

2. Functions

Exercise 2.1

1 A. Question

Give an example of a function

Which is one - one but not onto.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Now, Let, $f: \mathbb{N} \rightarrow \mathbb{N}$ given by $f(x) = x^2$

Check for Injectivity:

Let x,y be elements belongs to N i.e $x, y \in \mathbb{N}$ such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow x^2 - y^2 = 0$$

$$\Rightarrow (x - y)(x + y) = 0$$

As $x, y \in \mathbb{N}$ therefore $x + y > 0$

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = y$$

Hence f is One - One function

Check for Surjectivity:

Let y be element belongs to N i.e $y \in \mathbb{N}$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^2 = y$$

$$\Rightarrow x = \sqrt{y}$$

$\Rightarrow \sqrt{y}$ not belongs to N for non-perfect square value of y.

Therefore no non - perfect square value of y has a pre image in domain N.

Hence, $f: \mathbb{N} \rightarrow \mathbb{N}$ given by $f(x) = x^2$ is One - One but not onto.

1 B. Question

Give an example of a function

Which is not one - one but onto.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Now, Let, $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^3 - x$

Check for Injectivity:

Let x,y be elements belongs to R i.e $x, y \in \mathbb{R}$ such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^3 - x = y^3 - y$$

$$\Rightarrow x^3 - y^3 - (x - y) = 0$$

$$\Rightarrow (x - y)(x^2 + xy + y^2 - 1) = 0$$

$$\text{As } x^2 + xy + y^2 \geq 0$$

$$\Rightarrow \text{therefore } x^2 + xy + y^2 - 1 \geq -1$$

$$\Rightarrow x - y \neq 0$$

$$\Rightarrow x \neq y \text{ for some } x, y \in \mathbb{R}$$

Hence f is not One - One function

Check for Surjectivity:

Let y be element belongs to R i.e $y \in \mathbb{R}$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^3 - x = y$$

$$\Rightarrow x^3 - x - y = 0$$

Now, we know that for 3 degree equation has a real root

So, let $x = \alpha$ be that root

$$\Rightarrow \alpha^3 - \alpha = y$$

$$\Rightarrow f(\alpha) = y$$

Thus for clearly $y \in \mathbb{R}$, there exist $\alpha \in \mathbb{R}$ such that $f(x) = y$

Therefore f is onto

\Rightarrow Hence, $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^3 - x$ is not One - One but onto

1 C. Question

Give an example of a function

Which is neither one - one nor onto.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Now, Let, $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 5$

As we know

A constant function is neither one - one nor onto.

So, here $f(x) = 5$ is constant function

Therefore

$f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 5$ is neither one - one nor onto function.

2 A. Question

Which of the following functions from A to B are one - one and onto?

$$f_1 = \{(1, 3), (2, 5), (3, 7)\}; A = \{1, 2, 3\}, B = \{3, 5, 7\}$$

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Now, As given,

$$f_1 = \{(1, 3), (2, 5), (3, 7)\}$$

$$A = \{1, 2, 3\}, B = \{3, 5, 7\}$$

Thus we can see that,

Check for Injectivity:

Every element of A has a different image from B

Hence f is a One – One function

Check for Surjectivity:

Also, each element of B is an image of some element of A

Hence f is Onto.

2 B. Question

Which of the following functions from A to B are one – one and onto?

$$f_2 = \{(2, a), (3, b), (4, c)\}; A = \{2, 3, 4\}, B = \{a, b, c\}$$

Answer

TIP: – One – One Function: – A function $f: A \rightarrow B$ is said to be a one – one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One – One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: – A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co – domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Now, As given,

$$f_2 = \{(2, a), (3, b), (4, c)\}$$

$$A = \{2, 3, 4\}, B = \{a, b, c\}$$

Thus we can see that

Check for Injectivity:

Every element of A has a different image from B

Hence f is a One – One function

Check for Surjectivity:

Also, each element of B is an image of some element of A

Hence f is Onto.

2 C. Question

Which of the following functions from A to B are one – one and onto?

$$f_3 = \{(a, x), (b, x), (c, z), (d, z)\}; A = \{a, b, c, d\}, B = \{x, y, z\}$$

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Now, As given,

$$f_3 = \{(a, x), (b, x), (c, z), (d, z)\}$$

$$A = \{a, b, c, d\}, B = \{x, y, z\}$$

Thus we can clearly see that

Check for Injectivity:

Every element of A does not have different image from B

Since,

$$f_3(a) = x = f_3(b) \text{ and } f_3(c) = z = f_3(d)$$

Therefore f is not One - One function

Check for Surjectivity:

Also each element of B is not image of any element of A

Hence f is not Onto.

3. Question

Prove that the function $f: \mathbb{N} \rightarrow \mathbb{N}$, defined by $f(x) = x^2 + x + 1$ is one - one but not onto.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Now, $f: \mathbb{N} \rightarrow \mathbb{N}$ given by $f(x) = x^2 + x + 1$

Check for Injectivity:

Let x, y be elements belongs to \mathbb{N} i.e $x, y \in \mathbb{N}$ such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^2 + x + 1 = y^2 + y + 1$$

$$\Rightarrow x^2 - y^2 + x - y = 0$$

$$\Rightarrow (x - y)(x + y + 1) = 0$$

As $x, y \in \mathbb{N}$ therefore $x + y + 1 > 0$

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = y$$

Hence f is One - One function

Check for Surjectivity:

y be element belongs to \mathbb{N} i.e $y \in \mathbb{N}$ be arbitrary

Since for $y > 1$, we do not have any pre image in domain \mathbb{N} .

Hence, f is not Onto function.

4. Question

Let $A = \{-1, 0, 1\}$ and $f = \{(x, x^2) : x \in A\}$. Show that $f : A \rightarrow A$ is neither one - one nor onto.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Now, We have, $A = \{-1, 0, 1\}$ and $f = \{(x, x^2) : x \in A\}$.

To Prove: - $f : A \rightarrow A$ is neither One - One nor onto function

Check for Injectivity:

We can clearly see that

$$f(1) = 1$$

$$\text{and } f(-1) = 1$$

Therefore

$$f(1) = f(-1)$$

\Rightarrow Every element of A does not have different image from A

Hence f is not One - One function

Check for Surjectivity:

Since, $y = -1$ be element belongs to A

i.e $-1 \in A$ in co - domain does not have any pre image in domain A .

Hence, f is not Onto function.

5 A. Question

Classify the following functions as injection, surjection or bijection:

$f : \mathbb{N} \rightarrow \mathbb{N}$ given by $f(x) = x^2$

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

Now, $f: \mathbb{N} \rightarrow \mathbb{N}$ given by $f(x) = x^2$

Check for Injectivity:

Let x, y be elements belongs to \mathbb{N} i.e $x, y \in \mathbb{N}$ such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow x^2 - y^2 = 0$$

$$\Rightarrow (x - y)(x + y) = 0$$

As $x, y \in \mathbb{N}$ therefore $x + y > 0$

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = y$$

Hence f is One - One function

Check for Surjectivity:

Let y be element belongs to \mathbb{N} i.e $y \in \mathbb{N}$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^2 = y$$

$$\Rightarrow x = \sqrt{y}$$

$\Rightarrow \sqrt{y}$ not belongs to N for non-perfect square value of y .

Therefore no non - perfect square value of y has a pre-image in domain N .

Hence, f is not Onto function.

Thus, Not Bijective also.

5 B. Question

Classify the following functions as injection, surjection or bijection:

$f : Z \rightarrow Z$ given by $f(x) = x^2$

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of B i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

Now, $f : Z \rightarrow Z$ given by $f(x) = x^2$

Check for Injectivity:

Let $x_1, -x_1$ be elements belongs to Z i.e $x_1, -x_1 \in Z$ such that

So, from definition

$$\Rightarrow x_1 \neq -x_1$$

$$\Rightarrow (x_1)^2 = (-x_1)^2$$

$$\Rightarrow f(x_1)^2 = f(-x_1)^2$$

Hence f is not One - One function

Check for Surjectivity:

Let y be element belongs to Z i.e $y \in Z$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^2 = y$$

$$\Rightarrow x = \pm\sqrt{y}$$

$\Rightarrow \sqrt{y}$ not belongs to Z for non-perfect square value of y .

Therefore no non - perfect square value of y has a pre-image in domain Z .

Hence, f is not Onto function.

Thus, Not Bijective also

5 C. Question

Classify the following functions as injection, surjection or bijection:

$f : \mathbb{N} \rightarrow \mathbb{N}$ given by $f(x) = x^3$

Answer

TIP: – One – One Function: – A function $f: A \rightarrow B$ is said to be a one – one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One – One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: – A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co – domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: – A function $f: A \rightarrow B$ is said to be a bijection function if it is one – one as well as onto function.

Now, $f : \mathbb{N} \rightarrow \mathbb{N}$ given by $f(x) = x^3$

Check for Injectivity:

Let x, y be elements belongs to \mathbb{N} i.e $x, y \in \mathbb{N}$ such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^3 = y^3$$

$$\Rightarrow x^3 - y^3 = 0$$

$$\Rightarrow (x - y)(x^2 + y^2 + xy) = 0$$

As $x, y \in \mathbb{N}$ therefore $x^2 + y^2 + xy > 0$

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = y$$

Hence f is One – One function

Check for Surjectivity:

Let y be element belongs to \mathbb{N} i.e $y \in \mathbb{N}$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^3 = y$$

$$\Rightarrow x = \sqrt[3]{y}$$

$\Rightarrow \sqrt[3]{y}$ not belongs to \mathbb{N} for non-perfect cube value of y .

Since f attain only cubic number like 1, 8, 27, ...,

Therefore no non – perfect cubic values of y in \mathbb{N} (co – domain) has a pre-image in domain \mathbb{N} .

Hence, f is not onto function

Thus, Not Bijective also

5 D. Question

Classify the following functions as injection, surjection or bijection:

$f : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(x) = x^3$

Answer

TIP: – One – One Function: – A function $f: A \rightarrow B$ is said to be a one – one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One – One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: – A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of B i.e, if $f(A) = B$ or range of f is the co – domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: – A function $f: A \rightarrow B$ is said to be a bijection function if it is one – one as well as onto function.

Now, $f : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(x) = x^3$

Check for Injectivity:

Let x, y be elements belongs to \mathbb{Z} i.e $x, y \in \mathbb{Z}$ such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^3 = y^3$$

$$\Rightarrow x^3 - y^3 = 0$$

$$\Rightarrow x = y$$

Hence f is One – One function

Check for Surjectivity:

Let y be element belongs to \mathbb{Z} i.e $y \in \mathbb{Z}$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^3 = y$$

$$\Rightarrow x = \sqrt[3]{y}$$

$$\Rightarrow \sqrt[3]{y} \text{ not belongs to } \mathbb{Z} \text{ for non – perfect cube value of } y.$$

Since f attain only cubic number like 1, 8, 27,

Therefore no non – perfect cubic values of y in \mathbb{Z} (co – domain) have a pre-image in domain \mathbb{Z} .

Hence, f is not onto function

Thus, Not Bijective also

5 E. Question

Classify the following functions as injection, surjection or bijection:

$f : \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(x) = |x|$

Answer

TIP: – One – One Function: – A function $f: A \rightarrow B$ is said to be a one – one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One – One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: – A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of B i.e, if $f(A) = B$ or range of f is the co – domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: – A function $f: A \rightarrow B$ is said to be a bijection function if it is one – one as well as onto function.

Now, $f : \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(x) = |x|$

Check for Injectivity:

Let x,y be elements belongs to \mathbb{R} i.e $x, y \in \mathbb{R}$ such that

Case i

$$\Rightarrow x = y$$

$$\Rightarrow |x| = |y|$$

Case ii

$$\Rightarrow -x = y$$

$$\Rightarrow |-x| = |y|$$

$$\Rightarrow x = |y|$$

Hence from case i and case ii f is not One – One function

Check for Surjectivity:

Since f attain only positive values, for negative real numbers in \mathbb{R}

(co – domain) there is no pre-image in domain \mathbb{R} .

Hence, f is not onto function

Thus, Not Bijective also

5 F. Question

Classify the following functions as injection, surjection or bijection:

$f : \mathbb{Z} \rightarrow \mathbb{Z}$, defined by $f(x) = x^2 + x$

Answer

TIP: – One – One Function: – A function $f: A \rightarrow B$ is said to be a one – one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

Now, $f: Z \rightarrow Z$ given by $f(x) = x^2 + x$

Check for Injectivity:

Let x, y be elements belongs to Z i.e $x, y \in Z$ such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^2 + x = y^2 + y$$

$$\Rightarrow x^2 - y^2 + x - y = 0$$

$$\Rightarrow (x - y)(x + y + 1) = 0$$

Either $(x - y) = 0$ or $(x + y + 1) = 0$

Case i :

$$\text{If } x - y = 0$$

$$\Rightarrow x = y$$

Hence f is One - One function

Case ii :

$$\text{If } x + y + 1 = 0$$

$$\Rightarrow x + y = -1$$

$$\Rightarrow x \neq y$$

Hence f is not One - One function

Thus from case i and case ii f is not One - One function

Check for Surjectivity:

As $1 \in Z$

Let x be element belongs to Z i.e $x \in Z$ be arbitrary, then

$$\Rightarrow f(x) = 1$$

$$\Rightarrow x^2 + x = 1$$

$$\Rightarrow x^2 + x - 1 = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1+4}}{2}$$

Above value of x does not belong to Z

Therefore no values of x in Z (co - domain) have a pre-image in domain Z .

Hence, f is not onto function

Thus, Not Bijective also

5 G. Question

Classify the following functions as injection, surjection or bijection:

$f : Z \rightarrow Z$, defined by $f(x) = x - 5$

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of B i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

Now, $f : Z \rightarrow Z$ given by $f(x) = x - 5$

Check for Injectivity:

Let x, y be elements belongs to Z i.e $x, y \in Z$ such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x - 5 = y - 5$$

$$\Rightarrow x = y$$

Hence, f is One - One function

Check for Surjectivity:

Let y be element belongs to Z i.e $y \in Z$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x - 5 = y$$

$$\Rightarrow x = y + 5$$

Above value of x belongs to Z

Therefore for each element in Z (co - domain) there exists an element in domain Z .

Hence, f is onto function

Thus, Bijective function

5 H. Question

Classify the following functions as injection, surjection or bijection:

$f : R \rightarrow R$, defined by $f(x) = \sin x$

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

Now, $f: \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(x) = \sin x$

Check for Injectivity:

Let x, y be elements belongs to \mathbb{R} i.e $x, y \in \mathbb{R}$ such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow \sin x = \sin y$$

$$\Rightarrow x = n\pi + (-1)^n y$$

$$\Rightarrow x \neq y$$

Hence, f is not One - One function

Check for Surjectivity:

Let y be element belongs to \mathbb{R} i.e $y \in \mathbb{R}$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow \sin x = y$$

$$\Rightarrow x = \sin^{-1} y$$

Now, for $y > 1$ x not belongs to \mathbb{R} (Domain)

Hence, f is not onto function

Thus, It is also not Bijective function

5 I. Question

Classify the following functions as injection, surjection or bijection:

$f: \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(x) = x^3 + 1$

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of B i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

Now, Let, $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^3 + 1$

Check for Injectivity:

Let x, y be elements belongs to \mathbb{R} i.e $x, y \in \mathbb{R}$ such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^3 + 1 = y^3 + 1$$

$$\Rightarrow x^3 = y^3$$

$$\Rightarrow x = y$$

Hence f is One - One function

Check for Surjectivity:

Let y be element belongs to \mathbb{R} i.e $y \in \mathbb{R}$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^3 + 1 = y$$

Now, we know that for 3 degree equation has a real root

So, let $x = \alpha$ be that root

$$\Rightarrow \alpha^3 + 1 = y$$

$$\Rightarrow f(\alpha) = y$$

Thus for clearly $y \in \mathbb{R}$, there exist $\alpha \in \mathbb{R}$ such that $f(x) = y$

Therefore f is onto

Thus, It is also Bijective function

5 J. Question

Classify the following functions as injection, surjection or bijection:

$$f: \mathbb{R} \rightarrow \mathbb{R}, \text{ defined by } f(x) = x^3 - x$$

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of B i.e, if $f(A) = B$ or range of f is the co - domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

Now, Let, $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^3 + x$

Check for Injectivity:

Let x,y be elements belongs to \mathbb{R} i.e $x, y \in \mathbb{R}$ such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^3 - x = y^3 - y$$

$$\Rightarrow x^3 - y^3 - (x - y) = 0$$

$$\Rightarrow (x - y)(x^2 + xy + y^2 - 1) = 0$$

Hence f is not One - One function

Check for Surjectivity:

Let y be element belongs to \mathbb{R} i.e $y \in \mathbb{R}$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^3 - x = y$$

$$\Rightarrow x^3 - x - y = 0$$

Now, we know that for 3 degree equation has a real root

So, let $x = \alpha$ be that root

$$\Rightarrow \alpha^3 - \alpha = y$$

$$f(\alpha) = y$$

Thus for clearly $y \in \mathbb{R}$, there exist $\alpha \in \mathbb{R}$ such that $f(x) = y$

Therefore f is onto

Thus, It is not Bijective function

5 K. Question

Classify the following functions as injection, surjection or bijection:

$$f: \mathbb{R} \rightarrow \mathbb{R}, \text{ defined by } f(x) = \sin^2 x + \cos^2 x$$

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: – A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of B i.e, if $f(A) = B$ or range of f is the co – domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: – A function $f: A \rightarrow B$ is said to be a bijection function if it is one – one as well as onto function.

Now, $f: \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(x) = \sin^2 x + \cos^2 x$

Check for Injectivity and Check for Surjectivity

Let x be element belongs to \mathbb{R} i.e $x \in \mathbb{R}$ such that

So, from definition

$$\Rightarrow f(x) = \sin^2 x + \cos^2 x$$

$$\Rightarrow f(x) = \sin^2 x + \cos^2 x$$

$$\Rightarrow f(x) = 1$$

$$\Rightarrow f(x) = \text{constant}$$

We know that a constant function is neither One – One function nor onto function.

Thus, It is not Bijective function

5 L. Question

Classify the following functions as injection, surjection or bijection:

$$f: \mathbb{Q} - \{3\} \rightarrow \mathbb{Q}, \text{ defined by } f(x) = \frac{2x+3}{x-3}$$

Answer

TIP: – One – One Function: – A function $f: A \rightarrow B$ is said to be a one – one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One – One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: – A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of B i.e, if $f(A) = B$ or range of f is the co – domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: – A function $f: A \rightarrow B$ is said to be a bijection function if it is one – one as well as onto function.

$$\text{Now, } f: \mathbb{R} \rightarrow \mathbb{R} \text{ given by } f(x) = \frac{2x+3}{x-3}$$

Check for Injectivity:

Let x,y be elements belongs to \mathbb{Q} i.e $x, y \in \mathbb{Q}$ such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow \frac{2x+3}{x-3} = \frac{2y+3}{y-3}$$

$$\Rightarrow (2x+3)(y-3) = (2y+3)(x-3)$$

$$\Rightarrow 2xy - 6x + 3y - 9 = 2xy - 6y + 3x - 9$$

$$\Rightarrow -6x + 3y = -6y + 3x$$

$$\Rightarrow -6x + 3y + 6y - 3x = 0$$

$$\Rightarrow -9x + 9y = 0$$

$$\Rightarrow x = y$$

Thus, f is One - One function

Check for Surjectivity:

Let y be element belongs to Q i.e $y \in Q$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow \frac{2x+3}{x-3} = y$$

$$\Rightarrow 2x + 3 = y(x - 3)$$

$$\Rightarrow 2x + 3 = xy - 3y$$

$$\Rightarrow 2x - xy = -3(y + 1)$$

$$\Rightarrow x = \frac{-3(y+1)}{2-y}$$

Above value of x belongs to Q - [3] for y = 2

Therefore for each element in Q - [3] (co - domain), there does not exist an element in domain Q.

Hence, f is not onto function

Thus, Not Bijective function

5 M. Question

Classify the following functions as injection, surjection or bijection:

$f : Q \rightarrow Q$, defined by $f(x) = x^3 + 1$

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

Now, $f : \mathbb{Q} \rightarrow \mathbb{Q}$, defined by $f(x) = x^3 + 1$

Check for Injectivity:

Let x, y be elements belongs to \mathbb{Q} i.e $x, y \in \mathbb{Q}$ such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^3 + 1 = y^3 + 1$$

$$\Rightarrow x^3 = y^3$$

$$\Rightarrow x = y$$

Hence, f is One - One function

Check for Surjectivity:

Let y be element belongs to \mathbb{Q} i.e $y \in \mathbb{Q}$ be arbitrary, then

$$\Rightarrow x^3 + 1 = y$$

$$\Rightarrow x^3 + 1 - y = 0$$

Now, we know that for 3 degree equation has a real root

So, let $x = \alpha$ be that root

$$\Rightarrow \alpha^3 + 1 = y$$

$$\Rightarrow f(\alpha) = y$$

Thus for clearly $y \in \mathbb{Q}$, there exist $\alpha \in \mathbb{Q}$ such that $f(x) = y$

Therefore f is onto

Thus, It is a Bijective function

5 N. Question

Classify the following functions as injection, surjection or bijection:

$f : \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(x) = 5x^3 + 4$

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

Now, $f : \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(x) = 5x^3 + 4$

Check for Injectivity:

Let x, y be elements belongs to \mathbb{R} i.e $x, y \in \mathbb{R}$ such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow 5x^3 + 4 = 5y^3 + 4$$

$$\Rightarrow x^3 = y^3$$

$$\Rightarrow x = y$$

Hence, f is One - One function

Check for Surjectivity:

Let y be element belongs to \mathbb{R} i.e $y \in \mathbb{R}$ be arbitrary, then

$$\Rightarrow 5x^3 + 4 = y$$

$$\Rightarrow 5x^3 + 4 - y = 0$$

Now, we know that for 3 degree equation has a real root

So, let $x = \alpha$ be that root

$$\Rightarrow 5\alpha^3 + 4 = y$$

$$f(\alpha) = y$$

Thus for clearly $y \in \mathbb{R}$, there exist $\alpha \in \mathbb{R}$ such that $f(x) = y$

Therefore f is onto

Thus, It is a Bijective function

5 O. Question

Classify the following functions as injection, surjection or bijection:

$f : \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(x) = 3 - 4x$

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

Now, $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 3 - 4x$

Check for Injectivity:

Let x, y be elements belongs to \mathbb{R} i.e $x, y \in \mathbb{R}$ such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow 3 - 4x = 3 - 4y$$

$$\Rightarrow x = y$$

Hence, f is One - One function

Check for Surjectivity:

Let y be element belongs to R i.e. $y \in R$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow 3 - 4x = y$$

$$\Rightarrow x = \frac{3-y}{4}$$

Above value of x belongs to R

Therefore for each element in R (co - domain), there exists an element in domain R .

Hence, f is onto function

Thus, Bijective function

5 P. Question

Classify the following functions as injection, surjection or bijection:

$$f : R \rightarrow R, \text{ defined by } f(x) = 1 + x^2$$

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

Now, $f: R \rightarrow R$ given by $f(x) = 1 + x^2$

Check for Injectivity:

Let x, y be elements belongs to R i.e $x, y \in R$ such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^2 + 1 = y^2 + 1$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow \pm x = \pm y$$

Therefore, either $x = y$ or $x = -y$ or $x \neq y$

Hence f is not One - One function

Check for Surjectivity:

1 be element belongs to R i.e $1 \in R$ be arbitrary, then

$$\Rightarrow f(x) = 1$$

$$\Rightarrow x^2 + x = 1$$

$$\Rightarrow x^2 + x - 1 = 0$$

$$\Rightarrow x = \pm \sqrt{y-1}$$

Above value of x not belongs to R for $y < 1$

Therefore f is not onto

Thus, It is also not Bijective function

5 Q. Question

Classify the following functions as injection, surjection or bijection:

$$f: R \rightarrow R, \text{ defined by } f(x) = \frac{x}{x^2 + 1}$$

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

$$\text{Now, } f: R \rightarrow R \text{ given by } f(x) = \frac{x}{x^2 + 1}$$

Check for Injectivity:

Let x, y be elements belongs to R i.e. $x, y \in R$ such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow \frac{x}{x^2 + 1} = \frac{y}{y^2 + 1}$$

$$\Rightarrow xy^2 + x = yx^2 + y$$

$$\Rightarrow xy^2 + x - yx^2 - y = 0$$

$$\Rightarrow xy(y - x) + (x - y) = 0$$

$$\Rightarrow (x - y)(1 - xy) = 0$$

Case i :

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = y$$

f is One - One function

Case ii :

$$\Rightarrow 1 - xy = 0$$

$$\Rightarrow xy = 1$$

Thus from case i and case ii f is One - One function

Check for Surjectivity:

Let y be element belongs to R i.e $y \in \mathbb{R}$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow \frac{x}{x^2 + 1} = y$$

$$\Rightarrow x = x^2 y + y$$

$$\Rightarrow x - x^2 y = y$$

Above value of x belongs to R

Therefore for each element in R (co - domain) there exists an element in domain R.

Hence, f is onto function

Thus, Bijjective function

6. Question

If $f: A \rightarrow B$ is an injection such that range of $f = \{a\}$. Determine the number of elements in A.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Here, Range $\{f\} = \{a\}$

Since it is injective map, different elements have different images.

Thus A has only one element

7. Question

Show that the function $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{1\}$ given by $f(x) = \frac{x-2}{x-3}$ is a bijection.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

Now, $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = \frac{x-2}{x-3}$

To Prove: - $f(x) = \frac{x-2}{x-3}$ is a bijection

Check for Injectivity:

Let x, y be elements belongs to \mathbb{R} i.e. $x, y \in \mathbb{R}$ such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$$

$$\Rightarrow (x-2)(y-3) = (x-3)(y-2)$$

$$\Rightarrow xy - 3x - 2y + 6 = xy - 2x - 3y + 6$$

$$\Rightarrow -3x - 2y + 2x + 3y = 0$$

$$\Rightarrow -x + y = 0$$

$$\Rightarrow x = y$$

Hence, f is One - One function

Check for Surjectivity:

Let y be element belongs to \mathbb{R} i.e. $y \in \mathbb{R}$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow \frac{x-2}{x-3} = y$$

$$\Rightarrow x - 2 = xy - 3y$$

$$\Rightarrow x - xy = 2 - 3y$$

$$\Rightarrow x = \frac{2-3y}{1-y}$$

$x = \frac{2-3y}{1-y}$ is a real number for all $y \neq 1$.

Also, $\frac{2-3y}{1-y} \neq 2$ for any y

Therefore for each element in \mathbb{R} (co - domain), there exists an element in domain \mathbb{R} .

Hence, f is onto function

Thus, Bijection function

8 A. Question

Let $A = [-1, 1]$, Then, discuss whether the following functions from A to itself are one – one, onto or bijective:

$$f(x) = \frac{x}{2}$$

Answer

TIP: – One – One Function: – A function $f: A \rightarrow B$ is said to be a one – one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One – One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: – A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co – domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in B$ such that $f(a) = b$

Bijection Function: – A function $f: A \rightarrow B$ is said to be a bijection function if it is one – one as well as onto function.

Now, here $f: A \rightarrow A: A = [-1, 1]$ given by function is $f(x) = \frac{x}{2}$

Check for Injectivity:

Let x, y be elements belongs to A i.e. $x, y \in A$ such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow \frac{x}{2} = \frac{y}{2}$$

$$\Rightarrow 2x = 2y$$

$$\Rightarrow x = y$$

1 belongs to A then

$$f(1) = \frac{1}{2}$$

Not element of A co – domain

Hence, f is not One – One function

Check for Surjectivity:

Let y be element belongs to A i.e $y \in A$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow \frac{x}{2} = y$$

$$\Rightarrow x = 2y$$

Now,

1 belongs to A

$\Rightarrow x = 2$, which not belong to A co – domain

Hence, f is not onto function

Thus, It is not Bijective function

8 B. Question

Let $A = [-1, 1]$, Then, discuss whether the following functions from A to itself are one – one, onto or bijective:

$$g(x) = |x|$$

Answer

TIP: - One – One Function: - A function $f: A \rightarrow B$ is said to be a one – one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One – One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co – domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one – one as well as onto function.

Now, here $f: A \rightarrow A : A = [-1, 1]$ given by function is $g(x) = |x|$

Check for Injectivity:

Let x, y be elements belongs to A i.e $x, y \in A$ such that

$$\Rightarrow g(x) = g(y)$$

$$\Rightarrow |x| = |y|$$

$$\Rightarrow x = y$$

1 belongs to A then

$$\Rightarrow g(1) = 1 = g(-1)$$

Since, it has many element of A co – domain

Hence, g is not One – One function

Check for Surjectivity:

Let y be element belongs to A i.e $y \in A$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow \frac{x}{2} = y$$

$$\Rightarrow x = 2y$$

Now,

1 belongs to A

$$\Rightarrow x = 2, \text{ which not belong to } A \text{ co – domain}$$

Since g attain only positive values, for negative – 1 in A (co – domain) there is no pre-image in domain A .

Hence, g is not onto function

Thus, It is not Bijective function

8 C. Question

Let $A = [-1, 1]$, Then, discuss whether the following functions from A to itself are one – one, onto or bijective:

$$h(x) = x^2$$

Answer

TIP: – One – One Function: – A function $f: A \rightarrow B$ is said to be a one – one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One – One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: – A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co – domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in B$ such that $f(a) = b$

Bijection Function: – A function $f: A \rightarrow B$ is said to be a bijection function if it is one – one as well as onto function.

Now, here $f: A \rightarrow A: A = [-1, 1]$ given by function is $h(x) = x^2$

Check for Injectivity:

Let x, y be elements belongs to A i.e. $x, y \in A$ such that

$$\Rightarrow h(x) = h(y)$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow \pm x = \pm y$$

Since it has many elements of A co – domain

Hence, h is not One – One function

Check for Surjectivity:

Let y be element belongs to A i.e. $y \in A$ be arbitrary, then

$$\Rightarrow h(x) = y$$

$$\Rightarrow x^2 = y$$

$$\Rightarrow x = \pm\sqrt{y}$$

Since h have no pre-image in domain A.

Hence, h is not onto function

Thus, It is not Bijective function

9 A. Question

Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective:

$\{(x, y): x \text{ is a person, } y \text{ is the mother of } x\}$

Answer

TIP: – One – One Function: – A function $f: A \rightarrow B$ is said to be a one – one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One – One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: – A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co – domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Here, It is given (x, y): x is a person, y is the mother of x

As we know each person “x” has only one biological mother

Thus,

Given relation is a function

Since more than one person may have the same mother

Function, not One – One (injective) but Onto (Surjective)

9 B. Question

Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective:

$\{(a, b) : a \text{ is a person, } b \text{ is an ancestor of } a\}$

Answer

TIP: – One – One Function: – A function $f: A \rightarrow B$ is said to be a one – one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One – One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: – A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co – domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Here, It is given (a, b): a is a person, b is an ancestor of a

As we know any person “a” has more than one ancestor

Thus,

Given relation is not a function

10. Question

Let $A = \{1, 2, 3\}$. Write all one – one from A to itself.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

We have $A = \{1, 2, 3\}$

So all one - one functions from $A = \{1, 2, 3\}$ to itself are obtained by re - arranging elements of A.

Thus all possible one - one functions are:

$$f(1) = 1, f(2) = 2, f(3) = 3$$

$$f(1) = 2, f(2) = 3, f(3) = 1$$

$$f(1) = 3, f(2) = 1, f(3) = 2$$

$$f(1) = 1, f(2) = 3, f(3) = 2$$

$$f(1) = 3, f(2) = 2, f(3) = 1$$

$$f(1) = 2, f(2) = 1, f(3) = 3$$

11. Question

If $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x) = 4x^3 + 7$, show that f is a bijection.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

Now, $f: \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(x) = 4x^3 + 7$

To Prove : - $f: \mathbb{R} \rightarrow \mathbb{R}$ is bijective defined by $f(x) = 4x^3 + 7$

Check for Injectivity:

Let x,y be elements belongs to \mathbb{R} i.e $x, y \in \mathbb{R}$ such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow 4x^3 + 7 = 4y^3 + 7$$

$$\Rightarrow x^3 = y^3$$

$$\Rightarrow x = y$$

Hence, f is One - One function

Check for Surjectivity:

Let y be element belongs to \mathbb{R} i.e. $y \in \mathbb{R}$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow 4x^3 + 7 = y$$

$$\Rightarrow 4x^3 + 7 - y = 0$$

Now, we know that for 3 degree equation has a real root

So, let $x = \alpha$ be that root

$$\Rightarrow 4\alpha^3 + 7 = y$$

$$\Rightarrow f(\alpha) = y$$

Thus for clearly $y \in \mathbb{R}$, there exist $\alpha \in \mathbb{R}$ such that $f(x) = y$

Therefore f is onto

Thus, It is Bijective function

Hence Proved

12. Question

Show that the exponential function $f: \mathbb{R} \rightarrow \mathbb{R}$, given by $f(x) = e^x$, is one - one but not onto. What happens if the co - domain is replaced by \mathbb{R}_0^+ (set of all positive real numbers).

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Now, $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = e^x$

Check for Injectivity:

Let x, y be elements belongs to \mathbb{R} i.e $x, y \in \mathbb{R}$ such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow e^x = e^y$$

$$\Rightarrow \frac{e^x}{e^y} = 1$$

$$\Rightarrow e^{x-y} = 1$$

$$\Rightarrow e^{x-y} = e^0$$

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = y$$

Hence f is One - One function

Check for Surjectivity:

Here range of $f = (0, \infty) \neq \mathbb{R}$

Therefore f is not onto

Now if co - domain is replaced by \mathbb{R}_0^+ (set of all positive real numbers) i.e. $(0, \infty)$ then f becomes an onto function.

13. Question

Show that the logarithmic function $f : \mathbb{R}_+^0 \rightarrow \mathbb{R}$ given by $f(x) = \log_a x$, $a > 0$ is a bijection.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

To Prove : - Logarithmic function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ given by $f(x) = \log_a x$, $a > 0$ is a bijection.

Now, $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ given by $f(x) = \log_a x$, $a > 0$

Check for Injectivity:

Let x,y be elements belongs to \mathbb{R}_0^+ i.e $x, y \in \mathbb{R}_0^+$ such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow \log_a x = \log_a y$$

$$\Rightarrow \log_a x - \log_a y = 0$$

$$\Rightarrow \log_a \left(\frac{x}{y} \right) = 0$$

$$\Rightarrow \frac{x}{y} = 1$$

$$\Rightarrow x = y$$

Hence f is One - One function

Check for Surjectivity:

Let y be element belongs to \mathbb{R} i.e. $y \in \mathbb{R}$ be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow \log_a x = y$$

$$\Rightarrow x = a^y$$

Above value of x belongs to \mathbb{R}_0^+

Therefore, for all $y \in \mathbb{R}$ there exist $x = a^y$ such that $f(x) = y$.

Hence, f is Onto function.

Thus, it is Bijective also

14. Question

If $A = \{1, 2, 3\}$, show that a one - one function $f : A \rightarrow A$ must be onto.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in B$ such that $f(a) = b$

Now, $f: A \rightarrow A$ where $A = \{1, 2, 3\}$ and its a One - One function

To Prove: - A is Onto function

Since it is given that f is a One - One function,

Three elements of $A = \{1, 2, 3\}$ must be taken to 3 different elements of co - domain $A = \{1, 2, 3\}$ under f .

Thus by definition of Onto Function

f has to be Onto function.

Hence Proved

15. Question

If $A = \{1, 2, 3\}$, show that an onto function $f : A \rightarrow A$ must be one - one.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: – A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co – domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Now, $f : A \rightarrow A$ where $A = \{1, 2, 3\}$ and its an Onto function

To Prove: – A is a One – One function

Let's assume f is not Onto function,

Then,

There must be two elements let it be 1 and 2 in Domain $A = \{1, 2, 3\}$ whose images in co-domain $A = \{1, 2, 3\}$ is same.

Also, Image of 3 under f can be only one element.

Therefore,

Range set can have at most two elements in co – domain $A = \{1, 2, 3\}$

$\Rightarrow f$ is not an onto function

Hence it contradicts

$\Rightarrow f$ must be One – One function

Hence Proved

16. Question

Find the number of all onto functions from the set $A = \{1, 2, 3, \dots, n\}$ to itself.

Answer

TIP: –

Onto Function: – A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of A i.e, if $f(A) = B$ or range of f is the co – domain of f.

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Now, $f : A \rightarrow A$ where $A = \{1, 2, 3, \dots, n\}$

All onto function

It's a permutation of n symbols 1,2,3,...,n

Thus,

Total number of Onto maps from $A = \{1, 2, 3, \dots, n\}$ to itself =

Total number of permutations of n symbols 1,2,3,...,n.

17. Question

Give examples of two one – one functions f_1 and f_2 from R to R such that $f_1 + f_2 : R \rightarrow R$, defined by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ is not one – one.

Answer

TIP: – One – One Function: – A function $f: A \rightarrow B$ is said to be a one – one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$a = b \text{ for all } a, b \in A$$

Let, $f_1: \mathbb{R} \rightarrow \mathbb{R}$ and $f_2: \mathbb{R} \rightarrow \mathbb{R}$ be two functions given by (Examples)

$$f_1(x) = x$$

$$f_2(x) = -x$$

From above function it is clear that both are One - One functions

Now,

$$\Rightarrow (f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$\Rightarrow (f_1 + f_2)(x) = x - x$$

$$\Rightarrow (f_1 + f_2)(x) = 0$$

Therefore,

$f_1 + f_2 : \mathbb{R} \rightarrow \mathbb{R}$ is a function given by

$$(f_1 + f_2)(x) = 0$$

Since $f_1 + f_2$ is a constant function,

Hence it is not an One - One function.

18. Question

Give examples of two surjective function f_1 and f_2 from \mathbb{Z} to \mathbb{Z} such that $f_1 + f_2$ is not surjective.

Answer

TIP: -

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of B i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Let, $f_1: \mathbb{Z} \rightarrow \mathbb{Z}$ and $f_2: \mathbb{Z} \rightarrow \mathbb{Z}$ be two functions given by (Examples)

$$f_1(x) = x$$

$$f_2(x) = -x$$

From above function it is clear that both are Onto or Surjective functions

Now,

$$f_1 + f_2 : \mathbb{Z} \rightarrow \mathbb{Z}$$

$$\Rightarrow (f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$\Rightarrow (f_1 + f_2)(x) = x - x$$

$$\Rightarrow (f_1 + f_2)(x) = 0$$

Therefore,

$f_1 + f_2 : \mathbb{Z} \rightarrow \mathbb{Z}$ is a function given by

$$(f_1 + f_2)(x) = 0$$

Since $f_1 + f_2$ is a constant function,

Hence it is not an Onto/Surjective function.

19. Question

Show that if f_1 and f_2 are one - one maps from \mathbb{R} to \mathbb{R} , then the product $f_1 \times f_2 : \mathbb{R} \rightarrow \mathbb{R}$ defined by $(f_1 \times f_2)(x) = f_1(x)f_2(x)$ need not be one - one.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$a = b \text{ for all } a, b \in A$$

Let, $f_1: \mathbb{R} \rightarrow \mathbb{R}$ and $f_2: \mathbb{R} \rightarrow \mathbb{R}$ are two functions given by

$$f_1(x) = x$$

$$f_2(x) = x$$

From above function it is clear that both are One - One functions

Now, $f_1 \times f_2 : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\Rightarrow (f_1 \times f_2)(x) = f_1(x) \times f_2(x) = x^2$$

$$\Rightarrow (f_1 \times f_2)(x) = x^2$$

Also,

$$f(1) = 1 = f(-1)$$

Therefore,

f is not One - One

$\Rightarrow f_1 \times f_2 : \mathbb{R} \rightarrow \mathbb{R}$ is not One - One function.

Hence Proved

20. Question

Suppose f_1 and f_2 are non - zero one - one functions from \mathbb{R} to \mathbb{R} . Is $\frac{f_1}{f_2}$ necessarily one - one? Justify your

answer. Here, $\frac{f_1}{f_2} : \mathbb{R} \rightarrow \mathbb{R}$ is given by $\left(\frac{f_1}{f_2}\right)(x) = \frac{f_1(x)}{f_2(x)}$ for all $x \in \mathbb{R}$.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$a = b \text{ for all } a, b \in A$$

Let, $f_1: \mathbb{R} \rightarrow \mathbb{R}$ and $f_2: \mathbb{R} \rightarrow \mathbb{R}$ are two non - zero functions given by

$$f_1(x) = x^3$$

$$f_2(x) = x$$

From above function it is clear that both are One - One functions

Now, $\frac{f_1}{f_2}: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\Rightarrow \frac{f_1}{f_2}(x) = \frac{f_1(x)}{f_2(x)}$$

$$\Rightarrow \frac{f_1}{f_2}(x) = x^2 \text{ for all } x \in \mathbb{R}$$

Again,

$\frac{f_1}{f_2} = f(\text{let}): \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = x^2$$

Now,

$$\Rightarrow f(1) = 1 = f(-1)$$

Therefore,

f is not One - One

$$\Rightarrow \frac{f_1}{f_2}: \mathbb{R} \rightarrow \mathbb{R} \text{ is not One - One function.}$$

Hence it is not necessarily to $\frac{f_1}{f_2}$ be one - one function.

21 A. Question

Given $A = \{2, 3, 4\}$, $B = \{2, 5, 6, 7\}$. Construct an example of each of the following:

an injective map from A to B

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Now, $f: A \rightarrow B$, denotes a mapping such that

$$\Rightarrow f = \{(x, y): y = x + 3\}$$

It can be written as follows in roster form

$$f = \{(2, 5), (3, 6), (4, 7)\}$$

Hence this is injective mapping

21 B. Question

Given $A = \{2, 3, 4\}$, $B = \{2, 5, 6, 7\}$. Construct an example of each of the following:

a mapping from A to B which is not injective

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Now, $f: A \rightarrow B$, denotes a mapping such that

$$f = \{(2,2), (3,5), (4,5)\}$$

Hence this is not injective mapping

21 C. Question

Given $A = \{2, 3, 4\}$, $B = \{2, 5, 6, 7\}$. Construct an example of each of the following:

a mapping from A to B.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Now, $f: A \rightarrow B$, denotes a mapping such that

$$f = \{(2,2), (5,3), (6,4), (7,4)\}$$

Here it is clear that every first component is from B and second component is from A

Hence this is mapping from B to A

22. Question

Show that $f: \mathbb{R} \rightarrow \mathbb{R}$, given by $f(x) = x - [x]$, is neither one - one nor onto.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B.

So, $f: A \rightarrow B$ is One - One function

$$\Leftrightarrow a \neq b$$

$\Rightarrow f(a) \neq f(b)$ for all $a, b \in A$

$\Leftrightarrow f(a) = f(b)$

$\Rightarrow a = b$ for all $a, b \in A$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of B i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Now, $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x - [x]$

To Prove: - $f(x) = x - [x]$, is neither one - one nor onto

Check for Injectivity:

Let x be element belongs to \mathbb{Z} i.e $x \in \mathbb{Z}$ such that

So, from definition

$\Rightarrow f(x) = x - [x]$

$\Rightarrow f(x) = 0$ for $x \in \mathbb{Z}$

Therefore,

Range of $f = [0,1] \neq \mathbb{R}$

Hence f is not One - One function

Check for Surjectivity:

Since Range of $f = [0,1] \neq \mathbb{R}$

Hence, f is not Onto function.

Thus, it is neither One - One nor Onto function

Hence Proved

23. Question

Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be defined by

$$f(n) = \begin{cases} n+1, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even} \end{cases}$$

Show that f is a bijection.

Answer

TIP: - One - One Function: - A function $f: A \rightarrow B$ is said to be a one - one functions or an injection if different elements of A have different images in B .

So, $f: A \rightarrow B$ is One - One function

$\Leftrightarrow a \neq b$

$\Rightarrow f(a) \neq f(b)$ for all $a, b \in A$

$\Leftrightarrow f(a) = f(b)$

$\Rightarrow a = b$ for all $a, b \in A$

Onto Function: - A function $f: A \rightarrow B$ is said to be a onto function or surjection if every element of B i.e, if $f(A) = B$ or range of f is the co - domain of f .

So, $f: A \rightarrow B$ is Surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$

Bijection Function: - A function $f: A \rightarrow B$ is said to be a bijection function if it is one - one as well as onto function.

Now, suppose

$$f(n_1) = f(n_2)$$

If n_1 is odd and n_2 is even, then we have

$$\Rightarrow n_1 + 1 = n_2 - 2$$

$$\Rightarrow n_2 - n_1 = 2$$

Not possible

Suppose both n_1 even and n_2 is odd.

$$\text{Then, } f(n_1) = f(n_2)$$

$$\Rightarrow n_1 - 1 = n_2 + 1$$

$$\Rightarrow n_1 - n_2 = 2$$

Not possible

Therefore, both n_1 and n_2 must be either odd or even

Suppose both n_1 and n_2 are odd.

$$\text{Then, } f(n_1) = f(n_2)$$

$$\Rightarrow n_1 + 1 = n_2 + 1$$

$$\Rightarrow n_1 = n_2$$

Suppose both n_1 and n_2 are even.

$$\text{Then, } f(n_1) = f(n_2)$$

$$\Rightarrow n_1 - 1 = n_2 - 1$$

$$\Rightarrow n_1 = n_2$$

Then, f is One - One

Also, any odd number $2r + 1$ in the co - domain N will have an even number as image in domain N which is

$$\Rightarrow f(n) = 2r + 1$$

$$\Rightarrow n - 1 = 2r + 1$$

$$\Rightarrow n = 2r + 2$$

Any even number $2r$ in the co - domain N will have an odd number as image in domain N which is

$$\Rightarrow f(n) = 2r$$

$$\Rightarrow n + 1 = 2r$$

$$\Rightarrow n = 2r - 1$$

Thus f is Onto function.

Exercise 2.2

1 A. Question

Find gof and fog when $f: R \rightarrow R$ and $g: R \rightarrow R$ is defined by

$$f(x) = 2x + 3 \text{ and } g(x) = x^2 + 5$$

Answer

Since, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$

$f \circ g: \mathbb{R} \rightarrow \mathbb{R}$ and $g \circ f: \mathbb{R} \rightarrow \mathbb{R}$

Now, $f(x) = 2x + 3$ and $g(x) = x^2 + 5$

$$g \circ f(x) = g(2x + 3) = (2x + 3)^2 + 5$$

$$\Rightarrow g \circ f(x) = 4x^2 + 12x + 9 + 5 = 4x^2 + 12x + 14$$

$$f \circ g(x) = f(g(x)) = f(x^2 + 5) = 2(x^2 + 5) + 3$$

$$\Rightarrow f \circ g(x) = 2x^2 + 10 + 3 = 2x^2 + 13$$

Hence, $g \circ f(x) = 4x^2 + 12x + 14$ and $f \circ g(x) = 2x^2 + 13$

1 B. Question

Find $g \circ f$ and $f \circ g$ when $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(x) = 2x + x^2 \text{ and } g(x) = x^3$$

Answer

Since, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$

$f \circ g: \mathbb{R} \rightarrow \mathbb{R}$ and $g \circ f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = 2x + x^2 \text{ and } g(x) = x^3$$

$$\text{Now, } g \circ f(x) = g(f(x)) = g(2x + x^2)$$

$$g \circ f(x) = (2x + x^2)^3 = x^6 + 8x^3 + 6x^5 + 12x^4$$

$$\text{and } f \circ g(x) = f(g(x)) = f(x^3)$$

$$\Rightarrow f \circ g(x) = 2x^3 + x^6$$

$$\text{So, } g \circ f(x) = x^6 + 6x^5 + 12x^4 + 8x^3 \text{ and } f \circ g(x) = 2x^3 + x^6$$

1 C. Question

Find $g \circ f$ and $f \circ g$ when $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(x) = x^2 + 8 \text{ and } g(x) = 3x^3 + 1$$

Answer

Since, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$

$f \circ g: \mathbb{R} \rightarrow \mathbb{R}$ and $g \circ f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x^2 + 8 \text{ and } g(x) = 3x^3 + 1$$

$$\text{So, } g \circ f(x) = g(f(x))$$

$$g \circ f(x) = g(x^2 + 8)$$

$$g \circ f(x) = 3(x^2 + 8)^3 + 1$$

$$\Rightarrow g \circ f(x) = 3(x^6 + 512 + 24x^4 + 192x^2) + 1$$

$$\Rightarrow g \circ f(x) = 3x^6 + 72x^4 + 576x^2 + 1537$$

$$\text{Similarly, } f \circ g(x) = f(g(x))$$

$$\Rightarrow f \circ g(x) = f(3x^3 + 1)$$

$$\Rightarrow f \circ g(x) = (3x^3 + 1)^2 + 8$$

$$\Rightarrow \text{fog}(x) = (9x^6 + 1 + 6x^3) + 8$$

$$\Rightarrow \text{fog}(x) = 9x^6 + 6x^3 + 9$$

$$\text{So, } \text{gof}(x) = 3x^6 + 72x^4 + 576x^2 + 1537 \text{ and } \text{fog}(x) = 9x^6 + 6x^3 + 9$$

1 D. Question

Find gof and fog when $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(x) = x \text{ and } g(x) = |x|$$

Answer

Since, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$

$\text{fog}: \mathbb{R} \rightarrow \mathbb{R}$ and $\text{gof}: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x \text{ and } g(x) = |x|$$

$$\text{Now, } \text{gof}(x) = g(f(x)) = g(x)$$

$$\Rightarrow \text{gof}(x) = |x|$$

$$\text{and, } \text{fog}(x) = f(g(x)) = f(|x|) \Rightarrow \text{fog}(x) = |x|$$

$$\text{Hence, } \text{gof}(x) = \text{fog}(x) = |x|$$

1 E. Question

Find gof and fog when $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(x) = x^2 + 2x - 3 \text{ and } g(x) = 3x - 4$$

Answer

Since, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$

$\text{fog}: \mathbb{R} \rightarrow \mathbb{R}$ and $\text{gof}: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x^2 + 2x - 3 \text{ and } g(x) = 3x - 4$$

$$\text{Now, } \text{gof}(x) = g(f(x)) = g(x^2 + 2x - 3)$$

$$\text{gof}(x) = 3(x^2 + 2x - 3) - 4$$

$$\Rightarrow \text{gof}(x) = 3x^2 + 6x - 9 - 4$$

$$\Rightarrow \text{gof}(x) = 3x^2 + 6x - 13$$

$$\text{and, } \text{fog} = f(g(x)) = f(3x - 4)$$

$$\text{fog}(x) = (3x - 4)^2 + 2(3x - 4) - 3$$

$$= 9x^2 + 16 - 24x + 6x - 8 - 3$$

$$\therefore \text{fog}(x) = 9x^2 - 18x + 5$$

$$\text{Thus, } \text{gof}(x) = 3x^2 + 6x - 13 \text{ and } \text{fog}(x) = 9x^2 - 18x + 5$$

1 F. Question

Find gof and fog when $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(x) = 8x^3 \text{ and } g(x) = x^{1/3}$$

Answer

Since, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$

$\text{fog}: \mathbb{R} \rightarrow \mathbb{R}$ and $\text{gof}: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = 8x^3 \text{ and } g(x) = x^{\frac{1}{3}}$$

$$\text{Now, } \text{gof}(x) = g(f(x)) = g(8x^3)$$

$$\Rightarrow \text{gof}(x) = (8x^3)^{\frac{1}{3}}$$

$$\text{gof}(x) = 2x$$

$$\text{and, } \text{fog}(x) = f(g(x)) = f(x^{\frac{1}{3}})$$

$$= 8\left(x^{\frac{1}{3}}\right)^3$$

$$\text{fog}(x) = 8x$$

$$\text{Thus, } \text{gof}(x) = 2x \text{ and } \text{fog}(x) = 8x$$

2. Question

Let $f = \{(3, 1), (9, 3), (12, 4)\}$ and $g = \{(1, 3), (3, 3), (4, 9), (5, 9)\}$. Show that gof and fog are both defined, Also, find fog and gof .

Answer

$$\text{Let } f = \{(3,1), (9,3), (12,4)\} \text{ and}$$

$$g = \{(1,3), (3,3), (4,9), (5,9)\}$$

Now,

$$\text{range of } f = \{1, 3, 4\}$$

$$\text{domain of } f = \{3, 9, 12\}$$

$$\text{range of } g = \{3,9\}$$

$$\text{domain of } g = \{1, 3, 4, 5\}$$

$$\text{since, } \text{range of } f \subset \text{domain of } g$$

$$\therefore \text{gof is well defined.}$$

$$\text{Again, the range of } g \subseteq \text{domain of } f$$

$$\therefore \text{fog is well defined.}$$

$$\text{Finally, } \text{gof} = \{(3,3), (9,9), (12,9)\}$$

$$\text{fog} = \{(1,1), (3,1), (4,3), (5,3)\}$$

3. Question

Let $f = \{(1, -1), (4, -2), (9, -3), (16, 4)\}$ and $g = \{(-1, -2), (-2, -4), (-3, -6), (4, 8)\}$. Show that gof is defined while fog is not defined. Also, find gof .

Answer

We have,

$$f = \{(1, -1), (4, -2), (9, -3), (16,4)\} \text{ and}$$

$$g = \{(-1, -2), (-2, -4), (-3, -6), (4,8)\}$$

Now,

$$\text{Domain of } f = \{1,4,9,16\}$$

$$\text{Range of } f = \{-1, -2, -3, 4\}$$

$$\text{Domain of } g = \{-1, -2, -3,4\}$$

Range of $g = \{-2, -4, -6, 8\}$

Clearly range of $f = \text{domain of } g$

$\therefore \text{gof is defined.}$

but, range of $g \neq \text{domain of } f$ So, fog is not defined.

Now,

$$\text{gof}(1) = g(-1) = -2$$

$$\text{gof}(4) = g(-2) = -4$$

$$\text{gof}(9) = g(-3) = -6$$

$$\text{gof}(16) = g(4) = 8$$

$$\text{So, } \text{gof} = \{(1, -2), (4, -4), (9, -6), (16, 8)\}$$

4. Question

Let $A = \{a, b, c\}$, $B = \{u, v, w\}$ and let f and g be two functions from A to B and from B to A respectively defined as: $f = \{(a, v), (b, u), (c, w)\}$, $g = \{(u, b), (v, a), (w, c)\}$.

Show that f and g both are bijections and find fog and gof.

Answer

Given, $A = \{a, b, c\}$, $B = \{u, v, w\}$ and

$f = A \rightarrow B$ and $g: B \rightarrow A$ defined by

$$f = \{(a, v), (b, u), (c, w)\} \text{ and}$$

$$g = \{(u, b), (v, a), (w, c)\}$$

For both f and g , different elements of domain have different images

$\therefore f$ and g are one - one

Again, for each element in co - domain of f and g , there is a pre - image in the domain

$\therefore f$ and g are onto

Thus, f and g are bijective.

Now,

$$\text{gof} = \{(a, a), (b, b), (c, c)\} \text{ and}$$

$$\text{fog} = \{(u, u), (v, v), (w, w)\}$$

5. Question

Find fog (2) and gof (1) when: $f: \mathbb{R} \rightarrow \mathbb{R}; f(x) = x^2 + 8$ and $g: \mathbb{R} \rightarrow \mathbb{R}; g(x) = 3x^3 + 1$.

Answer

We have, $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2 + 8$ and

$g: \mathbb{R} \rightarrow \mathbb{R}$ given by $g(x) = 3x^3 + 1$

$$\text{fog}(x) = f(g(x)) = f(3x^3 + 1)$$

$$= (3x^3 + 1)^2 + 8$$

$$\text{fog}(2) = (3 \times 8 + 1)^2 + 8 = 625 + 8 = 633$$

Again,

$$\text{gof}(x) = g(f(x)) = g(x^2 + 8)$$

$$= 3(x^2 + 8)^3 + 1$$

$$\text{gof}(1) = 3(1 + 8)^3 + 1 = 2188$$

6. Question

Let \mathbb{R}^+ be the set of all non-negative real numbers. If $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are defined as $f(x) = x^2$ and $g(x) = \sqrt{x}$. Find fog and gof . Are they equal functions.

Answer

We have, $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ given by

$$f(x) = x^2$$

$g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ given by

$$g(x) = \sqrt{x}$$

$$\text{fog}(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$

Also,

$$\text{gof}(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = x$$

Thus,

$$\text{fog}(x) = \text{gof}(x)$$

They are equal functions as their domain and range are also equal.

7. Question

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$ and $g(x) = x + 1$. Show that $\text{fog} \neq \text{gof}$.

Answer

We have, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ are two functions defined by

$$f(x) = x^2 \text{ and } g(x) = x + 1$$

Now,

$$\text{fog}(x) = f(g(x)) = f(x + 1) = (x + 1)^2$$

$$\Rightarrow \text{fog}(x) = x^2 + 2x + 1 \dots\dots(i)$$

$$\text{gof}(x) = g(f(x)) = g(x^2) = x^2 + 1 \dots\dots(ii)$$

from (i) & (ii)

$$\text{fog} \neq \text{gof}$$

8. Question

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x + 1$ and $g(x) = x - 1$. Show that $\text{fog} = \text{gof} = I_{\mathbb{R}}$.

Answer

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ are defined as

$$f(x) = x + 1 \text{ and } g(x) = x - 1$$

Now,

$$\text{fog}(x) = f(g(x)) = f(x - 1) = x - 1 + 1$$

$$= x = I_{\mathbb{R}} \dots\dots(i)$$

Again,

$$f \circ g(x) = f(g(x)) = g(x + 1) = x + 1 - 1$$

$$= x = I_{\mathbb{R}} \dots \dots (ii)$$

from (i) & (ii)

$$f \circ g = g \circ f = I_{\mathbb{R}}$$

9. Question

Verify associativity for the following three mappings: $f: \mathbb{N} \rightarrow \mathbb{Z}_0$ (the set of non-zero integers), $g: \mathbb{Z}_0 \rightarrow \mathbb{Q}$ and $h: \mathbb{Q} \rightarrow \mathbb{R}$ given by $f(x) = 2x$, $g(x) = 1/x$ and $h(x) = e^x$.

Answer

We have, $f: \mathbb{N} \rightarrow \mathbb{Z}_0$, $g: \mathbb{Z}_0 \rightarrow \mathbb{Q}$ and $h: \mathbb{Q} \rightarrow \mathbb{R}$

$$\text{Also, } f(x) = 2x, \quad g(x) = \frac{1}{x} \text{ and } h(x) = e^x$$

Now, $f: \mathbb{N} \rightarrow \mathbb{Z}_0$ and $h \circ g: \mathbb{Z}_0 \rightarrow \mathbb{R}$

$$\therefore (h \circ g) \circ f: \mathbb{N} \rightarrow \mathbb{R}$$

Also, $g \circ f: \mathbb{N} \rightarrow \mathbb{Q}$ and $h: \mathbb{Q} \rightarrow \mathbb{R}$

$$\therefore h \circ (g \circ f): \mathbb{N} \rightarrow \mathbb{R}$$

Thus, $(h \circ g) \circ f$ and $h \circ (g \circ f)$ exist and are function from \mathbb{N} to set \mathbb{R} .

$$\text{Finally, } (h \circ g) \circ f(x) = (h \circ g)(f(x)) = (h \circ g)(2x)$$

$$= h\left(\frac{1}{2x}\right) = e^{\frac{1}{2x}}$$

$$\text{Now, } h \circ (g \circ f)(x) = h(g(2x)) = h\left(\frac{1}{2x}\right)$$

$$= e^{\frac{1}{2x}}$$

Hence, associativity verified.

10. Question

Consider $f: \mathbb{N} \rightarrow \mathbb{N}$, $g: \mathbb{N} \rightarrow \mathbb{N}$ and $h: \mathbb{N} \rightarrow \mathbb{R}$ defined as $f(x) = 2x$, $g(y) = 3y + 4$ and $h(z) = \sin z$ for all $x, y, z \in \mathbb{N}$. Show that $h \circ (g \circ f) = (h \circ g) \circ f$.

Answer

We have,

$$h \circ (g \circ f)(x) = h(g(f(x)))$$

$$= h(g(2x)) = h(3(2x) + 4)$$

$$= h(6x + 4) = \sin(6x + 4) \quad \forall x \in \mathbb{N}$$

$$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = (h \circ g)(2x)$$

$$= h(g(2x)) = h(3(2x) + 4)$$

$$= h(6x + 4) = \sin(6x + 4) \quad \forall x \in \mathbb{N}$$

This shows, $h \circ (g \circ f) = (h \circ g) \circ f$

11. Question

Give examples of two functions $f: \mathbb{N} \rightarrow \mathbb{N}$ and $g: \mathbb{N} \rightarrow \mathbb{N}$ such that $g \circ f$ is onto, but f is not onto.

Answer

Define $f: \mathbb{N} \rightarrow \mathbb{N}$ by, $f(x) = x + 1$ And, $g: \mathbb{N} \rightarrow \mathbb{N}$ by,

$$g(x) = \begin{cases} x - 1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$$

We first show that f is not onto.

For this, consider element 1 in co - domain \mathbb{N} . It is clear that this element is not an image of any of the elements in domain \mathbb{N} .

Therefore, f is not onto.

12. Question

Give examples of two functions $f: \mathbb{N} \rightarrow \mathbb{Z}$ and $g: \mathbb{Z} \rightarrow \mathbb{Z}$ such that $g \circ f$ is injective, but g is not injective.

Answer

Define $f: \mathbb{N} \rightarrow \mathbb{Z}$ as $f(x) = x$ and $g: \mathbb{N} \rightarrow \mathbb{N}$ as $g(x) = |x|$.

We first show that g is not injective.

It can be observed that:

$$g(-1) = |-1| = 1$$

$$g(1) = |1| = 1$$

Therefore, $g(-1) = g(1)$, but $-1 \neq 1$.

Therefore, g is not injective.

Now, $g \circ f: \mathbb{N} \rightarrow \mathbb{Z}$ is defined as $g \circ f(x) = g(f(x)) = g(x) = |x|$.

Let $x, y \in \mathbb{N}$ such that $g \circ f(x) = g \circ f(y)$.

$$\Rightarrow |x| = |y|$$

Since x and $y \in \mathbb{N}$ both are positive.

$$\therefore |x| = |y| \Rightarrow x = y$$

Hence, $g \circ f$ is injective

13. Question

If $f: A \rightarrow B$ and $g: B \rightarrow C$ are one - one functions show that $g \circ f$ is a one - one function.

Answer

We have, $f: A \rightarrow B$ and $g: B \rightarrow C$ are one - one functions.

Now we have to prove : $g \circ f: A \rightarrow C$ in one - one

let $x, y \in A$ such that

$$g \circ f(x) = g \circ f(y)$$

$$g(f(x)) = g(f(y))$$

$$f(x) = f(y) \text{ [As, } g \text{ in one - one]}$$

$$x = y \text{ [As, } f \text{ in one - one]}$$

$g \circ f$ is one - one function

14. Question

If $f: A \rightarrow B$ and $g: B \rightarrow C$ are onto functions show that $g \circ f$ is an onto function.

Answer

We have, $f: A \rightarrow B$ and $g: B \rightarrow C$ are onto functions.

Now, we need to prove: $\text{gof}: A \rightarrow C$ is onto.

let $y \in C$, then

$$\text{gof}(x) = y$$

$$g(f(x)) = y \dots\dots(i)$$

Since g is onto, for each element in C , there exists a preimage in B .

$$g(x)=y \dots\dots(ii)$$

From (i) & (ii)

$$f(x)=x$$

Since f is onto, for each element in B there exists a preimage in A

$$f(x)=x \dots\dots(iii)$$

From (ii) and (iii) we can conclude that for each $y \in C$, there exists a preimage in A such that $\text{gof}(x) = y$

$\therefore \text{gof}$ is onto.

Exercise 2.3

1 A. Question

Find fog and gof , if

$$f(x) = e^x, g(x) = \log_e x$$

Answer

$$f(x) = e^x \text{ and } g(x) = \log_e x$$

$$\text{Now, } \text{fog}(x) = f(g(x)) = f(\log_e x) = e^{\log_e x} = x$$

$$\Rightarrow \text{fog}(x) = x$$

$$\text{gof}(x) = g(f(x)) = g(e^x) = \log_e e^x = x$$

$$\Rightarrow \text{gof}(x) = x$$

Hence, $\text{fog}(x) = x$ and $\text{gof}(x) = x$

1 B. Question

Find fog and gof , if

$$f(x) = x^2, g(x) = \cos x$$

Answer

$$f(x) = x, g(x) = \cos x$$

Domain of f and Domain of $g = \mathbb{R}$

Range of $f = (0, \infty)$

Range of $g = (-1, 1)$

\therefore Range of $f \subset \text{domain of } g \Rightarrow \text{gof exist}$

Also, Range of $g \subset \text{domain of } f \Rightarrow \text{fog exist}$

Now,

$$\text{gof}(x) = g(f(x)) = g(x^2) = \cos x^2$$

And

$$\text{fog}(x) = f(g(x)) = f(\cos x) = \cos^2 x$$

$$\text{Hence, fog}(x) = \cos^2 x \text{ and } \text{gof}(x) = \cos^2 x$$

1 C. Question

Find fog and gof, if

$$f(x) = |x|, g(x) = \sin x$$

Answer

$$f(x) = |x| \text{ and } g(x) = \sin x$$

$$\text{Range of } f = (0, \infty) \subset \text{Domain of } g = \mathbb{R} \Rightarrow \text{gof exist}$$

$$\text{Range of } g = [-1, 1] \subset \text{Domain of } f = \mathbb{R} \Rightarrow \text{fog exist}$$

$$\text{Now, fog}(x) = f(g(x)) = f(\sin x) = |\sin x| \text{ and}$$

$$\text{gof}(x) = g(f(x)) = g(|x|) = \sin |x|$$

$$\text{Hence, fog}(x) = |\sin x| \text{ and } \text{gof}(x) = \sin |x|$$

1 D. Question

Find fog and gof, if

$$f(x) = x + 1, g(x) = e^x$$

Answer

$$f(x) = x + 1 \text{ and } g(x) = e^x$$

$$\text{Range of } f = \mathbb{R} \subset \text{Domain of } g = \mathbb{R} \Rightarrow \text{gof exist}$$

$$\text{Range of } g = (0, \infty) \subset \text{Domain of } f = \mathbb{R} \Rightarrow \text{fog exist}$$

Now,

$$\text{gof}(x) = g(f(x)) = g(x + 1) = e^{x+1}$$

And

$$\text{fog}(x) = f(g(x)) = f(e^x) = e^x + 1$$

$$\text{Hence, fog}(x) = e^x + 1 \text{ and } \text{gof}(x) = e^{x+1}$$

1 E. Question

Find fog and gof, if

$$f(x) = \sin^{-1} x, g(x) = x^2$$

Answer

$$f(x) = \sin^{-1} x \text{ and } g(x) = x^2$$

$$\text{Range of } f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \subset \text{Domain of } g = \mathbb{R} \Rightarrow \text{gof exist}$$

$$\text{Range of } g = (0, \infty) \subset \text{Domain of } f = \mathbb{R} \Rightarrow \text{fog exist}$$

Now,

$$\text{fog}(x) = f(g(x)) = f(x^2) = \sin^{-1} x^2 \text{ and}$$

$$\text{gof}(x) = g(f(x)) = g(\sin^{-1} x) = (\sin^{-1} x)^2$$

$$\text{Hence, fog}(x) = \sin^{-1} x^2 \text{ and } \text{gof}(x) = (\sin^{-1} x)^2$$

1 F. Question

Find fog and gof, if

$$f(x) = x + 1, g(x) = \sin x$$

Answer

$$f(x) = x + 1 \text{ and } g(x) = \sin x$$

Range of $f = \mathbb{R} \subset \text{Domain of } g = \mathbb{R} \Rightarrow \text{gof exists}$

Range of $g = [-1, 1] \subset \text{Domain of } f \Rightarrow \text{fog exists}$

Now,

$$\text{fog}(x) = f(g(x)) = f(\sin x) = \sin x + 1$$

And

$$\text{gof}(x) = g(f(x)) = g(x + 1) = \sin(x + 1)$$

Hence, $\text{fog}(x) = \sin x + 1$ and $\text{gof}(x) = \sin(x + 1)$

1 G. Question

Find fog and gof, if

$$f(x) = x + 1, g(x) = 2x + 3$$

Answer

$$f(x) = x + 1 \text{ and } g(x) = 2x + 3$$

Range of $f = \mathbb{R} \subset \text{Domain of } g = \mathbb{R} \Rightarrow \text{gof exists}$

Range of $g = \mathbb{R} \subset \text{Domain of } f \Rightarrow \text{fog exists}$

Now,

$$\text{fog}(x) = f(g(x)) = f(2x + 3) = (2x + 3) + 1 = 2x + 4 \text{ and}$$

$$\text{gof}(x) = g(f(x)) = g(x + 1) = 2(x + 1) + 3 = 2x + 5$$

So, $\text{fog}(x) = 2x + 4$ and $\text{gof}(x) = 2x + 5$

1 H. Question

Find fog and gof, if

$$f(x) = c, c \in \mathbb{R}, g(x) = \sin x^2$$

Answer

$$f(x) = c, c \in \mathbb{R} \text{ and}$$

$$g(x) = \sin x^2$$

Range of $f = \mathbb{R} \subset \text{Domain of } g = \mathbb{R} \Rightarrow \text{gof exists}$

Range of $g = [-1, 1] \subset \text{Domain of } f = \mathbb{R} \Rightarrow \text{fog exists}$

Now,

$$\text{gof}(x) = g(f(x)) = g(c) = \sin c^2 \text{ and}$$

$$\text{fog}(x) = f(g(x)) = f(\sin x^2) = c$$

Thus, $\text{gof}(x) = \sin c^2$ and $\text{fog}(x) = c$

1 I. Question

Find fog and gof, if

$$f(x) = x^2 + 2, \quad g(x) = 1 - \frac{1}{1-x}$$

Answer

$$f(x) = x^2 + 2 \text{ and } g(x) = 1 - \frac{1}{1-x}$$

Range of $f = (2, \infty) \subset \text{Domain of } g = \mathbb{R} \Rightarrow \text{gof exists}$

Range of $g = \mathbb{R} - \{-1\} \subset \text{Domain of } f = \mathbb{R} \Rightarrow \text{fog exists}$

Now,

$$\text{fog}(x) = f(g(x)) = f\left(-\frac{x}{1-x}\right) = \frac{x^2}{(1-x)^2} + 2 \text{ and}$$

$$\text{gof}(x) = g(f(x)) = g(x^2 + 2) = -\frac{x^2 + 2}{1 - (x^2 + 2)}$$

$$\text{gof}(x) = \frac{x^2 + 2}{(x^2 + 1)}$$

$$\text{Hence, fog}(x) = \frac{x^2}{(1-x)^2} + 2 \text{ and } \text{gof}(x) = -\frac{x^2 + 2}{1 - (x^2 + 2)}$$

2. Question

Let $f(x) = x^2 + x + 1$ and $g(x) = \sin x$. Show that $\text{fog} \neq \text{gof}$.

Answer

We have, $f(x) = x^2 + x + 1$ and $g(x) = \sin x$

Now,

$$\text{fog}(x) = f(g(x)) = f(\sin x)$$

$$\Rightarrow \text{fog}(x) = \sin^2 x + \sin x + 1$$

$$\text{Again, } \text{gof}(x) = g(f(x)) = g(x^2 + x + 1)$$

$$\Rightarrow \text{gof}(x) = \sin(x^2 + x + 1)$$

Clearly,

$$\text{fog} \neq \text{gof}$$

3. Question

If $f(x) = |x|$, prove that $\text{fof} = f$.

Answer

We have, $f(x) = |x|$

We assume the domain of $f = \mathbb{R}$ and range of $f = (0, \infty)$

Range of $f \subset \text{domain of } f$

$\therefore \text{fof exists,}$

Now,

$$\text{fof}(x) = f(f(x)) = f(|x|) = ||x|| = f(x)$$

$$\therefore \text{fof} = f$$

Hence proved.

4. Question

If $f(x) = 2x + 5$ and $g(x) = x^2 + 1$ be two real functions, then describe each of the following functions:

(i) fog

(ii) gof

(iii) fof

(iv) f^2

Also, show that $\text{fof} \neq f^2$.

Answer

$$f(x) = 2x + 5 \text{ and } g(x) = x^2 + 1$$

The range of $f = \mathbb{R}$ and range of $g = [1, \infty]$

The range of $f \subset \text{Domain of } g (\mathbb{R})$ and range of $g \subset \text{domain of } f (\mathbb{R})$

\therefore both fog and gof exist.

$$(i) \text{ fog}(x) = f(g(x)) = f(x^2 + 1)$$

$$= 2(x^2 + 1) + 5$$

$$\Rightarrow \text{fog}(x) = 2x^2 + 7$$

$$\text{Hence fog}(x) = 2x^2 + 7$$

$$(ii) \text{ gof}(x) = g(f(x)) = g(2x + 5)$$

$$= (2x + 5)^2 + 1$$

$$\text{gof}(x) = 4x^2 + 20x + 26$$

$$\text{Hence gof}(x) = 4x^2 + 20x + 26$$

$$(iii) \text{ fof}(x) = f(f(x)) = f(2x + 5)$$

$$= 2(2x + 5) + 5$$

$$\text{fof}(x) = 4x + 15$$

$$\text{Hence fof}(x) = 4x + 15$$

$$(iv) f^2(x) = [f(x)]^2 = (2x + 5)^2$$

$$= 4x^2 + 20x + 25$$

\therefore from (iii) and (iv)

$$\text{fof} \neq f^2$$

5. Question

If $f(x) = \sin x$ and $g(x) = 2x$ be two real functions, then describe gof and fog. Are these equal functions?

Answer

We have, $f(x) = \sin x$ and $g(x) = 2x$.

Domain of f and g is \mathbb{R}

Range of $f = [-1, 1]$, Range of $g = \mathbb{R}$

\therefore Range of $f \subset \text{Domain } g$ and Range of $g \subset \text{Domain } f$

fog and gof both exist.

$$\text{gof}(x) = g(f(x)) = g(\sin x)$$

$$\Rightarrow \text{gof}(x) = 2\sin x$$

$$\text{fog}(x) = f(g(x)) = f(2x) = \sin 2x$$

$$\therefore \text{gof} \neq \text{fog}$$

6. Question

Let f, g, h be real functions given by $f(x) = \sin x$, $g(x) = 2x$ and $h(x) = \cos x$. Prove that $\text{fog} = \text{go(fh)}$.

Answer

f, g and h are real functions given by $f(x) = \sin x$, $g(x) = 2x$ and

$$h(x) = \cos x$$

To prove: $\text{fog} = \text{go(fh)}$

L.H.S

$$\text{fog}(x) = f(g(x))$$

$$= f(2x) = \sin 2x$$

$$\Rightarrow \text{fog}(x) = 2\sin x \cos x \dots\dots(A)$$

R.H .S

$$\text{go(fh)}(x) = \text{go}(f(x).h(x))$$

$$= g(\sin x \cos x) = 2\sin x \cos x$$

$$\text{go(fh)}(x) = 2 \sin x \cos x \dots\dots(B)$$

from A and B

$$\text{fog}(x) = \text{go(fh)}(x)$$

Hence proved

7. Question

Let f be any real function and let g be a function given by $g(x) = 2x$. Prove that $\text{gof} = f + f$.

Answer

We are given that f is a real function and g is a function given by

$$g(x) = 2x$$

To prove; $\text{gof} = f + f$.

L.H.S

$$\text{gof}(x) = g(f(x)) = 2f(x)$$

$$= f + f = \text{R.H.S}$$

$$\text{gof} = f + f$$

Hence proved

8. Question

If $f(x) = \sqrt{1-x}$ and $g(x) = \log_e x$ are two real functions, then describe functions fog and gof .

Answer

$$f(x) = \sqrt{1-x}, g(x) = \log_e x$$

Domain of f and g are \mathbb{R} .

Range of $f = (-\infty, 1)$ Range of $g = (0, e)$

Range of $f \subset$ Domain of $g \Rightarrow \text{gof exists}$

Range of $g \subset$ Domain $f \Rightarrow \text{fog exists}$

$$\therefore \text{gof}(x) = g(f(x)) = g(\sqrt{1-x})$$

$$\therefore \text{gof}(x) = \log_e \sqrt{1-x}$$

Again

$$\text{fog}(x) = f(g(x)) = f(\log_e x)$$

$$\text{fog}(x) = \sqrt{1 - \log_e x}$$

9. Question

If $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$ and $g: [-1, 1] \rightarrow \mathbb{R}$ be defined as $f(x) = \tan x$ and $g(x) = \sqrt{1-x^2}$ respectively. Describe fog and gof.

Answer

$$f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R} \text{ and } g: [-1, 1] \rightarrow \mathbb{R} \text{ defined as } f(x) = \tan x \text{ and } g(x) = \sqrt{1-x^2}$$

Range of f : let $y = f(x)$

$$\Rightarrow y = \tan x$$

$$\Rightarrow x = \tan^{-1} y$$

$$\text{Since, } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), y \in (-\infty, \infty)$$

As Range of $f \subset$ Domain of g

\therefore gof exists.

Similarly, let $y = g(x)$

$$\Rightarrow y = \sqrt{1-x^2}$$

$$\Rightarrow x = \sqrt{1-y^2}$$

\therefore Range of g is $[-1, 1]$

As, Range of $g \subset$ Domain of f

Hence, fog also exists

Now,

$$\text{fog}(x) = f(g(x)) = f(\sqrt{1-x^2})$$

$$\Rightarrow \text{fog}(x) = \tan \sqrt{1-x^2}$$

Again,

$$\text{gof}(x) = g(f(x)) = g(\tan x)$$

$$\Rightarrow \text{gof}(x) = \sqrt{1 - \tan^2 x}$$

10. Question

If $f(x) = \sqrt{x+3}$ and $g(x) = x^2 + 1$ be two real functions, then find fog and gof.

Answer

$$f(x) = \sqrt{x+3}, g(x) = x^2 + 1$$

Now,

Domain of $f = [-3, \infty]$, domain of $g = (-\infty, \infty)$

Range of $f = [0, \infty)$, range of $g = [1, \infty)$

Then, range of $f \subset$ Domain of g and range of $g \subset$ Domain of f

Hence, $f \circ g$ and $g \circ f$ exists

Now,

$$f \circ g(x) = f(g(x)) = f(x^2 + 1)$$

$$\Rightarrow f \circ g(x) = \sqrt{x^2 + 4}$$

Again,

$$g \circ f(x) = g(f(x)) = g(\sqrt{x+3})$$

$$\Rightarrow g \circ f(x) = (\sqrt{x+3})^2 + 1$$

$$\Rightarrow g \circ f(x) = x + 4$$

11 A. Question

Let f be a real function given by $f(x) = \sqrt{x-2}$. Find each of the following:

$f \circ f$

Answer

$$\text{We have, } f(x) = \sqrt{x-2}$$

Clearly, domain of $f = [2, \infty]$ and range of $f = [0, \infty)$

We observe that range of f is not a subset of domain of f

$$\therefore \text{Domain of } (f \circ f) = \{x: x \in \text{Domain of } f \text{ and } f(x) \in \text{Domain of } f\}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \in [2, \infty)\}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 2\}$$

$$= \{x: x \in [2, \infty) \text{ and } x - 2 \geq 4\}$$

$$= \{x: x \in [2, \infty) \text{ and } x \geq 6\}$$

$$= [6, \infty)$$

Now,

$$f \circ f(x) = f(f(x)) = f(\sqrt{x-2}) = \sqrt{\sqrt{x-2}-2}$$

11 B. Question

Let f be a real function given by $f(x) = \sqrt{x-2}$. Find each of the following:

$f \circ f \circ f$

Answer

$$\text{We have, } f(x) = \sqrt{x-2}$$

Clearly, domain of $f = [2, \infty]$ and range of $f = [0, \infty)$

We observe that range of f is not a subset of domain of f

$$\therefore \text{Domain of (fof)} = \{x: x \in \text{Domain of } f \text{ and } f(x) \in \text{Domain of } f\}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \in [2, \infty)\}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 2\}$$

$$= \{x: x \in [2, \infty) \text{ and } x - 2 \geq 4\}$$

$$= \{x: x \in [2, \infty) \text{ and } x \geq 6\}$$

$$= [6, \infty)$$

Clearly, range of $f = [0, \infty) \not\subset \text{Domain of (fof)}$

$$\therefore \text{Domain of ((fof)of)} = \{x: x \in \text{Domain of } f \text{ and } f(x) \in \text{Domain of (fof)}\}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \in [6, \infty)\}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 6\}$$

$$= \{x: x \in [2, \infty) \text{ and } x - 2 \geq 36\}$$

$$= \{x: x \in [2, \infty) \text{ and } x \geq 38\}$$

$$= [38, \infty)$$

Now,

$$(\text{fof})(x) = f(f(x)) = f(\sqrt{x-2}) = \sqrt{\sqrt{x-2}-2}$$

$$(\text{fofof})(x) = (\text{fof})(f(x)) = (\text{fof})(\sqrt{x-2}) = \sqrt{\sqrt{\sqrt{x-2}-2}-2}$$

$\therefore \text{fofof} : [38, \infty) \rightarrow \mathbb{R}$ defined as

$$(\text{fofof})(x) = \sqrt{\sqrt{\sqrt{x-2}-2}-2}$$

11 C. Question

Let f be a real function given by $f(x) = \sqrt{x-2}$. Find each of the following:

$$(\text{fofof})(38)$$

Answer

$$\text{We have, } f(x) = \sqrt{x-2}$$

Clearly, domain of $f = [2, \infty)$ and range of $f = [0, \infty)$

We observe that range of f is not a subset of domain of f

$$\therefore \text{Domain of (fof)} = \{x: x \in \text{Domain of } f \text{ and } f(x) \in \text{Domain of } f\}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \in [2, \infty)\}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 2\}$$

$$= \{x: x \in [2, \infty) \text{ and } x - 2 \geq 4\}$$

$$= \{x: x \in [2, \infty) \text{ and } x \geq 6\}$$

$$= [6, \infty)$$

Clearly, range of $f = [0, \infty) \not\subset \text{Domain of (fof)}$

$$\therefore \text{Domain of ((fof)of)} = \{x: x \in \text{Domain of } f \text{ and } f(x) \in \text{Domain of (fof)}\}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \in [6, \infty)\}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 6\}$$

$$= \{x: x \in [2, \infty) \text{ and } x - 2 \geq 36\}$$

$$= \{x: x \in [2, \infty) \text{ and } x \geq 38\}$$

$$= [38, \infty)$$

Now,

$$(f \circ f)(x) = f(f(x)) = f(\sqrt{x-2}) = \sqrt{\sqrt{x-2}-2}$$

$$(f \circ f \circ f)(x) = (f \circ f)(f(x)) = (f \circ f)(\sqrt{x-2}) = \sqrt{\sqrt{\sqrt{x-2}-2}-2}$$

$\therefore f \circ f \circ f : [38, \infty) \rightarrow \mathbb{R}$ defined as

$$(f \circ f)(x) = f(f(x)) = f(\sqrt{x-2}) = \sqrt{\sqrt{x-2}-2}$$

$$(f \circ f \circ f)(x) = (f \circ f)(f(x)) = (f \circ f)(\sqrt{x-2}) = \sqrt{\sqrt{\sqrt{x-2}-2}-2}$$

$\therefore f \circ f \circ f : [38, \infty) \rightarrow \mathbb{R}$ defined as

$$(f \circ f \circ f)(x) = \sqrt{\sqrt{\sqrt{x-2}-2}-2}$$

$$(f \circ f \circ f)(38) = \sqrt{\sqrt{\sqrt{38-2}-2}-2} = \sqrt{\sqrt{\sqrt{36}-2}-2}$$

$$= \sqrt{\sqrt{6-2}-2} = \sqrt{\sqrt{4}-2} = \sqrt{2-2} = 0$$

11 D. Question

Let f be a real function given by $f(x) = \sqrt{x-2}$. Find each of the following:

$$f^2$$

Also, show that $f \circ f \neq f^2$.

Answer

$$\text{We have, } f(x) = \sqrt{x-2}$$

Clearly, domain of $f = [2, \infty]$ and range of $f = [0, \infty)$

We observe that range of f is not a subset of domain of f

\therefore Domain of $(f \circ f) = \{x: x \in \text{Domain of } f \text{ and } f(x) \in \text{Domain of } f\}$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \in [2, \infty)\}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 2\}$$

$$= \{x: x \in [2, \infty) \text{ and } x - 2 \geq 4\}$$

$$= \{x: x \in [2, \infty) \text{ and } x \geq 6\}$$

$$= [6, \infty)$$

Now,

$$(f \circ f)(x) = f(f(x)) = f(\sqrt{x-2}) = \sqrt{\sqrt{x-2}-2}$$

$\therefore f \circ f : [6, \infty) \rightarrow \mathbb{R}$ defined as

$$(f \circ f)(x) = \sqrt{\sqrt{x-2}-2}$$

$$f^2(x) = [f(x)]^2 = [\sqrt{x-2}]^2 = x-2$$

$\therefore f^2: [2, \infty) \rightarrow \mathbb{R}$ defined as

$$f^2(x) = x-2$$

$$\therefore f \circ f \neq f^2$$

12. Question

Let $f(x) = \begin{cases} 1+x, & 0 \leq x \leq 2 \\ 3-x, & 2 < x \leq 3 \end{cases}$. Find $f \circ f$.

Answer

$$f(x) = \begin{cases} 1+x & 0 \leq x \leq 2 \\ 3-x & 2 < x \leq 3 \end{cases}$$

Range of $f = [0, 3] \subset \text{Domain of } f$

$$\therefore f \circ f(x) = f(f(x)) = f\left(\begin{cases} 1+x & 0 \leq x \leq 2 \\ 3-x & 2 < x \leq 3 \end{cases}\right) = f\left(\begin{cases} 1+(1+x) & 0 \leq x \leq 1 \\ 3-(1+x) & 1 < x \leq 2 \\ 1+(3-x) & 2 < x \leq 3 \end{cases}\right)$$

$$\text{So, } f \circ f(x) = \begin{cases} 2+x & 0 \leq x \leq 1 \\ 2-x & 1 < x \leq 2 \\ 4-x & 2 < x \leq 3 \end{cases}$$

13. Question

If $f, g: \mathbb{R} \rightarrow \mathbb{R}$ be two functions defined as $f(x) = |x| + x$ and

$g(x) = |x|-x$ for all $x \in \mathbb{R}$. Then, find $f \circ g$ and $g \circ f$. Hence, find $f \circ g(-3)$,

$f \circ g(5)$ and $g \circ f(-2)$.

Answer

Domain of $f(x)$ and $g(x)$ is \mathbb{R} .

Range of $f(x) = [0, \infty)$ and range of $g(x) = [0, \infty)$

As, range of $f \subset \text{Domain of } g$ and range of $g \subset \text{Domain of } f$

So, $f \circ g$ and $g \circ f$ exists

Now,

$$f \circ g(x) = f(g(x)) = f(|x|-x)$$

$$\Rightarrow f \circ g(x) = ||x|-x| + |x|-x$$

$$\text{As, range of } g(x) \geq 0 \text{ so, } ||x|-x| = |x|-x$$

$$\text{So, } f \circ g(x) = ||x|-x| + |x|-x = |x|-x + |x|-x$$

$$\Rightarrow f \circ g(x) = 2(|x|-x)$$

Also,

$$g \circ f(x) = g(f(x)) = g(|x| + x) = ||x| + x| - (|x| + x)$$

$$\text{As, range of } f(x) \geq 0 \text{ so, } ||x| + x| = |x| + x$$

$$\text{So, } g \circ f(x) = ||x| + x| - (|x| + x) = |x| + x - (|x| + x) = 0$$

$$\text{Thus, } g \circ f(x) = 0$$

$$\text{Now, } \text{fog}(-3) = 2(|-3| - (-3)) = 2(3 + 3) = 6,$$

$$\text{fog}(5) = 2(|5| - 5) = 0, \text{gof}(-2) = 0$$

Exercise 2.4

1. Question

State with reasons whether the following functions have inverse:

(i) $f : [1, 2, 3, 4] \rightarrow \{10\}$ with $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

(ii) $g : \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$ with $g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

(iii) $h : \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$ with $h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

Answer

(i) $f : [1, 2, 3, 4] \rightarrow \{10\}$ with $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

Recall that a function is invertible only when it is both one-one and onto.

Here, we have $f(1) = 10 = f(2) = f(3) = f(4)$

Hence, f is not one-one.

Thus, the function f does not have an inverse.

(ii) $g : \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$ with $g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

Recall that a function is invertible only when it is both one-one and onto.

Here, we have $g(5) = 4 = g(7)$

Hence, g is not one-one.

Thus, the function g does not have an inverse.

(iii) $h : \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$ with $h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

Recall that a function is invertible only when it is both one-one and onto.

Here, observe that distinct elements of the domain $\{2, 3, 4, 5\}$ are mapped to distinct elements of the co-domain $\{7, 9, 11, 13\}$.

Hence, h is one-one.

Also, each element of the range $\{7, 9, 11, 13\}$ is the image of some element of $\{2, 3, 4, 5\}$.

Hence, h is also onto.

Thus, the function h has an inverse.

2. Question

Find f^{-1} if it exists for $f: A \rightarrow B$ where

(i) $A = \{0, -1, -3, 2\}; B = \{-9, -3, 0, 6\}$ & $f(x) = 3x$

(ii) $A = \{1, 3, 5, 7, 9\}; B = \{0, 1, 9, 25, 49, 81\}$ & $f(x) = x^2$

Answer

(i) $A = \{0, -1, -3, 2\}; B = \{-9, -3, 0, 6\}$ & $f(x) = 3x$

We have $f : A \rightarrow B$ and $f(x) = 3x$.

$$\Rightarrow f = \{(0, 3 \times 0), (-1, 3 \times (-1)), (-3, 3 \times (-3)), (2, 3 \times 2)\}$$

$$\therefore f = \{(0, 0), (-1, -3), (-3, -9), (2, 6)\}$$

Recall that a function is invertible only when it is both one-one and onto.

Here, observe that distinct elements of the domain $\{0, -1, -3, 2\}$ are mapped to distinct elements of the co-domain $\{0, -3, -9, 6\}$.

Hence, f is one-one.

Also, each element of the range $\{-9, -3, 0, 6\}$ is the image of some element of $\{0, -1, -3, 2\}$.

Hence, f is also onto.

Thus, the function f has an inverse.

We have $f^{-1} = \{(0, 0), (-3, -1), (-9, -3), (6, 2)\}$

(ii) $A = \{1, 3, 5, 7, 9\}$; $B = \{0, 1, 9, 25, 49, 81\}$ & $f(x) = x^2$

We have $f : A \rightarrow B$ and $f(x) = x^2$.

$\Rightarrow f = \{(1, 1^2), (3, 3^2), (5, 5^2), (7, 7^2), (9, 9^2)\}$

$\therefore f = \{(1, 1), (3, 9), (5, 25), (7, 49), (9, 81)\}$

Recall that a function is invertible only when it is both one-one and onto.

Here, observe that distinct elements of the domain $\{1, 3, 5, 7, 9\}$ are mapped to distinct elements of the co-domain $\{1, 9, 25, 49, 81\}$.

Hence, f is one-one.

However, the element 0 of the range $\{0, 1, 9, 25, 49, 81\}$ is not the image of any element of $\{1, 3, 5, 7, 9\}$.

Hence, f is not onto.

Thus, the function f does not have an inverse.

3. Question

Consider $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$ and $g : \{a, b, c\} \rightarrow \{\text{apple}, \text{ball}, \text{cat}\}$ defined as $f(1) = a$, $f(2) = b$, $f(3) = c$, $g(a) = \text{apple}$, $g(b) = \text{ball}$ and $g(c) = \text{cat}$. Show that f , g and $g \circ f$ are invertible. Find f^{-1} , g^{-1} , $(g \circ f)^{-1}$ and show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Answer

$f : \{1, 2, 3\} \rightarrow \{a, b, c\}$ and $f(1) = a$, $f(2) = b$, $f(3) = c$

$\Rightarrow f = \{(1, a), (2, b), (3, c)\}$

Recall that a function is invertible only when it is both one-one and onto.

Here, observe that distinct elements of the domain $\{1, 2, 3\}$ are mapped to distinct elements of the co-domain $\{a, b, c\}$.

Hence, f is one-one.

Also, each element of the range $\{a, b, c\}$ is the image of some element of $\{1, 2, 3\}$.

Hence, f is also onto.

Thus, the function f has an inverse.

We have $f^{-1} = \{(a, 1), (b, 2), (c, 3)\}$

$g : \{a, b, c\} \rightarrow \{\text{apple}, \text{ball}, \text{cat}\}$ and $g(a) = \text{apple}$, $g(b) = \text{ball}$, $g(c) = \text{cat}$

$\Rightarrow g = \{(a, \text{apple}), (b, \text{ball}), (c, \text{cat})\}$

Similar to the function f , g is also one-one and onto.

Thus, the function g has an inverse.

We have $g^{-1} = \{(\text{apple}, a), (\text{ball}, b), (\text{cat}, c)\}$

We know $(g \circ f)(x) = g(f(x))$

Thus, $\text{gof} : \{1, 2, 3\} \rightarrow \{\text{apple}, \text{ball}, \text{cat}\}$ and

$$(\text{gof})(1) = g(f(1)) = g(a) = \text{apple}$$

$$(\text{gof})(2) = g(f(2)) = g(b) = \text{ball}$$

$$(\text{gof})(3) = g(f(3)) = g(c) = \text{cat}$$

$$\Rightarrow \text{gof} = \{(1, \text{apple}), (2, \text{ball}), (3, \text{cat})\}$$

As the functions f and g , gof is also both one-one and onto.

Thus, the function gof has an inverse.

$$\text{We have } (\text{gof})^{-1} = \{(\text{apple}, 1), (\text{ball}, 2), (\text{cat}, 3)\}$$

Now, let us consider $f^{-1} \circ g^{-1}$.

$$\text{We know } (f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x))$$

Thus, $f^{-1} \circ g^{-1} : \{\text{apple}, \text{ball}, \text{cat}\} \rightarrow \{1, 2, 3\}$ and

$$(f^{-1} \circ g^{-1})(\text{apple}) = f^{-1}(g^{-1}(\text{apple})) = f^{-1}(a) = 1$$

$$(f^{-1} \circ g^{-1})(\text{ball}) = f^{-1}(g^{-1}(\text{ball})) = f^{-1}(b) = 2$$

$$(f^{-1} \circ g^{-1})(\text{cat}) = f^{-1}(g^{-1}(\text{cat})) = f^{-1}(c) = 3$$

$$\Rightarrow f^{-1} \circ g^{-1} = \{(\text{apple}, 1), (\text{ball}, 2), (\text{cat}, 3)\}$$

Therefore, we have $(\text{gof})^{-1} = f^{-1} \circ g^{-1}$.

4. Question

Let $A = \{1, 2, 3, 4\}$; $B = \{3, 5, 7, 9\}$; $C = \{7, 23, 47, 79\}$ and $f : A \rightarrow B$, $g : B \rightarrow C$ be defined as $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Express $(\text{gof})^{-1}$ and $f^{-1} \circ g^{-1}$ as the sets of ordered pairs and verify $(\text{gof})^{-1} = f^{-1} \circ g^{-1}$.

Answer

We have $f : A \rightarrow B$ & $f(x) = 2x + 1$

$$\Rightarrow f = \{(1, 2 \times 1 + 1), (2, 2 \times 2 + 1), (3, 2 \times 3 + 1), (4, 2 \times 4 + 1)\}$$

$$\therefore f = \{(1, 3), (2, 5), (3, 7), (4, 9)\}$$

Function f is clearly one-one and onto.

$$\text{Thus, } f^{-1} \text{ exists and } f^{-1} = \{(3, 1), (5, 2), (7, 3), (9, 4)\}$$

We have $g : B \rightarrow C$ & $g(x) = x^2 - 2$

$$\Rightarrow g = \{(3, 3^2 - 2), (5, 5^2 - 2), (7, 7^2 - 2), (9, 9^2 - 2)\}$$

$$\therefore g = \{(3, 7), (5, 23), (7, 47), (9, 79)\}$$

Function g is clearly one-one and onto.

$$\text{Thus, } g^{-1} \text{ exists and } g^{-1} = \{(7, 3), (23, 5), (47, 7), (79, 9)\}$$

$$\text{We know } (\text{gof})(x) = g(f(x))$$

Thus, $\text{gof} : A \rightarrow C$ and

$$(\text{gof})(1) = g(f(1)) = g(3) = 7$$

$$(\text{gof})(2) = g(f(2)) = g(5) = 23$$

$$(\text{gof})(3) = g(f(3)) = g(7) = 47$$

$$(\text{gof})(4) = g(f(4)) = g(9) = 79$$

$$\Rightarrow \text{gof} = \{(1, 7), (2, 23), (3, 47), (4, 79)\}$$

Clearly, gof is also both one-one and onto.

Thus, the function gof has an inverse.

We have $(\text{gof})^{-1} = \{(7, 1), (23, 2), (47, 3), (79, 4)\}$

Now, let us consider $f^{-1} \circ g^{-1}$.

We know $(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x))$

Thus, $f^{-1} \circ g^{-1} : C \rightarrow A$ and

$$(f^{-1} \circ g^{-1})(7) = f^{-1}(g^{-1}(7)) = f^{-1}(3) = 1$$

$$(f^{-1} \circ g^{-1})(23) = f^{-1}(g^{-1}(23)) = f^{-1}(5) = 2$$

$$(f^{-1} \circ g^{-1})(47) = f^{-1}(g^{-1}(47)) = f^{-1}(7) = 3$$

$$(f^{-1} \circ g^{-1})(79) = f^{-1}(g^{-1}(79)) = f^{-1}(9) = 4$$

$$\Rightarrow f^{-1} \circ g^{-1} = \{(7, 1), (23, 2), (47, 3), (79, 4)\}$$

Therefore, we have $(\text{gof})^{-1} = f^{-1} \circ g^{-1}$.

5. Question

Show that the function $f : Q \rightarrow Q$ defined by $f(x) = 3x + 5$ is invertible. Also, find f^{-1} .

Answer

We have $f : Q \rightarrow Q$ and $f(x) = 3x + 5$.

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that f is one-one.

Let $x_1, x_2 \in Q$ (domain) such that $f(x_1) = f(x_2)$

$$\Rightarrow 3x_1 + 5 = 3x_2 + 5$$

$$\Rightarrow 3x_1 = 3x_2$$

$$\therefore x_1 = x_2$$

So, we have $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Thus, function f is one-one.

Now, we will prove that f is onto.

Let $y \in Q$ (co-domain) such that $f(x) = y$

$$\Rightarrow 3x + 5 = y$$

$$\Rightarrow 3x = y - 5$$

$$\therefore x = \frac{y-5}{3}$$

Clearly, for every $y \in Q$, there exists $x \in Q$ (domain) such that $f(x) = y$ and hence, function f is onto.

Thus, the function f has an inverse.

We have $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found $f(x) = y \Rightarrow x = \frac{y-5}{3}$

Hence, $f^{-1}(y) = \frac{y-5}{3}$

Thus, $f(x)$ is invertible and $f^{-1}(x) = \frac{x-5}{3}$

6. Question

Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 4x + 3$ is invertible. Find the inverse of f .

Answer

We have $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = 4x + 3$.

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that f is one-one.

Let $x_1, x_2 \in \mathbb{R}$ (domain) such that $f(x_1) = f(x_2)$

$$\Rightarrow 4x_1 + 3 = 4x_2 + 3$$

$$\Rightarrow 4x_1 = 4x_2$$

$$\therefore x_1 = x_2$$

So, we have $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Thus, function f is one-one.

Now, we will prove that f is onto.

Let $y \in \mathbb{R}$ (co-domain) such that $f(x) = y$

$$\Rightarrow 4x + 3 = y$$

$$\Rightarrow 4x = y - 3$$

$$\therefore x = \frac{y-3}{4}$$

Clearly, for every $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ (domain) such that $f(x) = y$ and hence, function f is onto.

Thus, the function f has an inverse.

We have $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found $f(x) = y \Rightarrow x = \frac{y-3}{4}$

Hence, $f^{-1}(y) = \frac{y-3}{4}$

Thus, $f(x)$ is invertible and $f^{-1}(x) = \frac{x-3}{4}$

7. Question

Consider $f : \mathbb{R}^+ \rightarrow [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with f^{-1} of f given by $f^{-1}(x) = \sqrt{x-4}$, where \mathbb{R}^+ is the set of all non-negative real numbers.

Answer

We have $f : \mathbb{R}^+ \rightarrow [4, \infty)$ and $f(x) = x^2 + 4$.

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that f is one-one.

Let $x_1, x_2 \in \mathbb{R}^+$ (domain) such that $f(x_1) = f(x_2)$

$$\Rightarrow x_1^2 + 4 = x_2^2 + 4$$

$$\Rightarrow x_1^2 = x_2^2$$

$$\therefore x_1 = x_2 \text{ (} x_1 \neq -x_2 \text{ as } x_1, x_2 \in \mathbb{R}^+ \text{)}$$

So, we have $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Thus, function f is one-one.

Now, we will prove that f is onto.

Let $y \in [4, \infty)$ (co-domain) such that $f(x) = y$

$$\Rightarrow x^2 + 4 = y$$

$$\Rightarrow x^2 = y - 4$$

$$\therefore x = \sqrt{y - 4}$$

Clearly, for every $y \in [4, \infty)$, there exists $x \in \mathbb{R}^+$ (domain) such that $f(x) = y$ and hence, function f is onto.

Thus, the function f has an inverse.

$$\text{We have } f(x) = y \Rightarrow x = f^{-1}(y)$$

$$\text{But, we found } f(x) = y \Rightarrow x = \sqrt{y - 4}$$

$$\text{Hence, } f^{-1}(y) = \sqrt{y - 4}$$

$$\text{Thus, } f(x) \text{ is invertible and } f^{-1}(x) = \sqrt{x - 4}$$

8. Question

If $f(x) = \frac{4x+3}{6x-4}$, $x \neq \frac{2}{3}$, show that $(f \circ f)(x) = x$ for all $x \neq \frac{2}{3}$. What is the inverse of f ?

Answer

$$\text{We have } f(x) = \frac{4x+3}{6x-4}, x \neq \frac{2}{3}$$

$$\text{We know } (f \circ f)(x) = f(f(x))$$

$$\Rightarrow (f \circ f)(x) = f\left(\frac{4x+3}{6x-4}\right)$$

$$\Rightarrow (f \circ f)(x) = \frac{4\left(\frac{4x+3}{6x-4}\right) + 3}{6\left(\frac{4x+3}{6x-4}\right) - 4}$$

$$\Rightarrow (f \circ f)(x) = \frac{4(4x+3) + 3(6x-4)}{6(4x+3) - 4(6x-4)}$$

$$\Rightarrow (f \circ f)(x) = \frac{16x + 12 + 18x - 12}{24x + 18 - 24x + 16}$$

$$\Rightarrow (f \circ f)(x) = \frac{34x}{34}$$

$$\therefore (f \circ f)(x) = x$$

As $(f \circ f)(x) = x = I_x$ (the identity function), $f(x) = f^{-1}(x)$.

$$\text{Thus, } f^{-1}(x) = \frac{4x+3}{6x-4}$$

9. Question

Consider $f: \mathbb{R}^+ \rightarrow [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(x) = \frac{\sqrt{x+6}-1}{3}$.

Answer

We have $f: \mathbb{R}^+ \rightarrow [-5, \infty)$ and $f(x) = 9x^2 + 6x - 5$.

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that f is one-one.

Let $x_1, x_2 \in \mathbb{R}^+$ (domain) such that $f(x_1) = f(x_2)$

$$\Rightarrow 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$$

$$\Rightarrow 9x_1^2 + 6x_1 = 9x_2^2 + 6x_2$$

$$\Rightarrow 9x_1^2 - 9x_2^2 + 6x_1 - 6x_2 = 0$$

$$\Rightarrow 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow 9(x_1 - x_2)(x_1 + x_2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)[9(x_1 + x_2) + 6] = 0$$

$$\Rightarrow x_1 - x_2 = 0 \text{ (as } x_1, x_2 \in \mathbb{R}^+)$$

$$\therefore x_1 = x_2$$

So, we have $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Thus, function f is one-one.

Now, we will prove that f is onto.

Let $y \in [-5, \infty)$ (co-domain) such that $f(x) = y$

$$\Rightarrow 9x^2 + 6x - 5 = y$$

Adding 6 to both sides, we get

$$9x^2 + 6x - 5 + 6 = y + 6$$

$$\Rightarrow 9x^2 + 6x + 1 = y + 6$$

$$\Rightarrow (3x + 1)^2 = y + 6$$

$$\Rightarrow 3x + 1 = \sqrt{y + 6}$$

$$\Rightarrow 3x = \sqrt{y + 6} - 1$$

$$\therefore x = \frac{\sqrt{y + 6} - 1}{3}$$

Clearly, for every $y \in [-5, \infty)$, there exists $x \in \mathbb{R}^+$ (domain) such that $f(x) = y$ and hence, function f is onto.

Thus, the function f has an inverse.

We have $f(x) = y \Rightarrow x = f^{-1}(y)$

$$\text{But, we found } f(x) = y \Rightarrow x = \frac{\sqrt{y + 6} - 1}{3}$$

$$\text{Hence, } f^{-1}(y) = \frac{\sqrt{y + 6} - 1}{3}$$

$$\text{Thus, } f(x) \text{ is invertible and } f^{-1}(x) = \frac{\sqrt{x + 6} - 1}{3}$$

10. Question

If $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^3 - 3$, then prove that f^{-1} exists and find a formula for f^{-1} . Hence, find $f^{-1}(24)$

and $f^{-1}(5)$.

Answer

We have $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = x^3 - 3$.

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that f is one-one.

Let $x_1, x_2 \in \mathbb{R}$ (domain) such that $f(x_1) = f(x_2)$

$$\Rightarrow x_1^3 - 3 = x_2^3 - 3$$

$$\Rightarrow x_1^3 = x_2^3$$

$$\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$$

$$\Rightarrow x_1 - x_2 = 0 \text{ (as } x_1, x_2 \in \mathbb{R}^+)$$

$$\therefore x_1 = x_2$$

So, we have $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Thus, function f is one-one.

Now, we will prove that f is onto.

Let $y \in \mathbb{R}$ (co-domain) such that $f(x) = y$

$$\Rightarrow x^3 - 3 = y$$

$$\Rightarrow x^3 = y + 3$$

$$\therefore x = \sqrt[3]{y+3}$$

Clearly, for every $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ (domain) such that $f(x) = y$ and hence, function f is onto.

Thus, the function f has an inverse.

We have $f(x) = y \Rightarrow x = f^{-1}(y)$

$$\text{But, we found } f(x) = y \Rightarrow x = \sqrt[3]{y+3}$$

$$\text{Hence, } f^{-1}(y) = \sqrt[3]{y+3}$$

Thus, $f(x)$ is invertible and $f^{-1}(x) = \sqrt[3]{x+3}$

Hence, we have

$$f^{-1}(24) = \sqrt[3]{24+3} = \sqrt[3]{27} = 3$$

$$f^{-1}(5) = \sqrt[3]{5+3} = \sqrt[3]{8} = 2$$

Thus, $f^{-1}(24) = 3$ and $f^{-1}(5) = 2$.

11. Question

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined as $f(x) = x^3 + 4$. Is it a bijection or not? In case it is a bijection, find $f^{-1}(3)$.

Answer

We have $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = x^3 + 4$.

Recall that a function is a bijection only if it is both one-one and onto.

First, we will check if f is one-one.

Let $x_1, x_2 \in \mathbb{R}$ (domain) such that $f(x_1) = f(x_2)$

$$\Rightarrow x_1^3 + 4 = x_2^3 + 4$$

$$\Rightarrow x_1^3 = x_2^3$$

$$\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$$

As $x_1, x_2 \in \mathbb{R}$ and the second factor has no real roots,

$$x_1 - x_2 = 0$$

$$\therefore x_1 = x_2$$

So, we have $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Thus, function f is one-one.

Now, we will check if f is onto.

Let $y \in \mathbb{R}$ (co-domain) such that $f(x) = y$

$$\Rightarrow x^3 + 4 = y$$

$$\Rightarrow x^3 = y - 4$$

$$\therefore x = \sqrt[3]{y - 4}$$

Clearly, for every $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ (domain) such that $f(x) = y$ and hence, function f is onto.

Thus, the function f is a bijection and has an inverse.

We have $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found $f(x) = y \Rightarrow x = \sqrt[3]{y - 4}$

Hence, $f^{-1}(y) = \sqrt[3]{y - 4}$

Thus, $f(x)$ is invertible and $f^{-1}(x) = \sqrt[3]{x - 4}$

Hence, we have

$$f^{-1}(3) = \sqrt[3]{3 - 4} = \sqrt[3]{-1} = -1$$

Thus, $f^{-1}(3) = -1$.

12. Question

If $f : Q \rightarrow Q$, $g : Q \rightarrow Q$ are two functions defined by $f(x) = 2x$ and $g(x) = x + 2$, show that f and g are bijective maps. Verify that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Answer

We have $f : Q \rightarrow Q$ and $f(x) = 2x$.

Recall that a function is a bijection only if it is both one-one and onto.

First, we will prove that f is one-one.

Let $x_1, x_2 \in Q$ (domain) such that $f(x_1) = f(x_2)$

$$\Rightarrow 2x_1 = 2x_2$$

$$\therefore x_1 = x_2$$

So, we have $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Thus, function f is one-one.

Now, we will prove that f is onto.

Let $y \in Q$ (co-domain) such that $f(x) = y$

$$\Rightarrow 2x = y$$

$$\therefore x = \frac{y}{2}$$

Clearly, for every $y \in Q$, there exists $x \in Q$ (domain) such that $f(x) = y$ and hence, function f is onto.

Thus, the function f is a bijection and has an inverse.

$$\text{We have } f(x) = y \Rightarrow x = f^{-1}(y)$$

$$\text{But, we found } f(x) = y \Rightarrow x = \frac{y}{2}$$

$$\text{Hence, } f^{-1}(y) = \frac{y}{2}$$

$$\text{Thus, } f^{-1}(x) = \frac{x}{2}$$

Now, we have $g : Q \rightarrow Q$ and $g(x) = x + 2$.

First, we will prove that g is one-one.

Let $x_1, x_2 \in Q$ (domain) such that $g(x_1) = g(x_2)$

$$\Rightarrow x_1 + 2 = x_2 + 2$$

$$\therefore x_1 = x_2$$

$$\text{So, we have } g(x_1) = g(x_2) \Rightarrow x_1 = x_2.$$

Thus, function g is one-one.

Now, we will prove that g is onto.

Let $y \in Q$ (co-domain) such that $g(x) = y$

$$\Rightarrow x + 2 = y$$

$$\therefore x = y - 2$$

Clearly, for every $y \in Q$, there exists $x \in Q$ (domain) such that $g(x) = y$ and hence, function g is onto.

Thus, the function g is a bijection and has an inverse.

$$\text{We have } g(x) = y \Rightarrow x = g^{-1}(y)$$

$$\text{But, we found } g(x) = y \Rightarrow x = y - 2$$

$$\text{Hence, } g^{-1}(y) = y - 2$$

$$\text{Thus, } g^{-1}(x) = x - 2$$

$$\text{We have } (f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x))$$

$$\text{We found } f^{-1}(x) = \frac{x}{2} \text{ and } g^{-1}(x) = x - 2$$

$$\Rightarrow (f^{-1} \circ g^{-1})(x) = f^{-1}(x - 2)$$

$$\therefore (f^{-1} \circ g^{-1})(x) = \frac{x - 2}{2}$$

We know $(g \circ f)(x) = g(f(x))$ and $g \circ f : Q \rightarrow Q$

$$\Rightarrow (g \circ f)(x) = g(2x)$$

$$\therefore (g \circ f)(x) = 2x + 2$$

Clearly, $g \circ f$ is a bijection and has an inverse.

Let $y \in Q$ (co-domain) such that $(gof)(x) = y$

$$\Rightarrow 2x + 2 = y$$

$$\Rightarrow 2x = y - 2$$

$$\therefore x = \frac{y-2}{2}$$

We have $(gof)(x) = y \Rightarrow x = (gof)^{-1}(y)$

But, we found $(gof)(x) = y \Rightarrow x = \frac{y-2}{2}$

Hence, $(gof)^{-1}(y) = \frac{y-2}{2}$

Thus, $(gof)^{-1}(x) = \frac{x-2}{2}$

So, it is verified that $(gof)^{-1} = f^{-1}og^{-1}$.

13. Question

Let $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$. Consider the function $f : A \rightarrow B$ defined by $f(x) = \frac{x-2}{x-3}$. Show that f is one-one and onto and hence find f^{-1} .

Answer

We have $f : A \rightarrow B$ where $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$

$$f(x) = \frac{x-2}{x-3}$$

First, we will prove that f is one-one.

Let $x_1, x_2 \in A$ (domain) such that $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3}$$

$$\Rightarrow (x_1-2)(x_2-3) = (x_1-3)(x_2-2)$$

$$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$$

$$\Rightarrow -3x_1 - 2x_2 = -2x_1 - 3x_2$$

$$\Rightarrow -3x_1 + 2x_1 = 2x_2 - 3x_2$$

$$\Rightarrow -x_1 = -x_2$$

$$\therefore x_1 = x_2$$

So, we have $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Thus, function f is one-one.

Now, we will prove that f is onto.

Let $y \in B$ (co-domain) such that $f(x) = y$

$$\Rightarrow \frac{x-2}{x-3} = y$$

$$\Rightarrow \frac{(x-3)+1}{x-3} = y$$

$$\Rightarrow 1 + \frac{1}{x-3} = y$$

$$\Rightarrow \frac{1}{x-3} = y-1$$

$$\Rightarrow \frac{1}{y-1} = x-3$$

$$\Rightarrow x = 3 + \frac{1}{y-1}$$

$$\therefore x = \frac{3y-2}{y-1}$$

Clearly, for every $y \in B$, there exists $x \in A$ (domain) such that $f(x) = y$ and hence, function f is onto.

Thus, the function f has an inverse.

We have $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found $f(x) = y \Rightarrow x = \frac{3y-2}{y-1}$

Hence, $f^{-1}(y) = \frac{3y-2}{y-1}$

Thus, $f(x)$ is invertible and $f^{-1}(x) = \frac{3x-2}{x-1}$

14. Question

Consider the function $f : \mathbb{R}^+ \rightarrow [-9, \infty)$ given by $f(x) = 5x^2 + 6x - 9$. Prove that f is invertible with

$$f^{-1}(y) = \frac{\sqrt{54+5y}-3}{4}.$$

Answer

We have $f : \mathbb{R}^+ \rightarrow [-9, \infty)$ and $f(x) = 5x^2 + 6x - 9$.

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that f is one-one.

Let $x_1, x_2 \in \mathbb{R}^+$ (domain) such that $f(x_1) = f(x_2)$

$$\Rightarrow 5x_1^2 + 6x_1 - 9 = 5x_2^2 + 6x_2 - 9$$

$$\Rightarrow 5x_1^2 + 6x_1 = 5x_2^2 + 6x_2$$

$$\Rightarrow 5x_1^2 - 5x_2^2 + 6x_1 - 6x_2 = 0$$

$$\Rightarrow 5(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow 5(x_1 - x_2)(x_1 + x_2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)[5(x_1 + x_2) + 6] = 0$$

$$\Rightarrow x_1 - x_2 = 0 \text{ (as } x_1, x_2 \in \mathbb{R}^+)$$

$$\therefore x_1 = x_2$$

So, we have $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Thus, function f is one-one.

Now, we will prove that f is onto.

Let $y \in [-9, \infty)$ (co-domain) such that $f(x) = y$

$$\Rightarrow 5x^2 + 6x - 9 = y$$

$$\Rightarrow 5\left(x^2 + \frac{6}{5}x - \frac{9}{5}\right) = y$$

$$\Rightarrow x^2 + \frac{6}{5}x - \frac{9}{5} = \frac{y}{5}$$

$$\Rightarrow x^2 + \frac{6}{5}x = \frac{y+9}{5}$$

Adding $\frac{9}{25}$ to both sides, we get

$$\Rightarrow x^2 + \frac{6}{5}x + \frac{9}{25} = \frac{y+9}{5} + \frac{9}{25}$$

$$\Rightarrow \left(x + \frac{3}{5}\right)^2 = \frac{(5y+45) + 9}{25}$$

$$\Rightarrow \left(x + \frac{3}{5}\right)^2 = \frac{5y+54}{25}$$

$$\Rightarrow x + \frac{3}{5} = \sqrt{\frac{5y+54}{25}}$$

$$\Rightarrow x + \frac{3}{5} = \frac{\sqrt{5y+54}}{5}$$

$$\Rightarrow x = \frac{\sqrt{5y+54}}{5} - \frac{3}{5}$$

$$\therefore x = \frac{\sqrt{5y+54}-3}{5}$$

Clearly, for every $y \in [-9, \infty)$, there exists $x \in \mathbb{R}^+$ (domain) such that $f(x) = y$ and hence, function f is onto.

Thus, the function f has an inverse.

We have $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found $f(x) = y \Rightarrow x = \frac{\sqrt{5y+54}-3}{5}$

Hence, $f^{-1}(y) = \frac{\sqrt{5y+54}-3}{5}$

15. Question

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function defined as $f(x) = 9x^2 + 6x - 5$. Show that $f : \mathbb{N} \rightarrow S$, where S is the range of f , is invertible. Find the inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$.

Answer

We have $f : \mathbb{N} \rightarrow \mathbb{N}$ and $f(x) = 9x^2 + 6x - 5$.

We need to prove $f : \mathbb{N} \rightarrow S$ is invertible.

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that f is one-one.

Let $x_1, x_2 \in \mathbb{N}$ (domain) such that $f(x_1) = f(x_2)$

$$\Rightarrow 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$$

$$\Rightarrow 9x_1^2 + 6x_1 = 9x_2^2 + 6x_2$$

$$\Rightarrow 9x_1^2 - 9x_2^2 + 6x_1 - 6x_2 = 0$$

$$\Rightarrow 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow 9(x_1 - x_2)(x_1 + x_2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)[9(x_1 + x_2) + 6] = 0$$

$$\Rightarrow x_1 - x_2 = 0 \text{ (as } x_1, x_2 \in \mathbb{R}^+)$$

$$\therefore x_1 = x_2$$

So, we have $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Thus, function f is one-one.

Now, we will prove that f is onto.

Let $y \in S$ (co-domain) such that $f(x) = y$

$$\Rightarrow 9x^2 + 6x - 5 = y$$

Adding 6 to both sides, we get

$$9x^2 + 6x - 5 + 6 = y + 6$$

$$\Rightarrow 9x^2 + 6x + 1 = y + 6$$

$$\Rightarrow (3x + 1)^2 = y + 6$$

$$\Rightarrow 3x + 1 = \sqrt{y + 6}$$

$$\Rightarrow 3x = \sqrt{y + 6} - 1$$

$$\therefore x = \frac{\sqrt{y + 6} - 1}{3}$$

Clearly, for every $y \in S$, there exists $x \in N$ (domain) such that $f(x) = y$ and hence, function f is onto.

Thus, the function f has an inverse.

We have $f(x) = y \Rightarrow x = f^{-1}(y)$

$$\text{But, we found } f(x) = y \Rightarrow x = \frac{\sqrt{y+6}-1}{3}$$

$$\text{Hence, } f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$$

$$\text{Thus, } f(x) \text{ is invertible and } f^{-1}(x) = \frac{\sqrt{x+6}-1}{3}$$

Hence, we have

$$f^{-1}(43) = \frac{\sqrt{43+6}-1}{3} = \frac{\sqrt{49}-1}{3} = \frac{7-1}{3} = 2$$

$$f^{-1}(163) = \frac{\sqrt{163+6}-1}{3} = \frac{\sqrt{169}-1}{3} = \frac{13-1}{3} = 4$$

Thus, $f^{-1}(43) = 2$ and $f^{-1}(163) = 4$.

16. Question

Let $f : \mathbb{R} - \left\{ -\frac{4}{3} \right\} \rightarrow \mathbb{R}$ be a function defined as $f(x) = \frac{4x}{3x+4}$. Show that $f : \mathbb{R} - \left\{ -\frac{4}{3} \right\} \rightarrow \text{range}(f)$ is one-one and onto. Hence, find f^{-1} .

Answer

We have $f : \mathbb{R} - \left\{-\frac{4}{3}\right\} \rightarrow \mathbb{R}$ and $f(x) = \frac{4x}{3x+4}$

We need to prove $f : \mathbb{R} - \left\{-\frac{4}{3}\right\} \rightarrow \text{range}(f)$ is invertible.

First, we will prove that f is one-one.

Let $x_1, x_2 \in A$ (domain) such that $f(x_1) = f(x_2)$

$$\Rightarrow \frac{4x_1}{3x_1+4} = \frac{4x_2}{3x_2+4}$$

$$\Rightarrow (4x_1)(3x_2+4) = (3x_1+4)(4x_2)$$

$$\Rightarrow 12x_1x_2 + 16x_1 = 12x_1x_2 + 16x_2$$

$$\Rightarrow 16x_1 = 16x_2$$

$$\therefore x_1 = x_2$$

So, we have $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Thus, function f is one-one.

Now, we will prove that f is onto.

Let $y \in \text{range}(f)$ (co-domain) such that $f(x) = y$

$$\Rightarrow \frac{4x}{3x+4} = y$$

$$\Rightarrow 4x = 3xy + 4y$$

$$\Rightarrow 4x - 3xy = 4y$$

$$\Rightarrow x(4 - 3y) = 4y$$

$$\therefore x = \frac{4y}{4-3y}$$

Clearly, for every $y \in \text{range}(f)$, there exists $x \in A$ (domain) such that $f(x) = y$ and hence, function f is onto.

Thus, the function f has an inverse.

We have $f(x) = y \Rightarrow x = f^{-1}(y)$

$$\text{But, we found } f(x) = y \Rightarrow x = \frac{4y}{4-3y}$$

$$\text{Hence, } f^{-1}(y) = \frac{4y}{4-3y}$$

$$\text{Thus, } f(x) \text{ is invertible and } f^{-1}(x) = \frac{4x}{4-3x}$$

17. Question

If $f: \mathbb{R} \rightarrow (-1, 1)$ defined by $f(x) = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$ is invertible, find f^{-1} .

Answer

We have $f: \mathbb{R} \rightarrow (-1, 1)$ and $f(x) = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$

Given that f^{-1} exists.

Let $y \in (-1, 1)$ such that $f(x) = y$

$$\Rightarrow \frac{10^x - 10^{-x}}{10^x + 10^{-x}} = y$$

$$\Rightarrow \frac{10^x - \frac{1}{10^x}}{10^x + \frac{1}{10^x}} = y$$

$$\Rightarrow \frac{10^{2x} - 1}{10^{2x} + 1} = y$$

$$\Rightarrow 10^{2x} - 1 = y(10^{2x} + 1)$$

$$\Rightarrow 10^{2x} - 1 = 10^{2x}y + y$$

$$\Rightarrow 10^{2x} - 10^{2x}y = 1 + y$$

$$\Rightarrow 10^{2x}(1 - y) = 1 + y$$

$$\Rightarrow 10^{2x} = \frac{1 + y}{1 - y}$$

Taking \log_{10} on both sides, we get

$$\log_{10} 10^{2x} = \log_{10} \left(\frac{1 + y}{1 - y} \right)$$

$$\Rightarrow 2x \log_{10} 10 = \log_{10} \left(\frac{1 + y}{1 - y} \right)$$

$$\Rightarrow 2x = \log_{10} \left(\frac{1 + y}{1 - y} \right)$$

$$\therefore x = \frac{1}{2} \log_{10} \left(\frac{1 + y}{1 - y} \right)$$

We have $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found $f(x) = y \Rightarrow x = \frac{1}{2} \log_{10} \left(\frac{1 + y}{1 - y} \right)$

Hence, $f^{-1}(y) = \frac{1}{2} \log_{10} \left(\frac{1 + y}{1 - y} \right)$

Thus, $f^{-1}(x) = \frac{1}{2} \log_{10} \left(\frac{1 + x}{1 - x} \right)$

18. Question

If $f: \mathbb{R} \rightarrow (0, 2)$ defined by $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 1$ is invertible, find f^{-1} .

Answer

We have $f: \mathbb{R} \rightarrow (0, 2)$ and $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 1$

Given that f^{-1} exists.

Let $y \in (0, 2)$ such that $f(x) = y$

$$\Rightarrow \frac{e^x - e^{-x}}{e^x + e^{-x}} + 1 = y$$

$$\Rightarrow \frac{e^x - e^{-x} + (e^x + e^{-x})}{e^x + e^{-x}} = y$$

$$\Rightarrow \frac{2e^x}{e^x + e^{-x}} = y$$

$$\Rightarrow \frac{2e^x}{e^x + \frac{1}{e^x}} = y$$

$$\Rightarrow \frac{2e^{2x}}{e^{2x} + 1} = y$$

$$\Rightarrow 2e^{2x} = y(e^{2x} + 1)$$

$$\Rightarrow 2e^{2x} = e^{2x}y + y$$

$$\Rightarrow 2e^{2x} - e^{2x}y = y$$

$$\Rightarrow e^{2x}(2 - y) = y$$

$$\Rightarrow e^{2x} = \frac{y}{2 - y}$$

Taking \ln on both sides, we get

$$\ln e^{2x} = \ln\left(\frac{y}{2 - y}\right)$$

$$\Rightarrow 2x \ln e = \ln\left(\frac{y}{2 - y}\right)$$

$$\Rightarrow 2x = \ln\left(\frac{y}{2 - y}\right)$$

$$\therefore x = \frac{1}{2} \ln\left(\frac{y}{2 - y}\right)$$

We have $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found $f(x) = y \Rightarrow x = \frac{1}{2} \ln\left(\frac{y}{2 - y}\right)$

Hence, $f^{-1}(y) = \frac{1}{2} \ln\left(\frac{y}{2 - y}\right)$

Thus, $f^{-1}(x) = \frac{1}{2} \ln\left(\frac{x}{2 - x}\right)$

19. Question

Let $f : [-1, \infty) \rightarrow [-1, \infty)$ is given by $f(x) = (x + 1)^2 - 1$. Show that f is invertible. Also, find the set $S = \{x : f(x) = f^{-1}(x)\}$

Answer

We have $f : [-1, \infty) \rightarrow [-1, \infty)$ and $f(x) = (x + 1)^2 - 1$

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that f is one-one.

Let $x_1, x_2 \in [-1, \infty)$ (domain) such that $f(x_1) = f(x_2)$

$$\Rightarrow (x_1 + 1)^2 - 1 = (x_2 + 1)^2 - 1$$

$$\Rightarrow (x_1 + 1)^2 = (x_2 + 1)^2$$

$$\Rightarrow x_1^2 + 2x_1 + 1 = x_2^2 + 2x_2 + 1$$

$$\Rightarrow x_1^2 + 2x_1 = x_2^2 + 2x_2$$

$$\Rightarrow x_1^2 - x_2^2 + 2x_1 - 2x_2 = 0$$

$$\Rightarrow (x_1^2 - x_2^2) + 2(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)(x_1 + x_2) + 2(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)[x_1 + x_2 + 2] = 0$$

$$\Rightarrow x_1 - x_2 = 0 \text{ (as } x_1, x_2 \in \mathbb{R}^+)$$

$$\therefore x_1 = x_2$$

So, we have $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Thus, function f is one-one.

Now, we will prove that f is onto.

Let $y \in [-1, \infty)$ (co-domain) such that $f(x) = y$

$$\Rightarrow (x + 1)^2 - 1 = y$$

$$\Rightarrow (x + 1)^2 = y + 1$$

$$\Rightarrow x + 1 = \sqrt{y + 1}$$

$$\therefore x = \sqrt{y + 1} - 1$$

Clearly, for every $y \in [-1, \infty)$, there exists $x \in [-1, \infty)$ (domain) such that $f(x) = y$ and hence, function f is onto.

Thus, the function f has an inverse.

$$\text{We have } f(x) = y \Rightarrow x = f^{-1}(y)$$

$$\text{But, we found } f(x) = y \Rightarrow x = \sqrt{y + 1} - 1$$

$$\text{Hence, } f^{-1}(y) = \sqrt{y + 1} - 1$$

$$\text{Thus, } f(x) \text{ is invertible and } f^{-1}(x) = \sqrt{x + 1} - 1$$

Now, we need to find the values of x for which $f(x) = f^{-1}(x)$.

$$\text{We have } f(x) = f^{-1}(x)$$

$$\Rightarrow (x + 1)^2 - 1 = \sqrt{x + 1} - 1$$

$$\Rightarrow (x + 1)^2 = \sqrt{x + 1}$$

$$\text{We can write } (x + 1)^2 = (\sqrt{x + 1})^4$$

$$\Rightarrow (\sqrt{x + 1})^4 = \sqrt{x + 1}$$

On substituting $t = \sqrt{x + 1}$, we get

$$t^4 = t$$

$$\Rightarrow t^4 - t = 0$$

$$\Rightarrow t(t^3 - 1) = 0$$

$$\Rightarrow t(t - 1)(t^2 + t + 1) = 0$$

$t^2 + t + 1 \neq 0$ because this equation has no real root t .

$$\Rightarrow t = 0 \text{ or } t - 1 = 0$$

$$\Rightarrow t = 0 \text{ or } t = 1$$

Case - I: $t = 0$

$$\Rightarrow \sqrt{x+1} = 0$$

$$\Rightarrow x + 1 = 0$$

$$\therefore x = -1$$

Case - II: $t = 1$

$$\Rightarrow \sqrt{x+1} = 1$$

$$\Rightarrow x + 1 = 1$$

$$\therefore x = 0$$

Thus, $S = \{0, -1\}$

20. Question

Let $A = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$ and let $f : A \rightarrow A$, $g : A \rightarrow A$ be two functions defined by $f(x) = x^2$ and $g(x) = \sin \pi x/2$. Show that g^{-1} exists but f^{-1} does not exist. Also, find g^{-1} .

Answer

We have $f : A \rightarrow A$ where $A = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$ defined by $f(x) = x^2$.

Recall that a function is invertible only when it is both one-one and onto.

First, we will check if f is one-one.

Let $x_1, x_2 \in A$ (domain) such that $f(x_1) = f(x_2)$

$$\Rightarrow x_1^2 = x_2^2$$

$$\Rightarrow x_1^2 - x_2^2 = 0$$

$$\Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$$

$$\Rightarrow x_1 - x_2 = 0 \text{ or } x_1 + x_2 = 0$$

$$\therefore x_1 = \pm x_2$$

So, we have $f(x_1) = f(x_2) \Rightarrow x_1 = \pm x_2$.

This means that two different elements of the domain are mapped to the same element by the function f .

For example, consider $f(-1)$ and $f(1)$.

We have $f(-1) = (-1)^2 = 1$ and $f(1) = 1^2 = 1 = f(-1)$

Thus, f is not one-one and hence f^{-1} doesn't exist.

Now, let us consider $g : A \rightarrow A$ defined by $g(x) = \sin \frac{\pi x}{2}$

First, we will prove that g is one-one.

Let $x_1, x_2 \in A$ (domain) such that $g(x_1) = g(x_2)$

$$\Rightarrow \sin \frac{\pi x_1}{2} = \sin \frac{\pi x_2}{2}$$

$$\Rightarrow \frac{\pi x_1}{2} = \frac{\pi x_2}{2} \text{ (in the given range)}$$

$$\therefore x_1 = x_2$$

So, we have $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$.

Thus, function g is one-one.

Let $y \in A$ (co-domain) such that $g(x) = y$

$$\Rightarrow \sin \frac{\pi x}{2} = y$$

$$\Rightarrow \frac{\pi x}{2} = \sin^{-1} y$$

$$\Rightarrow \pi x = 2 \sin^{-1} y$$

$$\therefore x = \frac{2}{\pi} \sin^{-1} y$$

Clearly, for every $y \in A$, there exists $x \in A$ (domain) such that $g(x) = y$ and hence, function g is onto.

Thus, the function g has an inverse.

We have $g(x) = y \Rightarrow x = g^{-1}(y)$

But, we found $g(x) = y \Rightarrow x = \frac{2}{\pi} \sin^{-1} y$

Hence, $g^{-1}(y) = \frac{2}{\pi} \sin^{-1} y$

Thus, $g(x)$ is invertible and $g^{-1}(x) = \frac{2}{\pi} \sin^{-1} x$

21. Question

Let f be a function from \mathbb{R} to \mathbb{R} such that $f(x) = \cos(x + 2)$. Is f invertible? Justify your answer.

Answer

We have $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = \cos(x + 2)$.

Recall that a function is invertible only when it is both one-one and onto.

First, we will check if f is one-one.

Let $x_1, x_2 \in \mathbb{R}$ (domain) such that $f(x_1) = f(x_2)$

$$\Rightarrow \cos(x_1 + 2) = \cos(x_2 + 2)$$

As the cosine function repeats itself with a period 2π , we have

$$x_1 + 2 = x_2 + 2 \text{ or } x_1 + 2 = 2\pi + (x_2 + 2)$$

$$\therefore x_1 = x_2 \text{ or } x_1 = 2\pi + x_2$$

$$\text{So, we have } f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \text{ or } 2\pi + x_2$$

This means that two different elements of the domain are mapped to the same element by the function f .

For example, consider $f(0)$ and $f(2\pi)$.

$$\text{We have } f(0) = \cos(0 + 2) = \cos 2 \text{ and}$$

$$f(2\pi) = \cos(2\pi + 2) = \cos 2 = f(0)$$

Thus, f is not one-one.

Hence, f is not invertible and f^{-1} does not exist.

22. Question

If $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$, define any four bijections from A to B . Also, give their inverse function.

Answer

Given $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$.

We need to define bijections f_1, f_2, f_3 and f_4 from A to B .

Consider $f_1 = \{(1, a), (2, b), (3, c), (4, d)\}$

(1) f_1 is one-one because no two elements of the domain are mapped to the same element.

f_1 is also onto because each element in the co-domain has a pre-image in the domain.

Thus, f_1 is a bijection from A to B.

We have $f_1^{-1} = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$

Using similar explanation, we also have the following bijections defined from A to B -

(2) $f_2 = \{(1, b), (2, c), (3, d), (4, a)\}$

We have $f_2^{-1} = \{(b, 1), (c, 2), (d, 3), (a, 4)\}$

(3) $f_3 = \{(1, c), (2, d), (3, a), (4, b)\}$

We have $f_3^{-1} = \{(c, 1), (d, 2), (a, 3), (b, 4)\}$

(4) $f_4 = \{(1, d), (2, a), (3, b), (4, c)\}$

We have $f_4^{-1} = \{(d, 1), (a, 2), (b, 3), (c, 4)\}$

23. Question

Let A and B be two sets each with finite number of elements. Assume that there is an injective map from A to B and that there is an injective map from B to A. Prove that there is a bijection from A to B.

Answer

Given A and B are two finite sets. There are injective maps from both A to B and B to A.

Let f be the injective map defined from A to B.

Thus, we have f is one-one.

We also know that there is a one-one mapping from B to A.

This means that each element of B is mapped to a distinct element of A.

But, B is the co-domain of f and A is the domain of f.

So, every element of the co-domain of the function f has a pre-image in the domain of the function f.

Thus, f is also onto.

Therefore, f is a bijection as it is both one-one and onto.

Hence, there exists a bijection defined from A to B.

24. Question

If $f : A \rightarrow A$ and $g : A \rightarrow A$ are two bijections, then prove that

(i) $f \circ g$ is an injection

(ii) $f \circ g$ is a surjection

Answer

Given $f : A \rightarrow A$ and $g : A \rightarrow A$ are two bijections. So, both f and g are one-one and onto functions.

We know $(f \circ g)(x) = f(g(x))$

Thus, $f \circ g$ is also defined from A to A.

(i) First, we will prove that $f \circ g$ is an injection.

Let $x_1, x_2 \in A$ (domain) such that $(f \circ g)(x_1) = (f \circ g)(x_2)$

$$\Rightarrow f(g(x_1)) = f(g(x_2))$$

$$\Rightarrow g(x_1) = g(x_2) \text{ [since } f \text{ is one-one]}$$

$$\therefore x_1 = x_2 \text{ [since } g \text{ is one-one]}$$

$$\text{So, we have } (f \circ g)(x_1) = (f \circ g)(x_2) \Rightarrow x_1 = x_2.$$

Thus, function $f \circ g$ is an interjection.

(ii) Now, we will prove that $f \circ g$ is a surjection.

Let $z \in A$, the co-domain of $f \circ g$.

As f is onto, we have $y \in A$ (domain of f) such that $f(y) = z$.

However, as g is also onto and y belongs to the co-domain of g , we have $x \in A$ (domain of g) such that $g(x) = y$.

$$\text{Hence, } (f \circ g)(x) = f(g(x)) = f(y) = z.$$

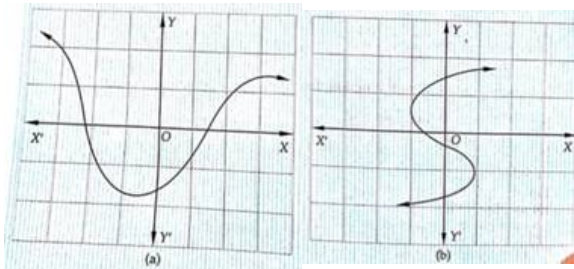
Here, x belongs to the domain of $f \circ g$ (A) and z belongs to the co-domain of $f \circ g$ (A).

Thus, function $f \circ g$ is a surjection.

Very short answer

1. Question

Which one of the following graphs represent a function?



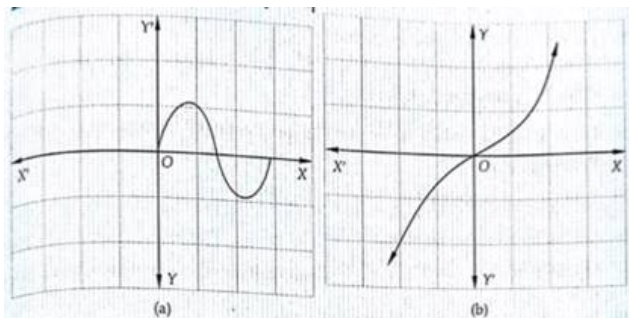
Answer

(a) It has a unique image therefore a function

(b) It has more than one image

2. Question

Which one of the following graphs represent a one-one function?



Answer

Formula:-

(i) A function $f: A \rightarrow B$ is one-one function or an injection if

$$f(x) \neq f(y)$$

$\Rightarrow x=y$ for all $x, y \in A$

or $f(x) \neq f(y)$

$\Rightarrow x \neq y$ for all $x, y \in A$

(a) It is not one-one function as it has same image on x axis

(b) It is one-one function as it has unique image

3. Question

If $A = \{1, 2, 3\}$ and $B = \{a, b\}$, write total number of functions from A to B.

Answer

Formula:-if A and B are two non-empty finite sets containing m and n

(i) Number of function from A to B = n^m

(ii) Number of one-one function from A to B = $\begin{cases} C_m^n \cdot m!, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(iii) Number of one-one and onto function from A to B = $\begin{cases} n!, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$

(iv) Number of onto function from A to B = $\sum_{r=1}^n (-1)^{n-r} C_r^n r^m$, if $m \geq n$

given: -

$A = \{1, 2, 3\}$ and $B = \{a, b\}$

$n(A)=3$, and $n(B)=2$

total number of functions = $2^3 = 8$

4. Question

If $A = \{a, b, c\}$ and $B = \{-2, -1, 0, 1, 2\}$, write total number of one-one functions from A to B.

Answer

Formula:-

(I) A function $f: A \rightarrow B$ is one-one function or an injection if

$f(x) = f(y)$

$\Rightarrow x=y$ for all $x, y \in A$

or $f(x) \neq f(y)$

$\Rightarrow x \neq y$ for all $x, y \in A$

(II) if A and B are two non-empty finite sets containing m and n

(i) Number of function from A to B = n^m

(ii) Number of one-one function from A to B = $\begin{cases} C_m^n \cdot m!, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(iii) Number of one-one and onto function from A to B = $\begin{cases} n!, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$

(iv) Number of onto function from A to B = $\sum_{r=1}^n (-1)^{n-r} C_r^n r^m$, if $m \geq n$

Let $f: A \rightarrow B$ be one-one function

$n(A)=3$ and $n(B)=5$

Using formula

Number of one-one function from A to B = $\begin{cases} C_m^n \cdot m!, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

$$\Rightarrow {}^3C_5 \cdot 5! = 60$$

5. Question

Write total number of one-one functions from set A = {1, 2, 3, 4} to set B = {a, b, c}.

Answer

Formula:-

(I) A function $f: A \rightarrow B$ is one-one function or an injection if

$$f(x) = f(y)$$

$$\Rightarrow x = y \text{ for all } x, y \in A$$

$$\text{or } f(x) \neq f(y)$$

$$\Rightarrow x \neq y \text{ for all } x, y \in A$$

(II) if A and B are two non-empty finite sets containing m and n

(i) Number of function from A to B = n^m

(ii) Number of one-one function from A to B = $\begin{cases} C_m^n \cdot m!, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(iii) Number of one-one and onto function from A to B = $\begin{cases} n!, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$

(iv) Number of onto function from A to B = $\sum_{r=1}^n (-1)^{n-r} C_r^n r^m$, if $m \geq n$

$$F(A) = 4 \text{ and } f(B) = 3$$

Using formula

Number of one-one function from A to B = $\begin{cases} C_m^n \cdot m!, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

$$\text{Number of one-one function from A to B} = 0$$

6. Question

If $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = x^2$, write $f^{-1}(25)$.

Answer

Formula:-

(i) A function $f: X \rightarrow Y$ is defined to be invertible, if there exists a function $g: Y \rightarrow X$

such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}

$$f(x) = y$$

$$\Rightarrow f^{-1}(y) = x$$

$$\Rightarrow x^2 = 25$$

$$\Rightarrow x = -5, 5$$

$$\Rightarrow f^{-1}(25) = \{-5, 5\}$$

7. Question

If $f: \mathbb{C} \rightarrow \mathbb{C}$ is defined by $f(x) = x^2$, write $f^{-1}(-4)$. Here, C denotes the set of all complex numbers.

Answer

Formula:-

(i) A function $f: X \rightarrow Y$ is defined to be invertible, if there exists a function $g: Y \rightarrow X$

such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}

$$f(x) = y$$

$$\Rightarrow f^{-1}(y) = x$$

$$\Rightarrow f(x) = -4$$

$$\Rightarrow x^2 = -4$$

$$\Rightarrow x = 2i, -2i$$

8. Question

If $f: \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = x^3$, write $f^{-1}(1)$.

Answer

Formula:-

(i) A function $f: X \rightarrow Y$ is defined to be invertible, if there exists a function $g: Y \rightarrow X$

such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}

$$f(x) = y$$

$$\Rightarrow f^{-1}(y) = x$$

$$\Rightarrow f^{-1}(1) = x$$

$$\Rightarrow f(x) = 1$$

$$\Rightarrow x^3 = 1$$

$$\Rightarrow x^3 - 1 = 0$$

$$\Rightarrow (x-1)(x^2+x+1) = 0$$

$$\Rightarrow x = 1$$

9. Question

Let \mathbb{C} denote the set of all complex numbers. A function $f: \mathbb{C} \rightarrow \mathbb{C}$ is defined by $f(x) = x^3$.

Write $f^{-1}(1)$.

Answer

Formula:-

(i) A function $f: X \rightarrow Y$ is defined to be invertible, if there exists a function $g: Y \rightarrow X$

such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}

$$f(x) = y$$

$$\Rightarrow f^{-1}(y) = x$$

$$\Rightarrow f^{-1}(1) = x$$

$$\Rightarrow f(x) = 1$$

$$\Rightarrow x^3=1$$

$$\Rightarrow x^3-1=0$$

$$\Rightarrow (x-1)(x^2+x+1)=0$$

$$\Rightarrow x=1, w, w^2$$

10. Question

Let f be a function from C (set of all complex numbers) to itself given by $f(x) = x^3$. Write $f^{-1}(-1)$.

Answer

Formula:-

(i) A function $f: X \rightarrow Y$ is defined to be invertible, if there exists a function $g: Y \rightarrow X$

such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}

$$f(x)=y$$

$$\Rightarrow f^{-1}(y)=x$$

$$\Rightarrow f(x)=-1$$

$$\Rightarrow f^{-1}(-1)=x$$

$$\Rightarrow x^3=-1$$

$$\Rightarrow x^3+1=0$$

$$\Rightarrow (x+1)(x^2-x+1)=0$$

$$\Rightarrow x=-1, -w, -w^2$$

11. Question

Let $f: R \rightarrow R$ be defined by $f(x) = x^4$, write $f^{-1}(1)$.

Answer

Formula:-

(i) A function $f: X \rightarrow Y$ is defined to be invertible, if there exists a function $g: Y \rightarrow X$

such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}

$$f(x)=y$$

$$\Rightarrow f^{-1}(y)=x$$

$$\Rightarrow f(x)=1$$

$$\Rightarrow f^{-1}(1)=x$$

$$\Rightarrow x^4=1$$

$$\Rightarrow x^4-1=0$$

$$\Rightarrow (x-1)(x^3+x^2+x+1)=0$$

$$\Rightarrow x=-1, 1$$

$$\Rightarrow f^{-1}(1) = \{-1, 1\}$$

12. Question

If $f : C \rightarrow C$ is defined by $f(x) = x^4$, $f^{-1}(1)$.

Answer

Formula:-

(i) A function $f : X \rightarrow Y$ is defined to be invertible, if there exists a function $g : Y \rightarrow X$

such that $gof = I_x$ and $fog = I_y$. The function g is called the inverse of f and is denoted by f^{-1}

$$f(x)=y$$

$$\Rightarrow f^{-1}(y)=x$$

$$\Rightarrow f(x)=1$$

$$\Rightarrow f^{-1}(1)=x$$

$$\Rightarrow x^4=1$$

$$\Rightarrow x^4-1=0$$

$$\Rightarrow (x-1)(x^2+1)=0$$

$$\Rightarrow x=-1, 1, i, -i$$

$$\Rightarrow f^{-1}(1) = \{-1, -i, 1, i\}$$

13. Question

If $f : R \rightarrow R$ is defined by $f(x) = x^2$, $f^{-1}(-25)$.

Answer

Formula:-

(i) A function $f : X \rightarrow Y$ is defined to be invertible, if there exists a function $g : Y \rightarrow X$

such that $gof = I_x$ and $fog = I_y$. The function g is called the inverse of f and is denoted by f^{-1}

$$f(x)=y$$

$$\Rightarrow f^{-1}(y)=x$$

$$\Rightarrow x^2=-25$$

but x should be Real number

$$f^{-1}(-25) = \emptyset$$

14. Question

If $f : C \rightarrow C$ is defined by $f(x) = (x - 2)^3$, write $f^{-1}(-1)$.

Answer

Formula:-

(i) A function $f : X \rightarrow Y$ is defined to be invertible, if there exists a function $g : Y \rightarrow X$

such that $gof = I_x$ and $fog = I_y$. The function g is called the inverse of f and is denoted by f^{-1}

$$f(x)=y$$

$$f^{-1}(y)=x$$

$$\Rightarrow (x - 2)^3 = -1$$

$$\Rightarrow x-2=-1, x-2=w \text{ and } x-2=-w^2$$

$$\Rightarrow x=1, -w+2, 2-w^2$$

$$\Rightarrow f^{-1}(25) = \{1, 2-w, 2-w^2\}$$

15. Question

If $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = 10x - 7$, then write $f^{-1}(x)$.

Answer

Formula:-

(i) A function $f: X \rightarrow Y$ is defined to be invertible, if there exists a function $g: Y \rightarrow X$

such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}

$$f^{-1}(x) = y$$

$$\Rightarrow f(y) = x$$

$$\Rightarrow 10y - 7 = x$$

$$\Rightarrow y = \frac{x+7}{10}$$

$$\Rightarrow f^{-1}(x) = \frac{x+7}{10}$$

16. Question

Let $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$ be a function defined by $f(x) = \cos[x]$. Write range (f).

Answer

Given:-

$$(i) f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$(ii) f(x) = \cos[x]$$

$$\text{Domain} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\text{For } f(x) = \cos[x]$$

$$\text{Range} = \{1, \cos 1, \cos 2\}$$

17. Question

If $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 3x - 4$ is invertible then write $f^{-1}(x)$.

Answer

Given:- (i) $f: \mathbb{R} \rightarrow \mathbb{R}$

$$(ii) f(x) = 3x - 4$$

Formula:-

(i) A function $f: X \rightarrow Y$ is defined to be invertible, if there exists a function $g: Y \rightarrow X$

such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}

$$\text{For } f^{-1}(x) = y$$

$$\Rightarrow f(y) = x$$

$$\Rightarrow 3y - 4 = x$$

$$\Rightarrow y = \frac{x+4}{3}$$

$$\Rightarrow f^{-1}(x) = \frac{x+4}{3}$$

18. Question

If $f : \mathbb{R} \rightarrow \mathbb{R}$, $g : \mathbb{R} \rightarrow \mathbb{R}$ are given by $f(x) = (x + 1)^2$ and $g(x) = x^2 + 1$, then write the value of $g \circ f(-3)$.

Answer

Formula:-

(I) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions.

Then, the composition of f and g , denoted by $g \circ f$, is defined as the function $g \circ f : A \rightarrow C$

given by $g \circ f(x) = g(f(x))$

Given:-

(i) $f : \mathbb{R} \rightarrow \mathbb{R}$

(ii) $g : \mathbb{R} \rightarrow \mathbb{R}$

(iii) $f(x) = (x + 1)^2$

(iv) $g(x) = x^2 + 1$

$g \circ f(-3) = f(g(-3))$

$\Rightarrow g \circ f(-3) = f((-3)^2 + 1)$

$\Rightarrow g \circ f(-3) = f(10)$

$\Rightarrow g \circ f(-3) = (10 + 1)^2$

$\Rightarrow g \circ f(-3) = 121$

19. Question

Let $A = \{x \in \mathbb{R} : -4 \leq x \leq 4 \text{ and } x \neq 0\}$ and $f : A \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{|x|}{x}$. Write the range of f .

Answer

Given:-

(i) $A = \{x \in \mathbb{R} : -4 \leq x \leq 4 \text{ and } x \neq 0\}$

(ii) $f : A \rightarrow \mathbb{R}$

(iii) $f(x) = \frac{|x|}{x}$

For $f(x) = \frac{|x|}{x}$

Range = $\{-1, 1\}$

20. Question

Let $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow A$ be defined by $f(x) = \sin x$. If f is a bijection, write set A .

Answer

Formula:-

(i) A function $f: A \rightarrow B$ is a bijection if it is one-one as well as onto

(ii) A function $f: A \rightarrow B$ is onto function or surjection if

Range (f) = co-domain(f)

Given:-

(i) $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(ii) $f(x) = \sin x$

(ii) f is bijection

For $f(x) = \sin x$

Codomain = range

Set $A = [-1, 1]$

21. Question

Let $f: \mathbb{R} \rightarrow \mathbb{R}^+$ be defined by $f(x) = a^x$, $a > 0$ and $a \neq 1$. Write $f^{-1}(x)$.

Answer

Given:-

(i) $f: \mathbb{R} \rightarrow \mathbb{R}^+$

(ii) $f(x) = a^x$, $a > 0$ and $a \neq 1$

Let

$f(y) = x$

$a^y = x$

$\Rightarrow y = \log_a x$

$\Rightarrow f^{-1}(x) = \log_a x$

22. Question

Let $f: \mathbb{R} - \{-1\} \rightarrow \mathbb{R} - \{1\}$ be given by $f(x) = \frac{x}{x+1}$. Write $f^{-1}(x)$.

Answer

Given:-

(i) $f: \mathbb{R} - \{-1\} \rightarrow \mathbb{R} - \{1\}$

(ii) $f(x) = \frac{x}{x+1}$

$F(y) = x$

$\Rightarrow \frac{y}{y+1} = x$

$\Rightarrow Y = xy + x$

$\Rightarrow y = \frac{x}{1-x}$

$\Rightarrow f^{-1} = \frac{x}{1-x}$

23. Question

Let $f : \mathbb{R} - \left\{-\frac{3}{5}\right\} \rightarrow \mathbb{R}$ be a function defined as $f(x) = \frac{2x}{5x+3}$.

Answer

Formula:-

(i) A function $f : X \rightarrow Y$ is defined to be invertible, if there exists a function $g : Y \rightarrow X$

such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}

Given:-

$$(i) f : \mathbb{R} - \left\{-\frac{3}{5}\right\} \rightarrow \mathbb{R}$$

$$(ii) f(x) = \frac{2x}{5x+3}$$

$$F(y) = x$$

$$\Rightarrow \frac{2y}{5x+3} = x$$

$$\Rightarrow 2y - 3x - 5xy = 0$$

$$\Rightarrow y = \frac{3x}{2-5x}$$

$$\Rightarrow f^{-1}(x) = \frac{3x}{2-5x}$$

24. Question

Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $g : \mathbb{R} \rightarrow \mathbb{R}$ be two functions defined by $f(x) = x^2 + x + 1$ and $g(x) = 1 - x^2$. Write $f \circ g(-2)$.

Answer

Formula :- (I) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions.

Then, the composition of f and g , denoted by $g \circ f$, is defined as the function $g \circ f : A \rightarrow C$

given by $g \circ f(x) = g(f(x))$

Given:-

$$(i) f : \mathbb{R} \rightarrow \mathbb{R}$$

$$(ii) g : \mathbb{R} \rightarrow \mathbb{R}$$

$$(iii) f(x) = x^2 + x + 1$$

$$(iv) g(x) = 1 - x^2$$

$$Fog(-2) = f(g(-2))$$

$$\Rightarrow Fog(-2) = f(1 - (-2)^2)$$

$$\Rightarrow Fog(-2) = f(-3)$$

$$\Rightarrow Fog(-2) = (-3)^2 - 3 + 1 = 7$$

25. Question

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = \frac{2x-3}{4}$. Write $f \circ f^{-1}(1)$.

Answer

Formula:-

(i) A function $f : X \rightarrow Y$ is defined to be invertible, if there exists a function $g : Y \rightarrow X$

such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}

(II) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions.

Then, the composition of f and g , denoted by $g \circ f$, is defined as the function $g \circ f : A \rightarrow C$

given by $g \circ f(x) = g(f(x))$

Given:-

(i) $f : \mathbb{R} \rightarrow \mathbb{R}$

$$(ii) f(x) = \frac{2x - 3}{4}$$

$$F(y) = x$$

$$\Rightarrow \frac{2y - 3}{4} = x$$

$$\Rightarrow 2y - 3 - 4x = 0$$

$$\Rightarrow y = \frac{4x + 3}{2}$$

Now

$$\Rightarrow f \circ f^{-1}(1) = f\left(\frac{7}{2}\right)$$

$$\Rightarrow f \circ f^{-1}(1) = \frac{7 - 3}{4} = 1$$

26. Question

Let f be an invertible real function. Write $(f^{-1} \text{ of } (1)) + (f^{-1} \text{ of } (2)) + \dots + (f^{-1} \text{ of } (100))$.

Answer

Formula:-

(i) A function $f : X \rightarrow Y$ is defined to be invertible, if there exists a function $g : Y \rightarrow X$

such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}

(II) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions.

Then, the composition of f and g , denoted by $g \circ f$, is defined as the function $g \circ f : A \rightarrow C$

given by $g \circ f(x) = g(f(x))$

Given:-

(i) f be an invertible real function

$$(f^{-1} \text{ of } (1)) + (f^{-1} \text{ of } (2)) + \dots + (f^{-1} \text{ of } (100))$$

$$= 1 + 2 + 3 + \dots + 100$$

$$= \frac{100(100 + 1)}{2} = 5050$$

27. Question

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b\}$ be two sets. Write total number of onto functions from A to B .

Answer

Formula:-

(I) A function $f: A \rightarrow B$ is onto function or surjection if

Range (f) = co-domain(f)

(II) if A and B are two non-empty finite sets containing m and n

(i) Number of function from A to B = n^m

(ii) Number of one-one function from A to B = $\begin{cases} C_m^n \cdot m!, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(iii) Number of one-one and onto function from A to B = $\begin{cases} n!, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$

(iv) Number of onto function from A to B = $\sum_{r=1}^n (-1)^{n-r} C_r^n r^m$, if $m \geq n$

Given:-

(i) A = {1, 2, 3, 4} = 4

(ii) B = {a, b} = 2

Using formula (iv)

Number of onto function from A to B = $\sum_{r=1}^n (-1)^{n-r} C_r^n r^m$, if $m \geq n$

Where $m=4, n=2$

$$\sum_{r=1}^n (-1)^{n-r} C_r^n r^m = (-1)^{2-1} C_1^2 (1)^4 + (-1)^{2-2} C_2^2 (2)^4$$

$$= -2 + 16 = 14$$

28. Question

Write the domain of the real function $f(x) = \sqrt{x - [x]}$.

Answer

$f(x) = \sqrt{x - [x]}$ where x is for all real number

Then,

domain = R

129. Question

Write the domain of the real function $f(x) = \sqrt{[x] - x}$.

Answer

$f(x) = \sqrt{[x] - x}$ where x is not for real number

Domain = \emptyset

30. Question

Write the domain of the real function $f(x) = \frac{1}{\sqrt{|x| - x}}$

Answer

$$f(x) = \frac{1}{\sqrt{|x| - x}}$$

When $x < 0$ negative

$$\frac{1}{\sqrt{|x|} - x} = \frac{1}{\sqrt{-x} - x}$$
$$= \frac{1}{\sqrt{-2x}}$$

When $x > 0$

$$\frac{1}{\sqrt{|x|} - x} = \frac{1}{\sqrt{x} - x} = \infty$$

Domain = $(-\infty, 0)$

31. Question

Write whether $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x + \sqrt{x^2}$ is one-one, many-one, onto or into.

Answer

(I) A function $f: A \rightarrow B$ is one-one function or an injection if

$$f(x) = f(y) \Rightarrow x = y \text{ for all } x, y \in A$$

$$\text{or } f(x) \neq f(y) \Rightarrow x \neq y \text{ for all } x, y \in A$$

(II) A function $f: A \rightarrow B$ is onto function or surjection if

$$\text{Range}(f) = \text{co-domain}(f)$$

(III) A function $f: A \rightarrow B$ is not onto function, then

$f: A \rightarrow A$ is always an onto function

Given:-

$$(i) f : \mathbb{R} \rightarrow \mathbb{R}$$

$$(ii) f(x) = x + \sqrt{x^2}$$

$$f(x) = x + \sqrt{x^2}$$

$$= x \pm x$$

$$= 0, 2x$$

Now putting $x=0$

$$F(0) = 0 + \sqrt{0^2} = 0$$

Again putting $x=-1$

$$F(-1) = -1 + \sqrt{-1^2} = 0$$

Hence f is many one

32. Question

If $f(x) = x + 7$ and $g(x) = x - 7$, $x \in \mathbb{R}$, write $f \circ g(7)$.

Answer

Formula:-

(i) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions.

Then, the composition of f and g , denoted by $g \circ f$, is defined as the function $g \circ f : A \rightarrow C$

given by $g \circ f(x) = g(f(x))$

Given:-

$$(i) f(x) = x + 7$$

$$(ii) g(x) = x - 7, x \in \mathbb{R}$$

$$Fog(7) = f(g(7))$$

$$\Rightarrow Fog(7) = f(7-7)$$

$$\Rightarrow Fog(7) = f(0)$$

$$\Rightarrow Fog(7) = 0 + 7$$

$$\Rightarrow Fog(7) = 7$$

33. Question

What is the range of the function $f(x) = \frac{|x-1|}{x-1}$?

Answer

$$f(x) = \frac{|x-1|}{x-1}$$

$$= \pm 1$$

Range of $f = \{-1, 1\}$

34. Question

If $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = (3 - x^3)^{1/3}$, then find $f \circ f(x)$.

Answer

Formula:-

(i) Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions.

Then, the composition of f and g , denoted by $g \circ f$, is defined as the function $g \circ f: A \rightarrow C$

given by $g \circ f(x) = g(f(x))$

Given:-

$$(i) f: \mathbb{R} \rightarrow \mathbb{R}$$

$$(ii) f(x) = (3 - x^3)^{1/3}$$

$$Fof(x) = f(f(x))$$

$$\Rightarrow fof(x) = f((3 - x^3)^{1/3})$$

$$\Rightarrow fof(x) = (3 - (3 - x^3))^{1/3}$$

$$\Rightarrow fof(x) = (x^3)^{1/3} = x$$

35. Question

If $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = 3x + 2$, find $f(f(x))$.

Answer

Given:-

$$(i) f: \mathbb{R} \rightarrow \mathbb{R}$$

$$F(f(x)) = f(3x+2)$$

$$\Rightarrow F(f(x)) = 3(3x+2)+2$$

$$\Rightarrow F(f(x)) = 9x+8$$

36. Question

Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B . State whether f is one-one or not.

Answer

Given:-

$$(i) A = \{1, 2, 3\}$$

$$(ii) B = \{4, 5, 6, 7\}$$

$$(iii) f = \{(1, 4), (2, 5), (3, 6)\}$$

each element has a unique image

hence, f is one-one

37. Question

If $f : \{5, 6\} \rightarrow \{2, 3\}$ and $g : \{2, 3\} \rightarrow \{5, 6\}$ are given by $f = \{(5, 2), (6, 3)\}$ and $g = \{(2, 5), (3, 6)\}$, find $f \circ g$.

Answer

Formula:-

(i) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions.

Then, the composition of f and g , denoted by $g \circ f$, is defined as the function $g \circ f : A \rightarrow C$

given by $g \circ f(x) = g(f(x))$

Given:-

$$(i) f : \{5, 6\} \rightarrow \{2, 3\}$$

$$(ii) g : \{2, 3\} \rightarrow \{5, 6\}$$

$$(iv) f = \{(5, 2), (6, 3)\}$$

$$(v) g = \{(2, 5), (3, 6)\}$$

for $f \circ g(2) = f(g(2))$

$$\Rightarrow f \circ g(2) = f(5)$$

$$\Rightarrow f \circ g(2) = 2$$

38. Question

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x) = 4x - 3$ for all $x \in \mathbb{R}$. Then write f^{-1} .

Answer

Formula:-

(i) A function $f : X \rightarrow Y$ is defined to be invertible, if there exists a function $g : Y \rightarrow X$

such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}

Given:-

$$(i) f : \mathbb{R} \rightarrow \mathbb{R}$$

$$(ii) f(x) = 4x - 3 \text{ for all } x \in \mathbb{R}.$$

$$f(x)=y$$

$$\Rightarrow 4x-3=y$$

$$\Rightarrow x = \frac{y+3}{4}$$

$$f^{-1}(y) = x = \frac{y+3}{4}$$

$$f^{-1}(x) = \frac{x+3}{4}$$

39. Question

Which one the following relations on $A = \{1, 2, 3\}$ is a function?

$f = \{(1, 3), (2, 3), (3, 2)\}$, $g = \{(1, 2), (1, 3), (3, 1)\}$.

Answer

Given:-

(i) $A = \{1, 2, 3\}$

(ii) $f = \{(1, 3), (2, 3), (3, 2)\}$

(iii) $g = \{(1, 2), (1, 3), (3, 1)\}$.

In case of set A and f

Every element in A has a unique image in f

So, f is a function

In case of set A and g

Only one element has image in g

So, g is not a function

40. Question

Write the domain of the real function f defined by $f(x) = \sqrt{25 - x^2}$.

Answer

$$f(x) = \sqrt{25 - x^2}$$

$$\Rightarrow 25 - x^2 \geq 0$$

$$\Rightarrow -(x+5)(x-5) \geq 0$$

$$\Rightarrow (x+5)(x-5) \leq 0$$

$$\Rightarrow x \leq -5 \text{ or } 5$$

$$\text{Domain} = [-5, 5]$$

41. Question

Let $A = \{a, b, c, d\}$ and $f : A \rightarrow A$ be given by $f = \{(a, b), (b, d), (c, a), (d, c)\}$, write f^{-1} .

Answer

Formula:-

(i) A function $f : X \rightarrow Y$ is defined to be invertible, if there exists a function $g : Y \rightarrow X$

such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}

(ii) A function $f: A \rightarrow B$ is onto function or surjection if

Range (f) = co-domain(f)

Given:-

(i) $A = \{a, b, c, d\}$

(ii) $f: A \rightarrow A$

(iii) $f = \{(a, b), (b, d), (c, a), (d, c)\}$

f is one-one since each element of A is assigned to distinct element of the set A . Also, f is onto since $f(A) = A$.

$f^{-1} = \{(b, a), (d, b), (a, c), (c, d)\}$.

42. Question

Let $f, g: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ for all $x \in \mathbb{R}$, respectively. Then, find $g \circ f$.

Answer

Formula:-

(i) Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions.

Then, the composition of f and g , denoted by $g \circ f$, is defined as the function $g \circ f: A \rightarrow C$

given by $g \circ f(x) = g(f(x))$

Given:-

(i) $f, g: \mathbb{R} \rightarrow \mathbb{R}$

(ii) $f(x) = 2x + 1$

(ii) $g(x) = x^2 - 2$ for all $x \in \mathbb{R}$

$g \circ f(x) = g(f(x))$

$\Rightarrow g \circ f(x) = g(2x + 1)$

$\Rightarrow g \circ f(x) = (2x + 1)^2 - 2$

$\Rightarrow g \circ f(x) = 4x^2 + 4x - 1$

43. Question

If the mapping $f: \{1, 3, 4\} \rightarrow \{1, 2, 5\}$ and $g: \{1, 2, 5\} \rightarrow \{1, 3\}$, given by

$f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$, write $f \circ g$.

Answer

Formula:-

(i) Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions.

Then, the composition of f and g , denoted by $g \circ f$, is defined as the function $g \circ f: A \rightarrow C$

given by $g \circ f(x) = g(f(x))$

Given:-

(i) $f: \{1, 3, 4\} \rightarrow \{1, 2, 5\}$

(ii) $g: \{1, 2, 5\} \rightarrow \{1, 3\}$

(iii) $f = \{(1, 2), (3, 5), (4, 1)\}$

(iv) $g = \{(2, 3), (5, 1), (1, 3)\}$

$$f \circ g(1) = f(g(1)) = f(3) = 5$$

$$f \circ g(2) = f(g(2)) = f(3) = 5$$

$$f \circ g(5) = f(g(5)) = f(1) = 2$$

$$\Rightarrow f \circ g = \{(1, 5), (2, 5), (5, 2)\}$$

44. Question

If a function $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ is described by $g(x) = \alpha x + \beta$, find the values of α and β .

Answer

Given:-

$$(i) g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$$

$$(ii) g(x) = \alpha x + \beta$$

For $x=1$ and $\alpha x + \beta$

$$g(1) = \alpha(1) + \beta = 1$$

$$\Rightarrow \alpha + \beta = 1$$

For $x=2$

$$g(2) = \alpha(2) + \beta = 3$$

$$\Rightarrow 2\alpha + \beta = 3$$

Similarly with $g(3)$ and $g(4)$

Using above value

$$\alpha = 2$$

$$\beta = 1$$

45. Question

If $f(x) = 4 - (x - 7)^3$, write $f^{-1}(x)$.

Answer

Formula:-

(i) A function $f : X \rightarrow Y$ is defined to be invertible, if there exists a function $g : Y \rightarrow X$

such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}

Given:-

$$(i) f(x) = 4 - (x - 7)^3$$

Let $f(x) = y$

$$y = 4 - (x - 7)^3$$

$$x = 7 + \sqrt[3]{4 - y}$$

$$f^{-1}(x) = 7 + \sqrt[3]{4 - x}$$

MCQ

1. Question

Mark the correct alternative in each of the following:

Let $A = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$, $B = \{x \in \mathbb{R} : x \geq 0\}$ and let $S = \{(x, y) \in A \times B : x^2 + y^2 = 1\}$ and $S_0 =$

$\{(x, y) \in A \times C : x^2 + y^2 = 1\}$ Then

- A. S defines a function from A to B
- B. S_0 defines a function from A to C
- C. S_0 defines a function from A to B
- D. S defines a function from A to C

Answer

Given that

$$A = \{x \in \mathbb{R} : -1 \leq x \leq 1\} = B$$

$$C = \{x \in \mathbb{R} : x \geq 0\}$$

$$S = \{(x, y) \in A \times B : x^2 + y^2 = 1\}$$

$$S_0 = \{(x, y) \in A \times C : x^2 + y^2 = 1\}$$

$$x^2 + y^2 = 1$$

$$\Rightarrow y^2 = 1 - x^2$$

$$\Rightarrow y = \sqrt{1 - x^2}$$

$$\therefore y \in B$$

Hence, S defines a function from A to B.

2. Question

Mark the correct alternative in each of the following:

$f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x + \sqrt{x^2}$ is

- A. injective B. surjective
- C. bijective D. none of these

Answer

Given function is $f: \mathbb{R} \rightarrow \mathbb{R}$ given

$$f(x) = x + \sqrt{x^2}$$

For this function if we take $x = 2$,

$$f(x) = 2 + \sqrt{4}$$

$$\Rightarrow f(x) = 4$$

For this function if we take $x = -2$,

$$f(x) = -2 + \sqrt{4}$$

$$\Rightarrow f(x) = 0$$

So, in general for every negative x , $f(x)$ will be always 0. There is no $x \in \mathbb{R}$ for which $f(x) \in (-\infty, 0)$.

Hence, it is neither injective nor surjective and so it is not bijective either.

3. Question

Mark the correct alternative in each of the following:

If $f: A \rightarrow B$ given by $3^{f(x)} + 2^{-x} = 4$ is a bijection, then

A. $A = \{x \in \mathbb{R} : -1 < x < \infty\}$, $B = \{x \in \mathbb{R} : 2 < x < 4\}$

B. $A = \{x \in \mathbb{R} : -3 < x < \infty\}$, $B = \{x \in \mathbb{R} : 0 < x < 4\}$

C. $A = \{x \in \mathbb{R} : -2 < x < \infty\}$, $B = \{x \in \mathbb{R} : 0 < x < 4\}$

D. none of these

Answer

Given that $f: A \rightarrow B$ given by $3^{f(x)} + 2^{-x} = 4$ is a bijection.

$$3^{f(x)} + 2^{-x} = 4$$

$$\Rightarrow 3^{f(x)} = 4 - 2^{-x}$$

$$\Rightarrow 4 - 2^{-x} \geq 0$$

$$\Rightarrow 4 \geq 2^{-x}$$

$$\Rightarrow 2 \geq -x$$

$$\Rightarrow x \geq -2$$

$$\text{So, } x \in (-2, \infty)$$

$$\text{But, for } x=0, f(x) = 1.$$

Hence, the correct option is none of these.

4. Question

Mark the correct alternative in each of the following:

The function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2^x + 2^{|x|}$ is

A. one-one and onto

B. many-one and onto

C. one-one and into

D. many-one and into

Answer

Given that $f: \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = 2^x + 2^{|x|}$

Here, for each value of x we will get different value of $f(x)$.

So, it is one-one.

Also, $f(x)$ is always positive for $x \in \mathbb{R}$.

There is no $x \in \mathbb{R}$ for which $f(x) \in (-\infty, 0)$.

So, it is into.

Hence, the given function is one-one and into.

5. Question

Mark the correct alternative in each of the following:

Let the function $f: \mathbb{R} - \{-b\} \rightarrow \mathbb{R} - \{1\}$ be defined by $f(x) = \frac{x+a}{x+b}$, $a \neq b$, then

A. f is one-one but not onto

B. f is onto but not one-one

C. f is both one-one and onto

D. none of these

Answer

Given that $f: \mathbb{R} - \{-b\} \rightarrow \mathbb{R} - \{1\}$ where

$$f(x) = \frac{x+a}{x+b}, a \neq b.$$

Here, $f(x) = f(y)$ only when $x=y$.

Hence, it is one-one.

Now, $f(x) = y$

$$\Rightarrow \frac{x+a}{x+b} = y$$

$$\Rightarrow x+a = y(x+b)$$

$$\Rightarrow x - yx = yb - a$$

$$\Rightarrow x = \frac{yb-a}{1-y}, y \neq 1$$

So, $x \in \mathbb{R} - \{1\}$

Hence, it is onto.

6. Question

Mark the correct alternative in each of the following:

The function $f: A \rightarrow B$ defined by $f(x) = -x^2 + 6x - 8$ is a bijection, if

A. $A = (-\infty, 5]$ and $B = (-\infty, 1]$

B. $A = [-3, \infty]$ and $B = (-\infty, 1]$

C. $A = (-\infty, 3]$ and $B = [1, \infty)$

D. $A = [3, \infty)$ and $B = [1, \infty)$

Answer

Given that $f: A \rightarrow B$ defined by $f(x) = -x^2 + 6x - 8$ is a bijection.

$$f(x) = -x^2 + 6x - 8$$

$$\Rightarrow f(x) = -(x^2 - 6x + 8)$$

$$\Rightarrow f(x) = -(x^2 - 6x + 8 + 1 - 1)$$

$$\Rightarrow f(x) = -(x^2 - 6x + 9 - 1)$$

$$\Rightarrow f(x) = -[(x-3)^2 - 1]$$

Hence, $x \in (-\infty, 5]$ and $f(x) \in (-\infty, 1]$

7. Question

Mark the correct alternative in each of the following:

Let $A = \{x \in \mathbb{R} : -1 \leq x \leq 1\} = B$. Then, the mapping $f: A \rightarrow B$ given by $f(x) = x|x|$ is

A. injective but not surjective

B. surjective but not injective

C. bijective

D. none of these

Answer

Given that $A = \{x \in \mathbb{R} : -1 \leq x \leq 1\} = B$. Then, the mapping $f: A \rightarrow B$ given by $f(x) = x|x|$.

For $x < 0$, $f(x) < 0$

$$\Rightarrow y = -x^2$$

$$\Rightarrow x = \sqrt{-y}, \text{ which is not possible for } x > 0.$$

Hence, f is one-one and onto.

\therefore the given function is bijective.

8. Question

Mark the correct alternative in each of the following:

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = [x]^2 + [x + 1] - 3$, where $[x]$ denotes the greatest integer less than or equal to x . Then, $f(x)$ is

- A. many-one and onto
- B. many-one and into
- C. one-one and into
- D. one-one and onto

Answer

Given that $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = [x]^2 + [x + 1] - 3$

As $[x]$ is the greatest integer so for different values of x , we will get same value of $f(x)$.

$[x]^2 + [x + 1]$ will always be an integer.

So, f is many-one.

Similarly, in this function co domain is mapped with at most one element of domain because for every $x \in \mathbb{R}$, $f(x) \in \mathbb{Z}$.

So, f is into.

9. Question

Mark the correct alternative in each of the following:

Let M be the set of all 2×2 matrices with entries from the set \mathbb{R} of real numbers. Then the function $f: M \rightarrow \mathbb{R}$ defined by $f(A) = |A|$ for every $A \in M$, is

- A. one-one and onto
- B. neither one-one nor onto
- C. one-one not one-one
- D. onto but not one-one

Answer

Given that M is the set of all 2×2 matrices with entries from the set \mathbb{R} of real numbers. Then the function $f: M \rightarrow \mathbb{R}$ defined by $f(A) = |A|$ for every $A \in M$.

If $f(a) = f(b)$

$$\Rightarrow |a| = |b|$$

But this does not mean that $a=b$.

So, f is not one-one.

As $a \neq b$ but $|a|=|b|$

So, f is onto.

10. Question

Mark the correct alternative in each of the following:

The function $f : [0, \infty) \rightarrow \mathbb{R}$ given by $f(x) = \frac{x}{x+1}$ is

- A. one-one and onto
- B. one-one but not onto
- C. onto but not one-one
- D. neither one-one nor onto

Answer

Given that $f: [0, \infty) \rightarrow \mathbb{R}$ where $f(x) = \frac{x}{x+1}$

Let $f(x) = f(y)$

$$\Rightarrow \frac{x}{x+1} = \frac{y}{y+1}$$

$$\Rightarrow xy + x = xy + y$$

$$\Rightarrow x = y$$

So, f is one-one.

Now, $y = f(x)$

$$\Rightarrow y = \frac{x}{x+1}$$

$$\Rightarrow xy + y = x$$

$$\Rightarrow y = x - xy$$

$$\Rightarrow \frac{y}{1-y} = x$$

Here, $y \neq 1$ i.e. $y \in \mathbb{R}$.

So, f is not onto.

11. Question

Mark the correct alternative in each of the following:

The range of the function $f(x) = {}^{7-x}P_{x-3}$ is

- A. $\{1, 2, 3, 4, 5\}$
- B. $\{1, 2, 3, 4, 5, 6\}$
- C. $\{1, 2, 3, 4\}$
- D. $\{1, 2, 3\}$

Answer

Given that $f(x) = {}^{7-x}P_{x-3}$

Here, $7-x \geq x-3$

$$\Rightarrow 10 \geq 2x$$

$$\Rightarrow 5 \geq x$$

So, domain = $\{3, 4, 5\}$

$$\text{Range} = \{{}^4P_0, {}^3P_1, {}^2P_2\} = \{1, 3, 2\}$$

12. Question

Mark the correct alternative in each of the following:

A function f from the set on natural numbers to integers defined by

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}.$$

- A. neither one-one nor onto
- B. one-one but not onto
- C. onto but not one-one
- D. one-one and onto both

Answer

Given that a function f from the set on natural numbers to integers where

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$$

For n is odd

Let $f(n) = f(m)$

$$\Rightarrow \frac{n-1}{2} = \frac{m-1}{2}$$

$$\Rightarrow n = m$$

For n is even

Let $f(n) = f(m)$

$$\Rightarrow \frac{-n}{2} = \frac{-m}{2}$$

$$\Rightarrow n = m$$

So, f is one-one.

Also, each element of y is associated with at least one element of x , so f is onto.

Hence, f is one-one and onto.

13. Question

Mark the correct alternative in each of the following:

Let f be an injective map with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$ such that exactly one of the following statements is correct and the remaining are false.

$$f(x) = 1, f(y) \neq 1, f(z) \neq 2.$$

The value of $f^{-1}(1)$ is

- A. x
- B. y
- C. z

D. none of these

Answer

Given that f is an injective map with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$.

Case-1

Let us assume that $f(x) = 1$ is true and $f(y) \neq 1, f(z) \neq 2$ is false.

Then $f(x) = 1, f(y) = 1$ and $f(z) = 2$.

This violates the injectivity of f because it is one-one.

Case-2

Let us assume that $f(y) \neq 1$ is true and $f(x) = 1, f(z) \neq 2$ is false.

Then $f(x) \neq 1, f(y) \neq 1$ and $f(z) = 2$.

This means there is no pre image of 1 which contradicts the fact that the range of f is $\{1, 2, 3\}$.

Case-3

Let us assume that $f(z) \neq 2$ is true and $f(x) = 1, f(y) \neq 1$ is false.

Then $f(z) \neq 2, f(y) = 1$ and $f(x) \neq 1$.

$$\Rightarrow f^{-1}(1) = y$$

14. Question

Mark the correct alternative in each of the following:

Which of the following functions from \mathbb{Z} to itself are bijections?

A. $f(x) = x^3$

B. $f(x) = x + 2$

C. $f(x) = 2x + 1$

D. $f(x) = x^2 + x$

Answer

a. $f(x) = x^3$

\Rightarrow For no value of $x \in \mathbb{Z}$, $f(x) = 2$.

Hence, it is not bijection.

b. $f(x) = x + 2$

If $f(x) = f(y)$

$$\Rightarrow x + 2 = y + 2$$

$$\Rightarrow x = y$$

So, f is one-one.

Also, $y = x + 2$

$$\Rightarrow x = y - 2 \in \mathbb{Z}$$

So, f is onto.

Hence, this function is bijection.

c. $f(x) = 2x + 1$

If $f(x) = f(y)$

$$\Rightarrow 2x + 1 = 2y + 1$$

$$\Rightarrow x = y$$

So, f is one-one.

$$\text{Also, } y = 2x + 1$$

$$\Rightarrow 2x = y - 1$$

$$\Rightarrow x = \frac{y-1}{2}$$

So, f is into because x can never be odd for any value of y .

$$\text{d. } f(x) = x^2 + x$$

For this function if we take $x = 2$,

$$f(x) = 4 + 2$$

$$\Rightarrow f(x) = 6$$

For this function if we take $x = -2$,

$$f(x) = 4 - 2$$

$$\Rightarrow f(x) = 2$$

So, in general for every negative x , $f(x)$ will be always 0. There is no $x \in \mathbb{R}$ for which $f(x) \in (-\infty, 0)$.

It is not bijection.

15. Question

Mark the correct alternative in each of the following:

Which of the following functions from $A = \{x : -1 \leq x \leq 1\}$ to itself are bijections?

$$\text{A. } f(x) = \frac{x}{2}$$

$$\text{B. } g(x) = \sin\left(\frac{\pi x}{2}\right)$$

$$\text{C. } h(x) = |x|$$

$$\text{D. } k(x) = x^2$$

Answer

Given that $A = \{x : -1 \leq x \leq 1\}$

$$\text{a. } f(x) = \frac{x}{2}$$

It is one-one but not onto.

$$\text{b. } g(x) = \sin\left(\frac{\pi x}{2}\right)$$

It is bijective as it is one-one and onto with range $[-1, 1]$.

$$\text{c. } h(x) = |x|$$

It is not one-one because $h(-1)=1$ and $h(1)=1$.

$$\text{d. } k(x) = x^2$$

It is not one-one because $k(-1)=1$ and $k(1)=1$.

16. Question

Mark the correct alternative in each of the following:

Let $A = \{x : -1 \leq x \leq 1\}$ and $f : A \rightarrow A$ such that $f(x) = x|x|$, then f is

- A. a bijection
- B. injective but not surjective
- C. surjective but not injective
- D. neither injective nor surjective

Answer

Given that $A = \{x : -1 \leq x \leq 1\}$ and $f : A \rightarrow A$ such that $f(x) = x|x|$.

For $x < 0$, $f(x) < 0$

$$\Rightarrow y = -x^2$$

$\Rightarrow x = \sqrt{-y}$, which is not possible for $x > 0$.

Hence, f is one-one and onto.

\therefore the given function is bijective.

17. Question

Mark the correct alternative in each of the following:

If the function $f : \mathbb{R} \rightarrow A$ given by $f(x) = \frac{x^2}{x^2 + 1}$ is a surjection, then $A =$

- A. \mathbb{R}
- B. $[0, 1]$
- C. $(0, 1]$
- D. $[0, 1)$

Answer

Given that $f : \mathbb{R} \rightarrow A$ such that $f(x) = \frac{x^2}{x^2 + 1}$ is a surjection.

$$f(x) = y$$

$$\Rightarrow y = \frac{x^2}{x^2 + 1}$$

$$\Rightarrow y(x^2 + 1) = x^2$$

$$\Rightarrow yx^2 + y = x^2$$

$$\Rightarrow yx^2 - x^2 = -y$$

$$\Rightarrow x^2 = \frac{y}{1 - y}$$

$$\Rightarrow x = \sqrt{\frac{y}{1 - y}}$$

$$\text{Here, } \frac{y}{1 - y} \geq 0$$

$$\text{So, } y \in [0, 1)$$

18. Question

Mark the correct alternative in each of the following:

If a function $f : [2, \infty) \rightarrow B$ defined by $f(x) = x^2 - 4x + 5$ is a bijection, then $B =$

- A. \mathbb{R}
- B. $[1, \infty)$
- C. $[4, \infty)$
- D. $[5, \infty)$

Answer

Given that a function $f : [2, \infty) \rightarrow B$ defined by $f(x) = x^2 - 4x + 5$ is a bijection.

Put $x = 2$ in $f(x)$,

$$f(x) = 2^2 - 4 \times 2 + 5$$

$$\Rightarrow f(x=2) = 4 - 8 + 5$$

$$\Rightarrow f(x=2) = 1$$

So, $B \in [1, \infty)$

19. Question

Mark the correct alternative in each of the following:

The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = (x - 1)(x - 2)(x - 3)$ is

- A. one-one but not onto
- B. onto but not one-one
- C. both one and onto
- D. neither one-one nor onto

Answer

Given that function $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = (x - 1)(x - 2)(x - 3)$

If $f(x) = f(y)$

Then

$$(x - 1)(x - 2)(x - 3) = (y - 1)(y - 2)(y - 3)$$

$$\Rightarrow f(1) = f(2) = f(3) = 0$$

So, f is not one-one.

$$y = f(x)$$

$\because x \in \mathbb{R}$ also $y \in \mathbb{R}$ so f is onto.

20. Question

Mark the correct alternative in each of the following:

The function $f : [-1/2, 1/2] \rightarrow [\pi/2, \pi/2]$ defined by $f(x) = \sin^{-1}(3x - 4x^3)$ is

- A. bijection
- B. injection but not a surjection
- C. surjection but not an injection
- D. neither an injection nor a surjection

Answer

Given that $f : [-1/2, 1/2] \rightarrow [\pi/2, \pi/2]$ where $f(x) = \sin^{-1}(3x - 4x^3)$

Put $x = \sin\theta$ in $f(x) = \sin^{-1}(3x - 4x^3)$

$$\Rightarrow f(x=\sin\theta) = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$$

$$\Rightarrow f(x) = \sin^{-1}(\sin 3\theta)$$

$$\Rightarrow f(x) = 3\theta$$

$$\Rightarrow f(x) = 3 \sin^{-1}x$$

$$\text{If } f(x) = f(y)$$

Then

$$3 \sin^{-1}x = 3 \sin^{-1}y$$

$$\Rightarrow x = y$$

So, f is one-one.

$$y = 3 \sin^{-1}x$$

$$\Rightarrow x = \sin \frac{y}{3}$$

$\therefore x \in \mathbb{R}$ also $y \in \mathbb{R}$ so f is onto.

Hence, f is bijection.

21. Question

Mark the correct alternative in each of the following:

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$. Then,

- A. f is a bijection
- B. f is an injection only
- C. f is surjection on only
- D. f is neither an injection nor a surjection

Answer

Given that $f: \mathbb{R} \rightarrow \mathbb{R}$ is a function defined as

$$f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$$

Here, $e^{|x|}$ is always positive whether x is negative or positive. So, we will get same values of $f(x)$ for different values of x .

Hence, it is not one-one and onto.

$\therefore f$ is neither an injection nor a surjection

22. Question

Mark the correct alternative in each of the following:

Let $f: \mathbb{R} - \{n\} \rightarrow \mathbb{R}$ be a function defined by $f(x) = \frac{x-m}{x-n}$, where $m \neq n$. Then,

- A. f is one-one onto
- B. f is one-one into
- C. f is many one onto

D. f is many one into

Answer

Given that $f : \mathbb{R} - \{n\} \rightarrow \mathbb{R}$ where

$$f(x) = \frac{x-m}{x-n}, \text{ such that } m \neq n$$

Let $f(x) = f(y)$

$$\Rightarrow \frac{x-m}{x-n} = \frac{y-m}{y-n}$$

$$\Rightarrow (x-m)(y-n) = (x-n)(y-m)$$

$$\Rightarrow xy - xn - my + mn = xy - xm - ny + mn$$

$$\Rightarrow x = y$$

So, f is one-one.

$$f(x) = \frac{x-m}{x-n}$$

$$\Rightarrow y = \frac{x-m}{x-n}$$

$$\Rightarrow y(x-n) = (x-m)$$

$$\Rightarrow xy - ny = x - m$$

$$\Rightarrow x(y-1) = ny - m$$

$$\Rightarrow x = \frac{ny-m}{y-1}, y \neq 1$$

For $y = 1$, no x is defined.

So, f is into.

23. Question

Mark the correct alternative in each of the following:

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = \frac{x^2-8}{x^2+2}$. Then, f is

A. one-one but not onto

B. one-one and onto

C. onto but not one-one

D. neither one-one nor onto

Answer

Given that $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function where

$$f(x) = \frac{x^2-8}{x^2+2}$$

Here, we can see that for negative as well as positive x we will get same value.

So, it is not one-one.

$$y = f(x)$$

$$\Rightarrow y = \frac{x^2-8}{x^2+2}$$

$$\Rightarrow y(x^2 + 2) = (x^2 - 8)$$

$$\Rightarrow x^2(y-1) = -2y - 8$$

$$\Rightarrow x = \sqrt{\frac{2y + 8}{1 - y}}$$

For $y = 1$, no x is defined.

So, f is not onto.

24. Question

Mark the correct alternative in each of the following:

$$f : \mathbb{R} \rightarrow \mathbb{R} \text{ is defined by } f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}} \text{ is}$$

- A. one-one but not onto
- B. one-one and onto
- C. onto but not one-one
- D. neither one-one nor onto

Answer

Given that $f : \mathbb{R} \rightarrow \mathbb{R}$ where

$$f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$$

Here, we can see that for negative as well as positive x we will get same value.

So, it is not one-one.

$$f(x) = y$$

By definition of onto, each element of y is not mapped to at least one element of x .

So, it is not onto.

25. Question

Mark the correct alternative in each of the following:

The function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$ is

- A. injective but not surjective
- B. surjective but not injective
- C. injective as well as surjective
- D. neither injective nor surjective

Answer

Given that $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$

Let $f(x) = y(x)$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow x = \pm y$$

So, it is not one-one.

$$f(x) = y$$

$$\Rightarrow x^2 = y$$

$$\Rightarrow x = \pm\sqrt{y}$$

But co domain is R.

Hence, f is neither injective nor surjective.

26. Question

Mark the correct alternative in each of the following:

A function f from the set of natural, numbers to the set of integers defined by

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$$

- A. neither one-one nor onto
- B. one-one but not onto
- C. onto but not one-one
- D. one-one and onto both

Answer

Given that a function f from the set on natural numbers to integers where

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$$

For n is odd

$$\text{Let } f(n) = f(m)$$

$$\Rightarrow \frac{n-1}{2} = \frac{m-1}{2}$$

$$\Rightarrow n = m$$

For n is even

$$\text{Let } f(n) = f(m)$$

$$\Rightarrow \frac{-n}{2} = \frac{-m}{2}$$

$$\Rightarrow n = m$$

So, f is one-one.

Also, each element of y is associated with at least one element of x, so f is onto.

Hence, f is one-one and onto.

27. Question

Mark the correct alternative in each of the following:

Which of the following functions from $A = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$ to itself are bijections?

A. $f(x) = |x|$

B. $f(x) = \sin \frac{\pi x}{2}$

C. $f(x) = \sin \frac{\pi x}{4}$

D. none of these

Answer

Given that $A = \{x : -1 \leq x \leq 1\}$

a. $f(x) = |x|$

It is not one-one because $f(-1) = 1$ and $f(1) = 1$.

b. $f(x) = \sin\left(\frac{\pi x}{2}\right)$

It is bijective as it is one-one and onto with range $[-1, 1]$.

28. Question

Mark the correct alternative in each of the following:

Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be given by $f(x) = \begin{cases} \frac{x}{2}, & \text{if } x \text{ is even} \\ 0, & \text{if } x \text{ is odd} \end{cases}$. Then, f is

- A. onto but not one-one
- B. one-one but not onto
- C. one-one and onto
- D. neither one-one nor onto

Answer

Given function $f: \mathbb{Z} \rightarrow \mathbb{Z}$ defined as

$$f(x) = \begin{cases} \frac{x}{2}, & \text{when } x \text{ is even} \\ 0, & \text{when } x \text{ is odd} \end{cases}$$

For $x = 3$, $f(x) = 0$

For $x = 5$, $f(x) = 0$

But $3 \neq 5$

So, f is not one-one.

$Y = f(x)$

$\because x \in \mathbb{R} \Rightarrow y \in \mathbb{R}$

$\therefore \text{Domain} = \text{Range}$

Hence, f is not one-one but onto.

29. Question

Mark the correct alternative in each of the following:

The function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 6^x + 6^{|x|}$ is

- A. one-one and onto
- B. many one and onto

- C. one-one and into
- D. many one and into

Answer

Given that function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 6^x + 6^{|x|}$

Let $f(x) = f(y)$

$$\Rightarrow 6^x + 6^{|x|} = 6^y + 6^{|y|}$$

Only when $x = y$

So, f is one-one.

Now for $y=f(x)$

y can never be negative which means for no $x \in \mathbb{R}$ y is negative.

So, f is not onto but into.

30. Question

Mark the correct alternative in each of the following:

Let $f(x) = x^2$ and $g(x) = 2^x$. Then the solution set of the equation $\text{fog}(x) = \text{gof}(x)$ is

- A. \mathbb{R}
- B. $\{0\}$
- C. $\{0, 2\}$
- D. none of these

Answer

Given that $f(x) = x^2$ and $g(x) = 2^x$.

Also, $\text{fog}(x) = \text{gof}(x)$

$$\Rightarrow f(2^x) = g(x^2)$$

$$\Rightarrow 2^{2^x} = 2^{x^2}$$

$$\Rightarrow 2x = x^2$$

$$\Rightarrow x^2 - 2x = 0$$

$$\Rightarrow x(x-2) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 2$$

31. Question

Mark the correct alternative in each of the following:

If $f: \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = 3x - 5$, then $f^{-1}(x)$

A. is given by $\frac{1}{3x-5}$

B. is given by $\frac{x+5}{3}$

C. does not exist because f is not one-one

D. does not exist because f is not onto

Answer

Given that $f: \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = 3x - 5$

To find $f^{-1}(x)$:

$$y = f(x)$$

$$\Rightarrow y = 3x - 5$$

$$\Rightarrow y + 5 = 3x$$

$$\Rightarrow x = \frac{y + 5}{3}$$

$$\Rightarrow f^{-1}(x) = \frac{x + 5}{3}$$

32. Question

Mark the correct alternative in each of the following:

If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin \sqrt{x})^2$, then

A. $f(x) = \sin^2 x, g(x) = \sqrt{x}$

B. $f(x) = \sin x, g(x) = |x|$

C. $f(x) = x^2, g(x) = \sin \sqrt{x}$

D. f and g cannot be determined

Answer

Given that $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin \sqrt{x})^2$

a. For $f(x) = \sin^2 x, g(x) = \sqrt{x}$

$$f(g(x)) = f(\sqrt{x}) = (\sin \sqrt{x})^2$$

$$g(f(x)) = g(\sin^2 x) = \sqrt{\sin^2 x} = |\sin x|$$

Correct

b. For $f(x) = \sin x, g(x) = |x|$

$$f(g(x)) = f(|x|) = \sin |x|$$

$$g(f(x)) = g(\sin x) = |\sin x|$$

Incorrect

c. $f(x) = x^2, g(x) = \sin \sqrt{x}$

$$f(g(x)) = f(\sin \sqrt{x}) = (\sin \sqrt{x})^2$$

$$g(f(x)) = g(x^2) = \sin |x|$$

Incorrect

33. Question

Mark the correct alternative in each of the following:

The inverse of the function $f: \mathbb{R} \rightarrow [x \in \mathbb{R} : x < 1]$ given by $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, is

A. $\frac{1}{2} \log \frac{1+x}{1-x}$

B. $\frac{1}{2} \log \frac{2+x}{2-x}$

C. $\frac{1}{2} \log \frac{1-x}{1+x}$

D. none of these

Answer

Given that $f: \mathbb{R} \rightarrow [x \in \mathbb{R} : x < 1]$ defined by

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Put $y = f(x)$

$$\Rightarrow y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\Rightarrow y = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\Rightarrow y(e^{2x} + 1) = e^{2x} - 1$$

$$\Rightarrow e^{2x}(y - 1) = -y - 1$$

$$\Rightarrow e^{2x} = \frac{y + 1}{1 - y}$$

$$\Rightarrow 2x = \log\left(\frac{y+1}{1-y}\right)$$

$$\Rightarrow x = \frac{1}{2} \log\left(\frac{y+1}{1-y}\right)$$

$$\text{So, } f^{-1}(x) = \frac{1}{2} \log\left(\frac{x+1}{1-x}\right)$$

34. Question

Mark the correct alternative in each of the following:

Let $A = \{x \in \mathbb{R} : x \geq 1\}$. The inverse of the function $f: A \rightarrow A$ given by $f(x) = 2^{x(x-1)}$, is

A. $\left(\frac{1}{2}\right)^{x(x-1)}$

B. $\frac{1}{2} \left\{ 1 + \sqrt{1 + 4 \log_2 x} \right\}$

C. $\frac{1}{2} \left\{ 1 - \sqrt{1 + 4 \log_2 x} \right\}$

D. not defined

Answer

Given that $A = \{x \in \mathbb{R} : x \geq 1\}$. The function $f: A \rightarrow A$ given by $f(x) = 2^{x(x-1)}$

Put $y = f(x)$

$$\Rightarrow y = 2^{x(x-1)}$$

$$\Rightarrow \log_2 y = x(x-1)$$

$$\Rightarrow \log_2 y = x^2 - x$$

$$\Rightarrow \log_2 y + \frac{1}{4} = x^2 - x + \frac{1}{4}$$

$$\Rightarrow \log_2 y + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$$

$$\Rightarrow \sqrt{\frac{4\log_2 y + 1}{4}} + \frac{1}{2} = x$$

$$\Rightarrow \frac{1 + \sqrt{4\log_2 y + 1}}{2} = x$$

$$f^{-1}(x) = \frac{1 + \sqrt{4\log_2 x + 1}}{2}$$

35. Question

Mark the correct alternative in each of the following:

Let $A = \{x \in \mathbb{R} : x \leq 1\}$ and $f : A \rightarrow A$ given by $f(x) = x(2 - x)$. Then, $f^{-1}(x)$ is

A. $1 + \sqrt{1 - x}$

B. $1 - \sqrt{1 - x}$

C. $\sqrt{1 - x}$

D. $1 \pm \sqrt{1 - x}$

Answer

Given that $A = \{x \in \mathbb{R} : x \leq 1\}$ and $f : A \rightarrow A$ given by $f(x) = x(2 - x)$.

$$y = f(x)$$

$$\Rightarrow y = x(2 - x)$$

$$\Rightarrow y = 2x - x^2$$

$$\Rightarrow y - 1 = 2x - x^2 - 1$$

$$\Rightarrow y - 1 = -(x^2 + 1 - 2x)$$

$$\Rightarrow (x - 1)^2 = 1 - y$$

$$\Rightarrow x = 1 - \sqrt{1 - y}$$

$$f^{-1}(x) = 1 - \sqrt{1 - x}$$

36. Question

Mark the correct alternative in each of the following:

Let $f(x) = \frac{1}{1-x}$. Then, $\{f \circ f\}(x)$

A. x for all $x \in \mathbb{R}$

B. x for all $x \in \mathbb{R} - \{1\}$

C. x for all $x \in \mathbb{R} - \{0, 1\}$

D. none of these

Answer

Given that $f(x) = \frac{1}{1-x}$

$$fof(x) = f\left(\frac{1}{1-x}\right), \text{ for } x \neq 1$$

$$\Rightarrow fof = \frac{1}{1 - \frac{1}{1-x}}$$

$$\Rightarrow fof = \frac{1-x}{1-x-1}$$

$$\Rightarrow fof = \frac{x-1}{x}$$

$$fofof(x) = f\left(\frac{x-1}{x}\right), \text{ for } x \neq 0$$

$$\Rightarrow fofof = \frac{1}{1 - \frac{x-1}{x}}$$

$$\Rightarrow fofof = \frac{x}{x-x+1}$$

$$\Rightarrow fofof = x \text{ for all } x \in \mathbb{R} - \{0, 1\}$$

37. Question

Mark the correct alternative in each of the following:

If the function $f: \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(x) = x - [x]$, where $[x]$ denotes the greatest integer less than or equal to x , then $f^{-1}(x)$ is

A. $\frac{1}{x - [x]}$

B. $[x] - x$

C. not defined

D. none of these

Answer

Given that $f: \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(x) = x - [x]$, where $[x]$ denotes the greatest integer less than or equal to x

We will have same value of f for different values of x .

So, the function is not one-one.

$\therefore f$ is not bijective

$\therefore f$ does not have inverse.

38. Question

Mark the correct alternative in each of the following:

If $F: [1, \infty) \rightarrow [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, then $f^{-1}(x)$ equals.

A. $\frac{x + \sqrt{x^2 - 4}}{2}$

B. $\frac{x}{1 + x^2}$

$$C. \frac{x - \sqrt{x^2 - 4}}{2}$$

$$D. 1 + \sqrt{x^2 - 4}.$$

Answer

Given that $f: [1, \infty) \rightarrow [2, \infty)$ defined as

$$f(x) = x + \frac{1}{x}$$

$$y = f(x)$$

$$\Rightarrow y = x + \frac{1}{x}$$

$$\Rightarrow y = \frac{x^2 + 1}{x}$$

$$\Rightarrow xy = x^2 + 1$$

$$\Rightarrow x^2 - xy + \frac{y^2}{4} = \frac{y^2}{4} - 1$$

$$\Rightarrow \left(x - \frac{y}{2}\right)^2 = \frac{y^2}{4} - 1$$

$$\Rightarrow x = \frac{y}{2} + \sqrt{\frac{y^2 - 4}{4}}$$

$$\Rightarrow x = \frac{y}{2} + \frac{1}{2}\sqrt{y^2 - 4}$$

$$f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$$

39. Question

Mark the correct alternative in each of the following:

$$\text{Let } g(x) = 1 + x - [x] \text{ and } f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}, \text{ where } [x] \text{ denotes the greatest integer less than or equal to } x.$$

Then for all x , $f(g(x))$ is equal to

A. x

B. 1

C. $f(x)$

D. $g(x)$

Answer

Given that $g(x) = 1 + x - [x]$ and

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

where $[x]$ denotes the greatest integer less than or equal to x .

(i) $-1 < x < 0$

$$g(x) = 1 + x - [x]$$

$$\Rightarrow g(x) = 1 + x + 1 \text{ } \{ \because [x] = -1 \}$$

$$\Rightarrow g(x) = 2 + x$$

$$f(g(x)) = f(2 + x)$$

$$\Rightarrow f(g(x)) = 1 + 2 + x - [2 + x]$$

$$\Rightarrow f(g(x)) = 3 + x - 2 - x$$

$$\Rightarrow f(g(x)) = 1$$

$$(ii) \ x = 0$$

$$f(g(x)) = f(1 + x - [x])$$

$$\Rightarrow f(g(x)) = 1 + 1 + x - [x] - [1 + x + [x]]$$

$$\Rightarrow f(g(x)) = 2 + 0 - 1$$

$$\Rightarrow f(g(x)) = 1$$

$$(iii) \ x > 1$$

$$f(g(x)) = f(1 + x - [x])$$

$$\Rightarrow f(g(x)) = f(x > 0) = 1$$

Hence, $f(g(x)) = 1$ for all cases.

40. Question

Mark the correct alternative in each of the following:

Let $f(x) = \frac{\alpha x}{x+1}$, $x \neq -1$. Then, for what value of α is $f(f(x)) = x$?

A. $\sqrt{2}$

B. $-\sqrt{2}$

C. 1

D. -1

Answer

Given that $f(x) = \frac{\alpha x}{x+1}$, $x \neq -1$ and $f(f(x)) = x$

$$\Rightarrow f\left(\frac{\alpha x}{x+1}\right) = x$$

$$\Rightarrow \frac{\alpha \frac{\alpha x}{x+1}}{\frac{\alpha x}{x+1} + 1} = x$$

$$\Rightarrow \frac{\alpha^2 x}{x+1} = x\left(\frac{\alpha x}{x+1} + 1\right)$$

$$\Rightarrow \alpha^2 x = x(\alpha x + x + 1)$$

$$\Rightarrow \alpha^2 = \alpha x + x + 1$$

$$\Rightarrow \alpha^2 - \alpha x = x + 1$$

On comparing $-\alpha x$ with x ,

We get $\alpha = -1$

41. Question

Mark the correct alternative in each of the following:

The distinct linear functions which map $[-1, 1]$ onto $[0, 2]$ are

A. $f(x) = x + 1, g(x) = -x + 1$

B. $f(x) = x - 1, g(x) = x + 1$

C. $f(x) = -x - 1, g(x) = x - 1$

D. none of these

Answer

a. $f(x) = x + 1, g(x) = -x + 1$

$$f(-1) = -1 + 1 = 0$$

$$f(1) = 1 + 1 = 2$$

$$\text{Also, } g(-1) = 1 + 1 = 2$$

$$g(1) = -1 + 1 = 0$$

These functions map $[-1, 1]$ onto $[0, 2]$.

b. $f(x) = x - 1, g(x) = x + 1$

$$f(-1) = -1 - 1 = -2$$

$$f(1) = 1 - 1 = 0$$

$$\text{Also, } g(-1) = -1 + 1 = 0$$

$$g(1) = 1 + 1 = 2$$

These functions do not map $[-1, 1]$ onto $[0, 2]$.

c. $f(x) = -x - 1, g(x) = x - 1$

$$f(-1) = 1 - 1 = 0$$

$$f(1) = -1 - 1 = -2$$

$$\text{Also, } g(-1) = -1 - 1 = -2$$

$$g(1) = 1 - 1 = 0$$

These functions do not map $[-1, 1]$ onto $[0, 2]$.

42. Question

Mark the correct alternative in each of the following:

Let $f : [2, \infty) \rightarrow X$ be defined by $f(x) = 4x - x^2$. Then, f is invertible, if $X =$

A. $[2, \infty)$

B. $(-\infty, 2]$

C. $(-\infty, 4]$

D. $[4, \infty)$

Answer

Given that $f : [2, \infty) \rightarrow X$ be defined by

$$f(x) = 4x - x^2$$

$$\text{Let } y = f(x)$$

$$\Rightarrow y = 4x - x^2$$

$$\Rightarrow -y + 4 = 4 - 4x + x^2$$

$$\Rightarrow 4 - y = (x - 2)^2$$

$$\Rightarrow x - 2 = \sqrt{4 - y}$$

$$\Rightarrow x = 2 + \sqrt{4 - y}$$

$$\text{So, } f^{-1}(x) = 2 + \sqrt{4 - x}$$

where $x < 4$

So, $x \in (-\infty, 4]$

43. Question

Mark the correct alternative in each of the following:

If $f : \mathbb{R} \rightarrow (-1, 1)$ is defined by $f(x) = \frac{-x|x|}{1+x^2}$, then $f^{-1}(x)$ equals

A. $\sqrt{\frac{|x|}{1-|x|}}$

B. $-\text{Sgn}(x) \sqrt{\frac{|x|}{1-|x|}}$

C. $-\sqrt{\frac{x}{1-x}}$

D. none of these

Answer

Given that $f : \mathbb{R} \rightarrow (-1, 1)$ is defined by

$$f(x) = \frac{-x|x|}{1+x^2}$$

Here for mod function we will consider three cases, $x = 0$, $x < 0$ and $x > 0$.

For $x < 0$

$$f(x) = \frac{-x(-x)}{1+x^2}$$

$$y = \frac{x^2}{1+x^2}$$

$$\Rightarrow y(1+x^2) = x^2$$

$$\Rightarrow x^2(1-y) = y$$

$$\Rightarrow x = -\sqrt{\frac{y}{1-y}}$$

$$\Rightarrow x = -\sqrt{\frac{|y|}{1-|y|}}, x < 0$$

Also, checking on $x > 0$ and $x = 0$ we find that

$$f^{-1}(x) = -\operatorname{sgn}(x) \sqrt{\frac{|y|}{1-|y|}},$$

44. Question

Mark the correct alternative in each of the following:

Let $[x]$ denote the greatest integer less than or equal to x . If $f(x) = \sin^{-1} x$, $g(x) = [x^2]$ and

$$h(x) = 2x, \frac{1}{2} \leq x \leq \frac{1}{\sqrt{2}}, \text{ then}$$

A. $\text{fogoh}(x) = \pi/2$

B. $\text{fogoh}(x) = \pi$

C. $\text{hofog} = \text{hogof}$

D. $\text{hofog} \neq \text{hogof}$

Answer

Given that $f(x) = \sin^{-1} x$, $g(x) = [x^2]$ and $h(x) = 2x, \frac{1}{2} \leq x \leq \frac{1}{\sqrt{2}}$

a. $\text{goh}(x) = g(2x)$

$$\Rightarrow \text{goh}(x) = [4x^2]$$

$$\text{fogoh}(x) = f([4x^2])$$

$$\Rightarrow \text{fogoh}(x) = \sin^{-1} [4x^2]$$

Hence, given option is incorrect.

b. Similarly, this option is also incorrect.

c. $\text{fog}(x) = f([x^2])$

$$\Rightarrow \text{fog}(x) = \sin^{-1} [x^2]$$

$$\text{hofog}(x) = h(\sin^{-1} [x^2])$$

$$\Rightarrow \text{hofog}(x) = 2(\sin^{-1} [x^2])$$

$$\text{gof}(x) = g(\sin^{-1} x)$$

$$\Rightarrow \text{gof}(x) = [(\sin^{-1} x)^2]$$

$$\text{hogof}(x) = h([(\sin^{-1} x)^2])$$

$$\Rightarrow \text{hogof}(x) = 2[(\sin^{-1} x)^2]$$

Hence, $\text{hogof}(x) \neq \text{hofog}(x)$

45. Question

Mark the correct alternative in each of the following:

If $g(x) = x^2 + x - 2$ and $\frac{1}{2} \text{gof}(x) = 2x^2 - 5x + 2$, then $f(x)$ is equal to

A. $2x - 3$

B. $2x + 3$

C. $2x^2 + 3x + 1$

D. $2x^2 - 3x - 1$

Answer

Given that $g(x) = x^2 + x - 2$ and

$$\frac{1}{2}g \circ f(x) = 2x^2 - 5x + 2$$

a. Let $f(x) = 2x - 3$

$$g \circ f(x) = g(2x - 3)$$

$$\Rightarrow g \circ f(x) = (2x - 3)^2 + 2x - 3 - 2$$

$$\Rightarrow g \circ f(x) = 4x^2 - 12x + 9 + 2x - 5$$

$$\Rightarrow g \circ f(x) = 4x^2 - 10x + 4$$

$$\frac{1}{2}g \circ f(x) = \frac{4x^2 - 10x + 4}{2}$$

$$\Rightarrow \frac{1}{2}g \circ f(x) = 2x^2 - 5x + 2$$

Hence, this option is the required value of $f(x)$.

b. Let $f(x) = 2x + 3$

$$g \circ f(x) = g(2x + 3)$$

$$\Rightarrow g \circ f(x) = (2x + 3)^2 + 2x + 3 - 2$$

$$\Rightarrow g \circ f(x) = 4x^2 + 12x + 9 + 2x + 1$$

$$\Rightarrow g \circ f(x) = 4x^2 + 14x + 10$$

$$\frac{1}{2}g \circ f(x) = \frac{4x^2 + 14x + 10}{2}$$

$$\Rightarrow \frac{1}{2}g \circ f(x) = 2x^2 + 7x + 10$$

Hence, this option is not the required value of $f(x)$.

c and d option are incorrect because their degree is more than 1. So, the degree of $g \circ f$ will be more than 2.

46. Question

Mark the correct alternative in each of the following:

If $f(x) = \sin^2 x$ and the composite function $g(f(x)) = |\sin x|$, then $g(x)$ is equal to

A. $\sqrt{x-1}$

B. \sqrt{x}

C. $\sqrt{x+1}$

D. $-\sqrt{x}$

Answer

Given that $f(x) = \sin^2 x$ and the composite function $g(f(x)) = |\sin x|$.

$$g(f(x)) = g(\sin^2 x)$$

a. If $g(x) = \sqrt{x-1}$

$$g(f(x)) = \sqrt{\sin^2 x - 1}$$

Hence, given option is incorrect.

b. If $g(x) = \sqrt{x}$

$$g(f(x)) = \sqrt{\sin^2 x}$$

$$\Rightarrow g(f(x)) = |\sin x|$$

Hence, given option is correct.

c. If $g(x) = \sqrt{x + 1}$

$$g(f(x)) = \sqrt{\sin^2 x + 1}$$

Hence, given option is incorrect.

d. If $g(x) = -\sqrt{x}$

$$g(f(x)) = -\sqrt{\sin^2 x}$$

$$\Rightarrow g(f(x)) = -\sin x$$

Hence, given option is incorrect.

47. Question

Mark the correct alternative in each of the following:

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = x^3 + 3$, then $f^{-1}(x)$ is equal to

A. $x^{1/3} - 3$

B. $x^{1/3} + 3$

C. $(x - 3)^{1/3}$

D. $x + 3^{1/3}$

Answer

Given that $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = x^3 + 3$

Then $f^{-1}(x)$:

$$y = f(x)$$

$$\Rightarrow y = x^3 + 3$$

$$\Rightarrow y - 3 = x^3$$

$$\Rightarrow x = \sqrt[3]{y - 3}$$

$$\text{So, } f^{-1}(x) = \sqrt[3]{x - 3}$$

48. Question

Mark the correct alternative in each of the following:

Let $f(x) = x^3$ be a function with domain $\{0, 1, 2, 3\}$. Then domain of f^{-1} is

A. $\{3, 2, 1, 0\}$

B. $\{0, -1, -2, -3\}$

C. $\{0, 1, 8, 27\}$

D. $\{0, -1, -8, -27\}$

Answer

Given that $f(x) = x^3$ be a function with domain $\{0, 1, 2, 3\}$.

Then range = $\{0, 1, 8, 27\}$

f can be written as $\{(0, 0), (1, 1), (2, 8), (3, 27)\}$

f^{-1} can be written as $\{(0, 0), (1, 1), (8, 2), (27, 3)\}$

So, the domain of f^{-1} is $\{0, 1, 8, 27\}$

49. Question

Mark the correct alternative in each of the following:

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2 - 3$. Then, f^{-1} is given by

A. $\sqrt{x+3}$

B. $\sqrt{x} + 3$

C. $x + \sqrt{3}$

D. none of these

Answer

Given that $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2 - 3$

For f^{-1} :

$$y = f(x)$$

$$\Rightarrow y = x^2 - 3$$

$$\Rightarrow x = \pm\sqrt{y+3}$$

$$f^{-1}(x) = \pm\sqrt{x+3}$$

50. Question

Mark the correct alternative in each of the following:

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = \tan x$. Then, $f^{-1}(1)$ is

A. $\frac{\pi}{4}$

B. $\left\{ n\pi + \frac{\pi}{4} : n \in \mathbb{Z} \right\}$

C. does not exist

D. none of these

Answer

Given that $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = \tan x$

For f^{-1} :

$$y = f(x)$$

$$\Rightarrow y = \tan x$$

$$\Rightarrow x = \tan^{-1} y$$

$$f^{-1} = \tan^{-1} x$$

$$\Rightarrow f^{-1}(x) = n\pi + \frac{\pi}{4}; n \in \mathbb{Z}$$

51. Question

Mark the correct alternative in each of the following:

$$\text{Let } f: \mathbb{R} \rightarrow \mathbb{R} \text{ be defined as } f(x) = \begin{cases} 2x, & \text{if } x > 3 \\ x^2, & \text{if } 1 < x \leq 3 \\ 3x, & \text{if } x \leq 1 \end{cases}$$

Then, find $f(-1) + f(2) + f(4)$

- A. 9
- B. 14
- C. 5
- D. none of these

Answer

Given that $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 2x, & \text{if } x > 3 \\ x^2, & \text{if } 1 < x \leq 3 \\ 3x, & \text{if } x \leq 1 \end{cases}$$

For $f(-1)$:

$$f(x) = 3x$$

$$\Rightarrow f(-1) = -3$$

For $f(2)$:

$$f(x) = x^2$$

$$\Rightarrow f(2) = 4$$

For $f(4)$:

$$f(x) = 2x$$

$$\Rightarrow f(4) = 8$$

$$f(-1) + f(2) + f(4) = -3 + 4 + 8$$

$$\Rightarrow f(-1) + f(2) + f(4) = 9$$

52. Question

Mark the correct alternative in each of the following:

Let $A = \{1, 2, \dots, n\}$ and $B = \{a, b\}$. Then the number of subjections from A into B is

- A. ${}^n P_2$
- B. $2^n - 2$
- C. 0
- D. none of these

Answer

Given that $A = \{1, 2, \dots, n\}$ and $B = \{a, b\}$

The number of functions from a set with n number of elements into a set of 2 number of elements = 2^n

But two functions can be many-one into functions.

Hence, answer is $2^n - 2$.

53. Question

Mark the correct alternative in each of the following:

If the set A contains 5 elements and the set B contains 6 elements, then the number of one-one and onto mappings from A to B is

- A. 720
- B. 120
- C. 0
- D. none of these

Answer

Given that set A contains 5 elements and set B contains 6 elements.

Number of one-one and onto mappings from A to B means bijections from A to B.

Number of bijections are possible only when $n(B) < n(A)$.

But here, $n(A) < n(B)$

So, the number of one-one and onto mappings from A to B is 0.

54. Question

Mark the correct alternative in each of the following:

If the set A contains 7 elements and the set B contains 10 elements, then the number one-one functions from A to B is

- A. ${}^{10}C_7$
- B. ${}^{10}C_7 \times 7!$
- C. 7^{10}
- D. 10^7

Answer

Given that set A contains 7 elements and set B contains 10 elements.

The number one-one functions from A to B is ${}^{10}C_7 \times 7!$.

55. Question

Mark the correct alternative in each of the following:

Let $f : \mathbb{R} - \left\{ \frac{3}{5} \right\} \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{3x+2}{5x-3}$. Then,

- A. $f^{-1}(x) = x$
- B. $f^{-1}(x) = -f(x)$
- C. $f \circ f(x) = x$
- D. $f^{-1}(x) = \frac{1}{19}f(x)$

Answer

Given that $f: \mathbb{R} - \left\{\frac{3}{5}\right\} \rightarrow \mathbb{R}$ defined as $f(x) = \frac{3x + 2}{5x - 3}$

For f^{-1} :

$$y = \frac{3x + 2}{5x - 3}$$

$$\Rightarrow y(5x - 3) = 3x + 2$$

$$\Rightarrow x(5y - 3) = 2 + 3y$$

$$\Rightarrow x = \frac{2 + 3y}{5y - 3}$$

$$\text{So, } f^{-1}(x) = \frac{2 + 3x}{5x - 3}$$

$$f \circ f(x) = f\left(\frac{3x + 2}{5x - 3}\right)$$

$$\Rightarrow f \circ f(x) = \frac{3\frac{3x+2}{5x-3} + 2}{5\frac{3x+2}{5x-3} - 3}$$

$$\Rightarrow f \circ f(x) = \frac{3(3x + 2) + 2(5x - 3)}{5(3x + 2) - 3(5x - 3)}$$

$$\Rightarrow f \circ f(x) = \frac{19x}{19}$$

$$\Rightarrow f \circ f(x) = x$$

Hence, option C is correct.