must also be done on the prefactor I_2 of right hand side matrix.

Note. Definition of inverse of a square matrix. A square matrix B is said to be inverse of a square matrix A if **AB = I** and **BA = I**. Then $B = A^{-1}$.

Remark. If the student is interested in finding A^{-1} by elementary column transformations, then he or she should start with A = AI and apply only column operations.

2. $\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

We know that $A = I_2A \quad \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Operate $R_1 \leftrightarrow R_2$ (to make $a_{11} = 1$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 2R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 2-2 & 1-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1-0 & 0-2 \end{bmatrix} A$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} A$$

Operate $R_2 \to (-1) R_2$ (to make $a_{22} = 1$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} A$$

Operate $R_1 \to R_1 - R_2$ (to make $a_{12} = 0$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} A$$

$\therefore$ By definition of inverse of a square matrix, $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

3. $\begin{bmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{2} & \mathbf{7} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Here $a_{11} = 1$. To make $a_{21} = 0$, let us operate $R_2 \to R_2 - 2R_1$.

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

$$\begin{array}{rll} R_2 \to 2 & & 7 \\ 2R_1 \to 2 & & 6 \\ - \quad - & & - \\ \hline \therefore R_2 - 2R_1 = 0 & & 1 \\ R_2 \to 0 & & 1 \\ 2R_1 \to 2 & & 0 \\ - \quad - & & - \\ \hline \therefore R_2 - 2R_1 = -2 & & 1 \end{array}$$

Now $a_{22} = 1$. To make a_{12} as zero, operate $R_1 \to R_1 - 3R_2$.

$$\Rightarrow \begin{bmatrix} 1-0 & 3-3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+6 & 0-3 \\ -2 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} A$$

$\therefore$ By definition, $A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$.

4. $\begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{5} & \mathbf{7} \end{bmatrix}$

Sol. Set $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Let us try to make $a_{11} = 1$. Operate $R_2 \to R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 2 & 3 \\ 5-4 & 7-6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0-2 & 1-0 \end{bmatrix} A \quad \Rightarrow \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

Now operate $R_1 \leftrightarrow R_2$ to make $a_{11} = 1$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} A$$

Operate $R_2 \leftrightarrow R_2 - 2R_1$ to make $a_{21} = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2-2 & 3-2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1+4 & 0-2 \end{bmatrix} A \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix} A$$

Operate $R_1 \to R_1 - R_2$ to make $a_{12} = 0$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} -2-5 & 1+2 \\ 5 & -2 \end{bmatrix}$$

$$\Rightarrow \quad I_2 = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} A \Rightarrow A^{-1} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$

Remark. In the above solution to make $a_{11} = 1$, we could also operate $R_1 \to \frac{1}{2} R_1$. But for the sake of convenience and to avoid lengthy computations, we should avoid multiplying by fractions.

5. $\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{7} & \mathbf{4} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$

We know that $A = I_2 A \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Let us try to make $a_{11} = 1$. Operate $R_2 \to R_2 - 3R_1$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 7-6 & 4-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0-3 & 1-0 \end{bmatrix} A \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} A$$

Operate $R_1 \to R_1 - R_2$ to make $a_{11} = 1$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} A$$

Now Operate $R_2 \to R_2 - R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} A$$

Now $a_{12} = 0$ and $a_{22} = 1$.

or $\quad I_2 = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} A$

$\therefore$ By definition of inverse of a square matrix, $A^{-1} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$.

6. $\begin{bmatrix} \mathbf{2} & \mathbf{5} \\ \mathbf{1} & \mathbf{3} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Operate $R_1 \leftrightarrow R_2$ to make $a_{11} = 1$;

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 2R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2-2 & 5-6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1-0 & 0-2 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} A$$

Operate $R_2 \to (-1)R_2$ to make $a_{22} = 1$;

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} A$$

Operate $R_1 \to R_1 - 3R_2$ (to make $a_{12} = 0$)

$$\Rightarrow \begin{bmatrix} 1-0 & 3-3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0+3 & 1-6 \\ -1 & 2 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} A$$

$\therefore$ By Definition, $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Using elementary transformations, find the inverse of each of the matrices, if it exists, in Exercises 7 to 14.

7. $\begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{5} & \mathbf{2} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Let us try to make $a_{11} = 1$.

Operate $R_1 \to 2R_1 \Rightarrow \begin{bmatrix} 6 & 2 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A$

Operate $R_1 \to R_1 - R_2$ (to make $a_{11} = 1$)

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 5R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0-10 & 1+5 \end{bmatrix} A \quad \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -10 & 6 \end{bmatrix} A$$

Operate $R_2 \to \frac{1}{2} R_2$ (to make $a_{22} = 1$)

$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} A$

Now a_{12} has already become zero. Therefore,

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}.$$

8. $\begin{bmatrix} \mathbf{4} & \mathbf{5} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Operate $R_1 \to R_1 - R_2$ (to make $a_{11} = 1$)

$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A$

Operate $R_2 \to R_2 - 3R_1$ (to make $a_{21} = 0$)

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 3-3 & 4-3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0-3 & 1+3 \end{bmatrix} A \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} A$

Now a_{22} has already become 1.

Operate $R_1 \to R_1 - R_2$ (to make $a_{12} = 0$)

$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} 1+3 & -1-4 \\ -3 & 4 \end{bmatrix} A$

$\Rightarrow \quad I_2 = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$ A. Therefore, $A^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$.

9. $\begin{bmatrix} \mathbf{3} & \mathbf{10} \\ \mathbf{2} & \mathbf{7} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Operate $R_1 \to R_1 - R_2$ (to make $a_{11} = 1$)

$\Rightarrow \quad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A$

Operate $R_2 \to R_2 - 2R_1$ (to make $a_{21} = 0$)

$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2-2 & 7-6 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0-2 & 1+2 \end{bmatrix} A \Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} A$

Now $a_{22} = 1$. Operate $R_1 \to R_1 - 3R_2$ (to make $a_{12} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+6 & -1-9 \\ -2 & 3 \end{bmatrix} A$$

$$\Rightarrow I_2 = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix} A \Rightarrow A^{-1} = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$$

10. $\begin{bmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{-4} & \mathbf{2} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Let us try to make $a_{11} = 1$

Operate $R_1 \to R_1 + R_2$.

$$\Rightarrow \begin{bmatrix} 3-4 & -1+2 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+1 \\ 0 & 1 \end{bmatrix} A \Rightarrow \begin{bmatrix} -1 & 1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A$$

Operate $R_1 \to (-1) R_1$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 + 4R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -4 & -3 \end{bmatrix} A$$

Operate $R_2 \to \left(-\frac{1}{2}\right) R_2$ (to make $a_{22} = 1$)

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & \frac{3}{2} \end{bmatrix} A$$

Operate $R_1 \to R_1 + R_2$ (to make $a_{12} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} A$$

$\therefore$ By definition of inverse of a matrix; $A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$.

11. $\begin{bmatrix} \mathbf{2} & \mathbf{-6} \\ \mathbf{1} & \mathbf{-2} \end{bmatrix}$.

Sol. Let A = $\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}A$

Operate $R_1 \leftrightarrow R_2$ (to make $a_{11} = 1$)

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}A$$

Operate $R_2 \to R_2 - 2R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 2-2 & -6+4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1-0 & 0-2 \end{bmatrix}A \quad \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}A$$

Operate $R_2 \to \left(-\frac{1}{2}\right)R_2$ (to make $a_{22} = 1$)

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & 1 \end{bmatrix}A$$

Operate $R_1 \to R_1 + 2R_2$ (to make $a_{12} = 0$)

$$\Rightarrow \begin{bmatrix} 1+0 & -2+2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0-1 & 1+2 \\ \frac{-1}{2} & 1 \end{bmatrix}A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}A \Rightarrow A^{-1} = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}.$$

12. $\begin{bmatrix} \mathbf{6} & \mathbf{-3} \\ \mathbf{-2} & \mathbf{1} \end{bmatrix}$.

Sol. Let A = $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$.

Here, A is a 2 × 2 matrix. So, we start with $A = I_2 A$

or $$\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Operating $R_1 \to 1/6\ R_1$ to make $a_{11} = 1$,

we have $$\begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & 1 \end{bmatrix} A$$

Operating $R_2 \to R_2 + 2R_1$ to make non-diagonal entry a_{21} below a_{11} as zero,

we have $$\begin{bmatrix} 1 & \frac{-1}{2} \\ -2+2 & 1-\frac{2}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0+\frac{2}{6} & 1+0 \end{bmatrix} A$$

or $$\begin{bmatrix} 1 & \frac{-1}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ \frac{1}{3} & 1 \end{bmatrix} A$$

Here, all entries in second row of left side matrix are zero.

$\therefore$ A^{-1} does not exist.

Note. If after doing one or more elementary row operations, we obtain all 0's in one or more rows of the left hand matrix A, then A^{-1} does not exist and we say A is not invertible.

13. $\begin{bmatrix} \mathbf{2} & \mathbf{-3} \\ \mathbf{-1} & \mathbf{2} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Operate $R_1 \to R_1 + R_2$ (to make $a_{11} = 1$)

$$\Rightarrow \begin{bmatrix} 2-1 & -3+2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+1 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 + R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} A$$

Now $a_{22} = 1$. Operate $R_1 \to R_1 + R_2$ (to make $a_{12} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I_2) = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} A$$

$\therefore$ By definition; $A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.

14. $\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{4} & \mathbf{2} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

We know that $A = I_2A \Rightarrow \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Operate $R_1 \to \frac{1}{2} R_1$ (to make $a_{11} = 1$)

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 4R_1$ (to make $a_{21} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 4-4 & 2-2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0-2 & 1-0 \end{bmatrix} A \Rightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -2 & 1 \end{bmatrix} A$$

Here one row (namely second row) of the matrix on L.H.S. contains zeros only.

Hence, A^{-1} does not exist.

Using elementary transformations, find the inverse of each of the matrices, if it exists, in Exercises 15 to 17.

15. $\begin{bmatrix} \mathbf{2} & \mathbf{-3} & \mathbf{3} \\ \mathbf{2} & \mathbf{2} & \mathbf{3} \\ \mathbf{3} & \mathbf{-2} & \mathbf{2} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

We know that $A = I_3A$ (we have taken I_3 because matrix A is of order 3×3)

$$\Rightarrow \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Let us try to make $a_{11} = 1$

Operate $R_1 \to R_1 - R_3$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_1 \to (-1) R_1$ to make $a_{11} = 1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$ (to make $a_{21} = 0$ and $a_{31} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 2-2 & 2-2 & 3+2 \\ 3-3 & -2-3 & 2+3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0+2 & 1-0 & 0-2 \\ 0+3 & 0-0 & 1-3 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 5 \\ 0 & -5 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 0 & -2 \end{bmatrix} A$$

Operate $R_2 \leftrightarrow R_3$ (to make a_{22} non-zero)

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 5 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -2 \\ 2 & 1 & -2 \end{bmatrix} \mathrm{A}$$

Operate $R_2 \to \left(-\frac{1}{5}\right) R_2$ to make $a_{22} = 1$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -\frac{3}{5} & 0 & \frac{2}{5} \\ 2 & 1 & -2 \end{bmatrix} \mathrm{A}$$

Operate $R_1 \to R_1 - R_2$ (to make $a_{12} = 0$). Here a_{32} is already zero.

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -1+\frac{3}{5} & 0-0 & 1-\frac{2}{5} \\ -\frac{3}{5} & 0 & \frac{2}{5} \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 0 & \frac{2}{5} \\ 2 & 1 & -2 \end{bmatrix} \mathrm{A}$$

Operate $R_3 \to \frac{1}{5} R_3$ (to make $a_{33} = 1$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix} \mathrm{A}$$

Operate $R_2 \to R_2 + R_3$ (to make $a_{23} = 0$). Here a_{13} is already zero.

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (= I_3) = \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix} \mathrm{A}$$

$$\therefore \quad \text{By definition} \quad A^{-1} = \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix}.$$

16. $\begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{-2} \\ \mathbf{-3} & \mathbf{0} & \mathbf{-5} \\ \mathbf{2} & \mathbf{5} & \mathbf{0} \end{bmatrix}$.

Sol. Let A = $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$

We know that A = I_3A

$$\Rightarrow \quad \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Here a_{11} is already 1.

Operate $R_2 \to R_2 + 3R_1$ and $R_3 \to R_3 - 2R_1$ (to make $a_{21} = 0$ and $a_{31} = 0$)

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ -3+3 & 0+9 & -5-6 \\ 2-2 & 5-6 & 0+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0+3 & 1+0 & 0+0 \\ 0-2 & 0-0 & 1-0 \end{bmatrix} A$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -11 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \leftrightarrow R_3$ to make a_{22} simpler entry

$$\Rightarrow \quad \begin{bmatrix} 1 & 3 & -2 \\ 0 & -1 & 4 \\ 0 & 9 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} A$$

Operate $R_2 \to (-1)\,R_2$ to make $a_{22} = 1$

$$\Rightarrow \quad \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -4 \\ 0 & 9 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix} A$$

Operate $R_1 \to R_1 - 3R_2$ to make $a_{12} = 0$ and $R_3 \to R_3 - 9R_2$ (to make $a_{32} = 0$)

$$\Rightarrow \begin{bmatrix} 1-0 & 3-3 & -2+12 \\ 0 & 1 & -4 \\ 0 & 9-9 & -11+36 \end{bmatrix} = \begin{bmatrix} 1-6 & 0-0 & 0+3 \\ 2 & 0 & -1 \\ 3-18 & 1-0 & 0+9 \end{bmatrix} A$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -4 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ 2 & 0 & -1 \\ -15 & 1 & 9 \end{bmatrix} A$$

Operate $R_3 \to \dfrac{1}{25} R_3$ to make $a_{33} = 1$.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ 2 & 0 & -1 \\ -\frac{15}{25} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A$$

Operate $R_1 \to R_1 - 10R_3$, (to make $a_{13} = 0$) and $R_2 \to R_2 + 4R_3$ (to make $a_{23} = 0$).

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (= I_3) = \begin{bmatrix} -5 + \frac{150}{25} & 0 - \frac{10}{25} & 3 - \frac{90}{25} \\ 2 - \frac{60}{25} & 0 + \frac{4}{25} & -1 + \frac{36}{25} \\ \frac{-15}{25} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A$$

$$\Rightarrow I_3 = \begin{bmatrix} 1 & \frac{-2}{5} & \frac{-3}{5} \\ \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A$$

$$\therefore \text{ By Definition, } A^{-1} = \begin{bmatrix} 1 & \frac{-2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}.$$

17. $\begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{5} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{3} \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

We know that $A = I_3 \; A \Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Let us try to make $a_{11} = 1$

Operate $R_2 \to R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_1 \leftrightarrow R_2$ (to make $a_{11} = 1$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 2R_1$ to make $a_{21} = 0$. Here a_{31} is already 0

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 5 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \leftrightarrow R_3$ (to make $a_{22} = 1$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -5 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -2 & 0 \end{bmatrix} A$$

Operate $R_1 \to R_1 - R_2$ to make $a_{12} = 0$ and $R_3 \to R_3 + 2R_2$ to make $a_{32} = 0$.

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 1 \\ 5 & -2 & 2 \end{bmatrix} A$$

Now $a_{33} = 1$

Operate $R_1 \to R_1 + R_3$ (to make $a_{13} = 0$) and $R_2 \to R_2 - 3R_3$ (to make $a_{23} = 0$)

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (= I_3) = \begin{bmatrix} -2+5 & 1-2 & -1+2 \\ 0-15 & 0+6 & 1-6 \\ 5 & -2 & 2 \end{bmatrix} A$$

or
$$I_3 = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

$\therefore$ By definition,
$$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}.$$

18. Matrices A and B will be inverse of each other only if

(A) AB = BA **(B) AB = BA = 0**

(C) AB = 0, BA = I **(D) AB = BA = I.**

Sol. Option (D) *i.e.,* **AB = BA = I** is correct answer by definition of inverse of a square matrix.

MISCELLANEOUS EXERCISE

1. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bA)^n = a^n I + na^{n-1} bA$ where I is the identity matrix of order 2 and $n \in N$.

Sol. Step I. When $n = 1$, $(aI + bA)^n = a^n I + na^{n-1} bA$

$\Rightarrow (aI + bA)^1 = aI + 1a^0 bA \Rightarrow aI + bA = aI + bA$ which is true.

$\therefore$ The result is true for $n = 1$.

Step II. Suppose the result is true for $n = k$.

i.e., let $(aI + bA)^k = a^k I + ka^{k-1} bA$ ...(*i*)

Step III. To prove that the result is true for $n = k + 1$.

Now $(aI + bA)^{k+1} = (aI + bA) . (aI + bA)^k$

$= (aI + bA)(a^k I + ka^{k-1} bA)$ [Using (*i*)]

$= a^{k+1} I^2 + ka^k bIA + a^k bAI + ka^{k-1} b^2A^2$

[By distributive property]

$= a^{k+1} I + ka^k bA + a^k bA + ka^{k-1} b^2 O.$

$$\left[\because\ I^2 = I, IA = AI = A \text{ and } A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O\right]$$

$= a^{k+1} I + (k + 1) a^k b A + O = a^{k+1} I + (k + 1) a^{(k+1)-1} bA$

$\Rightarrow$ The result is true for $n = k + 1$.

Hence, by the principle of mathematical induction, the result is true for all positive integers n.

2. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, prove that $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$ $n \in N$.

Sol. We shall prove the result by using principle of mathematical induction.

Given: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ ...(*i*)

Let $P(n)$: $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$ ...(*ii*)

Step I. Putting $n = 1$ in (*ii*),

Therefore, P(1): $A = \begin{bmatrix} 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

which is given to be true by (*i*).

$\therefore$ P(1) is true *i.e.,* Eqn. (*ii*) is true for $n = 1$.

Step II. Let P(*k*) be true *i.e.,* eqn. (*ii*) is true for $n = k$.

Putting $n = k$ in (*ii*), $A^k = \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}$ ...(*iii*)

Step III. Multiplying corresponding sides of eqn. (*iii*) by eqn. (*i*)

$$A^k \,.\, A^1 = \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Performing row by column multiplication on right side

$$\Rightarrow A^{k+1} = \begin{bmatrix} 3^{k-1}+3^{k-1}+3^{k-1} & 3^{k-1}+3^{k-1}+3^{k-1} & 3^{k-1}+3^{k-1}+3^{k-1} \\ 3^{k-1}+3^{k-1}+3^{k-1} & 3^{k-1}+3^{k-1}+3^{k-1} & 3^{k-1}+3^{k-1}+3^{k-1} \\ 3^{k-1}+3^{k-1}+3^{k-1} & 3^{k-1}+3^{k-1}+3^{k-1} & 3^{k-1}+3^{k-1}+3^{k-1} \end{bmatrix}$$

$$\Rightarrow \quad A^{k+1} = \begin{bmatrix} 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \end{bmatrix}$$

($\because$ $3^{k-1} + 3^{k-1} + 3^{k-1} = 3 \,.\, 3^{k-1}$ ($\because$ $x + x + x = 3x$)
$= 3^1 \,.\, 3^{k-1} = 3^{1+k-1} = 3^k$)

$\therefore$ Eqn. (*ii*) is true for $n = k + 1$ ($\because$ on putting $n = k + 1$ in (*ii*), we get the above equation)

i.e., P(k + 1) is true

$\therefore$ P(n) *i.e.,* eqn. (*ii*) is true for all natural n by P.M.I.

3. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ where *n* is any positive integer.

Sol. We prove the result by mathematical induction.

Step I. When $n = 1$, $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ ...(*i*)

$\Rightarrow \quad A^1 = \begin{bmatrix} 1+2 & -4 \\ 1 & 1-2 \end{bmatrix}$

or $\quad A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ which is true. $\Rightarrow$ The result is true for $n = 1$.

Step II. Suppose that equation (i) is true for $n = k$,

i.e., let $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$ ...(ii)

Step III. To prove that the result is true for $n = k + 1$, we have to show that

(Putting $n = k + 1$ in (i))

$$A^{k+1} = \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ (k+1) & 1-2(k+1) \end{bmatrix} = \begin{bmatrix} 3+2k & -4-4k \\ k+1 & -1-2k \end{bmatrix} \quad ...(iii)$$

Now $A^{k+1} = A^k A = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ [Using (ii)]

Performing row by column multiplication,

$$= \begin{bmatrix} 3+6k-4k & -4-8k+4k \\ 3k+1-2k & -4k-1+2k \end{bmatrix} = \begin{bmatrix} 3+2k & -4-4k \\ 1+k & -1-2k \end{bmatrix}$$

which is the same as (iii).

$\Rightarrow$ The result is true for $n = k + 1$.

Hence, by the principle of mathematical induction, the result is true for all positive integers n.

4. If A and B are symmetric matrices, prove that AB – BA is a skew symmetric matrix.

Sol. A and B are symmetric matrices

$\Rightarrow \quad A' = A$ and $B' = B$ ...(i)

Now $(AB - BA)' = (AB)' - (BA)'$ [$\because (P - Q)' = P' - Q'$]

$= B'A' - A'B'$ [Reversal Law]

$= BA - AB$ [Using (i)]

$= -(AB - BA)$

$\therefore$ $(AB - BA)$ is a skew symmetric matrix.

5. Show that the matrix B′AB is symmetric or skew symmetric according as A is symmetric or skew symmetric.

Sol. Now, $(B'AB)' = [B'(AB)]'$

$= (AB)'(B')'$ [$\because (CD)' = D'C'$]

or $(B'AB)' = B'A'B$ ...(i) [$\because (CD)' = D'C'$]

Case I. A is a symmetric matrix

$\therefore \quad A' = A$

Putting $A' = A$ in equation (i), $(B'AB)' = B'AB$

$\therefore \quad B'AB$ is a symmetric matrix.

Case II. A is a skew symmetric matrix.

$\therefore \quad A' = -A$

Putting $A' = -A$ in equation (i), $(B'AB)' = B'(-A)B = -B'AB$

$\therefore \quad B'AB$ is a skew symmetric matrix.

6. Find the values of x, y, z if the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$$ **satisfies the equation A′A = I.**

Sol. Given: $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$.

Therefore, $A' = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}' = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$

$\therefore$ $A'A = I$ (given)

$$\Rightarrow \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Here I is I_3 because) matrices A and A′ are matrices of order 3×3)

$$\Rightarrow \begin{bmatrix} 0 + x^2 + x^2 & 0 + xy - xy & 0 - xz + xz \\ 0 + xy - xy & 4y^2 + y^2 + y^2 & 2yz - yz - yz \\ 0 - zx + zx & 2yz - yz - yz & z^2 + z^2 + z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating corresponding entries, we have

$2x^2 = 1,$ $\quad 6y^2 = 1,$ $\quad 3z^2 = 1$

$\Rightarrow$ $x^2 = \frac{1}{2},$ $\quad y^2 = \frac{1}{6},$ $\quad z^2 = \frac{1}{3}$

$\Rightarrow$ $x = \pm\sqrt{\frac{1}{2}},$ $\quad y = \pm\sqrt{\frac{1}{6}},$ $\quad z = \pm\sqrt{\frac{1}{3}}$

$\therefore$ $x = \pm\frac{1}{\sqrt{2}},$ $\quad y = \pm\frac{1}{\sqrt{6}},$ $\quad z = \pm\frac{1}{\sqrt{3}}.$

7. For what value of x, $[1 \quad 2 \quad 1]\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = 0$?

Sol. Given: $\underset{1\times 3}{\underset{\downarrow}{[\,1 \quad 2 \quad 1\,]}}\underset{3\times 3}{\underset{\downarrow}{\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}}}\underset{3\times 1}{\underset{\downarrow}{\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix}}} = 0$

Orders 1×3 3×3 3×1

Multiplying first matrix with second matrix.

$\Rightarrow \quad [1 + 4 + 1 \quad 2 + 0 + 0 \quad 0 + 2 + 2]\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = 0$

$\Rightarrow \quad \underset{1\times 3}{\underset{\downarrow}{[\,6 \quad 2 \quad 4\,]}}\underset{3\times 1}{\underset{\downarrow}{\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix}}} = 0$

$\Rightarrow \quad [0 + 4 + 4x]_{1 \times 1} = 0 = [0]_{1 \times 1}$

Equating corresponding entries $0 + 4 + 4x = 0 \Rightarrow 4x = -4$

$\Rightarrow \quad x = \frac{-4}{4} = -1.$

8. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$.

Sol. Given: $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$\therefore A^2 = A\,.\,A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

L.H.S. $= A^2 - 5A + 7I = A^2 - 5A + 7I_2$

[$\because$ A is 2×2, therefore I is I_2]

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 8-15 & 5-5 \\ -5+5 & 3-10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -7+7 & 0+0 \\ 0+0 & -7+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 =$ R.H.S.

9. Find x, if $[x \quad -5 \quad -1] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = 0$.

Sol. Given: $\underset{\text{order } 1 \times 3}{\underbrace{[\, x \quad -5 \quad -1 \,]}} \underset{\text{order } 3 \times 3}{\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}}} \underset{\text{order } 3 \times 1}{\underbrace{\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix}}} = 0$

Multiplying first matrix with second matrix

$$[\, x - 0 - 2 \quad 0 - 10 - 0 \quad 2x - 5 - 3 \,] \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \quad \underset{\text{order } 1 \times 3}{\underbrace{[\, x - 2 \quad -10 \quad 2x - 8 \,]}} \underset{\text{order } 3 \times 1}{\underbrace{\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix}}}$$

$\Rightarrow \quad [(x - 2)x - 10(4) + (2x - 8)1] = 0$

$\Rightarrow \quad [x^2 - 2x - 40 + 2x - 8] = 0 \quad \Rightarrow \quad [x^2 - 48]_{1 \times 1} = [0]_{1 \times 1}$

Equating corresponding entries, $x^2 - 48 = 0$

$\Rightarrow \quad x^2 = 48 \quad \Rightarrow \quad x = \pm \sqrt{48} = \pm \sqrt{16 \times 3} = \pm 4\sqrt{3}$.

10. A manufacturer produces three products x, y, z which he sells in two markets. Annual sales are indicated below:

Market	**Products**		
I	**10,000**	**2,000**	**18,000**
II	**6,000**	**20,000**	**8,000**

(*a*) If unit sale prices of x, y and z are ₹ 2.50, ₹ 1.50 and ₹ 1.00, respectively, find the total revenue in each market with the help of matrix algebra.

(*b*) If the unit costs of the above three commodities are ₹ 2.00, ₹ 1.00 and 50 paise respectivley. Find the gross profit.

Sol. The matrix showing the production of the three items in market I and II can be shown by a 2 × 3 matrix.

Let A be this matrix, then

$$A = \begin{array}{c} \\ \mathrm{I} \\ \mathrm{II} \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 10,000 & 2,000 & 18,000 \\ 6,000 & 20,000 & 8,000 \end{bmatrix}_{2 \times 3} \end{array}$$

(a) Let B be the column matrix representing sale price of each unit of products x, y, z.

Then $\quad B = \begin{bmatrix} 2.5 \\ 1.5 \\ 1 \end{bmatrix}_{3 \times 1}$

We know that revenue (= sale price × number of items sold)

In matrix form,

$[\text{Revenue matrix}]_{2 \times 1} = A_{2 \times 3} \times B_{3 \times 1}$

$$\Rightarrow \begin{bmatrix} \text{Revenue from Market I} \\ \text{Revenue from Market II} \end{bmatrix}$$

$$= \begin{bmatrix} 10,000 & 2,000 & 18,000 \\ 6,000 & 20,000 & 8,000 \end{bmatrix}_{2 \times 3} \begin{bmatrix} 2.5 \\ 1.5 \\ 1 \end{bmatrix}_{3 \times 1}$$

$$= \begin{bmatrix} 25,000 + 3,000 + 18,000 \\ 15,000 + 30,000 + 8,000 \end{bmatrix}_{2 \times 1} = \begin{bmatrix} 46,000 \\ 53,000 \end{bmatrix}$$

Equating corresponding entries, we have the revenue collected by sale of all items in Market I = ₹ 46,000 and the revenue collected by sale of all items in Market II = ₹ 53,000.

(b) Let the cost matrix showing the cost of each unit of products x, y, z be given by the column matrix C (say)

$$C = \begin{bmatrix} 2 \\ 1 \\ 0.5 \end{bmatrix}_{3 \times 1}$$

Thus, the total cost of three items for each market is given by: (In general form)

[Cost matrix] = AC

$$= \begin{bmatrix} 10,000 & 2,000 & 18,000 \\ 6,000 & 20,000 & 8,000 \end{bmatrix}_{2 \times 3} \begin{bmatrix} 2 \\ 1 \\ 0.5 \end{bmatrix}_{3 \times 1}$$

$$= \begin{bmatrix} 20{,}000 + 2{,}000 + 9{,}000 \\ 12{,}000 + 20{,}000 + 4{,}000 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 31{,}000 \\ 36{,}000 \end{bmatrix}$$

$\therefore$ The profit collected in two markets is given in matrix form as

Profit matrix = Revenue matrix – Cost matrix

$$= \begin{bmatrix} 46{,}000 \\ 53{,}000 \end{bmatrix} - \begin{bmatrix} 31{,}000 \\ 36{,}000 \end{bmatrix} = \begin{bmatrix} 15{,}000 \\ 17{,}000 \end{bmatrix}$$

Hence, the gross profit in both the markets

= ₹ 15,000 + ₹ 17,000 = ₹ 32,000.

11. Find the matrix X so that

$$\mathbf{X} \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{bmatrix} = \begin{bmatrix} \mathbf{-7} & \mathbf{-8} & \mathbf{-9} \\ \mathbf{2} & \mathbf{4} & \mathbf{6} \end{bmatrix}.$$

Sol. Given: $$\underset{\substack{\downarrow \\ \text{order} \\ m \times n \text{ (say)}}}{X} \underset{\substack{\downarrow \\ \text{order} \\ 2 \times 3}}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} = \underset{\substack{\downarrow \\ \text{order } 2 \times 3}}{\begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}} \quad ...(i)$$

$\therefore$ $n = 2$ (because numbers of columns in pre-matrix of product must be equal to number of rows in post matrix)

and so L.H.S. matrix is of order $m \times 3$. Again R.H.S. matrix is of order 2×3. Therefore, $m = 2$ (By definition of equal matrices)

$\therefore$ Therefore, matrix X is of order $m \times n$ *i.e.,* 2×2.

Let $$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad ...(ii)$$

Putting this value of X in eqn. (*i*),

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a + 4b & 2a + 5b & 3a + 6b \\ c + 4d & 2c + 5d & 3c + 6d \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

Equating corresponding entries, we have

$a + 4b = -7$ ...(*iii*) $c + 4d = 2$ ...(*vi*)

$2a + 5b = -8$ ...(*iv*) $2c + 5d = 4$ ...(*vii*)

$3a + 6b = -9$ ...(*v*) $3c + 6d = 6$ ...(*viii*)

Let us solve eqns. (*iii*) and (*iv*) for *a* and *b*

Eqn. (*iii*) $\times$ 2 gives $2a + 8b = -14$

Eqn. (*iv*) is $2a + 5b = -8$

$\quad - \quad - \quad\quad +$

Subtracting, $3b = -6 \Rightarrow b = -\dfrac{6}{3} = -2$

Putting $b = -2$ in (*iii*), $a - 8 = -7 \Rightarrow a = -7 + 8 = 1$

Putting $a = 1$ and $b = -2$ in eqn. (*v*), $3 - 12 = -9$

$\Rightarrow$ $-9 = -9$ which is true. $\therefore$ values of $a = 1$ and $b = -2$ exist.

Now let us solve eqns. (*vi*) and (*vii*) for *c* and *d*.

Eqn. (*vi*) $\times$ 2 gives $2c + 8d = 4$

Eqn. (*vii*) is $2c + 5d = 4$

$\quad - \quad - \quad -$

Subtracting, $3d = 0$ $\Rightarrow$ $d = \dfrac{0}{3} = 0$

Putting $d = 0$ in (*vi*), $c = 2$

Putting $c = 2$ and $d = 0$ in (*viii*), $6 = 6$ which is true.

$\therefore$ values of $c = 2$ and $d = 0$ exist.

Putting these values of a, b, c, d in (*ii*), matrix $X = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$.

12. If A and B are square matrices of the same order such that AB = BA, then prove by induction that $AB^n = B^nA$. Further, prove that $(AB)^n = A^nB^n$ for all $n \in N$.

Sol. Given: $AB = BA$ ...(*i*)

Let $P(n)$: $AB^n = B^nA$ ...(*ii*)

We have been asked to prove eqn. (*ii*) by P.M.I.

(Even if not asked, we would have proved it by P.M.I.)

Step I. For $n = 1$. From eqn. (*ii*), P(1) : becomes AB = BA which is given to be true by eqn. (*i*)

$\therefore$ P(1) is true *i.e.,* eqn. (*ii*) is true for $n = 1$

Step II. Let $P(k)$ be true *i.e.,* eqn. (*ii*) is true for $n = k$.

$\therefore$ Putting $n = k$ in (*ii*), we have $AB^k = B^kA$ ...(*iii*)

Step III. Post-multiplying both sides of eqn. (*iii*) by B,

We have $AB^kB = B^kAB$

or $A \cdot B^{k+1} = B^k AB$

Putting AB = BA from (*i*) in R.H.S., we have

$A B^{k+1} = B^kBA$ $\Rightarrow$ $AB^{k+1} = B^{k+1} A$

$\therefore$ Eqn. (*ii*) is true for $n = k + 1$

($\because$ On putting $n = k + 1$ in (*ii*), we get the above result)

$\therefore$ $P(k + 1)$ is true.

$\therefore$ $P(n)$ *i.e.,* eqn. (*ii*) is true for all $n \in N$ by P.M.I.

13. If $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is such that $A^2 = I$; then

(A) $1 + \alpha^2 + \beta\gamma = 0$ **(B) $1 - \alpha^2 + \beta\gamma = 0$**

(C) $1 - \alpha^2 - \beta\gamma = 0$ **(D) $1 + \alpha^2 - \beta\gamma = 0$.**

Sol. Given: $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ and $A^2 = I\ (= I_2)$ | $\because$ A is 2×2

$\Rightarrow$ $A \cdot A = I_2$ $\Rightarrow$ $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} \alpha^2 + \beta\gamma & \alpha\beta - \alpha\beta \\ \alpha\gamma - \gamma\alpha & \beta\gamma + \alpha^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \alpha^2 + \beta\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating corresponding entries, we have $\alpha^2 + \beta\gamma = 1$

$\therefore \quad 1 - \alpha^2 - \beta\gamma = 0.$

Therefore, option (C) is the correct answer.

14. If the matrix A is both symmetric and skew symmetric, then

(A) A is a diagonal matrix (B) A is a zero matrix

(C) A is a square matrix (D) None of these.

Sol. Because A is symmetric, therefore $A' = A$ ...(*i*)

Because A is skew-symmetric, therefore $A' = -A$ ...(*ii*)

Putting $A' = A$ from (*i*) in (*ii*), $A = -A \Rightarrow A + A = 0$

$$\Rightarrow \quad 2A = 0 \Rightarrow A = \frac{0}{2} = 0$$

i.e., A is a zero matrix. $\therefore$ Option (B) is correct answer.

Note: It may be noted that if A and B are square matrices of the same order, then

$$(A + B)^2 \neq A^2 + B^2 + 2AB \text{ always.}$$

But if matrices A and B commute *i.e.,* $AB = BA$, then

$(A + B)^2 = A^2 + B^2 + 2AB$ and also $(A + B)^3 = A^3 + B^3 + 3AB(A + B)$

15. If A is a square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to

(A) A (B) I – A (C) I (D) 3A.

Sol. Given: $A^2 = A$ ...(*i*)

Multiplying both sides by A, $A^3 = A^2 = A$ (By (*i*)) ...(*ii*)

The given expression $= (I + A)^3 - 7A$

$$= I^3 + A^3 + 3IA(I + A) - 7A$$

[We know that $AI = IA$, therefore using above note we can apply $(A + B)^3 = A^3 + B^3 + 3AB(A + B)$]

$$= I^3 + A^3 + 3I^2A + 3IA^2 - 7A$$

Putting $A^2 = A$ from (*i*) and $A^3 = A$ from (*ii*) and

$I^3 = I$ and $I^2 = I$ (Because $I^n = I$ always for all $n \in N$)

$$= I + A + 3IA + 3IA - 7A$$

$$= I + A + 3A + 3A - 7A \quad (\because \ AI = A \text{ and } IA = A)$$

$$= I + 7A - 7A = I$$

Hence, option (C) is the correct answer.