CHAPTER 35

Calculus of Variations

35.1 INTRODUCTION

The calculus of variations is a powerful technique for the solution of problems in dynamics of rigid bodies, optimization of orbits and vibration problems. The subject primarily concerns with finding maximum or minimum value of a definite integral involving a certain function. It is something beyond finding stationary values of a given function. Only an elementary exposition of the subject is given here with the sole aim of introducing the student to a topic whose importance is fast growing in science and engineering.

Before proceeding further, the student should revise § 5.12 concerning maxima and minima of functions of several variables.

35.2 FUNCTIONALS

Consider the problem of finding a curve through two points (x_1, y_1) and (x_2, y_2) whose length is a minimum (Fig. 35.1). It is same as determining the curve $y = y(x)$ for which $y(x_1) = y_1$, $y(x_2) = y_2$ such that $\int_{x_1}^{x_2} \sqrt{(1 + y'^2)}\, dx$ is a minimum.

In general terms, we wish to find the curve $y = y(x)$ where $y(x_1) = y_1$ and $y(x_2) = y_2$ such that for a given function $f(x, y, y')$,

$$\int_{x_1}^{x_2} f(x, y, y')\, dx \text{ is a stationery value or an extremum.} \qquad ...(1)$$

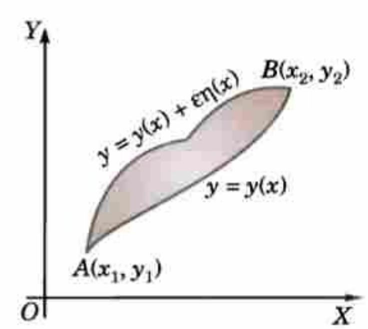

Fig. 35.1

An integral such as (1), which assumes a definite value for functions of the type $y = y(x)$ is called a **functional.**

In differential calculus, we deal with the problems of maxima and minima of functions. The calculus of variations is however, concerned with maximizing or minimizing functionals.

35.3 EULER'S EQUATION

A necessary condition for

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx \textit{ to be an extremum is that}$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{y}} - \frac{\mathbf{d}}{\mathbf{dx}}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{y}'}\right) = \mathbf{0}$$

This is called *Euler's equation.*

Proof. Let $y = y(x)$ be the curve joining points $A(x_1, y_1)$, $B(x_2, y_2)$ which makes I an extremum. Let

$$y = y(x) + \varepsilon\eta(x) \quad ...(1)$$

be a neighbouring curve joining these points so that at A, $\eta(x_1) = 0$ and at B, $\eta(x_2) = 0$. ...(2)

The value of I along (1) is $I = \int_{x_1}^{x_2} f[x, y(x) + \varepsilon\eta(x), y'(x) + \varepsilon\eta'(x)]\, dx$

This being a function of ε, is a maximum or minimum for $\varepsilon = 0$, when

$$\frac{dI}{d\varepsilon} = 0 \text{ at } \varepsilon = 0 \quad ...(3)$$

$\therefore$ Differentiating I under the integral sign by Leibnitz's rule (p. 139), we have

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varepsilon} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \varepsilon}\right) dx \quad ...(4)$$

But ε being independent of x, $\frac{\partial x}{\partial \varepsilon} = 0$. Also from (1), $\frac{\partial y}{\partial \varepsilon} = \eta(x)$ and $\frac{\partial y'}{\partial \varepsilon} = \eta'(x)$.

Substituting these values in (4), we get $\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\right] dx$

Integrating the second term on the right by parts, we have

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x)\, dx + \left[\left|\frac{\partial f}{\partial y'}\eta(x)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\, dx\right]$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\, dx \quad \text{[By (2)]}$$

Since this has to be zero by (3),

$$\therefore \quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad ...(I)$$

which is the desired *Euler's equation.*

Obs. 1. Other forms of Euler's equation.

(*a*) Since f is a function of x, y, y', we have $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx}$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \quad ...(5)$$

and
$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial f}{\partial y'}y'' \quad ...(6)$$

Subtracting (6) from (5), we get $\frac{df}{dx} - \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$

or
$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial x} = y'\left\{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right\} = 0 \quad \text{[By (I)]}$$

Hence
$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial x} = 0 \quad ...(II)$$

which is another form of (*I*).

(*b*) Again since $\frac{\partial f}{\partial y'}$ is also a function of x, y, y', say : $\psi(x, y, y')$.

$$\therefore \quad \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{d\psi}{dx} = \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} + \frac{\partial \psi}{\partial y'}\frac{dy'}{dx}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right) y' + \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right) y'' = \frac{\partial^2 f}{\partial x \partial y'} + y' \frac{\partial^2 f}{\partial y \partial y'} + y'' \frac{\partial^2 f}{\partial y'^2}$$

Substituting this in (*I*), we get $\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - y' \frac{\partial^2 f}{\partial y \partial y'} - y'' \frac{\partial^2 f}{\partial y'^2} = 0$ (*V.T.U., 2001*) ...(*III*)

which is an extended form of (*I*).

Obs. 2. The above problem can easily be extended to the integral

$$= \int_{x_1}^{x_2} f(x, y_1, y_2 \dots y_n, y_1', y_2', \dots y_n')\, dx$$

involving n functions $y_1, y_2, \dots y_n$ of x. Then the necessary condition for this integral to be stationary is

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0, \quad i = 1, 2, \dots, n \qquad \text{...(IV)}$$

These are Euler's equations for the n functions.

35.4 SOLUTIONS OF EULER'S EQUATION

Every solution of the Euler's equation which satisfies the boundary conditions, is called an *extremal* or a *stationary function* of the problem. The extremal can easily be obtained in the following cases :

(1) *When f is independent of x*

We have $\partial f/\partial x = 0$ and Euler's equation (*II*) above becomes $\frac{d}{dx}\left(f - y' \frac{\partial f}{\partial y'}\right) = 0$

Integrating, we get $f - y' \frac{\partial f}{\partial y'}$ = constant. This directly gives a solution of Euler's equation.

(2) *When f is independent of y*

We have $\partial f/\partial y = 0$ and Euler's equation (*I*) reduces to $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

Integrating $\frac{\partial f}{\partial y}$ = constant which gives a solution directly.

(3) *When f is independent of y'.*

We have $\partial f/\partial y' = 0$ and the equation (*I*) becomes $\frac{\partial f}{\partial y} = 0$ which gives the desired solution.

(4) *When f is independent of x and y*

We have $$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y'} = 0, \frac{\partial^2 f}{\partial y \partial y'} = 0.$$

Then the equation (*III*) above becomes $y'' \frac{\partial^2 f}{\partial y'^2} = 0$.

If $\frac{\partial^2 f}{\partial y'^2} \neq 0$, it reduces to $y'' = 0$ which gives a solution of the form $y = ax + b$.

Example 35.1. *Find the extremals of the functional* $\int_{x_0}^{x_1} (y'^2/x^3)\, dx$. (*V.T.U., 2003*)

Solution. We have $f = y'^2/x^3$ which is independent of y *i.e.*, $\partial f/\partial y = 0$.

Also $$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{d}{dx}\left(\frac{2y'}{x^3}\right) = 2\,\frac{x^3 y'' - y'.3x^2}{x^6} = \frac{2}{x^4}(xy'' - 3y')$$

$\therefore$ Euler's equation reduces to $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

i.e., $$\frac{2}{x^4}(xy'' - 3y') = 0 \quad \text{or} \quad y''/y' = 3/x$$

Integrating both sides, $\int \frac{y''}{y'} dy = 3 \int \frac{dx}{x} + \log c$

i.e., $\log y' = 3 \log x + \log c$ or $y' = cx^3$

Hence $y = cx^4/4 + c'$ or $y = c_1x^4 + c_2$

This is the required extremal.

Example 35.2. *Prove that the shortest distance between two points in a plane is a straight line.*
(V.T.U., 2003 S ; Bhopal, 2003)

Solution. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given points and s the arc length of a curve connecting them (Fig. 35.2). Then

$$s = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{(1 + y'^2)}\ dx$$

Now s will be minimum if it satisfies Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Here $f = \sqrt{(1 + y'^2)}$ which is independent of y *i.e.,* $\partial f/\partial y = 0$

$$\therefore \quad \frac{d}{dx}\left\{\frac{\partial}{\partial y'}\sqrt{(1 + y'^2)}\right\} = 0 \quad \text{or} \quad \frac{d}{dx}\left\{\frac{y'}{\sqrt{(1 + y'^2)}}\right\} = 0.$$

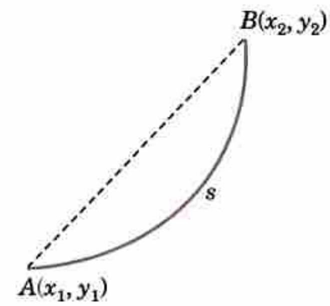

Fig. 35.2

$\therefore$ On integration, we have $y'/\sqrt{(1 + y'^2)}$ = constant y' = constant, m say.

Integrating, we get $y = mx + c$, which is a straight line, the constants m and c are determined from the fact that the straight line passes through A and B.

Example 35.3. *Find the curve passing through the points (x_1, y_1) and (x_2, y_2) which when rotated about the x-axis gives a minimum surface area.* *(V.T.U., 2009)*

Solution. In Fig. 35.3, the surface area $= \int_{x_1}^{x_2} 2\pi y\ ds$

$$= 2\pi \int_{x_1}^{x_2} y\sqrt{(1 + y'^2)}\, dx. \text{ This has to be minimum.}$$

Since $f = y\sqrt{(1 + y'^2)}$ is independent of x, therefore, Euler's equation reduces to

$$f - y'\frac{\partial f}{\partial y'} = \text{constant, } c : \text{say} \qquad [\text{By § 35.4 (1)}]$$

$$\therefore \quad y\sqrt{(1 + y'^2)} - y'\frac{\partial}{\partial y'}\left\{y\sqrt{(1 + y'^2)}\right\} = c$$

i.e.,
$$y\sqrt{(1 + y'^2)} - y'\left\{\frac{y}{2}(1 + y'^2)^{-1/2} \cdot 2y'\right\} = c$$

or
$$y/\sqrt{(1 + y'^2)} = c \quad \text{or} \quad y' = \frac{dy}{dx} = \frac{\sqrt{(y^2 - c^2)}}{c}$$

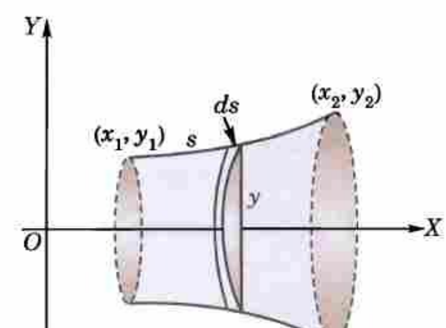

Fig. 35.3

Separating the variables and integrating, we have

$$\int \frac{dy}{\sqrt{(y^2 - c^2)}} = \int \frac{dx}{c} + c' \quad \text{or} \quad \cosh^{-1}\left(\frac{y}{c}\right) = \frac{x + a}{c}$$

i.e.,
$$y = c\cosh\left(\frac{x + a}{c}\right)$$

which is a *catenary*. The constants a and c are determined from the points (x_1, y_1) and (x_2, y_2).

Obs. This problem is also important in connection with *soap films* which are known to have shapes with minimum surface areas.

Example 35.4. *Find the path on which a particle in the absence of friction, will slide from one point to another in the shortest time under the action of gravity.* (V.T.U., 2004)

Solution. Let the particle start sliding on the curve OP_1 from O with zero velocity (Fig. 35.4). At time t, let the particle be $P(x, y)$ such that arc $OP = s$.

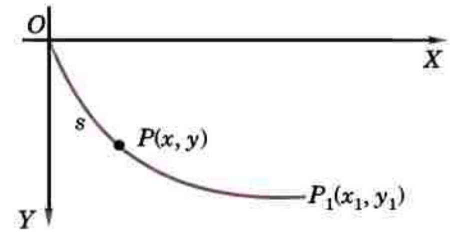

Fig. 35.4

By the principle of work and energy, we have

K.E. at P – K.E. at O = Work done in moving the particle from O to P.

$$\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 - 0 = mgy$$

or $$ds/dt = \sqrt{(2gy)} \qquad ...(i)$$

Thus the time taken by the particle to move from O to P_1 is

$$T = \int_0^T dt = \int_0^{x_1} \frac{ds}{\sqrt{(2gy)}} = \frac{1}{\sqrt{(2g)}} \int_0^{x_1} \frac{\sqrt{(1+y'^2)}}{\sqrt{y}}\,dx$$

Here $f = \sqrt{[(1+y'^2)/y]}$ is independent of x.

∴ Euler's equation reduces to $f - y'\,\partial f/\partial y'$ = constant, c : say

i.e., $$\frac{\sqrt{(1+y'^2)}}{\sqrt{y}} - y'\frac{\partial}{\partial y'}\left\{\frac{\sqrt{(1+y'^2)}}{\sqrt{y}}\right\} = c \quad \text{or} \quad \frac{\sqrt{(1+y'^2)}}{\sqrt{y}} - y'\left\{\frac{y'}{\sqrt{(1+y'^2)}\sqrt{y}}\right\} = c$$

or $$\sqrt{[y(1+y'^2)]} = 1/c = \sqrt{a}\text{, say.}$$

Solving for y', we have $$y' = \frac{dy}{dx} = \sqrt{\left(\frac{a-y}{y}\right)}$$

Separating the variables and integrating, we get

$$\int_0^x dx = \int_0^y \sqrt{\left(\frac{y}{a-y}\right)}\,dy \qquad [\text{Put } y = a\sin^2\theta] \quad ...(i)$$

or $$x = \int_0^\theta \sqrt{\left(\frac{a\sin^2\theta}{a - a\sin^2\theta}\right)}\, 2a\sin\theta\cos\theta\, d\theta$$

$$= a\int_0^\theta 2\sin^2\theta\, d\theta = a\int_0^\theta (1-\cos 2\theta)\, d\theta = \frac{a}{2}(2\theta - \sin 2\theta) \qquad ...(ii)$$

Writing $a/2 = b$ and $2\theta = \phi$, equations (*ii*) and (*i*) become $x = b(\phi - \sin\phi)$, $y = b(1-\cos\phi)$ which is a *cycloid*. The constant b is found from the fact that the curve goes through (x_1, y_1).

Obs. This is the well-known **brachistochrone problem** which derives its name from the Greek words 'brachistos' meaning shortest and 'chronos' meaning time. It was proposed by John Bernoulli in 1696 and its solution formed the basis for the study of the 'Calculus of Variations'. (V.T.U., 2006)

Example 35.5. *Find the curves on which the functional* $\int_0^1 [(y')^2 + 12xy]\,dx$ *with* $y(0) = 0$ *and* $y(1) = 1$ *can be extremised.* (V.T.U., 2010)

Solution. We have $$f = y'^2 + 12xy$$

∴ $$\partial f/\partial y = 12x\,;\ \frac{\partial f}{\partial y'} = 2y'\,;\ \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 2y''$$

Hence the Euler's equation $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ becomes

$$12x - 2y'' = 0 \quad i.e., \quad y'' = 6x \qquad ...(i)$$

Integrating (i), $y' = 3x^2 + C$ ...(ii) and $y = x^3 + Cx + C'$ ...(iii)

Using the boundary conditions, when $x = 0, y = 0$ (iii) gives $C' = 0$.

When $x = 1, y = 1$, (iii) again gives $C = 0$.

Hence (iii) reduces to $y = x^3$ which is the only curve on which extremum can be attained.

Example 35.6. *On which curve the functional* $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)dy$ *with* $y(0) = 0$ *and* $y(\pi/2) = 0$, *be extremized?* (V.T.U., 2006)

Solution. Let $f = y'^2 - y^2 + 2xy$ so that $\frac{\partial f}{\partial y} = 0 - 2y + 2x$

and
$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{d}{dx}(2y') = 2y''$$

$\therefore$ Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes

$$-2y + 2x - 2y'' = 0 \quad \text{or} \quad y'' + y = x \quad \text{or} \quad (D^2 + 1)\,y = x$$

Its A.E. $D^2 + 1 = 0$ gives $D = \pm i$.

$$\therefore \quad \text{C.F.} = c_1 \cos x + c_2 \sin x$$

and
$$\text{P.I.} = \frac{1}{D^2 + 1} x = (1 + D^2)^{-1} x = (1 - D^2)\,x = x$$

Thus
$$y = c_1 \cos x + c_2 \sin x + x \quad ...(i)$$

Using boundary conditions : when $x = 0, y = 0$, (i) gives $c_1 = 0$;

when $x = \pi/2, y = 0$, (i) gives $0 = c_2 + \pi/2$, *i.e.*, $c_2 = -\pi/2$.

Hence (i) reduces to $y = x - \frac{\pi}{2} \sin x$, which is the only curve on which the given functional can be extremized.

Example 35.7. *Solve the variational problem*

$$\delta \int_1^2 [x^2 (y')^2 + 2y(x + y)]\, dx = 0, \text{ given } y(1) = y(2) = 0. \quad \text{(V.T.U., 2006)}$$

Solution. Let $f = x^2 (y')^2 + 2xy + 2y^2$ so that $\frac{\partial f}{\partial y} = 2x + 4y$, $\frac{\partial f}{\partial y} = 2x^2y'$

$\therefore$ Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes

$$2x + 4y - \frac{d}{dx}(2x^2y') = 0 \quad \text{or} \quad 2x + 4y - (2x^2y'' + 4xy') = 0$$

or
$$x^2y'' + 2xy' - 2y = x. \text{ This is Cauchy's homogeneous linear (§ 13.9)}$$

Putting $x = e^t$, it reduces to $(D^2 + D - 2)\,y = e^t$.

Its solution is
$$y = c_1 e^t + c_2 e^{-2t} + \frac{1}{3} te^t \quad \text{or} \quad y = c_1 x + \frac{c_2}{x^2} + \frac{1}{3} x \log x \quad ...(i)$$

Since $y(1) = 0$, we have $c_1 + c_2 = 0$

and
$$y(2) = 0 \text{ gives } 0 = 2\,c_1 + \frac{1}{4}c_2 + \frac{2}{3} \log 2$$

Solving these equations, we get $c_1 = -c_2 = \frac{-8}{21} \log 2$.

Putting the values of c_1 and c_2 in (i), we get

$$y = \frac{1}{21}\{8 \log 2\,(x^{-2} - x) + 7x \log x\}$$

which is the required solution.

35.5 GEODESICS

A geodesic on a surface is a curve along which the distance between any two points of the surface is a minimum. To find the geodesics on a surface is a variational problem involving the conditional extremum. This problem was first studied by *Jacob Bernoulli* in 1698 and its general method of solution was given by *Euler*.

Example 35.8. *Show that the geodesics on a plane are straight lines.* (V.T.U., 2009)

Solution. Let $y = y(x)$ be a curve joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the xy-plane. Then the length of a curve joining A and B is given by

$$s = \int_A^B \frac{ds}{dx}\,dx = \int_{x_1}^{x_2} \sqrt{[1+(dy/dx)^2]}\,dx \quad i.e., \quad s = \int_{x_1}^{x_2} \sqrt{(1+y'^2)}\,dx$$

The geodesic on the xy-plane is the curve $y = y(x)$ for which s is minimum.

We have $f(x, y, y') = \sqrt{(1+y'^2)}$ which depends on y' only. Hence the Euler's equation.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \text{ yields}$$

$$\frac{d}{dx}\left\{\frac{2y'}{2\sqrt{(1+y'^2)}}\right\} = 0 \quad i.e., \quad y''\sqrt{(1+y'^2)} - \frac{y'_2 2y'y''}{2\sqrt{(1+y'^2)}} = 0$$

i.e.

$$y''(1+y'^2) - y'^2 y'' = 0 \quad i.e., \quad \frac{d^2y}{dx^2} = 0$$

Integrating twice, we get $y = c_1 x + c_2$

which is a straight line.

Hence *the geodesics on a plane are straight lines.*

Example 35.9. *Find the geodesics on a right circular cylinder of radius a.*

Solution. In cylindrical coordinates (ρ, ϕ, z) we have

$$x = r\cos\phi,\; y = r\sin\phi,\; z = z. \qquad (p.\ 357$$

$\therefore$ The element of arc on a right circular cylinder of radius a, is given by

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 = (d\rho)^2 + (\rho\,d\phi)^2 + dz^2 = a^2\,d\phi^2 + dz^2 \qquad [\because\ \rho = a \text{ and } d\rho = 0]$$

or

$$ds = \sqrt{\{a^2 + (dz/d\phi)^2\}}\,.\,d\phi \quad \text{or} \quad s = \int_{\phi_1}^{\phi_2} \sqrt{(a^2+z'^2)}\,d\phi \qquad ...(i)$$

Now the geodesic for the given cylinder is the curve for which s is minimum. Here $f = \sqrt{(a^2+z'^2)}$, which is a function of ϕ and z' while z is absent.

$\therefore$ Euler's equation for the functional (i) reduces to

$$\frac{\partial f}{\partial z'} = \text{constant} \quad \text{or} \quad \frac{z'}{\sqrt{(a^2+z'^2)}} = c, \text{ say.}$$

This simplifies to $(z')^2$ = constant or $dz/d\phi = c_1$, say.

Integrating, $z = c_1\phi + c_2$

This is the desired geodesics on a circular cylinder which is a circular helix. (Example 8.3, *p.* 318)

Example 35.10. *Show that the geodesics on a sphere of radius a are its great circles.*

Solution. In spherical coordinates (r, θ, ϕ), we have

$$x = r\sin\theta\cos\phi,\; y = r\sin\theta\sin\phi,\; z = r\cos\theta. \qquad (p.\ 359) \quad ...(i)$$

$\therefore$ The arc element on a sphere of radius a, is given by

$$ds^2 = dr^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2 = a^2\,d\theta^2 + (a\sin\theta)^2\,d\phi^2 \qquad [\because\ r = a,\ dr = 0$$

or

$$ds = a\sqrt{[1+\sin^2\theta\,(d\phi/d\theta)^2]}\,d\theta, \quad \text{or} \quad s = a\int_{\theta_1}^{\theta_2}\sqrt{[1+\sin^2\theta\,.\,(\phi')^2]}\,d\theta$$

Now the geodesic on the sphere $r = a$ is the curve for which s is minimum. Here $f = a\sqrt{(1+\sin^2\theta\,.\,\phi'^2)}$ which is a function of θ and ϕ' while ϕ is absent.

$\therefore$ Euler's equation reduces to $\partial f/\partial\phi'$ = constant.

$$\frac{\partial f}{\partial\phi'} = \frac{a\sin^2\theta\,.\,\phi'}{\sqrt{(1+\sin^2\theta\,.\,\phi'^2)}} = \text{constant.}$$

or $$\frac{\sin^2\theta\,.\,\phi'}{\sqrt{(1+\sin^2\theta\,.\,\phi'^2)}} = c \text{ (say)} \quad\text{or}\quad \sin^2\theta\,(\sin^2\theta - c^2)\,\phi'^2 = c^2$$

or $$\frac{d\phi}{d\theta} = \frac{c}{\sin\theta\sqrt{(\sin^2\theta - c^2)}} = \frac{c\,\text{cosec}^2\theta}{\sqrt{(1-c^2\,\text{cosec}^2\theta)}}$$

Integrating $$\phi = \int\frac{c\,\text{cosec}^2\theta\,.\,d\theta}{\sqrt{[(1-c^2)-(c\cot\theta)^2]}} + c' = -\sin^{-1}\left\{\frac{c\cot\theta}{\sqrt{(1-c^2)}}\right\} + c'$$

or $$\cot\theta = A\cos\phi + B\sin\phi \quad\text{or}\quad a\cos\theta = Aa\sin\theta\cos\phi + Ba\sin\theta\sin\phi$$

or $$z = Ax + By$$ [By (*i*) when $r = a$]

This is a plane through the centre (0, 0, 0) of the sphere which cuts the sphere along a great circle. Hence the required *geodesics are the arcs of the great circles.*

PROBLEMS 35.1

1. Solve the Euler's equation for the following functionals :

   (*i*) $\int_{x_0}^{x_1}(x+y')\,y'\,dx$ (*ii*) $\int_{x_0}^{x_1}(1+x^2y')\,y'\,dx$. (*V.T.U., 2004*)

2. Show that the general solution of the Euler's equation for the integral

$$\int_a^b \frac{1}{y}\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx \text{ is } (x-h)^2 + y^2 = k^2.$$

Find the extremals of the following functionals :

3. $\int_{x_1}^{x_2}(y^2 + y'^2 + 2ye^x)\,dx$. (*V.T.U., 2004*)

4. $\int_0^{\pi/2}(y^2 + y'^2 - 2y\sin x)\,dx,\ y(0) = y(\pi/2) = 0$. (*V.T.U., 2008*)

5. $\int_0^{\pi}(y'^2 - y^2 + 4y\cos x)\,dx;\ y(0) = 0,\ y(\pi) = 0$. (*V.T.U., 2008 S*)

6. $\int_{x_0}^{x_1}\frac{1+y^2}{y'^3}\,dx$. 7. $\int_1^2\frac{x^3}{y'^2}\,dx\ ;\ y(1) = 0,\ y(2) = 3$.

8. $\int_1^2\frac{\sqrt{(1+y'^2)}}{x}\,dx\ ;\ y(1) = 0,\ y(2) = 1$. (*Madurai, M.E., 2000 S*)

9. Solve the variational problem $\delta\int_0^{\pi/2}[y^2 - (y')^2]\,dx$ under the conditions $y(0) = 0,\ y(\pi/2) = 2$. (*V.T.U., 2010*)

10. A heavy cable hangs freely under gravity between two fixed points. Show that the shape of the cable is a catenary.

11. A particle is moving with a force perpendicular to and proportional to its distance from the line of zero velocity. Show that the path of quickest descent (brachistochrone) is a circle.

12. Find the geodesics on a *right circular cone* of semi-vertical angle α.

35.6 ISOPERIMETRIC PROBLEMS

In certain problems, it is necessary to make a given integral.

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx \quad ...(1)$$

maximum or minimum while keeping another integral

$$J = \int_{x_1}^{x_2} g(x, y, y')\, dx \quad ...(2)$$

constant. Such problems involve one or more constraint conditions, just as $J = a$ constant. A typical example of this type is that of finding a closed curve of a given perimeter and maximum area. This being one of the earliest problems to engage attention, we often refer to problems of this type as *isoperimetric problems.*

Such problems are generally solved by the method of Lagrange multipliers. To extremize (1), we multiply (2) by λ and add to (1) where λ is the Lagrange multiplier. Then the necessary condition for the integral $\int_{x_1}^{x_2} H\, dx$ to be an extremum is $\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$ where $H = f + \lambda g$. The values of the two constants of integration and the parameter λ are determined from the three conditions namely : the two boundary conditions and the integral J having given constant value.

Example 35.11. *Find the plane curve of fixed perimeter and maximum area.* *(V.T.U., 2000 S)*

Solution. Let l be the fixed perimeter of a plane curve between the points with abscissae x_1 and x_2 (Fig. 35.5). Then

$$l = \int_{x_1}^{x_2} \sqrt{(1 + y'^2)}\, dx \quad ...(i)$$

Also the area between the curve and the x-axis is

$$A = \int_{x_1}^{x_2} y\, dx \quad ...(ii)$$

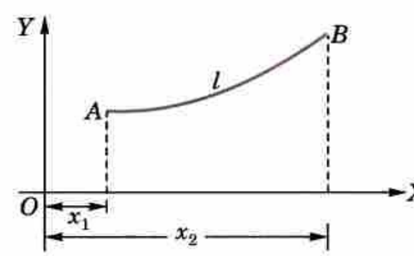

Fig. 35.5

We have to maximize (*ii*) subject to constraint (*i*).

$\therefore$ Taking $f = y$ and $g = \sqrt{(1 + y'^2)}$, we write $H = f + \lambda g = y + \lambda\sqrt{(1 + y'^2)}$

Now H must satisfy the Euler's equation

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \quad \therefore \quad 1 - \frac{d}{dx}\left[\frac{\lambda y'}{\sqrt{(1 + y'^2)}}\right] = 0$$

Integrating w.r.t. x, we have $x - \lambda y'/\sqrt{(1 + y'^2)} = 0$

Solving for y', we get $\quad y' = \dfrac{x - a}{\sqrt{[\lambda^2 - (x - a)^2]}}$

Integrating again, $\quad y = \sqrt{[\lambda^2 - (x - a)^2]} + b$ *i.e.* $(x - a)^2 + (y - b)^2 = \lambda^2$ which is a circle.

Example 35.12. *Prove that the sphere is the solid figure of revolution which, for a given surface area, has maximum volume.* *(V.T.U., 2006 ; Madras, 2000 S)*

Solution. Consider the arc OPA of the curve which rotates about the x-axis as shown in Fig. 35.6.

Then surface area $\quad S = \int_{x=0}^{a} 2\pi y\, ds = \int_0^a 2\pi y\sqrt{(1 + y'^2)}\, dx$

and volume of the solid so formed $\quad V = \int_0^a \pi y^2\, dx.$

Here we have to maximize V subject to fixed S. Taking $f = \pi y^2$ and $g = 2\pi y\sqrt{(1+y'^2)}$, we write $H = f + \lambda g = \pi y^2 + 2\pi\lambda y\sqrt{(1+y'^2)}$.

Now H has to satisfy Euler's equation. But it does not contain x explicitly.

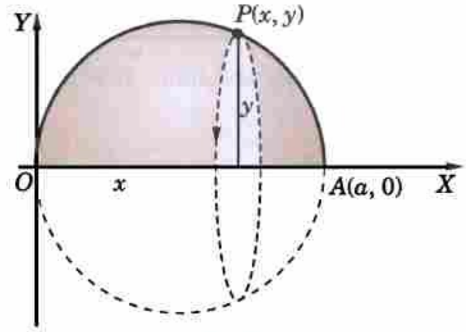

Fig. 35.6

$$\therefore \qquad H - y'\frac{\partial H}{\partial y'} = \text{constant, } c : \text{say}$$

i.e., $$\pi y^2 + 2\pi\lambda y\sqrt{(1+y'^2)} - y' \cdot 2\pi\lambda y\frac{y'}{\sqrt{(1+y'^2)}} = c$$

or $$\pi y^2 + \frac{2\pi\lambda y}{\sqrt{(1+y'^2)}} = c \qquad ...(i)$$

Since the curve passes through O and A for which $y = 0$, (i) gives $c = 0$.

$$\therefore \qquad y + 2\lambda/\sqrt{(1+y'^2)} = 0$$

Solving for y', $$y'\left(=\frac{dy}{dx}\right) = \frac{\sqrt{(4\lambda^2 - y^2)}}{y}$$

Separating the variables and integrating, we get

$$\int dx = \int \frac{y\,dy}{\sqrt{(4\lambda^2 - y^2)}} + k \quad \text{or} \quad x = k - \sqrt{(4\lambda^2 - y^2)} \qquad ...(ii)$$

When $x = 0, y = 0 \quad \therefore \quad k = 2\lambda$

$\therefore$ (ii) becomes $(x - 2\lambda)^2 + y^2 = (2\lambda)^2$ which is a circle with centre $(2\lambda, 0)$ and radius 2λ.

Hence the figure formed by the revolution of given arc is a sphere.

PROBLEMS 35.2

1. Find a function $y(x)$ for which $\int_0^1 (x^2 + y'^2)\,dx$ is stationary, given that $\int_0^1 y^2\,dx = 2$; $y(0) = 0, y(1) = 0$. *(Madras, 2000 S)*
2. Find the extremals of the isoperimetric problem $v[y(x)] = \int_{x_0}^{x_1} y'^2\,dx$ given that $\int_{x_0}^{x_1} y\,dx = c$, a constant.
3. Show that the curve c of given length l which minimizes the curved surface area of the solid generated by the revolution of c about the x-axis is a catenary. *(V.T.U., 2000 S)*
4. Find the extremal of the functional $I = \int_0^\pi [(y')^2 - y^2]\,dx$ under the conditions $y(0) = 0, y(\pi) = 1$ and subject to the constraint $\int_0^\pi y\,dx = 1$.
5. Prove that the extremal of the isoperimetric problem $v[y(x)] = \int_1^4 y_2'\,dx, y(1) = 3, y(4) = 24$, subject to the condition $\int_1^4 y\,dx = 36$ is a parabola. *(Madras M.E., 2000)*

35.7 SEVERAL DEPENDENT VARIABLES

We now extend the variational problem of § 35.3 to a problem with several variables as functions of a single independent variable *i.e.*, *A necessary condition for*

$$I = \int_{x_1}^{x_2} f(x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_n')\,dx \qquad ...(1)$$

to be an extremum is that $\quad \dfrac{\partial \mathbf{f}}{\partial \mathbf{y}_i} - \dfrac{\mathbf{d}}{\mathbf{dx}}\left(\dfrac{\partial \mathbf{f}}{\partial \mathbf{y}_i'}\right) = \mathbf{0},\ i = 1, 2, \ldots, n$ ...(2)

Let $y_1, y_2, \ldots, y_n$ satisfy the boundary conditions $y_i(x_1) = y_{i1}, y_i(x_2) = y_{i2}$

Consider arbitrary functions $\eta_1(x), \eta_2(x), \ldots, \eta_n(x)$ which are all zero on the boundary *i.e.*,

$$\eta_i(x_1) = 0 = \eta_i(x_2)$$

Replacing $y_1, y_2, \ldots$ by $y_1 + \varepsilon_1\eta_1, y_2 + \varepsilon_2\eta_2, \ldots$ in (1), we get

$$I(\varepsilon) = \int_{x_1}^{x_2} f\,(x, y_1 + \varepsilon_1\eta_1, y_2 + \varepsilon_2\eta_2, \ldots, y_1' + \varepsilon_1\eta_1', y_2' + \varepsilon_2\eta_2', \ldots)\ dx \qquad \ldots(3)$$

This is a function of the parameters $\varepsilon_1, \varepsilon_2, \ldots$ and reduces to (1) for $\varepsilon_1 = \varepsilon_2 = \ldots = 0$.

To find the stationary value of (1), we find the stationary value of $I(\varepsilon)$ for $\varepsilon_1 = \varepsilon_2 = \ldots = 0$, $I(\varepsilon)$ will have a stationary value when

$$\frac{\partial I(\varepsilon)}{\partial \varepsilon_1} = 0,\ \frac{\partial I(\varepsilon)}{\partial \varepsilon_2} = 0, \ldots$$

Writing $\quad f = f\,(x, y_1, y_2, \ldots, y_1', y_2', \ldots)$

and $\quad F = f(x, y_1 + \varepsilon_1\eta_1, y_2 + \varepsilon_2\eta_2, \ldots, y_1' + \varepsilon_1\eta_1', y_2' + \varepsilon_2\eta_2', \ldots).$

(3) becomes $I(\varepsilon) = \displaystyle\int_{x_1}^{x_2} F\,dx.$

x_1, x_2 being constants independent of ε_1, differentiating under the integral sign, we get

$$\frac{\partial I(\varepsilon)}{\partial \varepsilon_1} = \int_{x_1}^{x_2} \frac{\partial F}{\partial \varepsilon_1}\,dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y_1}\eta_1 + \frac{\partial F}{\partial y_1}\eta_1'\right) dx$$

$\therefore \quad \dfrac{\partial I(\varepsilon)}{\partial \varepsilon_1} = 0$, when $\varepsilon_1 = \varepsilon_2 = \ldots = 0$ gives

$$\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y_1}\eta_1 + \frac{\partial f}{\partial y_1'}\eta_1'\right) = 0$$

Integrating by parts the second term, we get

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y_1}\eta_1 + \left|\frac{\partial f}{\partial y_1'}\eta_1\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y_1'}\right)\eta_1\,(x)\,dx = 0$$

i.e.,
$$\int_{x_1}^{x_2} \left\{\frac{\partial f}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_1'}\right)\right\}\eta_1(x)\,dx = 0 \qquad [\because\ \eta_1(x_1) = 0 = \eta_1(x_2)]$$

Since this equation must hold good for all values of $\eta_1(x)$, we get

$$\frac{\partial f}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_1'}\right) = 0$$

Similarly $\dfrac{\partial I(\varepsilon)}{\partial \varepsilon_2} = 0$ when $\varepsilon_1 = \varepsilon_2 = \ldots = 0$, will give

$$\frac{\partial f}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_2'}\right) = 0 \text{ and so on.}$$

All these conditions give a system of Euler's equation (2). A solution of these equations leads to the desired curves.

Example 35.13. *Show that the functional* $\displaystyle\int_0^{\pi/2} \left\{2xy + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} dt$ *such that* $x(0) = 0$, $x(\pi/2) = -1$, $y(0) = 0$, $y(\pi/2) = 1$ *is stationary for* $x = -\sin t$, $y = \sin t$.

Solution. Euler's equations are $\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial x'}\right) = 0$ ...(*i*)

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad ...(ii)$$

Here $f = 2xy + x'^2 + y'^2$. $\therefore$ $\frac{\partial f}{\partial x} = 2y,\ \frac{\partial f}{\partial x'} = 2x',\qquad \frac{\partial f}{\partial y} = 2x,\ \frac{\partial f}{\partial y'} = 2y'$

(*i*) becomes $2y - \frac{d}{dt}(2x') = 0$ *i.e.*, $2y - 2\frac{d^2x}{dt^2} = 0$ or $\frac{d^2x}{dt^2} = y$ ...(*iii*)

(*ii*) becomes $2x - \frac{d}{dt}(2y') = 0$ *i.e.*, $2x - 2\frac{d^2y}{dt^2} = 0$ or $\frac{d^2y}{dt^2} = x$ ...(*iv*)

Now to solve these simultaneous differential equations, we differentiate (*iii*) twice,

$$\frac{d^4x}{dt^4} = \frac{d^2y}{dt^2} = x \quad \text{[By (iv)]}$$

or $(D^4 - 1)\,x = 0$ which is a linear equation with constant coefficients.

Its solution is $x = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$ ...(*v*)

From (*iii*), $y = x'' = c_1 e^x + c_2 e^{-x} - c_3 \cos x - c_4 \sin x$ ...(*vi*)

Since $x = 0$ when $t = 0$ $\therefore$ $0 = c_1 + c_2 + c_3$ ...(*vii*)

$y = 0$ when $t = 0$ $\therefore$ $0 = c_1 + c_2 - c_3$ ...(*viii*)

Also $x = -1$ when $t = \pi/2$ $\therefore$ $-1 = c_1 e^{\pi/2} + c_2 e^{-\pi/2} + c_4$ ...(*ix*)

$y = 1$ when $t = \pi/2$ $\therefore$ $1 = c_1 e^{\pi/2} + c_2 e^{-\pi/2} - c_4$ ...(*x*)

Adding (*vii*) and (*viii*), $c_1 + c_2 = 0$

Adding (*ix*) and (*x*), $c_1 e^{\pi/2} + c_2 e^{-\pi/2} = 0$

Solving these equations, we get $c_1 = c_2 = 0$.

From (*viii*), $c_3 = c_1 + c_2 = 0$. From (*ix*), $c_4 = -1$.

Hence from (*v*), $x = -\sin x$ and from (*vi*), $y = \sin x$.

35.8 FUNCTIONALS INVOLVING HIGHER ORDER DERIVATIVES

A necessary condition for

$$I = \int_{x_1}^{x_2} f(x, y, y', y'')\, dx \quad ...(1)$$

to be extremum is $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0.$

Let $y(x)$ be the function which makes (1) stationary and satisfies the boundary conditions

$$y(x_1) = y_1, y(x_2) = y_2, y'(x_1) = y_1' \text{ and } y'(x_2) = y_2'.$$

Consider the differentiable function $\eta(x)$ such that

$$\eta(x_1) = 0 = \eta(x_2) \quad \text{and} \quad \eta'(x_1) = 0 = \eta'(x_2) \quad ...(2)$$

Replacing y by $y + \varepsilon\eta$ in (1), we get

$$I(\varepsilon) = \int_{x_1}^{x_2} f(x, y + \varepsilon\eta, y' + \varepsilon\eta', y'' + \varepsilon\eta'')\, dx \quad ...(3)$$

This is a function of the parameter ε and reduces to (1) for $\varepsilon = 0$.

To find the stationary value of (1), we find the stationary value of $I(\varepsilon)$ for $\varepsilon = 0$. But $I(\varepsilon)$ will have a stationary value when $dI(\varepsilon)/d\varepsilon = 0$.

Writing $f = f(x, y, y', y'')$ and $F = f(x, y + \varepsilon\eta, y' + \varepsilon\eta', y'' + \varepsilon\eta'')$.

(3) becomes $I(\varepsilon) = \int_{x_1}^{x_2} F\ dx$

x_1, x_2 being constants independent of ε, differentiating under the integral sign, we get

$$\frac{dI(\varepsilon)}{d\varepsilon} = \int_{x_1}^{x_2} \frac{dF}{d\varepsilon}\, dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' + \frac{\partial F}{\partial y''}\eta'' \right) dx$$

$\therefore$ $$\frac{dI(\varepsilon)}{d\varepsilon} = 0 \text{ when } \varepsilon = 0 \text{ gives } \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' + \frac{\partial f}{\partial y''}\eta'' \right) dx = 0$$

Integrating by parts once the second term and twice the third term, we get

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta\, dx + \left| \frac{\partial f}{\partial y'}\eta \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right).\eta\, dx + \left| \frac{\partial f}{\partial y''}\eta' - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right).\eta \right|_{x_1}^{x_2}$$

$$+ \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right).\eta\, dx = 0$$

or $$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) \right] \eta(x)\, dx = 0 \qquad \text{[By (2)]}$$

Since this equation must hold good for all values of $\eta(x)$, we get

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$$

In general, *a necessary condition for the functional* $I = \int_{x_1}^{x_2} f(x, y, y', y'', \ldots y^{(n)})\, dx$ *to be stationary will be*

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \ldots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^{(n)}}\right) = 0$$

which is called the *Euler-Poisson equation* and its solutions are called *extremals.*

Example 35.14. *Show that the curve which extremizes the functional* $I = \int_0^{\pi/4} (y''^2 - y^2 + x^2)\, dx$ *under the conditions* $y(0) = 0,\ y'(0) = 1,\ y(\pi/4) = y'(\pi/4) = 1/\sqrt{2}$ *is* $y = \sin x$. *(Madras M.E., 2000 S)*

Solution. The Euler-Poisson equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0 \qquad \ldots(i)$$

Here $$f = (y'')^2 - y^2 + x^2,\ \frac{\partial f}{\partial y} = -2y,\ \frac{\partial f}{\partial y'} = 0,\ \frac{\partial f}{\partial y''} = 2y''$$

$\therefore$ (i) becomes $-2y + \dfrac{d^2}{dx^2}(2y'') = 0$ or $y^{iv} - y = 0$ or $(D^4 - 1)\, y = 0$

Its solution is $y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$ $\ldots(ii)$

$\therefore$ $y'(x) = c_1 e^x - c_2 e^{-x} - c_3 \sin x + c_4 \cos x$ $\ldots(iii)$

Applying the given boundary conditions to (ii) and (iii), we get

$$0 = y(0) = c_1 + c_2 + c_3,\ 1 = y'(0) = c_1 - c_2 + c_4$$

$$\frac{1}{\sqrt{2}} = y(\pi/4) = c_1 e^{\pi/4} + c_2 e^{-\pi/4} + \frac{1}{\sqrt{2}}c_3 + \frac{1}{\sqrt{2}}c_4 \qquad \ldots(iv)$$

$$\frac{1}{\sqrt{2}} = y'(\pi/4) = c_1 e^{\pi/4} - c_2 e^{-\pi/4} - \frac{1}{\sqrt{2}}c_3 + \frac{1}{\sqrt{2}}c_4$$

Solving the equation (iv), we get $c_1 = c_2 = c_3 = 0,\ c_4 = 1$. Hence the required curve is $y = \sin x$.

PROBLEMS 35.3

1. Show that the functional $\int_0^1 \left\{2x + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} dt$, such that $x(0) = 1, y(0) = 1; x(1) = 1.5, y(1) = 1$ is stationary for $x = 1 + t^2/2, y = t$.
2. Find the extremals of the functional $v(y, z) = \int_{x_0}^{x_1} (2yz - 2y^2 + y'^2 - z'^2)\, dx$ (*V.T.U., M.E., 2006*)
3. Find a function $y(x)$ such that $\int_0^\pi y^2\, dx = 1$ which makes $\int_0^\pi y''^2\, dx$ a minimum if $y(0) = 0 = y(\pi), y''(0) = 0 = y''(\pi)$.
4. Find the extremals of the functional $\int_0^{\pi/2} (y''^2 - y^2 + x^2)\, dx$ that satisfies the conditions $y(0) = 1, y'(0) = 0, y(\pi/2) = 0, y'(\pi/2) = -1$. (*Madras, M.E. 2000 S*)
5. Find the extremals of the functional $\int_{-a}^{a} \left(\lambda y + \frac{1}{2}\mu y''^2\right) dx$ which satisfies the boundary conditions $y(-a) = 0, y'(-a) = 0, y(a) = 0, y'(a) = 0$.
6. Find the extremals of the following functionals :

   (*i*) $v[y(x)] = \int_{x_0}^{x_1} (16y^2 - y''^2 + x^2)\, dx$ (*Nagpur, 1997*) (*ii*) $v[y(x)] = \int_{x_0}^{x_1} (2xy + y'''^2)\, dx$.

35.9 APPROXIMATE SOLUTION OF BOUNDARY VALUE PROBLEMS — Rayleigh-Ritz Method

In § 35.3, we have seen that the solution of Euler's differential equation alongwith boundary conditions amounts to extremising a certain definite integral. This fact provides a technique of solving a boundary value problem approximately by assuming a trial solution satisfying the given boundary conditions and then extremising the integral whose integrand is found from the given differential equation.

To solve a boundary value problem of Rayleigh-Ritz method, we try to write the given differential equation as the Euler's equation of some variational problem. Then we reduce this variational problem to a minimizing problem assuming an approximate solution in the form

$$\bar{y}(x) = y_0(x) + \Sigma c_i\, \phi_i(x) \qquad ...(1)$$

where the *trial functions* $\phi_i(x)$ satisfy the boundary conditions and $\phi_i(x) = 0$ on the boundary C of its region R.

Let the integral to be extremised be $I = \int_a^b f(y, y', x)\, dx$ ...(2)

such that $y(a) = A$ and $y(b) = B$.

Substituting (1) in (2) by replacing y in $\bar{y}$ in I, giving $\bar{I}$ as a function of the unknowns c_i. Then c's become parameters which are so determined as to extremise $\bar{I}$. This requires

$$\frac{\partial \bar{I}}{\partial c_i} = 0, \qquad i = 1, 2, ...$$

Solving these equations, we get the values of c_i, which when substituted in (1) give the desired solution.

Example 35.15. *Solve the boundary value problem $y'' - y + x = 0$ $(0 \le x \le 1)$, $y(0) = y(1) = 0$ by Rayleigh-Ritz method.*

Solution. Given differential equation is $y'' - y + x = 0$ ...(*i*)

Its solution is equivalent to extremising the integral $I = \int_0^1 F(x, y, y')\, dx$

where $$F(x, y, y') = 2xy - y^2 - y'^2, \qquad ...(ii)$$

since the Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ gives (*i*).

Assume that the *trial function* is $\bar{y}(x) = c_0 + c_1 x + c_2 x^2$ ...(*iii*)

To satisfy $y(0) = 0, y(1) = 0$, we require $c_0 = 0, c_2 = -c_1$.

$\therefore$ (*iii*) becomes $\quad \bar{y}(x) = c_1 x(1-x)$ ...(*iv*)

Substituting $\bar{y}$ and $\bar{y}'$ in I, we have

$$\bar{I} = \int_0^1 [2x\bar{y} - \bar{y}^2 - (\bar{y}')^2]\,dx = \int_0^1 [2c_1(x^2 - x^3) - c_1^2(x-x^2)^2 - c_1^2(1-x)^2]\,dx$$

$$= \frac{1}{6}c_1 - \frac{11}{30}c_1^2$$

Its stationary values are given by $dI/dc_1 = 0$. $\quad \therefore \quad \frac{1}{6} - \frac{11}{15}c_1 = 0 \quad$ *i.e.*, $\quad c_1 = \frac{5}{22}$.

Thus the approximate solution is $\bar{y}(x) = \frac{5}{22}x(1-x)$ ...(*v*)

35.10 WEIGHTED RESIDUAL METHOD—Galerkin's Method

The starting point of this method is to guess a solution to the differential equation which satisfies the boundary conditions. This *trial solution* will contain certain parameters which can be adjusted to minimize the errors so that the trial solution is as close to the exact solution as possible.

Consider the boundary value problem

$$y'' = f(y, y', x) \text{ with } y(a) = A \text{ and } y(b) = B \qquad ...(1)$$

We write the differential equation as $R = \bar{y}'' - f(\bar{y}, \bar{y}', x)$ ...(2)

where R is the residual of the equation ($R = 0$ for the exact solution $y(x)$ only which will satisfy the boundary conditions).

Consider the trial solution as $\bar{y}(x) = c_1\,\phi_1(x) + c_2\,\phi_2(x) + \ldots$

where $\bar{y}(a) = A$ and $\bar{y}(b) = B$. The trial solution is differentiated twice and is substituted in (2).

To find $c_1, c_2, \ldots$, we weight the residual by trial functions $\phi_1(x), \phi_2(x), \ldots$ and set the integrals to zero. Thus we have $\int_c R\,\phi_1(x)\,dx = 0, \int_c R\,\phi_2(x)\,dx = 0, \ldots$

These lead to simultaneous equations in the unknowns.

Having found $c_1, c_2, \ldots$, the approximate solution $\bar{y}(x)$ is obtained.

Example 35.16. *Use Galerkin's method to solve the boundary value problem of Example 35.14. Compare your approximate solution with the exact solution.*

Solution. The residual is $\qquad R = \bar{y}'' - \bar{y} + x$ ...(*i*)

To find the trial solution which satisfies the boundary conditions, we derive a Lagrangian polynomial (§ 28.8) which passes through the points :

x	:	0	1/2	1
y	:	0	c	0

The resulting polynomial is $\bar{y}(x) = 4cx(1-x)$, so that $\phi(x) = x(1-x)$.

Substituting $\bar{y}(x), \bar{y}''(x)$ in (*i*), we get $R = 4cx^2 + (1-4c)x - 8c$

Thus $\quad \int R\,\phi(x)\,dx = 0$ gives $\int_0^1 [4cx^2 + (1-4c)x - 8c]\,x(1-x)\,dx = 0$ whence $c = 5/88$.

Hence the approximate solution is $\bar{y}(x) = \frac{5}{22}x(1-x)$ which is same as found in Example 34.14.

Exact solution. Rewriting the given equation as $(D^2 - 1)y = -x$,

we find its solution as $y = c_1 e^x + c_2 e^{-x} + x$

Since $y(0) = 0$ and $y(1) = 0$, therefore $c_2 = -c_1 = 1/(e - e^{-1})$.

Hence the exact solution is $y = x - \dfrac{e^x - e^{-x}}{e - e^{-1}}$

The approximate and the exact solutions for some values of x are given below for comparison :

x	*Approx. Sol.*	*Exact Sol.*
0.25	0.043	0.035
0.50	0.057	0.057
0.75	0.043	0.05

Obs. To obtain a trial solution containing two unknown parameters, we derive a Lagrangian polynomial which passes through the points :

x	:	0	1/3	2/3	1
y	:	0	c_1	c_2	0

More the undetermined parameters, the more accurate is the solution, but it involves more labour to find their values.

Example 35.17. *Find the approximate deflection of a simply supported beam under a uniformly distributed load w Fig. 35.7, using Galerkin's method.*

Solution. The differential equation governing the deflection $y(x)$ of the beam is

$$EI\frac{d^4y}{dx^4} - w = 0,\ 0 < x < l \qquad (i)\ [\S\ 14.8]$$

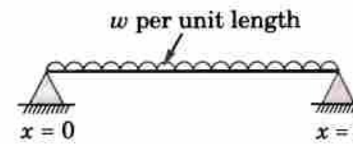

Fig. 35.7

The boundary conditions to be satisfied are

$$y\ (x = 0) = y\ (x = l) = 0 \quad \text{(deflection is zero at ends)} \qquad ...(ii)$$

$$\frac{d^2y}{dx^2}(x = 0) = \frac{d^2y}{dx^2}(x = l) = 0 \quad \text{(bending moment zero at ends)} \qquad ...(iii)$$

We assume the trial solution $\bar{y}(x) = c_1 \sin(\pi x/l) + c_2 \sin(3\pi x/l)$, which satisfies the boundary conditions (ii) and (iii).

Substituting the trial solution in (i), we obtain the residual

$$R = EIc_1(\pi/l)^4 \sin(\pi x/l) + EIc_2(3\pi/l)^4 \sin(3\pi x/l) - w$$

Thus $\quad \int_0^l R\ .\ \sin(\pi x/l) = 0 \quad$ and $\quad \int_0^l R\ .\ \sin(3\pi x/l) = 0$

Computing these integrals, we get

$$EIc_1(\pi/l)^4\ l/2 - w\ .\ 2l/\pi = 0,\ EIc_2(3\pi/l)^4\ l/2 - w\ .\ 2l/3\pi = 0$$

Solving these, we obtain $\quad c_1 = \dfrac{4wl^4}{\pi^5 EI}$ and $c_2 = \dfrac{4wl^4}{243\pi^5 EI}$.

Hence the deflection of the beam is given by

$$\bar{y}(x) = \frac{4wl^4}{\pi^5 EI}\left\{\sin\left(\frac{\pi x}{l}\right) + \frac{1}{243}\sin\left(\frac{3\pi x}{l}\right)\right\}.$$

PROBLEMS 35.4

1. Solve the boundary value problem :
   $$y'' + y + x = 0\ (0 \le x \le 1),\ y(0) = y(1) = 0 \text{ by}$$
   (i) Rayleigh-Ritz method, $\quad (ii)$ Galerkin's method. Compare your solution with the exact solution.
2. Using Galerkin's method, solve the boundary value problem $y'' = 3x + 4y$; $y(0) = 0$, $y(1) = 1$.
3. *Apply Galerkin's method to the boundary value problem* $y'' + y + x = 0$ $(0 \le x \le 1)$; $y(0) = y(1) = 0$, *to find the coefficients of the approximate solution* $\bar{y}(x) = c_1x(1 - x) + c_2x^2(1 - x)$.

[**Hint.** Substituting $\bar{y}(x)$, $\bar{y}''(x)$ in the given equation replacing y, y'' by $\bar{y}, y''$, we get the residual $R = -2c_1 + c_2 (2-6x) + x(1-x)(c_1 + c_2 x) + x$

Thus $$\int_0^1 R \,.\, x(1-x)\,dx = 0 \text{ and } \int_0^1 R \,.\, x^2(1-x)\,dx = 0.$$

Computing these integrals, we get

$$\frac{3}{10}c_1 + \frac{3}{20}c_2 = \frac{1}{12}, \frac{3}{20}c_1 + \frac{13}{305}c_2 = \frac{1}{20}.$$

Solving these, we obtain $c_1 = 71/369$, $c_2 = 7/41$.]

4. Using Ritz method, find an approximate solution of the problem $y'' - y + 4xe^x = 0$, $y'(0) - y(0) = 1$, $y'(1) + y(1) = -e$.
5. Solve the boundary value problem : $y'' + (1 + x^2)y + 1 = 0$, $y(-1) = y(1) = 0$, by taking the approximate solution $\bar{y}(x) = c_1(1-x^2) + c_2x^2(1-x^2)$ and using (*i*) Ritz method, (*ii*) Galerkin's method.
6. Given the boundary value problem : $y'' + \pi^2 y = x$, $y(0) = 1$, $y(1) = -0.9$.

   Use Galerkin's method to estimate $y(0.5)$, taking the trial solution :
   $$y = 1 - 1.9x + c_1 x(1-x) + c_2 x^2(1-x).$$
7. Using Galerkin's method, obtain an approximate solution of the boundary value problem :
   $$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + y = x, \; y(0) = 0, \; y(1) = 1,$$
   in the form $\bar{y}(x) = x + x(1-x)(c_1 + c_2 x)$.
8. Of all the parabolas which pass through (0, 0) and (1, 1), determine the one, which when rotated about the x-axis, generates a solid of revolution with least possible volume between $x = 0$ and $x = 1$.

   [**Hint.** Take the parabola as $y = x + cx(1-x)$.]
9. *Using Rayleigh-Ritz method, find the potential at any point due to a charged sphere of radius a.*

   [**Hint.** Potential at a distance r from the centre of the sphere is $\phi = \phi_0 (r/a)^k$, where ϕ_0 is the value of ϕ for $r = a$ and $k < 0$ so that $\phi \to 0$ as $r \to \infty$.

   Electrostatic field due to charged sphere being *conservative*, electrostatic intensity $E = \nabla\phi$.

   Also potential energy for unit volume $= \frac{1}{8\pi}E^2$

   $\therefore$ Total potential energy of the field in the entire region R exterior to the given sphere is

   $$V = \frac{1}{8\pi}\iiint_R E^2\,dx\,dy\,dz = \frac{1}{8\pi}\iiint_R \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] dx\,dy\,dz$$
   $$= \frac{1}{8\pi}\int_a^\infty \int_0^\pi \int_0^{2\pi} \left(\frac{\partial\phi}{\partial r}\right)^2 r^2 \sin\theta\,d\phi\,d\theta\,dr.$$

   Electrostatic field will be in stable equilibrium if V is minimum, *i.e.*, $dV/dp = 0$ and $d^2V/dp^2 > 0$.

   This gives $k = -1$. Hence $\phi = \phi_0\, a/r$.]

35.11 HAMILTON'S PRINCIPLE*

An important concept of mathematical physics is *Hamilton's principle* which states that *the time integral of the difference between the kinetic and potential energies of a dynamical system is stationary*

Consider a particle of mass m moving from a fixed origin O under the effect of a force F (Fig. 35.8). At any time t, let its position vector be $\mathbf{R}$. Then by Newton's second law,

$$\frac{md^2\mathbf{R}}{dt^2} = \mathbf{F} \qquad \text{...(1)}$$

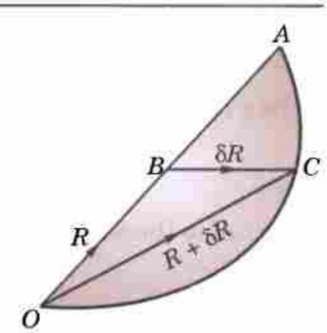

Fig. 35.8

* Named after the Irish mathematician *William Rowan Hamilton* (1805–1865) who is known for his contributions to dynamics.

Let the natural path OBA of the particle be changed to another path OCA, end points remaining the same. Let this variation in path, often called *virtual displacement*, be $\delta\mathbf{R}$. Then the work done during this displacement is

$$\delta W = \mathbf{F} \cdot \delta\mathbf{R} = \frac{md^2\mathbf{R}}{dt^2} \cdot \delta\mathbf{R} \qquad \text{[By (1)]}$$

Also the kinetic energy of the particle is

$$T = \frac{1}{2} m \left(\frac{d\mathbf{R}}{dt}\right)^2$$

$$\therefore \qquad \delta T = m \frac{d\mathbf{R}}{dt} \cdot \delta\left(\frac{d\mathbf{R}}{dt}\right) = m \frac{d\mathbf{R}}{dt} \cdot \frac{d}{dt}(\delta\mathbf{R})$$

Thus $$\delta T + \delta W = m \frac{d\mathbf{R}}{dt} \cdot \frac{d}{dt}(\delta\mathbf{R}) + m \frac{d^2\mathbf{R}}{dt^2} \cdot \delta\mathbf{R} = m \frac{d}{dt}\left(\frac{d\mathbf{R}}{dt} \cdot \delta\mathbf{R}\right)$$

Integrating both sides w.r.t. t from t_0 to t_1, we get

$$\int (\delta T + \delta W)\, dt = m \left| \frac{d\mathbf{R}}{dt} \cdot \delta\mathbf{R} \right|_{t_0}^{t_1} = 0 \qquad \text{...(2)} \quad [\because \ \delta\mathbf{R} = 0 \text{ at } t_0 \text{ and } t_1]$$

If the force field is conservative, there exists a potential V such that $W = -V$. Then (2) takes the form

$$\int (\delta T - \delta V)\, dt = 0 \quad \text{or} \quad \delta \int (T - V)\, dt = 0 \quad i.e., \quad \int (T - V)\, dt \qquad \text{...(3)}$$

is stationary. This proves the Hamilton's principle for a particle. Its derivation can be extended to a system of particles by summation and to a rigid body by integration. Hence the principle is true for any dynamical system.

Obs. It can be easily shown that *the integral (3) is a minimum along the natural path* as compared to that along any other path joining O to A.

Def. *The energy difference $T - V = L$ is called the* **kinetic potential** *or* the **Lagrangian function.**

35.12 LAGRANGE'S EQUATION

In a dynamical system, the position of a body can be specified by the quantities $q_1, q_2, \dots q_n$ which are called the *generalised coordinates.*

The potential energy V, being a function of position only depends on these generalised coordinates q_i. The kinetic energy T, however, depends upon q_i and the velocities dq_i/dt (*i.e.*, $\dot{q}_i$) $i = 1, 2, \dots n$. Therefore, Lagrangian function $L = T - V$ is also a function of q_1 and $\dot{q}_i$, $i = 1, 2, \dots n$.

Thus by Hamilton's principle, the system moves so that $\int_{t_0}^{t_1} L\, dt$ is stationary.

$\therefore$ Euler's equation must hold good, *i.e.*, $\dfrac{\partial L}{\partial q_i} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) = 0, i = 1, 2, \dots n.$

These are called *Lagrange's equations* which determine the motion of the system.

Example 35.18. *A mass, suspended at the end of a light spring having spring constant k, is set into vertical motion. Use Lagrange's equation, to find the equation of motion of the mass.*

Solution. At any time t, let the displacement of m from the equilibrium position O be x (Fig. 35.9). Then the kinetic energy of P is

$$T = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2$$

Also the work done during its fall from O to P is

$$W = \int_0^x (mg - kx)\, dx = mgx - \tfrac{1}{2} kx^2.$$

If V is the potential energy of the mass at P, then

$$V = -W = \tfrac{1}{2} kx^2 - mgx$$

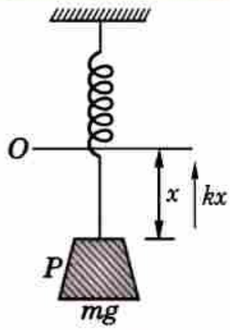

Fig. 35.9

$\therefore$ Lagrangian $\qquad L = T - V = \frac{1}{2} m\dot{x}^2 + mgx - \frac{1}{2}kx^2.$

Thus the Lagrange's equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

becomes $\qquad (mg - kx) - \frac{d}{dt}(m\dot{x}) = 0 \quad \text{or} \quad m\frac{d^2x}{dt^2} = mg - kx$

which is the required equation of motion.

Example 35.19. *Apply Lagrange's equations, to show that the equations of motion of the double pendulum of Fig. 35.10 are given by*

$$(m_1 + m_2)\, l_1 \ddot{\theta}_1 + m_2\, l_2 \ddot{\theta}_2 + (m_1 + m_2)\, g\theta_1 = 0$$

and $\qquad l_1\ddot{\theta}_1 + l_2\ddot{\theta}_2 + g\theta_2 = 0$

for small angles θ_1, θ_2. *(Punjab, M.E., 1997)*

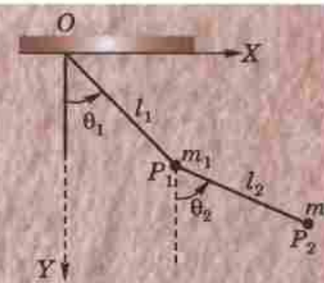

Fig. 35.10

Solution. At any time t, let the masses m_1, m_2 be at $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ (Fig. 35.10) so that

$$\left.\begin{aligned} x_1 &= l_1 \sin\theta_1,\ y_1 = l_1\cos\theta_1 \\ x_2 &= l_1\sin\theta_1 + l_2\sin\theta_2,\ y_2 = l_1\cos\theta_1 + l_2\cos\theta_2 \end{aligned}\right\} \qquad ...(i)$$

Then total kinetic energy is

$$T = \tfrac{1}{2}m_1(\dot{x}_1^{\,2} + \dot{y}_1^{\,2}) + \tfrac{1}{2}m_2(\dot{x}_2^{\,2} + \dot{y}_2^{\,2})$$

$$= \tfrac{1}{2}(m_1 + m_2)\, l_1^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) \qquad \text{[Using } (i)]$$

Also total potential energy is

$$V = m_1 g l_1 (1 - \cos\theta_1) + m_2 g(l_1 + l_2 - l_1\cos\theta_1 - l_2\cos\theta_2)$$

$\therefore$ Lagrangian $\qquad L = T - V = \frac{1}{2}(m_1 + m_2)\, l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2$

$$+ m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) - (m_1 + m_2)\, g l_1(1 - \cos\theta_1) - m_2 g l_2(1 - \cos\theta_2) \qquad ...(ii)$$

Thus the Lagrange's equation corresponding to θ_1, is

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = 0, \text{ becomes}$$

$$-m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - (m_1 + m_2)\, g l_1\sin\theta_1 - \frac{d}{dt}\,[(m_1 + m_2)\, l_1^2\,\dot{\theta}_1 + m_2 l_1 l_2\dot{\theta}_2\cos(\theta_1 + \theta_2)] = 0$$

or $\qquad (m_1 + m_2)\, l_1\ddot{\theta}_1 + m_2\, l_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2 l_2\dot{\theta}_2^{\,2}\sin(\theta_1 - \theta_2) + (m_1 + m_2)\, g\sin\theta_1 = 0 \qquad ...(iii)$

Similarly from (ii), Lagrange's equation corresponding to θ_2, *i.e.*,

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \theta_2}\right) = 0$$

becomes $\qquad m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - m_2\, g l_2\sin\theta_2 - \frac{d}{dt}[m_2 l_2^2\,\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1\cos(\theta_1 - \theta_2)] = 0$

or $\qquad l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1\cos(\theta_1 - \theta_2) - l_1\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + g\sin\theta_2 = 0 \qquad ...(iv)$

Now $\dot{\theta}_1$ and $\dot{\theta}_2$ being small, retaining first order terms only, (iii) and (iv) reduce to

$$(m_1 + m_2)\, l_1\ddot{\theta}_1 + m_2\, l_2\ddot{\theta}_2 + (m_1 + m_2)\, g\theta_1 = 0 \quad \text{and} \quad l_1\ddot{\theta}_1 + l_2\ddot{\theta}_2 + g\theta_2 = 0.$$

PROBLEMS 35.5

1. In a single pendulum, a mass m is suspended by a light rod of length l and the system vibrates in a plane. Using Lagrange's equation, show that $\ddot{\theta} + (g/l) \sin \theta = 0$.

   Show that if θ is small, the period of oscillation is $2\pi \sqrt{(l/g)}$.

2. Two masses m_1 and m_2 are connected by an inextensible string which passes over a fixed pulley. Using Lagrange's equations, show that the acceleration of either mass is numerically

$$= (m_1 - m_2)\, g/(m_1 + m_2).$$

3. A perfectly flexible rope of uniform density per unit length is suspended with its end points fixed. Show that it assumes the shape of a catenary.

4. A bead of mass m from rest slides without friction under gravity along a wire inclined at an angle α to the vertical and rotating with constant angular velocity ω. Show that in times t, the bead has slided through a distance

$$\frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} \cosh (\omega t \sin \alpha - 1).$$