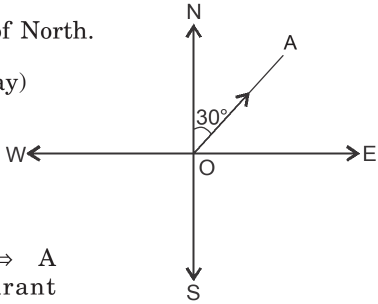
Exercise 10.1

1. Represent graphically a displacement of 40 km, 30° east of north.

Sol. Displacement 40 km, 30° East of North.



$\Rightarrow$ Displacement vector $\overrightarrow{OA}$ (say)

such that $|\overrightarrow{OA}| = 40$ (given)

and vector $\overrightarrow{OA}$ makes an angle 30° with North in East-North quadrant.

Note. $\alpha°$ South of West $\Rightarrow$ A vector in South-West quadrant making an angle of $\alpha°$ with West.

2. Check the following measures as scalars and vectors:

(i) **10 kg** ***(ii)*** **2 meters north-west** ***(iii)*** **40°**

(iv) **40 Watt** ***(v)*** **10^{-19} coulomb** ***(vi)*** **20 m/sec^2.**

Sol. (*i*) 10 kg is a measure of mass and therefore a **scalar.**

($\because$ 10 kg has no direction, it is magnitude only).

(*ii*) 2 meters North-West is a measure of velocity (*i.e.,* has magnitude and direction both) and hence is a **vector.**

(*iii*) 40° is a measure of angle *i.e.,* is magnitude only and, therefore, a scalar.

(*iv*) 40 Watt is a measure of power (*i.e.,* 40 watt has no direction) and, therefore, a scalar.

(*v*) 10^{-19} coulomb is a measure of electric charge (*i.e.,* is magnitude only) and, therefore, a scalar.

(*vi*) 20 m/sec^2 is a measure of acceleration *i.e.,* is a measure of rate of change of velocity and hence is a vector.

3. Classify the following as scalar and vector quantities:

(i) **time period** ***(ii)*** **distance** ***(iii)*** **force**

(iv) **velocity** ***(v)*** **work done.**

Sol. (*i*) Time-scalar (*ii*) Distance-scalar (*iii*) Force-vector

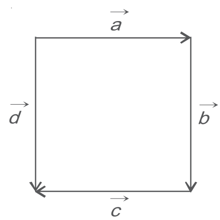
(*iv*) Velocity-vector (*v*) Work done-scalar.

4. In the adjoining figure, (a square), identify the following vectors.

(i) **Coinitial**

(ii) **Equal**

(iii) **Collinear but not equal.**



Sol. (*i*) $\vec{a}$ and $\vec{d}$ have same initial point and, therefore, coinitial vectors.

(*ii*) $\vec{b}$ and $\vec{d}$ have same direction and same magnitude. Therefore, $\vec{b}$ and $\vec{d}$ are equal vectors.

(*iii*) $\vec{a}$ and $\vec{c}$ have parallel supports, so that they are collinear. Since they have opposite directions, they are not equal. Hence $\vec{a}$ and $\vec{c}$ are collinear but not equal.

5. Answer the following as true or false.

(*i*) $\vec{a}$ and $-\vec{a}$ are collinear.

(*ii*) Two collinear vectors are always equal in magnitude.

(*iii*) Two vectors having same magnitude are collinear.

(*iv*) Two collinear vectors having the same magnitude are equal.

Sol. (*i*) True.

(*ii*) False. ($\because \vec{a}$ and $2\vec{a}$ are collinear vectors but $|2\vec{a}| = 2|\vec{a}|$)

(*iii*) False.

($\because |\hat{i}| = |\hat{j}| = 1$ but $\hat{i}$ and $\hat{j}$ are vectors along x-axis (OX) and y-axis (OY) respectively.)

(*iv*) False.

($\because$ Vectors $\vec{a}$ and $-\vec{a}$ $(= (-1)\vec{a} = m\vec{a})$ are collinear vectors and $|\vec{a}| = |-\vec{a}|$ but we know that $\vec{a} \neq -\vec{a}$ because their directions are opposite).

Note. Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if

(*i*) $|\vec{a}| = |\vec{b}|$ (*ii*) $\vec{a}$ and $\vec{b}$ have same (like) direction.

Exercise 10.2

1. Compute the magnitude of the following vectors:

$\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}$,

$\vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$.

Sol. Given: $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.

Therefore, $|\vec{a}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{1+1+1} = \sqrt{3}$.

$\vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}$.

Therefore, $|\vec{b}| = \sqrt{4 + 49 + 9} = \sqrt{62}$.

$\vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$.

Therefore, $|\vec{c}| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{-1}{\sqrt{3}}\right)^2}$

$= \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = \sqrt{\frac{3}{3}} = \sqrt{1} = 1.$

2. Write two different vectors having same magnitude.

Sol. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$.

Clearly, $\vec{a} \neq \vec{b}$. ($\because$ Coefficients of $\hat{i}$ and $\hat{j}$ are same in vectors $\vec{a}$ and $\vec{b}$ but coefficients of $\hat{k}$ in $\vec{a}$ and $\vec{b}$ are unequal as $1 \neq -1$).

But $|\vec{a}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{1+1+1} = \sqrt{3}$

and $|\vec{b}| = \sqrt{1+1+1} = \sqrt{3}$ $\quad\therefore\quad |\vec{a}| = |\vec{b}|$.

Remark. In this way, we can construct an infinite number of possible answers.

3. Write two different vectors having same direction.

Sol. Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ ...(*i*)

and $\vec{b} = 2(\hat{i} + 2\hat{j} + 3\hat{k})$ ...(*ii*)

$= 2\vec{a}$ [By (*i*)]

$\therefore \quad \vec{b} = m\vec{a}$ where $m = 2 > 0$.

$\therefore$ Vectors $\vec{a}$ and $\vec{b}$ have the same direction.

But $\vec{b} \neq \vec{a}$ $[\because \vec{b} = 2\vec{a} \Rightarrow |\vec{b}| = |2||\vec{a}| = 2|\vec{a}| \neq |\vec{a}|]$

Remark. In this way, we can construct an infinite number of possible answers.

4. Find the values of x and y so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal.

Sol. Given: $2\hat{i} + 3\hat{j} = x\hat{i} + y\hat{j}$.

Comparing coefficients of $\hat{i}$ and $\hat{j}$ on both sides, we have

$$x = 2 \text{ and } y = 3.$$

5. Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (– 5, 7).

Sol. Let $\vec{AB}$ be the vector with initial point A(2, 1) and terminal point B(– 5, 7).

$\Rightarrow$ P.V. (Position Vector) of point A is (2, 1) = $2\hat{i} + \hat{j}$ and P.V. of point B is (– 5, 7) = $-5\hat{i} + 7\hat{j}$.

$\therefore$ $\vec{AB}$ = P.V. of point B – P.V. of point A

$= (-5\hat{i} + 7\hat{j}) - (2\hat{i} + \hat{j}) = -5\hat{i} + 7\hat{j} - 2\hat{i} - \hat{j}$

$\Rightarrow$ $\vec{AB} = -7\hat{i} + 6\hat{j}$.

$\therefore$ By definition, scalar components of the vectors $\vec{AB}$ are coefficients of $\hat{i}$ and $\hat{j}$ in $\vec{AB}$ *i.e.,* – 7 and 6 and vector components of the vector $\vec{AB}$ are $-7\hat{i}$ and $6\hat{j}$.

6. Find the sum of the vectors:

$$\vec{a} = \hat{i} - 2\hat{j} + \hat{k},\ \vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$$

and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

Sol. Given: $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$

and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

Adding $\vec{a} + \vec{b} + \vec{c} = 0\hat{i} - 4\hat{j} - \hat{k} = -4\hat{j} - \hat{k}$.

7. Find the unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$.

Sol. We know that a unit vector in the direction of the vector

$$\vec{a} = \hat{i} + \hat{j} + 2\hat{k} \text{ is } \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{1+1+4}}$$

$$\Rightarrow \quad \hat{a} = \frac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}.$$

8. Find the unit vector in the direction of the vector $\overrightarrow{PQ}$ where P and Q are the points (1, 2, 3) and (4, 5, 6) respectively.

Sol. Because points P and Q are P(1, 2, 3) and Q(4, 5, 6) (given),

therefore, position vector of point P = $\overrightarrow{OP} = 1\hat{i} + 2\hat{j} + 3\hat{k}$

and position vector of point Q = $\overrightarrow{OQ} = 4\hat{i} + 5\hat{j} + 6\hat{k}$
where O is the origin.

$\therefore \quad \overrightarrow{PQ}$ = Position vector of point Q – Position vector of point P

$$= \overrightarrow{OQ} - \overrightarrow{OP} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

Therefore, a unit vector in the direction of vector $\overrightarrow{PQ}$

$$= \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{\sqrt{9 + 9 + 9 = 27 = 9 \times 3}}$$

$$= \frac{3(\hat{i} + \hat{j} + \hat{k})}{3\sqrt{3}} = \frac{(\hat{i} + \hat{j} + \hat{k})}{\sqrt{3}} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}.$$

9. For given vectors $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$; find the unit vector in the direction of $\vec{a} + \vec{b}$.

Sol. Given: Vectors $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{j}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$

$\therefore \quad \vec{a} + \vec{b} = 2\hat{i} - \hat{j} + 2\hat{k} - \hat{i} + \hat{j} - \hat{k} = \hat{i} + 0\hat{j} + \hat{k}$

$\therefore \quad |\vec{a} + \vec{b}| = \sqrt{(1)^2 + (0)^2 + (1)^2} = \sqrt{2}$

$\therefore$ A unit vector in the direction of $\vec{a} + \vec{b}$ is

$$\frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \frac{\hat{i} + 0\hat{j} + \hat{k}}{\sqrt{2}} = \frac{\hat{i} + \hat{k}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}.$$

10. Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units.

Sol. Let $\vec{a} = 5\hat{i} - \hat{j} + 2\hat{k}$.

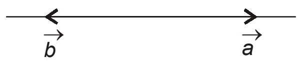
$\therefore$ A vector in the direction of vector $\vec{a}$ which has magnitude 8 units

$$= 8\hat{a} = 8\frac{\vec{a}}{|\vec{a}|} = \frac{8(5\hat{i} - \hat{j} + 2\hat{k})}{\sqrt{25+1+4}}$$

$$= \frac{8}{\sqrt{30}}(5\hat{i} - \hat{j} + 2\hat{k}) = \frac{40}{\sqrt{30}}\hat{i} - \frac{8}{\sqrt{30}}\hat{j} + \frac{16}{\sqrt{30}}\hat{k}.$$

11. Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.

Sol. Let $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ ...(*i*)



and $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k}$

$= -2(2\hat{i} - 3\hat{j} + 4\hat{k}) = -2\vec{a}$ [By (*i*)]

$\Rightarrow$ $\vec{b} = -2\vec{a} = m\vec{a}$ where $m = -2 < 0$

$\therefore$ Vectors $\vec{a}$ and $\vec{b}$ are collinear (unlike because $m = -2 < 0$).

12. Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.

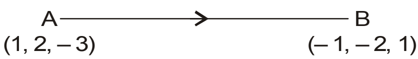
Sol. The given vector is $(\vec{a}) = \hat{i} + 2\hat{j} + 3\hat{k}$

$$= \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}} = \frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$$

We know that direction cosines of a vector $\vec{a}$ are coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{a}$ *i.e.,* $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$.

13. Find the direction cosines of the vector joining the points A(1, 2, – 3) and B(– 1, – 2, 1) directed from A to B.

Sol. **Given:** Points A(1, 2, – 3) and B(– 1, – 2, 1).



$\Rightarrow$ P.V. (Position Vector, $\vec{OA}$) of point A is A(1, 2, – 3) $= \hat{i} + 2\hat{j} - 3\hat{k}$

and P.V. of point B is B(– 1, – 2, 1) $= -\hat{i} - 2\hat{j} + \hat{k}$.

$\therefore$ Vector $\overrightarrow{AB}$ (directed from A to B)

= P.V. of point B – P.V. of point A

$= -\hat{i} - 2\hat{j} + \hat{k} - (\hat{i} + 2\hat{j} - 3\hat{k})$

$= -\hat{i} - 2\hat{j} + \hat{k} - \hat{i} - 2\hat{j} + 3\hat{k} = -2\hat{i} - 4\hat{j} + 4\hat{k}$

$\therefore$ AB $= |\overrightarrow{AB}| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{4 + 16 + 16} = 6$

$\therefore$ A unit vector along $\overrightarrow{AB} = \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$

$= \dfrac{-2\hat{i} - 4\hat{j} + 4\hat{k}}{6} = -\dfrac{2}{6}\hat{i} - \dfrac{4}{6}\hat{j} + \dfrac{4}{6}\hat{k} = \dfrac{-1}{3}\hat{i} - \dfrac{2}{3}\hat{j} + \dfrac{2}{3}\hat{k}$.

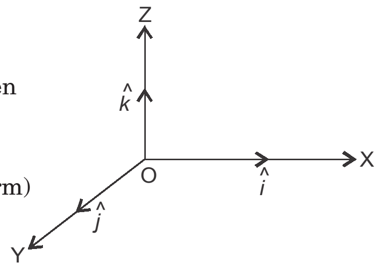
We know that Direction Cosines of the vector $\overrightarrow{AB}$ are the coefficients of $\hat{i}, \hat{j}, \hat{k}$ in a unit vector along $\overrightarrow{AB}$ *i.e.,* $\dfrac{-1}{3}, \dfrac{-2}{3}, \dfrac{2}{3}$.

14. Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY and OZ.

Sol. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.

Let us find angle θ_1 (say) between vector $\vec{a}$ and OX ($\Rightarrow \hat{i}$)

($\because \hat{i}$ represents OX in vector form)



$\therefore \cos\theta_1 = \dfrac{\vec{a}\cdot\hat{i}}{|\vec{a}||\hat{i}|}$

$\Rightarrow \cos\theta_1 = \dfrac{(\hat{i} + \hat{j} + \hat{k}).(\hat{i} + 0\hat{j} + 0\hat{k})}{|\hat{i} + \hat{j} + \hat{k}||\hat{i} + 0\hat{j} + 0\hat{k}|}$

$\Rightarrow \cos\theta_1 = \dfrac{1(1) + 1(0) + 1(0)}{\sqrt{1+1+1}\sqrt{1+0+0}} = \dfrac{1}{\sqrt{3}} \Rightarrow \theta_1 = \cos^{-1}\dfrac{1}{\sqrt{3}}$

Similarly, angle θ_2 between vectors $\vec{a}$ and $\hat{j}$ (OY) is $\cos^{-1}\dfrac{1}{\sqrt{3}}$

and angle θ_3 between vectors $\vec{a}$ and $\hat{k}$ (OZ) is also $\cos^{-1}\dfrac{1}{\sqrt{3}}$.

$\therefore \theta_1 = \theta_2 = \theta_3$.

$\therefore$ Vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ is equally inclined to OX, OY and OZ

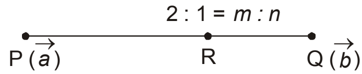
15. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2 : 1 (*i*) internally (*ii*) externally.

Sol. P.V. of point P is $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$

and P.V. of point Q is $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$

(*i*) Therefore P.V. of point R dividing PQ internally (*i.e.,* R lies within the segment PQ) in the ratio 2 : 1 (= $m : n$) (= PR : QR)

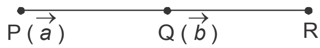
is $\dfrac{\boldsymbol{m\vec{b} + n\vec{a}}}{\boldsymbol{m + n}}$



$$= \frac{2(-\hat{i} + \hat{j} + \hat{k}) + \hat{i} + 2\hat{j} - \hat{k}}{2 + 1} = \frac{-2\hat{i} + 2\hat{j} + 2\hat{k} + \hat{i} + 2\hat{j} - \hat{k}}{3}$$

$$= \frac{-\hat{i} + 4\hat{j} + \hat{k}}{3} = \frac{-1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}.$$

(*ii*) P.V. of point R dividing PQ externally (*i.e.,* R lies outside PQ and to the right of point Q because ratio 2 : 1 = $\frac{2}{1}$ > 1 as PR is 2 times PQ *i.e.,* $\left.\frac{\text{PR}}{\text{QR}} = \frac{2}{1}\right)$ is $\dfrac{\boldsymbol{m\vec{b} - n\vec{a}}}{\boldsymbol{m - n}}$



$$= \frac{2(-\hat{i} + \hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} - \hat{k})}{2 - 1}$$

$$= -2\hat{i} + 2\hat{j} + 2\hat{k} - \hat{i} - 2\hat{j} + \hat{k} = -3\hat{i} + \hat{k}.$$

Remark. In the above question 15(*ii*), had R been dividing PQ externally in the ratio 1 : 2; then R will lie to the left of point P and $\frac{\text{PR}}{\text{QR}} = \frac{1}{2}$.

16. Find the position vector of the mid-point of the vector joining the points P(2, 3, 4) and Q(4, 1, – 2).

Sol. Given: Point P is (2, 3, 4) and Q is (4, 1, – 2).

$\Rightarrow$ P.V. of point P(2, 3, 4) is $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

and P.V. of point Q(4, 1, – 2) is $\vec{b} = 4\hat{i} + \hat{j} - 2\hat{k}$.

$\therefore$ P.V. of mid-point R of PQ is $\dfrac{\boldsymbol{\vec{a} + \vec{b}}}{\boldsymbol{2}}$.

[By Formula of Internal division]

$$= \frac{2\hat{i} + 3\hat{j} + 4\hat{k} + 4\hat{i} + \hat{j} - 2\hat{k}}{2} = \frac{6\hat{i} + 4\hat{j} + 2\hat{k}}{2} = 3\hat{i} + 2\hat{j} + \hat{k}.$$

17. Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$, respectively form the vertices of a right-angled triangle.

Sol. **Given:** P.V. of points A, B, C respectively are $\vec{a}\ (= \overrightarrow{OA}) = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b}\ (= \overrightarrow{OB}) = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c}\ (= \overrightarrow{OC}) = \hat{i} - 3\hat{j} - 5\hat{k}$, where O is the origin.

Step I. $\therefore$ $\overrightarrow{AB}$ = P.V. of point B – P.V. of point A

$= 2\hat{i} - \hat{j} + \hat{k} - (3\hat{i} - 4\hat{j} - 4\hat{k}) \quad = 2\hat{i} - \hat{j} + \hat{k} - 3\hat{i} + 4\hat{j} + 4\hat{k}$

or $\overrightarrow{AB} = -\hat{i} + 3\hat{j} + 5\hat{k}$ ...(*i*)

$\overrightarrow{BC}$ = P.V. of point C – P.V. of point B

$= (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = \hat{i} - 3\hat{j} - 5\hat{k} - 2\hat{i} + \hat{j} - \hat{k}$

$= -\hat{i} - 2\hat{j} - 6\hat{k}$ ...(*ii*)

$\overrightarrow{AC}$ = P.V. of point C – P.V. of point A

$= \hat{i} - 3\hat{j} - 5\hat{k} - (3\hat{i} - 4\hat{j} - 4\hat{k}) = \hat{i} - 3\hat{j} - 5\hat{k} - 3\hat{i} + 4\hat{j} + 4\hat{k}$

$= -2\hat{i} + \hat{j} - \hat{k}$ ...(*iii*)

Adding (*i*) and (*ii*),

$\overrightarrow{AB} + \overrightarrow{BC} = -\hat{i} + 3\hat{j} + 5\hat{k} - \hat{i} - 2\hat{j} - 6\hat{k}$

$-2\hat{i} + \hat{j} - \hat{k} = \overrightarrow{AC}$ [By (*iii*)]

$\therefore$ By Triangle Law of addition of Vectors, Points A, B, C are the Vertices of a triangle or points A, B, C are collinear.

Step II.

From (*i*) AB = $|\overrightarrow{AB}| = \sqrt{1 + 9 + 25} = \sqrt{35}$

From (*ii*), BC = $|\overrightarrow{BC}| = \sqrt{1 + 4 + 36} = \sqrt{41}$

From (*iii*), AC = $|\overrightarrow{AC}| = \sqrt{4 + 1 + 1} = \sqrt{6}$

We can observe that (Longest side BC)$^2 = (\sqrt{41})^2 = 41 = 35 + 6$
$= AB^2 + AC^2$

$\therefore$ Points A, B, C are the vertices of a right-angled triangle.

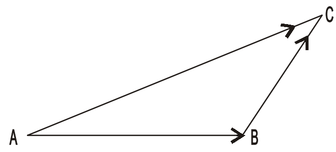
18. In triangle ABC (Fig. below), which of the following is not true:

(A) $\vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$

(B) $\vec{AB} + \vec{BC} - \vec{AC} = \vec{0}$

(C) $\vec{AB} + \vec{BC} - \vec{CA} = \vec{0}$

(D) $\vec{AB} - \vec{CB} + \vec{CA} = \vec{0}$



Sol. Option (C) is not true.

Because we know by Triangle Law of Addition of vectors that

$$\vec{AB} + \vec{BC} = \vec{AC}, \textit{ i.e.,} \quad \boxed{\vec{AB} + \vec{BC} = -\vec{CA}}$$

$$\Rightarrow \vec{AB} + \vec{BC} - \vec{AC} = \vec{0} \quad \boxed{\Rightarrow \vec{AB} + \vec{BC} + \vec{CA} = \vec{0}}$$

But for option (C), $\vec{AB} + \vec{BC} - \vec{CA} = \vec{AC} + \vec{AC} = 2\vec{AC} \neq \vec{0}$.

Option (D) is same as option (A).

19. If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect:

(A) $\vec{b} = \lambda\vec{a}$**, for some scalar λ.** **(B)** $\vec{a} = \pm\vec{b}$

(C) the respective components of $\vec{a}$ and $\vec{b}$ are proportional

(D) both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes.

Sol. Option (D) is not true because two collinear vectors can have **different** directions and also different magnitudes.

The options (A) and (C) are true by definition of collinear vectors.

Option (B) is a particular case of option (A) (taking $\lambda = \pm 1$).

Exercise 10.3

1. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitude $\sqrt{3}$ and 2, respectively having $\vec{a} \cdot \vec{b} = \sqrt{6}$.

Sol. Given: $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{6}$

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$. We know that

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Putting values, $\cos\theta = \dfrac{\sqrt{6}}{\sqrt{3}(2)}$

$$= \frac{\sqrt{6}}{\sqrt{3}\sqrt{4}} = \frac{\sqrt{6}}{\sqrt{12}} = \sqrt{\frac{6}{12}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} \qquad \therefore \quad \theta = \frac{\pi}{4}.$$

2. Find the angle between the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.

Sol. Given: Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$.

$\therefore |\vec{a}| = \sqrt{1+4+9} = \sqrt{14}$ $\left| \because |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2} \right.$

and $|\vec{b}| = \sqrt{9+4+1} = \sqrt{14}$

Also, $\vec{a} \cdot \vec{b}$ = Product of coefficients of $\hat{i}$ + Product of coefficient of $\hat{j}$ + Product of coefficients of $\hat{k}$

$$= 1(3) + (-2)(-2) + 3(1) = 3 + 4 + 3 = 10$$

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

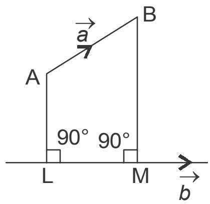
We know that $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{10}{\sqrt{14}\sqrt{14}} = \dfrac{10}{14} = \dfrac{5}{7}$

$\therefore \quad \theta = \cos^{-1}\dfrac{5}{7}$.

3. Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$.

Sol. Let $\vec{a} = \hat{i} - \hat{j} = \hat{i} - \hat{j} + 0\hat{k}$

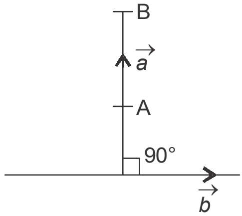
and $\vec{b} = \hat{i} + \hat{j} = \hat{i} + \hat{j} + 0\hat{k}$



Projection of vector $\vec{a}$ and $\vec{b}$

$= \text{Length LM} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

$= \dfrac{(1)(1)+(-1)(1)+0(0)}{\sqrt{(1)^2+(1)^2+0^2}} = \dfrac{1-1+0}{\sqrt{2}} = \dfrac{0}{\sqrt{2}} = 0.$



Remark. If projection of vector $\vec{a}$ on $\vec{b}$ is zero, then vector $\vec{a}$ is perpendicular to vector $\vec{b}$.

4. Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$.

Sol. Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$

We know that projection of vector $\vec{a}$ on vector $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

$$= \frac{1(7)+3(-1)+7(8)}{\sqrt{(7)^2+(-1)^2+(8)^2}} = \frac{7-3+56}{\sqrt{49+1+64}} = \frac{60}{\sqrt{114}}.$$

5. Show that each of the given three vectors is a unit vector: $\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$, $\frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$, $\frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$.
Also show that they are mutually perpendicular to each other.

Sol. Let $\vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}) = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$ ...(*i*)

$\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}$ ...(*ii*)

$\vec{c} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k}) = \frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{3}{7}\hat{k}$ ...(*iii*)

$\therefore \quad |\vec{a}| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}}$

$$= \sqrt{\frac{49}{49}} = \sqrt{1} = 1$$

$$|\vec{b}| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(\frac{-6}{7}\right)^2 + \left(\frac{2}{7}\right)^2} = \sqrt{\frac{9}{49} + \frac{36}{49} + \frac{4}{49}} = \sqrt{\frac{49}{49}}$$

$$= \sqrt{1} = 1$$

$$|\vec{c}| = \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(\frac{-3}{7}\right)^2} = \sqrt{\frac{36}{49} + \frac{4}{49} + \frac{9}{49}} = \sqrt{\frac{49}{49}}$$

$$= \sqrt{1} = 1$$

$\therefore$ Each of the three given vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is a unit vector.

From (*i*) and (*ii*),

$$\vec{a} \cdot \vec{b} = \left(\frac{2}{7}\right) \cdot \left(\frac{3}{7}\right) + \left(\frac{3}{7}\right) \left(\frac{-6}{7}\right) + \left(\frac{6}{7}\right) \left(\frac{2}{7}\right)$$

$$[\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3]$$

$$= \frac{6}{49} - \frac{18}{49} + \frac{12}{49} = \frac{6-18+12}{49} = \frac{0}{49} = 0$$

$\therefore$ $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

From (*ii*) and (*iii*),

$$\vec{b} \cdot \vec{c} = \left(\frac{3}{7}\right) \left(\frac{6}{7}\right) + \left(\frac{-6}{7}\right) \left(\frac{2}{7}\right) + \frac{2}{7} \left(\frac{-3}{7}\right)$$

$$= \frac{18}{49} - \frac{12}{49} - \frac{6}{49} = \frac{18-12-6}{49} = \frac{0}{49} = 0$$

$\therefore$ $\vec{b}$ and $\vec{c}$ are perpendicular to each other.

From (*i*) and (*iii*),

$$\vec{a} \cdot \vec{c} = \frac{2}{7} \left(\frac{6}{7}\right) + \frac{3}{7} \left(\frac{2}{7}\right) + \left(\frac{6}{7}\right)\left(\frac{-3}{7}\right)$$

$$= \frac{12}{49} + \frac{6}{49} - \frac{18}{49} = \frac{12+6-18}{49} = \frac{0}{49} = 0$$

$\therefore$ $\vec{a}$ and $\vec{c}$ are perpendicular to each other.

Hence, $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular vectors.

6. Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.

Sol. Given: $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$ ...(*i*)

$\Rightarrow \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} - \vec{b}\cdot\vec{b} = 8$

$\Rightarrow |\vec{a}|^2 - \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{b} - |\vec{b}|^2 = 8$

[$\because$ We know that $\vec{a}\cdot\vec{a} = |\vec{a}|^2$ and $\vec{b}\cdot\vec{b} = |\vec{b}|^2$ and $\vec{b}\cdot\vec{a} = \vec{a}\cdot\vec{b}$]

$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 8$ ...(*ii*)

Putting $|\vec{a}| = 8|\vec{b}|$ from (*i*) in (*ii*), $64|\vec{b}|^2 - |\vec{b}|^2 = 8$

or $(64 - 1)|\vec{b}|^2 = 8$ $\Rightarrow 63|\vec{b}|^2 = 8$

$\Rightarrow |\vec{b}|^2 = \frac{8}{63}$ $\Rightarrow |\vec{b}| = \sqrt{\frac{8}{63}} = \sqrt{\frac{4 \times 2}{9 \times 7}}$

($\because$ Length *i.e.,* modulus of a vector is never negative.)

$\Rightarrow |\vec{b}| = \frac{2}{3}\sqrt{\frac{2}{7}}$

Putting this value of $|\vec{b}|$ in (*i*),

$$|\vec{a}| = 8\left(\frac{2}{3}\sqrt{\frac{2}{7}}\right) = \frac{16}{3}\sqrt{\frac{2}{7}}.$$

7. Evaluate the product $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$.

Sol. The given expression $= (3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$

$= (3\vec{a})\cdot(2\vec{a}) + (3\vec{a})\cdot(7\vec{b}) - (5\vec{b})\cdot(2\vec{a}) - (5\vec{b})\cdot(7\vec{b})$

$= 6\vec{a}\cdot\vec{a} + 21\vec{a}\cdot\vec{b} - 10\vec{b}\cdot\vec{a} - 35\vec{b}\cdot\vec{b}$

$= 6|\vec{a}|^2 + 21\vec{a}\cdot\vec{b} - 10\vec{a}\cdot\vec{b} - 35|\vec{b}|^2$

[$\because \vec{a}\cdot\vec{a} = |\vec{a}|^2$ and $\vec{b}\cdot\vec{b} = |\vec{b}|^2$ and $\vec{b}\cdot\vec{a} = \vec{a}\cdot\vec{b}$]

$= 6|\vec{a}|^2 + 11\vec{a}\cdot\vec{b} - 35|\vec{b}|^2$.

8. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.

Sol. Given: $|\vec{a}| = |\vec{b}|$ and angle θ (say) between $\vec{a}$ and $\vec{b}$ is

$60°$ and their scalar (*i.e.,* dot) product $= \frac{1}{2}$

i.e., $\vec{a} \cdot \vec{b} = \frac{1}{2}$

$\Rightarrow \quad |\vec{a}|\,|\vec{b}| \cos\theta = \frac{1}{2}$ $\quad [\because \vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta]$

Putting $|\vec{b}| = |\vec{a}|$ (given) and $\theta = 60°$ (given), we have

$|\vec{a}|\,|\vec{a}| \cos 60° = \frac{1}{2} \quad \Rightarrow \quad |\vec{a}|^2 \left(\frac{1}{2}\right) = \frac{1}{2}$

Multiplying by 2, $|\vec{a}|^2 = 1 \quad \Rightarrow \quad |\vec{a}| = 1$ ...(*i*)

($\because$ Length of a vector is never negative)

$\therefore \quad |\vec{b}| = |\vec{a}| = 1$ [By (*i*)]

$\therefore \quad |\vec{a}| = 1$ and $|\vec{b}| = 1$.

9. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$.

Sol. Given: $\vec{a}$ is a unit vector $\Rightarrow |\vec{a}| = 1$ ...(*i*)

Also given $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$

$\Rightarrow \quad \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{a} - \vec{a} \cdot \vec{x} - \vec{a} \cdot \vec{a} = 12$

$\Rightarrow \quad |\vec{x}|^2 + \vec{a} \cdot \vec{x} - \vec{a} \cdot \vec{x} - |\vec{a}|^2 = 12$

$\Rightarrow \quad |\vec{x}|^2 - |\vec{a}|^2 = 12$

Putting $|\vec{a}| = 1$ from (*i*), $|\vec{x}|^2 - 1 = 12$

$\Rightarrow \quad |\vec{x}|^2 = 13 \quad \Rightarrow \quad |\vec{x}| = \sqrt{13}$.

($\because$ Length of a vector is never negative.)

10. If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

Sol. Given : $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$

and $\vec{c} = 3\hat{i} + \hat{j}$.

Now, $\vec{a} + \lambda\vec{b} = 2\hat{i} + 2\hat{j} + 3\hat{k} + \lambda(-\hat{i} + 2\hat{j} + \hat{k})$

$= 2\hat{i} + 2\hat{j} + 3\hat{k} - \lambda\hat{i} + 2\lambda\hat{j} + \lambda\hat{k}$

$\Rightarrow \quad \vec{a} + \lambda\vec{b} = (2 - \lambda)\hat{j} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}$

Again given $\vec{c} = 3\hat{i} + \hat{j} = 3\hat{i} + \hat{j} + 0\hat{k}$.

Because vector $\vec{a} + \lambda\vec{a}$ is perpendicular to $\vec{c}$, therefore,

$$(\vec{a} + \lambda\vec{b}) . \vec{c} = 0$$

i.e., Product of coefficients of $\hat{i}$ + ... = 0

$\Rightarrow$ $(2 - \lambda)3 + (2 + 2\lambda)1 + (3 + \lambda)0 = 0$

$\Rightarrow$ $6 - 3\lambda + 2 + 2\lambda = 0$ $\Rightarrow$ $-\lambda + 8 = 0$

$\Rightarrow$ $-\lambda = -8$ $\Rightarrow$ $\lambda = 8$.

11. Show that $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$, for any two non-zero vectors $\vec{a}$ and $\vec{b}$.

Sol. Let $\vec{c} = |\vec{a}|\vec{b} + |\vec{b}|\vec{a} = l\vec{b} + m\vec{a}$

where $l = |\vec{a}|$ and $m = |\vec{b}|$

Let $\vec{d} = |\vec{a}|\vec{b} - |\vec{b}|\vec{a} = l\vec{b} - m\vec{a}$

Now, $\vec{c}.\vec{d} = (l\vec{b} + m\vec{a}) . (l\vec{b} - m\vec{a})$

$= l^2\,\vec{b}.\vec{b} - lm\,\vec{b}.\vec{a} + lm\,\vec{a}.\vec{b} - m^2\vec{a}\,\vec{a}$

$= l^2|\vec{b}|^2 - lm\,\vec{a}.\vec{b} + lm\,\vec{a}.\vec{b} - m^2|\vec{a}|^2 = l^2|\vec{b}|^2 - m^2|\vec{a}|$

Putting $l = |\vec{a}|$ and $m = |\vec{b}|$,

$$= |\vec{a}|^2|\vec{b}|^2 - |\vec{b}|^2|\vec{a}|^2 = 0$$

i.e., $\vec{c}.\vec{d} = 0$

$\therefore$ Vectors $\vec{c}$ and $\vec{d}$ are perpendicular to each other.

12. If $\vec{a}.\vec{a} = 0$ and $\vec{a}.\vec{b} = 0$, then what can be concluded about the vector $\vec{b}$?

Sol. Given: $\vec{a}.\vec{a} = 0 \Rightarrow |\vec{a}|^2 = 0 \Rightarrow |\vec{a}| = 0$ ...(*i*)

($\Rightarrow$ $\vec{a}$ is a zero vector by definition of zero vector.)

Again given $\vec{a}.\vec{b} = 0$ $\Rightarrow$ $|\vec{a}||\vec{b}|\cos\theta = 0$

Putting $|\vec{a}| = 0$ from (*i*), we have $0|\vec{b}|\cos\theta = 0$

i.e., 0 = 0 for all (any) vectors $\vec{b}$. $\therefore$ $\vec{b}$ can be any vector.

Note. $(\vec{a} + \vec{b} + \vec{c})^2 = (\vec{a} + (\vec{b} + \vec{c}))^2$

$= \vec{a}^{\,2} + (\vec{b} + \vec{c})^2 + 2\vec{a} . (\vec{b} + \vec{c})$

[$\because$ $(\vec{A} + \vec{B})^2 = \vec{A}^{\,2} + \vec{B}^{\,2} + 2\vec{A}.\vec{B}$]

$= \vec{a}^{\,2} + \vec{b}^{\,2} + \vec{c}^{\,2} + 2\vec{b}.\vec{c} + 2\vec{a}.\vec{b} + 2\vec{a}.\vec{c}$

Using $\vec{a}.\vec{c} = \vec{c}.\vec{a}$

or $(\vec{a} + \vec{b} + \vec{c})^2 = \vec{a}^{\,2} + \vec{b}^{\,2} + \vec{c}^{\,2} + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a})$

13. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$.

Sol. Because $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors, therefore,

$|\vec{a}| = 1$, $|\vec{b}| = 1$ and $|\vec{c}| = 1$. ...(*i*)

Again given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

Squaring both sides $(\vec{a} + \vec{b} + \vec{c})^2 = 0$

Using formula of **Note** above

$\Rightarrow \quad \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$

or $\quad |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$

Putting $|\vec{a}| = 1$, $|\vec{b}| = 1$, $|\vec{c}| = 1$ from (*i*),

$$1 + 1 + 1 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$$

$$\Rightarrow \quad 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = -3$$

Dividing both sides by 2, $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = \dfrac{-3}{2}$.

14. If either vector $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a}.\vec{b} = 0$. But the converse need not be true. Justify your answer with an example.

Sol. Case I. Vector $\vec{a} = \vec{0}$. Therefore, by definition of zero vector,

$|\vec{a}| = 0$ ...(*i*)

$\therefore \quad \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = 0\ (|\vec{b}|\cos\theta)$ [By (*i*)]

$= 0$

Case II. Vector $\vec{b} = \vec{0}$. Proceeding as above we can prove that $\vec{a}.\vec{b} = 0$

But the converse is not true.

Let us justify it with an example.

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$. Therefore, $|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \neq 0$.

Therefore $\vec{a} \neq \vec{0}$ (By definition of Zero Vector)

Let $\quad \vec{b} = \hat{i} + \hat{j} - 2\hat{k}$.

Therefore, $|\vec{b}| = \sqrt{(1)^2 + (1)^2 + (-2)^2} = \sqrt{6} \neq 0$.

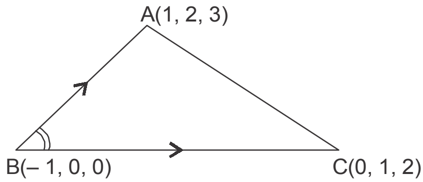
Therefore, $\quad \vec{b} \neq \vec{0}$.

But $\quad \vec{a}.\vec{b} = 1(1) + 1(1) + 1(-2) = 1 + 1 - 2 = 0$

So here $\vec{a}.\vec{b} = 0$ but neither $\vec{a} = \vec{0}$ nor $\vec{b} = \vec{0}$.

15. If the vertices A, B, C of a triangle ABC are (1, 2, 3), (– 1, 0, 0) and (0, 1, 2), respectively, then find ∠ABC.

Sol. Given: Vertices A, B, C of a triangle are A(1, 2, 3), B(– 1, 0, 0) and C(0, 1, 2) respectively.



∴ Position vector (P.V.) of point A (=s $\vec{OA}$) = (1, 2, 3)

$= \hat{i} + 2\hat{j} + 3\hat{k}$

Position vector (P.V.) of point B (= $\vec{OB}$) = (– 1, 0, 0)

$= -\hat{i} + 0\hat{j} + 0\hat{k}$

and position vector (P.V.) of point C (= $\vec{OC}$) = (0, 1, 2)

$= 0\hat{i} + \hat{j} + 2\hat{k}$

We can see from the above figure that ∠ABC is the angle between the vectors $\vec{BA}$ and $\vec{BC}$

Now $\vec{BA}$ = P.V. of terminal point A – P.V. of initial point B

$= \hat{i} + 2\hat{j} + 3\hat{k} - (-\hat{i} + 0\hat{j} + 0\hat{k})$

$= \hat{i} + 2\hat{j} + 3\hat{k} + \hat{i} - 0\hat{j} - 0\hat{k} = 2\hat{i} + 2\hat{j} + 3\hat{k}$ ...(*i*)

and $\vec{BC}$ = P.V. of point C – P.V. of point B

$= 0\hat{i} + \hat{j} + 2\hat{k} - (-\hat{i} + 0\hat{j} + 0\hat{k})$

$= 0\hat{i} + \hat{j} + 2\hat{k} + \hat{i} - 0\hat{j} - 0\hat{k} = \hat{i} + \hat{j} + 2\hat{k}$ ...(*ii*)

We know that $\cos \angle ABC = \dfrac{\vec{BA} . \vec{BC}}{|\vec{BA}||\vec{BC}|}$ | $\cos\theta = \dfrac{\vec{a} . \vec{b}}{|\vec{a}||\vec{b}|}$

Using (*i*) and (*ii*)

$$= \frac{2(1) + 2(1) + 3(2)}{\sqrt{4+4+9}\,\sqrt{1+1+4}} = \frac{10}{\sqrt{17}\sqrt{6}} = \frac{10}{\sqrt{102}}$$

∴ $\angle ABC = \cos^{-1} \dfrac{10}{\sqrt{102}}$.

16. Show that the points A(1, 2, 7), B(2, 6, 3) and C(3, 10, – 1) are collinear.

Sol. Given points are A(1, 2, 7), B(2, 6, 3) and C(3, 10, – 1).

$\Rightarrow$ P.V.'s $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ of points A, B, C are

$$\overrightarrow{OA} = (1, 2, 7) = \hat{i} + 2\hat{j} + 7\hat{k}$$

$$\overrightarrow{OB} = (2, 6, 3) = 2\hat{i} + 6\hat{j} + 3\hat{k}$$

and $\overrightarrow{OC} = (3, 10, -1) = 3\hat{i} + 10\hat{j} - \hat{k}$

$\therefore$ $\overrightarrow{AB}$ = P.V. of terminal point B – P.V. of initial point A

$$= 2\hat{i} + 6\hat{j} + 3\hat{k} - (\hat{i} + 2\hat{j} + 7\hat{k})$$

$$= 2\hat{i} + 6\hat{j} + 3\hat{k} - \hat{i} - 2\hat{j} - 7\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k} \quad \ldots(i)$$

and $\overrightarrow{AC}$ = P.V. of point C – P.V. of point A

$$= 3\hat{i} + 10\hat{j} - \hat{k} - (\hat{i} + 2\hat{j} + 7\hat{k})$$

$$= 3\hat{i} + 10\hat{j} - \hat{k} - \hat{i} - 2\hat{j} - 7\hat{k}$$

$$= 2\hat{i} + 8\hat{j} - 8\hat{k} = 2(\hat{i} + 4\hat{j} - 4\hat{k})$$

$\Rightarrow$ $\overrightarrow{AC} = 2\overrightarrow{AB}$ [By (*i*)]

$\Rightarrow$ Vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear or parallel. $\mid \because \vec{a} = m\vec{b}$

$\Rightarrow$ Points A, B, C are collinear.

($\because$ Vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ have a common point A and hence can't be parallel.)

Remark. When we come to exercise 10.4 and learn that Exercise, we have a second solution for proving points A, B, C to be collinear:

Prove that $\overrightarrow{AB} \times \overrightarrow{AC} = \vec{0}$.

17. Show that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ form the vertices of a right angled triangle.

Sol. Let the given (position) vectors be P.V.'s of the points A, B, C respectively.

P.V. of point A is $2\hat{i} - \hat{j} + \hat{k}$ and

P.V. of point B is $\hat{i} - 3\hat{j} - 5\hat{k}$ and

P.V. of point C is $3\hat{i} - 4\hat{j} - 4\hat{k}$.

$\therefore \quad \overrightarrow{AB}$ = P.V. of point B – P.V. of point A

$= \hat{i} - 3\hat{j} - 5\hat{k} - (2\hat{i} - \hat{j} + \hat{k}) = \hat{i} - 3\hat{j} - 5\hat{k} - 2\hat{i} + \hat{j} - \hat{k}$

$= -\hat{i} - 2\hat{j} - 6\hat{k}$ ...(*i*)

and $\overrightarrow{BC}$ = P.V. of point C – P.V. of point B

$= 3\hat{i} - 4\hat{j} - 4\hat{k} - (\hat{i} - 3\hat{j} - 5\hat{k}) = 3\hat{i} - 4\hat{j} - 4\hat{k} - \hat{i} + 3\hat{j} + 5\hat{k}$

$= 2\hat{i} - \hat{j} + \hat{k}$ ...(*ii*)

and $\overrightarrow{AC}$ = P.V. of point C – P.V. of point A

$= 3\hat{i} - 4\hat{j} - 4\hat{k} - (2\hat{i} - \hat{j} + \hat{k}) = 3\hat{i} - 4\hat{j} - 4\hat{k} - 2\hat{i} + \hat{j} - \hat{k}$

$= \hat{i} - 3\hat{j} - 5\hat{k}$ ...(*iii*)

Adding (*i*) and (*ii*), we have

$\overrightarrow{AB} + \overrightarrow{BC} = -\hat{i} - 2\hat{j} - 6\hat{k} + 2\hat{i} - \hat{j} + \hat{k}$

$= \hat{i} - 3\hat{j} - 5\hat{k} = \overrightarrow{AC}$ [By (*iii*)]

$\therefore$ By Triangle Law of addition of vectors, points A, B, C are the vertices of a triangle ABC or points A, B, C are collinear.

Now from (*i*) and (*ii*), $\overrightarrow{AB} . \overrightarrow{BC} = (-1)(2) + (-2)(-1) + (-6)(1)$
$= -2 + 2 - 6 = -6 \neq 0$

From (*ii*) and (*iii*), $\overrightarrow{BC} . \overrightarrow{AC} = 2(1) + (-1)(-3) + 1(-5)$
$= 2 + 3 - 5 = 0$

$\Rightarrow \quad \overrightarrow{BC}$ is perpendicular to $\overrightarrow{AC}$

$\Rightarrow$ Angle C is 90°. $\quad \therefore \quad \Delta$ABC is right angled at point C.

$\therefore$ Points A, B, C are the vertices of a right angled triangle.

18. If $\vec{a}$ is a non-zero vector of magnitude 'a' and λ is a non-zero scalar, then $\lambda\vec{a}$ is a unit vector if

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $a = |\lambda|$ (D) $a = \frac{1}{|\lambda|}$

Sol. Given: $\vec{a}$ is a non-zero vector of magnitude a

$\Rightarrow \quad |\vec{a}| = 1$ ...(*i*)

Also given: $\lambda \neq 0$ and $\lambda\vec{a}$ is a unit vector.

$\Rightarrow \quad |\lambda\vec{a}| = 1 \qquad \Rightarrow \quad |\lambda||\vec{a}| = 1$

$\Rightarrow \quad |\lambda|\, a = 1 \qquad \Rightarrow \quad a = \frac{1}{|\lambda|}$

$\therefore$ Option (D) is the correct answer.

Exercise 10.4

1. Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.

Sol. Given: $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.

Therefore, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$

$\left[\because \text{ If } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \text{ and } \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k};\right.$ then $\left.\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\right]$

Expanding along first row,

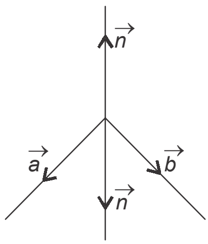
$$\vec{a} \times \vec{b} = \hat{i}\begin{vmatrix} -7 & 7 \\ -2 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 7 \\ 3 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -7 \\ 3 & -2 \end{vmatrix}$$

$$\Rightarrow \quad \vec{a} \times \vec{b} = \hat{i}(-14 + 14) - \hat{j}(2 - 21) + \hat{k}(-2 + 21)$$

$$= 0\hat{i} + 19\hat{j} + 19\hat{k}$$

$$\therefore \quad |\vec{a} \times \vec{b}| = \sqrt{0^2 + (19)^2 + (19)^2} = \sqrt{2(19)^2} = \sqrt{2}\,(19) = 19\sqrt{2}.$$

Result: We know that $\vec{n} = \vec{a} \times \vec{b}$ is a vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$.
Therefore, a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$ is



$$\hat{n} = \pm\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}. \qquad \left[\because \hat{A} = \frac{\vec{A}}{|\vec{A}|}\right]$$

2. Find a unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

Sol. Given: $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

Adding, $\vec{c} = \vec{a} + \vec{b} = 4\hat{i} + 4\hat{j} + 0\hat{k}$

Subtracting $\vec{d} = \vec{a} - \vec{b} = 2\hat{i} + 0\hat{j} + 4\hat{k}$

Therefore, $\vec{n} = \vec{c} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$

Expanding along first row $= \hat{i}(16 - 0) - \hat{j}(16 - 0) + \hat{k}(0 - 8)$

$\Rightarrow \quad \vec{n} = 16\hat{i} - 16\hat{j} - 8\hat{k}$

$\therefore |\vec{n}| = \sqrt{(16)^2 + (-16)^2 + (-8)^2} = \sqrt{256 + 256 + 64} = \sqrt{576} = 24.$

Therefore, a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$\hat{n} = \pm \frac{\vec{n}}{|\vec{n}|} = \pm \frac{(16\hat{i} - 16\hat{j} - 8\hat{k})}{24}$$

$$= \pm \left(\frac{16}{24}\hat{i} - \frac{16}{24}\hat{j} - \frac{8}{24}\hat{k}\right) = \pm \left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}\right).$$

3. If a unit vector $\hat{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\hat{a}$.

Sol. Let $\hat{a} = x\hat{i} + y\hat{j} + z\hat{k}$ be a unit vector ...(*i*)

$\Rightarrow \quad |\hat{a}| = 1 \Rightarrow \sqrt{x^2 + y^2 + z^2} = 1$

Squaring both sides, $x^2 + y^2 + z^2 = 1$ ...(*ii*)

Given: Angle between vectors $\hat{a}$ and $\hat{i} = \hat{i} + 0\hat{j} + 0\hat{k}$ is $\frac{\pi}{3}$.

$\therefore \quad \cos\frac{\pi}{3} = \frac{\hat{a}.\hat{i}}{|\hat{a}||\hat{i}|}$ $\quad \left[\because \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right]$

$\Rightarrow \quad \frac{1}{2} = \frac{x(1) + y(0) + z(0)}{(1)(1)}$ or $\frac{1}{2} = x$ ...(*iii*)

Again, **Given:** Angle between vectors $\hat{a}$ and $\hat{j} = 0\hat{i} + \hat{j} + 0\hat{k}$ is $\frac{\pi}{4}$.

$$\therefore \quad \cos\frac{\pi}{4} = \frac{\hat{a}\cdot\hat{j}}{|\hat{a}||\hat{j}|} \quad \Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{x(0)+y(1)+z(0)}{(1)(1)}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} = y \qquad \text{...}(iv)$$

Again, **Given:** Angle between vectors $\hat{a}$ and $\hat{k} = 0\hat{i} + 0\hat{j} + \hat{k}$ is θ where θ is acute.

$$\therefore \quad \cos\theta = \frac{\hat{a}\cdot\hat{k}}{|\hat{a}||\hat{k}|} = \frac{x(0)+y(0)+z(1)}{(1)(1)} = z \qquad \text{...}(v)$$

Putting values of x, y and z from (iii), (iv) and (v) in (ii),

$$\frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$$

$$\Rightarrow \quad \cos^2\theta = 1 - \frac{1}{4} - \frac{1}{2} = \frac{4-1-2}{4} = \frac{1}{4} \quad \Rightarrow \quad \cos\theta = \pm\frac{1}{2}$$

But θ is acute angle (given)

$\Rightarrow$ $\cos\theta$ is positive and hence $= \frac{1}{2} = \cos\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$

From (v), $z = \cos\theta = \frac{1}{2}$

Putting values of x, y, z in (i), $\hat{a} = \frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}$

$\therefore$ Components of $\hat{a}$ are coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ in $\hat{a}$

i.e., $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$ and acute angle $\theta = \frac{\pi}{3}$.

4. Show that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2\vec{a} \times \vec{b}$.

Sol. L.H.S. $= (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$

$= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$

$= \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} - \vec{0}$

$[\because \vec{a} \times \vec{a} = \vec{0},\ \vec{b} \times \vec{b} = \vec{0}$ and $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}]$

$= 2\vec{a} \times \vec{b}$ = R.H.S.

5. Find λ and μ if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.

Sol. Given: $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$

$$\Rightarrow \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = \vec{0}$$

Expanding along first row,

$$\hat{i}(6\mu - 27\lambda) - \hat{j}(2\mu - 27) + \hat{k}(2\lambda - 6) = \vec{0} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

Comparing coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ on both sides, we have

$$6\mu - 27\lambda = 0 \quad ...(i)$$
$$2\mu - 27 = 0 \quad ...(ii)$$
and
$$2\lambda - 6 = 0 \quad ...(iii)$$

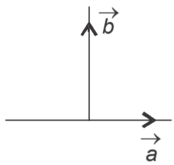
From (ii), $2\mu = 27 \Rightarrow \mu = \frac{27}{2}$

From (iii), $2\lambda = 6 \Rightarrow \lambda = \frac{6}{2} = 3$

Putting $\lambda = 3$ and $\mu = \frac{27}{2}$ in (i), $6\left(\frac{27}{2}\right) - 27(3) = 0$

or $81 - 81 = 0$ or $0 = 0$ which is true. $\therefore$ $\lambda = 3$ and $\mu = \frac{27}{2}$.

6. Given that $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?



Sol. Given: $\vec{a} \cdot \vec{b} = 0 \Rightarrow |\vec{a}|\,|\vec{b}| \cos\theta = 0$

$\Rightarrow$ Either $|\vec{a}| = 0$

or $|\vec{b}| = 0$ or $\cos\theta = 0$ ($\Rightarrow \theta = 90°$)

$\Rightarrow$ Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

or vector $\vec{a}$ is perpendicular to $\vec{b}$. ...(i)

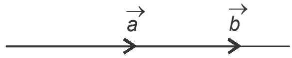
($\because$ By definition, vector $\vec{a}$ is zero vector if and only if $|\vec{a}| = 0$)

Again given $\vec{a} \times \vec{b} = \vec{0} \quad \Rightarrow |\vec{a} \times \vec{b}| = 0$

$\Rightarrow |\vec{a}|\,|\vec{b}| \sin\theta = 0$

$[\because |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta]$

$\Rightarrow$ Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$ or $\sin\theta = 0$ ($\Rightarrow$ $\theta = 0$)



$\Rightarrow$ Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or vectors $\vec{a}$ and $\vec{b}$ are collinear (or parallel) vectors. ...(*ii*)

We know from common sense that vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other as well as are parallel (or collinear) is impossible. ...(*iii*)

$\therefore$ From (*i*), (*ii*) and (*iii*), either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

$\therefore$ $\vec{a}\cdot\vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$

$\Rightarrow$ Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

7. Let the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ be given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

Then show that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

Sol. Given: Vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\therefore \quad \vec{b} + \vec{c} = (b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}$$

$$\text{L.H.S.} = \vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

[By Property of Determinants]

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \text{R.H.S.}$$

8. If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.

Sol. Given: Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

$\therefore \quad |\vec{a}| = |\vec{0}| = 0$ or $|\vec{b}| = |\vec{0}| = 0$ ...(*i*)

$\therefore \quad |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta = 0 \ (\sin\theta) = 0$ [By (*i*)]

$\therefore \quad \vec{a} \times \vec{b} = \vec{0}$ (By definition of zero vector)

But the converse is not true.

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ $\quad \therefore \quad |\vec{a}| = \sqrt{1+1+1} = \sqrt{3} \neq 0.$

$\therefore \quad \vec{a}$ is a non-zero vector.

Let $|\vec{b}| = 2(\hat{i} + \hat{j} + \hat{k}) = 2\hat{i} + 2\hat{j} + 2\hat{k}$

$\therefore |\vec{b}| = \sqrt{4+4+4}$ or $|\vec{b}| = \sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3} \neq 0.$

$\therefore \quad \vec{b}$ is a non-zero vector.

But $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix}$

Taking 2 common from R_3, $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \vec{0}$

($\because R_2$ and R_3 are identical)

9. Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

Sol. Vertices of ΔABC are A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

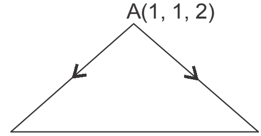
$\therefore$ Position Vector (P.V.) of point A is (1, 1, 2) $= \hat{i} + \hat{j} + 2\hat{k}$

P.V. of point B is (2, 3, 5)

$= 2\hat{i} + 3\hat{j} + 5\hat{k}$

P.V. of point C is (1, 5, 5)

$= \hat{i} + 5\hat{j} + 5\hat{k}$



$\therefore \quad \vec{AB}$ = P.V. of point B – P.V. of point A

$= 2\hat{i} + 3\hat{j} + 5\hat{k} - (\hat{i} + \hat{j} + 2\hat{k})$

$= 2\hat{i} + 3\hat{j} + 5\hat{k} - \hat{i} - \hat{j} - 2\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}$

and $\vec{AC}$ = P.V. of point C – P.V. of point A

$= \hat{i} + 5\hat{j} + 5\hat{k} - (\hat{i} + \hat{j} + 2\hat{k}) = \hat{i} + 5\hat{j} + 5\hat{k} - \hat{i} - \hat{j} - 2\hat{k}$

$= 0\hat{i} + 4\hat{j} + 3\hat{k}$

$\therefore \quad \overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$

$= \hat{i}(6 - 12) - \hat{j}(3 - 0) + \hat{k}(4 - 0) = -6\hat{i} - 3\hat{j} + 4\hat{k}$

We know that **area of triangle ABC**

$= \frac{1}{2} |\overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}}| = \frac{1}{2}\sqrt{36+9+16} \quad | \sqrt{x^2+y^2+z^2}$

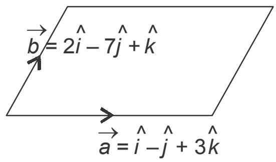
$= \frac{1}{2}\sqrt{61}$ sq. units.

10. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Sol. Given: Vectors representing two adjacent sides of a parallelogram are

$\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$

and $\quad \vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.



$\therefore \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$

$= \hat{i}(-1 + 21) - \hat{j}(1 - 6) + \hat{k}(-7 + 2) = 20\hat{i} + 5\hat{j} - 5\hat{k}$

We know that **area of parallelogram = $|\vec{a} \times \vec{b}|$**

$= \sqrt{400+25+25} = \sqrt{450} = \sqrt{25 \times 9 \times 2}$

$= 5(3)\sqrt{2} = 15\sqrt{2}$ square units.

Note. Area of parallelogram whose **diagonal vectors** are $\vec{\alpha}$ and $\vec{\beta}$ is $\frac{1}{2} |\vec{\alpha} \times \vec{\beta}|$.

11. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$, $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{4}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{2}$.

Sol. Given: $|\vec{a}| = 3$, $|\vec{b}| = \frac{\sqrt{2}}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector.

$\Rightarrow \quad |\vec{a} \times \vec{b}| = 1 \qquad \Rightarrow \quad |\vec{a}|\,|\vec{b}| \sin\theta = 1$

where θ is the angle between vectors $\vec{a}$ and $\vec{b}$.

Putting values of $|\vec{a}|$ and $|\vec{b}|$, $3\left(\dfrac{\sqrt{2}}{3}\right) \sin\theta = 1$

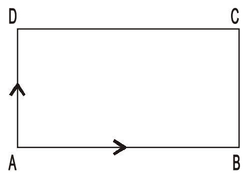
$\Rightarrow \quad \sqrt{2}\sin\theta = 1 \quad \Rightarrow \quad \sin\theta = \dfrac{1}{\sqrt{2}} = \sin\dfrac{\pi}{4} \quad \Rightarrow \quad \theta = \dfrac{\pi}{4}$

$\therefore$ Option (B) is the correct answer.

12. Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$, respectively, is

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

Sol. Given: ABCD is a rectangle.



We know that $\vec{AB}$ = P.V. of point B – P.V. of point A

$= \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k} - \left(-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$

$= \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k} + \hat{i} - \frac{1}{2}\hat{j} - 4\hat{k} \quad = 2\hat{i} + 0\hat{j} + 0\hat{k}$

$\therefore \quad AB = |\vec{AB}| = \sqrt{4+0+0} = \sqrt{4} = 2$

and $\vec{AD}$ = P.V. of point D – P.V. of point A

$= -\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k} - \left(-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$

$= -\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k} + \hat{i} - \frac{1}{2}\hat{j} - 4\hat{k} = -\hat{j} = 0\hat{i} - \hat{j} + 0\hat{k}$

$\therefore \quad AD = |\vec{AD}| = \sqrt{0+1+0} = \sqrt{1} = 1$

$\therefore$ Area of rectangle ABCD = (AB)(AD) (= Length × Breadth)

= 2(1) = 2 sq. units

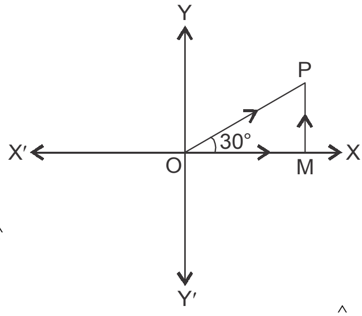
$\therefore$ Option (C) is the correct answer.

or Area of rectangle ABCD = $\left|\vec{AB} \times \vec{AD}\right|$.

MISCELLANEOUS EXERCISE

1. Write down a unit vector in XY-plane making an angle of 30° with the positive direction of x-axis.

Sol. Let $\overrightarrow{OP}$ be the **unit** vector in XY-plane such that $\angle XOP = 30°$



Therefore, $|\overrightarrow{OP}| = 1$

i.e., $OP = 1$ ...(*i*)

By Triangle Law of Addition of vectors,

In ΔOMP, $\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$

$= (OM)\hat{i} + (MP)\hat{j}$

$\left[\because \hat{a} = \frac{\vec{a}}{|\vec{a}|} \Rightarrow \vec{a} = |\vec{a}|\,\hat{a}\right.$ and unit vector along OX is $\hat{i}$

and along OY is $\hat{j}$]

$\Rightarrow \overrightarrow{OP} = OP\,\frac{OM}{OP}\,\hat{i} + OP\,\frac{MP}{OP}\,\hat{j}$

(Dividing and multiplying by OP in R.H.S.)

$= (1)(\cos 30°)\,\hat{i} + (1)(\sin 30°)\,\hat{j}$ [$\because$ By (*i*), OP = 1]

$\Rightarrow$ unit vector $\overrightarrow{OP} = (\cos 30)\,i + (\sin 30°)\,j$ ... (*ii*)

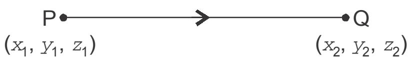
$\Rightarrow \overrightarrow{OP} = \frac{\sqrt{3}}{2}\,\hat{i} + \frac{1}{2}\,\hat{j}$.

Remark: From Eqn. (*ii*) of above solution, we can generalise the following result.

A unit vector along a line making an angle θ with positive x-axis is (cos θ) $\hat{i}$ + (sin θ) $\hat{j}$

2. Find the scalar components and magnitude of the vector joining the points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2).

Sol. Given points are $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.



$\Rightarrow$ P.V. (Position vector) of point P is

$(x_1, y_1, z_1) = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

and P.V. of point Q is $(x_2, y_2, z_2) = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\therefore$ Vector $\overrightarrow{PQ}$, the vector joining the points P and Q.

$= \text{P.V. of terminal point Q} - \text{P.V. of initial point P}$

$= x_2\hat{i} + y_2\hat{j} + z_2\hat{k} - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$

$= x_2\hat{i} + y_2\hat{j} + z_2\hat{k} - x_1\hat{i} - y_1\hat{j} - z_1\hat{k}$

$\Rightarrow \overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$\therefore$ Scalar components of the vector $\overrightarrow{PQ}$ are the coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\overrightarrow{PQ}$ *i.e.,* $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$

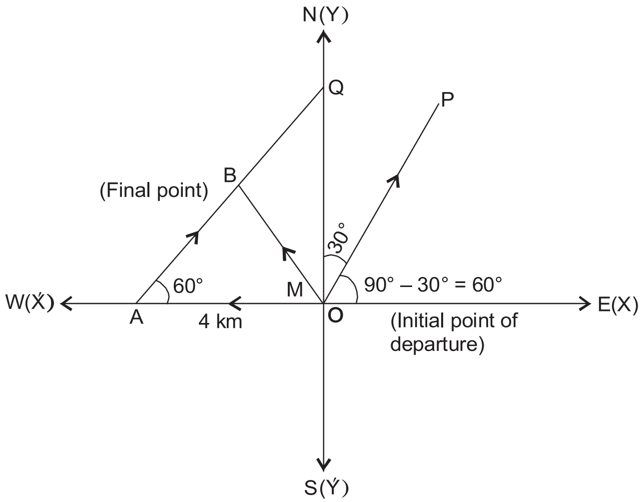
and magnitude of vector $\overrightarrow{PQ}$

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. | $\sqrt{x^2 + y^2 + z^2}$

3. A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Sol. Let us take the initial point of departure as origin.

Let the girl walk a distance OA = 4 km towards west.



Through the point A draw a line AQ parallel to a line OP (which is 30° east of North *i.e.,* in East-North quadrant making an angle of 30° with North)

Let the girl walk a distance AB = 3 km (given) along this direction $\overrightarrow{OQ}$ (given). $\therefore \overrightarrow{OA} = 4(-\hat{i})$[$\because$Vector $\overrightarrow{OA}$ is along

OX′)]

$$= -4\hat{i} \quad ...(i)$$

We know that (By Remark Q.N. 1 of this miscellaneous exercise) a unit vector along $\overrightarrow{AQ}$ (or $\overrightarrow{AB}$) making an angle $\theta = 60°$ with positive x-axis is $(\cos\theta)\hat{i} + (\sin\theta)\hat{j} = (\cos 60°)\hat{i} + (\sin 60°)\hat{j}$

$$= \frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}.$$

$\therefore \quad \overrightarrow{AB} = |\overrightarrow{AB}|$ (A unit vector along $\overrightarrow{AB}$) $\left| \because \vec{a} = |\vec{a}|\hat{a} \right.$

$$= 3\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) = \frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j} \quad ...(ii)$$

$\therefore$ Girl's displacement from her initial point O of departure (to final point B) $= \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ (By Triangle Law of Addition of vectors)

$$= -4\hat{i} + \left(\frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}\right) = \left(-4 + \frac{3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

[By (i)] [By (ii)]

$$= \frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}.$$

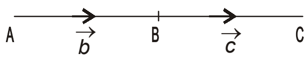
4. If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.

Sol. The result is not true (always).

Given: $\vec{a} = \vec{b} + \vec{c}$.

$\therefore$ Either the vectors $\vec{a}, \vec{b}, \vec{c}$ are collinear or vectors $\vec{a}, \vec{b}, \vec{c}$ form the sides of a triangle.

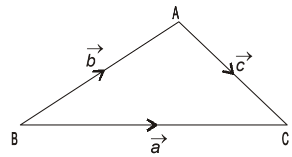
Case I. Vectors $\vec{a}, \vec{b}, \vec{c}$ are collinear.



Let $\vec{a} = \overrightarrow{AC}$, $\vec{b} = \overrightarrow{AB}$ and $\vec{c} = \overrightarrow{BC}$,

then $\vec{a} = \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{b} + \vec{c}$.

Also, $|\vec{a}| = AC = AB + BC = |\vec{b}| + |\vec{c}|$.



Case II. Vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ form a triangle.

Here also by Triangle Law of vectors,

$$\vec{a} = \vec{b} + \vec{c}$$

But $|\vec{a}| < |\vec{b}| + |\vec{c}|$

($\because$ Each side of a triangle is less than sum of the other two sides)

$\therefore$ $|(\vec{a}) = \vec{b} + \vec{c}| = |\vec{b}| + |\vec{c}|$ is true only when vectors $\vec{b}$ and $\vec{c}$ are collinear vectors.

5. Find the value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

Sol. Because $x(\hat{i} + \hat{j} + \hat{k}) = x\hat{i} + x\hat{j} + x\hat{k}$ is a unit vector (given)

Therefore, $|x\hat{i} + x\hat{j} + x\hat{k}| = 1$

$\therefore$ $\sqrt{x^2 + x^2 + x^2} = 1$

[$\because$ $x\hat{i} + y\hat{j} + z\hat{k} = \sqrt{x^2 + y^2 + z^2}$]

Squaring both sides $3x^2 = 1$ or $x^2 = \frac{1}{3}$ $\therefore$ $x = \pm \frac{1}{\sqrt{3}}$.

6. Find a vector of magnitude 5 units and parallel to the resultant of the vectors

$$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k} \text{ and } \vec{b} = \hat{i} - 2\hat{j} + \hat{k}.$$

Sol. Given: Vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Let vector $\vec{c}$ be the resultant of vectors $\vec{a}$ and $\vec{b}$.

$\therefore$ $\vec{c} = \vec{a} + \vec{b} = 2\hat{i} + 3\hat{j} - \hat{k} + \hat{i} - 2\hat{j} + \hat{k}$.

$$= 3\hat{i} + \hat{j} + 0\hat{k}.$$

$\therefore$ Required vector of magnitude 5 units and parallel (or collinear) to resultant vector $\vec{c} = \vec{a} + \vec{b}$ is

$$5\hat{c} = 5\frac{\vec{c}}{|\vec{c}|} = 5\left(\frac{3\hat{i} + \hat{j} + 0\hat{k}}{\sqrt{9+1+0}}\right)$$

$$= \frac{5}{\sqrt{10}}(3\hat{i} + \hat{j}) = \frac{5}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}}(3\hat{i} + \hat{j})$$

$$= \frac{5}{10}\sqrt{10}(3\hat{i} + \hat{j}) = \frac{\sqrt{10}}{2}(3\hat{i} + \hat{j}) = \frac{3}{2}\sqrt{10}\,\hat{i} + \frac{\sqrt{10}}{2}\hat{j}.$$

7. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.

Sol. Given: Vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$

and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$.

Let $\vec{d} = 2\vec{a} - \vec{b} + 3\vec{c}$

$= 2(\hat{i} + \hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k}) + 3(\hat{i} - 2\hat{j} + \hat{k})$

$= 2\hat{i} + 2\hat{j} + 2\hat{k} - 2\hat{i} + \hat{j} - 3\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k}$

$\therefore \quad \hat{d} = 3\hat{i} - 3\hat{j} + 2\hat{k}$ $\therefore$ A unit vector parallel to the vector

$\vec{d} = 3\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$\hat{d} = \frac{\vec{d}}{|\vec{d}|} = \frac{3\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{9+9+4} = \sqrt{22}} = \frac{3}{\sqrt{22}}\hat{i} - \frac{3}{\sqrt{22}}\hat{j} + \frac{2}{\sqrt{22}}\hat{k}.$$

8. Show that the points A(1, – 2, – 8), B(5, 0, – 2) and C(11, 3, 7) are collinear and find the ratio in which B divides AC.

Sol. Given: Points A(1, – 2, – 8), B(5, 0, – 2) and C(11, 3, 7).

i.e., Position vectors of points A, B, C are

$\vec{OA}$ (= A(1, – 2, – 8)) $= \hat{i} - 2\hat{j} - 8\hat{k}$

$\vec{OB}$ (= B(5, 0, – 2)) $= 5\hat{i} + 0\hat{j} - 2\hat{k} = 5\hat{i} - 2\hat{k}$

and $\vec{OC}$ (= C(11, 3, 7)) $= 11\hat{i} + 3\hat{j} + 7\hat{k}$

$\therefore \quad \vec{AB}$ = P.V. of point B – P.V. of point A

$= 5\hat{i} - 2\hat{k} - (\hat{i} - 2\hat{j} - 8\hat{k}) = 5\hat{i} - 2\hat{k} - \hat{i} + 2\hat{j} + 8\hat{k}$

or $\vec{AB} = 4\hat{i} + 2\hat{j} + 6\hat{k}$

$\therefore \quad AB = |\vec{AB}| = \sqrt{16+4+36} = \sqrt{56} = \sqrt{4 \times 14} = 2\sqrt{14}$

and $\vec{BC}$ = P.V. of point C – P.V. of point B

$= 11\hat{i} + 3\hat{j} + 7\hat{k} - (5\hat{i} - 2\hat{k}) = 11\hat{i} + 3\hat{j} + 7\hat{k} - 5\hat{i} + 2\hat{k}$

$= 6\hat{i} + 3\hat{j} + 9\hat{k}$

$\therefore$ BC = $|\overrightarrow{BC}|$ = $\sqrt{36+9+81}$ = $\sqrt{126}$ = $\sqrt{9 \times 14}$ = $3\sqrt{14}$

$\overrightarrow{AC}$ = P.V. of point C – P.V. of point A

$= 11\hat{i} + 3\hat{j} + 7\hat{k} - (\hat{i} - 2\hat{j} - 8\hat{k})$

$= 11\hat{i} + 3\hat{j} + 7\hat{k} - \hat{i} + 2\hat{j} + 8\hat{k} = 10\hat{i} + 5\hat{j} + 15\hat{k}$

$\therefore$ AC = $|\overrightarrow{AC}|$ = $\sqrt{100+225+225}$ = $\sqrt{350}$ = $\sqrt{25 \times 14}$ = $5\sqrt{14}$

Now, $\overrightarrow{AB} + \overrightarrow{BC} = 4\hat{i} + 2\hat{j} + 6\hat{k} + 6\hat{i} + 3\hat{j} + 9\hat{k}$

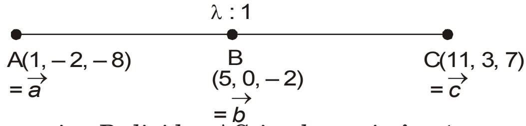
$= 10\hat{i} + 5\hat{j} + 15\hat{k} = \overrightarrow{AC}$

$\therefore$ Points A, B, C are either collinear or are the vertices of ΔABC.

Again AB + BC = $2\sqrt{14} + 3\sqrt{14} = (2+3)\sqrt{14} = 5\sqrt{14}$ = AC

$\therefore$ Points A, B, C are collinear.

Now to find the ratio in which B divides AC



Let the point B divides AC in the ratio λ : 1.

$\therefore$ By section formula, P.V. of point B is $\dfrac{\lambda\vec{c} + 1\vec{a}}{\lambda + 1}$

$\Rightarrow$ $(5, 0, -2) = \dfrac{\lambda(11, 3, 7) + (1, -2, -8)}{\lambda + 1}$

Cross-multiplying,

$(\lambda + 1)(5\hat{i} + 0\hat{j} - 2\hat{k}) = \lambda(11\hat{i} + 3\hat{j} + 7\hat{k}) + (\hat{i} - 2\hat{j} - 8\hat{k})$

$\Rightarrow 5(\lambda + 1)\hat{i} - 2(\lambda + 1)\hat{k} = 11\lambda\hat{i} + 3\lambda\hat{j} + 7\lambda\hat{k} + \hat{i} - 2\hat{j} - 8\hat{k}$

$\Rightarrow (5\lambda + 5)\hat{i} - (2\lambda + 2)\hat{k} = (11\lambda + 1)\hat{i} + (3\lambda - 2)\hat{j} + (7\lambda - 8)\hat{k}$

Comparing coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ on both sides, we have

$5\lambda + 5 = 11\lambda + 1,\ 0 = 3\lambda - 2,\ -(2\lambda + 2) = 7\lambda - 8$

$\Rightarrow -6\lambda = -4,\ -3\lambda = -2,\ -2\lambda - 2 = 7\lambda - 8\ (\Rightarrow\ -9\lambda = -6)$

$\Rightarrow \lambda = \dfrac{4}{6} = \dfrac{2}{3},\ \lambda = \dfrac{2}{3},\ \lambda = \dfrac{6}{9} = \dfrac{2}{3}$

All three values of λ are same.

$\therefore$ Required ratio is $\lambda : 1 = \dfrac{2}{3} : 1 = 2 : 3$.

9. Find the position vector of a point R which divides the line joining the two points P and Q whose position vectors are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ externally in the ratio 1 : 2. Also, show that P is the middle point of line segment RQ.

Sol. We know that position vector of the point R dividing the join of P and Q externally in the ratio 1 : 2 = $m : n$ is given by

$$\vec{c} = \frac{m\vec{b} - n\vec{a}}{m - n} = \frac{1(\vec{a} - 3\vec{b}) - 2(2\vec{a} + \vec{b})}{1 - 2}$$

$$= \frac{\vec{a} - 3\vec{b} - 4\vec{a} - 2\vec{b}}{1 - 2} = \frac{-3\vec{a} - 5\vec{b}}{-1} = 3\vec{a} + 5\vec{b}$$

Again position vector of the middle point of the line segment RQ

$$= \frac{\text{P.V. of point R} + \text{P.V. of point Q}}{2} = \frac{3\vec{a} + 5\vec{b} + \vec{a} - 3\vec{b}}{2} = \frac{4\vec{a} + 2\vec{b}}{2}$$

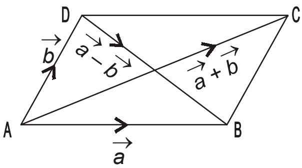
$= 2\vec{a} + \vec{b}$ = P.V. of point P. (given)

$\therefore$ Point P is the middle point of the line segment RQ.

10. Two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

Sol. Let ABCD be a parallelogram.

Given: The vectors representing two adjacent sides of this parallelogram are say



$$\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$$

and $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$

Formula: $\therefore$ Vectors along the diagonals $\overrightarrow{AC}$ and $\overrightarrow{DB}$ of the parallelogram are

$$\vec{a} + \vec{b} \text{ and } \vec{a} - \vec{b}$$

i.e., $\vec{a} + \vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k} + \hat{i} - 2\hat{j} - 3\hat{k}$

$$= 3\hat{i} - 6\hat{j} + 2\hat{k}$$

and $\vec{a} - \vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k} - (\hat{i} - 2\hat{j} - 3\hat{k})$

$= 2\hat{i} - 4\hat{j} + 5\hat{k} - \hat{i} + 2\hat{j} + 3\hat{k} = \hat{i} - 2\hat{j} + 8\hat{k}$

$\therefore$ Unit vectors parallel to (or along) diagonals are

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} \text{ and } \frac{\vec{a}-\vec{b}}{|\vec{a}-\vec{b}|} = \frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{9+36+4}=\sqrt{49}=7} \text{ and } \frac{\hat{i}-2\hat{j}+8\hat{k}}{\sqrt{1+4+64}=\sqrt{69}}$$

Let us find area of parallelogram

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix} = \hat{i}(12+10) - \hat{j}(-6-5) + \hat{k}(-4+4)$$

$$= 22\hat{i} + 11\hat{j} + 0\hat{k}$$

We know that area of parallelogram $= |\vec{a} \times \vec{b}|$

$$= \sqrt{(22)^2 + (11)^2 + 0^2} = \sqrt{484+121} = \sqrt{605}$$

$$= \sqrt{5 \times 121} = \sqrt{121 \times 5} = 11\sqrt{5} \text{ sq. units.}$$

11. Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Sol. Let l, m, n be the direction cosines of a vector equally inclined to the axes OX, OY, OZ.

$\therefore$ A unit vector along the given vector is

$\hat{a} = l\hat{i} + m\hat{j} + n\hat{k}$ and $|\hat{a}| = 1$

$\Rightarrow \sqrt{l^2+m^2+n^2} = 1 \quad \therefore \quad l^2 + m^2 + n^2 = 1$ ...(*i*)

Let the given vector (for which unit vector is $\hat{a}$) make equal angles (given) θ, θ, θ (say) with **OX** ($\Rightarrow \hat{i}$), **OY** ($\Rightarrow \hat{j}$) and **OZ** ($\Rightarrow \hat{k}$)

$\therefore$ The given vector is in positive octant OXYZ and hence θ is acute. ...(*ii*)

$\therefore$ For angle θ between $\hat{a}$ and $\hat{i}$,

$$\cos\theta = \frac{\hat{a}\cdot\hat{i}}{|\hat{a}||\hat{i}|} = \frac{(l\hat{i}+m\hat{j}+n\hat{k})\cdot(\hat{i}+0\hat{j}+0\hat{k})}{(1)(1)}$$

or $\cos\theta = l(1) + m(0) + n(0) = l$

or $l = \cos\theta$ ...(*iii*)

Similarly, for angle θ between $\hat{a}$ and $\hat{j}$, $m = \cos\theta$ ...(*iv*)

Similarly, for angle θ between $\hat{a}$ and $\hat{k}$, $n = \cos\theta$ ...(*v*)

Putting these values of l, m, n from (*iii*), (*iv*) and (*v*) in (*i*), we have

$$\cos^2\theta + \cos^2\theta + \cos^2\theta = 1 \quad \Rightarrow \quad 3\cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = \frac{1}{3} \quad \Rightarrow \quad \cos\theta = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}}$$

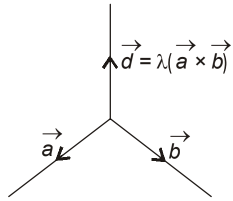
$\therefore \cos\theta = \frac{1}{\sqrt{3}}$ ($\because$ By (*ii*), θ is acute and hence cos θ is positive)

Putting $\cos\theta = \frac{1}{\sqrt{3}}$ in (*ii*), (*iii*) and (*iv*), direction cosines of the required vector are $l, m, n = \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

12. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c}\cdot\vec{d} = 15$.

Sol. Given: Vectors are $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$

and $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$



By definition of cross-product of two vectors, $\vec{a} \times \vec{b}$ is *a* vector perpendicular to both $\vec{a}$ and $\vec{b}$.

Hence, vector $\vec{d}$ which is also perpendicular to both $\vec{a}$ and $\vec{b}$ is $\vec{d} = \lambda(\vec{a} \times \vec{b})$ where $\lambda = 1$ or some other scalar.

Therefore, $\vec{d} = \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$

Expanding along first row, $= \lambda[\hat{i}(28 + 4) - \hat{j}(7 - 6) + \hat{k}(-2 - 12)]$

or $\vec{d} = \lambda[32\hat{i} - \hat{j} - 14\hat{k}]$ ...(*i*)

or $\vec{d} = 32\lambda\hat{i} - \lambda\hat{j} - 14\lambda\hat{k}$

To find λ: Given: $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$

Also given $\vec{c}\cdot\vec{d} = 15$

$\Rightarrow\ 2(32\lambda) + (-1)(-\lambda) + 4(-14\lambda) = 15$

$\Rightarrow\ 64\lambda + \lambda - 56\lambda = 15 \quad \Rightarrow\ 9\lambda = 15 \quad \Rightarrow\ \lambda = \frac{5}{3}$

Putting $\lambda = \frac{5}{3}$ in (*i*), required vector

$$\vec{d} = \frac{5}{3}(32\hat{i} - \hat{j} - 14\hat{k}) = \frac{1}{3}(160\hat{i} - 5\hat{j} - 70\hat{k}).$$

13. The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ.

Sol. Given: Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ ...(*i*)

$\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$

$\therefore$ $\vec{b} + \vec{c}$ (= $\vec{d}$ (say)) = $(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}$

$\therefore$ $\hat{d}$, a unit vector along $\vec{b} + \vec{c} = \vec{d}$ is

$$\hat{d} = \frac{\vec{d}}{|\vec{d}|} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 36 + 4} = \sqrt{4 + \lambda^2 + 4\lambda + 40}}$$

or $$\hat{d} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}$$

$$= \frac{(2+\lambda)}{\sqrt{\lambda^2 + 4\lambda + 44}}\hat{i} + \frac{6}{\sqrt{\lambda^2 + 4\lambda + 44}}\hat{j} - \frac{2}{\sqrt{\lambda^2 + 4\lambda + 44}}\hat{k}$$ ...(*ii*)

Given: Scalar (*i.e.*, Dot) Product of $\vec{a}$ and $\hat{d}$ *i.e.*, = $\vec{a} \cdot \hat{d} = 1$

$\therefore$ From (*i*) and (*ii*),

$$\frac{1(2+\lambda)}{\sqrt{\lambda^2 + 4\lambda + 44}} + \frac{1(6)}{\sqrt{\lambda^2 + 4\lambda + 44}} + \frac{1(-2)}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$

Multiplying by L.C.M. = $\sqrt{\lambda^2 + 4\lambda + 44}$,

$2 + \lambda + 6 - 2 = \sqrt{\lambda^2 + 4\lambda + 44}$ $\Rightarrow$ $\lambda + 6 = \sqrt{\lambda^2 + 4\lambda + 44}$

Squaring both sides $(\lambda + 6)^2 = \lambda^2 + 4\lambda + 44$

$\Rightarrow$ $\lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 44$

$\Rightarrow$ $8\lambda = 8$ $\Rightarrow$ $\lambda = 1$.

14. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitude, show that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}, \vec{b}, \vec{c}$.

Sol. Given: $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitude.

$\therefore$ $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = 0$, $\vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{b} = 0$,

$\vec{a} \cdot \vec{a} = \vec{a} \cdot \vec{c} = 0$ ... (*i*) and $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ (say) ... (*ii*)

Let vector $\vec{d} = \vec{a} + \vec{b} + \vec{c}$ make angles θ_1, θ_2, θ_3 with vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively.

$$\therefore \quad \cos\theta_1 = \frac{\vec{d}\cdot\vec{a}}{|\vec{d}+\vec{a}|} = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$= \frac{\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{|\vec{a}|^2+0+0}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} \qquad \text{[By } (i)\text{]}$$

$$\Rightarrow \quad \cos\theta_1 = \frac{|\vec{a}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} \qquad \dots (iii)$$

Let us now find $|\vec{a} + \vec{b} + \vec{c}|$.

We know that $|\vec{a} + \vec{b} + \vec{c}|^2 = (\vec{a} + \vec{b} + \vec{c})^2$

$= \vec{a}^{\,2} + (\vec{b} + \vec{c})^2 + 2\vec{a}\cdot(\vec{b} + \vec{c})$

$[\because (\vec{A} + \vec{B})^2 = \vec{A}^{\,2} + \vec{B}^{\,2} + 2\vec{A}\cdot\vec{B}]$

$= \vec{a}^{\,2} + \vec{b}^{\,2} + \vec{c}^{\,2} + 2\vec{b}\cdot\vec{c} + 2\vec{a}\cdot\vec{b} + 2\vec{a}\cdot\vec{c}$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b}\cdot\vec{c} + 2\vec{a}\cdot\vec{b} + 2\vec{a}\cdot\vec{c}$

Putting values from (*i*) and (*ii*)

$$|\vec{a} + \vec{b} + \vec{c}|^2 = \lambda^2 + \lambda^2 + \lambda^2 + 0 + 0 + 0 = 3\lambda^2$$

$$\therefore \quad |\vec{a} + \vec{b} + \vec{c}| = \sqrt{3\lambda^2} = \lambda\sqrt{3}$$

Putting this value of $|\vec{a} + \vec{b} + \vec{c}| = \lambda\sqrt{3}$ and $|\vec{a}| = \lambda$ from (*ii*) in (*iii*), $\cos\theta_1 = \frac{\lambda}{\lambda\sqrt{3}} = \frac{1}{\sqrt{3}}$ $\quad\therefore\quad \theta_1 = \cos^{-1}\frac{1}{\sqrt{3}}$

Similarly, $\theta_2 = \cos^{-1}\frac{1}{\sqrt{3}}$ and $\theta_3 = \cos^{-1}\frac{1}{\sqrt{3}}$

$$\therefore \quad \theta_1 = \theta_2 = \theta_3 \left(= \cos^{-1}\frac{1}{\sqrt{3}}\right)$$

$\therefore$ Vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

15. Prove that $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$, if and only if $\vec{a}, \vec{b}$ are perpendicular, given $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$.

Sol. We know that $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$

$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{b} + |\vec{b}|^2$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} \qquad \text{...}(i)$$

For If part: Given: $\vec{a}$ and $\vec{b}$ are perpendicular

$\Rightarrow \quad \vec{a}\cdot\vec{b} = 0$

Putting $\vec{a}\cdot\vec{b} = 0$ in (*i*), we have

$$(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$$

For Only if part:

Given: $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$

Putting this value in L.H.S. eqn. (*i*), we have

$$|\vec{a}|^2 + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$$

$\Rightarrow \quad 0 = 2\vec{a}\cdot\vec{b} \qquad \Rightarrow \quad \vec{a}\cdot\vec{b} = \frac{0}{2} = 0$

But $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$ (given).

$\therefore$ Vector $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

16. Choose the correct answer:

If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\cdot\vec{b} \geq 0$ only when

(A) $0 < \theta < \frac{\pi}{2}$ (B) $0 \leq \theta \leq \frac{\pi}{2}$ (C) $0 < \theta < \pi$ (D) $0 < \theta \leq \pi$

Sol. Given: $\vec{a}\cdot\vec{b} \geq 0$

$\Rightarrow \quad |\vec{a}|\,|\vec{b}| \cos\theta \geq 0 \qquad \Rightarrow \quad \cos\theta \geq 0$

[$\because$ $|\vec{a}|$ and $|\vec{b}|$ being lengths of vectors are always ≥ 0]

and this is true only for option (B) out of the given options $\left(\because \text{ For option (A)} \quad 0 < \theta < \frac{\pi}{2}, \cos\theta > 0\right)$.

17. Choose the correct answer:

Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a} + \vec{b}$ is a unit vector if

(A) $\theta = \frac{\pi}{4}$ (B) $\theta = \frac{\pi}{3}$ (C) $\theta = \frac{\pi}{2}$ (D) $\theta = \frac{2\pi}{3}$.

Sol. Given: $\vec{a}$, $\vec{b}$ and $\vec{a} + \vec{b}$ are unit vectors

$\Rightarrow \quad |\vec{a}| = 1, |\vec{b}| = 1$ and $|\vec{a} + \vec{b}| = 1$

Now, squaring both sides of $|\vec{a} + \vec{b}| = 1$, we have

$$|\vec{a} + \vec{b}|^2 = 1 \qquad \Rightarrow \quad (\vec{a} + \vec{b})^2 = 1$$

$\Rightarrow \qquad \vec{a}^{\,2} + \vec{b}^{\,2} + 2\vec{a}\cdot\vec{b} = 1$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}|\,|\vec{b}|\cos\theta = 1$

where θ is the given angle between vectors $\vec{a}$ and $\vec{b}$.

Putting $|\vec{a}| = 1$ and $|\vec{b}| = 1$, we have $1 + 1 + 2\cos\theta = 1$

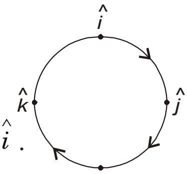
$\Rightarrow \quad 2\cos\theta = -1 \qquad \Rightarrow \quad \cos\theta = \dfrac{-1}{2} = -\cos 60°$

$\Rightarrow \quad \cos\theta = \cos(180° - 60°) \qquad \Rightarrow \quad \cos\theta = \cos 120°$

$\Rightarrow \qquad \theta = 120° = 120 \times \dfrac{\pi}{180} = \dfrac{2\pi}{3}$

$\therefore$ Option (D) is the correct answer.

Very Important Results



(1) $\hat{i}\cdot\hat{i} = |\hat{i}|^2 = 1,\ \hat{j}\cdot\hat{j} = 1,\ \hat{k}\cdot\hat{k} = 1.$

(2) $\hat{i}\times\hat{i} = \vec{0}$, $\hat{j}\times\hat{j} = \vec{0}$ and $\hat{k}\times\hat{k} = \vec{0}$.

(3) $\hat{i}\cdot\hat{j} = 0 = \hat{j}\cdot\hat{i},\ \hat{j}\cdot\hat{k} = 0 = \hat{k}\cdot\hat{j},\ \hat{i}\cdot\hat{k} = 0 = \hat{k}\cdot\hat{i}$.

(4) $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i}$ and $\hat{k}\times\hat{i} = \hat{j}$.

18. Choose the correct answer:

The value of $\hat{i}\cdot(\hat{j}\times\hat{k}) + \hat{j}\cdot(\hat{i}\times\hat{k}) + \hat{k}\cdot(\hat{i}\times\hat{j})$ is
(A) 0 (B) – 1 (C) 1 (D) 3

Sol. $\hat{i}\cdot(\hat{j}\times\hat{k}) + \hat{j}\cdot(\hat{i}\times\hat{k}) + \hat{k}\cdot(\hat{i}\times\hat{j})$

$= \hat{i}\cdot\hat{i} + \hat{j}\cdot(-\hat{j}) + \hat{k}\cdot\hat{k}$

$(\because \hat{i}\times\hat{k} = -\hat{k}\times\hat{i} = -\hat{j})$

$= 1 - 1 + 1 = 1$

$\therefore$ Option (C) is the correct answer.

19. If θ be the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$, when θ is equal to

(A) 0 (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{2}$ (D) π

Sol. Given: $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$

$\Rightarrow \quad |\vec{a}|\,|\vec{b}|\,|\cos\theta| = |\vec{a}|\,|\vec{b}|\sin\theta$

$(\because \quad \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow |\vec{a}\cdot\vec{b}| = |\vec{a}||\vec{b}|\,|\cos\theta|)$

Dividing both sides by $|\vec{a}||\vec{b}|$, we have $|\cos\theta| = \sin\theta$

and this equation is true only for option (B) namely $\theta = \dfrac{\pi}{4}$ out of the given options.

$$\left[\because \quad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and also } \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

$\therefore$ Option (B) is the correct option.