Chapter : 16. CONDITIONAL IDENTITIES INVOLVING THE ANGLES OF A TRIANGLE

Exercise : 16

Question: 1

If A + B + C = π,

Solution:

= sin 2A + sin 2B – sin 2C

= 2 sin (B + C) cos A + 2 sin (A + C) cos B - 2 sin (A + B) cos Cusing formula,sin (A + B) = sin A cos B + cos A sin B= sin 2A + sin 2B - sin 2C

Using formula

sin2A = 2sinAcosA= 2sinAcosA + 2sinBcosB - 2sinCcosC

since A + B + C = π

$\rightarrow B + C = 180 - A$

And sin(π – A) = sinA= 2sin(B + C)cos A + 2sin(A + C)cosB - 2sin(A + B)cosC= 2 (sin B cos C + cos B sin C) cos A + 2(sinAcosC + cosAsinC)cosB - 2(sinAcosB + cosAsinB)cosC= 2cosAsinBcosC + 2cosAcosBsinC + 2sinAcosBcosC + 2cosAcosBsinC– 2sinAcosBcosC – 2cosAsinBcosC= 2cosAcosBsinC + 2cosAcosBsinC= 4cosAcosBsinC

= R.H.S

Question: 2

If A + B + C = π,

Solution:

= cos2A – (cos2B + cos2C)

Using formula

$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= \cos 2A - \left\{2\cos\left(\frac{2B+2C}{2}\right)\cos\left(\frac{2B-2C}{`2}\right)\right\}$

= cos2A - {2cos(B+C)cos(B-C)}

since A + B + C = π

$\rightarrow B + C = 180 - A$

= cos2A – {2cos(π – A)cos(B-C)}

And cos(π – A) = -cosA

= cos2A – {-2cosAcos(B-C)}

= cos2A + 2cosAcos(B-C)

Using cos2A = 2cos^{2}A -1

= 2cos^{2}A – 1 + 2cosAcos(B-C)

= 2cosA{cosA + cos(B-C)} – 1

Using , $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= 2\cos A\left\{2\cos\left(\frac{A+B-C}{2}\right)\cos\left(\frac{A+C-B}{`2}\right)\right\} - 1$

$= 2\cos A\left\{2\cos\left(\frac{\pi-C-C}{2}\right)\cos\left(\frac{\pi-B-B}{`2}\right)\right\}-1$

As, $\cos\left(\frac{\pi}{2}-A\right)=\sin A$

$= 2\cos A\left\{2\cos\left(\frac{\pi}{2}-\frac{2C}{2}\right)\cos\left(\frac{\pi}{`2}-\frac{2B}{2}\right)\right\}-1$

= 2cosA{2sinCsinB} – 1

= 4cosAsinBsinC – 1

= R.H.S

Question: 3

If A + B + C = π,

Solution:

= cos2A – cos2B + cos2C

Using,

$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$

$= \cos 2A - \left\{2\sin\left(\frac{2B+2C}{2}\right)\sin\left(\frac{2B-2C}{`2}\right)\right\}$

= cos2A - {2sin(B+C)sin(B-C)}

since A + B + C = π

$\rightarrow B+C=180-A$

And sin(π – A) = sinA

= cos2A – {2sin(π – A)sin(B-C)}

= cos2A – {2sinAsin(B-C)}

= cos2A - 2sinAsin(B-C)

Using , cos2A = 1 – 2sin^{2}A

= -2sin^{2}A + 1 – 2sinAsin(B-C)

= -2sinA{sinA + sin(B-C)} + 1

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= -2\sin A\left\{2\sin\left(\frac{A+B-C}{2}\right)\cos\left(\frac{A+C-B}{`2}\right)\right\}+1$

$= -2\sin A\left\{2\sin\left(\frac{\pi-C-C}{2}\right)\cos\left(\frac{\pi-B-B}{`2}\right)\right\}+1$

$= -2\sin A\left\{2\sin\left(\frac{\pi}{2}-\frac{2C}{2}\right)\cos\left(\frac{\pi}{`2}-\frac{2B}{2}\right)\right\}+1$

As, $\sin\left(\frac{\pi}{2}-A\right)=\cos A$

= -2sinA{2cosCsinB} + 1

= -4sinAcosBsinC + 1

= R.H.S

Question: 4

If A + B + C = π,

Solution:

= sinA + sinB + sinC

Using,

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= \sin A + \left\{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{`2}\right)\right\}$

since A + B + C = π

$\rightarrow B + C = 180 - A$

And,

$\sin\left(\frac{\pi}{2} - A\right) = \cos A$

$= \sin A + \{2\sin\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

$= \sin A + \{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

Using , sin2A = 2sinAcosA

$= 2\sin\frac{A}{2}\cos\frac{A}{2} + \{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

$= 2\cos\frac{A}{2}\{\sin\frac{A}{2} + \cos\left(\frac{B-C}{2}\right)\}$

$\rightarrow B + C = 180 - A$

And,

$\sin\left(\frac{\pi}{2} - A\right) = \cos A$

$= 2\cos\frac{A}{2}\{\cos(\frac{B+C}{2}) + \cos\left(\frac{B-C}{2}\right)\}$

$= 2\cos\frac{A}{2}\{2\cos\left(\frac{B}{2}\right)\cos(\frac{C}{2})\}$

$= 4\cos\frac{A}{2}\cos\left(\frac{B}{2}\right)\cos(\frac{C}{2})$

= R.H.S

Question: 5

If A + B + C = π,

Solution:

= cosA + cosB + cosC

Using ,

$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos(\frac{A-B}{2})$

$= \cos A + \left\{2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{`2}\right)\right\}$

since A + B + C = π

$\rightarrow B + C = 180 - A$

And,

$\cos\left(\frac{\pi}{2} - A\right) = \sin A$

$= \cos A + \{2\cos\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

$= \cos A + \{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

Using , $\cos 2A = 1 - 2\sin^2 A$

$= 1 - 2\sin^2\frac{A}{2} + \{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

$= 2\sin\frac{A}{2}\left\{-\sin\frac{A}{2} + \cos\left(\frac{B-C}{2}\right)\right\} + 1$

$= 2\sin\frac{A}{2}\left\{\cos\left(\frac{-B-C}{2}\right) + \cos\left(\frac{B-C}{2}\right)\right\} + 1$

$= 2\sin\frac{A}{2}\left\{2\cos\left(\frac{-C}{2}\right)\cos\left(\frac{-B}{2}\right)\right\} + 1$

$= 4\sin\frac{A}{2}\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) + 1$

= R.H.S

Question: 6

If A + B + C = π,

Solution:

= sin2A + sin2B + sin2C

Using,

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

Sin2A = 2sinAcosA

= 2sinAcosA + 2sin(B+C)cos(B - C) since A + B + C = π

$\rightarrow B + C = 180 - A$

= 2sinAcosA + 2sin(π - A)cos(B - C) = 2sinAcosA + 2sinAcos(B - C)= 2sinA{cosA + cos (B-C)}(
but cos A = cos { 180 - (B + C) } = - cos (B + C)

And now using $\cos A - \cos B = 2\sin(\frac{A+B}{2})\sin(\frac{-A+B}{2})$ = 2sinA{2sinBsinC}= 4sinAsinBsinC

$= 32\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}$

Now,

= sinA + sinB + sinC

Using,

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= \sin A + \left\{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{`2}\right)\right\}$

$= \sin A + \{2\sin\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

$= \sin A + \{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

$= 2\sin\frac{A}{2}\cos\frac{A}{2} + \{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\}$

$= 2\cos\frac{A}{2}\{\sin\frac{A}{2} + \cos\left(\frac{B-C}{2}\right)\}$

$= 2\cos\frac{A}{2}\{\cos(\frac{B+C}{2}) + \cos\left(\frac{B-C}{2}\right)\}$

$= 2\cos\frac{A}{2}\{2\cos\left(\frac{B}{2}\right)\cos(\frac{C}{2})\}$

$= 4\cos\frac{A}{2}\cos\left(\frac{B}{2}\right)\cos(\frac{C}{2})$

Therefore,

$= \frac{32\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$

$= 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

= R.H.S

Question: 7

If A + B + C = π,

Solution:

= sin (B + C – A) + sin (C + A – B) – sin (A + B – C)

Using,

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

= 2sinC cos(B-A) – sin(A+B-C)

since A + B + C = π

$\rightarrow B + A = 180 - C$

= 2sinCcos(B-A) – sin(π – C – C)

= 2sinCcos(B-A) – sin2C

Since , sin2A = 2sinAcosA,

= 2sinCcos(B-A) – 2sinCcosC

= 2sinC{cos(B-A) – cosC}

Using ,

$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$

$= 2\sin C\{2\sin(\frac{B-A+C}{2})\sin(\frac{C-B+A}{2})\}$

$= 2\sin C\{2\sin(\frac{\pi-A-A}{2})\sin(\frac{\pi-B-B}{2})\}$

= 4cosAcosBsinC

= R.H.S

Question: 8

If A + B + C = π,

Solution:

$= \frac{\cos A}{\sin B\sin C} + \frac{\cos B}{\sin C\sin A} + \frac{\cos C}{\sin A\sin B}$

Taking L.C.M

$= \frac{\cos A\sin A + \cos B\sin B + \cos C\sin C}{\sin B\sin C\sin A}$

Multiplying and divide the above equation by 2, we get

$= \frac{2\cos A\sin A + 2\cos B\sin B + 2\cos C\sin C}{2\sin B\sin C\sin A}$

Since , sin2A = 2sinAcosA

$= \frac{\sin 2A + \sin 2B + \sin 2C}{2\sin B\sin C\sin A}$

NOW,

= sin2A + sin2B + sin2C

= 2sinAcosA + 2sin(B+C)cos(B - C) since A + B + C = π

$\rightarrow B + A = 180 - C$

= 2sinAcosA + 2sin(π - A)cos(B - C) = 2sinAcosA + 2sinAcos(B - C)= 2sinA{cosA + cos (B-C)}(but cos A = cos { 180 - (B + C) } = - cos (B + C)

And now using $\cos A - \cos B = 2\sin(\frac{A+B}{2})\sin(\frac{-A+B}{2})$ = 2sinA{2sinBsinC}= 4sinAsinBsinC

Putting the above value in the equation, we get

$= \frac{4\sin A\sin B\sin C}{2\sin B\sin C\sin A}$

= 2

= R.H.S

Question: 9

If A + B + C = π,

Solution:

= $\cos^2 A + \cos^2 B + \cos^2 C$

Using formula ,

$\frac{1+\cos 2A}{2} = \cos^2 A$

$= \frac{1+\cos 2A}{2} + \frac{1+\cos 2B}{2} + \frac{1+\cos 2C}{2}$

$= \frac{1 + \cos 2A + 1 + \cos 2B + 1 + \cos 2C}{2}$

$= \frac{3 + \cos 2A + \cos 2B + \cos 2C}{2}$

Using ,

$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= \frac{3 + \cos 2A + 2\cos(\frac{2B+2C}{2})\cos(\frac{2B-2C}{2})}{2}$

$= \frac{3 + \cos 2A + 2\cos(B+C)\cos(B-C)}{2}$

Using , since A + B + C = π

$\rightarrow B + C = 180 - A$

And, cos(π – A) = -cosA

$= \frac{3 + \cos 2A + 2\cos(\pi - A)\cos(B-C)}{2}$

$= \frac{3 + \cos 2A - 2\cos(A)\cos(B-C)}{2}$

Using $\cos 2A = 2\cos^2 A - 1$

$= \frac{3 + 2\cos^2 A - 1 - 2\cos(A)\cos(B-C)}{2}$

$= \frac{2 + 2\cos^2 A - 2\cos(A)\cos(B-C)}{2}$

$= 1 + \cos^2 A - \cos A\cos(B-C)$

$= 1 + \cos A\{\cos A - \cos(B-C)\}$

Using ,

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

$$= 1 + \cos A\left(2\sin\left(\frac{A+B-C}{2}\right)\sin\left(\frac{B-C-A}{2}\right)\right)$$

Since , A + B + C = π

$$= 1 + \cos A\left(2\sin\left(\frac{\pi-C-C}{2}\right)\sin\left(\frac{B-(\pi-B)}{2}\right)\right)$$

$$= 1 + \cos A\left(2\cos C\sin\left(\frac{B}{2} - \frac{\pi}{2}\right)\right)$$

$= 1 - 2\cos A\cos C\cos C$

= R.H.S

Question: 10

If A + B + C = π,

Solution:

$= \sin^2 A - \sin^2 B + \sin^2 C$

Using formula ,

$$\frac{1-\cos 2A}{2} = \sin^2 A$$

$$= \frac{1-\cos 2A}{2} - \frac{1-\cos 2B}{2} + \frac{1-\cos 2C}{2}$$

$$= \frac{1-\cos 2A - 1 + \cos 2B + 1 - \cos 2C}{2}$$

$$= \frac{1-\cos 2A + \cos 2B - \cos 2C}{2}$$

Using ,

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

$$= \frac{1-\cos 2A + \left\{2\sin\left(\frac{2B+2C}{2}\right)\sin\left(\frac{2C-2B}{2}\right)\right\}}{2}$$

$$= \frac{1-\cos 2A + 2\sin(B+C)\sin(C-B)}{2}$$

since A + B + C = π

$\rightarrow B + C = 180 - A$

And $\sin(\pi - A) = \sin A$

$$= \frac{1-\cos 2A + 2\sin(\pi-A)\sin(C-B)}{2}$$

$$= \frac{1-\cos 2A + 2\sin A\sin(C-B)}{2}$$

Using , $\cos 2A = 1 - 2\sin^2 A$

$$= \frac{1-1+2\sin^2 A + 2\sin A\sin(C-B)}{2}$$

$$= \frac{2\sin A\{\sin A + \sin(C-B)\}}{2}$$

$= \frac{2\sin A\{\sin A + \sin(C - B)\}}{2}$

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= \frac{2\sin A\{2\sin\left(\frac{A+C-B}{2}\right)\cos\left(\frac{A-C+B}{2}\right)\}}{2}$

$= \frac{1 - 2\sin A\{2\sin\left(\frac{\pi - B - B}{2}\right)\cos\left(\frac{\pi - C - C}{2}\right)\}}{2}$

$= \frac{2\sin A\{2\sin\left(\frac{\pi}{2} - \frac{2B}{2}\right)\cos\left(\frac{\pi}{2} - \frac{2C}{2}\right)\}}{2}$

As, $\sin\left(\frac{\pi}{2} - A\right) = \cos A$

$= \frac{2\sin A\{2\cos B\sin C\}}{2}$

$= 2\sin A\cos B\sin C$

= R.H.S

Question: 11

If A + B + C = π,

Solution:

$= \sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2}$

Using formula ,

$\frac{1 - \cos 2A}{2} = \sin^2 A$

$= \frac{1 - \cos A}{2} + \frac{1 - \cos B}{2} + \frac{1 - \cos C}{2}$

$= \frac{1 - \cos A + 1 - \cos B + 1 - \cos C}{2}$

$= \frac{3 - \cos A - \cos B - \cos C}{2}$

Using ,

$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= \frac{3 - \cos A - \{2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\}}{2}$

$= \frac{3 - \cos A - 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2}$

Using , since A + B + C = π

$\rightarrow B + C = 180 - A$

And, cos(π – A) = -cosA

$= \frac{3 - \cos A - 2\cos\left(\frac{\pi}{2} - \frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2}$

$= \frac{3 - \cos A - 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2}$

Using , cos2A = 1 – 2sin^{2}A

$= \frac{3 - 1 + 2\sin^2\frac{A}{2} - 2\sin\frac{A}{2}\cos\frac{B-C}{2}}{2}$

$= \frac{2-2\sin\frac{A}{2}\{\sin\frac{A}{2}-\cos\left(\frac{B-C}{2}\right)\}}{2}$

since A + B + C = π

and Using ,

$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$

$= \frac{2-2\sin\frac{A}{2}\{2\sin\left(\frac{\frac{B+C}{2}+\frac{B-C}{2}}{2}\right)\sin\left(\frac{\frac{B+C}{2}-\left(\frac{B-C}{2}\right)}{2}\right)\}}{2}$

$= \frac{2-2\sin\frac{A}{2}\{2\sin\left(\frac{\frac{2B}{2}}{2}\right)\sin\left(\frac{\frac{2C}{2}}{2}\right)\}}{2}$

Using , since A + B + C = π

$= \frac{2-2\sin\frac{A}{2}\{2\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\}}{2}$

$= 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

= R.H.S

Question: 12

If A + B + C = π,

Solution:

= tan 2A + tan 2B + tan 2C

Since A + B + C = π

A + B = π – C

2A + 2B = 2π – 2C

Tan (2A+2B) = tan (2π – 2C)

Since tan (2π – C) = -tan C

Tan (2A + 2B) = -tan 2C

Now using formula,

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\frac{\tan 2A + \tan 2B}{1 - \tan 2A \tan 2B} = -\tan 2C$

Tan 2A + tan 2B = -tan 2C + tan 2C tan 2B tan 2A

Tan 2A + tan 2B + tan 2C = tan 2A tan 2B tan 2C

= R.H.S