15. Mean Value Theorems

Exercise 15.1

1 A. Question

Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals :

f(x) = 3 + (x – 2)$^{2/3}$ on [1, 3]

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

(i) Given function is:

$\Rightarrow f(x) = 3 + (x-2)^{\frac{2}{3}}$ on [1,3]

Let us check the differentiability of the function f(x).

Let's find the derivative of f(x),

$$\Rightarrow f'(x) = \frac{d}{dx}\left(3 + (x-2)^{\frac{2}{3}}\right)$$

$$\Rightarrow f'(x) = \frac{d(3)}{dx} + \frac{d\left((x-2)^{\frac{2}{3}}\right)}{dx}$$

$$\Rightarrow f'(x) = 0 + \frac{2}{3}(x-2)^{\frac{2}{3}-1}$$

$$\Rightarrow f'(x) = \frac{2}{3}(x-2)^{-\frac{1}{3}}$$

$$\Rightarrow f'(x) = \frac{2}{3(x-2)^{\frac{1}{3}}}$$

Let's the differentiability at the value of x = 2

$$\Rightarrow \lim_{x\to 2} f'(x) = \lim_{x\to 2} \frac{2}{3(x-2)^{\frac{1}{3}}}$$

$$\Rightarrow \lim_{x\to 2} f'(x) = \frac{2}{3(2-2)^{\frac{1}{3}}}$$

$$\Rightarrow \lim_{x\to 2} f'(x) = \frac{2}{3(0)}$$

$$\Rightarrow \lim_{x\to 2} f'(x) = \text{undefined}$$

∴ f is not differentiable at x = 2, so it is not differentiable in the closed interval (1,3).

So, Rolle's theorem is not applicable for the function f on the interval [1,3].

1 B. Question

Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals :

f(x = [x] for – 1 ≤ x ≤ 1, where [x] denotes the greatest integer not exceeding x

Answer

Given function is:

⇒ f(x) = [x], – 1≤x≤1 where [x] denotes the greatest integer not exceeding x.

Let us check the continuity of the function 'f'.

Here in the interval x∈[– 1,1], the function has to be Right continuous at x = 1 and left continuous at x = 1.

⇒ $\lim_{x\to 1+} f(x) = \lim_{x\to 1+} [x]$

⇒ $\lim_{x\to 1+} f(x) = \lim_{x\to 1+h} [x]$ where h>0.

⇒ $\lim_{x\to 1+} f(x) = \lim_{h\to 0} 1$

⇒ $\lim_{x\to 1+} f(x) = 1$ ......(1)

⇒ $\lim_{x\to 1-} f(x) = \lim_{x\to 1-} [x]$

⇒ $\lim_{x\to 1-} f(x) = \lim_{x\to 1-h} [x]$, where h>0

⇒ $\lim_{x\to 1-} f(x) = \lim_{h\to 0} 0$

⇒ $\lim_{x\to 1-} f(x) = 0$ ......(2)

From (1) and (2), we can see that the limits are not the same so, the function is not continuous in the interval [– 1,1].

∴ Rolle's theorem is not applicable for the function f in the interval [– 1,1].

1 C. Question

Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals :

f(x) = sin 1/x for – 1 ≤ x ≤ 1

Answer

Given function is:

⇒ $f(x) = \sin\left(\frac{1}{x}\right)$ for – 1≤x≤1

Let us check the continuity of the function 'f' at the value of x = 0.

We can not directly find the value of limit at x = 0, as the function is not valid at x = 0. So, we take the limit on either sides and x = 0, and we check whether they are equal or not.

Right – Hand Limit:

⇒ $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \sin\left(\frac{1}{x}\right)$

We assume that the limit $\lim_{h\to 0} \sin\left(\frac{1}{h}\right) = k$, k∈[– 1,1].

⇒ $\lim_{x\to 0+} f(x) = \lim_{x\to 0+h} \sin\left(\frac{1}{x}\right)$, where h>0

⇒ $\lim_{x\to 0+} f(x) = \lim_{h\to 0} \sin\left(\frac{1}{h+0}\right)$

⇒ $\lim_{x\to 0+} f(x) = \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$

⇒ $\lim_{x\to 0+} f(x) = k$ ...... (1)

Left – Hand Limit:

$\Rightarrow \lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0-} \sin\left(\frac{1}{x}\right)$

$\Rightarrow \lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0-h} \sin\left(\frac{1}{x}\right)$, where h>0

$\Rightarrow \lim\limits_{x\to 0-} f(x) = \lim\limits_{h\to 0} \sin\left(\frac{1}{0-h}\right)$

$\Rightarrow \lim\limits_{x\to 0-} f(x) = \lim\limits_{h\to 0} \sin\left(\frac{1}{-h}\right)$

$\Rightarrow \lim\limits_{x\to 0-} f(x) = \lim\limits_{h\to 0} - \sin\left(\frac{1}{h}\right)$

$\Rightarrow \lim\limits_{x\to 0-} f(x) = -\lim\limits_{h\to 0} \sin\left(\frac{1}{h}\right)$

$\Rightarrow \lim\limits_{x\to 0-} f(x) = -k$ ......(2)

From (1) and (2), we can see that the Right hand and left – hand limits are not equal, so the function ‘f’ is not continuous at x = 0.

∴ Rolle’s theorem is not applicable to the function ‘f’ in the interval [– 1,1].

1 D. Question

Discuss the applicability of Rolle’s theorem for the following functions on the indicated intervals :

$f(x) = 2x^2 - 5x + 3$ on [1, 3]

Answer

Given function is:

⇒ $f(x) = 2x^2 - 5x + 3$ on [1,3]

Since given function ‘f’ is a polynomial. So, it is continuous and differentiable every where.

Now, we find the values of function at the extremum values.

⇒ $f(1) = 2(1)^2 - 5(1) + 3$

⇒ f(1) = 2 – 5 + 3

⇒ f(1) = 0 ......(1)

⇒ $f(3) = 2(3)^2 - 5(3) + 3$

⇒ f(3) = 2(9)–15 + 3

⇒ f(3) = 18 – 12

⇒ f(3) = 6 ......(2)

From (1) and (2), we can say that,

$f(1) \neq f(3)$

∴ Rolle’s theorem is not applicable for the function f in interval [1,3].

1 E. Question

Discuss the applicability of Rolle’s theorem for the following functions on the indicated intervals :

$f(x) = x^{2/3}$ on [– 1, 1]

Answer

Given function is:

⇒ $f(x) = x^{\frac{2}{3}}$ on [– 1,1]

Let’s find the derivative of the given function:

$\Rightarrow f'(x) = \frac{d\left(x^{\frac{2}{3}}\right)}{dx}$

$\Rightarrow f'(x) = \frac{2}{3}x^{\frac{2}{3}-1}$

$\Rightarrow f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$

$\Rightarrow f'(x) = \frac{2}{3x^{\frac{1}{3}}}$

Let’s check the differentiability of the function at x = 0.

$\Rightarrow \lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{2}{3x^{\frac{1}{3}}}$

$\Rightarrow \lim_{x\to 0} f'(x) = \frac{2}{3(0)^{\frac{1}{3}}}$

$\Rightarrow \lim_{x\to 0} f'(x) = \text{undefined}$

Since the limit for the derivative is undefined at x = 0, we can say that f is not differentiable at x = 0.

∴ Rolle’s theorem is not applicable to the function ‘f’ on [– 1,1].

1 F. Question

Discuss the applicability of Rolle’s theorem for the following functions on the indicated intervals :

$$f(x)s = \begin{cases} -4x+5, & 0 \le x \le 1 \\ 2x-3, & 1 < x \le 2 \end{cases}$$

Answer

Given function is:

$\Rightarrow f(x) = \begin{cases} -4x + 5, 0 \le x \le 1 \\ 2x - 3, 1 < x \le 2 \end{cases}$

Let’s check the continuity at x = 1 as the equation of function changes.

Left – Hand Limit:

$\Rightarrow \lim_{x\to 1-} f(x) = \lim_{x\to 1-} -4x + 5$

$\Rightarrow \lim_{x\to 1-} f(x) = -4(1) + 5$

$\Rightarrow \lim_{x\to 1-} f(x) = 1$ ......(1)

Right – Hand Limit:

$\Rightarrow \lim_{x\to 1+} f(x) = \lim_{x\to 1+} 2x - 3$

$\Rightarrow \lim_{x\to 1+} f(x) = 2(0) - 3$

$\Rightarrow \lim_{x\to 1+} f(x) = -1$ ......(2)

From (1) and (2), we can see that the values of both side limits are not equal. So, the function ‘f’ is not continuous at x = 1.

∴ Rolle’s theorem is not applicable to the function ‘f’ in the interval [0,2].

2 A. Question

Verify Rolle’s theorem for each of the following functions on the indicated intervals :

$f(x) = x^2 - 8x + 12$ on [2, 6]

Answer

First let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

$\Rightarrow f(x) = x^2 - 8x + 12$ on [2,6]

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R.

Let us find the values at extremums:

$\Rightarrow f(2) = 2^2 - 8(2) + 12$

$\Rightarrow f(2) = 4 - 16 + 12$

$\Rightarrow f(2) = 0$

$\Rightarrow f(6) = 6^2 - 8(6) + 12$

$\Rightarrow f(6) = 36 - 48 + 12$

$\Rightarrow f(6) = 0$

∴ f(2) = f(6), Rolle’s theorem applicable for function ‘f’ on [2,6].

Let’s find the derivative of f(x):

$\Rightarrow f'(x) = \frac{d(x^2 - 8x + 12)}{dx}$

$\Rightarrow f'(x) = \frac{d(x^2)}{dx} - \frac{d(8x)}{dx} + \frac{d(12)}{dx}$

$\Rightarrow f'(x) = 2x - 8 + 0$

$\Rightarrow f'(x) = 2x - 8$

We have f’(c) = 0 $c \in (2,6)$, from the definition given above.

$\Rightarrow f'(c) = 0$

$\Rightarrow 2c - 8 = 0$

$\Rightarrow 2c = 8$

$\Rightarrow c = \frac{8}{2}$

$\Rightarrow C = 4 \in (2,6)$

∴ Rolle’s theorem is verified.

2 B. Question

Verify Rolle’s theorem for each of the following functions on the indicated intervals :

$f(x) = x^2 - 4x + 3$ on [1, 3]

Answer

First let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given function is:

⇒ $f(x) = x^2 - 4x + 3$ on [1,3]

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R.

Let us find the values at extremums:

⇒ $f(1) = 1^2 - 4(1) + 3$

⇒ $f(1) = 1 - 4 + 3$

⇒ $f(1) = 0$

⇒ $f(3) = 3^2 - 4(3) + 3$

⇒ $f(3) = 9 - 12 + 3$

⇒ $f(3) = 0$

∴ f(1) = f(3), Rolle's theorem applicable for function 'f' on [1,3].

Let's find the derivative of f(x):

⇒ $f'(x) = \frac{d(x^2 - 4x + 3)}{dx}$

⇒ $f'(x) = \frac{d(x^2)}{dx} - \frac{d(4x)}{dx} + \frac{d(3)}{dx}$

⇒ $f'(x) = 2x - 4 + 0$

⇒ $f'(x) = 2x - 4$

We have f'(c) = 0 c∈(1,3), from the definition given above.

⇒ $f'(c) = 0$

⇒ $2c - 4 = 0$

⇒ $2c = 4$

⇒ $c = \frac{4}{2}$

⇒ C = 2∈(1,3)

∴ Rolle's theorem is verified.

2 C. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$f(x) = (x - 1)(x - 2)^2$ on [1, 2]

Answer

First let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

⇒ $f(x) = (x - 1)(x - 2)^2$ on [1,2]

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e, on R.

Let us find the values at extremums:

⇒ $f(1) = (1 - 1)(1 - 2)^2$

⇒ $f(1) = 0(1)^2$

⇒ $f(1) = 0$

⇒ $f(2) = (2 - 1)(2 - 2)^2$

⇒ $f(2) = 1.0^2$

⇒ $f(2) = 0$

∴ f(1) = f(2), Rolle’s theorem applicable for function ‘f’ on [1,2].

Let’s find the derivative of f(x):

⇒ $f'(x) = \frac{d((x-1)(x-2)^2)}{dx}$

Differentiating using UV rule

⇒ $f'(x) = (x-2)^2 \times \frac{d(x-1)}{dx} + (x-1) \times \frac{d((x-2)^2)}{dx}$

⇒ $f'(x) = ((x - 2)^2 \times 1) + ((x - 1) \times 2 \times (x - 2))$

⇒ $f'(x) = x^2 - 4x + 4 + 2(x^2 - 3x + 2)$

⇒ $f'(x) = 3x^2 - 10x + 8$

We have f’(c) = 0 c∊(1,2), from the definition given above.

⇒ $f'(c) = 0$

⇒ $3c^2 - 10c + 8 = 0$

⇒ $c = \frac{10 \pm \sqrt{(-10)^2 - (4 \times 3 \times 8)}}{2 \times 3}$

⇒ $c = \frac{10 \pm \sqrt{100 - 96}}{6}$

⇒ $c = \frac{10 \pm 2}{6}$

⇒ $c = \frac{12}{6}$ or $c = \frac{8}{6}$

⇒ $c = \frac{4}{3} \in (1,2)$ (neglecting the value 2)

∴ Rolle’s theorem is verified.

2 D. Question

Verify Rolle’s theorem for each of the following functions on the indicated intervals :

$f(x) = x(x - 1)^2$ on [0, 1]

Answer

First let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given function is:

$\Rightarrow f(x) = x(x - 1)^2$ on [0,1]

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e, on R.

Let us find the values at extremums:

$\Rightarrow f(0) = 0(0 - 1)^2$

$\Rightarrow f(0) = 0$

$\Rightarrow f(1) = 1(1 - 1)^2$

$\Rightarrow f(1) = 0^2$

$\Rightarrow f(1) = 0$

∴ f(0) = f(1), Rolle's theorem applicable for function 'f' on [0,1].

Let's find the derivative of f(x):

$\Rightarrow f'(x) = \frac{d(x(x-1)^2)}{dx}$

Differentiating using UV rule,

$\Rightarrow f'(x) = (x-1)^2 \times \frac{d(x)}{dx} + x\frac{d((x-1)^2)}{dx}$

$\Rightarrow f'(x) = ((x - 1)^2 \times 1) + (x \times 2 \times (x - 1))$

$\Rightarrow f'(x) = (x - 1)^2 + 2(x^2 - x)$

$\Rightarrow f'(x) = x^2 - 2x + 1 + 2x^2 - 2x$

$\Rightarrow f'(x) = 3x^2 - 4x + 1$

We have f'(c) = 0 $c \in (0,1)$, from the definition given above.

$\Rightarrow f'(c) = 0$

$\Rightarrow 3c^2 - 4c + 1 = 0$

$\Rightarrow c = \frac{4 \pm \sqrt{(-4)^2 - (4 \times 3 \times 1)}}{2 \times 3}$

$\Rightarrow c = \frac{4 \pm \sqrt{16-12}}{6}$

$\Rightarrow c = \frac{4 \pm \sqrt{4}}{6}$

$\Rightarrow c = \frac{6}{6}$ or $c = \frac{2}{6}$

⇒ $c = \frac{1}{3} \epsilon (0,1)$

∴ Rolle's theorem is verified.

2 E. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$f(x) = (x^2 - 1)(x - 2)$ on [– 1, 2]

Answer

First let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given function is:

⇒ $f(x) = (x^2 - 1)(x - 2)$ on [– 1,2]

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e, on R.

Let us find the values at extremums:

⇒ $f(-1) = ((-1)^2 - 1)(-1 - 2)$

⇒ $f(-1) = (1 - 1)(-3)$

⇒ $f(-1) = (0)(-3)$

⇒ $f(-1) = 0$

⇒ $f(2) = (2^2 - 1)(2 - 2)$

⇒ $f(2) = (4 - 1)(0)$

⇒ $f(2) = 0$

∴ f(– 1) = f(2), Rolle's theorem applicable for function 'f' on [– 1,2].

Let's find the derivative of f(x):

⇒ $f'(x) = \frac{d((x^2-1)(x-2))}{dx}$

Differentiating using UV rule,

⇒ $f'(x) = (x-2) \times \frac{d(x^2-1)}{dx} + (x^2-1)\frac{d(x-2)}{dx}$

⇒ $f'(x) = ((x - 2)\times 2x) + ((x^2 - 1)\times 1)$

⇒ $f'(x) = 2x^2 - 4x + x^2 - 1$

⇒ $f'(x) = 2x^2 - 4x - 1$

We have f'(c) = 0 c∊(– 1,2), from the definition given above.

⇒ $f'(c) = 0$

⇒ $2c^2 - 4c - 1 = 0$

⇒ $c = \frac{4 \pm \sqrt{(-4)^2-(4\times 2\times -1)}}{2\times 2}$

$\Rightarrow c = \frac{4 \pm \sqrt{16+8}}{4}$

$\Rightarrow c = \frac{4 \pm \sqrt{24}}{4}$

$\Rightarrow c = \frac{4+2\sqrt{6}}{4}$ or $c = \frac{4-2\sqrt{6}}{4}$

$\Rightarrow c = 1 + \frac{\sqrt{6}}{2}$ or $c = 1 - \frac{\sqrt{6}}{2}$

$\Rightarrow c = 1 - \frac{\sqrt{6}}{2} \epsilon (- 1,2)$

∴ Rolle’s theorem is verified.

2 F. Question

Verify Rolle’s theorem for each of the following functions on the indicated intervals :

$f(x) = x(x - 4)^2$ on [0, 4]

Answer

First let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

$\Rightarrow f(x) = x(x - 4)^2$ on [0,4]

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R.

Let us find the values at extremums:

$\Rightarrow f(0) = 0(0 - 4)^2$

$\Rightarrow f(0) = 0$

$\Rightarrow f(4) = 4(4 - 4)^2$

$\Rightarrow f(4) = 4(0)^2$

$\Rightarrow f(4) = 0$

∴ f(0) = f(4), Rolle’s theorem applicable for function ‘f’ on [0,4].

Let’s find the derivative of f(x):

$\Rightarrow f'(x) = \frac{d(x(x-4)^2)}{dx}$

Differentiating using UV rule,

$\Rightarrow f'(x) = (x-4)^2 \times \frac{d(x)}{dx} + x\frac{d((x-4)^2)}{dx}$

$\Rightarrow f'(x) = ((x - 4)^2 \times 1) + (x \times 2 \times (x - 4))$

$\Rightarrow f'(x) = (x - 4)^2 + 2(x^2 - 4x)$

$\Rightarrow f'(x) = x^2 - 8x + 16 + 2x^2 - 8x$

⇒ $f'(x) = 3x^2 - 16x + 16$

We have $f'(c) = 0$ $c \epsilon (0,4)$, from the definition given above.

⇒ $f'(c) = 0$

⇒ $3c^2 - 16c + 16 = 0$

⇒ $c = \frac{16 \pm \sqrt{(-16)^2 - (4 \times 3 \times 16)}}{2 \times 3}$

⇒ $c = \frac{16 \pm \sqrt{256 - 192}}{6}$

⇒ $c = \frac{16 \pm \sqrt{64}}{6}$

⇒ $c = \frac{8}{6}$ or $c = \frac{24}{6}$

⇒ $c = \frac{8}{6} \epsilon (0,4)$

∴ Rolle's theorem is verified.

2 G. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$f(x) = x(x - 2)^2$ on [0, 2]

Answer

First let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given function is:

⇒ $f(x) = x(x - 2)^2$ on [0,2]

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e, on R.

Let us find the values at extremums:

⇒ $f(0) = 0(0 - 2)^2$

⇒ $f(0) = 0$

⇒ $f(2) = 2(2 - 2)^2$

⇒ $f(2) = 2(0)^2$

⇒ $f(2) = 0$

∴ f(0) = f(2), Rolle's theorem applicable for function 'f' on [0,2].

Let's find the derivative of f(x):

⇒ $f'(x) = \frac{d(x(x-2)^2)}{dx}$

Differentiating using UV rule,

⇒ $f'(x) = (x - 2)^2 \times \frac{d(x)}{dx} + x\frac{d((x-2)^2)}{dx}$

$\Rightarrow f'(x) = ((x - 2)^2 \times 1) + (x \times 2 \times (x - 2))$

$\Rightarrow f'(x) = (x - 2)^2 + 2(x^2 - 2x)$

$\Rightarrow f'(x) = x^2 - 4x + 4 + 2x^2 - 4x$

$\Rightarrow f'(x) = 3x^2 - 8x + 4$

We have f'(c) = 0 $c \in (0,1)$, from the definition given above.

$\Rightarrow f'(c) = 0$

$\Rightarrow 3c^2 - 8c + 4 = 0$

$\Rightarrow c = \frac{8 \pm \sqrt{(-8)^2 - (4 \times 3 \times 4)}}{2 \times 3}$

$\Rightarrow c = \frac{8 \pm \sqrt{64 - 48}}{6}$

$\Rightarrow c = \frac{8 \pm \sqrt{16}}{6}$

$\Rightarrow c = \frac{12}{6} \text{ or } c = \frac{6}{6}$

$\Rightarrow c = 1 \in (0,2)$

∴ Rolle's theorem is verified.

2 H. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$f(x) = x^2 + 5x + 6$ on [– 3, – 2]

Answer

First let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given function is:

$\Rightarrow f(x) = x^2 + 5x + 6$ on [– 3, – 2]

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R.

Let us find the values at extremums:

$\Rightarrow f(-3) = (-3)^2 + 5(-3) + 6$

$\Rightarrow f(-3) = 9 - 15 + 6$

$\Rightarrow f(-3) = 0$

$\Rightarrow f(-2) = (-2)^2 + 5(-2) + 6$

$\Rightarrow f(-2) = 4 - 10 + 6$

$\Rightarrow f(-2) = 0$

∴ f(– 3) = f(– 2), Rolle's theorem applicable for function 'f' on [– 3, – 2].

Let's find the derivative of f(x):

$\Rightarrow f'(x) = \frac{d(x^2 + 5x + 6)}{dx}$

$\Rightarrow f'(x) = \frac{d(x^2)}{dx} + \frac{d(5x)}{dx} + \frac{d(6)}{dx}$

$\Rightarrow f'(x) = 2x + 5 + 0$

$\Rightarrow f'(x) = 2x + 5$

We have f'(c) = 0 c∈(– 3, – 2), from the definition given above.

$\Rightarrow f'(c) = 0$

$\Rightarrow 2c + 5 = 0$

$\Rightarrow 2c = -5$

$\Rightarrow c = -\frac{5}{2}$

⇒ C = – 2.5∈(– 3, – 2)

∴ Rolle's theorem is verified.

3 A. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

f(x) = cos 2 (x – π/4) on [0, π/2]

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given function is :

$\Rightarrow f(x) = \cos 2\left(x - \frac{\pi}{4}\right) \text{ on } \left[0, \frac{\pi}{2}\right]$

We know that cosine function is continuous and differentiable on R.

Let's find the values of the function at an extremum,

$\Rightarrow f(0) = \cos 2\left(0 - \frac{\pi}{4}\right)$

$\Rightarrow f(0) = \cos 2\left(-\frac{\pi}{4}\right)$

$\Rightarrow f(0) = \cos\left(-\frac{\pi}{2}\right)$

We know that cos(– x) = cosx

$\Rightarrow f(0) = 0$

$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{4}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 0$

We got $f(0) = f\left(\frac{\pi}{2}\right)$, so there exist a $c \epsilon \left(0, \frac{\pi}{2}\right)$ such that f'(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d\left(\cos 2\left(x - \frac{\pi}{4}\right)\right)}{dx}$

$\Rightarrow f'(x) = -\sin\left(2\left(x - \frac{\pi}{4}\right)\right)\frac{d\left(2\left(x - \frac{\pi}{4}\right)\right)}{dx}$

$\Rightarrow f'(x) = -2\sin 2\left(x - \frac{\pi}{4}\right)$

We have f'(c) = 0,

$\Rightarrow -2\sin 2\left(c - \frac{\pi}{4}\right) = 0$

$\Rightarrow c - \frac{\pi}{4} = 0$

$\Rightarrow c = \frac{\pi}{4} \epsilon \left(0, \frac{\pi}{2}\right)$

∴ Rolle's theorem is verified.

3 B. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

f(x) = sin 2x on [0, π/2]

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given fuction is:

⇒ f(x) = sin2x on $\left[0, \frac{\pi}{2}\right]$

We know that sine function is continuous and differentiable on R.

Let's find the values of function at extremum,

⇒ f(0) = sin2(0)

⇒ f(0) = sin0

⇒ f(0) = 0

$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin 2\left(\frac{\pi}{2}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin(\pi)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 0$

We got $f(0) = f\left(\frac{\pi}{2}\right)$, so there exist a $c\epsilon\left(0, \frac{\pi}{2}\right)$ such that f’(c) = 0.

Let’s find the derivative of f(x)

⇒ $f'(x) = \frac{d(\sin 2x)}{dx}$

⇒ $f'(x) = \cos 2x \frac{d(2x)}{dx}$

⇒ f’(x) = 2cos2x

We have f’(c) = 0,

⇒ 2cos2c = 0

⇒ $2c = \frac{\pi}{2}$

⇒ $c = \frac{\pi}{4}\epsilon\left(0, \frac{\pi}{2}\right)$

∴ Rolle’s theorem is verified.

3 C. Question

Verify Rolle’s theorem for each of the following functions on the indicated intervals :

f(x) = cos 2x on [– π/4, π/4]

Answer

First, let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

⇒ cos2x on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

We know that cosine function is continuous and differentiable on R.

Let’s find the values of the function at an extremum,

⇒ $f\left(-\frac{\pi}{4}\right) = \cos 2\left(-\frac{\pi}{4}\right)$

⇒ $f(0) = \cos\left(-\frac{\pi}{2}\right)$

We know that cos(– x) = cosx

⇒ f(0) = 0

⇒ $f\left(\frac{\pi}{4}\right) = \cos 2\left(\frac{\pi}{4}\right)$

⇒ $f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)$

⇒ $f\left(\frac{\pi}{2}\right) = 0$

We got $f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right)$, so there exist a $c\epsilon\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ such that f’(c) = 0.

Let’s find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(\cos 2x)}{dx}$

$\Rightarrow f'(x) = -\sin 2x \frac{d(2x)}{dx}$

⇒ f’(x) = – 2sin2x

We have f’(c) = 0,

⇒ – 2sin2c = 0

⇒ 2c = 0

$\Rightarrow c = \frac{\pi}{4} \epsilon \left(0, \frac{\pi}{2}\right)$

∴ Rolle’s theorem is verified.

3 D. Question

Verify Rolle’s theorem for each of the following functions on the indicated intervals :

f(x) = e^x sin x on [0, π]

Answer

First, let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

⇒ f(x) = e^xsinx on [0,π]

We know that exponential and sine functions are continuous and differentiable on R.

Let’s find the values of the function at an extremum,

⇒ f(0) = e^0sin(0)

⇒ f(0) = 1×0

⇒ f(0) = 0

$\Rightarrow f(\pi) = e^{\pi}\sin(\pi)$

$\Rightarrow f(\pi) = e^{\pi} \times 0$

$\Rightarrow f(\pi) = 0$

We got $f(0) = f(\pi)$, so there exist a $c \epsilon (0, \pi)$ such that f’(c) = 0.

Let’s find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(e^x \sin x)}{dx}$

$\Rightarrow f'(x) = \sin x \frac{d(e^x)}{dx} + e^x \frac{d(\sin x)}{dx}$

⇒ f’(x) = e^x(sinx + cosx)

We have f’(c) = 0,

⇒ $e^c(\sin c + \cos c) = 0$

⇒ sinc + cosc = 0

⇒ $\frac{1}{\sqrt{2}}\sin c + \frac{1}{\sqrt{2}}\cos c = 0$

⇒ $\sin\left(\frac{\pi}{4}\right)\sin c + \cos\left(\frac{\pi}{4}\right)\cos c = 0$

⇒ $\cos\left(c - \frac{\pi}{4}\right) = 0$

⇒ $c - \frac{\pi}{4} = \frac{\pi}{2}$

⇒ $c = \frac{3\pi}{4} \epsilon (0, \pi)$

∴ Rolle’s theorem is verified.

3 E. Question

Verify Rolle’s theorem for each of the following functions on the indicated intervals :

f(x) = e^x cos x on [– π/2, π/2]

Answer

First, let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

⇒ f(x) = e^xcosx on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

We know that exponential and cosine functions are continuous and differentiable on R.

Let’s find the values of the function at an extremum,

⇒ $f\left(-\frac{\pi}{2}\right) = e^{-\frac{\pi}{2}}\cos\left(-\frac{\pi}{2}\right)$

⇒ $f\left(-\frac{\pi}{2}\right) = e^{-\frac{\pi}{2}} \times 0$

⇒ $f\left(-\frac{\pi}{2}\right) = 0$

⇒ $f\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}}\cos\left(\frac{\pi}{2}\right)$

⇒ $f(\pi) = e^{\frac{\pi}{2}} \times 0$

⇒ $f(\pi) = 0$

We got $f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)$, so there exist a $c\epsilon\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that f’(c) = 0.

Let’s find the derivative of f(x)

⇒ $f'(x) = \frac{d(e^x \cos x)}{dx}$

$\Rightarrow f'(x) = \cos x \frac{d(e^x)}{dx} + e^x \frac{d(\cos x)}{dx}$

$\Rightarrow f'(x) = e^x(- \sin x + \cos x)$

We have f'(c) = 0,

$\Rightarrow e^c(- \sin c + \cos c) = 0$

$\Rightarrow - \sin c + \cos c = 0$

$\Rightarrow \frac{-1}{\sqrt{2}} \sin c + \frac{1}{\sqrt{2}} \cos c = 0$

$\Rightarrow -\sin\left(\frac{\pi}{4}\right) \sin c + \cos\left(\frac{\pi}{4}\right) \cos c = 0$

$\Rightarrow \cos\left(c + \frac{\pi}{4}\right) = 0$

$\Rightarrow c + \frac{\pi}{4} = \frac{\pi}{2}$

$\Rightarrow c = \frac{\pi}{4} \epsilon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

∴ Rolle's theorem is verified.

3 F. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

f(x) = cos 2x on [0, π]

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given function is:

⇒ f(x) = cos 2x on [0,π]

We know that cosine function is continuous and differentiable on R.

Let's find the values of function at extremum,

⇒ f(0) = cos2(0)

⇒ f(0) = cos(0)

⇒ f(0) = 1

⇒ f(π) = cos2(π)

⇒ f(π) = cos(2π)

⇒ f(π) = 1

We got $f(0) = f(\pi)$, so there exist a $c \epsilon (0, \pi)$ such that f'(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(\cos 2x)}{dx}$

$\Rightarrow f'(x) = -\sin 2x \frac{d(2x)}{dx}$

⇒ f'(x) = – 2sin2x

We have f'(c) = 0,

⇒ – 2sin2c = 0

⇒ 2c = 0

$\Rightarrow c = \frac{\pi}{4} \epsilon (0, \pi)$

∴ Rolle's theorem is verified.

3 G. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = \frac{\sin x}{e^x} \text{ on } 0 \leq x \leq \pi$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given function is:

$\Rightarrow f(x) = \frac{\sin x}{e^x}$ on [0,π]

This can be written as

⇒ f(x) = e^{-x}sinx on [0,π]

We know that exponential and sine functions are continuous and differentiable on R.

Let's find the values of the function at an extremum,

⇒ f(0) = e^{-0}sin(0)

⇒ f(0) = 1×0

⇒ f(0) = 0

$\Rightarrow f(\pi) = e^{-\pi} \sin(\pi)$

$\Rightarrow f(\pi) = e^{-\pi} \times 0$

$\Rightarrow f(\pi) = 0$

We got $f(0) = f(\pi)$, so there exist a $c \epsilon (0, \pi)$ such that f'(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(e^{-x} \sin x)}{dx}$

$\Rightarrow f'(x) = \sin x \frac{d(e^{-x})}{dx} + e^{-x} \frac{d(\sin x)}{dx}$

⇒ f’(x) = sinx(– e^{-x}) + e^{-x}(cosx)

⇒ f’(x) = e^{-x}(– sinx + cosx)

We have f’(c) = 0,

⇒ e^{-c}(– sinc + cosc) = 0

⇒ – sinc + cosc = 0

⇒ $-\frac{1}{\sqrt{2}}\text{sinc} + \frac{1}{\sqrt{2}}\text{cosc} = 0$

⇒ $-\sin\left(\frac{\pi}{4}\right)\text{sinc} + \cos\left(\frac{\pi}{4}\right)\text{cosc} = 0$

⇒ $\cos\left(c + \frac{\pi}{4}\right) = 0$

⇒ $c + \frac{\pi}{4} = \frac{\pi}{2}$

⇒ $c = \frac{\pi}{4}\epsilon(0,\pi)$

∴ Rolle’s theorem is verified.

3 H. Question

Verify Rolle’s theorem for each of the following functions on the indicated intervals :

f(x) = sin 3x on [0, π]

Answer

First, let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

⇒ f(x) = sin3x on [0,π]

We know that sine function is continuous and differentiable on R.

Let’s find the values of function at extremum,

⇒ f(0) = sin3(0)

⇒ f(0) = sin0

⇒ f(0) = 0

⇒ f(π) = sin3(π)

⇒ f(π) = sin(3π)

⇒ f(π) = 0

We got $f(0) = f(\pi)$, so there exist a $c\epsilon(0,\pi)$ such that f’(c) = 0.

Let’s find the derivative of f(x)

⇒ $f'(x) = \frac{d(\sin 3x)}{dx}$

$\Rightarrow f'(x) = \cos 3x \frac{d(3x)}{dx}$

⇒ f’(x) = 3cos3x

We have f’(c) = 0,

⇒ 3cos3c = 0

$\Rightarrow 3c = \frac{\pi}{2}$

$\Rightarrow c = \frac{\pi}{6} \epsilon (0, \pi)$

∴ Rolle’s theorem is verified.

3 I. Question

Verify Rolle’s theorem for each of the following functions on the indicated intervals :

$f(x) = e^{1-x^2} \text{ on } [-1, 1]$

Answer

First, let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

$\Rightarrow f(x) = e^{1-x^2}$ on [– 1,1]

We know that exponential function is continuous and differentiable over R.

Let’s find the value of function f at extremums,

$\Rightarrow f(-1) = e^{1-(-1)^2}$

$\Rightarrow f(-1) = e^{1-1}$

⇒ f(– 1) = e^0

⇒ f(– 1) = 1

$\Rightarrow f(1) = e^{1-1^2}$

$\Rightarrow f(1) = e^{1-1}$

⇒ f(1) = e^0

⇒ f(1) = 1

We got f(– 1) = f(1) so, there exists a cϵ(– 1,1) such that f’(c) = 0.

Let’s find the derivative of the function f:

$\Rightarrow f'(x) = \frac{d\left(e^{1-x^2}\right)}{dx}$

$\Rightarrow f'(x) = e^{1-x^2} \frac{d(1-x^2)}{dx}$

$\Rightarrow f'(x) = e^{1-x^2}(-2x)$

We have f’(c) = 0

$\Rightarrow e^{1-c^2}(-2c) = 0$

⇒ 2c = 0

⇒ c = 0ϵ[– 1,1]

∴ Rolle’s theorem is verified.

3 J. Question

Verify Rolle’s theorem for each of the following functions on the indicated intervals :

f(x) = log(x^2 + 2) – log 3 on [–1, 1]

Answer

First, let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

⇒ f(x) = log(x^2 + 2) – log3 on [– 1,1]

We know that logarithmic function is continuous and differentiable in its own domain.

We check the values of the function at the extremum,

⇒ f(– 1) = log((– 1)^2 + 2) – log3

⇒ f(– 1) = log(1 + 2) – log3

⇒ f(– 1) = log3 – log3

⇒ f(– 1) = 0

⇒ f(1) = log(1^2 + 2) – log3

⇒ f(1) = log(1 + 2) – log3

⇒ f(1) = log3 – log3

⇒ f(1) = 0

We have got f(– 1) = f(1). So, there exists a c such that cϵ(– 1,1) such that f’(c) = 0.

Let’s find the derivative of the function f,

$\Rightarrow f'(x) = \frac{d(\log(x^2+2)-\log 3)}{dx}$

$\Rightarrow f'(x) = \frac{1}{x^2+2}\frac{d(x^2+2)}{dx} - 0$

$\Rightarrow f'(x) = \frac{2x}{x^2+2}$

We have f’(c) = 0

$\Rightarrow \frac{2c}{c^2+2} = 0$

⇒ 2c = 0

⇒ c = 0ϵ(– 1,1)

∴ Rolle's theorem is verified.

3 K. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

f(x) = sin x + cos x on [0, π/2]

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given function is:

⇒ f(x) = sinx + cosx on $\left[0, \frac{\pi}{2}\right]$

We know that sine and cosine functions are continuous and differentiable on R.

Let's the value of function f at extremums:

⇒ f(0) = sin(0) + cos(0)

⇒ f(0) = 0 + 1

⇒ f(0) = 1

⇒ $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)$

⇒ $f\left(\frac{\pi}{2}\right) = 1 + 0$

⇒ $f\left(\frac{\pi}{2}\right) = 1$

We have got $f(0) = f\left(\frac{\pi}{2}\right)$. So, there exists a c$\epsilon\left(0, \frac{\pi}{2}\right)$ such that f'(c) = 0.

Let's find the derivative of the function 'f'.

⇒ $f'(x) = \frac{d(\sin x + \cos x)}{dx}$

⇒ f'(x) = cosx – sinx

We have f'(c) = 0

⇒ cosc – sinc = 0

⇒ $\frac{1}{\sqrt{2}}\cos c - \frac{1}{\sqrt{2}}\sin c = 0$

⇒ $\sin\left(\frac{\pi}{4}\right)\cos c - \cos\left(\frac{\pi}{4}\right)\sin c = 0$

⇒ $\sin\left(\frac{\pi}{4} - c\right) = 0$

⇒ $\frac{\pi}{4} - c = 0$

$\Rightarrow c = \frac{\pi}{4} \epsilon \left(0, \frac{\pi}{2}\right)$

∴ Rolle’s theorem is verified.

3 L. Question

Verify Rolle’s theorem for each of the following functions on the indicated intervals :

f(x) = 2 sin x + sin 2x on [0, π]

Answer

First, let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

⇒ f(x) = 2sinx + sin2x on [0,π]

We know that sine function continuous and differentiable over R.

Let’s check the values of function f at the extremums

⇒ f(0) = 2sin(0) + sin2(0)

⇒ f(0) = 2(0) + 0

⇒ f(0) = 0

⇒ f(π) = 2sin(π) + sin2(π)

⇒ f(π) = 2(0) + 0

⇒ f(π) = 0

We have got f(0) = f(π). So, there exists a c∊(0,π) such that f’(c) = 0.

Let’s find the derivative of function ‘f’.

$\Rightarrow f'(x) = \frac{d(2\sin x + \sin 2x)}{dx}$

$\Rightarrow f'(x) = 2\cos x + \cos 2x \frac{d(2x)}{dx}$

⇒ f’(x) = 2cosx + 2cos2x

⇒ f’(x) = 2cosx + 2(2cos^{2}x – 1)

⇒ f’(x) = 4cos^{2}x + 2cosx – 2

We have f’(c) = 0,

⇒ 4cos^{2}c + 2cosc – 2 = 0

⇒ 2cos^{2}c + cosc – 1 = 0

⇒ 2cos^{2}c + 2cosc – cosc – 1 = 0

⇒ 2cosc(cosc + 1) – 1(cosc + 1) = 0

⇒ (2cosc – 1)(cosc + 1) = 0

$\Rightarrow \cos c = \frac{1}{2}$ or $\cos c = -1$

$\Rightarrow c = \frac{\pi}{3} \epsilon (0, \pi)$

∴ Rolle’s theorem is verified.

3 M. Question

Verify Rolle’s theorem for each of the following functions on the indicated intervals :

$$f(x) = \frac{x}{2} - \sin\frac{\pi x}{6} \text{ on } [-1, 0]$$

Answer

First, let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

$\Rightarrow f(x) = \frac{x}{2} - \sin\left(\frac{\pi x}{6}\right)$ on [– 1,0]

We know that sine function is continuous and differentiable over R.

Let’s check the values of ‘f’ at an extremum

$\Rightarrow f(-1) = \frac{-1}{2} - \sin\left(\frac{\pi(-1)}{6}\right)$

$\Rightarrow f(-1) = -\frac{1}{2} - \sin\left(\frac{-\pi}{6}\right)$

$\Rightarrow f(-1) = -\frac{1}{2} - \left(-\frac{1}{2}\right)$

⇒ f(– 1) = 0

$\Rightarrow f(0) = \frac{0}{2} - \sin\left(\frac{\pi(0)}{6}\right)$

$\Rightarrow f(0) = 0 - \sin(0)$

⇒ f(0) = 0 – 0

⇒ f(0) = 0

We have got f(– 1) = f(0). So, there exists a c϶(– 1,0) such that f’(c) = 0.

Let’s find the derivative of the function ‘f’

$\Rightarrow f'(x) = \frac{d\left(\frac{x}{2} - \sin\left(\frac{\pi x}{6}\right)\right)}{dx}$

$\Rightarrow f'(x) = \frac{1}{2} - \cos\left(\frac{\pi x}{6}\right)\frac{d\left(\frac{\pi x}{6}\right)}{dx}$

$\Rightarrow f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\left(\frac{\pi x}{6}\right)$

We have f’(c) = 0

$\Rightarrow \frac{1}{2} - \frac{\pi}{6}\cos\left(\frac{\pi c}{6}\right) = 0$

$\Rightarrow \frac{\pi}{6}\cos\left(\frac{\pi c}{6}\right) = \frac{1}{2}$

$\Rightarrow \cos\left(\frac{\pi c}{6}\right) = \frac{1}{2} \times \frac{6}{\pi}$

$\Rightarrow \cos\left(\frac{\pi c}{6}\right) = \frac{3}{\pi}$

$\Rightarrow \frac{\pi c}{6} = \cos^{-1}\left(\frac{3}{\pi}\right)$

$\Rightarrow c = \frac{6}{\pi}\cos^{-1}\left(\frac{3}{\pi}\right)$

Cosine is positive between $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, for our convenience we take the interval to be $-\frac{\pi}{2} \leq \theta \leq 0$, since the values of the cosine repeats.

We know that $\frac{3}{\pi}$ value is nearly equal to 1. So, the value of the c nearly equal to 0.

So, we can clearly say that c϶(– 1,0).

∴ Rolle's theorem is verified.

3 N. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = \frac{6x}{\pi} - 4\sin^2 x \text{ on} \left[0, \frac{\pi}{2}\right]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given function is:

$\Rightarrow f(x) = \frac{6x}{\pi} - 4\sin^2 x \text{ on } \left[0, \frac{\pi}{6}\right]$

We know that sine function is continuous and differentiable over R.

Let's check the values of function 'f' at the extremums,

$\Rightarrow f(0) = \frac{6(0)}{\pi} - 4\sin^2(0)$

⇒ f(0) = 0 – 4(0)

⇒ f(0) = 0

$\Rightarrow f\left(\frac{\pi}{6}\right) = \frac{6\left(\frac{\pi}{6}\right)}{\pi} - 4\sin^2\left(\frac{\pi}{6}\right)$

$\Rightarrow f\left(\frac{\pi}{6}\right) = \frac{\pi}{\pi} - 4\left(\frac{1}{2}\right)^2$

$\Rightarrow f\left(\frac{\pi}{6}\right) = 1 - 4\left(\frac{1}{4}\right)$

$\Rightarrow f\left(\frac{\pi}{6}\right) = 1 - 1$

$\Rightarrow f\left(\frac{\pi}{6}\right) = 0.$

We got $f(0) = f\left(\frac{\pi}{6}\right)$. So, there exists a $c\epsilon\left(0,\frac{\pi}{6}\right)$ such that f'(c) = 0.

Let's find the derivative of function 'f.'

$\Rightarrow f'(x) = \frac{d\left(\frac{6x}{\pi} - 4\sin^2 x\right)}{dx}$

$\Rightarrow f'(x) = \frac{6}{\pi} - 4 \times 2\sin x \times \frac{d(\sin x)}{dx}$

$\Rightarrow f'(x) = \frac{6}{\pi} - 8\sin x(\cos x)$

$\Rightarrow f'(x) = \frac{6}{\pi} - 4(2\sin x\cos x)$

$\Rightarrow f'(x) = \frac{6}{\pi} - 4\sin 2x$

We have f'(c) = 0

$\Rightarrow \frac{6}{\pi} - 4\sin 2c = 0$

$\Rightarrow 4\sin 2c = \frac{6}{\pi}$

$\Rightarrow \sin 2c = \frac{6}{4\pi}$

We know $\frac{6}{4\pi} < \frac{1}{2}$

$\Rightarrow \sin 2c < \frac{1}{2}$

$\Rightarrow 2c < \sin^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow 2c < \frac{\pi}{6}$

$\Rightarrow c < \frac{\pi}{12} \epsilon \left(0,\frac{\pi}{6}\right)$

∴ Rolle's theorem is verified.

3 O. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$f(x) = 4^{\sin x}$ on [0, π]

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given function is:

$\Rightarrow f(x) = 4^{\sin x}$ on [0,π]

We that sine function is continuous and differentiable over R.

Let’s check the values of function ‘f’ at extremums

⇒ $f(0) = 4^{\sin(0)}$

⇒ $f(0) = 4^0$

⇒ $f(0) = 1$

⇒ $f(\pi) = 4^{\sin\pi}$

⇒ $f(\pi) = 4^0$

⇒ $f(\pi) = 1$

We got $f(0) = f(\pi)$. So, there exists a $c \epsilon (0,\pi)$ such that $f'(c) = 0$.

Let’s find the derivative of ‘f’

⇒ $f'(x) = \frac{d(4^{\sin x})}{dx}$

⇒ $f'(x) = 4^{\sin x} \log 4 \frac{d(\sin x)}{dx}$

⇒ $f'(x) = 4^{\sin x} \log 4 \cos x$

We have $f'(c) = 0$

⇒ $4^{\sin c}\log 4\cos c = 0$

⇒ $\cos c = 0$

⇒ $c = \frac{\pi}{2} \epsilon (0,\pi)$

∴ Rolle’s theorem is verified.

3 P. Question

Verify Rolle’s theorem for each of the following functions on the indicated intervals :

$f(x) = x^2 - 5x + 4$ on [1, 4]

Answer

First, let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

⇒ $f(x) = x^2 - 5x + 4$ on [1,4]

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R.

Let us find the values at extremums:

⇒ $f(1) = 1^2 - 5(1) + 4$

⇒ $f(1) = 1 - 5 + 4$

⇒ $f(1) = 0$

⇒ $f(4) = 4^2 - 5(4) + 4$

⇒ $f(4) = 16 - 20 + 4$

⇒ $f(4) = 0$

∴ We got f(1) = f(4). So, there exists a $c \in (1,4)$ such that $f'(c) = 0$.

Let's find the derivative of f(x):

⇒ $f'(x) = \frac{d(x^2 - 5x + 4)}{dx}$

⇒ $f'(x) = \frac{d(x^2)}{dx} - \frac{d(5x)}{dx} + \frac{d(4)}{dx}$

⇒ $f'(x) = 2x - 5 + 0$

⇒ $f'(x) = 2x - 5$

We have $f'(c) = 0$

⇒ $f'(c) = 0$

⇒ $2c - 5 = 0$

⇒ $2c = 5$

⇒ $c = \frac{5}{2}$

⇒ $C = 2.5 \in (1,4)$

∴ Rolle's theorem is verified.

3 Q. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = \sin^4 x + \cos^4 x \text{ on } \left[0, \frac{\pi}{2}\right]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given function is:

⇒ $f(x) = \sin^4 x + \cos^4 x$ on $\left[0, \frac{\pi}{2}\right]$

We know that sine and cosine functions are continuous and differentiable functions over R.

Let's find the value of function 'f' at extremums

⇒ $f(0) = \sin^4(0) + \cos^4(0)$

⇒ $f(0) = (0)^4 + (1)^4$

⇒ $f(0) = 0 + 1$

⇒ $f(0) = 1$

$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin^4\left(\frac{\pi}{2}\right) + \cos^4\left(\frac{\pi}{2}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 1^4 + 0^4$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 1 + 0$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 1$

We got $f(0) = f\left(\frac{\pi}{2}\right)$. So, there exists a $c \epsilon \left(0, \frac{\pi}{2}\right)$ such that f'(c) = 0.

Let's find the derivative of the function 'f'.

$\Rightarrow f'(x) = \frac{d(\sin^4 x + \cos^4 x)}{dx}$

$\Rightarrow f'(x) = 4\sin^3 x\frac{d(\sin x)}{dx} + 4\cos^3 x\frac{d(\cos x)}{dx}$

$\Rightarrow f'(x) = 4\sin^3 x\cos x – 4\cos^3 x\sin x$

$\Rightarrow f'(x) = 4\sin x\cos x(\sin^2 x – \cos^2 x)$

$\Rightarrow f'(x) = 2(2\sin x\cos x)(– \cos 2x)$

$\Rightarrow f'(x) = – 2(\sin 2x)(\cos 2x)$

$\Rightarrow f'(x) = – \sin 4x$

We have f'(c) = 0

$\Rightarrow – \sin 4c = 0$

$\Rightarrow \sin 4c = 0$

$\Rightarrow 4c = 0$ or π

$\Rightarrow c = \frac{\pi}{4} \epsilon \left(0, \frac{\pi}{2}\right)$

∴ Rolle's theorem is verified.

3 R. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

f(x) = sin x – sin 2x on [0, π]

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given function is:

$\Rightarrow f(x) = \sin x – \sin 2x$ on $[0,\pi]$

We know that sine function is continuous and differentiable over R.

Let's check the values of the function 'f' at the extremums.

⇒ f(0) = sin(0)–sin2(0)

⇒ f(0) = 0 – sin(0)

⇒ f(0) = 0

⇒ f(π) = sin(π) – sin2(π)

⇒ f(π) = 0 – sin(2π)

⇒ f(π) = 0

We got f(0) = f(π). So, there exists a c∊$(0,\pi)$ such that f’(c) = 0.

Let’s find the derivative of the function ‘f’

⇒ $f'(x) = \frac{d(\sin x - \sin 2x)}{dx}$

⇒ $f'(x) = \cos x - \cos 2x \frac{d(2x)}{dx}$

⇒ $f'(x) = \cos x - 2\cos 2x$

⇒ f’(x) = cosx – 2(2cos^{2}x – 1)

⇒ f’(x) = cosx – 4cos^{2}x + 2

We have f’(c) = 0

⇒ cosc – 4cos^{2}c + 2 = 0

⇒ $\cos c = \frac{-1 \pm \sqrt{(1)^2 - (4 \times -4 \times 2)}}{2 \times -4}$

⇒ $\cos c = \frac{-1 \pm \sqrt{1 + 33}}{-8}$

⇒ $c = \cos^{-1}(\frac{-1 \pm \sqrt{33}}{-8})$

We can see that c∊(0,π)

∴ Rolle’s theorem is verified.

4. Question

Using Rolle’s theorem, find points on the curve y = 16 – x^2, x∊[– 1,1], where the tangent is parallel to the x – axis.

Answer

First, let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

⇒ y = 16 – x^2, x∊[– 1,1]

We know that polynomial function is continuous and differentiable over R.

Let’s check the values of ‘y’ at extremums

$\Rightarrow y(-1) = 16 - (-1)^2$

$\Rightarrow y(-1) = 16 - 1$

$\Rightarrow y(-1) = 15$

$\Rightarrow y(1) = 16 - (1)^2$

$\Rightarrow y(1) = 16 - 1$

$\Rightarrow y(1) = 15$

We got y(– 1) = y(1). So, there exists a $c \in (-1,1)$ such that f’(c) = 0.

We know that for a curve g, the value of the slope of the tangent at a point r is given by g’(r).

Let’s find the derivative of curve y

$\Rightarrow y' = \frac{d(16-x^2)}{dx}$

$\Rightarrow y' = -2x$

We have y’(c) = 0

$\Rightarrow -2c = 0$

$\Rightarrow c = 0 \in (-1,1)$

Value of y at x = 1 is

$\Rightarrow y = 16 - 0^2$

$\Rightarrow y = 16$

∴ The point at which the curve y has a tangent parallel to x – axis (since the slope of x – axis is 0) is (0,16).

5 A. Question

At what points on the following curves, is the tangent parallel to the x–axis?

$y = x^2$ on [–2, 2]

Answer

First, let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

$\Rightarrow y = x^2$ on [– 2,2]

We know that polynomials are continuous and differentiable over R.

Let’s check the values of y at the extremums

$\Rightarrow y(-2) = (-2)^2$

$\Rightarrow y(-2) = 4$

$\Rightarrow y(2) = (2)^2 \Rightarrow y(2) = 4$

We got y(– 2) = y(2). So, there exists a c such that f’(c) = 0.

For a curve g to have a tangent parallel to x – axis at point r, the criteria to be satisfied is g’(r) = 0.

⇒ y’(x) = 0

⇒ $\frac{d(x^2)}{dx} = 0$

⇒ 2x = 0

⇒ x = 0

The value of y is

⇒ y = $(0)^2$

⇒ y = 0

The point at which the curve has tangent parallel to x – axis is (0,0).

5 B. Question

At what points on the following curves, is the tangent parallel to the x–axis?

$y = e^{1-x^2}$ on [–1, 1]

Answer

First, let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

⇒ $y = e^{1-x^2}$on [– 1,1]

We know that exponential functions are continuous and differentiable over R.

Let’s check the values of y at the extremums

⇒ $y(-1) = e^{1-(-1)^2}$

⇒ y(– 1) = $e^{1 - 1}$

⇒ y(– 1) = e^0⇒ y(– 1) = 1

⇒ $y(1) = e^{1-1^2}$

⇒ y(1) = $e^{1 - 1}$

⇒ y(1) = e^0

⇒ y(1) = 1

We got y(– 1) = y(1). So, there exists a c such that f’(c) = 0.

For a curve g to have a tangent parallel to the x – axis at point r, the criteria to be satisfied is g’(r) = 0.

⇒ y’(x) = 0

⇒ $\frac{d(e^{1-x^2})}{dx} = 0$

$\Rightarrow e^{1-x^2}\frac{d(1-x^2)}{dx} = 0$

$\Rightarrow e^{1-x^2}(-2x) = 0$

⇒ 2x = 0

⇒ x = 0

The value of y is

$\Rightarrow y = e^{1-0^2}$

⇒ y = $e^{1 - 0}$

⇒ y = e^1

⇒ y = e

The point at which the curve has a tangent parallel to the x – axis is (0,e).

5 C. Question

At what points on the following curves, is the tangent parallel to the x–axis?

y = 12(x + 1) (x – 2) on [–1, 2]

Answer

First, let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

⇒ y = 12(x + 1)(x – 2) on [– 1,2]

We know that polynomials are continuous and differentiable over R.

Let’s check the values of y at the extremums

⇒ y(– 1) = 12(– 1 + 1)(– 1 – 2)

⇒ y(– 1) = 12(0)(– 3)

⇒ y(– 1) = 0

⇒ y(2) = 12(2 + 1)(2 – 2)⇒ y(2) = 12(3)(0)

⇒ y(2) = 0

We got y(– 1) = y(2). So, there exists a c such that f’(c) = 0.

For a curve g to have a tangent parallel to the x – axis at point r, the criteria to be satisfied is g’(r) = 0.

⇒ y’(x) = 0

$\Rightarrow \frac{d(12(x+1)(x-2))}{dx} = 0$

$\Rightarrow 12\left((x+1)\frac{d(x-2)}{dx} + (x-2)\frac{d(x+1)}{dx}\right) = 0$

⇒ ((x + 1)×1) + ((x – 2)×1) = 0

$\Rightarrow$ x + 1 + x – 2 = 0

$\Rightarrow$ 2x – 1 = 0

$\Rightarrow$ 2x = 1

$\Rightarrow x = \frac{1}{2}$

The value of y is

$\Rightarrow y = 12\left(\frac{1}{2} + 1\right)\left(\frac{1}{2} - 2\right)$

$\Rightarrow y = 12\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right)$

$\Rightarrow$ y = – 27

The point at which the curve has tangent parallel to x – axis is $\left(\frac{1}{2}, -27\right)$.

6. Question

If f:[– 5,5]→R is differentiable and if f'(x) doesn't vanish anywhere, then prove that f(– 5)≠f(5).

Answer

Given that f is continuous and differentiable in the interval [– 5,5].

It is also given that f'(x) doesn't vanish anywhere.

According to Rolle's theorem for a differentiable function on [a,b] will have atleast one c∊(a,b) such that f'(c) = 0, if the following condition had satisfied:

$\Rightarrow$ f(a) = f(b).

According to the problem it is given for any value of x, say r the values never equals to zero.

$\Rightarrow$ f'(r)≠0

This is possible when Rolle's theorem is not applicable.

Let us Recap the Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

First, two conditions are satisfied according to the problem, so the only condition that cannot be satisfied is (c).

So, we can clearly say that f(– 5)≠f(5).

7 A. Question

Examine if the Rolle's theorem applies to anyone of the following functions:

f(x) = [x] for x∊[5,9]

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f'(c) = 0.

Given function is:

⇒ f(x) = [x] for x∈[5,9]

Let us check the continuity of the function 'f'.

Here in the interval x∈[5,9], the function has to be Right continuous at x = 5 and left continuous at x = 5.

Right Hand Limit:

⇒ $\lim_{x \to 5+} f(x) = \lim_{x \to 5+} [x]$

⇒ $\lim_{x \to 5+} f(x) = \lim_{x \to 5+h} [x]$ where h>0.

⇒ $\lim_{x \to 5+} f(x) = \lim_{h \to 0} 5$

⇒ $\lim_{x \to 5+} f(x) = 5$ ......(1)

Left Hand Limit:

⇒ $\lim_{x \to 5-} f(x) = \lim_{x \to 5-} [x]$

⇒ $\lim_{x \to 5-} f(x) = \lim_{x \to 5-h} [x]$, where h>0

⇒ $\lim_{x \to 5-} f(x) = \lim_{h \to 0} 4$

⇒ $\lim_{x \to 5-} f(x) = 4$ ......(2)

From (1) and (2), we can clearly see that the limits are not same so, the function is not continuous in the interval [5,9].

∴ Rolle's theorem is not applicable for the function f in the interval [5,9].

7 B. Question

Examine if the Rolle's theorem applies to anyone of the following functions:

f(x) = [x] for x∈[– 2,2]

Answer

Given function is:

⇒ f(x) = [x] for x∈[– 2,2]

Let us check the continuity of the function 'f'.

Here in the interval x∈[– 2,1], the function has to be Right continuous at x = 2 and left continuous at x = 2.

Right Hand Limit:

⇒ $\lim_{x \to 2+} f(x) = \lim_{x \to 2+} [x]$

⇒ $\lim_{x \to 2+} f(x) = \lim_{x \to 2+h} [x]$ where h>0.

⇒ $\lim_{x \to 2+} f(x) = \lim_{h \to 0} 2$

⇒ $\lim_{x \to 2+} f(x) = 2$ ......(1)

Left Hand Limit:

⇒ $\lim_{x \to 2-} f(x) = \lim_{x \to 2-} [x]$

$\Rightarrow \lim\limits_{x \to 2-} f(x) = \lim\limits_{x \to 2-h} [x]$, where h>0

$\Rightarrow \lim\limits_{x \to 2-} f(x) = \lim\limits_{h \to 0} 1$

$\Rightarrow \lim\limits_{x \to 2-} f(x) = 1$ ......(2)

From (1) and (2), we can see that the limits are not the same so, the function is not continuous in the interval [– 2,2].

∴ Rolle’s theorem is not applicable for the function f in the interval [– 2,2].

8. Question

It is given that the Rolle’s theorem holds for the function f(x) = $x^3 + bx^2 + cx$, x∈[1,2] at the point x = 4/3. Find the values of b and c.

Answer

First, let us write the conditions for the applicability of Rolle’s theorem:

For a Real valued function ‘f’:

a) The function ‘f’ needs to be continuous in the closed interval [a,b].

b) The function ‘f’ needs differentiable on the open interval (a,b).

c) f(a) = f(b)

Then there exists at least one c in the open interval (a,b) such that f’(c) = 0.

Given function is:

⇒ f(x) = $x^3 + bx^2 + cx$, x∈[1,2]

According to the problem the Rolle’s theorem holds for the function ‘f’ at $x = \frac{4}{3}$.

We can say that $f'\left(\frac{4}{3}\right) = 0$.

Let’s find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(x^3 + bx^2 + cx)}{dx}$

$\Rightarrow f'(x) = \frac{d(x^3)}{dx} + \frac{d(bx^2)}{dx} + \frac{d(cx)}{dx}$

⇒ f’(x) = $3x^2 + 2bx + c$

We have $f'\left(\frac{4}{3}\right) = 0$

$\Rightarrow 3\left(\frac{4}{3}\right)^2 + 2b\left(\frac{4}{3}\right) + c = 0$

$\Rightarrow 3\left(\frac{16}{9}\right) + b\left(\frac{8}{3}\right) + c = 0$

$\Rightarrow \frac{16}{3} + \frac{8b}{3} + c = 0$

⇒ 8b + 3c = – 16 ...... (1)

We also have f(1) = f(2)

⇒ $(1)^3 + b(1)^2 + c(1) = (2)^3 + b(2)^2 + c(2)$

⇒ 1 + b(1) + c = 8 + b(4) + 2c

⇒ 3b + c = – 7 ......(2)

On solving (1) and (2), we get

⇒ $b = -5$ and $c = 8$

∴ The values of b and c is – 5 and 8.

Exercise 15.2

1 A. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$f(x) = x^2 - 1$ on [2, 3]

Answer

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$f(x) = x^2 - 1$ on [2, 3]

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [2, 3] and differentiable in (2, 3). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point c∈(2, 3) such that:

$$f'(c) = \frac{f(3) - f(2)}{3 - 2}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(2)}{1}$$

$f(x) = x^2 - 1$

Differentiating with respect to x:

$f'(x) = 2x$

For f'(c), put the value of x=c in f'(x):

$f'(c)= 2c$

For f(3), put the value of x=3 in f(x):

$f(3)= (3)^2 - 1$

$= 9 - 1$

$= 8$

For f(2), put the value of x=2 in f(x):

$f(2)= (2)^2 - 1$

$= 4 - 1$

$= 3$

∴ $f'(c) = f(3) - f(2)$

⇒ $2c = 8 - 3$

$\Rightarrow 2c = 5$

$\Rightarrow c = \frac{5}{2} \in (2, 3)$

Hence, Lagrange’s mean value theorem is verified.

1 B. Question

Verify Lagrange’s mean value theorem for the following functions on the indicated intervals. In each find a point ‘c’ in the indicated interval as stated by the Lagrange’s mean value theorem :

$f(x) = x^3 - 2x^2 - x + 3$ on [0, 1]

Answer

Lagrange’s mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$

This theorem is also known as First Mean Value Theorem.

$f(x) = x^3 - 2x^2 - x + 3$ on [0, 1]

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [0, 1] and differentiable in (0, 1). So both the necessary conditions of Lagrange’s mean value theorem is satisfied.

Therefore, there exists a point $c \in (0, 1)$ such that:

$f'(c) = \frac{f(1) - f(0)}{1 - 0}$

$\Rightarrow f'(c) = \frac{f(1) - f(0)}{1}$

$f(x) = x^3 - 2x^2 - x + 3$

Differentiating with respect to x:

$f'(x) = 3x^2 - 2(2x) - 1$

$= 3x^2 - 4x - 1$

For f’(c), put the value of x=c in f’(x):

$f'(c)= 3c^2 - 4c - 1$

For f(1), put the value of x=1 in f(x):

$f(1)= (1)^3 - 2(1)^2 - (1) + 3$

$= 1 - 2 - 1 + 3$

$= 1$

For f(0), put the value of x=0 in f(x):

$f(0)= (0)^3 - 2(0)^2 - (0) + 3$

$= 0 - 0 - 0 + 3$

$= 3$

$\therefore f'(c) = f(1) - f(0)$

$\Rightarrow 3c^2 - 4c - 1 = 1 - 3$

$\Rightarrow 3c^2 - 4c = 1 + 1 - 3$

$\Rightarrow 3c^2 - 4c = -1$

$\Rightarrow 3c^2 - 4c + 1 = 0$

$\Rightarrow 3c^2 - 3c - c + 1 = 0$

$\Rightarrow 3c(c - 1) - 1(c - 1) = 0$

$\Rightarrow (3c - 1)(c - 1) = 0$

$\Rightarrow c = \frac{1}{3}, 1$

$\Rightarrow c = \frac{1}{3} \in (0, 1)$

Hence, Lagrange's mean value theorem is verified.

1 C. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$f(x) = x(x - 1)$ on $[1, 2]$

Answer

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$

This theorem is also known as First Mean Value Theorem.

$f(x) = x(x - 1)$ on $[1, 2]$

$= x^2 - x$

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [1, 2] and differentiable in (1, 2). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (1, 2)$ such that:

$f'(c) = \frac{f(2) - f(1)}{2 - 1}$

$\Rightarrow f'(c) = \frac{f(2) - f(1)}{1}$

$f(x) = x^2 - x$

Differentiating with respect to x:

$f'(x) = 2x - 1$

For f'(c), put the value of x=c in f'(x):

$f'(c) = 2c - 1$

For f(2), put the value of x=2 in f(x):

$f(2) = (2)^2 - 2$

$= 4 – 2$

$= 2$

For f(1), put the value of x=1 in f(x):

$f(1)= (1)^2 – 1$

$= 1 – 1$

$= 0$

$\therefore f'(c) = f(2) – f(1)$

$\Rightarrow 2c – 1 = 2 – 0$

$\Rightarrow 2c = 2 + 1$

$\Rightarrow 2c = 3$

$$\Rightarrow c = \frac{3}{2} \in (1, 2)$$

Hence, Lagrange's mean value theorem is verified.

1 D. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$f(x) = x^2 – 3x + 2$ on $[– 1, 2]$

Answer

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)−f(a)=f'(c)(b−a)$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$f(x) = x^2 – 3x + 2$ on $[– 1, 2]$

Every polynomial function is **continuous** everywhere on $(−\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [– 1, 2] and differentiable in (– 1, 2). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (- 1, 2)$ such that:

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(-1)}{2 + 1}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(-1)}{3}$$

$f(x) = x^2 – 3x + 2$

Differentiating with respect to x:

$f'(x) = 2x – 3$

For f'(c), put the value of x=c in f'(x):

$f'(c)= 2c – 3$

For f(2), put the value of x=2 in f(x):

$f(2)= (2)^2 – 3(2) + 2$

$= 4 – 6 + 2$

$= 0$

For f(– 1), put the value of x= – 1 in f(x):

$f(– 1) = (– 1)^2 – 3(– 1) + 2$

$= 1 + 3 + 2$

$= 6$

$$f'(c)=\frac{f(2)-f(-1)}{3}$$

$$\Rightarrow 2c-3=\frac{0-6}{3}$$

$$\Rightarrow 2c=\frac{-6}{3}+3$$

$\Rightarrow 2c = – 2 + 3$

$\Rightarrow 2c= – 1$

$$\Rightarrow c=\frac{-1}{2}\in(-1,2)$$

Hence, Lagrange’s mean value theorem is verified.

1 E. Question

Verify Lagrange’s mean value theorem for the following functions on the indicated intervals. In each find a point ‘c’ in the indicated interval as stated by the Lagrange’s mean value theorem :

$f(x) = 2x^2 – 3x + 1$ on [1, 3]

Answer

Lagrange’s mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)−f(a)=f′(c)(b−a)$

$$\Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a}$$

This theorem is also known as First Mean Value Theorem.

$f(x) = 2x^2 – 3x + 1$ on [1, 3]

Every polynomial function is **continuous** everywhere on $(−\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [1, 3] and differentiable in (1, 3). So both the necessary conditions of Lagrange’s mean value theorem is satisfied.

Therefore, there exists a point $c\in(1, 3)$ such that:

$$f'(c)=\frac{f(3)-f(1)}{3-1}$$

$$\Rightarrow f'(c)=\frac{f(3)-f(1)}{2}$$

$f(x) = 2x^2 – 3x + 1$

Differentiating with respect to x:

$f'(x) = 2(2x) – 3$

$= 4x – 3$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$f'(c)= 4c – 3$

For $f(3)$, put the value of $x=3$ in $f(x)$:

$f(3)= 2(3)^2 – 3(3) + 1$

$= 2(9) – 9 + 1$

$= 18 – 8 = 10$

For $f(1)$, put the value of $x=1$ in $f(x)$:

$f(1) = 2(1)^2 – 3(1) + 1$

$= 2(1) – 3 + 1$

$= 2 – 2 = 0$

$$f'(c) = \frac{f(3) - f(1)}{2}$$

$$\Rightarrow 4c - 3 = \frac{10 - 0}{2}$$

$$\Rightarrow 4c = \frac{10}{2} + 3$$

$\Rightarrow 4c = 5 + 3$

$\Rightarrow 4c= 8$

$$\Rightarrow c = \frac{8}{4} = 2 \in (1, 3)$$

Hence, Lagrange’s mean value theorem is verified.

1 F. Question

Verify Lagrange’s mean value theorem for the following functions on the indicated intervals. In each find a point ‘c’ in the indicated interval as stated by the Lagrange’s mean value theorem :

$f(x) = x^2 – 2x + 4$ on $[1, 5]$

Answer

Lagrange’s mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)−f(a)=f'(c)(b−a)$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$f(x) = x^2 – 2x + 4$ on $[1, 5]$

Every polynomial function is **continuous** everywhere on $(−\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [1, 5] and differentiable in (1, 5). So both the necessary conditions of Lagrange’s mean value theorem is satisfied.

Therefore, there exists a point $c \in (1, 5)$ such that:

$$f'(c)=\frac{f(5)\ -\ f(1)}{5-1}$$

$$\Rightarrow f'(c)=\frac{f(5)\ -\ f(1)}{4}$$

$f(x) = x^2 – 2x + 4$

Differentiating with respect to x:

$f'(x) = 2x – 2$

For f'(c), put the value of x=c in f'(x):

$f'(c)= 2c – 2$

For f(5), put the value of x=5 in f(x):

$f(5)= (5)^2 – 2(5) + 4$

$= 25 – 10 + 4$

$= 19$

For f(1), put the value of x=1 in f(x):

$f(1) = (1)^2 – 2(1) + 4$

$= 1 – 2 + 4$

$= 3$

$$f'(c)=\frac{f(5)\ -\ f(1)}{4}$$

$$\Rightarrow 2c-2=\frac{19-3}{4}$$

$$\Rightarrow 2c=\frac{16}{4}+2$$

$\Rightarrow 2c = 4 + 2$

$\Rightarrow 2c= 6$

$$\Rightarrow c=\frac{6}{2}=3\in(1,\,5)$$

Hence, Lagrange's mean value theorem is verified.

1 G. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$f(x) = 2x – x^2$ on [0, 1]

Answer

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$$\Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a}$$

This theorem is also known as First Mean Value Theorem.

$f(x) = 2x – x^2$ on [0, 1]

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [0, 1] and differentiable in (0, 1). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (0, 1)$ such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$\Rightarrow f'(c) = f(1) - f(0)$

$f(x) = 2x - x^2$

Differentiating with respect to x:

$f'(x) = 2 - 2x$

For f'(c), put the value of x=c in f'(x):

$f'(c) = 2 - 2c$

For f(1), put the value of x=1 in f(x):

$f(1) = 2(1) - (1)^2$

$= 2 - 1$

$= 1$

For f(0), put the value of x=0 in f(x):

$f(0) = 2(0) - (0)^2$

$= 0 - 0$

$= 0$

$f'(c) = f(1) - f(0)$

$\Rightarrow 2 - 2c = 1 - 0$

$\Rightarrow -2c = 1 - 2$

$\Rightarrow -2c = -1$

$$\Rightarrow c = \frac{-1}{-2} = \frac{1}{2} \in (0, 1)$$

Hence, Lagrange's mean value theorem is verified.

1 H. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$f(x) = (x - 1)(x - 2)(x - 3)$ on [0, 4]

Answer

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b) - f(a) = f'(c)(b - a)$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$f(x) = (x - 1)(x - 2)(x - 3)$ on [0, 4]

$= (x^2 – x – 2x + 3) (x – 3)$

$= (x^2 – 3x + 3) (x – 3)$

$= x^3 – 3x^2 + 3x – 3x^2 + 9x – 9$

$= x^3 – 6x^2 + 12x – 9$ on [0, 4]

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [0, 4] and differentiable in (0, 4). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (0, 4)$ such that:

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$$

$f(x) = x^3 – 6x^2 + 12x – 9$

Differentiating with respect to x:

$f'(x) = 3x^2 – 6(2x) + 12$

$= 3x^2 – 12x + 12$

For f'(c), put the value of x=c in f'(x):

$f'(c)= 3c^2 – 12c + 12$

For f(4), put the value of x=4 in f(x):

$f(4)= (4)^3 – 6(4)^2 + 12(4) – 9$

$= 64 – 96 + 48 – 9$

$= 7$

For f(0), put the value of x=0 in f(x):

$f(0)= (0)^3 – 6(0)^2 + 12(0) – 9$

$= 0 – 0 + 0 – 9$

$= – 9$

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$$\Rightarrow 3c^2 - 12c + 12 = \frac{7 - (-9)}{4}$$

$$\Rightarrow 3c^2 - 12c + 12 = \frac{7 + 9}{4}$$

$$\Rightarrow 3c^2 - 12c + 12 = \frac{16}{4}$$

$\Rightarrow 3c^2 – 12c + 12 = 4$

$\Rightarrow 3c^2 – 12c + 8 = 0$

For quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow c = \frac{-(-12) \pm \sqrt{(-12)^2 - 4\times3\times8}}{2\times3}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{48}}{6}$$

$$\Rightarrow c = \frac{12 \pm 4\sqrt{3}}{6}$$

$$\Rightarrow c = \frac{12}{6} \pm \frac{4\sqrt{3}}{6}$$

$$\Rightarrow c = 2 \pm \frac{2\sqrt{3}}{3}$$

$$\Rightarrow c = 2 + \frac{2\sqrt{3}}{3}, 2 - \frac{2\sqrt{3}}{3} \in c$$

Hence, Lagrange’s mean value theorem is verified.

1 I. Question

Verify Lagrange’s mean value theorem for the following functions on the indicated intervals. In each find a point ‘c’ in the indicated interval as stated by the Lagrange’s mean value theorem :

$\sqrt{25 - x^2}$ on [- 3, 4]

Answer

Lagrange’s mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$f(x) = \sqrt{25 - x^2}$ on [– 3, 4]

Here,

$\sqrt{25 - x^2} > 0$

$\Rightarrow 25 - x^2 > 0$

$\Rightarrow x^2 < 25$

$\Rightarrow -5 < x < 5$

$\Rightarrow \sqrt{25 - x^2}$ has unique values for all $x \in (-5, 5)$

∴ f(x) is continuous in [– 3, 4]

$$f(x) = (25 - x^2)^{\frac{1}{2}}$$

Differentiating with respect to x:

$$f'(x) = \frac{1}{2}(25 - x^2)^{\left(\frac{1}{2} - 1\right)} \frac{d(25 - x^2)}{dx}$$

$$\Rightarrow f'(x)=\frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)$$

$$\Rightarrow f'(x)=\frac{-2x}{2(25-x^2)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x)=\frac{-2x}{2(25-x^2)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x)=\frac{-x}{\sqrt{25-x^2}}$$

Here also,

$$\sqrt{25-x^2}>0$$

$\Rightarrow -5 < x < 5$

$\therefore$ f(x) is differentiable in (– 3, 4)

So both the necessary conditions of Lagrange’s mean value theorem is satisfied.

Therefore, there exists a point $c\in(-3, 4)$ such that:

$$f'(c)=\frac{f(4)-f(-3)}{4-(-3)}$$

$$\Rightarrow f'(c)=\frac{f(4)-f(-3)}{4+3}$$

$$\Rightarrow f'(c)=\frac{f(4)-f(-3)}{7}$$

$$f(x)=(25-x^2)^{\frac{1}{2}}$$

On differentiating with respect to x:

$$f'(x)=\frac{-x}{\sqrt{25-x^2}}$$

For f’(c), put the value of x=c in f’(x):

$$f'(c)=\frac{-c}{\sqrt{25-c^2}}$$

For f(4), put the value of x=4 in f(x):

$$f(4)=\left(25-4^2\right)^{\frac{1}{2}}$$

$$\Rightarrow f(4)=(25-16)^{\frac{1}{2}}$$

$$\Rightarrow f(4)=(9)^{\frac{1}{2}}$$

$\Rightarrow$ f(4) = 3

For f(– 3), put the value of x= – 3 in f(x):

$$f(-3)=\left(25-(-3)^2\right)^{\frac{1}{2}}$$

$$\Rightarrow f(-3)=(25-9)^{\frac{1}{2}}$$

$\Rightarrow f(-3)=(16)^{\frac{1}{2}}$

⇒ f(– 3) = 4

$f'(c)=\frac{f(4)-f(-3)}{7}$

$\Rightarrow \frac{-c}{\sqrt{25-c^2}}=\frac{3-4}{7}$

$\Rightarrow \frac{-c}{\sqrt{25-c^2}}=\frac{-1}{7}$

$\Rightarrow -7c=-\sqrt{25-c^2}$

Squaring both sides:

$\Rightarrow (-7c)^2=\left(-\sqrt{25-c^2}\right)^2$

$\Rightarrow 49c^2 = 25 - c^2$

$\Rightarrow 50c^2 = 25$

$\Rightarrow c^2=\frac{25}{50}$

$\Rightarrow c^2=\frac{1}{2}$

$\Rightarrow c=\pm\frac{1}{\sqrt{2}}\in(-3,4)$

Hence, Lagrange’s mean value theorem is verified.

1 J. Question

Verify Lagrange’s mean value theorem for the following functions on the indicated intervals. In each find a point ‘c’ in the indicated interval as stated by the Lagrange’s mean value theorem :

$f(x) = \tan^{-1} x$ on [0, 1]

Answer

Lagrange’s mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$\Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a}$

This theorem is also known as First Mean Value Theorem.

$f(x) = \tan^{-1} x$ on [0, 1]

$\tan^{-1} x$ has unique value for all x between 0 and 1.

∴ f(x) is continuous in [0, 1]

$f(x) = \tan^{-1} x$

Differentiating with respect to x:

$f'(x)=\frac{1}{1+x^2}$

x^2 always has value greater than 0.

⇒ $1 + x^2 > 0$

∴ f(x) is differentiable in (0, 1)

So both the necessary conditions of Lagrange’s mean value theorem is satisfied.

Therefore, there exists a point $c \in (-3, 4)$ such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

⇒ $f'(c) = f(1) - f(0)$

$f(x) = \tan^{-1} x$

Differentiating with respect to x:

$$f'(x) = \frac{1}{1+x^2}$$

For f’(c), put the value of x=c in f’(x):

$$f'(c) = \frac{1}{1+c^2}$$

For f(1), put the value of x=1 in f(x):

$f(1) = \tan^{-1} 1$

$$\Rightarrow f(1) = \frac{\pi}{4}$$

For f(0), put the value of x=0 in f(x):

$f(0) = \tan^{-1} 0$

⇒ $f(0) = 0$

$f'(c) = f(1) - f(0)$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\pi}{4} - 0$$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\pi}{4}$$

$$\Rightarrow 4 = \pi(1+c^2)$$

$$\Rightarrow 4 = \pi + \pi c^2$$

$$\Rightarrow -\pi c^2 = \pi - 4$$

$$\Rightarrow c^2 = \frac{\pi - 4}{-\pi}$$

$$\Rightarrow c^2 = \frac{4 - \pi}{\pi}$$

$$\Rightarrow c = \sqrt{\frac{4}{\pi} - 1} \approx 0.52 \in (0, 1)$$

Hence, Lagrange’s mean value theorem is verified.

1 K. Question

Verify Lagrange’s mean value theorem for the following functions on the indicated intervals. In each find a point ‘c’ in the indicated interval as stated by the Lagrange’s mean value theorem :

$f(x)=x+\frac{1}{x}$ on $[1, 3]$

Answer

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$\Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a}$

This theorem is also known as First Mean Value Theorem.

$f(x)=x+\frac{1}{x}$ on $[1, 3]$

f(x) has unique values for all $x\in(1, 3)$

∴ f(x) is continuous in [1, 3]

$f(x)=x+\frac{1}{x}$ on $[1, 3]$

Differentiating with respect to x:

$f'(x)=1+(-1)(x)^{-2}$

$\Rightarrow f'(x)=1-\frac{1}{x^2}$

$\Rightarrow f'(x)=\frac{x^2-1}{x^2}$

Here,

$x^2\neq0$

⇒ f'(x) exists for all values except 0

∴ f(x) is differentiable in (1, 3)

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c\in(1, 3)$ such that:

$f'(c)=\frac{f(3)-f(1)}{3-1}$

$\Rightarrow f'(c)=\frac{f(3)-f(1)}{2}$

$f(x)=x+\frac{1}{x}$

On differentiating with respect to x:

$f'(x)=\frac{x^2-1}{x^2}$

For f'(c), put the value of x=c in f'(x):

$f'(c)=\frac{c^2-1}{c^2}$

For f(3), put the value of x=3 in f(x):

$$f(3)=3+\frac{1}{3}$$

$$\Rightarrow f(3)=\frac{9+1}{3}$$

$$\Rightarrow f(3)=\frac{10}{3}$$

For f(1), put the value of x=1 in f(x):

$$f(1)=1+\frac{1}{1}$$

⇒ f(1) = 2

$$\Rightarrow f'(c)=\frac{f(3)\ -\ f(1)}{2}$$

$$\Rightarrow \frac{c^2-1}{c^2}=\frac{\frac{10}{3}-2}{2}$$

$$\Rightarrow 2(c^2-1)=\ c^2\left(\frac{10}{3}-2\right)$$

$$\Rightarrow 2(c^2-1)=\ c^2\left(\frac{10-6}{3}\right)$$

$$\Rightarrow 2(c^2-1)=\ c^2\left(\frac{4}{3}\right)$$

⇒ $6(c^2 - 1) = 4c^2$

⇒ $6c^2 - 6 = 4c^2$

⇒ $6c^2 - 4c^2 = 6$

⇒ $2c^2 = 6$

$$\Rightarrow c^2=\frac{6}{2}$$

⇒ $c^2 = 3$

$$\Rightarrow c=\pm\sqrt{3}\in(-3,4)$$

Hence, Lagrange’s mean value theorem is verified.

1 L. Question

Verify Lagrange’s mean value theorem for the following functions on the indicated intervals. In each find a point ‘c’ in the indicated interval as stated by the Lagrange’s mean value theorem :

$f(x) = x (x + 4)^2$ on [0, 4]

Answer

Lagrange’s mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$$\Rightarrow\ f'(c)=\frac{f(b)-f(a)}{b-a}$$

This theorem is also known as First Mean Value Theorem.

$f(x) = x (x + 4)^2$ on [0, 4]

$= x\ [(x)^2+2(4)(x)+(4)^2]$

$= x(x^2 + 8x + 16)$

$= x^3 + 8x^2 + 16x$ on [0, 4]

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [0, 4] and differentiable in (0, 4). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (0, 4)$ such that:

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$$

$f(x) = x^3 + 8x^2 + 16x$

Differentiating with respect to x:

$f'(x) = 3x^2 + 8(2x) + 16$

$= 3x^2 + 16x + 16$

For f'(c), put the value of x=c in f'(x):

$f'(c) = 3c^2 + 16c + 16$

For f(4), put the value of x=4 in f(x):

$f(4) = (4)^3 + 8(4)^2 + 16(4)$

$= 64 + 128 + 64$

$= 256$

For f(0), put the value of x=0 in f(x):

$f(0) = (0)^3 + 8(0)^2 + 16(0)$

$= 0 + 0 + 0$

$= 0$

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$$\Rightarrow 3c^2 + 16c + 16 = \frac{256 - 0}{4}$$

$$\Rightarrow 3c^2 + 16c + 16 = \frac{256}{4}$$

$\Rightarrow 3c^2 + 16c + 16 = 64$

$\Rightarrow 3c^2 + 16c + 16 - 64 = 0$

$\Rightarrow 3c^2 + 16c - 48 = 0$

For quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow c = \frac{-(16) \pm \sqrt{(16)^2 - 4 \times 3 \times (-48)}}{2 \times 3}$$

$$\Rightarrow c = \frac{-16 \pm \sqrt{256+576}}{6}$$

$$\Rightarrow c = \frac{-16 \pm \sqrt{832}}{6}$$

$$\Rightarrow c = \frac{-16 \pm 8\sqrt{13}}{6}$$

$$\Rightarrow c = \frac{-16}{6} \pm \frac{8\sqrt{13}}{6}$$

$$\Rightarrow c = \frac{-8}{3} \pm \frac{4\sqrt{13}}{3}$$

$$\Rightarrow c = \frac{-8}{3} + \frac{4\sqrt{13}}{3}, \frac{-8}{3} - \frac{4\sqrt{13}}{3} \in c$$

Hence, Lagrange's mean value theorem is verified.

1 M. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = x\sqrt{x^2 - 4} \text{ on } [2, 4]$$

Answer

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

f(b)−f(a)=f′(c)(b−a)

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$f(x) = \sqrt{x^2 - 4}$ on [2, 4]

Here,

$\sqrt{x^2 - 4} > 0$

$\Rightarrow x^2 - 4 > 0$

$\Rightarrow x^2 > 4$

⇒ f(x) exists for all values expect (– 2, 2)

∴ f(x) is continuous in [2, 4]

$f(x) = \sqrt{x^2 - 4}$

Differentiating with respect to x:

$$f'(x) = \frac{1}{2}(x^2 - 4)^{\left(\frac{1}{2} - 1\right)} \frac{d(x^2 - 4)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{2}(x^2 - 4)^{-\frac{1}{2}}(2x)$$

$$\Rightarrow f'(x) = \frac{2x}{2\,(x^2 - 4)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x) = \frac{x}{\sqrt{x^2 - 4}}$$

Here also,

$\sqrt{x^2 - 4} > 0$

⇒ f'(x) exists for all values of x except (2, – 2)

∴ f(x) is differentiable in (2, 4)

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point c∈(2, 4) such that:

$$f'(c) = \frac{f(4) - f(2)}{4 - 2}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(2)}{2}$$

$$f(x) = \sqrt{x^2 - 4}$$

On differentiating with respect to x:

$$f'(x) = \frac{x}{\sqrt{x^2 - 4}}$$

For f'(c), put the value of x=c in f'(x):

$$f'(c) = \frac{c}{\sqrt{c^2 - 4}}$$

For f(4), put the value of x=4 in f(x):

$$f(4) = \sqrt{4^2 - 4}$$

$$\Rightarrow f(4) = (16 - 4)^{\frac{1}{2}}$$

$$\Rightarrow f(4) = \sqrt{12}$$

$$\Rightarrow f(4) = 2\sqrt{3}$$

For f(2), put the value of x=2 in f(x):

$$f(2) = \sqrt{2^2 - 4}$$

$$\Rightarrow f(2) = (4 - 4)^{\frac{1}{2}}$$

$$\Rightarrow f(2) = 0$$

$$\Rightarrow f'(c) = \frac{f(4) - f(2)}{2}$$

$$\Rightarrow \frac{c}{\sqrt{c^2 - 4}} = \frac{2\sqrt{3} - 0}{2}$$

$\Rightarrow \frac{c}{\sqrt{c^2-4}} = \sqrt{3}$

$\Rightarrow c = (\sqrt{3})\sqrt{c^2-4}$

Squaring both sides:

$\Rightarrow (c)^2 = \left((\sqrt{3})\sqrt{c^2-4}\right)^2$

$\Rightarrow c^2 = 3(c^2 - 4)$

$\Rightarrow c^2 = 3c^2 - 12$

$\Rightarrow - 2c^2 = - 12$

$\Rightarrow c^2 = \frac{-12}{-2}$

$\Rightarrow c^2 = 6$

$\Rightarrow c = \pm\sqrt{6}$

$\Rightarrow c = \sqrt{6} \in (2, 4)$

Hence, Lagrange’s mean value theorem is verified.

1 N. Question

Verify Lagrange’s mean value theorem for the following functions on the indicated intervals. In each find a point ‘c’ in the indicated interval as stated by the Lagrange’s mean value theorem :

$f(x) = x^2 + x - 1$ on [0, 4]

Answer

Lagrange’s mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$

This theorem is also known as First Mean Value Theorem.

$f(x) = x^2 + x - 1$ on [0, 4]

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [0, 4] and differentiable in (0, 4). So both the necessary conditions of Lagrange’s mean value theorem is satisfied.

Therefore, there exists a point $c \in (0, 4)$ such that:

$f'(c) = \frac{f(4) - f(0)}{4-0}$

$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$

$f(x) = x^2 + x - 1$

Differentiating with respect to x:

$f'(x) = 2x + 1$

For f’(c), put the value of x=c in f’(x):

$f'(c) = 2c + 1$

For f(4), put the value of x=4 in f(x):

$f(4) = (4)^2 + 4 - 1$

$= 16 + 4 - 1$

$= 19$

For f(0), put the value of x=0 in f(x):

$f(0) = (0)^2 + 0 - 1$

$= 0 + 0 - 1$

$= -1$

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$$\Rightarrow 2c + 1 = \frac{19 - (-1)}{4}$$

$$\Rightarrow 2c + 1 = \frac{20}{4}$$

$\Rightarrow 2c + 1 = 5$

$\Rightarrow 2c = 5 - 1$

$\Rightarrow 2c = 4$

$$\Rightarrow c = \frac{4}{2} = 2 \in (0, 4)$$

Hence, Lagrange's mean value theorem is verified.

1 O. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$f(x) = \sin x - \sin 2x - x$ on $[0, \pi]$

Answer

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b) - f(a) = f'(c)(b - a)$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$f(x) = \sin x - \sin 2x - x$ on $[0, \pi]$

sin x and cos x functions are **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (0, \pi)$ such that:

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$$

$$\Rightarrow f'(c) = \frac{f(\pi) - f(0)}{\pi}$$

$f(x) = \sin x - \sin 2x - x$

Differentiating with respect to x:

$f(x) = \sin x - \sin 2x - x$

$\Rightarrow f'(x) = \cos x - \cos 2x \frac{d(2x)}{dx} - 1$

$\Rightarrow f'(x) = \cos x - 2\cos 2x - 1$

For f'(c), put the value of x=c in f'(x):

$f'(c) = \cos c - 2\cos 2c - 1$

For f(π), put the value of x=π in f(x):

$f(\pi) = \sin \pi - \sin 2\pi - \pi$

$= 0 - 0 - \pi$

$= -\pi$

For f(0), put the value of x=0 in f(x):

$f(0) = \sin 0 - \sin 2(0) - 0$

$= \sin 0 - \sin 0 - 0$

$= 0 - 0 - 0$

$= 0$

$f'(c) = \frac{f(\pi) - f(0)}{\pi}$

$\Rightarrow \cos c - 2\cos 2c - 1 = \frac{-\pi - 0}{\pi}$

$\Rightarrow \cos c - 2\cos 2c - 1 = -1$

$\Rightarrow \cos c - 2(2\cos^2 c - 1) = -1 + 1$

$\Rightarrow \cos c - 4\cos^2 c + 2 = 0$

$\Rightarrow 4\cos^2 c - \cos c - 2 = 0$

For quadratic equation, $ax^2 + bx + c = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow \cos c = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 4 \times (-2)}}{2 \times 4}$

$\Rightarrow \cos c = \frac{1 \pm \sqrt{1 + 32}}{8}$

$\Rightarrow \cos c = \frac{1 \pm \sqrt{33}}{8}$

$\Rightarrow c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right) \in (0, \pi)$

Hence, Lagrange's mean value theorem is verified.

1 P. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$f(x) = x^3 – 5x^2 – 3x$ on [1, 3]

Answer

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$f(x) = x^3 – 5x^2 – 3x$ on [1, 3]

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [1, 3] and differentiable in (1, 3). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (1, 3)$ such that:

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(1)}{2}$$

$f(x) = x^3 – 5x^2 – 3x$

Differentiating with respect to x:

$f'(x) = 3x^2 – 5(2x) – 3$

$= 3x^2 – 10x – 3$

For f'(c), put the value of x=c in f'(x):

$f'(c)= 3c^2 – 10c – 3$

For f(3), put the value of x=3 in f(x):

$f(3)= (3)^3 – 5(3)^2 – 3(3)$

$= 27 – 45 – 9$

$= – 27$

For f(1), put the value of x=1 in f(x):

$f(1)= (1)^3 – 5(1)^2 – 3(1)$

$= 1 – 5 – 3$

$= – 7$

$$f'(c) = \frac{f(3) - f(1)}{2}$$

$$\Rightarrow 3c^2 - 10c - 3 = \frac{(-27) - (-7)}{2}$$

$$\Rightarrow 3c^2 - 10c - 3 = \frac{-27+7}{2}$$

$\Rightarrow 3c^2 - 10c - 3 = \frac{-20}{2}$

$\Rightarrow 3c^2 - 10c - 3 = -10$

$\Rightarrow 3c^2 - 10c - 3 + 10 = 0$

$\Rightarrow 3c^2 - 10c + 7 = 0$

$\Rightarrow 3c^2 - 7c - 3c + 7 = 0$

$\Rightarrow c(3c - 7) - 1(3c - 7) = 0$

$\Rightarrow (3c - 7)(c - 1) = 0$

$\Rightarrow c = \frac{7}{3}, 1$

$\Rightarrow c = \frac{7}{3} \in (1, 3)$

Hence, Lagrange's mean value theorem is verified.

2. Question

Discuss the applicability of Lagrange's mean value theorem for the function f(x) = |x| on [– 1, 1].

Answer

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$

This theorem is also known as First Mean Value Theorem.

f(x) = |x| on [– 1, 1]

So f(x) can be defined as $= \begin{cases} -x, & x<0 \\ x, & x\geq 0 \end{cases}$

For differentiability at x=0,

LHD $= \lim_{x\to 0^-} \frac{f(0-h) - f(0)}{-h}$

{Since f(x)= – x, x<0}

$= \lim_{x\to 0^-} \frac{-(0-h) - 0}{-h}$

$= \lim_{x\to 0^-} \frac{h - 0}{-h}$

$= \lim_{x\to 0^-} \frac{h}{-h}$

$= -1$

RHD $= \lim_{x\to 0^+} \frac{f(0-h) - f(0)}{-h}$

{Since f(x)= x, x>0}

$= \lim_{x\to 0^-} \frac{(0-h) - 0}{-h}$

$$= \lim_{x \to 0^-} \frac{-h-0}{-h}$$

$$= \lim_{x \to 0^-} \frac{-h}{-h}$$

= 1

LHD$\neq$RHD

⇒ f(x) is not differential at x=0

∴ Lagrange’s mean value theorem is not applicable for the function f(x) = |x| on [– 1, 1].

3. Question

Show that the Lagrange's mean value theorem is not applicable to the function f(x) = 1/x on [–1, 1].

Answer

Lagrange’s mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

f(b)−f(a)=f′(c)(b−a)

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

This theorem is also known as First Mean Value Theorem.

$f(x) = \frac{1}{x}$ on $[-1, 1]$

Here,

$x \neq 0$

⇒ f(x) exists for all values of x except 0

⇒ f(x) is discontinuous at x=0

∴ f(x) is not continuous in [– 1, 1]

Hence the lagrange’s mean value theorem is not applicable to the

function $f(x) = \frac{1}{x}$ on $[-1, 1]$

4. Question

"Verify the hypothesis and conclusion of Lagrange’s mean value theorem for the function

$$f(x) = \frac{1}{4x-1}, 1 < x < 4.$$

Answer

Lagrange’s mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

f(b)−f(a)=f′(c)(b−a)

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

This theorem is also known as First Mean Value Theorem.

Let $f(x) = \frac{1}{4x-1}$ on $[1, 4]$

4x – 1>0

$\Rightarrow$ f(x) has unique values for all x except $\frac{1}{4}$

∴ f(x) is continuous in [1, 4]

$$f(x) = \frac{1}{4x - 1}$$

Differentiating with respect to x:

$$f'(x) = (-1)(4x - 1)^{-2}(4)$$

$$\Rightarrow f'(x) = -\frac{4}{(4x - 1)^2}$$

Here,

$\Rightarrow$ 4x – 1>0

$\Rightarrow$ f'(x) has unique values for all x except $\frac{1}{4}$

∴ f(x) is differentiable in (1, 4)

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point c∈(1, 4) such that:

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(1)}{3}$$

$$f(x) = \frac{1}{4x - 1}$$

On differentiating with respect to x:

$$f'(x) = -\frac{4}{(4x - 1)^2}$$

For f'(c), put the value of x=c in f'(x):

$$f'(c) = -\frac{4}{(4c - 1)^2}$$

For f(4), put the value of x=4 in f(x):

$$f(4) = \frac{1}{4(4) - 1}$$

$$\Rightarrow f(4) = \frac{1}{16 - 1}$$

$$\Rightarrow f(4) = \frac{1}{15}$$

For f(1), put the value of x=1 in f(x):

$$f(1) = \frac{1}{4(1) - 1}$$

$$\Rightarrow f(1) = \frac{1}{4 - 1}$$

$$\Rightarrow f(1) = \frac{1}{3}$$

$$\Rightarrow f'(c)=\frac{f(4)-f(1)}{3}$$

$$\Rightarrow -\frac{4}{(4c-1)^2}=\frac{\frac{1}{15}-\frac{1}{3}}{3}$$

$$\Rightarrow -3(4)=(4c-1)^2\left(\frac{1}{15}-\frac{1}{3}\right)$$

$$\Rightarrow -12=(4c-1)^2\left(\frac{3-15}{45}\right)$$

$$\Rightarrow -12=(4c-1)^2\left(\frac{-12}{45}\right)$$

$$\Rightarrow -12\times\frac{45}{-12}=(4c-1)^2$$

$\Rightarrow (4c - 1)^2 = 45$

$$\Rightarrow (4c-1)=\pm\sqrt{45}$$

$$\Rightarrow (4c-1)=\pm 3\sqrt{5}$$

$$\Rightarrow c=\frac{\pm3\sqrt{5}+1}{4}$$

$$\Rightarrow c=\frac{3\sqrt{5}+1}{4}\approx 1.92\in(1,4)$$

Hence, Lagrange's mean value theorem is verified.

5. Question

Find a point on the parabola $y = (x - 4)^2$, where the tangent is parallel to the chord joining (4, 0) and (5, 1).

Answer

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$$\Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a}$$

This theorem is also known as First Mean Value Theorem.

Let $f(x) = (x - 4)^2$ on [4, 5]

This interval [a, b] is obtained by x – coordinates of the points of the chord.

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [4, 5] and differentiable in (4, 5). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c\in(4, 5)$ such that:

$$f'(c)=\frac{f(5)-f(4)}{5-4}$$

$$\Rightarrow f'(c)=\frac{f(5)-f(4)}{1}$$

$f(x) = (x - 4)^2$

Differentiating with respect to x:

$$f'(x)=2(x-4)\frac{d(x-4)}{dx}$$

⇒ f’(x) = 2(x – 4)(1)

⇒ f’(x) = 2(x – 4)

For f’(c), put the value of x=c in f’(x):

f’(c)= 2(c – 4)

For f(5), put the value of x=5 in f(x):

f(5)= $(5 – 4)^2$

= $(1)^2$

= 1

For f(4), put the value of x=4 in f(x):

f(4) = $(4 – 4)^2$

= $(0)^2$

= 0

f’(c) = f(5) – f(4)

⇒ 2(c – 4) = 1 – 0

⇒ 2c – 8 = 1

⇒ 2c = 1 + 8

$$\Rightarrow c=\frac{9}{2}=4.5\in(4,5)$$

We know that, the value of c obtained in Lagrange’s Mean value Theorem is nothing but the value of x – coordinate of the point of the contact of the tangent to the curve which is parallel to the chord joining the points (4, 0) and (5, 1).

Now, Put this value of x in f(x) to obtain y:

y = $(x – 4)^2$

$$\Rightarrow y=\left(\frac{9}{2}-4\right)^2$$

$$\Rightarrow y=\left(\frac{9-8}{2}\right)^2$$

$$\Rightarrow y=\left(\frac{1}{2}\right)^2$$

$$\Rightarrow y=\frac{1}{4}$$

Hence, the required point is $\left(\frac{9}{2},\frac{1}{4}\right)$

6. Question

Find a point on the curve y = x^2 + x, where the tangent is parallel to the chord joining (0, 0) and (1, 2).

Answer

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$\Rightarrow f'(c)=\dfrac{f(b)-f(a)}{b-a}$

This theorem is also known as First Mean Value Theorem.

Let $f(x) = x^2 + x$ on [0, 1]

This interval [a, b] is obtained by x – coordinates of the points of the chord.

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [0, 1] and differentiable in (0, 1). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c\in(0, 1)$ such that:

$f'(c)=\dfrac{f(1)-f(0)}{1-0}$

$\Rightarrow f'(c)=\dfrac{f(1)-f(0)}{1}$

$f(x) = x^2 + x$

Differentiating with respect to x:

$f'(x) = 2x + 1$

For f'(c), put the value of x=c in f'(x):

$f'(c)= 2c + 1$

For f(1), put the value of x=1 in f(x):

$f(1)= (1)^2 + 1$

$= 1 + 1$

$= 2$

For f(0), put the value of x=0 in f(x):

$f(0) = (0)^2 + 0$

$= 0 + 0$

$= 0$

$f'(c) = f(1) – f(0)$

$\Rightarrow 2c + 1= 2 – 0$

$\Rightarrow 2c = 2 – 1$

$\Rightarrow 2c = 1$

$\Rightarrow c = \dfrac{1}{2} = 0.5\in(0, 1)$

We know that the value of c obtained in Lagrange's Mean value Theorem is nothing but the value of x – coordinate of the point of the contact of the tangent to the curve which is parallel to the chord joining the points (0, 0) and (1, 2).

Now, put this value of x in f(x) to obtain y:

$y = x^2 + x$

$\Rightarrow y = \left(\frac{1}{2}\right)^2 + \frac{1}{2}$

$\Rightarrow y = \frac{1}{4} + \frac{1}{2}$

$\Rightarrow y = \frac{1+2}{4}$

$\Rightarrow y = \frac{3}{4}$

Hence, the required point is $\left(\frac{1}{2}, \frac{3}{4}\right)$

7. Question

Find a point on the parabola $y = (x - 3)^2$, where the tangent is parallel to the chord joining (3, 0) and (4, 1).

Answer

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$

This theorem is also known as First Mean Value Theorem.

Let $f(x) = (x - 3)^2$ on [3, 4]

This interval [a, b] is obtained by x – coordinates of the points of the chord.

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [3, 4] and differentiable in (3, 4). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point c∈(4, 5) such that:

$f'(c) = \frac{f(4) - f(3)}{4 - 3}$

$\Rightarrow f'(c) = \frac{f(4) - f(3)}{1}$

$f(x) = (x - 3)^2$

Differentiating with respect to x:

$f'(x) = 2(x - 3)\frac{d(x - 3)}{dx}$

$\Rightarrow f'(x) = 2(x - 3)(1)$

$\Rightarrow f'(x) = 2(x - 3)$

For f'(c), put the value of x=c in f'(x):

$f'(c) = 2(c - 3)$

For f(4), put the value of x=4 in f(x):

$f(4) = (4 - 3)^2$

$= (1)^2$

= 1

For f(3), put the value of x=3 in f(x):

$f(3) = (3 – 3)^2$

$= (0)^2$

= 0

f’(c) = f(4) – f(3)

⇒ 2(c – 3) = 1 – 0

⇒ 2c – 6 = 1

⇒ 2c = 1 + 6

$\Rightarrow c = \frac{7}{2} = 3.5 \in (3, 4)$

We know that, the value of c obtained in Lagrange’s Mean value Theorem is nothing but the value of x – coordinate of the point of the contact of the tangent to the curve which is parallel to the chord joining the points (3, 0) and (4, 1).

Now, Put this value of x in f(x) to obtain y:

$y = (x – 3)^2$

$\Rightarrow y = \left(\frac{7}{2} - 3\right)^2$

$\Rightarrow y = \left(\frac{7-6}{2}\right)^2$

$\Rightarrow y = \left(\frac{1}{2}\right)^2$

$\Rightarrow y = \frac{1}{4}$

Hence, the required point is $\left(\frac{7}{2}, \frac{1}{4}\right)$

8. Question

Find points on the curve $y = x^3 – 3x$, where the tangent to the curve is parallel to the chord joining (1, – 2) and (2, 2).

Answer

Lagrange’s mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

f(b)−f(a)=f′(c)(b−a)

$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$

This theorem is also known as First Mean Value Theorem.

Let $f(x) = x^3 – 3x$ on [1, 2]

This interval [a, b] is obtained by x – coordinates of the points of the chord.

Every polynomial function is **continuous** everywhere on (−∞, ∞) and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [1, 2] and differentiable in (1, 2). So both the necessary conditions of Lagrange’s mean value theorem is satisfied.

Therefore, there exists a point $c \in (1, 2)$ such that:

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(1)}{1}$$

$f(x) = x^3 – 3x$

Differentiating with respect to x:

$f'(x) = 3x^2 – 3$

For f’(c), put the value of x=c in f’(x):

$f'(c)= 3c^2 – 3$

For f(2), put the value of x=2 in f(x):

$f(2)= (2)^3 – 3(2)$

$= 8 – 6$

$= 2$

For f(1), put the value of x=1 in f(x):

$f(1) = (1)^3 – 3(1)$

$= 1 – 3$

$= – 2$

$f'(c) = f(2) – f(1)$

$\Rightarrow 3c^2 – 3 = 2 – (– 2)$

$\Rightarrow 3c^2 – 3 = 2 + 2$

$\Rightarrow 3c^2 = 4 + 3$

$$\Rightarrow c^2 = \frac{7}{3}$$

$$\Rightarrow c = \pm\sqrt{\frac{7}{3}}$$

$$\Rightarrow c = \sqrt{\frac{7}{3}} \in (1, 2)$$

We know that, the value of c obtained in Lagrange’s Mean value Theorem is nothing but the value of x – coordinate of the point of the contact of the tangent to the curve which is parallel to the chord joining the points (1, – 2) and (2, 2).

Now, Put this value of x in f(x) to obtain y:

$y = x^3 – 3x$

$$\Rightarrow y = \left(\sqrt{\frac{7}{3}}\right)^3 - 3\left(\sqrt{\frac{7}{3}}\right)$$

$$\Rightarrow y = \frac{7}{3}\left(\sqrt{\frac{7}{3}}\right) - 3\left(\sqrt{\frac{7}{3}}\right)$$

$$\Rightarrow y = \left(\sqrt{\frac{7}{3}}\right)\left(\frac{7}{3} - 3\right)$$

$$\Rightarrow y = \left(\sqrt{\frac{7}{3}}\right)\left(\frac{7-9}{3}\right)$$

$$\Rightarrow y = \frac{-2}{3}\left(\sqrt{\frac{7}{3}}\right)$$

Hence, the required point is $\left(\sqrt{\frac{7}{3}}, \frac{-2}{3}\left(\sqrt{\frac{7}{3}}\right)\right)$

9. Question

Find a point on the curve $y = x^3 + 1$ where the tangent is parallel to the chord joining (1, 2) and (3, 28).

Answer

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

$f(b)-f(a)=f'(c)(b-a)$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

Let $f(x) = x^3 + 1$ on [1, 3]

This interval [a, b] is obtained by x – coordinates of the points of the chord.

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, f(x) is a polynomial function. So it is continuous in [1, 3] and differentiable in (1, 3). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (1, 3)$ such that:

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(1)}{2}$$

$f(x) = x^3 + 1$

Differentiating with respect to x:

$f'(x) = 3x^2$

For f'(c), put the value of x=c in f'(x):

$f'(c) = 3c^2$

For f(3), put the value of x=3 in f(x):

$f(3) = (3)^3 + 3$

= 27 + 3

= 30

For f(1), put the value of x=1 in f(x):

$f(1) = (1)^3 + 3$

= 1 + 3

= 4

$$f'(c) = \frac{f(3) - f(1)}{2}$$

$$\Rightarrow 3c^2 = \frac{30 - 4}{2}$$

$$\Rightarrow 3c^2 = \frac{26}{2}$$

$\Rightarrow 3c^2 = 13$

$$\Rightarrow c^2 = \frac{13}{3}$$

$$\Rightarrow c = \pm\sqrt{\frac{13}{3}}$$

$$\Rightarrow c = \sqrt{\frac{13}{3}} \in (1, 3)$$

We know that, the value of c obtained in Lagrange's Mean value Theorem is nothing but the value of x – coordinate of the point of the contact of the tangent to the curve which is parallel to the chord joining the points (1, 2) and (3, 28).

Now, Put this value of x in f(x) to obtain y:

$y = x^3 + 1$

$$\Rightarrow y = \left(\sqrt{\frac{13}{3}}\right)^3 + 1$$

Hence, the required point is $\left(\sqrt{\frac{13}{3}}, \left(\sqrt{\frac{13}{3}}\right)^3 + 1\right)$

10. Question

Let C be a curve defined parametrically as $x = a\cos^3\theta$, $y = a\sin^3\theta$, $0 \leq \theta \leq \pi/2$. Determine a point P on C, where the tangent to C is parallel to the chord joining the points (a, 0) and (0, a).

Answer

sin x and cos x functions are **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

$x = a\cos^3\theta$

$$\Rightarrow \cos^3\theta = \frac{x}{a}$$

$\Rightarrow \cos\theta = \left(\frac{x}{a}\right)^{\frac{1}{3}}$

$y = a \sin^3 \theta$

$\Rightarrow \sin^3\theta = \frac{y}{a}$

$\Rightarrow \sin\theta = \left(\frac{y}{a}\right)^{\frac{1}{3}}$

We know that,

$\sin^2\theta + \cos^2\theta = 1$

$\therefore \left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{a}\right)^{\frac{2}{3}} = 1$

$\Rightarrow (x)^{\frac{2}{3}} + (y)^{\frac{2}{3}} = (a)^{\frac{2}{3}}$

$\Rightarrow (y)^{\frac{2}{3}} = (a)^{\frac{2}{3}} - (x)^{\frac{2}{3}}$

$\Rightarrow y = \left[(a)^{\frac{2}{3}} - (x)^{\frac{2}{3}}\right]^{\frac{3}{2}}$

Let $f(x) = \left[(a)^{\frac{2}{3}} - (x)^{\frac{2}{3}}\right]^{\frac{3}{2}}$

Therefore, there exists a point $c \in (0, a)$ such that:

$f'(c) = \frac{f(a) - f(0)}{a - 0}$

$\Rightarrow f'(c) = \frac{f(a) - f(0)}{a}$

$x = a\cos^3\theta$

$\Rightarrow \frac{dx}{d\theta} = \frac{d(a\cos^3\theta)}{d\theta}$

$\Rightarrow \frac{dx}{d\theta} = 3a\cos^2\theta \times \frac{d(\cos\theta)}{d\theta}$

$\Rightarrow \frac{dx}{d\theta} = 3a\cos^2\theta \times (-\sin\theta)$

$y = a\sin^3\theta$

$\Rightarrow \frac{dy}{d\theta} = \frac{d(a\sin^3\theta)}{d\theta}$

$\Rightarrow \frac{dy}{d\theta} = 3a\sin^2\theta \times \frac{d(\sin\theta)}{d\theta}$

$\Rightarrow \frac{dy}{d\theta} = 3a\sin^2\theta \times (\cos\theta)$

$\frac{dy}{dx} = \left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{3a\sin^2\theta \times (\cos\theta)}{3a\cos^2\theta \times (-\sin\theta)}\right)$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = -\tan\theta$$

For f'(c), put the value of x=c in f'(x):

$f'(c) = -\tan\theta$

$$f(x) = \left[(a)^{\frac{2}{3}} - (x)^{\frac{2}{3}}\right]^{\frac{3}{2}}$$

For f(a), put the value of x=a in f(x):

$$f(a) = \left[(a)^{\frac{2}{3}} - (a)^{\frac{2}{3}}\right]^{\frac{3}{2}}$$

$= 0$

For f(0), put the value of x=0 in f(x):

$$f(0) = \left[(a)^{\frac{2}{3}} - (0)^{\frac{2}{3}}\right]^{\frac{3}{2}}$$

$$= \left[(a)^{\frac{2}{3}}\right]^{\frac{3}{2}}$$

$= a$

$$f'(c) = \frac{f(a) - f(0)}{a}$$

$$\Rightarrow -\tan\theta = \frac{0 - a}{a}$$

$$\Rightarrow -\tan\theta = \frac{-a}{a}$$

$\Rightarrow -\tan\theta = -1$

$\Rightarrow \tan\theta = 1$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Now put the value of θ in the function of x and y:

$x = a\cos^3\theta$

$$\Rightarrow x = a\cos^3\left(\frac{\pi}{4}\right)$$

$$\Rightarrow x = a\left(\frac{1}{\sqrt{2}}\right)^3$$

$$\Rightarrow x = \frac{a}{2\sqrt{2}}$$

Similarly,

$y = a\sin^3\theta$

$$\Rightarrow y = a\sin^3\left(\frac{\pi}{4}\right)$$

$$\Rightarrow y = a\left(\frac{1}{\sqrt{2}}\right)^3$$

$$\Rightarrow y = \frac{a}{2\sqrt{2}}$$

So the required point is $\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$.

11. Question

Using Lagrange's mean value theorem, prove that

(b – a) $\sec^2$ a < tan b – tan a < (b – a) $\sec^2$ b, where 0 < a < b < π/2.

Answer

Let f(x) = tan x on [a, b]

We know that, tan x function is **continuous** and **differentiable** on

$\left(0, \frac{\pi}{2}\right)$. Since a and b lie between 0 and $\frac{\pi}{2}$, tan x is continuous and

differentiable on (a, b).

Lagrange's mean value theorem states that if a function f(x) is continuous on a closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point x=c on this interval, such that

f(b)−f(a)=f′(c)(b−a)

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

f(x) = tan x

Differentiating with respect to x:

f'(x) = $\sec^2$ x

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \sec^2 c = \frac{\tan b - \tan a}{b - a}$$

Since c lies between a and b

⇒ a < c < b

⇒ $\sec^2$ a < $\sec^2$ c < $\sec^2$ b

$$\Rightarrow \sec^2 a < \frac{\tan b - \tan a}{b - a} < \sec^2 b$$

⇒ $\sec^2$ a (b – a)<tan b – tan a< $\sec^2$ b (b – a)

Hence Proved

MCQ

1. Question

Mark the correct alternative in the following:

If the polynomial equation $a_0x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... a_2x^2 + a_1x + a_0 = 0$

n being a positive integer, has two different real roots α and β, then between α and β, the equation $n\ a_nx^{n-1} + (n-1)\ a_{n-1}x^{n-2} + ... + a_1 = 0$ has

A.exactly one root

B. almost one root

C. at least one root

D. no root

Answer

As the polynomial, $na_nx^{n-1} + (n-1)\ a_{n-1}x^{n-2} + ... + a_1 = 0$ is a derivative of the polynomial $a_0x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ...a_2x^2 + a_1x + a_0 = 0$ --------- (i)

Putting x = 0 in equation (i),

$f(0) = a_0 < 0$, {Y – Intercept of the graph is negative}

On the other hand, ∵ an > 0 and 'n' is even, the leading term on x^n, is positive for only x.

For |x| to be large, the term anx_n will dominate, so

$\lim_{x\to-\infty} f(x) = +\infty$

$\lim_{x\to+\infty} f(x) = +\infty$

If $\lim_{x\to+\infty} f(x) = +\infty$, there must exist

Same number α < 0, where f(α > 0)

∵ $f(0) = a_0 < 0$,

∴ α < β < 0, such that f(β) = 0

Also, there is some value 0 < α,

Where f(a) & so there exists,

0 < b < a with f(b) = 0

Additionally, the polynomial function (equation (i)) is continuous everywhere in R and consequently derivative in R.

∴ $a_0x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ...a_2x^2 + a_1x + a_0 = 0$ is continuous on α,β and derivative on α,β.

Thus, it satisfies both the conditions of Rolle' s Theorem.

As per the Rolle's Theorem, between any two roots of a function f(x), there exists at least one root of its derivative.

Thus, the equation $na_nx^{n-1} + (n-1)\ a_{n-1}x^{n-2} + ... + a_1 = 0$ will have at least one root between α & β.

Hence, Option (C) is the answer.

2. Question

Mark the correct alternative in the following:

If 4a + 2b + c = 0, then the equation $3ax^2 + 2bx + c = 0$ has at least one real root lying in the interval.

A.(0, 1)

B. (1, 2)

C. (0, 2)

D. none of these

Answer

Let f(x) = ax3 + bx2 + cx + d --------- (i)

∴ f(0) = d

$f(2) = a(2)^3 + b(2)^2 + c(2) + d$

= 8a + 4b + 2c + d

= 2(4a + 2b + c) + d

∵ 4a + 2b + c = 0 {Given}

= 2 (0) + d

= 0 + d

= d

f is continuous in closed interval [0, 2] and f is derivable in the open interval (0, 2).

Also, f(0) = f(2)

As per Rolle’s Theorem,

f’(α) = 0 for 0 < α < 2

$f'(x) = 3ax^2 + 2bx + c$

$\therefore f'(\alpha) = 3a\alpha^2 + 2b(\alpha) + c$

$3a\alpha^2 + 2b(\alpha) + c = 0$

Hence equation (i) has at least one root in the interval (0, 2).

Thus, f’(x) must have one root in the interval (0, 2).

Hence, Option (C) is the answer.

3. Question

Mark the correct alternative in the following:

For the function $f(x) = x + \frac{1}{x}, x \in [1, 3]$, the value of c for the Lagrange’s mean value theorem is

A.1

B. $\sqrt{3}$

C. 2

D. none of these

Answer

$f(x) = x + \frac{1}{x}$

$= \frac{x^2 + 1}{x}$

It shows that f(x) is continuous on 1, 3 and derivable on 1, 3.

So, both the conditions of Lagrange’s Theorem are satisfied.

Consequently, there exists c Є 1, 3 such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$= \frac{f(3) - f(1)}{2}$$

$$f(x) = \frac{x^2 + 1}{x}$$

$$f'(x) = \frac{x^2 - 1}{x^2} \left\{ \because f(x) = x + \frac{1}{x}, f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2} \right\}$$

$$f(1) = \frac{x^2 + 1}{x}$$

$$= \frac{1^2 + 1}{1}$$

$$= \frac{1 + 1}{1}$$

$$= 2$$

$$f(3) = \frac{x^2 + 1}{x}$$

$$= \frac{3^2 + 1}{3}$$

$$= \frac{9 + 1}{1}$$

$$= \frac{10}{3}$$

$$f'(x) = \frac{f(3) - f(1)}{3 - 1}$$

$$= \frac{f(3) - f(1)}{2}$$

$$\therefore \frac{x^2 - 1}{x^2}$$

$$= \frac{\frac{10}{3} - 2}{2}$$

$$= \frac{\frac{10 - 6}{3}}{2}$$

$$= \frac{4}{6}$$

$$= \frac{2}{3}$$

$$\frac{x^2 - 1}{x^2} = \frac{2}{3}$$

$$3x^2 - 3 = 2x^2$$

$$x^2 = 3$$

$x = \pm\sqrt{3}$Hence, $c = \sqrt{3}$ ∈ (1, 3) such that $f'(c) = \frac{f(3)-f(1)}{3-1}$.

Hence, Option (B) is the answer.

4. Question

Mark the correct alternative in the following:

If from Lagrange's mean value theorem, we have $\frac{f'(x) = f'(b) - f(a)}{b-a}$, then

A. a < x_1 ≤ b

B. a ≤ x_1< b

C. a < x_1< b

D. a ≤ x_1 ≤ b

Answer

$\because f'(x) = \frac{f(b) - f(a)}{b - a}$

In the Lagrange's Mean Value Theorem, c ∈ (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

f is continuous on [a, b] and differentiable on (a, b), then there exists a real number x ∈ (a, b)

So, in the case of x_1, $f'(x_1) = \frac{f(b)-f(a)}{b-a}$, then x_1∈ (a, b)

∴ a <x_1 < b

Hence, Option (C) is the answer.

5. Question

Mark the correct alternative in the following:

Rolle's theorem is applicable in case of ϕ (x) = $a^{\sin x}$, a > 0 in

A. any interval

B. the interval [0, π]

C. the interval (0, π/2)

D. none of these

Answer

Φ(x) = $a^{\text{Sin } x}$, a > 0

Differentiating the above-mentioned function, with respect to 'x',

Φ'(x) = log a (cos x $a^{\text{Sin } x}$)

∴ Φ'(c) = log a (cos c $a^{\text{Sin } c}$)

Let Φ'(c) = 0

log a (cos c $a^{\text{Sin } c}$) = 0

cos c $a^{\text{Sin } c}$ = 0

cos c = 0

$\text{Cos} c = \text{Cos}\frac{\pi}{2}$

$\therefore c = \frac{\pi}{2}$

Also, the above-mentioned function, is derivable and continuous on the interval [0, π].

Thus, here Rolle's Theorem is applicable on the above mentioned function in the interval [0,π].

Hence, Option (B) is the answer.

6. Question

Mark the correct alternative in the following:

The value of e in Rolle's theorem show $f(x) = 2x^3 - 5x^2 - 4x + 3$, is $x \in [1/3, 3]$

A.2

B. $-\frac{1}{3}$

C. –2

D. $\frac{2}{3}$

Answer

$f(x) = 2x^3 - 5x^2 - 4x + 3$

$f'(x) = 6x^2 - 10x - 4$

$f'(c) = 6c^2 - 10c - 4$

$\therefore f'(c) = 0$

$\therefore 6c^2 - 10c - 4 = 0$

$3c^2 - 5c - 2 = 0$

$3c^2 + c - 6c - 2 = 0$

$c (3c + 1) - 2 (3c + 1) = 0$

$(3c + 1) (c - 2) = 0$

c = 2 or $c = -\frac{1}{3}$

$\therefore c = 2 \in (\frac{1}{3}, 3)$

Thus, as per Rolle's Theorem, $c = 2 \in (\frac{1}{3}, 3)$.

So, the required value of c = 2

Hence, Option (A) is the answer.

7. Question

Mark the correct alternative in the following:

`The value tangent to the curve y = x log x is parallel to the chord joining the points (1, 0) and (e, e), the value of is

A. $e^{1/1-e}$

B. $e^{(e-1)(2e-1)}$

C. $e^{\frac{2e-1}{e-1}}$

D. $\frac{e-1}{e}$

Answer

y = x log x

Differentiating the function with respect to ‘x’,

$$\frac{dy}{dx} = 1 + \log x$$

Slope of tangent to the curve = 1 + log x

And, slope of the chord joining the points, (1, 0) & (e, e)

$$m = \frac{e}{e-1}$$

The tangent to the curve is parallel to the chord joining the points, (1, 0) & (e, e)

∴ m = 1 + log x

$$\frac{e}{e-1} = 1 + \log x$$

$$\log x = \frac{e}{e-1} - 1$$

$$\log x = \frac{e-e+1}{e-1}$$

$$\log x = \frac{1}{e-1}$$

$$x = e^{\frac{1}{1-e}}$$

Hence, Option (A) is the answer.

8. Question

Mark the correct alternative in the following:

The value of c in Rolle’s theorem for the function $f(x) = \frac{x(x+1)}{e^x}$ defined on [–1, 0] is

A.0.5

B. $\frac{1+\sqrt{5}}{2}$

C. $\frac{1-\sqrt{5}}{2}$

D. –0.5

Answer

$$f(x) = \frac{\{x(x+1)\}}{e^x}$$

Differentiating the function with respect to ‘x’,

$$f'(x) = \frac{e^x(2x+1) - x(x+1)e^x}{(e^x)^2}$$

$$f'(x) = \frac{e^x[(2x+1) - x(x+1)]}{(e^x)^2}$$

$$f'(x) = \frac{[(2x+1) - x(x+1)]}{e^x}$$

$$f'(x) = \frac{2x+1-x^2-x}{e^x}$$

$$f'(x) = \frac{x+1-x^2}{e^x}$$

$$f'(x) = \frac{-x^2+x+1}{e^x}$$

$$\therefore f'(c) = \frac{-c^2+c+1}{e^c}$$

$$\therefore f'(c) = 0$$

$$\frac{-c^2+c+1}{e^c} = 0$$

$$-c^2+c+1 = 0$$

$$c^2-c-1 = 0$$

Using Sridharacharya Formula,

In a general equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore c = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$c = \frac{1 \pm \sqrt{1+4}}{2}$$

$$c = \frac{1 \pm \sqrt{5}}{2}$$

$\therefore c = \frac{1-\sqrt{5}}{2} \in$ (-1, 0)

So the required value of $c = \frac{1-\sqrt{5}}{2} \in$ (-1, 0).

Hence, Option (C) is the answer.

9. Question

Mark the correct alternative in the following:

The value of c in Lagrange’ mean value theorem for the function f(x) = x (x – 2) where x$\in$ [1, 2] is

A.1

B. $\frac{1}{2}$

C. $\frac{2}{3}$

D. $\frac{3}{2}$

Answer

f(x) = x (x – 2)

f(x) = x^2 – 2x

Since, a polynomial function is always continuous and differentiable.

As f(x) is a polynomial function so it is always continuous on 1, 2 and differentiable on 1, 2.

∴ f(x) satisfies both the conditions of Lagrange's Theorem on 1, 2.

So, a real number has to exist c ∈ 1, 2, such that

$$f'(c) = \frac{\{f(2) - f(1)\}}{(2 - 1)}$$

$$= \frac{\{f(2) - f(1)\}}{1}$$

∵ f(x) = x^2 – 2x

f'(x) = 2x – 2

∴ f(1) = 1^2 – 2(1)

= 1 – 2

= -1

f(2) = 2^2 – 2(2)

= 4 – 4

= 0

$$\therefore f'(x) = \frac{\{f(2) - f(1)\}}{1}$$

$$f'(x) = \frac{0 - 1}{1}$$

$$= \frac{0 + 1}{1}$$

= 1

So, 2x – 2 = 1

2x = 3

$$x = \frac{3}{2}$$

∴ $c = \frac{3}{2}$ ∈ (1, 2)

Hence, Option (D) is the answer.

10. Question

Mark the correct alternative in the following:

The value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in the interval $\left[0, \sqrt{3}\right]$ is

A.1

B. –1

C. $\frac{3}{2}$

D. $\frac{1}{3}$

Answer

$f(x) = x^3 - 3x$

The above mentioned polynomial function is continuous and derivable in R.

∴ the function is continous on [0, $\sqrt{3}$] and derivable on [0, $\sqrt{3}$].

Differentiating the function with respect to x,

$f(x) = x^3 - 3x$

$f'(x) = 3x^2 - 3$

∴ $f'(c) = 3c^2 - 3$

∴ $f'(c) = 0$

$3c^2 - 3 = 0$

$c^2 - 1 = 0$

$c^2 = 1$

$c = \pm 1$

Hence, c = 1 ∈ [0, $\sqrt{3}$], as per the condition of Rolle's Theorem.

The required value is c = 3.

Hence, Option (A) is the answer.

11. Question

Mark the correct alternative in the following:

If $f(x) = e^x \sin x$ in [0, π], then c in Rolle's theorem is

A. $\frac{\pi}{6}$

B. $\frac{\pi}{4}$

C. $\frac{\pi}{2}$

D. $\frac{3\pi}{4}$

Answer

As, $f(x) = e^x \text{Sin } x$

Differentiating the function with respect to 'x',

$f'(x) = e^x \text{Cos } x + \text{Sin } x \, e^x$

$\therefore f'(c) = e^c \text{Cos } c + \text{Sin } c \, e^c$

As, $e^x \text{Cos } x$ is continuous and derivable in R.

$\therefore$ it is contionous on $[0, \pi]$ and derivable on $(0, \pi)$.

$f(0) = e^0 \text{Sin } (0)$

$= 0$

$f(\pi) = e^{\pi} \text{Sin}\pi$

$= e^{\pi}(0)$

$= 0$

$\therefore f'(c) = 0$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(\pi) - f(0)}{\pi - 0}$$

$$= \frac{0 - 0}{\pi}$$

$= 0$

$e^c \text{Cos } c + \text{Sin } c \, e^c = 0$

$e^c (\text{Cos } c + \text{Sin } c) = 0$

$\text{Cos } c + \text{Sin } c = 0$ -------- (i)

$\text{Cos } c = - \text{Sin } c$

$$\text{Cos } c = -\text{Cos}\left(\frac{\pi}{2} - c\right)$$

$$\text{Cos } c = \cos\left(\pi + \left(\frac{\pi}{2} - c\right)\right)$$

$$\text{Cos } c = \cos\left(\pi + \frac{\pi}{2} - c\right)$$

$$\text{Cos } c = \cos\left(\frac{3\pi}{2} - c\right)$$

$$c = \frac{3\pi}{2} - c$$

$$2c = \frac{3\pi}{2}$$

$$c = \frac{3\pi}{4}$$

$\therefore c = \frac{3\pi}{4} \in (0, \pi)$

Hence, Option (D) is the answer.

Very short answer

1. Question

If $f(x) = Ax^2 + Bx + C$ is such that $f(a) = f(b)$, then write the value of c in Rolle's theorem.

Answer

$f(x) = Ax^2 + Bx + C$

Differentiating the above-mentioned function with respect to 'x',

$f'(x) = 2Ax + B$

$f'(c) = 2Ac + B$

$f'(c) = 0$

$\therefore 2Ac + B = 0$

$c = -\frac{B}{2A}$ -------- (i)

As $f(a) = f(b)$,

$f(a) = Aa^2 + Ba + C$

$f(b) = Ab^2 + Bb + C$

$Aa^2 + Ba + C = Ab^2 + Bb + C$

$Aa^2 + Ba = Ab^2 + Bb$

$A (a^2 – b^2) + B (a – b) = 0$

$A (a + b) (a – b) + B (a – b) = 0$

$(a – b) \{A (a + b) + B\} = 0$

a = b, $A = -\frac{B}{a+b}$

$a + b = -\frac{B}{A}$ {As $a \neq b$}

From equation (i)

$$c = \frac{a+b}{2}$$

Hence the required value is $= \frac{a+b}{2}$.

2. Question

State Rolle's theorem.

Answer

Rolle's Theorem is stated as below: -

Let 'f', be a real valued function defined on the closed interval a, b such that

i. It is continuous in the closed interval [a, b].

ii. It is differentiable in the open interval (a, b).

iii. f(a) = f(b)

Then there exists a real number c ∈ (a, b) such that f'(c) = 0.

3. Question

State Lagrange’s mean value theorem.

Answer

Lagrange’s Mean Value Theorem is stated as below :-

Let f(x) be a function defined on a, b such that

i. It is continuous on a, b and

ii. It is differentiable on a, b.

Then there exists a real number c ∈ a, b such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

4. Question

If the value of c prescribed in Rolle’s theorem for the function f(x) = 2x (x – 3)^n on the interval $\left[0, 2\sqrt{3}\right]$ is $\frac{3}{4}$, write the value of n (a positive integer).

Answer

f(x) = 2x (x – 3)^n

Differentiating the above-mentioned function with respect to ‘x’,

f’(x) = 2 [xn (x – 3)$^{n-1}$ + (x – 3)^n]

$$f'(x) = 2(x-3)^n\left[\frac{xn}{x-3}+1\right]$$

$$\therefore f'(c) = 2(c-3)^n\left[\frac{cn}{c-3}+1\right]$$

$$\because f'\left(\frac{3}{4}\right) = 0$$

$$\therefore 2-\left(\frac{9}{4}\right)^n\left[\frac{\frac{3}{4}n}{-\frac{9}{4}}+1\right] = 0$$

$$2-\left(\frac{9}{4}\right)^n\left[-\frac{n}{3}+1\right] = 0$$

$$-\frac{n}{3}+1 = 0$$

$$\frac{(-n+3)}{3} = 0$$

-n + 3 = 0

-n = -3

n = 3

Hence, the required value of ‘n’ is 3.

5. Question

Find the value of c prescribed by Lagrange’s mean value theorem for the function $f(x) = \sqrt{x^2-4}$ defined on [2, 3].

Answer

$$f(x) = \sqrt{x^2-4}$$

f(x) will exist, if

$x^2 - 4 \geq 0$

$x^2 \geq 4$

$\therefore$ x ≤ -2 or x ≥ 2

$\because$ for each x ∈ [2, 3], the function f(x) has a unique definite value, f(x) is continuous on (2, 3).

$$f'(x) = \frac{1}{2\sqrt{x^2 - 4}}(2x)$$

$$= \frac{x}{2\sqrt{x^2 - 4}}$$

Exists for all x ∈ (2, 3).

So, f(x) is differentiable on (2,3).

Hence, both the conditions of Lagrange's Theorem are satisfied.

Consequently, there exists c ∈ (2, 3) such that,

$$f'(c) = \frac{f(3) - f(2)}{3 - 2}$$

$$= \frac{f(3) - f(2)}{1}$$

$$f(x) = \sqrt{x^2 - 4}$$

$$f(3) = \sqrt{3^2 - 4}$$

$$= \sqrt{9 - 4}$$

$$= \sqrt{5}$$

$$f(2) = \sqrt{2^2 - 4}$$

$$= \sqrt{4 - 4}$$

$$= 0$$

$$f'(x) = \frac{x}{\sqrt{x^2 - 4}}$$

$$\therefore f'(x) = \frac{f(3) - f(2)}{3 - 2}$$

$$f'(x) = \frac{\sqrt{5} - 0}{1}$$

$$\frac{x}{\sqrt{x^2 - 4}} = \sqrt{5}$$

Squaring both sides,

$$\frac{x^2}{(\sqrt{x^2 - 4})^2} = \left(\sqrt{5}\right)^2$$

$$\frac{x^2}{x^2 - 4} = 5$$

$5x^2 - 20 - x^2 = 0$

$4x^2 = 20$

$x^2 = 5$

$x = \pm \sqrt{5}$

Hence, c = √5 ∈ (2, 3) such that $f'(c) = \frac{f(3)-f(2)}{3-2}$

Hence the above explanation verifies the Lagrange's Theorem.

1. Question

What are the values of 'a' for which $f(x) = a^x$ is increasing on R?

Answer

$f(x) = a^x$

$f'(x) = a^x \log a$

∵ f(x) is increasing on R.

∴ f'(x) > 0

∴ $a^x \log a > 0$

∵ Logarithmic function is defined for positive values of a.

∴ a > 0

∴ $a^x > 0$

∵ $a^x \log a > 0$

∴ It can be possible when $\log a^x > 0$ & log a > 0 or $a^x < 0$ & log a < 0.

log a > 0

∴ a > 1

Hence, f(x) is increasing when a > 1.

2. Question

What are the values of 'a' for which $f(x) = a^x$ is decreasing on R?

Answer

$f(x) = a^x$

$f'(x) = a^x \log a$

∵ f(x) is decreasing on R.

∴ $f'(x) < 0, \forall x \in R$

∴ $a^x \log a < 0, \forall x \in R$

∵ Logarithmic function is not defined for negative values of a.

∴ $a^x > 0$

∴ $a^x \log a < 0$ can be possible when $\log a < 0, \forall x \in R$.

∴ 1 > a > 0

Hence the function f(x), whose values are 1 > a > 0.

3. Question

Write the set of values of ‘a’ for which $f(x) = \log_a x$ is increasing in its domain.

Answer

$f(x) = \log_a x$

Let $x_1, x_2 \in (0, \infty)$ such that $x_1 < x_2$.

$\because$ the function here is a logarithmic function, so either $a > 1$ or $1 > a > 0$.

Case – 1

Let $a > 1$

$x_1 < x_2$

$\therefore \log_a x_1 < \log_a x_2$

$\therefore f(x_1) < f(x_2)$

$\therefore x_1 < x_2$ & $f(x_1) < f(x_2), \forall x_1, x_2 \in (0, \infty)$

Hence, f(x)is increasing on $(0, \infty)$.

Case – 2

Let, $1 > a > 0$

$x_1 < x_2$

$\therefore \log_a x_1 > \log_a x_2$

$\therefore f(x_1) > f(x_2)$

$\therefore x_1 < x_2$ & $f(x_1) > f(x_2), \forall x_1, x_2 \in (0, \infty)$

Thus, for $a > 1$, f(x) is increasing in its domain.

4. Question

Write the set of values of ‘a’ for which $f(x) = \log_a x$ is decreasing in its domain.

Answer

$f(x) = \log_a x$

Domain of the above mentioned function is $(0, \infty)$

Let $x_1, x_2 \in (0, \infty)$ such that $x_1 < x_2$.

$\because$ the function here is a logarithmic function, so either $a > 1$ or $1 > a > 0$.

Case – 1

Let $a > 1$

$x_1 < x_2$

$\therefore \log_a x_1 < \log_a x_2$

$\therefore f(x_1) < f(x_2)$

$\therefore x_1 < x_2$ & $f(x_1) < f(x_2), \forall x_1, x_2 \in (0, \infty)$

Hence, f(x)is increasing on $(0, \infty)$.

Case – 2

Let $1 > a > 0$

$x_1 < x_2$

$\therefore \log_a x_1 > \log_a x_2$

$\therefore f(x_1) > f(x_2)$

$\therefore x_1 < x_2$ & $f(x_1) > f(x_2), \forall\ x_1, x_2 \in (0, \infty)$

Hence, f(x)is decreasing on $(0, \infty)$.

Thus, for $1 > a > 0$, f(x) is decreasing in its domain.

5. Question

Find ‘a’ for which f(x) = a (x + sinx) + a is increasing on R.

Answer

f(x) = a (x + Sin x) + a

f’(x) = a (1 + Cos x) + 0

f’(x) = a (1 + Cos x)

For f(x), to be increasing, it must have,

f’(x) > 0

$\therefore$ a (1 + Cos x) > 0 -------------- (i)

$\because -1 \leq \text{Cos } x \leq 1, \forall x \in R$

$\therefore 0 \leq (1 + \text{Cos } x) \leq 2, \forall x \in R$

$\therefore$ a > 0 {From eq. (i)}

$\therefore a \in (0, \infty)$

Hence the required set of values is $a \in (0, \infty)$.

6. Question

Find the vales of ‘a’ for which the function f(x) = sinx – ax + 4 is increasing function on R.

Answer

f(x) = Sin x – ax + 4

f’(x) = Cos x – a + 0

f’(x) = Cos x – a

$\because$ f(x) is increasing on R.

f’(x) > 0

$\therefore$ Cos x – a > 0

Cos x > a

$\because \text{Cos } x \geq -1, \forall x \in R$

$\therefore a < -1$

$\therefore a \in (-\infty, -1)$

Hence the required set of values is $a \in (-\infty, -1)$.

7. Question

Find the set of values of ‘b’ for which f(x) = b(x + cosx) + 4 is decreasing on R.

Answer

$f(x) = b (x + \text{Cos } x) + 4$

$f'(x) = b (1 - \text{Sin } x) + 0$

$f'(x) = b (1 - \text{Sin } x)$

$\because$ f(x) is decreasing on R.

$\therefore f'(x) < 0$

$b (1 - \text{Sin } x) < 0$

$\because \text{Sin } x \leq 1$

$1 - \text{Sin } x \geq 0$

$\because b (1 - \text{Sin } x) < 0$ & $1 - \text{Sin } x \geq 0$

$\therefore b < 0$

$\therefore b \in (-\infty, 0)$

Hence the required set of values is $b \in (-\infty, 0)$.

8. Question

Find the set of values of a'a' for which f(x) = x + cosx + ax + b is increasing on R.

Answer

$f(x) = x + \text{Cos } x + ax + b$

$f'(x) = 1 - \text{Sin } x + a + 0$

$f'(x) = 1 - \text{Sin } x + a$

For, f(x) to be increasing, it must have

$f'(x) > 0$

$\therefore 1 - \text{Sin } x + a > 0$

$1 > \text{Sin } x - a$

$\text{Sin } x < a + 1$

$\because$ the maximum value of Sin x is 1.

Also, $1 < a + 1$

$a > 0$

$\therefore a \in (0, \infty)$

Hence the required set of values is $a \in (0, \infty)$.

9. Question

Write the set of values of k for which f(x) = kx – sinx is increasing on R.

Answer

$f(x) = kx - \text{Sin } x$

$f'(x) = k - \text{Cos } x$

For, f(x) to be increasing, it must have

$f'(x) > 0$

∴ k – Cos x > 0

k > Cos x

Cos x < k

∵ the minimum value of Xos x is 1.

Also, Cos x < k

∴ The minimum value of k is 1.

∴ k ϵ (1, ∞)

Hence the required set of values is k ϵ (1, ∞).

10. Question

If g(x) is a decreasing function on R and f(x) = $\tan^{-1}$ {g(x)}. State whether f(x) is increasing or decreasing on R.

Answer

∵ g(x) is decreasing on R.

f(x) = $\tan^{-1}x$ is an increasing function.

f o g(x) = f(g(x)) = $\tan^{-1}$(g(x))

is a decreasing function

Composite of two functions,

f(x)	g(x)	Composite
↑	↑	↑
↓	↓	↑
↑	↓	↓
↓	↑	↓

Glossary –

↑ - Increasing

↓ - For decreasing f(x).

∴ $x_1 < x_2$

∴ g (x_1) > g (x_2)

Applying, $\tan^{-1}$ on both the sides, of the mentioned equation,

$\therefore \tan^{-1}\{g(x_1)\} > \tan^{-1}\{g(x_2)\}$

$\therefore f(x_1) > f(x_2)$

Hence it is decreasing on R.

11. Question

Write the set of values of a for which the function f(x) = ax + b is decreasing for all $x \in R$.

Answer

f(x) = ax + b

f'(x) = a + 0

f'(x) = a

For, f(x) to be decreasing, it must have

f'(x) < 0

$\therefore$ a < 0

$\therefore a \in (-\infty, 0)$

Hence the required set of values is $a \in (-\infty, 0)$.

12. Question

Write the interval in which f(x) = sin x + cos x, $x \in [0, \pi/2]$ is increasing.

Answer

f(x) = Sin x + Cos x, $x \in [0, \frac{\pi}{2}]$

f'(x) = Cos x – Sin x

For, f(x) to be increasing, it must have

f'(x) > 0

Cos x – Sin x > 0

Sin x < Cos x

$\frac{\text{Sin x}}{\text{Cos x}} < 1$

tan x < 1

$\tan x < \tan\frac{\pi}{4}$

$\therefore x \in \left[0, \frac{\pi}{4}\right)$

Hence, the required interval is $x \in \left[0, \frac{\pi}{4}\right)$.

13. Question

State whether f(x) = tanx – x is increasing or decreasing its domain.

Answer

f(x) = tan x – x

$f'(x) = Sec^2x - 1$

$Sec^2x - 1 \geq 0$

$Sec^2x \geq 1$

∵ $1 + \tan^2 x = \sec^2 x$

∴ $1 + \tan^2 x \geq 1$

$\tan^2 x \geq 0$

∴ $\tan^2 x \geq 0\ \forall x \in [0, 2\pi]$

∴ f(x) is increasing its domain.

14. Question

Write the set of values of a for which $f(x) = \cos x + a^2 x + b$ is strictly increasing on R.

Answer

$f(x) = \cos x + a^2 x + b$

$f'(x) = -\sin x + a^2 + 0$

$f'(x) = a^2 - \sin x$

∵ f(x) is strictly increasing on R

∴ $f'(x) > 0, \forall x \in R$

∴ $a^2 - \sin x > 0, \forall x \in R$

∴ $a^2 > \sin x, \forall x \in R$

∵ Maximum value of Sin x is 1.

∴ $a^2 > \sin x$, a^2 is always greater than 1.

$a^2 > 1$

$a^2 - 1 > 0$

$(a + 1)(a - 1) > 0$

∴ $a \in (-\infty, -1) \cup (1, \infty)$