CHAPTER 29

Finite Differences and Interpolation

1. Finite differences. 2. Differences of a polynomial. 3. Factorial notation 4. Relations between the operators. 5. To find one or more missing terms. 6. Newton's interpolation formulae. 7. Central difference interpolation formulae—Gauss's interpolation formulae ; Stirling's formula ; Bessel's formula ; Everett's formula. 8. Choice of an interpolation formula. 9. Interpolation with unequal intervals. 10. Lagrange's formula. 11. Divided differences. 12. Newton's divided difference formula. 13. Inverse interpolation. 14. Objective Type of Questions.

29.1 FINITE DIFFERENCES

Suppose we are given the following values of $y = f(x)$ for a set of values of x :

x :	x_0	x_1	$x_2 \ldots x_n$.
y :	y_0	y_1	$y_2 \ldots y_n$.

Then the process of finding the values of y corresponding to any value of $x = x_i$ between x_0 and x_n is called *interpolation*. Thus *interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable* while the process of computing the value of the function outside the given range is called *extrapolation*. The study of the interpolation is based on the concept of differences of a function which we proceed to discuss. For a detailed study, the reader should refer to author's book *'Numerical Methods in Engineering and Science'.*

Suppose that the function $y = f(x)$ is tabulated for the equally spaced values $x = x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$ giving $y = y_0, y_1, y_2, \ldots, y_n$. To determine the values of $f(x)$ or $f'(x)$ for some intermediate values of x, the following three types of differences are found useful :

(1) **Forward differences.** The differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1}$ respectively are called the *first forward differences* where Δ is the *forward difference operator*. Thus the first forward differences are $\Delta y_r = y_{r+1} - y_r$.

Similarly, the second forward differences are defined by

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$$

In general, $$\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$$

defines the *pth forward differences.*

These differences are systematically set out as follows in what is called a *Forward Difference Table.*

In a difference table, x is called the *argument* and y the *function* or the *entry* y_0, the first entry is called the *leading term* and $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ etc. are called the *leading differences.*

Obs. Any higher order forward difference can be expressed in terms of the entries.

We have $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0) = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

The coefficients occurring on the right hand side being the binomial coefficient, we have in general,

$$\Delta^n y_0 = y_n - {}^nc_1 y_{n-1} + {}^nc_2 y_{n-2} - \ldots + (-1)^n y_0$$

Forward Difference Table

Value of x	*Value of y*	*1st. diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		Δy_0				
$x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_0 + 5h$	y_5					

(2) Backward differences. The differences $y_1 - y_0, y_2 - y_1, ..., y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, ..., \nabla y_n$ respectively, are called the *first backward differences* where ∇ is the *backward difference operator*. Similarly we define higher order backward differences. Thus we have

$$\nabla y_r = y_r - y_{r-1}, \nabla^2 y_r = \nabla y_r - \nabla y_{r-1},$$
$$\nabla^3 y_r = \nabla^2 y_r - \nabla^2 y_{r-1} \text{ etc.}$$

The differences are exhibited in the following :

Backward Difference Table

Value of x	*Value of y*	*1st. diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		∇y_1				
$x_0 + h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
$x_0 + 2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
$x_0 + 3h$	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
$x_0 + 4h$	y_4		$\nabla^2 y_5$			
		∇y_5				
$x_0 + 5h$	y_5					

(3) Central differences. Sometimes it is convenient to employ another system of differences known as *central differences.* In this system, the *central difference operator* δ is defined by the relations :

$$y_1 - y_0 = \delta y_{1/2}, y_2 - y_1 = \delta y_{3/2}, ..., y_n - y_{n-1} = \delta y_{n-1/2}$$

Similarly, higher order central differences are defined as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1, \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2, ...,$$
$$\delta^2 y_2 - \delta^2 y_1 = \delta^3 y_{3/2} \text{ and so on.}$$

These differences are shown in the following :

Central Difference Table

Value of x	*Value of y*	*1st. diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		$\delta y_{1/2}$				
$x_0 + h$	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
$x_0 + 2h$	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
$x_0 + 3h$	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
$x_0 + 4h$	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
$x_0 + 5h$	y_5					

We see from this table that the central differences on the same horizontal line have the same suffix. Also the differences of odd order are known only for half values of the suffix and those of even order for only integral values of the suffix.

It is often required to find the mean of adjacent values in the same column of differences. We denote this mean by μ. Thus

$$\mu\delta y_1 = \tfrac{1}{2}(\delta y_{1/2} + \delta y_{3/2}),\ \mu\delta^2 y_{3/2} = \tfrac{1}{2}(\delta^2 y_1 + \delta^2 y_2) \text{ etc.}$$

Obs. The reader should note that it is only the notation which changes and not the differences.

$$y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{1/2}.$$

Of all the interpolation formulae, those involving central differences are most useful in practice as the coefficients in such formulae decrease much more rapidly.

Example 29.1. *Evaluate (i) $\Delta \tan^{-1} x$ (ii) $\Delta(e^x \log 2x)$ (iii) $\Delta(x^2/\cos 2x)$ (iv) $\Delta^2 \cos 2x$.* (P.T.U., 2001)

Solution. (*i*) $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$

$$= \tan^{-1}\left\{\frac{x+h-x}{1+(x+h)x}\right\} = \tan^{-1}\left\{\frac{h}{1+hx+x^2}\right\}$$

(*ii*)
$$\Delta(e^x \log 2x) = e^{x+h}\log 2(x+h) - e^x \log 2x$$
$$= e^{x+h}\log 2(x+h) - e^{x+h}\log 2x + e^{x+h}\log 2x - e^x \log 2x$$
$$= e^{x+h}\log\frac{x+h}{x} + (e^{x+h} - e^x)\log 2x$$
$$= e^x\left[e^h \log\left(1+\frac{h}{x}\right) + (e^h - 1)\log 2x\right]$$

(*iii*)
$$\Delta\left(\frac{x^2}{\cos 2x}\right) = \frac{(x+h)^2}{\cos 2(x+h)} - \frac{x^2}{\cos 2x} = \frac{(x+h)^2\cos 2x - x^2\cos 2(x+h)}{\cos 2(x+h)\cos 2x}$$
$$= \frac{[(x+h)^2 - x^2]\cos 2x + x^2[\cos 2x - \cos 2(x+h)]}{\cos 2(x+h)\cos 2x}$$
$$= \frac{(2hx+h^2)\cos 2x + 2x^2\sin(h)\sin(2x+h)}{\cos 2(x+h)\cos 2x}$$

(*iv*)
$$\Delta^2 \cos 2x = \Delta\{\cos 2(x+h) - \cos 2x\}$$
$$= \Delta\cos 2(x+h) - \Delta\cos 2x$$
$$= [\cos 2(x+2h) - \cos 2(x+h)] - [\cos 2(x+h) - \cos 2x]$$
$$= -2\sin(2x+3h)\sin h + 2\sin(2x+h)\sin h$$
$$= -2\sin h\,[\sin(2x+3h) - \sin(2x+h)]$$
$$= -2\sin h\,[2\cos(2x+2h)\sin h] = -4\sin^2 h\cos(2x+2h).$$

Example 29.2. *Evaluate (i) $\Delta^2\left(\dfrac{5x+12}{x^2+5x+16}\right)$* (Mumbai, 2003) *(ii) $\Delta^2(ab^x)$ (iii) $\Delta^n(e^x)$ interval of differencing being unity.* (Rohtak, 2003)

Solution. (*i*)
$$\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right) = \Delta^2\left\{\frac{5x+12}{(x+2)(x+3)}\right\} = \Delta^2\left\{\frac{2}{x+2} + \frac{3}{x+3}\right\}$$
$$= \Delta\left\{\Delta\left(\frac{2}{x+2}\right) + \Delta\left(\frac{3}{x+3}\right)\right\} = \Delta\left\{2\left(\frac{1}{x+3} - \frac{1}{x+2}\right) + 3\left(\frac{1}{x+4} - \frac{1}{x+3}\right)\right\}$$
$$= -2\Delta\left\{\frac{1}{(x+2)(x+3)}\right\} - 3\Delta\left\{\frac{1}{(x+3)(x+4)}\right\}$$
$$= -2\left\{\frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)}\right\} - 3\left\{\frac{1}{(x+4)(x+5)} - \frac{1}{(x+3)(x+4)}\right\}$$

$$= \frac{4}{(x+2)(x+3)(x+4)} + \frac{6}{(x+3)(x+4)(x+5)} = \frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}$$

(*ii*) $\Delta(ab^x) = a\,\Delta(b^x) = a(b^{x+1} - b^x) = ab^x(b-1)$

$\Delta^2(ab^x) = \Delta[\Delta ab^x] = a(b-1)\,\Delta(b^x)$

$= a(b-1)(b^{x+1} - b^x) = a(b-1)^2 - b^x.$

(*iii*) $\Delta e^x = e^{x+1} - e^x = (e-1)e^x$

$\Delta^2 e^x = \Delta(\Delta e^x) = \Delta[(e-1)\,e^x]$

$= (e-1)\,\Delta e^x = (e-1)(e-1)\,e^x = (e-1)^2\,e^x$

Similarly $\Delta^3 e^x = (e-1)^3 e^x, \Delta^4 e^x = (e-1)^4 e^x, \ldots$ and $\Delta^n e^x = (e-1)^n e^x$.

29.2 DIFFERENCES OF A POLYNOMIAL

The nth differences of a polynomial of the nth degree are constant and all higher order differences are zero.

Let the polynomial of the nth degree in x, be

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \ldots + k(x+h) + l$$

$\therefore$ $\Delta f(x) = f(x+h) - f(x)$

$= a[(x+h)^n - x^n] + b[(x+h)^{n-1} - x^{n-1}] + \ldots + kh$

$= anhx^{n-1} + b'x^{n-2} + c'x^{n-3} + \ldots + k'x + l'$ ...(1)

where $b', c', \ldots, l'$ are new constant coefficients.

Thus the first differences of a polynomial of the nth degree is a polynomial of degree $(n-1)$.

Similarly $\Delta^2 f(x) = \Delta[f(x+h) - f(x)] = \Delta f(x+h) - \Delta f(x)$

$= anh[(x+h)^{n-1} - x^{n-1}] + b'\,[(x+h)^{n-2} - x^{n-2}] + \ldots + k'h$

$= an(n-1)\,h^2 x^{n-2} + b''x^{n-3} + c''\,x^{n-4} + \ldots + k''$, [by (1)]

$\therefore$ the second differences represent a polynomial of degree $(n-2)$.

Continuing this process, for the nth differences we get a polynomial of degree zero *i.e.*

$$\Delta^n f(x) = an(n-1)(n-2)\ldots 1\,.\,h^n = an\,!\,h^n \qquad \ldots(2)$$

which is a constant. Hence the $(n+1)$th and higher differences of a polynomial of nth degree will be zero.

Obs. The converse of this theorem is also true *i.e. if the nth differences of a function tabulated at equally spaced intervels are constant, the function is a polynomial of degree n.* This fact is important in numerical analysis as it enables us to approximate a function by a polynomial of nth degree, if its nth order differences become nearly constant.

Example 29.3. *Evaluate* $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$.

Solution. $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)] = \Delta^{10}[abcd\,x^{10} + (\ )\,x^9 + (\ )\,x^8 + \ldots + 1]$

$= abcd\,\Delta^{10}(x^{10})$ $[\because\ \Delta^{10}(x^n) = 0 \text{ for } n < 10]$

$= abcd\,(10\,!).$ [by (2) above]

29.3 (1) FACTORIAL NOTATION

A product of the form $x(x-1)(x-2)\ldots(x-r+1)$ is denoted by $[x]^r$ and is called a **factorial.**

In particular $[x] = x,\ [x]^2 = x(x-1)$

$[x]^3 = x(x-1)(x-2)$, etc.

In general $[x]^n = x(x-1)(x-2)\ldots(x-n+1)$

In case, the interval of differencing is h, then

$$[x]^n = x(x-h)(x-2h)\ldots(x - \overline{n-1}h)$$

which is called a *Factorial polynomial or function.*

The factorial notation is of special utility in the theory of finite differences. It helps in finding the successive differences of a polynomial directly by simple rule of differentiation.

The result of differencing $[x]^r$ is analogous to that of differentiating x^r.

(2) To express a polynomial in the factorial notation

(*i*) *arrange the coefficients of the powers of x in descending order, replacing missing powers by zeros ;*

(*ii*) *using detached coefficients divide by x, x – 1, x – 2, etc. successively.*

Obs. Every polynomial of degree n can be expressed as a factorial polynomial of the same degree and *vice versa*.

Example 29.4. *Express* $y = 2x^3 - 3x^2 + 3x - 10$ *in a factorial notation and hence show that* $\Delta^3 y = 12$.

(*Bhopal, 2007 ; P.T.U., 2005*)

Solution. First method : Let $y = A[x]^3 + B[x]^2 + C[x] + D$.

Then

	x^3	x^2	x	
1	2	-3	3	$-10 = D$
	—	2	-1	
2	2	-1	$2 = C$	
	—	4		
3	2	$3 = B$		
	—			
	$2 = A$			

Hence $\qquad y = 2[x]^3 + 3[x]^2 + 2[x] - 10$

$\therefore \qquad \Delta y = 2 \times 3[x]^2 + 3 \times 2[x] + 2$

$\qquad \Delta^2 y = 6 \times 2[x] + 6$

$\qquad \Delta^3 y = 12$, which shows that the third differences of y are constant, as they should be.

Obs. *The coefficient of the highest power of x remains unchanged while transforming a polynomial to factorial notation.*

Second method (*Direct method*) :

Let
$$y = 2x^3 - 3x^2 + 3x - 10$$
$$= 2x(x-1)(x-2) + Bx(x-1) + Cx + D$$

Putting $x = 0, -10 = D$

Putting $x = 1, 2 - 3 + 3 - 10 = C + D$

$\therefore \quad C = -8 - D = -8 + 10 = 2$

Putting $x = 2, 16 - 12 + 6 - 10 = 2B + 2C + D$

$$\therefore \qquad B = \frac{1}{2}(-2C - D) = \frac{1}{2}(-4 + 10) = 3.$$

Hence
$$y = 2x(x-1)(x-2) + 3x(x-1) + 2x - 10 = 2[x]^3 + 3[x]^2 + 2[x] - 10$$
$$\therefore \qquad \Delta y = 2 \times 3[x]^2 + 3 \times 2[x] + 2, \Delta^2 y = 6 \times 2[x] + 6, \Delta^3 y = 12.$$

Example 29.5. *Find the missing values in the following table :*

x :	45	50	55	60	65
y :	3.0	—	2.0	—	– 2.4

(*Bhopal, 2007 ; V.T.U., 2001*)

Solution. The difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	$y_0 = 3$			
		$y_1 - 3$		
50	y_1		$5 - 2y_1$	
		$2 - y_1$		$3y_1 + y_3 - 9$
55	$y_2 = 2$		$y_1 + y_3 - 4$	
		$y_3 - 2$		$3.6 - y_1 - 3y_3$
60	y_3	$-2.4 - y_3$	$-0.4 - 2y_3$	
65	$y_4 = -2.4$			

As only three entries y_0, y_2, y_4 are given, the function y can be represented by a second degree polynomial.

$\therefore$ $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

i.e., $3y_1 + y_3 = 9$; $y_1 + 3y_3 = 3.6$

Solving these, we get $y_1 = 2.925$, $y_3 = 0.225$.

Otherwise : As only three entries $y_0 = 3, y_2 = 2, y_4 = -2.4$ are given, the function y can be represented by a second degree polynomial.

$\therefore$ $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

i.e., $(E-1)^3 y_0 = 0$ and $(E-1)^3 y_1 = 0$

i.e., $(E^3 - 3E^2 + 3E - 1)\, y_0 = 0$ and $(E^3 - 3E^2 + 3E - 1) y_1 = 0$

i.e., $y_3 - 3y_2 + 3y_1 - y_0 = 0$

$y_4 - 3y_3 + 3y_2 - y_1 = 0$

i.e., $y_3 + 3y_1 = 9$; $3y_3 + y_1 = 3.6$

Solving these, we get $y_1 = 2.925$, $y_3 = 0.225$.

Example 29.6. *Assuming that the following values of y belong to a polynomial of degree 4, compute the next three values :*

x :	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
y :	*1*	*– 1*	*1*	*– 1*	*1*	—	—	—

Solution. We construct the following difference table from the given data :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$y_0 = 1$				
		-2			
1	$y_1 = -1$		4		
		2		-8	
2	$y_2 = 1$		-4		16
		-2		8	
3	$y_3 = -1$		4		16
		2		$\Delta^3 y_2,$	
4	$y_4 = 1$		$\Delta^2 y_3$		16
		Δy_4		$\Delta^2 y_3,$	
5	y_5		$\Delta^2 y_4$		16
		Δy_5		$\Delta^3 y_4,$	
6	y_6		$\Delta^2 y_5$		
		Δy_6			
7	y_7				

Since the values of y belong to a polynomial of degree 4, the fourth differences must be constant. But $\Delta^4 y = 16$.

$\therefore$ The other fourth order differences must also be 16. Thus

$$\Delta^4 y_1 = 16 = \Delta^3 y_2 - \Delta^3 y_1$$

i.e.,
$$\Delta^3 y_2 = \Delta^3 y_1 + \Delta^4 y_1 = 8 + 16 = 24$$
$$\Delta^2 y_3 = \Delta^2 y_2 + \Delta^3 y_2 = 4 + 24 = 28$$
$$\Delta y_4 = \Delta y_3 + \Delta^2 y_3 = 2 + 28 = 30$$

and
$$y_5 = y_4 + \Delta y_4 = 1 + 30 = 31$$

Similarly starting with $\Delta^4 y_2 = 16$, we get

$$\Delta^3 y_3 = 40, \Delta^2 y_4 = 68, \Delta y_5 = 98, y_6 = 129.$$

Starting with $\Delta^4 y_3 = 16$, we obtain

$$\Delta^3 y_4 = 56, \Delta^2 y_5 = 124, \Delta y_6 = 222, y_7 = 351.$$

PROBLEMS 29.1

1. Construct the table of differences for the data below :

x :	0	1	2	3	4
$f(x)$:	1.0	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 f(2)$.

2. If $u_0 = 3, u_1 = 12, u_2 = 18, u_3 = 2000, u_4 = 100$, calculate Δu_0.
3. Show that $\Delta^3 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$.
4. Form the table of backward differences of the function
$$f(x) = x^3 - 3x^2 - 5x - 7 \text{ for } x = -1, 0, 1, 2, 3, 4, 5.$$
5. Form a table of differences for the function
$$f(x) = x^3 + 5x - 7 \text{ for } x = -1, 0, 1, 2, 3, 4, 5$$
Continue the table to obtain $f(6)$.
6. Extend the following table to two more terms on either side by constructing the difference table :

x :	$-.2$	0.0	0.2	0.4	0.6	0.8	1.0
y :	2.6	3.0	3.4	4.28	7.08	14.2	29.0

7. Show that

(i) $\Delta\left[\frac{1}{f(x)}\right] = \frac{-\Delta f(x)}{f(x)\, f(x+1)}$; (Raipur, 2005) (ii) $\Delta \log f(x) = \log\left\{1 + \frac{\Delta f(x)}{f(x)}\right\}$.

8. Evaluate :

(i) $\Delta(x + \cos x)$ (ii) $\Delta \tan^{-1}\left(\frac{n-1}{n}\right)$ (iii) $\Delta\left\{\frac{1}{x(x+4)(x+6)}\right\}$ *(Madras, 2001)*

(iv) $\Delta^2\left(\frac{1}{x^2+5x+6}\right)$ *(P.T.U., 2001)*

9. Evaluate :

(i) $\Delta(e^{3x} \log 2x)$ (ii) $\Delta(2^x/x\,!)$ (iii) $\Delta^n(a^x)$ *(Burdwan, 2003)* (iv) $\Delta^n\left(\frac{1}{x}\right)$.

10. If $f(x) = e^{ax+b}$, show that its leading differences form a geometric progression. *(Mumbai, 2003)*
11. Prove that

(i) $y_3 = y_2 + \Delta y_1 + \Delta^2 y_0 + \Delta^3 y_0$ (ii) $\nabla^2 y_8 = y_8 - 2y_7 + y_6$; $\delta^2 y_5 = y_6 - 2y_5 + y_4$.

12. Evaluate :

(i) $\Delta^3 [(1-x)(1-2x)(1-3x)]$

(ii) $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)]$, if the interval of differencing is 2.

13. Express $x^3 - 2x^2 + x - 1$ into factorial polynomial. Hence show that $\Delta^4 f(x) = 0$ *(P.T.U., 2001)*
14. Express $u = x^4 - 12x^3 + 24x^2 - 30x + 9$ and its successive differences in factorial notation. Hence show that $\Delta^5 u = 0$.
15. Find the first and second differences of $x^4 - 6x^3 + 11x^2 - 5x + 8$ with $h = 1$. Show that the fourth difference is constant.
16. Obtain the function whose first difference is $2x^3 + 3x^2 - 5x + 4$.
17. Find the first term of the series whose second and subsequent terms are 8, 3, 0, -1, 0.
18. If $u(x)$ and $v(x)$ be two functions of x, prove that

(i) $\Delta[u(x)\, v(x)] = u(x)\, \Delta v(x) + v(x+1)\, \Delta u(x)$. (ii) $\Delta\left\{\frac{u(x)}{v(x)}\right\} = \frac{v(x)\, \Delta u(x) - u(x)\, \Delta v(x)}{v(x)\, v(x+1)}$.

29.4 (1) OTHER DIFFERENCE OPERATORS

We have already introduced the operators Δ, ∇ and δ. Besides these, there are the operators E and μ, which we define below :

(i) **Shift operator E** *is the operation of increasing the argument x by h* so that
$$Ef(x) = f(x+h),\ E^2 f(x) = f(x+2h),\ E^3 f(x) = f(x+3h) \text{ etc.}$$
The inverse operator E^{-1} is defined by $E^{-1} f(x) = f(x-h)$

If y_x is the function $f(x)$, then $Ey_x = y_{x+h}$, $E^{-1}y_x = y_{x-h}$, $E^n y_x = y_{x+nh}$, where n may be any real number.

(*ii*) **Averaging operator** μ is defined by the equation $\mu y_x = \frac{1}{2}(y_{x+h/2} + y_{x-h/2})$

Obs. *In the difference calculus, Δ and E are regarded as the fundamental operators and ∇, δ, μ can be expressed in terms of these.*

(2) Relations between the operators. We shall now establish the following identities :

(*i*) $\mathbf{\Delta = E - 1}$ (*ii*) $\mathbf{\nabla = 1 - E^{-1}}$

(*iii*) $\mathbf{\delta = E^{1/2} - E^{-1/2}}$ (*iv*) $\mathbf{\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})}$

(*v*) $\mathbf{\Delta = E\nabla = \nabla E = \delta E^{1/2}}$ (*vi*) $\mathbf{E = e^{hD}}$.

Proofs. (*i*) $\Delta y_x = y_{x+h} - y_x = Ey_x - y_x = (E-1)\,y_x$.

This shows that the operators Δ and E are connected by the symbolic relation

$$\Delta = E - 1 \quad \text{or} \quad E = 1 + \Delta$$

(*ii*) $\nabla y_x = y_x - y_{x-h} = y_x - E^{-1} y_x = (1 - E^{-1})y_x$

$\therefore$ $\nabla = 1 - E^{-1}$ or $E = (1 - \nabla)^{-1}$

(*iii*) $\delta y_x = y_{x+h/2} - y_{x-h/2} = E^{1/2}y_x - E^{-1/2}y_x = (E^{1/2} - E^{-1/2})y_x$

$\therefore$ $\delta = E^{1/2} - E^{-1/2}$.

(*iv*) $\mu y_x = \frac{1}{2}(y_{x+h/2} + y_{x-y/2}) = \frac{1}{2}(E^{1/2}y_x + E^{-1/2}y_x) = \frac{1}{2}(E^{1/2} + E^{-1/2})\,y_x$

$\therefore$ $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$.

(*v*) $E\nabla y_x = E(y_x - y_{x-h}) = Ey_x - Ey_{x-h} = y_{x+h} - y_x = \Delta y_x \quad \therefore \quad E\nabla = \Delta$

Also $\nabla E y_x = \nabla y_{x+h} = y_{x+h} - y_x = \Delta y_x \quad \therefore \quad \nabla E = \Delta$

$\delta E^{1/2} y_x = \delta y_{x+h/2} = y_{x+h/2+h/2} - y_{x+h/2-h/2} = y_{x+h} - y_x = \Delta y_x$

$\therefore$ $\delta E^{1/2} = \Delta$

Hence $\Delta = E\nabla = \nabla E = \delta E^{1/2}$.

(*vi*) $Ef(x) = f(x+h)$

$$= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots$$ [By Taylor's series]

$$= f(x) + hD\,f(x) + \frac{h^2}{2!}D^2 f(x) + \dots = \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \dots\dots\right) f(x) = e^{hD} f(x)$$

$\therefore$ $E = e^{hD}$

Cor. 1. $E = 1 + \Delta = e^{hD}$.

2. $D = \frac{1}{h}\log(1+\Delta) = \frac{1}{h}\left(\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \dots\right)$ (*Burdwan, 2003*)

Note. *A table showing the symbolic relations between the various operators is given below for ready reference. To prove such relations between the operators, always express each operator in terms of the fundamental operator E.*

(3) Relations between the various operators

In terms of	E	Δ	∇	δ	hD
E		$\Delta + 1$	$(1+\nabla)^{-1}$	$1 + \frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	e^{hD}
Δ	$E - 1$	—	$(1-\nabla)^{-1} - 1$	$\frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	$e^{hD} - 1$
∇	$1 - E^{-1}$	$1 - (1+)^{-1} - 1$	—	$-\frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	$1 - e^{-hD}$
δ	$E^{1/2} - E^{-1/2}$	$\Delta(1+\Delta)^{-1/2}$	$\nabla(1-\nabla)^{-1/2}$	—	$2\sinh(hD/2)$
μ	$\frac{1}{2}(E^{1/2} + E^{-1/2})$	$(1+\Delta/2)(1+\Delta)^{-1/2}$	$(1+\nabla/2)(1+\nabla)^{-1/2}$	$\sqrt{(1+\delta^2/4)}$	$\cosh(hD/2)$
hD	$\log E$	$\log(1+\Delta)$	$\log(1-)^{-1}$	$2\sinh^{-1}(\delta/2)$	

Example 29.7. *Prove that*

$$e^x = \left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x}, \text{ the interval of differencing being } h. \qquad \textit{(Bhopal, 2009)}$$

Solution. Since $\left(\dfrac{\Delta^2}{E}\right)e^x = \Delta^2 . E^{-1}e^x = \Delta^2 e^{x-h} = \Delta^2 e^x . e^{-h} = e^{-h}\Delta^2 e^x$

$\therefore$ $$\text{R.H.S.} = e^{-h}\Delta^2 e^x . \frac{Ee^x}{\Delta^2 e^x} = e^{-h} Ee^x = e^{-h} . e^{x+h} = e^x.$$

Example 29.8. *Prove with the usual notations, that*

(i) $hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$ *(Rohtak, 2005)*

(ii) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$ *(Bhopal, 2009 ; U.P.T.U., 2009)*

(iii) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$ *(iv)* $\Delta^3 y_2 = \nabla^3 y_5$.

Solution. *(i)* We know that $e^{hD} = E = 1 + \Delta \quad \therefore \quad hD = \log(1+\Delta)$

Also $hD = \log E = -\log(E^{-1}) = -\log(1-\nabla)$ $\qquad [\because \ E^{-1} = 1 - \nabla]$

We have proved that $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$ and $\delta = E^{1/2} - E^{-1/2}$

$\therefore$ $$\mu\delta = \tfrac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \tfrac{1}{2}(E - E^{-1}) = \tfrac{1}{2}(e^{hD} - e^{-hD}) = \sinh(hD)$$

i.e. $hD = \sinh^{-1}(\mu\delta)$.

Hence $hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$

(ii) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = (E^{1/2} + E^{-1/2})E^{1/2} = E + 1 = 1 + \Delta + 1 = 2 + \Delta$.

(iii) $\frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$

$$= \tfrac{1}{2}(E^{1/2} - E^{-1/2})^2 + (E^{1/2} - E^{-1/2})\sqrt{[1 + (E^{1/2} - E^{-1/2})^2/4]}$$
$$= \tfrac{1}{2}(E + E^{-1} - 2) + (E^{1/2} - E^{-1/2})\sqrt{[(E + E^{-1} + 2)/4]}$$
$$= \tfrac{1}{2}(E + E^{-1} - 2) + \tfrac{1}{2}(E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2})$$
$$= \tfrac{1}{2}[(E + E^{-1} - 2) + (E - E^{-1})] = \tfrac{1}{2}(2E - 2) = E - 1 = \Delta.$$

(iv) $\Delta^3 y_2 = (E-1)^3 y_2$ $\qquad [\because \ \Delta = E - 1]$

$= (E^3 - 3E^2 + 3E - 1)\,y_2 = y_5 - 3y_4 + 3y_3 - y_2$ ...(1)

$\nabla^3 y_5 = (1 - E^{-1})^3 y_5$ $\qquad [\because \ \Delta = 1 - E^{-1}]$

$= (1 - 3E^{-1} + 3E^{-2} - E^{-3})\,y_5 = y_5 - 3y_4 + 3y_3 - y_2$ ...(2)

From (1) and (2), $\Delta^3 y_2 = \nabla^3 y_5$.

29.5 TO FIND ONE OR MORE MISSING TERMS

When one or more values of $y = f(x)$ corresponding to the equidistant values of x are missing, we can find these using any of the following two methods :

First method : We assume the missing term or terms as a, b etc. and form the difference table. Assuming the last difference as zero, we solve these equations for a, b. These give the missing term/terms.

Second method : If n entries of y are given, $f(x)$ can be represented by a $(n-1)$th degree polynomial *i.e.*, $\Delta^n y = 0$. Since $\Delta = E - 1$, therefore $(E-1)^n y = 0$. Now expanding $(E-1)^n$ and substituting the given values, we obtain the missing term/terms.

Example 29.9. *Find the missing term in the table :*

x :	2	3	4	5	6
y :	45.0	49.2	54.1	...	67.4

(U.P.T.U., 2008)

Solution. Let the missing term be a. Then the difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	$45.0\ (= y_0)$				
		4.2			
3	$49.2\ (= y_1)$		0.7		
		49.9		$a - 59.7$	
4	$54.1\ (= y_2)$		$a - 59.0$		$240.2 - 4a$
		$a - 54.1$		$180.5 - 3a$	
5	$a(= y_3)$		$121.5 - a$		
		$67.4 - a$			
6	$67.4\ (= y_4)$				

We know that $\Delta^4 y = 0$ *i.e.*, $240.2 - 4a = 0$.

Hence $a = 60.05$.

Otherwise: As only four entries y_0, y_1, y_2, y_3 are given, therefore $y = f(x)$ can be represented by a third degree polynomial.

$\therefore$ $\Delta^3 y$ = constant or $\Delta^4 y = 0$ *i.e.*, $(E - 1)^4 = 0$

i.e., $(E^4 - 4E^3 + 6E^2 - 4E + 1) = 0$ or $y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

Let the missing entry y_3 be a so that

$$67.4 - 4a + 6(54.1) - 4(49.2) + 45 = 0 \text{ or } - 4a = - 240.2$$

Hence $a = 60.05$.

Example 29.10. *Find the missing values in the following data :*

x :	*45*	*50*	*55*	*60*	*65*
y :	*3.0*	*...*	*2.0*	*...*	*– 2.4*

(Bhopal, 2007)

Solution. Let the missing value be a, b. Then the difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	$3(= y_0)$			
		$a - 3$		
50	$a(= y_1)$		$5 - 2a$	
		$2 - a$		$3a + b - 9$
55	$2(= y_2)$		$b + a - 4$	
		$b - 2$		$3.6 - a - 36$
60	$b(= y_3)$		$- 0.4 - 2b$	
		$- 2.4 - b$		
65	$- 2.4\ (= y_4)$			

As only three entries y_0, y_2, y_4 are given, y can be represented by a second degree polynomial having third differences as zero.

$$\therefore \quad \Delta^3 y_0 = 0 \text{ and } \Delta^3 y_1 = 0$$

i.e., $$3a + b = 9, a + 3b = 3.6$$

Solving these, we get $a = 2.925, b + 0.0225$.

Otherwise. As only three entries $y_0 = 3, y_2 = 2, y_4 = - 2.4$ are given, y can be represented by a second degree polynomial having third differences as zero.

$$\therefore \quad \Delta^3 y_0 = 0 \quad \text{and} \quad \Delta^3 y_1 = 0$$

i.e., $$(E - 1)^3 y_0 = 0 \text{ and } (E - 1)^3 y_1 = 0$$

i.e., $$(E^3 - 3E^3 + 3E - 1) y_0 = 0 \, ; (E^3 - 3E^2 + 3E - 1) \, . \, y_1 = 0$$

or $$y_3 - 3y_2 + 3y_1 - y_0 = 0 \, ; y_4 - 3y_3 + 3y_2 - y_1 = 0$$

or $$y_3 + 3y_1 = 9 \, ; 3y_3 + y_1 = 3.6$$

Solving three, we get $y_1 = 2.925, y_2 = 0.225$.

Example 29.11. *If $y_{10} = 3, y_{11} = 6, y_{12} = 11, y_{13} = 18, y_{14} = 27$, find y_4.* *(Mumbai, 2005)*

Solution. Taking y_{14} as u_0, we are required to find y_4 *i.e.*, u_{-10}. Then the difference table is

x	u	Δu	$\Delta^2 u$	
x_{-4}	$y_{10} = u_{-4} = 3$			
		3		
x_{-3}	$y_{11} = u_{-3} = 6$		2	
		5		0
x_{-2}	$y_{12} = u_{-2} = 11$		2	
		7		0
x_{-1}	$y_{13} = u_{-1} = 18$		2	
		9		
x_0	$y_{14} = u_0 = 27$			

Then
$$y_4 = u_{-10} = (E^{-1})^{10}\, u_0 = (1-\nabla)^{10}\, u_8$$
$$= \left(1 - 10\nabla + \frac{10.9}{2}\nabla^2 - \frac{10.9.8}{1.2.3}\nabla^3 + \ldots\right)u_0$$
$$= u_0 - 10\nabla u_0 + 45\nabla^2 u_0 - 120\nabla^3 u_0$$
$$= 27 - 10 \times 9 + 45 \times 2 - 120 \times 0 = 27.$$

Example 29.12. *If y_x is a polynomial for which fifth difference is constant and $y_1 + y_7 = -7845$, $y_2 + y_6 = 686$, $y_3 + y_5 = 1088$, find y_4.* *(Mumbai, 2004)*

Solution. Starting with y_1 instead of y_0, we note that $\Delta^6 y_1 = 0$ [$\because$ $\Delta^5 y_1$ is constant.

i.e.,
$$(E-1)^6\, y_1 = (E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)\, y_1 = 0$$
$\therefore$
$$y_7 - 6y_6 + 15y_5 - 20y_4 + 15y_3 - 6y_2 + y_1 = 0$$
or
$$(y_7 + y_1) - 6(y_6 + y_2) + 15(y_5 + y_3) - 20y_4 = 0$$
i.e.
$$y_4 = \frac{1}{20}\,[(y_1 + y_7) - 6(y_2 + y_6) + 15(y_3 + y_5)]$$
$$= \frac{1}{20}\,[-784 - 6(686) + 15(1088)] = 571.$$

Example 29.13. *Prove the following identities :*

(*i*) $u_1 x + u_2 x^2 + u_3 x^3 + \ldots = \dfrac{x}{1-x} u_1 + \left(\dfrac{x}{1-x}\right)^2 \Delta u_1 + \left(\dfrac{x}{1-x}\right)^3 \Delta^2 u_1 + \ldots$

(*ii*) $u_0 + \dfrac{u_1 x}{1!} + \dfrac{u_2 x^2}{2!} + \dfrac{u_3 x^3}{3!} + \ldots = e^x \left(u_0 + x\Delta u_0 + \dfrac{x^2}{2!}\Delta^2 u_0 + \dfrac{x^3}{3!}\Delta^3 u_0 + \ldots\right)$.

Solution. (*i*) L.H.S. $= xu_1 + x^2 E u_1 + x^3 E^2 u_1 + \ldots = x(1 + xE + x^2E^2 + \ldots)\, u_1$ [$\because$ $u_{x+h} = E^h u_x$]

$$= x \,.\, \frac{1}{1 - xE}\, u_1, \text{ taking sum of infinite G.P.}$$

$$= x\left[\frac{1}{1 - x(1+\Delta)}\right] u_1 \qquad [\because\ E = 1 + \Delta]$$

$$= x\left(\frac{1}{1 - x - x\Delta}\right) u_1 = \frac{x}{1-x}\left(1 - \frac{x\Delta}{1-x}\right)^{-1} u_1 = \frac{x}{1-x}\left(1 + \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x^2)} + \ldots\ldots\right) u_1$$

$$= \frac{x}{1-x}\, u_1 + \frac{x^2}{(1-x)^2}\,\Delta u_1 + \frac{x^2}{(1-x)^3}\,\Delta^2 u_1 + \ldots = \text{R.H.S.}$$

(ii) $$\text{L.H.S.} = u_0 + \frac{x}{1!}Eu_0 + \frac{x^2}{2!}E^2u_0 + \frac{x^3}{3!}E^3u_0 + \ldots$$

$$= \left(1 + \frac{xE}{1!} + \frac{x^2E^2}{2!} + \frac{x^3E^3}{3!} + \ldots\right)u_0 = e^{xE}\,u_0 = e^{x(1+\Delta)}\,u_0$$

$$= e^x \,.\, e^{x\Delta}\,u_0 = e^x\left(1 + \frac{x\Delta}{1!} + \frac{x^2\Delta^2}{2!} + \frac{x^3\Delta^3}{3!} + \ldots\right)u_0$$

$$= e^x\left(u_0 + \frac{x}{1!}\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \ldots\right) = \text{R.H.S.}$$

PROBLEMS 29.2

1. Explain the difference between $\left(\frac{\Delta^2}{E}\right)u_x$ and $\frac{\Delta^2 u_x}{Eu_x}$. (Madras, 2003)

2. Evaluate taking h as the interval of differencing :

   (i) $\frac{\Delta^2}{E}\sin x$ (ii) $\left(\frac{\Delta^2}{E}\right)x^4$, $(h = 1)$ (W.B.T.U., 2005)

   (iii) $\left(\frac{\Delta^2}{E}\right)\sin(x+h) + \frac{\Delta^2 \sin(x+h)}{E\sin(x+h)}$ (iv) $(\Delta + \nabla)^2(x^2 + x)$, $(h = 1)$.

3. With the usual notations, show that

   (i) $\nabla = 1 - e^{-hD}$ (ii) $D = \frac{2}{h}\sinh^{-1}\left(\frac{\delta}{2}\right)$

   (iii) $(1 + \Delta)(1 - \nabla) = 1$. (iv) $\Delta - \nabla = \nabla\Delta = \delta^2$. (Mumbai, 2005)

4. Prove that

   (i) $\delta = \Delta(1 + \Delta)^{-1/2} = \nabla(1 - \nabla)^{-1/2}$ (ii) $\mu^2 = 1 + \frac{\delta^2}{4}$ (U.P.T.U, 2009)

   (iii) $\delta(E^{1/2} + E^{-1/2}) = \Delta E^{-1} + \Delta$ (iv) $\nabla = \Delta E^{-1} = E^{-1}\Delta = 1 - E^{-1}$

5. Show that (i) $\mu\delta = \frac{1}{2}(\Delta + \nabla)$ (ii) $1 + \delta^2/2 = \sqrt{(1 + \delta^2\mu^2)}$ (U.P.T.U., MCA, 2008)

   (iii) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$ (U.P.T.U., 2009) (iv) $\nabla^2 = h^2D^2 - h^3D^3 + \frac{7}{12}h^4D^4 - \ldots\ldots$

6. Prove that

   (i) $\nabla^r f_k = \Delta^r f_{k-r}$ (ii) $\Delta f_k^{\,2} = (f_k + f_{k+1})\,\Delta f_k$ (J.N.T.U., MCA, 2006)

   (iii) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{\left(1 + \frac{\delta^2}{4}\right)}$ (iv) $E^{1/2} = (1 + \delta^2/4)^{1/2} + \delta/2$.

7. Prove that $\nabla y_{n+1} = h\left(1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \ldots\right)y'_n$.

8. The following table gives the values of y which is a polynomial of degree five. It is known that $f(3)$ is in error. Correct the error.

x:	0	1	2	3	4	5	6
y:	1	2	33	254	1025	3126	7777

(Mumbai, 2004)

9. Estimate the missing term in the following table :

x :	0	1	2	3	4
$f(x)$:	1	3	9	—	81

(S.V.T.U., 2007)

10. Find the missing terms of the following data :

x :	1	1.5	2	2.5	3	3.5	4
$f(x)$:	6	?	10	20	?	1.5	5

(U.P.T.U., 2010)

11. Find the missing values in the following table :

x :	0	1	2	3	4	5	6
y :	5	11	22	40	...	140	...

(V.T.U., 2006)

12. If $u_{13} = 1, u_{14} = -3, u_{15} = -1, u_{16} = 13$ find u_8. *(Mumbai, 2004)*
13. Evaluate y_4 from the following data (stating the assumptions you make) :
$$y_0 + y_8 = 1.9243, y_1 + y_7 = 1.9590, y_2 + y_6 = 1.9823, y_3 + y_5 = 1.9956.$$ *(Mumbai, 2003)*
14. Using the method of separation of symbols, prove that
(*i*) $u_0 + u_1 + u_2 + \ldots\ldots + u_n = {}^{n+1}C_1 u_0 + {}^{n+1}C_3 \Delta^2 u_0 + \ldots\ldots + {}^{n+1}C_{n+1} \Delta^n u_0$.
(*ii*) $y_x = y_n - {}^{n-x}C_1 \Delta y_{n-1} + {}^{n-x}C_2 \Delta^2 y_{n-2} - \ldots\ldots + (-1)^{n-x} \Delta^{n-x} y_{n-(n-x)}$.
15. Using the method of finite differences, sum the following series :
(*i*) 2.5 + 5.8 + 8.11 + 11.14 + ... to n terms.
(*ii*) 1.2.3 + 2.3.4 + 3.4.5 + ... to n terms.
16. Prove that $u_0 + u_1 x + u_2 x^2 + \ldots\ldots \infty = \dfrac{u_0}{1-x} + \dfrac{x\Delta u_0}{(1-x)^2} + \dfrac{x^2\Delta^2 u_0}{(1-x)^3} + \ldots\ldots \infty$
Hence sum the series $1.2 + 2.3x + 3.4x^2 + \ldots\ldots \infty$.

29.6 NEWTON'S INTERPOLATION FORMULAE*

We now derive two important interpolation formulae by means of the forward and backward differences of a function. These formulae are often employed in engineering and scientific problems.

(1) Newton's forward interpolation formula. Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number.

For any real number p, we have defined E such that
$$E^p f(x) = f(x + ph)$$
$$\therefore \quad y_p = f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^{-p} y_0 \qquad [\because E = 1 + \Delta]$$
$$= \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right\} y_0 \qquad \text{[Using Binomial theorem]}$$
i.e.,
$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \qquad \ldots(1)$$

It is called **Newton's forward interpolation formula** as (1) contains y_0 and the forward differences of y_0.

Obs. *This formula is used for interpolating the values of y near the beginning of a set of tabulated values and extrapolating values of y a little backward (i.e. to the left) of y_0.*

(2) Newton's backward interpolation formula. Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number. Then we have
$$y_p = f(x_n + ph) = E^p f(x_n) = (1 - \nabla)^{-p} y_n \qquad [\because E^{-1} = 1 - \nabla]$$
$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^2 + \ldots\right] y_n \qquad \text{[Using Binomial theorem]}$$
i.e.,
$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots \qquad \ldots(2)$$

It is called *Newton's backward interpolation formula as* (2) contains y_n and backward differences of y_n.

Obs. *This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little ahead* (to the right) *of y_n.*

Example 29.14. *The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earth's surface :*

x = height :	100	150	200	250	300	350	400
y = distance :	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when (i) x = 218 ft (Madras, 2003 S) (ii) 410 ft. *(V.T.U., 2002)*

---

*See foot note p.466.

Solution. The difference table is as under :

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	13.03		– 0.39		
		2.01		0.15	
200	**15.04**		– 0.24		– 0.07
		1.77		0.08	
250	16.81		**– 0.16**		– 0.05
		1.61		**0.03**	
300	18.42		– 0.13		**– 0.01**
		1.48		**0.02**	
350	19.90		**– 0.11**		
		1.37			
400	**21.27**				

(*i*) If we take $x_0 = 200$, then $y_0 = 15.04$, $\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$, $\Delta^3 y_0 = 0.03$ etc.

Since $x = 218$ and $h = 50$, $\therefore\ p = \dfrac{x - x_0}{h} = \dfrac{18}{50} = 0.36$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{218} = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \dots$$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)}{2}(-0.16) + \frac{0.36(-0.64)(-1.64)}{6}(0.03) + \dots$$

$$= 15.04 + 0.637 + 0.018 + 0.001 + \dots = 15.696 \quad \textit{i.e.}, \text{ 15.7 nautical miles}$$

(*ii*) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$\therefore$ taking $x_n = 400$, $p = \dfrac{x - x_n}{h} = \dfrac{10}{50} = 0.2$

Using the line of backward differences

$$y_n = 21.27,\ \nabla y_n = 1.37,\ \nabla^2 y_n = -0.11,\ \nabla^3 y_n = 0.02 \text{ etc.}$$

$\therefore$ newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{1.2.3.}\nabla^3 y_{400} + \dots$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2}(-0.11) + \dots = 21.53 \text{ nautical miles.}$$

Example 29.15. *From the following table, estimate the number of students who obtained marks between 40 and 45 :*

Marks :	*30—40*	*40—50*	*50—60*	*60—70*	*70—80*
No. of Students :	*31*	*42*	*51*	*35*	*31*

(V.T.U., 2011 S ; S.V.T.U., 2007 ; Madras, 2006)

Solution. First we prepare the cumulative frequency table, as follows :

Marks less than (x) :	40	50	60	70	80
No. of Students (y_x) :	31	73	124	159	190

Now the difference table is

x	y	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	**31**				
		42			
50	73		**9**		
		51		**– 25**	
60	124		– 16		37
		35		12	
70	159		– 4		
		31			
80	190				

We shall find y_{45} *i.e.* number of students with marks less than 45.

Taking $x_0 = 40, x = 45$, we have $p = \dfrac{x - x_0}{h} = \dfrac{5}{10} = 0.5$ $[\because \ h = 10]$

$\therefore$ using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_{40} + \ldots$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(0.5)(-1.5)}{6} \times (-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 37$$

$= 47.87$, on simplification.

$\therefore$ the number of students with marks less than 45 is 47.87 *i.e.*, 48.

But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45 = 48 – 31 = 17.

Example 29.16. *Find the cubic polynomial which takes the following values :*

x :	0	1	2	3
$f(x)$:	1	2	1	10

Hence or otherwise evaluate f (4). (Bhopal, 2009 ; Rohtak, 2005 ; W.B.T.U., 2005)

Solution. The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	**1**			
		1		
1	2		**– 2**	
		– 1		**12**
2	1		10	
		9		
3	10			

We take $x_0 = 0$ and $p = \dfrac{x - 0}{h} = x$ $[\because \ h = 1]$

$\therefore$ using Newton's forward interpolation formula, we get

$$f(x) = f(0) + \frac{x}{1}\Delta f(0) + \frac{x(x-1)}{1.2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 f(0)$$

$$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$= 2x^3 - 7x^2 + 6x + 1$, which is the required polynomial.

To compute $f(4)$, we take $x_n = 3, x = 4$ so that $p = \dfrac{x - x_n}{h} = 1$ $[\because \ h = 1]$

Using Newton's backward interpolation formula, we get

$$f(4) = f(3) + p\nabla f(3) + \frac{p(p+1)}{1.2}\nabla^2 f(3) + \frac{p(p+1)(p+2)}{1.2.3}\nabla^3 f(3)$$

$= 10 + 9 + 10 + 12 + 41.$

which is the same value as that obtained by substituting $x = 4$ in the cubic polynomial above.

Obs. The above example shows that *if a tabulated function is a polynomial, then interpolation and extrapolation give the same values.*

Example 29.17. *In the table below, the values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series :*

x :	3	4	5	6	7	8	9
y :	4.8	8.4	14.5	23.6	36.2	52.8	73.9

(Anna, 2007)

Solution. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3.0		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4.0		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

To find the first term, use Newton's forward interpolation formula with $x_0 = 3, x = 1, h = 1$ and $p = -2$. We have

$$y(1) = 4.8 + \frac{(-2)}{1} \times 3.6 + \frac{(-2)(-3)}{1.2} \times 2.5 + \frac{(-2)(-3)(-4)}{1.2.3} \times 0.5 = 3.1$$

To obtain the tenth term, use Newton's backward interpolation formula with $x_n = 9, x = 10, h = 1$ and $p = 1$. This gives

$$y(10) = 73.9 + \frac{1}{1} \times 21.1 + \frac{1(2)}{1.2} \times 4.5 + \frac{1(2)(3)}{1.2.3} \times 0.5 = 100.$$

PROBLEMS 29.3

1. Using Newton's forward formula, find the value of $f(1.6)$, if

x :	1	1.4	1.8	2.2
$f(x)$:	3.49	4.82	5.96	6.5

(*J.N.T.U., 2006*)

2. State Newton's interpolation formula and use it to calculate the value of exp (1.85), given the following table :

x :	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$f(x)$:	5.474	6.050	6.686	7.389	8.166	9.025	9.974

(*Kottayam, 2005*)

3. If $f(1.15) = 1.0723, f(1.20) = 1.0954, f(1.25) = 1.1180$ and $f(1.30) = 1.1401$, find $f(1.28)$.
4. Given sin 45° = 0.7071, sin 50° = 0.7660, sin 55° = 0.8192, sin 60° = 0.8660, find sin 52°, using Newton's forward formula.
5. From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age of 46 :

Age :	45	50	55	60	65
Premium (in rupees) :	114.84	96.16	83.32	74.48	68.48

(*U.P.T.U., 2010*)

6. The area A of a circle of diameter d is given for the following values :

d :	80	85	90	95	100
A :	5026	5674	6362	7088	7854

(*V.T.U., 2010*)

Calculate the area of a circle of diameter 105.

7. Estimate the value of $f(22)$ and $f(42)$ from the following available data :

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

(*J.N.T.U., 2007*)

8. From the following table :

$x°$:	10	20	30	40	50	60	70	80
cos x :	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1737

Calculate cos 25° and cos 73° using Gregory Newton formulae. (*U.P.T.U., 2006*)

9. Find the number of men getting wages below Rs. 15 from the following data :

Wages in Rs. :	0—10	10—20	20—30	30—40
Frequency :	9	30	35	42

(*Nagarjuna, 2001*)

10. Find the polynomial interpolating the data :

x :	0	1	2
$f(x)$:	0	5	2

(*U.P.T.U., 2008*)

11. Construct Newton's forward interpolation polynomial for the following data :

x :	4	6	8	10
y :	1	3	8	16

(*Madras, 2006*)

Hence evaluate y for $x = 5$.

12. Construct the difference table for the following data :

x :	0.1	0.3	0.5	0.7	0.9	1.1	1.3
$f(x)$:	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Evaluate $f(0.6)$. (*J.N.T.U., 2007*)

13. Estimate from following table $f(3.8)$ to three significant figures using Gregory Newton backward interpolation formula:

x :	0	1	2	3	4
$f(x)$:	1	1.5	2.2	3.1	4.6

(*U.P.T.U., 2009*)

14. The following table gives the population of a town during the last six censuses. Estimate the increase in the population during the period from 1976 to 1978 :

Year :	1941	1951	1961	1971	1981	1991
Population (in thousands) :	12	15	20	27	39	52

(*U.P.T.U., 2009*)

15. In the following table, the values of y are consecutive terms of a series of which 12.5 is the 5th term. Find the first and tenth terms of the series.

x :	3	4	5	6	7	8	9
y :	2.7	6.4	12.5	21.6	34.3	51.2	72.9

(*P.T.U., 2001*)

16. Given $u_1 = 40$, $u_3 = 45$, $u_5 = 54$, find u_2 and u_4. (*Nagarjuna, 2003 S*)
17. If $u_{-1} = 10$, $u_1 = 8$, $u_2 = 10$, $u_4 = 50$, find u_0 and u_3.
18. Given $y_0 = 3, y_1 = 12, y_2 = 81, y_3 = 200, y_4 = 100, y_5 = 8$, without forming the difference table, find $\Delta^5 y_0$.

29.7 CENTRAL DIFFERENCE INTERPOLATION FORMULAE

In the preceding section, we derived Newton's forward and backward interpolation formulae which are applicable for interpolation near the beginning and end of tabulated values. Now we shall develop central difference formulae which are best suited for interpolation near the middle of the table.

If x takes the values $x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h$ and the corresponding values of $y = f(x)$ are $y_{-2}, y_{-1}, y_0, y_1, y_2$, then we can write the difference table in the two notations as follows :

x	y	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*
$x_0 - 2h$	y_{-2}				
		$\Delta y_{-2} (= \delta y_{-3/2})$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} (= \delta^2 y_{-1})$		
		$\Delta y_{-1} (= \delta y_{-1/2})$		$\Delta^3 y_{-2} (= \delta^3 y_{-1/2})$	
x_0	y_0		$\Delta^2 y_{-1} (\delta^2 y_0)$		$\Delta^4 y_{-2} (= \delta^4 y_0)$
		$\Delta y_0 (= \delta y_{1/2})$		$\Delta^3 y_{-1} (= \delta^3 y_{1/2})$	
$x_0 + h$	y_1		$\Delta^2 y_0 (= \delta^2 y_1)$		
		$\Delta y_1 (= \delta y_{3/2})$			
$x_0 + 2h$	y_2				

(1) Gauss's forward interpolation formula. The Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \ldots \quad \ldots(1)$$

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

i.e., $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$ ...(2)

Similarly $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ ...(3)

$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$ etc. ...(4)

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

i.e., $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$

Similarly $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc. ...(5)

Substituting for $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0$ ... from (2), (3), (4) ... in (1), we get

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{1.2.3}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$
$$+ \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$

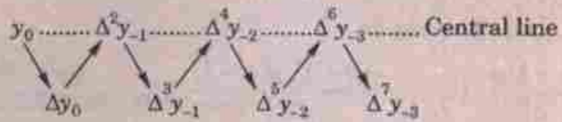
Hence $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots \text{ [Using (5)]}$$

which is called *Gauss's forward interpolation formula.*

Cor. *In the central differences notation, this formula will be*

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2} + \frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 y_0 + \ldots$$

Obs. 1. *It employs odd differences just below the central line and even difference on the central line as shown below:*



Obs. 2. This formula is used to interpolate the values of y for p $(0 < p < 1)$ measured forwardly from the origin.

(2) Gauss's backward interpolation formula. The Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \ldots \quad \ldots(1)$$

We have $\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$

i.e., $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$ ...(2)

Similarly $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$ ...(3)

$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ etc. ...(4)

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

i.e., $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$ ...(5)

Similarly $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc. ...(6)

Substituting for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$, ... from (2), (3), (4) in (1), we get

$$y_p = y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{1.2}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{1.2.3}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$
$$+ \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$
$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1.2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1.2.3}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{1.2.3.4}\Delta^4 y_{-1}$$
$$+ \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}\Delta^5 y_{-1} + \ldots$$

$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1.2}\Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{1.2.3}(\Delta^3 y_{-2} + \Delta^4 y_{-2})$$

$$+ \frac{(p+1)\,p\,(p-1)(p-2)}{1.2.3.4}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \ldots \quad \text{[Using (5) and (6)]}$$

Hence $$y_p + y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)\,p\,(p-1)}{4!}\Delta^4 y_{-2} + \ldots$$

which is called *Gauss's backward interpolation formula.*

Cor. *In the central differences notation, this formula will be*

$$y_p = y_0 + p\delta y_{-1/2} + \frac{(p+1)p}{2!}\delta^2 y_0 + \frac{(p+1)\,p(p-1)}{3!}\delta^3 y_{-1/2} + \frac{(p+2)(p+1)\,p\,(p-1)}{4!}\delta^4 y_0 + \ldots$$

Obs. 1. This formula contains odd differences above the central line and even differences on the central line as shown below :

$$\begin{array}{ccccccc} & \Delta y_{0-1} & & \Delta^3 y_{-2} & & \Delta^5 y_{-3} & \\ y_0 \cdots\cdots & & \Delta^2 y_{-1} \cdots\cdots & & \Delta^4 y_{-2} \cdots\cdots & & \Delta^6 y_{-3} \cdots\cdots \text{Central line} \end{array}$$

Obs. 2. It is used to interpolate the values of y for a negative value of p lying between -1 and 0.

Obs. 3. Gauss's forward and backward formulae are not of much practical use. However, these serve as intermediate steps for obtaining the important formulae of the following sections.

(3) **Stirling's formula.*** Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(1)$$

Gauss's backward interpolation formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)\,p\,(p-1)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(2)$$

Taking the mean of (1) and (2), we obtain

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p\,(p^2-1)}{3!} \times \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2\,(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(3)$$

which is called *Stirling's formula*.

Cor. *In the central differences notation, (3) takes the form*

$$y_p = y_0 + p\,\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \frac{p\,(p^2+1^2)}{3!}\mu\delta^3 y_0 + \frac{p^2\,(p^2-1^2)}{4!}\delta^4 y_0 + \ldots \quad \ldots(4)$$

for $$\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2}) = \mu\delta y_0$$

$$\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) = \frac{1}{2}(\delta^3 y_{1/2} + \delta^3 y_{-1/2}) = \mu\delta^3 y_0 \text{ etc.}$$

Obs. *This formula involves means of the odd differences just above and below the central line and even differences on this line* as shown below :

$$\ldots y_0 \ldots \begin{pmatrix}\Delta y_{-1}\\ \Delta y_0\end{pmatrix} \ldots \Delta^2 y_{-1} \ldots \begin{pmatrix}\Delta^3 y_{-2}\\ \Delta^3 y_{-1}\end{pmatrix} \ldots \Delta^4 y_{-2} \ldots \begin{pmatrix}\Delta^5 y_{-3}\\ \Delta^5 y_{-2}\end{pmatrix} \ldots \Delta^6 y_{-3} \ldots \text{ Central line}$$

(4) **Bessel's formula.**** Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} \ldots \quad \ldots(1)$$

*Named after the Scottish mathematicians *James Stirling* (1692-1770).

**See footnote p. 550.

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

i.e., $\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1}$ ...(2)

Similarly $\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2}$ etc. ...(3)

Now (1) can be written as

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{1}{2}\Delta^2 y_{-1} + \frac{1}{2}\Delta^2 y_{-1}\right) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \frac{p(p^2-1)(p-2)}{4!}\left(\frac{1}{2}\Delta^4 y_{-2} + \frac{1}{2}\Delta^4 y_{-2}\right) + \dots$$

$$= y_0 + p\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p-1)}{2!}(\Delta^2 y_0 - \Delta^3 y_{-1}) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!} \times (\Delta^4 y_{-1} - \Delta^5 y_{-2}) + \dots \quad \text{[Using (2), (3) etc.]}$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)}{2!} \times \left(\frac{p+1}{3} - \frac{1}{2}\right)\Delta^3 y_{-1} + \frac{p(p^2-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \dots$$

Hence $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-1/2)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \dots \quad \dots(4)$$

which is known as the *Bessel's formula.*

Cor. *In the central differences notation, (4) becomes*

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\mu\delta^2 y_{1/2} + \frac{(p-1/2)\,p(p-1)}{3!}\delta^3 y_{1/2} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\mu\delta^4 y_{1/2} + \dots \quad \dots(5)$$

for $\frac{1}{2}(\Delta^2 y_{-1} + \Delta^2 y_0) = y\delta^2 y_{1/2}$, $\frac{1}{2}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) = \mu\delta^4 y_{1/2}$ etc.

Obs. *This is a very useful formula for practical purposes. It involves odd differences below the central line and means of even differences of and below his line as shown below :*

$$\begin{array}{llllllll} y_0 & \left\{\begin{matrix}\Delta^2 y_{-1} \\ \\ \Delta^2 y_0\end{matrix}\right\} & & \left\{\begin{matrix}\Delta^4 y_{-2} \\ \\ \Delta^4 y_{-1}\end{matrix}\right\} & & \left\{\begin{matrix}\Delta^6 y_{-3} \\ \\ \Delta^6 y_{-2}\end{matrix}\right\} & \text{— Central line} \\ & \Delta y_0 & \Delta^3 y_{-1} & & \Delta^5 y_{-2} & & \end{array}$$

(5) Everett's formula. Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)\,p(p-1)(p-2)}{5!}\Delta^5 y_{-2} + \dots \quad \dots(1)$$

We eliminate the odd difference in (1) by using the relations

$$\Delta y_0 = y_1 - y_0,\ \Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1},\ \Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2} \text{ etc.}$$

Then (1) becomes

$$y_p = y_0 + p(y_1 - y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}(\Delta^2 y_0 - \Delta^2 y_{-1})$$

$$+ \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)\,p(p-1)(p-2)}{5!} \times (\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \dots$$

$$= (1-p)\,y_0 + py_1 - \frac{p(p-1)(p-2)}{3!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^2 y_0$$

$$- \frac{(p+1)\,p\,(p-1)(p-2)(p-3)}{5!}\,\Delta^4 y_{-2} + \frac{(p+2)(p+1)\,p\,(p-1)(p-2)}{5!}\,\Delta^4 y_{-1} - \dots$$

To change the terms with negative sign, putting $p = 1 - q$, we obtain

$$y_p = q y_0 + \frac{q(q^2-1^2)}{3!}\,\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\,\Delta^4 y_{-2} + \dots\dots + p y_1 + \frac{p(p^2-1^2)}{3!}\,\Delta^2 y_0 + \frac{p\,(p^2-1^2)(p^2-2^2)}{5!}\,\Delta^4 y_{-1} + \dots$$

This is known as *Everett's formula.*

Obs. *This formula is extensively used and involves only even differences on and below the central line* as shown below :

y_0 ...... $\Delta^2 y_{-1}$ ...... $\Delta^4 y_{-2}$ ...... $\Delta^6 y_{-3}$ ...... Central line

y_1 $\quad \Delta^2 y_0$ $\quad \Delta^4 y_{-1}$ $\quad \Delta^6 y_{-2}$

29.8 CHOICE OF AN INTERPOLATION FORMULA

The coefficients in the central difference formulae are smaller and converge faster than those in Newton's formulae. After a few terms, the coefficients in the Stirling's formula decrease more rapidly than those of the Bessel's formula and the coefficients of Bessel's formula decrease more rapidly than those of Newton's formula. As much, whenever possible, *central difference formulae should be used in preference to Newton's formulae.*

The right choice of an interpolation formula however, depends on the position of the interpolated value in the given data.

The following rules will be found useful :

1. *To find a tabulated value near the beginning of the table, use Newton's forward formula.*
2. *To find a value near the end of the table, use Newton's backward formula.*
3. *To find an interpolated value near the centre of the table, use either Stirling's or Bessel's or Everett's formula.*

If interpolation is required for p lying between –1/4 and 1/4, prefer Stirling's formula.

If interpolation is desired for p lying between 1/4 and 3/4, use Bessel's or Everett's formula.

Example 29.18. *Find f(22) from the Gauss forward formula :*

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

(J.N.T.U., 2007)

Solution. Taking $x_0 = 25$, $h = 5$, we have to find the value of $f(x)$ for $x = 22$.

i.e., for $\quad p = \dfrac{x - x_0}{h} = \dfrac{22-25}{5} = -0.6$

The difference table is as follows :

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
20	–1	354 $(= y_{-1})$					
			–22				
25	0	332 $(= y_0)$		–19			
			–41		29		
30	1	291 $(= y_1)$		10		–37	
			–31		–8		45
35	2	260 $(= y_2)$		2		8	
			–29		0		
40	3	231 $(= y_3)$		2			
			–27				
45	4	204 $(= y_4)$					

Gauss forward formula is

$$y^p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta y_{-1} + \frac{(p+1)\,p\,(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)\,p\,(p-1)(p-2)}{4!}\Delta^4 y_{-2} + (p+1)(p-1)(p-2)(p+2)\,\Delta^5 y_{-2}$$

$\therefore$

$$f(22) = 332 + (0.6)(-41) + \frac{(-0.6)(-0.6-1)}{2!}(-19) + \frac{(-0.6+1)(-0.6)(-0.6-1)}{3!}(-8)$$

$$+ \frac{(-0.6-1)(-0.6)(-0.6-1)(-0.6-2)}{4!}(-37)$$

$$+ \frac{(-0.6+1)(-0.6)(-0.6-1)(-0.6-2)(-0.6+2)}{5!}(45)$$

$$= 332 + 24.6 - 9.12 + 1.5392 - 0.5241$$

Hence $f(22) = 347.983.$

Example 29.19. *Interpolate by means of Gauss's backward formula, the population of a town for the year 1974, given that :*

Year :	*1939*	*1949*	*1959*	*1969*	*1979*	*1989*
Population (in thousands) :	*12*	*15*	*20*	*27*	*39*	*52*

(Kottayam, 2005 ; Madras, 2003)

Solution. Taking $x_0 = 1969$, $h = 10$, the population of the town is to be found for $p = \dfrac{1974-1969}{10} = 0.5.$

The central difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
1939	−3	12					
			3				
1949	−2	15		2			
			5		0		
1959	−1	20		2		3	
			7		3		−10
1969	0	27		5		−7	
			12		−4		
1979	1	39		1			
			13				
1989	2	52					

Gauss's backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)\,p}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+1)(p+1)\,p\,(p-1)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{(p+2)(p+1)\,p\,(p-1)(p-2)}{5!}\Delta^5 y_3 + \ldots$$

i.e.,

$$y_5 = 27 + (0.5)(7) + \frac{(1.5)(.5)}{2}(5) + \frac{(1.5)(.5)(-.5)}{6}(3) + \frac{(2.5)(1.5)(-.5)}{24}(-7)$$

$$+ \frac{(2.5)(1.5)(.5)(-.5)(-1.5)}{120}(-10)$$

$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2743 - 0.1172 = 32.345 \text{ thousands approx.}$$

Example 29.20. *Given*

θ° :	*0*	*5*	*10*	*15*	*20*	*25*	*30*
tan θ :	*0*	*0.0875*	*0.1763*	*0.2679*	*0.3640*	*0.4663*	*0.5774*

Using Stirling's formula, estimate the value of tan 16°. (Anna, 2005)

Solution. Taking the origin at $\theta° = 15°$, $h = 5°$ and $p = \dfrac{\theta - 15}{5}$, we have the following central difference table :

p						
−3	0.0000					
		0.08575				
−2	0.0875		0.0013			
		0.0888				
−1	0.1763		0.0028			
		0.0916		0.0015		
0	0.2679		0.0045		0.0002	
		0.0961		0.0017		−0.0002
1	0.3640		0.0062		0.0000	
		0.1023		0.0017		0.0009
2	0.4663		0.0088		0.0009	
		0.1111		0.0026		
3	0.5774					

At $\theta = 16°$, $\quad p = \dfrac{16 - 15}{5} = 0.2$

Stirling's formula is

$$y_p = y_0 + \frac{p}{1} \cdot \frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \cdot \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \dots$$

$$\therefore \quad y_{0.2} = 0.2679 + (0.2)\left(\frac{0.0916 + 0.0961}{2}\right) + \frac{(0.2)^2}{2!}(0.0045) + \dots$$

$$= 0.2679 + 0.01877 + 0.00009 + \dots = 0.28676$$

Hence $\quad \tan 16° = 0.28676.$

Example 29.21. *Employ Stirling's formula to compute* $y_{12.2}$ *from the following table* ($y_x = 1 + \log_{10} \sin x$) :

$x°$:	*10*	*11*	*12*	*13*	*14*
$10^5 y_x$:	*23,967*	*28,060*	*31,788*	*35,209*	*38,368*

(V.T.U., 2004)

Solution. Taking the origin at $x_0 = 12°$, $h = 1$ and $p = x - 12$, we have the following central table :

p	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
−2	0.23967				
		0.04093			
−1	0.28060		−0.00365		
		0.03728		0.00058	
0	0.31788		−0.00307		−0.00013
		0.034121		−0.00045	
1	0.35209		−0.00062		
		0.03159			
2	0.38368				

At $x = 12.2$, $p = 0.2$. (As p lies between − 1/4 and 1/4, the use of Stirling's formula will be quite suitable.)

Stirling's formula is

$$y_p = y_0 + \frac{p}{1}\cdot\frac{\Delta y_{-1} + \Delta y_{-0}}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\cdot\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots$$

When $p = 0.2$, we have

$$\therefore \quad y_{0.2} = 0.31788 + 0.2\left(\frac{0.03728 + 0.03421}{2}\right) + \frac{(0.2)^2}{2}(-0.00307)$$

$$+ \frac{(0.2)[(0.2)^2 - 1]}{6}\left(\frac{0.00058 - 0.00045}{2}\right) + \frac{(0.2)^2[(0.2)^2 - 1]}{24}(-0.00013)$$

$$= 0.31788 + 0.00715 - 0.00006 - 0.000002 + 0.0000002 = 0.32497.$$

Example 29.22. *Apply Bessel's formula to obtain y_{25}, given $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$.* *(S.V.T.U., 2007 ; V.T.U., 2000 S)*

Solution. Taking the origin at $x_0 = 24$, $h = 4$, we have $p = \frac{1}{4}(x - 24)$.

$\therefore$ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$
–1	2854			
		308		
0	*3162*		*74*	
		382		*–8*
1	3544		*66*	
		448		
2	3992			

At $x = 25$, $p = (25 - 24)/4 = 1/4$. (As p lies between 1/4 and 3/4, the use of Bessel's formula will yield accurate esult.)

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-1/2)\,p\,(p-1)}{3!}\Delta^3 y_{-1} + \ldots \qquad \ldots(1)$$

When $p = 0.25$, we have

$$y_p = 3162 + 0.25 \times 382 + \frac{0.25(-0.75)}{2}\left(\frac{74 + 66}{2}\right) + \frac{(-0.25)\,0.25\,(-0.75)}{6}(-8)$$

$$= 3162 + 95.5 - 6 - 5625 - 0.0625 = 3250.875 \text{ approx.}$$

Example 29.23. *Apply Bessel's formula to find the value of f(27.5) from the table :*

x :	25	26	27	28	29	30
$f(x)$:	4.000	3.846	3.704	3.571	3.448	3.333

(U.P.T.U., 2009)

Solution. Taking the origin at $x_0 = 27$, $h = 1$, we have $p = x - 27$

The central difference table is

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
25	–2	4.000				
			–0.154			
26	–1	3.846		0.012		
			–0.142		–0.003	
27	0	3.704		0.009		0.004
			–0.133		–0.001	
28	1	3.571		0.010		–0.001
			–0.123		–0.002	
29	2	3.448		0.008		
			–0.115			
30	3	3.333				

At $x = 27.5$, $p = 0.5$ (As p lies between 1/4 and 3/4, the use of Bessel's formula will yield accurate result)
Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)\,p(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \ldots$$

When $p = 0.5$, we have

$$y_p = 3.704 - \frac{(0.5)(0.5-1)}{2}\left(\frac{0.009 + 0.010}{2}\right) + 0$$
$$+ \frac{(0.5+1)(0.5)(0.5-1)(0.5-2)}{24}\left(\frac{-0.001 - 0.004}{2}\right)$$
$$= 3.704 - 0.11875 - 0.00006 = 3.585$$

Hence $f(27.5) = 3.585$.

Example 29.24. *Given the table*

x :	310	320	330	340	350	360
$\log x$:	2.49136	2.50515	2.51851	2.53148	2.54407	2.55630

find the value of log 337.5 by Everett's formula.

Solution. Taking the origin at $x_0 = 330$ and $h = 10$, we have $p = \dfrac{x - 330}{10}$

$\therefore$ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
−2	2.49136					
		0.01379				
−1	2.50515		−0.00043			
		0.01336		0.00004		
0	2.51881		−0.00039		−0.00003	
		0.01297		0.00001		0.00004
1	2.53148		−0.00038		0.00001	
		0.01259		0.00002		
2	2.54407		−0.00036			
		0.01223				
3	2.55630					

To evaluate log 337.5 *i.e.* for $x = 337.5$, $p = \dfrac{337.5 - 330}{10} = 0.75$

(As $p > 0.5$ and $= 0.75$, Everett's formula will be quite suitable)
Everett's formula is

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots + py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0$$
$$+ \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots$$
$$= 0.25 \times 2.51851 + \frac{0.25\,(0.0625-1)}{6} \times (-0.00039) + \frac{0.25\,(0.0625-1)(0.0625-4)}{120}$$
$$\times (-0.00003) + 0.75 \times 2.53148 + \frac{0.75\,(0.5625-1)}{6} \times (-0.00038)$$
$$+ \frac{0.75\,(0.5625-1)(0.5625-4)}{120} \times (0.00001)$$
$$= 0.62963 + 0.00002 - 0.0000002 + 1.89861 + 0.00002 + 0.0000001 = 2.52828 \text{ nearly.}$$

PROBLEMS 29.4

1. Using Gauss's forward formula, evaluate $f(3.75)$ from the table :

x:	2.5	3.0	3.5	4.0	4.5	5.0
y:	24.145	22.043	20.225	18.644	17.262	16.047

(*Bhopal, 2002 ; Madras, 2000*)

2. Using Gauss's backward difference formula, find $y(8)$ from the following table :

x:	0	5	10	15	20	25
y:	7	11	14	18	24	32

(*J.N.T.U., 2007*)

3. Using Gauss's backward formula, estimate the number of persons earning wages between Rs. 60 and Rs. 70 from the following data :

Wages (₹) :	Below 40	40–60	60–80	80–100	100–120
No. of persons : (in thousands)	250	120	100	70	50

(*Madras, 2000*)

4. From the following table :

x :	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x :	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

(*U.P.T.U., 2006*)

Find $e^{1.17}$, using Gauss forward formula.

5. The pressure p of wind corresponding to velocity v is given by the following data. Estimate p when $y = 25$.

v:	10	20	30	40
p:	1.1	2	4.4	7.9

6. Using Stirling's formula find y_{35}, given $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$, where y_x represents the number of persons at age x years in a life table. (*Nagarjuna, 2003 S*)

7. Employ Bessel's formula to find the value of F at $x = 1.95$, given that

x:	1.7	1.8	1.9	2.0	2.1	2.2	2.3
F:	2.979	3.144	3.283	3.391	3.463	3.997	4.491

Which other interpolation formula can be used here ? Which is more appropriate ? Give reasons.

8. Calculate the value of $f(1.5)$ using Bessel's interpolation formula, from the following table :

x :	0	1	2	3
$f(x)$:	3	6	12	15

(*U.P.T.U., 2008*)

9. Apply Everett's formula to obtain u_{25}, given $u_{20} = 854$, $u_{24} = 3162$, $u_{28} = 3544$, $u_{32} = 3992$. (*S.V.T.U., 2007*)

10. Using Everett's formula, evaluate $f(30)$, if $f(20) = 2854$, $f(28) = 3162$, $f(36) = 7088$, $f(44) = 7984$ (*U.P.T.U., 2006*)

11. Given the table :

x :	310	320	330	340	350	360
$\log x$:	2.4914	2.5052	2.5185	2.5315	2.5441	2.5563

Find the value of log 337.5 by Gauss's, Stirling's and Bessel's formulae.

29.9 INTERPOLATION WITH UNEQUAL INTERVALS

The various interpolation formulae derived so far possess the disadvantages of being applicable only to equally spaced values of the argument. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. Now we shall study two such formulae :

(*i*) Lagrange's interpolation formula

(*ii*) Newton's general interpolation formula with divided differences.

29.10 LAGRANGE'S INTERPOLATION FORMULA

If $y = f(x)$ takes the value $y_0, y_1, \ldots, y_n$ corresponding to $x = x_0, x_1, \ldots, x_n$, then

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}\, y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}\, y_1 + \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}\, y_n \quad \ldots(1)$$

This is known as *Lagrange's interpolation formula for unequal intervals.*

Proof. Let $y = f(x)$ be a function which takes the values $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$. Since there are $n + 1$ pairs of values of x and y, we can represent $f(x)$ by a polynomial in x of degree n. Let this polynomial be of the form

$$y = f(x) = a_0(x - x_1)(x - x_2) \ldots (x - x_n) + a_1(x - x_0)(x - x_2) \ldots (x - x_n)$$
$$+ a_2(x - x_0)(x - x_1)(x - x_3) \ldots (x - x_n) + \ldots + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1}) \quad \ldots(2)$$

Putting $x = x_0, y = y_0$, in (2), we get

$$y_0 = a_0(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)$$
$$a_0 = y_0/[(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)]$$

Similarly putting $x = x_1, y = y_1$ in (2), we have $a_1 = y_1/[(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)]$

Proceeding the same way, we find $a_2, a_3 \ldots a_n$

Substituting the values of $a_0, a_1, \ldots, a_n$ in (2), we get (1).

Obs. Lagranges interpolation formula (1) for n points is a polynomial of degree $(n - 1)$ which is known as *Lagrangian polynomial* and is very simple to implement on a computer.

This formula can also be used to split the given function into partial fractions.

For on dividing both sides of (1) by $(x - x_0)(x - x_1) \ldots (x - x_n)$, we get

$$\frac{f(x)}{(x_0 - x_0)(x_0 - x_1)\ldots(x_0 - x_n)} = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} \cdot \frac{1}{x - x_0}$$
$$+ \frac{y_0}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} \cdot \frac{1}{x - x_1} + \ldots + \frac{y_n}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} \cdot \frac{1}{x - x_n}.$$

Example 29.25. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1492	2366	5202

evaluate f (9), using (i) Lagrange's formula. (Anna, 2006)

Solution. (*i*) Here $x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17$

and $y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202.$

Putting $x = 9$ and substituting the above values in Lagrange's formula, we get

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$
$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$
$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202 = -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810.$$

Example 29.26. *Find the polynomial f(x) by using Lagrange's formula and hence find f (3) for*

x :	0	1	2	5
$f(x)$:	2	3	12	147

(Anna, 2005)

Solution. Here $x_0 = 0, x_1, x_2 = 2, x_3 = 5$

and $y_0 = 2, y_1 = 3, y_2 = 12, y_3 = 147$

Lagrange's formula is

$$y = \frac{(x - x_1)(x - x_2)\ldots(x - x_3)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)\ldots(x - x_3)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_3)} y_1$$
$$+ \frac{(x - x_0)(x - x_1)\ldots(x - x_3)}{(x_2 - x_0)(x_2 - x_1)\ldots(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_2)\ldots(x - x_5)}{(x_3 - x_0)(x_3 - x_2)\ldots(x_3 - x_2)} y_3$$
$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$
$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

Hence $\quad f(x) = x^3 + x^2 - x + 2$

$\therefore \quad f(3) = 27 + 9 - 3 + 2 = 35.$

Example 29.27. *A curve passes through the point (0, 18), (1, 10), (3, – 18) and (6, 90). Find the slope of the curve at x = 2.* (J.N.T.U., 2009)

Solution. Here $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 6$ and $y_0 = 18, y_1 = 10, y_2 = -18, y_3 = 90$

Since the values of x are unequally spaced, we use the Lagrange's formula :

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-1)(x-3)(x-6)}{(0-1)(0-2)(0-6)}(18) + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}(10) + \frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}(90)$$

$$= (-x^3 + 10x^2 - 27x + 18) + (x^3 - 9x^2 + 18x) + (x^3 - 7x^2 + 6x) + (x^3 - 4x^2 + 3x)$$

i.e., $\quad y = 2x^3 - 10x^2 + 18$

Thus the slope of the curve at $(x = 2) = \left(\frac{dy}{dx}\right)_{x=2}$

$$= (6x^2 - 20x)_{x=2} = -16.$$

Example 29.28. *Using Lagrange's formula, express the function* $\frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)}$ *as a sum of partial fractions.*

Solution. Let us evaluate $y = 3x^2 + x + 1$ for $x = 1, x = 2$ and $x = 3$

These values are

x :	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$
y :	$y_0 = 5$	$y_1 = 15$	$y_2 = 31$

Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-1)}(31)$$

Substituting the above values, we get

$$= \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$

$$= 2.5(x-2)(x-3) - 15(x-1)(x-3) + 15.5(x-1)(x-2)$$

Thus $\quad \frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)} = \frac{2.5(x-2)(x-3) - 15(x-1)(x-3) + 15.5(x-1)(x-2)}{(x-1)(x-2)(x-3)}$

$$= \frac{2.5}{x-1} - \frac{15}{x-2} + \frac{15.5}{x-3}.$$

Example 29.29. *Find the distance moved by a particle and its acceleration at the end of 4 seconds, if the time verses velocity data is as follows :*

t :	0	1	3	4
v :	21	15	12	10

Solution. Since the values of t are not equispaced, we use Lagrange's formula :

$$v = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)} v_0 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)} v_1$$

$$+ \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)} v_2 + \frac{(t-t_0)(t-t_2)(t-t_5)}{(t_3-t_0)(t_3-t_2)(t_3-t_2)} v_3$$

$$v = \frac{(t-1)(t-3)(t-4)}{(-1)(-2)(-4)}(21) + \frac{t(t-3)(t-4)}{(1)(-2)(-3)}(15) + \frac{t(t-1)(t-4)}{(3)(2)(-1)}(12) + \frac{t(t-1)(t-3)}{(4)(3)(1)}(10)$$

i.e., $$v = \frac{1}{12}(-5t^3 + 38t^2 - 105t + 252)$$

$\therefore$ Distance moved $s = \int_0^4 v\,dt = \frac{1}{12}\int_0^4 (-5t^3 + 38t^2 - 105t + 252)\,dt$ $\quad \left[\because\ v = \frac{ds}{dt}\right]$

$$= \frac{1}{12}\left(-\frac{5t^4}{4} + \frac{38t^3}{3} - \frac{105t^2}{2} + 252t\right)_0^4$$

$$= \frac{1}{12}\left(-320 + \frac{2432}{3} - 840 + 1008\right) = 54.9$$

Also acceleration $= \frac{dv}{dt} = \frac{1}{2}(-15t^2 + 76t - 105 + 0)$

Hence acceleration at $(t = 4) = \frac{1}{12}(-15(16) + 76(4) - 105) = -3.4.$

PROBLEMS 29.5

1. Use Lagrange's interpolation formula to find the value of y when $x = 10$, if the following values of x and y are given :

x :	5	6	9	11
y :	12	13	14	16

(*U.P.T.U., 2009 ; J.N.T.U., 2008*)

2. Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$, find by using Lagrange's formula, the value of $\log_{10} 656$. (*Hazaribagh, 2009*)

3. The following are the measurements T made on a curve recorded by oscilograph representing a change of current I due to a change in the conditions of an electric current.

T :	1.2	2.0	2.5	3.0
I :	1.36	0.58	0.34	0.20

Using Lagrange's formula, find I at $T = 16$. (*J.N.T.U., 2009*)

4. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data :

Year :	1997	1999	2001	2002
Profit in Lakhs of ₹ :	43	65	159	248

(*Anna, 2004*)

5. Use Lagrange's formula to find the form of $f(x)$, given

x :	0	2	3	6
$f(x)$:	648	704	729	792

(*Madras, 2003 S*)

6. If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$, $y(6) = 132$, find the Lagrange's interpolation polynomial that takes the same values as y at the given points. (*V.T.U., 2006*)

7. Given $f(0) = -18$, $f(1) = 0$, $f(3) = 0$, $f(5) = -248$, $f(6) = 0$, $f(9) = 13104$, find $f(x)$. (*Nagarjuna, 2003*)

8. Find the missing term in the following table using interpolation

x :	1	2	4	5	6
y :	14	15	5	...	9

9. Using Lagrange's formula, express the function $\frac{x^2 + x - 3}{x^3 - 2x^2 - x + 2}$ as sum of partial fractions.

29.11 DIVIDED DIFFERENCES

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. The labour of recomputing the interpolation coefficients is saved by using Newton's general interpolation formula which employs what are called 'divided differences'. Before deriving this formula, we shall first define these differences.

If $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots$ be given points, then the *first divided difference* for the arguments, x_0, x_1 is defined by the relation $[x_0, x_1] = \dfrac{y_1 - y_0}{x_1 - x_0}$.

Similarly $[x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $[x_2, x_3] = \dfrac{y_3 - y_2}{x_3 - x_2}$ etc.

The *second divided difference* for x_0, x_1, x_2 is defined as $[x_0, x_1, x_2] = \dfrac{[x_1 - x_2] - [x_0, x_1]}{x_2 - x_0}$

The *third divided difference* for x_0, x_1, x_2, x_3 is defined as

$[x_0, x_1, x_2, x_3] = \dfrac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$ and so on.

Obs. 1. *The divided differences are symmetrical in their arguments i.e. independent of the order of the arguments.*

For it is easy to write $[x_0, x_1] = \dfrac{y_0}{x_0 - x_1} + \dfrac{y_1}{x_1 - x_0} = [x_1, x_0]$ $[x_0, x_1, x_2]$

$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} \cdot \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$= [x_1, x_2, x_0]$ or $[x_2, x_0, x_1]$ and so on.

Obs. 2. *The nth divided differences of a polynomial of the nth degree are constant.*

Let the arguments be equally spaced so that, $x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$. Then

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} = \frac{1}{2h}\left\{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right\}$$

$= \dfrac{1}{2!\,h^2}\,\Delta^2 y_0$ and in general, $[x_0, x_1, x_2, \ldots x_n] = \dfrac{1}{n!\,h^n}\,\Delta^n y_0$.

If the tabulated function is a nth degree polynomial, then $\Delta^n y_0$ will be constant. Hence the nth divided differences will also be constant.

29.12 NEWTON'S DIVIDED DIFFERENCE FORMULA

Let $y_0, y_1, \ldots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$. Then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

so that $\qquad y = y_0 + (x - x_0)\,[x, x_0] \qquad \ldots(1)$

Again $\qquad [x, x_0, x_1] = \dfrac{[x, x_0] - [x_0, x_1]}{x - x_1}$

which gives $\qquad [x, x_0] = [x_0, x_1] + (x - x_1) + (x - x_1)\,[x, x_0, x_1]$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x, x_0, x_1] \qquad \ldots(2)$$

Also $\qquad [x, x_0, x_1, x_2] = \dfrac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$

which gives $[x, x_0, x_1,] = [x_0, x_1, x_2] + (x, x_2)\,[x, x_0, x_1, x_2]$

Substituting this value of $[x, x_0, x_1]$ in (2), we obtain

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\,[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$y = f(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \dots + (x - x_0)(x - x_1)\dots(x - x_n)[x, x_0, x_1, \dots, x_n] \quad \dots(3)$$

which is called *Newton's general interpolation formula with divided differences.*

Example 29.30. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202,

evaluate f (9), using Newton's divided difference formula. (V.T.U., 2010 ; P.T.U., 2005)

Solution. The divided difference table is

x	y	1st divided differences	2nd divided differences	3rd divided differences
5	150			
		$\frac{392-150}{7-5} = 121$		
7	392		$\frac{265-121}{11-5} = 24$	
		$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$
11	1452		$\frac{457-265}{13-7} = 32$	
		$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$
13	2366		$\frac{709-457}{17-11} = 42$	
		$\frac{5202-2366}{17-13} = 709$		
17	5202			

Taking $x = 9$ in the Newton's divided difference formula, we obtain

$$f(9) = 150 + (9 - 5) \times 121 + (9 - 5)(9 - 7) \times 24 + (9 - 5)(9 - 7)(9 - 11) \times 1$$
$$= 150 + 484 + 192 - 16 = 810.$$

Example 29.31. *Determine f (x) as a polynomial in x for the following data :*

x :	– 4	– 1	0	2	5
$f(x)$:	1245	33	5	9	1335

(V.T.U., 2007)

Solution. The divided differences table is

x	$f(x)$	1st divided differences	2nd divided differences	3rd divided differences	4th divided differences
– 4	1245				
		– 404			
– 1	33		94		
		– 28		– 14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

Applying Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \ldots$$
$$= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94)$$
$$+ (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2)(3)$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$

PROBLEMS 29.6

1. Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$. (*U.P.T.U., 2005*)
2. Use Newton's divided difference method to compute $f(5.5)$ from the following data :

x :	0	1	4	5	6
$f(x)$:	1	14	15	6	3

(*U.P.T.U., 2010*)

3. Using Newton's divided difference formula, evaluate $f(8)$ and $f(15)$ given :

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

(*U.P.T.U., MCA, 2009, V.T.U., 2008*)

4. Obtain the Newton's divided difference interpolation polynomial and hence find $f(6)$:

x :	3	7	9	10
$f(x)$:	168	120	72	63

(*U.P.T.U., 2007*)

5. Using Newton's divided difference interpolation, find the polynomial of the given data :

x :	−1	0	1	3
$f(x)$:	2	1	0	−1

(*Anna, 2007*)

6. For the following table, find $f(x)$ as a polynomial in x using Newton's divided difference formula:

x :	5	6	9	11
$f(x)$:	12	13	14	16

7. Using the following table, find $f(x)$ as a polynomial in

x :	−1	0	3	6	7
$f(x)$:	3	−6	39	822	1611

(*U.P.T.U., 2009*)

8. Find the missing term in the following table using Newton's divided difference formula

x :	0	1	2	3	4
y :	1	3	9	...	81

29.13 INVERSE INTERPOLATION

So far, given a set of values of x and y, we have been finding the values of y corresponding to a certain value of x. On the other hand, the process of estimating the value of x for a value of y (which is not in the table) is called the *inverse interpolation*.

Lagrange's formula is merely a relation between two variables either of which may be taken as the independent variable. Therefore, on inter-changing x and y in the Lagrange's formula, we obtain

$$x = \frac{(y - y_1)(y - y_2)\ldots(y - y_n)}{(y_0 - y_1)(y_0 - y_2)\ldots(y_0 - y_n)}x_0 + \frac{(y - y_0)(y - y_2)\ldots(y - y_n)}{(y_1 - y_0)(y_1 - y_2)\ldots(y_1 - y_n)}x_1$$
$$+ \ldots + \frac{(y - y_0)(y - y_1)\ldots(y - y_{n-1})}{(y_n - y_0)(y_n - y_1)\ldots(y_n - y_{n-1})}x_n \quad \ldots(1)$$

which is used for inverse interpolation.

Example 29.32. *The following table gives the values of x and y :*

x :	1.2	2.1	2.8	4.1	4.9	6.2
y :	4.2	6.8	9.8	13.4	15.5	19.6

Find the value of x corresponding to $y = 12$, using Lagrange's technique. (V.T.U., 2009)

Solution. Here $x_0 = 1.2, x_1 = 2.1, x_2 = 2.8, x_3 = 4.1, x_4 = 4.9, x_5 = 6.2$

and $y_0 = 4.2, y_1 = 6.8, y_2 = 9.8, y_3 = 13.4, y_4 = 15.5, y_5 = 19.6$

Taking $y = 12$, the above formula (1) gives

$$x = \frac{(12-6.8)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(4.2-6.8)(4.2-9.8)(4.2-13.4)(4.2-15.5)(4.2-19.6)} \times 1.2$$

$$+ \frac{(12-4.2)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(6.8-4.2)(6.8-9.8)(6.8-13.4)(6.8-15.5)(6.8-19.6)} \times 2.1$$

$$+ \frac{(12-4.2)(12-6.8)(12-13.4)(12-15.5)(12-19.6)}{(9.8-4.2)(9.8-6.8)(9.8-13.4)(9.8-15.5)(9.8-19.6)} \times 2.8$$

$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-15.5)(12-19.6)}{(13.4-4.2)(13.4-6.8)(13.4-9.8)(13.4-15.5)(13.4-19.6)} \times 4.1$$

$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-19.6)}{(15.5-4.2)(15.5-6.8)(15.5-9.8)(15.5-13.4)(15.5-19.6)} \times 4.9$$

$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-15.5)}{(19.6-4.2)(19.6-6.8)(19.6-9.8)(19.6-13.4)(19.6-15.5)} \times 6.2$$

$$= 0.022 - 0.234 + 1.252 + 3.419 - 0.964 + 0.055 = 3.55.$$

Example 29.33. *Apply Lagrange's formula inversely to obtain a root of the equation $f(x) = 0$, given that $f(30) = -30$, $f(34) = -13$, $f(38) = 3$, and $f(42) = 18$.* (V.T.U., 2009 S)

Solution. Here $x_0 = 30, x_1 = 34, x_2 = 38, x_3 = 42$

and $y_0 = -30, y_1 = -13, y_2 = 3, y_3 = 18$

It is required to find x corresponding to $y = f(x) = 0$.

Taking $y = 0$, the Lagrange's formula gives,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y-y_2)(y_1-y_3)} x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} x_3$$

$$= \frac{13(-3)(-18)}{(-17)(-33)(-48)} \times 30 + \frac{30(-3)(-18)}{17(-16)(-31)} \times 34 + \frac{30(13)(-18)}{33(16)(-15)} \times 38 + \frac{30(13)(-3)}{48(31)(15)} \times 42$$

$$= -0.782 + 6.532 + 33.682 - 2.202 = 37.23$$

Hence the desired root of $f(x) = 0$ is 37.23.

PROBLEMS 29.7

1. Apply Lagrange's method to find the value of x when $f(x) = 15$ from the given data :

x :	5	6	9	11
$f(x)$:	12	13	14	16

(Madras, 2000)

2. Obtain the value of t when $A = 85$ from the following table, using Lagrange's method :

t :	2	5	8	14
A :	94.8	87.9	81.3	68.7

29.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 29.8

Select the correct answer or fill up the blanks in the following problems :

1. Newton's backward interpolation formula is ......
2. Bessel's formula is most appropriate when p lies between
   (a) -0.25 and 0.25 (b) 0.25 and 0.75 (c) 0.75 and 1.00.

3. From the divided difference table for the following data :

x :	5	15	22
y :	7	36	160

4. Interpolation is the technique of estimating the value of a function for any ....
5. Bessel's formula for interpolation is ....
6. The 4th divided differences for x_0, x_1, x_2, x_3, x_4 = ...
7. Stirling's formula is best suited for p lying between ....
8. Newton's divided differences formula is ....
9. Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$, Lagrange's interpolation formula is ....
10. If $f(0) = 1$, $f(2) = 5$, $f(3) = 10$ and $f(x) = 14$, then $x = 0$ ....
11. Gauss forward interpolation formula involves
    (a) even differences above the central line and odd differences on the central line
    (b) even differences below the central line and odd differences on the central line
    (c) odd differences below the central line and even differences on the central line
    (d) odd differences above the central line and even differences on the central line.
12. If $y(1) = 4$, $y(3) = 12$, $y(4) = 19$ and $y(x) = 7$ find x using Lagrange's formula.
13. Extrapolation is defined as ....
14. The second divided difference of $f(x) = 1/x$, with arguments, a, b, c, is....... . *(Anna, 2007)*
15. Gauss-forward interpolation formula is used to interpolate values of y for
    (a) $0 < p < 1$ (b) $-1 < p < 0$
    (c) $0 < p < -\alpha$ (d) $-\alpha < p < 0$.
16. Given

x :	0	1	3	4
y :	-12	0	6	12

Using Lagrange's formula, a polynomial that can be fitted to the data is ....
17. The nth divided difference of a polynomial of degree n is
    (a) zero (b) a constant
    (c) a variable (d) none of these.
18. If h is the interval of differencing, $\Delta^2 x^3$ = ...