CHAPTER 7

Multiple Integrals and Beta, Gamma Functions

7.1 DOUBLE INTEGRALS

The definite integral $\int_a^b f(x)\,dx$ is defined as the limit of the sum

$$f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + \ldots + f(x_n)\,\delta x_n,$$

where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \ldots$ tends to zero. A double integral is its counterpart in two dimensions.

Consider a function $f(x, y)$ of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas $\delta A_1, \delta A_2, \ldots, \delta A_n$. Let (x_r, y_r) be any point within the rth elementary area δA_r. Consider the sum

$$f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \ldots + f(x_n, y_n)\,\delta A_n, \text{ i.e., } \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

The limit of this sum, if it exists, as the number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the *double integral of f(x, y) over the region R* and is written as $\iint_R f(x, y)\,dA$.

Thus
$$\iint_R f(x, y)\,dA = \underset{\substack{n \to \infty \\ \delta A \to 0}}{\text{Lt}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \qquad \ldots(1)$$

The utility of double integrals would be limited if it were required to take limit of sums to evaluate them. However, there is another method of evaluating double integrals by successive single integrations.

For purpose of evaluation, (1) is expressed as the repeated integral $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\,dxdy$.

Its value is found as follows :

(*i*) *When y_1, y_2 are functions of x and x_1, x_2 are constants,* $f(x, y)$ is first integrated w.r.t. y keeping x fixed between limits y_1, y_2 and then resulting expression is integrated w.r.t. x within the limits x_1, x_2 *i.e.,*

$$\boxed{I_1 = \int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x, y)\,dy}\,dx}$$

where integration is carried from the inner to the outer rectangle.

Figure 7.1 illustrates this process. Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. PQ is a vertical strip of width dx.

Then the inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant), while the outer rectangle integral corresponds to the sliding of the edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

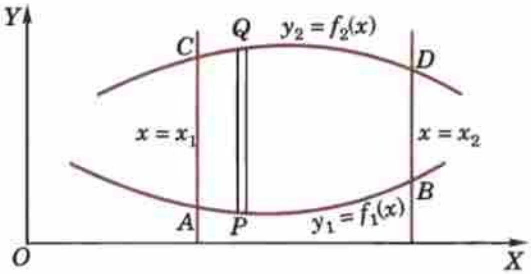

Fig. 7.1

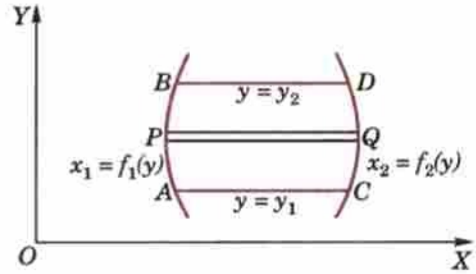

Fig. 7.2

(ii) When x_1, x_2 are functions of y and y_1, y_2 are constants, $f(x, y)$ is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. y between the limits y_1, y_2, *i.e.*,

$$I_2 = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)\, dx}\, dy}$$ which is geometrically illustrated by Fig. 7.2.

Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is a horizontal strip of width dy.

Then inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

(iii) When both pairs of limits are constants, the region of integration is the rectangle $ABDC$ (Fig. 7.3).

In I_1, we integrate along the vertical strip PQ and then slide it from AC to BD.

In I_2, we integrate along the horizontal strip $P'Q'$ and then slide it from AB to CD.

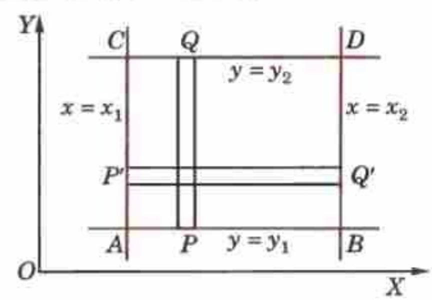

Fig. 7.3

Here obviously $I_1 = I_2$.

Thus *for constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.*

Example 7.1. *Evaluate* $\int_0^5 \int_0^{x^2} x(x^2 + y^2)\, dxdy$.

Solution. $$I = \int_0^5 dx \int_0^{x^2} (x^3 + xy^3)\, dy = \int_0^5 \left[x^3 y + x \cdot \frac{y^3}{3}\right]_0^{x^2} dx = \int_0^5 \left[x^3 \cdot x^2 + x \cdot \frac{y^6}{3}\right] dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx = \left| \frac{x^6}{6} + \frac{x^8}{24} \right|_0^5 = 5^6 \left[\frac{1}{6} + \frac{5^2}{24}\right] = 18880.2 \text{ nearly.}$$

Example 7.2. *Evaluate* $\iint_A xy\, dx\, dy$, *where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.*

Solution. The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at $L(2a, a)$. Figure 7.4 shows the domain A which is the area OML.

Integrating first over a vertical strip PQ, *i.e.*, w.r.t. y from $P(y = 0)$ to $Q(y = x^2/4a)$ on the parabola and then w.r.t. x from $x = 0$ to $x = 2a$, we have

$$\iint_A xy\, dx\, dy = \int_0^{2a} dx \int_4^{x^2/4a} xy\, dy = \int_0^{2a} x \left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

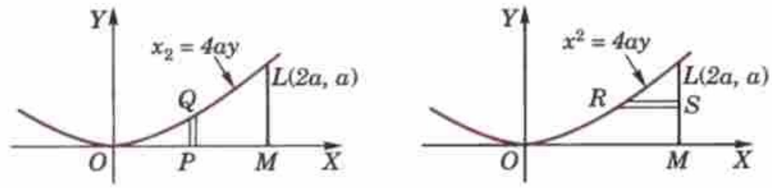

Fig. 7.4

$$= \frac{1}{32a^2}\int_0^{2a} x^5\,dx = \frac{1}{32a^2}\left|\frac{x^6}{6}\right|_0^{2a} = \frac{a^4}{3}.$$

Otherwise integrating first over a horizontal strip *RS*, *i.e.*, w.r.t. *x* from, $R\ (x = 2\sqrt{ay})$ on the parabola to $S(x = 2a)$ and then w.r.t. *y* from $y = 0$ to $y = a$, we get

$$\iint_A xy\,dx\,dy = \int_0^a dx \int_{2\sqrt{(ay)}}^{2a} xy\,dx = \int_0^a y\left[\frac{x^2}{2}\right]_{2\sqrt{(ay)}}^{2a} dy$$

$$= 2a\int_0^a (ay - y^2)\,dy = 2a\left[\frac{ay^2}{2} - \frac{y^3}{3}\right]_0^a = \frac{a^4}{3}.$$

Example 7.3. *Evaluate* $\iint_R x^2\,dx\,dy$ *where R is the region in the first quadrant bounded by the lines* $x = y$, $y = 0$, $x = 8$ *and the curve* $xy = 16$.

Solution. The line *AL* $(x = 8)$ intersects the hyperbola $xy = 16$ at $A\ (8, 2)$ while the line $y = x$ intersects this hyperbola at $B\ (4, 4)$. Figure 7.5 shows the region *R* of integration which is the area *OLAB*. To evaluate the given integral, we divide this area into two parts *OMB* and *MLAB*.

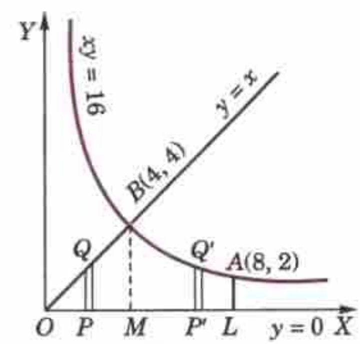

Fig. 7.5

$$\therefore \quad \iint_R x^2\,dxdy = \int_{x \text{ at } O}^{x \text{ at } M}\int_{y \text{ at } P}^{y \text{ at } Q} x^2\,dxdy + \int_{x \text{ at } M}^{x \text{ at } L}\int_{y \text{ at } P'}^{y \text{ at } Q'} x^2\,dxdy$$

$$= \int_0^4\int_0^x x^2\,dxdy + \int_4^8\int_0^{16/x} x^2\,dxdy$$

$$= \int_0^4 x^2\,dx\left|\,y\,\right|_0^x + \int_4^8 x^2\,dx\left|\,y\,\right|_0^{16/x}$$

$$= \int_0^4 x^3\,dx + \int_4^8 16x\,dx = \left|\frac{x^4}{4}\right|_0^4 + 16\left|\frac{x^2}{2}\right|_4^8 = 448$$

7.2 CHANGE OF ORDER OF INTEGRATION

In a double integral with variable limits, the change of order of integration changes the limit of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration quite often facilitates the evaluation of a double integral. The following examples will make these ideas clear.

Example 7.4. *By changing the order of integration of* $\int_0^\infty\int_0^\infty e^{-xy}\sin px\,dxdy$, *show that*

$$\int_0^\infty \frac{\sin px}{x}\,dx = \frac{\pi}{2}.$$

(U.P.T.U., 2004)

Solution. $\int_0^\infty\int_0^\infty e^{-xy}\sin px\,dxdy = \int_0^\infty\left(\int_0^\infty e^{-xy}\sin px\,dx\right)dy$

$$= \int_0^\infty \left| -\frac{e^{-xy}}{p^2+y^2}(p\cos px + y\sin px)\right|_0^\infty dy$$

$$= \int_0^\infty \frac{p}{p^2+y^2}\,dy = \left|\tan^{-1}\left(\frac{y}{p}\right)\right|_0^\infty = \frac{\pi}{2} \qquad \text{...}(i)$$

On changing the order of integration, we have

$$\int_0^\infty \int_0^\infty e^{-xy}\sin px\,dx\,dy = \int_0^\infty \sin px\left\{\int_0^\infty e^{-xy}\,dy\right\}dx$$

$$= \int_0^\infty \sin px\left|\frac{e^{-xy}}{-x}\right|_0^\infty dx = \int_0^\infty \frac{\sin px}{x}\,dx \qquad \text{...}(ii)$$

Thus from (*i*) and (*ii*), we have $\int_0^\infty \frac{\sin px}{x}\,dx = \frac{\pi}{2}$.

Example 7.5. *Change the order of integration in the integral*

$$I = \int_{-a}^{a}\int_0^{\sqrt{(a^2-y^2)}} f(x,y)\,dx\,dy.$$

Solution. Here the elementary strip is parallel to x-axis (such as PQ) and extends from $x = 0$ to $x = \sqrt{(a^2-y^2)}$ (*i.e.*, to the circle $x^2+y^2=a^2$) and this strip slides from $y = -a$ to $y = a$. This shaded semi-circular area is, therefore, the region of integration (Fig. 7.6).

On changing the order of integration, we first integrate w.r.t. y along a vertical strip RS which extends from R [$y = -\sqrt{(a^2-y^2)}$] to S [$y = \sqrt{(a^2-y^2)}$]. To cover the given region, we then integrate w.r.t. x from $x = 0$ to $x = a$.

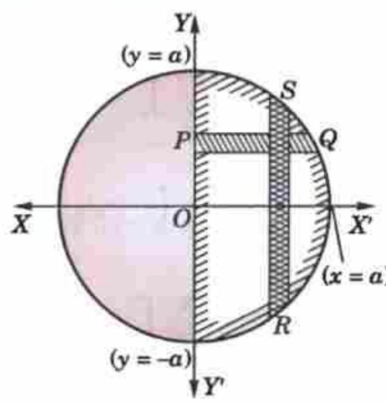

Fig. 7.6

Thus $$I = \int_0^a dx \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\,dy$$

or $$= \int_0^a \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\,dydx.$$

Example 7.6. *Evaluate* $\int_0^1\int_{e^x}^{e} dydx/\log y$ *by changing the order of integration.*

Solution. Here the integration is first w.r.t. y from P on $y = e^x$ to Q on the line $y = e$. Then the integration is w.r.t. x from $x = 0$ to $x = 1$, giving the shaded region ABC (Fig. 7.7).

On changing the order of integration, we first integrate w.r.t. x from R on $x = 0$ to S on $x = \log y$ and then w.r.t. y from $y = 1$ to $y = e$.

Thus $$\int_0^1\int_{e^x}^{e}\frac{dydx}{\log y} = \int_1^e\int_0^{\log y}\frac{dxdy}{\log y}$$

$$= \int_1^e \frac{dy}{\log y}\left|x\right|_0^{\log y} = \int_1^e dy = \left|y\right|_1^e = e-1.$$

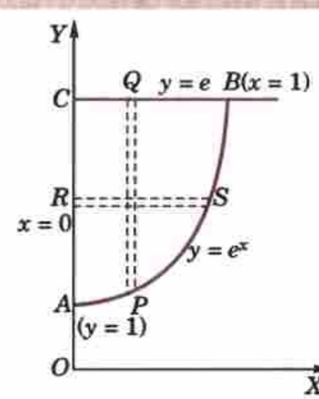

Fig. 7.7

Example 7.7. *Change the order of integration in* $I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx$ *and hence evaluate.*

(Nagpur, 2009 ; P.T.U., 2009 S)

Solution. Here integration is first w.r.t. y and P on the parabola $x^2 = 4ay$ to Q on the parabola $y^2 = 4ax$ and then w.r.t. x from $x = 0$ to $x = 4a$ giving the shaded region of integration (Fig. 7.8).

On changing the order of integration, we first integrate w.r.t. x from R to S, then w.r.t. y from $y = 0$ to $y = 4a$

$$\therefore \quad I = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dxdy = \int_0^{4a} dy \left| x \right|_{y^2/4a}^{2\sqrt{ay}} = \int_0^{4a} (2\sqrt{ay} - y^2/4a)\, dy$$

$$= \left| 2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a} \right|_0^{4a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

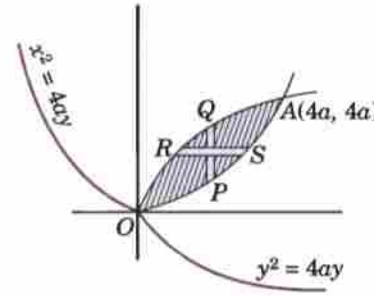

Fig. 7.8

Example 7.8. *Change the order of integration and hence evaluate*

$$I = \int_0^a \int_{\sqrt{ax}}^a \frac{y^2\, dxdy}{\sqrt{(y^4 - a^2x^2)}} \qquad \textit{(S.V.T.U., 2006 S)}$$

Solution. Here integration is first w.r.t. y from P on the parabola $y^2 = ax$ to Q on the line $y = a$, then w.r.t. x from $x = 0$ to $x = a$, giving the shaded region OAB of integration (Fig. 7.9).

On changing the order of integration, we first integrate w.r.t. x from R to S, then w.r.t. y from $y = 0$ to $y = a$.

$$\therefore \quad I = \int_0^a \int_0^{y^2/a} \frac{y^2\, dy}{\sqrt{(y^4 - a^2x^2)}}\, dx = \frac{1}{a}\int_0^a \int_0^{y^2/a} y^2\, dy \frac{dx}{\sqrt{[(y^2/a)^2 - x^2]}}\, dx$$

$$= \frac{1}{a}\int_0^a y^2 dy \left| \sin^{-1}\left(\frac{xa}{y^2}\right) \right|_0^{y^2/a} = \frac{1}{a}\int_0^a y^2 dy\ [\sin^{-1}(1) - \sin^{-1}(0)]$$

$$= \frac{\pi}{2a}\int_0^a y^2 dy = \frac{\pi}{2a}\left|\frac{y^3}{3}\right|_0^a = \frac{\pi a^2}{6}.$$

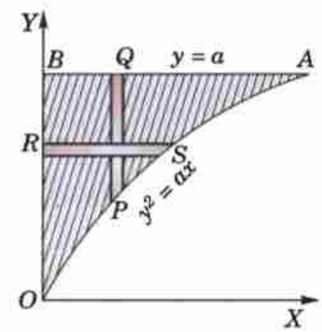

Fig. 7.9

Example 7.9. *Change the order of integration in* $I = \int_0^1 \int_{x^2}^{2-x} xy\, dxdy$ *and hence evaluate the same.*

(Bhopal, 2008 ; V.T.U., 2008 ; S.V.T.U., 2007 ; P.T.U., 2005 ; U.P.T.U., 2005)

Solution. Here the integration is first w.r.t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from $x = 0$ to $x = 1$, giving the region of integration as the curvilinear triangle OAB (shaded) in Fig. 7.10.

On changing the order of integration, we first integrate w.r.t. x along a horizontal strip $P'Q'$ and that requires the splitting up of the region OAB into two parts by the line AC ($y = 1$), *i.e.*, the curvilinear triangle OAC and the triangle ABC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = \sqrt{y}$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

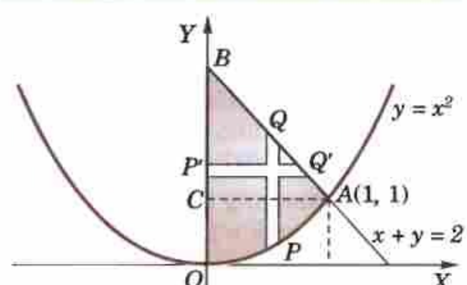

Fig. 7.10

$$I_1 = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx$$

For the region ABC, the limits of integration for x are from $x = 0$ to $x = 2 - y$ and those for y are from $y = 1$ to $y = 2$. So the contribution to I from the region ABC is

$$I_2 = \int_1^2 dy \int_0^{2-y} xy\, dx.$$

Hence, on reversing the order of integration,

$$I = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx + \int_1^2 dy \int_0^{2-y} xy\, dx$$

$$= \int_0^1 dy \left| \frac{x^2}{2} \cdot y \right|_0^{\sqrt{y}} + \int_1^2 dy \left| \frac{x^2}{2} \cdot y \right|_0^{2-y} = \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2\, dy = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

Example 7.10. *Change the order of integration in* $I = \int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x}{\sqrt{(x^2+y^2)}}\, dx$ *and hence evaluate it.*

(J.N.T.U., 2005 ; Rohtak, 2003)

Solution. Here the integration is first w.r.t. y along PQ which extends from P on the line $y = x$ to Q on the circle $y = \sqrt{(2 \,.\, x^2)}$. Then PQ slides from $y = 0$ to $y = 1$, giving the region of integration OAB as in Fig. 7.11.

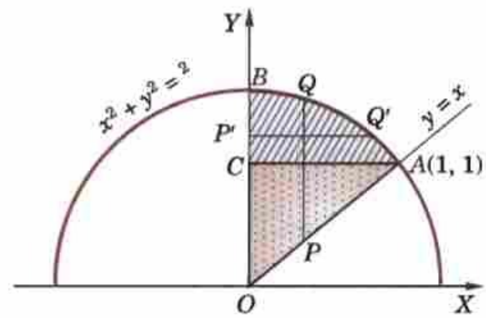

Fig. 7.11

On changing the order of integration, we first integrate w.r.t. x from P' to Q' and that requires splitting the region OAB into two parts OAC and ABC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = 1$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

$$I_1 = \int_0^1 dy \int_0^y \frac{x}{\sqrt{(x^2+y^2)}}\, dx.$$

For the region ABC, the limits of integration for x are 0 to $\sqrt{(2-y^2)}$ and these for y are from 1 to $\sqrt{2}$. So the contribution to I from the region ABC is

$$I_2 = \int_1^{\sqrt{2}} dy \int_0^{\sqrt{(2-y^2)}} \frac{x}{\sqrt{(x^2+y^2)}}\, dx$$

Hence $$I = \int_0^1 \left| (x^2+y^2)^{1/2} \right|_0^y dy + \int_1^{\sqrt{2}} \left| (x^2+y^2)^{1/2} \right|_0^{\sqrt{(2-y^2)}} dy$$

$$= \int_0^1 (\sqrt{2}-1)\, y\, dy + \int_1^{\sqrt{2}} \sqrt{(2-y)}\, dy = \frac{1}{2}(\sqrt{2}-1) + \sqrt{2}\sqrt{(2-1)} - \frac{1}{2} = 1 - 1/\sqrt{2}.$$

7.3 DOUBLE INTEGRALS IN POLAR COORDINATES

To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta)\, dr\, d\theta$, we first integrate w.r.t. r between limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

Figure 7.12 illustrates the process geometrically.

Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$. PQ is a wedge of angular thickness $\delta\theta$.

Then $\int_{r_1}^{r_2} f(r, \theta)\, dr$ indicates that the integration is along PQ from P to Q while the integration w.r.t. θ corresponds to the turning of PQ from AC to BD.

Thus the whole region of integration is the area $ACDB$. The order of integration may be changed with appropriate changes in the limits.

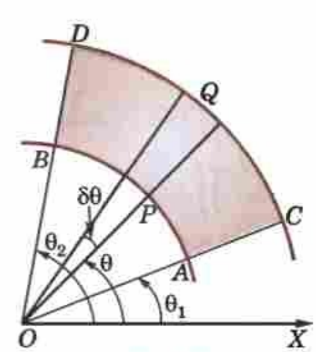

Fig. 7.12

Example 7.11. *Evaluate* $\iint r \sin\theta \, dr \, d\theta$ *over the cardioid* $r = a(1 - \cos\theta)$ *above the initial line.*

(Kerala, 2005)

Solution. To integrate first w.r.t. r, the limits are from 0 ($r = 0$) to P [$r = a(1 - \cos\theta)$] and to cover the region of integration R, θ varies from 0 to π (Fig. 7.13).

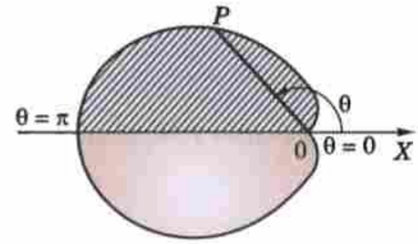

Fig. 7.13

$$\therefore \quad \iint_R r \sin\theta \, dr d\theta = \int_0^{\pi} \sin\theta \left[\int_0^{r = a(1-\cos\theta)} r dr \right] d\theta$$

$$= \int_0^{\pi} \sin\theta \, d\theta \left| \frac{r^2}{2} \right|_0^{a(1-\cos\theta)} = \frac{a^2}{2} \int_0^{\pi} (1 - \cos\theta)^2 . \sin\theta \, d\theta$$

$$= \frac{a^2}{2} \left| \frac{(1-\cos\theta)^3}{3} \right|_0^{\pi} = \frac{a^2}{2} \cdot \frac{8}{3} = \frac{4a^2}{3}.$$

Example 7.12. *Calculate* $\iint r^3 \, dr \, d\theta$ *over the area included between the circles* $r = 2\sin\theta$ *and* $r = 4\sin\theta$.

Solution. Given circles $r = 2\sin\theta$ ...(*i*)

and $r = 4\sin\theta$ ...(*ii*)

are shown in Fig. 7.14. The shaded area between these circles is the region of integration.

If we integrate first w.r.t. r, then its limits are from $P(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region θ varies from 0 to π. Thus the required integral is

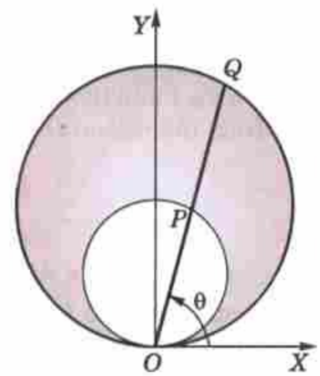

Fig. 7.14

$$I = \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 \, dr = \int_0^{\pi} d\theta \left[\frac{r^4}{4} \right]_{2\sin\theta}^{4\sin\theta}$$

$$= 60 \int_0^{\pi} \sin^4\theta \, d\theta = 60 \times 2 \int_0^{\pi/2} \sin^4\theta \, d\theta = 120 \times \frac{3.1}{4.2} \cdot \frac{\pi}{2} = 22.5\,\pi.$$

PROBLEMS 7.1

Evaluate the following integrals (1–7) :

1. $\int_1^2 \int_1^3 xy^2 \, dxdy$.

2. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) \, dx \, dy$. *(V.T.U., 2000)*

3. $\int_0^1 \int_0^x e^{x/y} \, dxdy$. *(P.T.U., 2005)*

4. $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dydx}{1 + x^2 + y^2}$ *(Rajasthan, 2005)*

5. $\iint xy \, dxdy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$. *(Rajasthan, 2006)*

6. $\iint (x + y)^2 dxdy$ over the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$. *(Kurukshetra, 2009 S ; U.P.T.U., 2004 S)*

7. $\iint xy(x + y) \, dxdy$ over the area between $y = x^2$ and $y = x$. *(V.T.U., 2010)*

Evaluate the following integrals by changing the order of integration (8–15) :

8. $\int_0^a \int_y^a \frac{xdxdy}{x^2 + y^2}$. *(Bhopal, 2008)*

9. $\int_0^3 \int_1^{\sqrt{(4-y)}} (x + y) \, dxdy$. *(V.T.U., 2005 ; Anna, 2003 S ; Delhi, 2002)*

10. $\int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x\,dydx}{\sqrt{(x^2+y^2)}}$. (P.T.U., 2010 ; Marathwada, 2008 ; U.P.T.U., 2006)

11. $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{(a^2-y^2)}} \log (x^2+y^2)\,dxdy\ (a > 0)$.

12. $\int_0^1 \int_x^{\sqrt{x}} xy\,dydx$. (V.T.U., 2010)

13. $\int_0^a \int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} xy\,dx\,dy$. (Anna, 2009)

14. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dydx$. (Bhopal, 2009 ; S.V.T.U., 2009 ; V.T.U., 2007)

15. $\int_0^\infty \int_0^x xe^{-x^2/y}\,dydx$. (S.V.T.U., 2006 ; U.P.T.U., 2005 ; V.T.U., 2004)

16. Sketch the region of integration of the following integrals and change the order of integrations,

(i) $\int_0^{2a} \int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} f(x)\,dxdy$ (Rajasthan, 2006) (ii) $\int_0^{ae^{\pi/4}} \int_{2\log(r/a)}^{\pi/2} f(r,\theta)\,r\,drd\theta$.

17. Show that $\iint_R r^2 \sin\theta\,drd\theta = 2a^2/3$, where R is the semi-circle $r = 2a\cos\theta$ above the initial line.

18. Evaluate $\iint \frac{r\,drd\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$. (Rohtak, 2006 S ; P.T.U., 2005)

19. Evaluate $\iint r^3\,drd\theta$ over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

(Anna, 2009 ; Madras, 2006)

7.4 AREA ENCLOSED BY PLANE CURVES

(1) *Cartesian coordinates.*

Consider the area enclosed by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = x_1$, $x = x_2$ [Fig. 7.15 (*a*)].

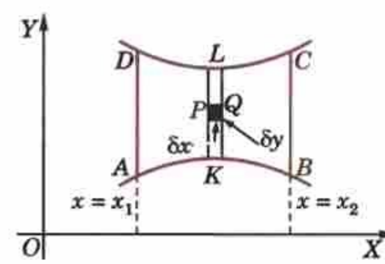

Fig. 7.15(*a*)

Divide this area into vertical strips of width δx. If $P(x, y)$, $Q(x + \delta x, y + \delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x\delta y$.

$\therefore$ area of strip $KL = \underset{\delta y \to 0}{\text{Lt}} \Sigma\, \delta x\, \delta y$.

Since for all rectangles in this strip δx is the same and y varies from $y = f_1(x)$ to $y = f_2(x)$.

$$\therefore \quad \text{area of the strip } KL = \delta x \underset{\delta y \to 0}{\text{Lt}} \sum_{f_1(x)}^{f_2(x)} dy = \delta x \int_{f_1(x)}^{f_2(x)} dy.$$

Now adding up all such strips from $x = x_1$ to $x = x_2$, we get the area $ABCD$

$$= \underset{\delta x \to 0}{\text{Lt}} \sum_{x_1}^{x_2} \delta x \cdot \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dxdy$$

Similarly, dividing the area $A'B'C'D$ [Fig. 7.15(*b*)] into horizontal strips of width δy, we get the area $A'B'C'D'$.

$$= \int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} dxdy$$

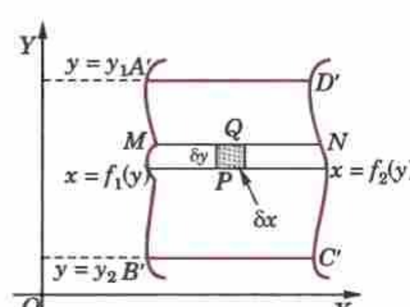

Fig. 7.15 (*b*)

(2) *Polar coordinates.*

Consider an area A enclosed by a curve whose equation is in polar coordinates.

Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points. Mark circular areas of radii r and $r + \delta r$ meeting OQ in R and OP (produced) in S (Fig. 7.16).

Since arc $PR = r\delta\theta$ and $PS = \delta r$.

$\therefore$ area of the curvilinear rectangle $PRQS$ is approximately $= PR \,.\, PS = r\delta\theta \,.\, \delta r$.

If the whole area is divided into such curvilinear rectangles, the sum $\Sigma\Sigma r\delta\theta\delta r$ taken for all these rectangles, gives in the limit the area A.

Hence $$A = \underset{\substack{\delta r \to 0 \\ \delta\theta \to 0}}{\text{Lt}} \Sigma\Sigma r\delta\theta\delta r = \iint r d\theta dr$$

where the limits are to be so chosen as to cover the entire area.

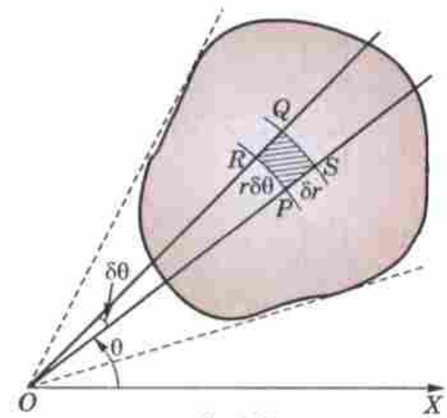

Fig. 7.16

Example 7.13. *Find the area of a plate in the form of a quadrant of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ *(V.T.U., 2001 ; Osmania, 2000 S)*

Solution. Dividing the area into vertical strips of width δx, y varies from $K(y = 0)$ to $L[y = b\sqrt{(1 - x^2/b^2)}]$ and then x varies from 0 to a (Fig. 7.17).

$\therefore$ required area

$$= \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy = \int_0^a dx \,[y]_0^{b\sqrt{(1-x^2/a^2)}}$$

$$= \frac{b}{a}\int_0^a \sqrt{(a^2 - x^2)}\, dx = \pi ab/4.$$

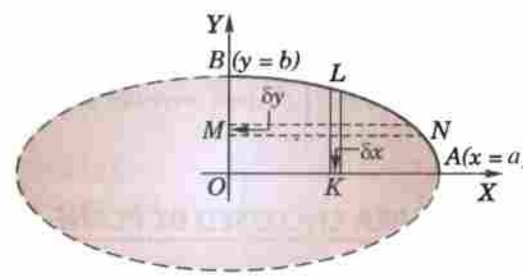

Fig. 7.17

Otherwise, dividing this area into horizontal strips of width δy, x varies from $M(x = 0)$ to $N[x = a\sqrt{(1 - y^2/b^2)}\,]$ and then y varies from 0 to b.

$$\therefore \text{ required area } = \int_0^b dy \int_0^{a\sqrt{(1-y^2/b^2)}} dx = \int_0^b dy\,[x]_0^{a\sqrt{(1-y^2/b^2)}}$$

$$= \frac{a}{b}\int_0^b \sqrt{(b^2 - y^2)}\, dy = \pi ab/4.$$

Obs. The change of the order of integration does not in any way affect the value of the area.

Example 7.14. *Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.*

(Kerala, 2005 ; Rohtak, 2003)

Solution. Solving the equations $y^2 = 4ax$ and $x^2 = 4ay$, it is seen that the parabolas intersect at O (0, 0) and A ($4a$, $4a$). As such for the shaded area between these parabolas (Fig. 7.18) x varies from 0 to $4a$ and y varies from P to Q *i.e.*, from $y = x^2/4a$ to $y = 2\sqrt{(ax)}$. Hence the *required area*

$$= \int_0^{4a}\int_{x^2/4a}^{2\sqrt{(ax)}} dydx = \int_0^{4a} (2\sqrt{(ax)} - x^2/4a)\, dx$$

$$= \left| 2\sqrt{a} \cdot \frac{2}{3}x^{3/2} - \frac{1}{4a}\cdot\frac{x^3}{3}\right|_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$

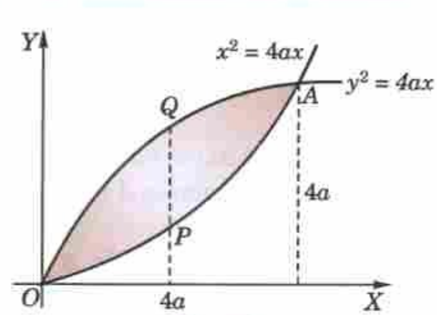

Fig. 7.18

Example 7.15. *Calculate the area included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote.*

Solution. The curve is symmetrical about the initial line and has an asymptote $r = a \sec\theta$ (Fig. 7.19).

Draw any line OP cutting the curve at P and its asymptote at P'. Along this line, θ is constant and r varies from $a \sec\theta$ at P' to $a(\sec\theta + \cos\theta)$ at P. Then to get the upper half of the area, θ varies from 0 to $\pi/2$.

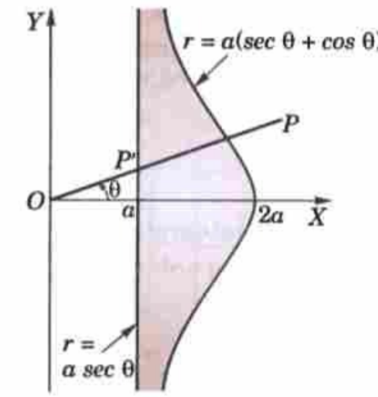

Fig. 7.19

$$\therefore \quad \text{required area} = 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi/2}(2+\cos^2\theta)\,d\theta = 5\pi a^2/4.$$

Example 7.16. *Find the area lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.*

Solution. In Fig. 7.20, $ABODA$ represents the cardioid $r = a(1 + \cos\theta)$ and $CBA'DC$ is the circle $r = a$.

Required area (shaded) = 2 (area $ABCA$)

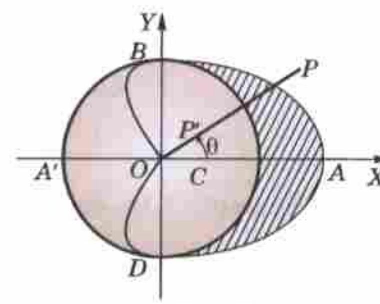

Fig. 7.20

$$= 2\int_0^{\pi/2}\int_{r=OP'}^{r=OP} r\,d\theta\,dr = 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)}(rdr)\,d\theta$$

$$= 2\int_0^{\pi/2}\left|\frac{r^2}{2}\right|_a^{a(1+\cos\theta)} d\theta = a^2\int_0^{\pi/2}[(1+\cos\theta)^2-1]\,d\theta$$

$$= a^2\int_0^{\pi/2}(\cos^2\theta + 2\cos\theta)\,d\theta = a^2\left(\frac{1}{2}\cdot\frac{\pi}{2}+2\right) = \frac{a^2}{4}(\pi+8).$$

PROBLEMS 7.2

1. Find, by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.
2. Find the area lying between the parabola $y = x^2$ and the line $x + y - z = 0$. *(Anna, 2009)*
3. By double integration, find the whole area of the curve $a^2x^2 = y^3(2a - y)$. *(U.P.T.U., 2001)*
4. Find, by double integration, the area enclosed by the curves $y = 3x/(x^2 + 2)$ and $4y = x^2$. *(J.N.T.U., 2005)*
5. Find, by double integration, the area of the lemniscate $r^2 = a^2 \cos 2\theta$. *(Madras, 2000 S)*
6. Find, by double integration, the area lying inside the circle $r = a \sin\theta$ and outside the cardioid $r = a(1 - \cos\theta)$. *(Anna 2009 ; Mumbai, 2006)*
7. Find the area lying inside the cardioid $r = 1 + \cos\theta$ and outside the parabola $r(1 + \cos\theta) = 1$.
8. Find the area common to the circles $r = a\cos\theta$, $r = a\sin\theta$ by double integration. *(Mumbai, 2007)*

7.5 TRIPLE INTEGRALS

Consider a function $f(x, y, z)$ defined at every point of the 3-dimensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. Let (x_r, y_r, z_r) be any point within the rth sub-division δV_r. Consider the sum

$$\sum_{r=1}^{\infty} f(x_r, y_r, z_r)\delta V_r .$$

The limit of this sum, if it exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the *triple integral of f(x, y, z) over the region V* and is denoted by

$$\iiint f(x, y, z)\,dV.$$

For purposes of evaluation, it can also be expressed as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\,dxdydz.$$

If x_1, x_2 are constants ; y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or functions of x and y, then this integral is evaluated as follows :

First $f(x, y, z)$ is integrated w.r.t. z between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t. y between the limits y_1 and y_2 keeping x constant. The result just obtained is finally integrated w.r.t. x from x_1 to x_2.

Thus
$$I = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1(x)}^{y_2(x)} \boxed{\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z) dz} dy} dx}$$

where the integration is carried out from the innermost rectangle to the outermost rectangle.

The order of integration may be different for different types of limits.

Example 7.17. *Evaluate* $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x + y + z)\, dxdydz.$ *(J.N.T.U., 2006 ; Cochin, 2005)*

Solution. Integrating first w.r.t. y keeping x and z constant, we have

$$I = \int_{-1}^{1} \int_{0}^{z} \left| xy + \frac{y^2}{2} + yz \right|_{x-z}^{x+z} dx\, dz = \int_{-1}^{1} \int_{0}^{z} \left[(x + z)(2z) + \frac{1}{2} 4xz \right] dx\, dz$$

$$= 2 \int_{-1}^{1} \left| \frac{x^2 z}{2} + z^2 x + \frac{x^2}{2} z \right|_{0}^{z} dz = 2 \int_{-1}^{1} \left(\frac{z^3}{2} + z^3 + \frac{z^3}{2} \right) dz = 4 \left| \frac{z^4}{4} \right|_{-1}^{1} = 0.$$

Example 7.18. *Evaluate* $\int_{0}^{1} \int_{0}^{\sqrt{(1-x^2)}} \int_{0}^{\sqrt{(1-x^2-y^2)}} xyz\, dxdydz.$ *(V.T.U., 2003 S)*

Solution. We have

$$I = \int_0^1 x \left[\int_0^{\sqrt{(1-x^2)}} y \left\{ \int_0^{\sqrt{(1-x^2-y^2)}} z\, dz \right\} dy \right] dx = \int_0^1 x \left\{ \int_0^{\sqrt{(1-x^2)}} y \,.\left| \frac{z^2}{2} \right|_0^{\sqrt{(1-x^2-y^2)}} dy \right\} dx$$

$$= \int_0^1 x \left\{ \int_0^{\sqrt{(1-x^2)}} y \,.\, \frac{1}{2}(1 - x^2 - y^2)\, dy \right\} dx = \frac{1}{2} \int_0^1 x \left| (1 - x^2)\frac{y^2}{2} - \frac{y^4}{4} \right|_0^{\sqrt{(1-x^2)}} dx$$

$$= \frac{1}{8} \int_0^1 [(1 - x^2)^2 \,.\, 2x - (1 - x^2)^4 \,.\, x]\, dx = \frac{1}{8} \int_0^1 (x - 2x^3 + x^5)\, dx$$

$$= \frac{1}{8} \left| \frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6} \right|_0^1 = \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right) = \frac{1}{48}.$$

PROBLEMS 7.3

Evaluate the following integrals :

1. $\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2)\, dx\, dy\, dz.$ *(Anna, 2009)*
2. $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dx\, dy\, dz$ *(S.V.T.U., 2009 ; V.T.U. 2000)*
3. $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$ *(Nagpur, 2009)*
4. $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx.$ *(V.T.U., 2010 ; Kurukshetra, 2009 S ; J.N.T.U., 2005)*
5. $\int_0^{\log 2} \int_0^x \int_0^{x + \log y} e^{x+y+z}\, dx\, dy\, dz.$ *(Bhopal, 2008)*
6. $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dz\, dx\, dy.$ *(S.V.T.U., 2008 ; Rohtak, 2005)*
7. $\int_0^{\pi/2} \int_0^{a \sin \theta} \int_0^{\frac{a^2 - r^2}{a}} r dz\, dr\, d\theta.$ *(V.T.U., 2009)*

7.6 VOLUMES OF SOLIDS

(1) **Volumes as double integrals.** Consider a surface $z = f(x, y)$. Let the orthogonal projection on XY-plane of its portion S' be the area S (Fig. 7.21).

Divide S into elementary rectangles of area $\delta x\, \delta y$ by drawing lines parallel to X and Y-axes. With each of these rectangles as base, erect a prism having its length parallel to OZ.

$\therefore$ volume of this prism between S and the given surface $z = f(x, y)$ is $z\delta x\delta y$.

Hence the volume of the solid cylinder on S as base, bounded by the given surface with generators parallel to the Z-axis.

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\text{Lt}} \Sigma\Sigma\, z\delta x\delta y$$

$$= \iint z\, dx\, dy \quad \text{or} \quad \iint f(x, y)\, dx\, dy$$

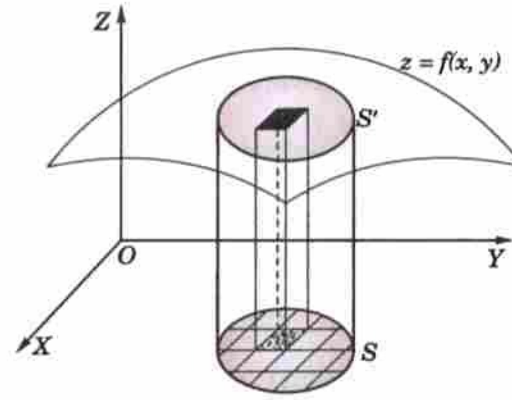

Fig. 7.21

where the integration is carried over the area S.

Obs. While using polar coordinates, divide S into elements of area $r\delta\theta\, \delta r$.

$\therefore$ replacing $dxdy$ by $r\delta\theta\delta r$, we get the required volume $= \iint zr\, d\theta\, dr$.

Example 7.19. *Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.* (S.V.T.U., 2007 ; Cochin, 2005 ; Madras, 2000 S)

Solution. From Fig. 7.22, it is self-evident that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the XY-plane. To cover the shaded half of this circle, x varies from 0 to $\sqrt{(4 - y^2)}$ and y varies from -2 to 2.

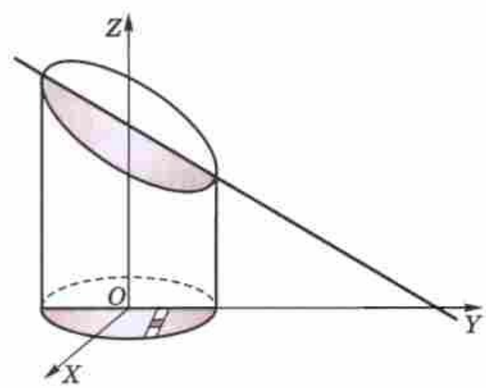

Fig. 7.22

$\therefore$ Required volume

$$= 2\int_{-2}^{2} \int_{0}^{\sqrt{(4-y^2)}} z\, dx\, dy = 2\int_{-2}^{2} \int_{0}^{\sqrt{(4-y^2)}} (4 - y)\, dx\, dy$$

$$= 2\int_{-2}^{2} (4 - y)\,[x]_0^{\sqrt{(4-y^2)}}\, dy = 2\int_{-2}^{2} (4 - y)\sqrt{(4 - y^2)}\, dy$$

$$= 2\int_{-2}^{2} 4\sqrt{(4 - y^2)}\, dy - 2\int_{-2}^{2} y\sqrt{(4 - y^2)}\, dy$$

$$= 8\int_{-2}^{2} \sqrt{(4 - y^2)}\, dy$$ [The second term vanishes as the integrand is an odd function.]

$$= 8\left|\frac{y\sqrt{(4-y)^2}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right|_{-2}^{2} = 16\pi.$$

(2) Volume as triple integral

Divide the given solid by planes parallel to the coordinate planes into rectangular parallelopipeds of volume $\delta x\, \delta y\, \delta z$ (Fig. 7.23).

$\therefore$ the total volume $= \underset{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}}{\text{Lt}} \Sigma\Sigma\Sigma\, \delta x\, \delta y\, \delta z$

$$= \iiint dx\, dy\, dz$$

with appropriate limits of integration.

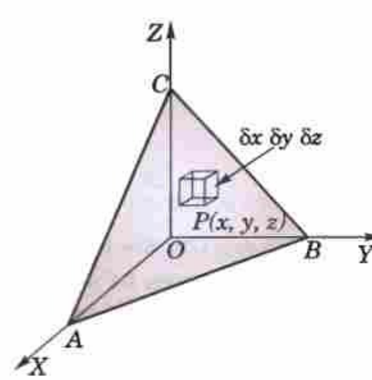

Fig. 7.23

Example 7.20. *Calculate the volume of the solid bounded by the planes $x = 0, y = 0, x + y + z = a$ and $z = 0$.* (P.T.U., 2009)

Solution. Volume required $= \int_0^a \int_0^{a-x} \int_0^{a-x-y} dz\, dy\, dx$

$$= \int_0^a \int_0^{a-x} (a - x - y)\, dy\, dx = \int_0^a \left| (a-x)y - \frac{y^2}{2} \right|_0^{a-x} dx$$

$$= \int_0^a \left\{ (a-x)^2 - \frac{(a-x)^2}{2} \right\} dx = \frac{1}{2} \int_0^a (a-x)^2\, dx = \frac{1}{2} \left| -\frac{(a-x)^3}{3} \right|_0^a = \frac{a^3}{6}.$$

Example 7.21. *Find the volume of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. (Anna, 2009 ; P.T.U., 2006 ; Kottayam, 2005)

Solution. Let $OABC$ be the positive octant of the given ellipsoid which is bounded by the planes OAB $(z = 0)$, OBC $(x = 0)$, OCA $(y = 0)$ and the surface ABC, *i.e.*,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Divide this region R into rectangular parallellopipeds of volume $\delta x \delta y \delta z$. Consider such an element at $P(x, y, z)$. (Fig. 7.24)

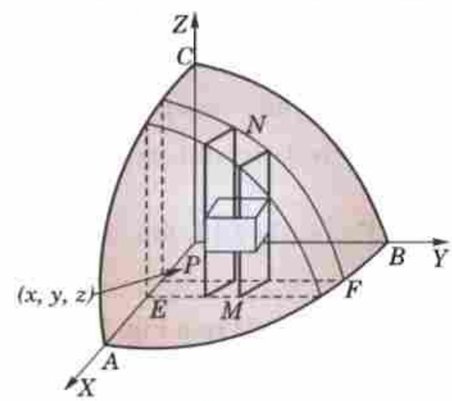

Fig. 7.24

$\therefore$ the required volume $= 8 \iiint_R dx\, dy\, dz.$

In this region R,

(*i*) z varies from 0 to MN where

$$MN = c\sqrt{(1 - x^2/a^2 - y^2/b^2)}.$$

(*ii*) y varies from 0 to EF, where $EF = b\sqrt{(1 - x^2/a^2)}$ from the equation of the ellipse OAB, *i.e.*,

$$x^2/a^2 + y^2/b^2 = 1.$$

(*iii*) x varies from 0 to $OA = a$.

Hence the volume of the whole ellipsoid

$$= 8 \int_0^a \int_0^{b\sqrt{(1-x^2/a^2)}} \int_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}} dx\, dy\, dz = 8 \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy \mid z \mid_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}}$$

$$= 8c \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} \sqrt{(1 - x^2/a^2 - y^2/b^2)}\, dy$$

$$= \frac{8c}{b} \int_0^a dx \int_0^{\rho} \sqrt{(\rho^2 - y^2)}\, dy \quad \text{when } \rho = b\sqrt{1 - x^2/a^2}.$$

$$= \frac{8c}{b} \int_0^a dx \left[\frac{y\sqrt{(\rho^2 - y^2)}}{2} + \frac{\rho^2}{2} \sin^{-1} \frac{y}{\rho} \right]_0^{\rho} = \frac{8c}{b} \int_0^a \frac{b^2}{2}\left(1 - \frac{x^2}{a^2}\right) \frac{\pi}{2}\, dx$$

$$= 2\pi bc \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = 2\pi bc \left| x - \frac{x^3}{3a^2} \right|_0^a = \frac{4\pi abc}{3}.$$

Otherwise. See Problem 27 page 292.

(3) Volumes of solids of revolution

Consider an elementary area $\delta x \delta y$ at the point $P(x, y)$ of a plane area A. (Fig. 7.25)

As this elementary area revolves about x-axis, we get a ring of volume

$$= \pi[(y + \delta y)^2 - y^2]\, \delta x = 2\pi y\, \delta x\, \delta y,$$

nearly to the first powers of δy.

Hence the total volume of the solid formed by the revolution of the area A about x-axis.

$$= \iint_A 2\pi y \, dxdy.$$

In polar coordinates, the above formula for the volume becomes

$$\iint_A 2\pi r \sin\theta \, . \, r d\theta dr, \text{ i.e. } \iint_A 2\pi r^2 \sin\theta \, d\theta dr$$

Similarly, the volume of the solid formed by the revolution of the area A about y-axis $= \iint_A 2\pi x \, dx \, dy$.

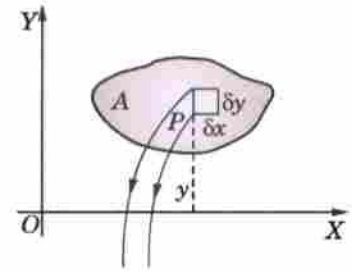

Fig. 7.25

Example 7.22. *Calculate by double integration, the volume generated by the revolution of the cardioid $r = a\,(1 - \cos\theta)$ about its axis.*

Solution. Required volume

$$= \int_0^{\pi} \int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta$$

$$= 2\pi \int_0^{\pi} \left| \frac{r^3}{3} \right|_0^{a(1-\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi} (1 - \cos\theta)^3 \, . \sin\theta \, d\theta = \frac{2\pi a^3}{3} \left| \frac{(1-\cos\theta)^4}{4} \right|_0^{\pi} = \frac{8\pi a^3}{3}.$$

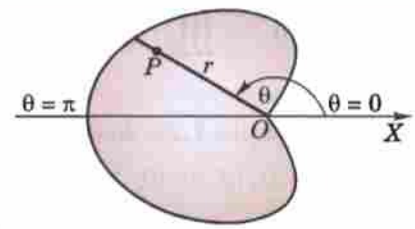

Fig. 7.26

7.7 CHANGE OF VARIABLES

An appropriate choice of co-ordinates quite often facilitates the evaluation of a double or a triple integral. By changing the variables, a given integral can be transformed into a simpler integral involving the new variables.

(1) **In a double integral,** let the variables x, y be changed to the new variables u, v by the transformation.

$$x = \phi\,(u, v), y = \psi\,(u, v)$$

where $\phi(u, v)$ and $\psi(u, v)$ are continuous and have continuous first order derivatives in some region R'_{uv} in the uv-plane which corresponds to the region R_{xy} in the xy-plane. Then

$$\iint_{R_{xy}} f(x, y) \, dxdy = \iint_{R'_{uv}} f[\phi(u, v), \psi(u, v)] \, | \, J \, | \, dudv \quad ...(1)$$

where $$J = \frac{\partial(x, y)}{\partial(u, v)} \; (\neq 0)$$

is the *Jacobian of transformation** from (x, y) to (u, v) coordinates.

(2) **For triple integrals,** the formula corresponding to (1) is

$$\iiint_{R_{xyz}} f(x, y, z) \, dxdydz = \iiint_{R'_{uvw}} f[x(u, v, w), y(u, v, w), z(u, v, w)] \, | \, J \, | \, dudvdw$$

where $$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} \; (\neq 0)$$

is the *Jacobian of transformation* from (x, y, z) to (u, v, w) coordinates.

Particular cases :

(*i*) ***To change cartesian coordinates (x, y) to polar coordinates (r, θ),*** *we have* $x = r\cos\theta, y = r\sin\theta$ *and*

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = r$$ [Ex. 5.25, p. 216]

$$\therefore \quad \iint_{R_{xy}} f(x, y) \, dx \, dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta) \, . \, r \, dr \, d\theta.$$

* See footnote page 215.

(*ii*) ***To change rectangular coordinates (x, y, z) to cylindrical coordinates*** (ρ, ϕ, z) — *Fig. 8.27, we have*

$$x = \rho \cos \phi, y = \rho \sin \phi, z = z$$

and $$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho \qquad \text{[Ex. 5.25]}$$

Then $$\iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R'_{\rho\phi z}} f(\rho \cos \phi, \rho \sin \phi, z)\, .\, \rho d\rho d\phi dz.$$

(*iii*) ***To change rectangular coordinates (x, y, z) to spherical polar coordinates (r,*** θ, ϕ)—*Fig. 8.28, we have*

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$$

and $$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta \qquad \text{[Ex. 5.25]}$$

Then $$\iiint_{R_{xyz}} f(x,y,z)\, dxdydz = \iiint_{R'_{r\theta\phi}} f(r \sin \theta \cos \phi, r \sin \theta, \sin \phi, r \cos \theta)\, .\, r^2 \sin \theta\, dr d\theta d\phi$$

Example 7.23. *Evaluate* $\iint_R (x + y)^2\, dxdy$, *where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation* $u = x + y$ *and* $v = x - 2y$. *(U.P.T.U., 2004)*

Solution. The region R, *i.e.*, parallelogram $ABCD$ in the xy-plane becomes the region R', *i.e.*, rectangle $A'B'C'D'$ in the uv-plane as shown in Fig. 7.27, by taking

$$u = x + y \quad \text{and} \quad v = x - 2y \qquad ...(i)$$

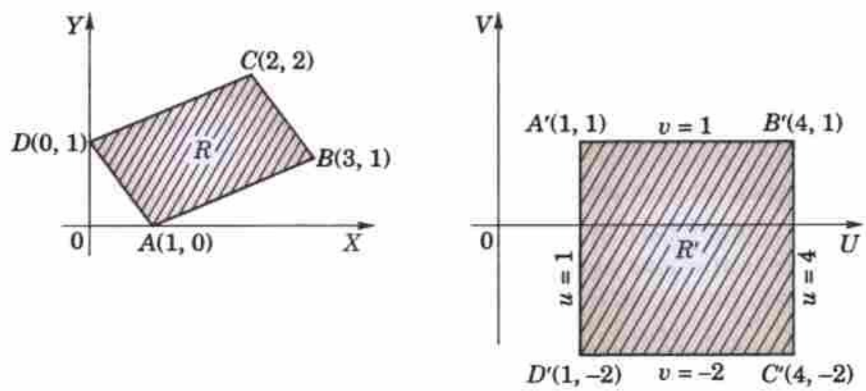

Fig. 7.27

From (*i*), we have $$x = \frac{1}{3}(2u + v), y = \frac{1}{3}(u - v)$$

$\therefore$ $$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = -\frac{1}{3}$$

Hence, the given integral

$$= \iint_{R'} u^2\, |J|\, dudv = \int_1^4 \int_{-2}^1 u^2 \cdot \frac{1}{3} \cdot dudv = \frac{1}{3} \left| \frac{u^3}{3} \right|_1^4 \cdot |\, v\, |_{-2}^1 = 21.$$

Example 7.24. *Evaluate* $\iint_D xy\sqrt{(1 - x - y)}\, dx\, dy$ *where D is the region bounded by* $x = 0, y = 0$ *and* $x + y = 1$ *using the transformation* $x + y = u, y = uv$. *(Marathwada, 2008)*

Solution. We have $x = u - uv$, $y = uv$

$$\therefore \qquad J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1-v & v \\ -u & u \end{vmatrix} = u.$$

Also when $x = 0$, $u = 0$, $v = 1$; when $y = 0$, $u = 0$, $v = 0$ and when $x + y = 1$, $u = 1$

$\therefore$ the limits of u are from 0 to 1 and limits of v are from 0 to 1.

Thus $$\iint_D xy\sqrt{(1-x-y)}\,dxdy = \int_0^1\int_0^1 u(1-v)\,uv\,(1-u)^{1/2}\,|J|\,dudv$$

$$= \int_0^1\int_0^1 u^3(1-u)^{1/2}\,v(1-v)\,du\,dv$$

where $u = \sin^2\theta$
$du = 2\sin\theta\cos\theta\,d\theta$
$u = 0, \theta = 0$
$u = 1, \theta = \pi/2$

$$= \int_0^1 u^3(1-u)^{1/2}\,du \times \int_0^1 v(1-v)\,dv$$

$$= \int_0^{\pi/2} \sin^6\theta\cos\theta\,.\,2\sin\theta\cos d\theta \times \left|\frac{v^2}{2} - \frac{v^3}{3}\right|_0^1$$

$$= 2\int_0^{\pi/2}\sin^7\theta\cos^2\theta\,d\theta\left(\frac{1}{6}\right) = \frac{1}{3}\cdot\frac{6\,.\,1}{9\,.\,7\,.\,5\,.\,3} = \frac{2}{945}.$$

Example 7.25. *Evaluate* $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dxdy$ *by changing to polar coordinates.* (Anna, 2003)

Hence show that $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi/2}$. (Madras, 2003 ; U.P.T.U., 2003 ; J.N.T.U., 2000)

Solution. The region of integration being the first quadrant of the xy-plane, r varies from 0 to ∞ and θ varies from 0 to $\pi/2$. Hence,

$$I = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dxdy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,rdrd\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left\{\int_0^\infty e^{-r^2}(-2r)\,dr\right\}d\theta = -\frac{1}{2}\int_0^{\pi/2}\left|e^{-r^2}\right|_0^\infty d\theta = \frac{1}{2}\int_0^{\pi/2}d\theta = \frac{\pi}{4}. \qquad ...(i)$$

Also $$I = \int_0^\infty e^{-x^2}\,dx \times \int_0^\infty e^{-y^2}\,dy = \left\{\int_0^\infty e^{-x^2}\,dx\right\}^2 \qquad ...(ii)$$

Thus, from (i) and (ii), we have $\int_0^\infty e^{-x^2} = \sqrt{\pi/2}$. ...(iii)

Example 7.26. *Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$.*

Solution. The required volume is found by integrating $z = (x^2 + y^2)/a$ over the circle $x^2 + y^2 = 2ay$.

Changing to polar coordinates in the xy-plane, we have $x = r\cos\theta$, $y = r\sin\theta$ so that $z = r^2/a$ and the polar equation of the circle is $r = 2a\sin\theta$.

To cover this circle, r varies from 0 to $2a\sin\theta$ and θ varies from 0 to π. (Fig. 7.28)

Hence the required volume

$$= \int_0^\pi\int_0^{2a\sin\theta} z\,.\,r\,d\theta\,dr = \frac{1}{a}\int_0^\pi d\theta\int_0^{2a\sin\theta} r^3\,dr$$

$$= \frac{1}{a}\int_0^\pi d\theta\left|\frac{r^4}{4}\right|_0^{2a\sin\theta} = 4a^3\int_0^\pi \sin^4\theta\,d\theta = \frac{3\pi a^3}{2}.$$

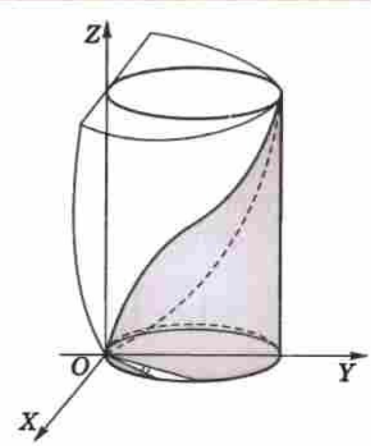

Fig. 7.28

Example 7.27. *Find, by triple integration, the volume of the sphere $x^2 + y^2 + z^2 = a^2$.*
(*Bhopal, 2009 ; Madras, 2006 ; V.T.U., 2003 S*)

Solution. Changing to polar spherical coordinates by putting

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta$$

we have $dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

Also the volume of the sphere is 8 times the volume of its portion in the positive octant for which r varies from 0 to a, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ volume of the sphere

$$= 8 \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin\theta\, dr\, d\theta\, d\phi = 8 \int_0^a r^2\, dr \,.\int_0^{\pi/2} .\sin\theta\, d\theta \,.\int_0^{\pi/2} d\phi$$

$$= 8 \,.\left|\frac{r^3}{3}\right|_0^a .\left| -\cos\theta \right|_0^{\pi/2} .\frac{\pi}{2} = 4\pi \,.\frac{a^3}{3}\,.(-0+1) = \frac{4}{3}\pi a^3.$$

Example 7.28. *Find the volume of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ay$.*
(*Rohtak, 2003*)

Solution. The required volume is easily found by changing to cylindrical coordinates (ρ, ϕ, z). We therefore, have

$$x = \rho\cos\phi,\ y = \rho \sin\phi,\ z = z$$

and

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

Then the equation of the sphere becomes $\rho^2 + z^2 = a^2$ and that of cylinder becomes $\rho = a \sin\phi$.

The volume inside the cylinder bounded by the sphere is twice the volume shown shaded in the Fig. 7.29 for which z varies from 0 to $\sqrt{(a^2 - \rho^2)}$, ρ varies from 0 to $a \sin\phi$ and ϕ varies from 0 to π.

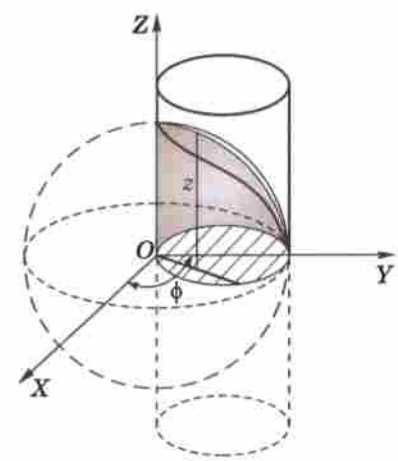

Fig. 7.29

Hence the required volume $= 2 \int_0^{\pi} \int_0^{a\sin\phi} \int_0^{\sqrt{(a^2-\rho^2)}} \rho dz d\rho d\phi$

$$= 2 \int_0^{\pi} \int_0^{a\sin\phi} \rho\sqrt{(a^2 - \rho^2)}\, d\rho d\phi = 2 \int_0^{\pi} \left| -\frac{1}{3}(a^2 - \rho^2)^{3/2} \right|_0^{a\sin\phi} d\phi$$

$$= \frac{2a^3}{3} \int_0^{\pi} (1 - \cos^3\phi)\, d\phi = \frac{2a^3}{9}(3\pi - 4).$$

Example 7.29. *Evaluate* $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_{\sqrt{(x^2+y^2)}}^1 \frac{dz\, dy\, dx}{\sqrt{(x^2 + y^2 + z^2)}}$.
(*V.T.U., 2008*)

Solution. We change to spherical polar coordinates (r, θ, ϕ), so that

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

and

$$J = r^2 \sin\theta,\ x^2 + y^2 + z^2 = r^2.$$

The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$ bounded by the plane $z = 1$ in the positive octant (Fig. 7.30). Hence θ varies from 0 to $\pi/4$, r varies from 0 to $\sec\theta$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ given integral becomes

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sec\theta} \frac{1}{r}.r^2 \sin\theta\, dr\, d\theta\, d\phi = \int_0^{\pi/2} d\phi \int_0^{\pi/4} \left|\frac{r^2}{2}\right|_0^{\sec\theta} \sin\theta\, d\theta$$

$$= \frac{\pi}{2}\int_0^{\pi/4} \frac{\sec^2\theta}{2} \sin\theta\, d\theta = \frac{\pi}{4}\int_0^{\pi/4} \sec\theta\tan\theta\, d\theta = \frac{\pi}{4}\left|\sec\theta\right|_0^{\pi/4} = \frac{(\sqrt{2}-1)\pi}{4}.$$

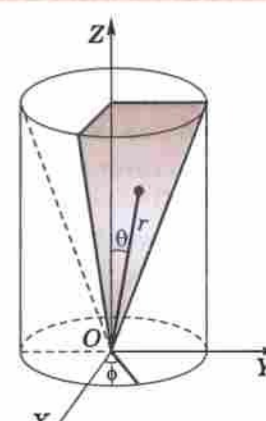

Fig. 7.30

Example 7.30. *Find the volume of the solid surrounded by the surface*

$$(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1.$$ *(Hissar, 2005 S)*

Solution. Changing the variables, x, y, z to X, Y, Z where, $(x/a)^{1/3} = X, (y/b)^{1/3} = Y, (z/c)^{1/3} = Z$

i.e., $x = aX^3, y = bY^3, z = cZ^3$ so that $J = \partial(x, y, z)/\partial(X, Y, Z) = 27\, abc\, X^2Y^2Z^2$.

$\therefore$ required volume $= \iiint dx\, dy\, dz = 27\, abc \iiint X^2Y^2Z^2\, dX\, dY dZ$

taken throughout the sphere $X^2 + Y^2 + Z^2 = 1$. ...(*i*)

Now change X, Y, Z to spherical polar coordinates r, θ, ϕ so that $X = r \sin\theta \cos\phi$, $Y = r \sin\theta \sin\phi$, $Z = r\cos\theta$, and $\partial(X, Y, Z)/\partial(r, \theta, \phi) = r^2 \sin\theta$. To describe the positive octant of the sphere (*i*), r varies from 0 to 1, θ from 0 to $\pi/2$ and ϕ from 0 to $\pi/2$.

$\therefore$ required volume $= 27abc \times 8 \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin^2\theta \cos^2\phi \times r^2 \sin^2\theta \sin^2\phi \,.\, r^2\cos^2\theta \,.\, r^2 \sin\theta\, dr d\theta d\phi$

$$= 216\, abc \int_0^1 r^8\, dr \int_0^{\pi/2} \sin^4\theta \cos^2\theta\, d\theta \int_0^{\pi/2} \sin^2\phi \cos^2\phi\, d\phi = 4\pi\, abc/35.$$

PROBLEMS 7.4

Evaluate the following integrals by changing to polar co-ordinates :

1. $\int_0^1 \int_0^{\sqrt{(1-y^2)}} (x^2 + y^2)\, dy\, dx$. (*P.T.U., 2010*)

2. $\int_0^2 \int_0^{\sqrt{(2x-x^2)}} \dfrac{x\, dx\, dy}{x^2 + y^2}$ (*Anna, 2009*)

3. $\int_0^{4a} \int_{y^2/4a}^{y} \dfrac{x^2 - y^2}{x^2 + y^2}\, dx dy$ (*Mumbai, 2006*)

4. $\iint xy(x^2 + y^2)^{n/2}\, dx dy$ over the positive quadrant of $x^2 + y^2 = 4$, supposing $n + 3 > 0$. (*S.V.T.U., 2007*)

5. $\iint \dfrac{dx dy}{(1 + x^2 + y^2)^2}$ over one loop of the lemniscate $(x^2 + y^2) = x^2 - y^2$. (*Mumbai, 2007*)

6. Transform the following to cartesian form and hence evaluate $\int_0^{\pi} \int_0^{a} r^3 \sin\theta \cos\theta\, dr d\theta$. (*P.T.U., 2005*)

7. $\iint y^2\, dx\, dy$ over the area outside $x^2 + y^2 - ax = 0$ and inside $x^2 + y^2 - 2ax = 0$. (*Mumbai, 2006*)

8. By using the transformation $x + y = u, y = uv$, show that $\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dy dx = \frac{1}{2}(e - 1)$. (*P.T.U., 2003*)

9. Transform $\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\dfrac{\sin\phi}{\sin\theta}}\, d\phi\, d\theta$ by the substitution $x = \sin\phi\cos\theta, y = \sin\phi\sin\theta$ and show that its value is π. (*U.P.T.U., 2001*)

Evaluate the following integrals by changing to spherical coordinates :

10. $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \dfrac{dx\, dy\, dz}{\sqrt{(1 - x^2 - y^2 - z^2)}}$. (*V.T.U., 2006 ; Kottayam, 2005*)

11. $\iiint_V \dfrac{dx\, dy\, dz}{x^2 + y^2 + z^2}$ where V is the volume of the sphere $x^2 + y^2 + z^2 = a^2$. (*Anna, 2009*)

12. Evaluate $\iiint \dfrac{dx\, dy\, dz}{(1 + x + y + z)^3}$ over the volume of the tetrahedron $x = 0, y = 0, z = 0, x + y + z = 1$. (*Mumbai, 2007*)

13. Show that $\iiint \dfrac{dx\, dy\, dz}{\sqrt{(a^2 - x^2 - y^2 - z^2)}} = \dfrac{\pi^2 a^3}{8}$, the integral being extended for all the values of the variables for which the expression is real. (*U.T.U., 2010*)

14. $\iiint z^2\, dx\, dy\, dz$, taken over the volume bounded by the surfaces $x^2 + y^2 = a^2, x^2 + y^2 = z$ and $z = 0$.

15. Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$. *(I.S.M., 2001)*
16. Find the volume bounded by the xy-plane, the paraboloid $2z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$. *(Raipur, 2005)*
17. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cone $x^2 + y^2 = z^2$.
18. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$. *(S.V.T.U., 2006)*
19. Find the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = 0$ and $z = x$. *(U.P.T.U., 2006)*
20. Find the volume enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$. *(Marathwada, 2008)*
21. Find the volume of the cylinder $x^2 + y^2 - 2ax = 0$, intercepted between the paraboloid $x^2 + y^2 = 2az$ and the xy-plane.
22. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.
23. Find the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$ and $y = -a$, $y = a$.
24. Prove, by using a double integral that the volume generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about its axis is $8\pi a^3/3$. *(V.T.U., 2000)*
25. Evaluate $\iiint (x + y + z)\, dx\, dy\, dz$ over the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$. [See Fig. 7.34]
26. Find the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. *(Burdwan, 2003)*
27. Work out example 7.21 by changing the variables.

7.8 AREA OF A CURVED SURFACE

Consider a point P of the surface $S : z = f(x, y)$. Let its projection on the xy-plane be the region A. Divide it into area elements by drawing lines parallel to the axes of X and Y. (Fig. 7.31).

On the element $\delta x \delta y$ as base, erect a cylinder having generators parallel to OZ and meeting the surface S in an element of area δS.

As $\delta x \delta y$ is the projection of δS on the xy-plane,

$\therefore$ $\delta x \delta y = \delta S \cdot \cos\gamma$, where γ is the angle between the xy-plane and the tangent plane to S at P, *i.e.*, it is the angle between the Z-axis and the normal to S at $P (= \angle Z'PN)$.

Now since the direction cosines of the normal to the surface $F(x, y, z) = 0$ proportional to

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}.$$

$\therefore$ the direction cosines of the normal to $S[F = f(x, y) - z]$ are proportional to $-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}$, 1 and those of the z-axis are 0, 0, 1.

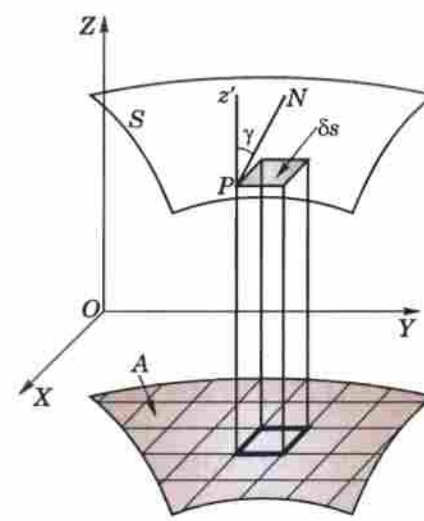

Fig. 7.31

Hence $\quad \cos\gamma = \dfrac{1}{\sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}} \quad \therefore \quad \delta S = \dfrac{\delta x\, \delta y}{\cos\gamma} = \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, \delta x\, \delta y$

Hence
$$S = \underset{\delta S \to 0}{\text{Lt}} \sum \delta S = \iint_A \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, dx dy$$

Similarly, if B and C be the projections of S on the yz-and zx-planes respectively, then

$$S = \iint_B \sqrt{\left[\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1\right]}\, dy\, dz$$

and
$$S = \iint_C \sqrt{\left[\left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 + 1\right]}\, dz\, dx.$$

Example 7.31. *Find the area of the portion of the cylinder $x^2 + z^2 = 4$ lying inside the cylinder $x^2 + y^2 = 4$.*

Solution. Figure 7.32 shows one-eighth of the required area. Its projection on the xy-plane is a quadrant circle $x^2 + y^2 = 4$.

For the cylinder $x^2 + z^2 = 4$, ...(*i*)

we have $\dfrac{\partial z}{\partial x} = -\dfrac{x}{z}, \dfrac{\partial z}{\partial y} = 0.$

so that $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1 = \dfrac{x^2 + z^2}{z^2} = \dfrac{4}{4 - x^2}.$

Hence the required surface area = 8 (surface area of the upper portion of (*i*) lying within the cylinder $x^2 + y^2 = 4$ in the positive octant)

$$= 8\int_0^2 \int_0^{\sqrt{(4-x^2)}} \frac{2}{\sqrt{(4-x^2)}}\, dxdy = 16\int_0^2 dx = 32 \text{ sq. units.}$$

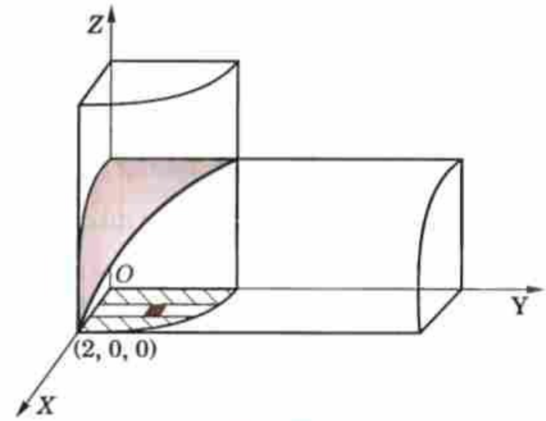

Fig. 7.32

Example 7.32. *Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$.*

Solution. Figure 7.33 shows one-fourth of the required area. Its projection on the xy-plane is the semi-circle $x^2 + y^2 = 3y$ bounded by the Y-axis.

For the sphere

$$x^2 + y^2 + z^2 = 9, \frac{\partial z}{\partial x} = -\frac{x}{z} \text{ and } \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\therefore \quad \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = (x^2 + y^2 + z^2)/z^2$$

$$= \frac{9}{9 - x^2 - y^2} = \frac{9}{9 - r^2} \quad \text{when} \quad x = r\cos\theta, y = r\sin\theta.$$

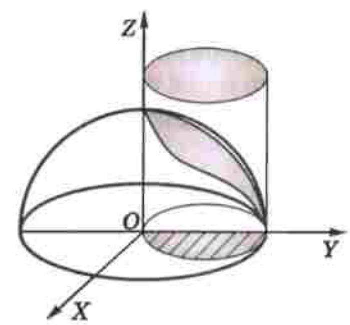

Fig. 7.33

Using polar coordinates, the required area is found by integrating $3/\sqrt{(9 - r^2)}$ over the semi-circle $r = 3\sin\theta$, for which r varies from 0 to $3\sin\theta$ and θ varies from 0 to $\pi/2$.

Hence the required surface area

$$= 4\int_0^{\pi/2}\int_0^{3\sin\theta} \frac{3}{\sqrt{(9-r^2)}}\, r d\theta\, dr = -6\int_0^{\pi/2} \left| \frac{\sqrt{(9-r^2)}}{1/2} \right|_0^{3\sin\theta} d\theta$$

$$= 36\int_0^{\pi/2} (1 - \cos\theta)\, d\theta = 36 \left| \theta - \sin\theta \right|_0^{\pi/2} = 18(\pi - 2) \text{ sq. units.}$$

PROBLEMS 7.5

1. Show that the surface area of the sphere $x^2 + y^2 + z^2 = a^2$ is $4\pi a^2$.
2. Find the area of the portion of the cylinder $x^2 + y^2 = 4y$ lying inside the sphere $x^2 + y^2 + z^2 = 16$.
3. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ax$.
4. Find the area of the surface of the cone $x^2 + y^2 = z^2$ cut off by the surface of the cylinder $x^2 + y^2 = a^2$ above the xy-plane.
5. Compute the area of that part of the plane $x + y + z = 2a$ which lies in the first octant and is bounded by the cylinder $x^2 + y^2 = a^2$. (*Burdwan, 2003*)

7.9 CALCULATION OF MASS

(a) **For a plane lamina,** if the surface density at the point $P(x, y)$ be $\rho = f(x, y)$ then the elementary mass at $P = \rho\delta x\delta y$.

$\therefore$ *total mass of the lamina* $= \iint \rho \, dx \, dy$ ...(i)

with integrals embracing the whole area of the lamina.

In polar coordinates, taking $\rho = \phi(r, \theta)$ at the point $P(r, \theta)$,

total mass of the lamina $= \iint \rho r \, d\theta \, dr$ ...(ii)

(b) **For a solid,** if the density at the point $P(x, y, z)$ be $\rho = f(x, y, z)$, then

total mass of the solid $= \iiint \rho \, dx \, dy \, dz$ with appropriate limits of integration.

Example 7.33. *Find the mass of the tetrahedron bounded by the coordinates planes and the plane* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, *the variable density* $\rho = \mu xyz$. (Rohtak, 2003 ; U.P.T.U., 2003)

Solution. Elementary mass at $P = \mu xyz \, . \, \delta x \, \delta y \, \delta z$.

$\therefore$ the whole mass $= \iiint \mu xyz \, dx \, dy \, dz$,

the integrals embracing the whole volume $OABC$ (Fig. 7.34). The limits for z are from 0 to $z = c(1 - x/a - y/b)$.

The limits for y are from 0 to $y = b(1 - x/a)$ and limits for x are from 0 to a.

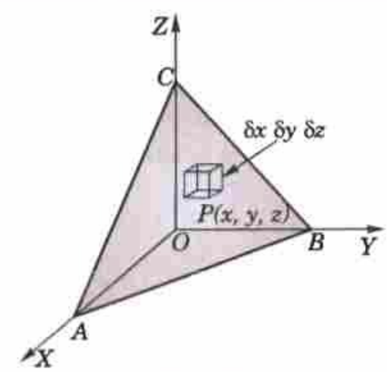

Fig. 7.34

Hence the required mass

$$= \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} \mu xyz \, dz \, dy \, dx$$

$$= \mu \int_0^a \int_0^{b(1-x/a)} xy \left| z^2/2 \right|_0^{c(1-x/a-y/b)} dydz$$

$$= \mu \int_0^a \int_0^{b(1-x/a)} xy \, . \, \frac{c^2}{2}\left(1 - \frac{x}{a} - \frac{y}{b}\right)^2 dydx$$

$$= \frac{\mu c^2}{2} \int_0^a \int_0^{b(1-x/a)} x \, . \left[\left(1-\frac{x}{a}\right)^2 y - 2\left(1-\frac{x}{a}\right)\frac{y^2}{b} + \frac{y^3}{b^2}\right] dy \, dx$$

$$= \frac{\mu c^2}{2} \int_0^a x \left| \left(1-\frac{x}{a}\right)^2 \frac{y^2}{2} - 2\left(1-\frac{x}{a}\right)\frac{y^3}{3b} + \frac{y^4}{4b^2} \right|_0^{b(1-x/a)} dx$$

$$= \frac{\mu c^2}{2} \int_0^a x \left[\frac{b^2}{2}\left(1-\frac{x}{a}\right)^4 - \frac{2b^2}{3}\left(1-\frac{x}{a}\right)^4 + \frac{b^2}{4}\left(1-\frac{x}{a}\right)^4\right] dx = \frac{\mu b^2 c^2}{24} \int_0^a x\,(1 - x/a)^4 \, dx = \frac{\mu a^2 b^2 c^2}{720}.$$

7.10 CENTRE OF GRAVITY

(a) **To find the C.G. $(\bar{x}, \bar{y})$ of a plane lamina,** take the element of mass $\rho\delta x\delta y$ at the point $P(x, y)$. Then

$$\bar{x} = \frac{\iint x\rho \, dxdy}{\iint \rho \, dxdy}, \; \bar{y} = \frac{\iint y \, \rho dxdy}{\iint \rho \, dxdy}$$ with integrals embracing the whole lamina.

While using polar coordinates, take the elementary mass as $\rho r\delta\theta\delta r$ at the point $P(r, \theta)$ so that $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \quad \bar{x} = \frac{\iint r\cos\theta \, \rho r \, d\theta dr}{\iint \rho \, rd\theta dr}, \; \bar{y} = \frac{\iint r\sin\theta \, \rho \, rd\theta dr}{\iint \rho \, rd\theta dr}$$

(b) **To find the C.G. $(\bar{x}, \bar{y}, \bar{z})$ of a solid,** take an element of mass $\rho\delta x\delta y\delta z$ enclosing the point $P(x, y, z)$. *Then*

$$\bar{x} = \frac{\iiint x\rho\, dxdydz}{\iiint \rho\, dxdydz},\ \bar{y} = \frac{\iiint y\rho\, dxdydz}{\iiint \rho\, dxdydz} \text{ and } \bar{z} = \frac{\iiint z\rho\, dxdydz}{\iiint \rho\, dxdydz}.$$

Example 7.34. *Find by double integration, the centre of gravity of the area of the cardioid $r = a(1 + \cos\theta)$.*

Solution. The cardioid being symmetrical about the initial line, its C.G. lies on OX, *i.e.*, $\bar{y} = 0$ (Fig. 7.35).

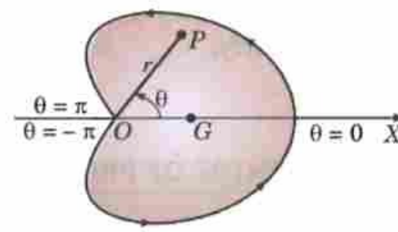

Fig. 7.35

$$\therefore \quad \bar{x} = \frac{\iint \rho\, r\cos\theta\, .\, rd\theta dr}{\iint \rho\, rd\theta dr} = \frac{\int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} \cos\theta.\, r^2 dr\, .\, d\theta}{\int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} rdr\, .\, d\theta}$$

$$= \frac{\int_{-\pi}^{\pi} \cos\theta \left| \frac{r^3}{3} \right|_0^{a(1+\cos\theta)} d\theta}{\int_{-\pi}^{\pi} \left| \frac{r^2}{2} \right|_0^{a(1+\cos\theta)} d\theta} = \frac{2a}{3} . \frac{\int_{-\pi}^{\pi} \cos\theta\,(1+\cos\theta)^3\, d\theta}{\int_{-\pi}^{\pi} (1+\cos\theta)^2\, d\theta}$$

$$= \frac{2a}{3} . \frac{2 . \int_0^{\pi} (3\cos^2\theta + \cos^4\theta)\, d\theta}{2 . \int_0^{\pi} (1+\cos^2\theta)\, d\theta} \qquad \left\{ \because \int_{-\pi}^{\pi} \cos^n\theta\, d\theta = 2\int_0^{\pi} \cos^n\theta\, d\theta \text{ or } 0 \text{ according as } n \text{ in even or odd.} \right\}$$

$$= \frac{2a}{3} . \frac{2 . \int_0^{\pi/2} (3\cos^2\theta + \cos^4\theta)\, d\theta}{2 . \int_0^{\pi/2} (1+\cos^2\theta)\, d\theta} \text{ (as the powers of } \cos\theta \text{ are even)} = \frac{2a}{3} . \frac{3 . \frac{1}{2} . \frac{\pi}{2} + \frac{3 . 1}{4 . 2} . \frac{\pi}{2}}{\frac{\pi}{2} + \frac{1}{2} . \frac{\pi}{2}} = \frac{5a}{6}$$

Hence the C.G. of the cardioid is at $G(5a/6, 0)$.

Example 7.35. *Using double integration, find the C.G. of a lamina in the shape of a quadrant of the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$, the density being $\rho = kxy$, where k is a constant.*

Solution. Let $G(\bar{x}, \bar{y})$ be the C.G. of the lamina OAB (Fig. 7.36), so that

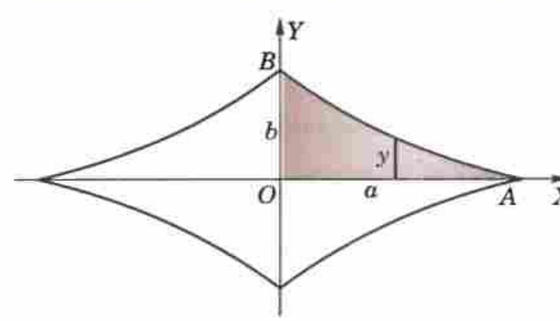

Fig. 7.36

$$\bar{x} = \frac{\iint kxy\, .\, xdxdy}{\iint kxy\, .\, dxdy} = \frac{\iint x^2 y\, dxdy}{\iint xy\, dxdy}$$

where the integrals are taken over the area OAB so that y varies from 0 to y (to be found from the equation of the curve in terms of x) and then x varies from 0 to a.

Thus

$$\bar{x} = \frac{\int_0^a \int_0^y x^2 y\, dydx}{\int_0^a \int_0^y xy\, dydx} = \frac{\int_0^a x^2 . \left| y^2/2 \right|_0^y dx}{\int_0^a x . \left| y^2/2 \right|_0^y dx} = \frac{\int_0^a x^2y^2\, dx}{\int_0^a xy^2\, dx}$$

For any point on the curve, we have

$$x = a\cos^3\theta,\ y = b\sin^3\theta \text{ so that}$$
$$dx = -3a\cos^2\theta\sin\theta\, d\theta.$$

Also when $\quad x = 0, \theta = \pi/2$; when $x = a, \theta = 0$.

Hence $$\bar{x} = \frac{\int_{\pi/2}^{0} a^2 \cos^6 \theta \,.\, b^2 \sin^6 \theta \,.\, (-3a \cos^2 \theta \sin \theta)\, d\theta}{\int_{\pi/2}^{0} a \cos^3 \theta \,.\, b^2 \sin^6 \theta \,.\, (-3a \cos^2 \theta \sin \theta)\, d\theta}$$

$$= a \frac{\int_0^{\pi/2} \sin^7 \theta \cos^8 \theta \, d\theta}{\int_0^{\pi/2} \sin^7 \theta \cos^5 \theta \, d\theta} = \frac{128}{429} a$$

Similarly, $$\bar{y} = \frac{\int_0^a \int_0^y kxy \,.\, y \, dxdy}{\int_0^a \int_0^y kxy \,.\, dxdy} = \frac{128}{429} b.$$ Hence the required C.G. is $G\left(\frac{128}{429}a, \frac{128}{429}b\right)$.

7.11 CENTRE OF PRESSURE

Consider plane area A immersed vertically in a homogeneous liquid. Take the line of intersection of the given plane with the free surface of the liquid as the x-axis and any line lying in this plane and perpendicular to it downwards as the y-axis (Fig. 7.37).

If p be the pressure at the point $P(x, y)$ of the area A, then the pressure on an elementary area $\delta x \delta y$ at P is $p\delta x \delta y$ which is normal to the plane.

$\therefore$ the resultant pressure on $A = \iint p dxdy$.

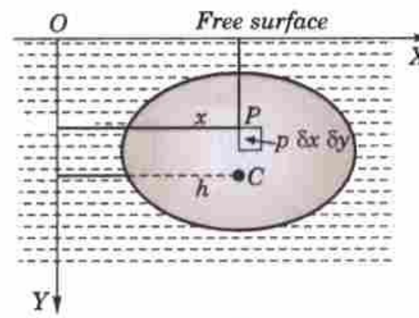

Fig. 7.37

If this resultant pressure acting at $C(h, k)$ is equivalent to pressure at various points such as $p\delta x \delta y$ distributed over the whole area A, then C is called the *centre of pressure*.

$\therefore$ taking the moment of the resultant pressure at C and the sum of the moments of the individual pressures such as $p\delta x \delta y$ at $P(x, y)$ about the y-axis, we get

$$h \iint p \, dxdy = \iint x \,.\, p dxdy, \quad i.e., \; h = \iint x \,.\, dxdy \Big/ \iint p dxdy$$

Similarly, taking moments about x-axis, we have

$$k = \iint y \,.\, p dxdy \Big/ \iint p dxdy$$ with integrals embracing the whole of the area A.

While using polar coordinates, replace x by $r \cos \theta$, y by $r \sin \theta$ and $dxdy$ by $r\, d\theta dr$ in the above formulae.

Example 7.36. *A horizontal boiler has a flat bottom and its ends are plane and semi-circular. If it is just full of water, show that the depth of the centre of pressure of either end is 0.7 × total depth approximately.*

Solution. Let the semi-circle $x^2 + y^2 = a^2$ represent an end of the given boiler (Fig. 7.38). By symmetry, its centre of pressure lies on OY.

If w be the weight of water per unit volume, then the pressure p at the point $P(x, y) = w(a - y)$.

$\therefore$ the height k of the C.P. above OX, is given by

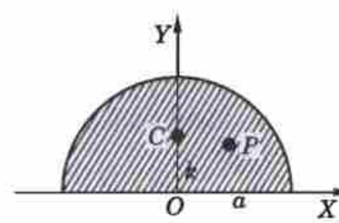

Fig. 7.38

$$k = \frac{\iint y \cdot p dxdy}{\iint p dxdy} = \frac{\int_{-a}^{a} \int_0^{\sqrt{(a^2 - x^2)}} w(a - y) y \, dy \cdot dx}{\int_{-a}^{a} \int_0^{\sqrt{(a^2 - x^2)}} w(a - y) \, dy \cdot dx}$$

$$= \frac{\int_{-a}^{a} \left| ay^2/2 - y^3/3 \right|_0^{\sqrt{(a^2 - x^2)}} dx}{\int_{-a}^{a} \left| ay - y^2/2 \right|_0^{\sqrt{(a^2 - x^2)}} dx} = \frac{\int_{-a}^{a} \left[\frac{a}{2}(a^2 - x^2) - \frac{1}{3}(a^2 - x^2)^{3/2}\right] dx}{\int_{-a}^{a} \left[a(a^2 - x^2)^{1/2} - \frac{1}{2}(a^2 - x^2)\right] dx}$$

Now put $x = a \sin \theta$, so that $dx = a \cos \theta \, d\theta$.

Also when $x = -a$, $\theta = -\pi/2$, and when $x = a$, $\theta = \pi/2$.

$$\therefore \qquad k = \frac{\int_{-\pi/2}^{\pi/2} \left[\frac{a^3}{2}\cos^2\theta - \frac{a^3}{3}\cos^3\theta\right] a\cos\theta \, d\theta}{\int_{-\pi/2}^{\pi/2} \left[a^2\cos\theta - \frac{a^2}{2}\cos^2\theta\right] a\cos\theta \, d\theta}$$

$$= \frac{a}{3} \cdot \frac{2\int_0^{\pi/2} (3\cos^3\theta - 2\cos^4\theta)\, d\theta}{2\int_0^{\pi/2} (2\cos^2\theta - \cos^3\theta)\, d\theta} = \frac{a}{4}\left(\frac{16 - 3\pi}{3\pi - 4}\right) = 0.3a \text{ nearly.}$$

Hence, the depth of the C.P. $= a - k = 0.7a$ approximately.

PROBLEMS 7.6

1. A lamina is bounded by the curves $y = x^2 - 3x$ and $y = 2x$. If the density at any point is given by λxy, find by double integration, the mass of the lamina.
2. Find the mass of a lamina in the form of cardioid $r = a(1 + \cos \theta)$ whose density at any point varies as the square of its distance from the initial line.
3. Find the mass of a solid in the form of the positive octant of the sphere $x^2 + y^2 + z^2 = 9$, if the density at any point is $2xyz$.
4. Find the centroid of the area enclosed by the parabola $y^2 = 4ax$, the axis of x and its latus-rectum.
5. The density at any point (x, y) of a lamina is $\sigma(x + y)/a$ where σ and a are constants. The lamina is bounded by the lines $x = 0, y = 0, x = a, y = b$. Find the position of its centre of gravity.
6. Find the centroid of a loop of the lemniscate $r^2 = a^2 \cos 2\theta$.
7. A plane in the form of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$ is of small but varying thickness, the thickness at any point being proportional to the product of the distances of that point from the axes ; show that the coordinates of the centroid are $(8a/15, 8b/15)$. *(Nagpur, 2009)*
8. In a semi-circular disc bounded by a diameter OA, the density at any point varies as the distance from O ; find the position of the centre of gravity.
9. Find the centroid of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$, the density at any point varying as its distance from the face $z = 0$.
10. Find $\bar{x}$ where $(\bar{x}, \bar{y}, \bar{z})$ is the centroid of the region R bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0, y = 0, y = 6, z = 0$. (Assume that the density is constant).
11. If the density at any point of the solid octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ varies as xyz, find the coordinates of the C.G. of the solid. *(P.T.U., 2005)*
12. A horizontal boiler has a flat bottom and its ends consist of a square 1 metre wide surmounted by an isosceles triangle of height 0.5 metre. Determine the depth of the centre of pressure of either end when the boiler is just full.
13. A quadrant of a circle is just, immersed vertically in a heavy homogeneous liquid with one edge in the surface. Find the centre of pressure.
14. Find the depth of the centre of pressure of a square lamina immersed in the liquid, with one vertex in the surface and the diagonal vertical.
15. Find the centre of pressure of a triangular lamina immersed in a homogeneous liquid with one side in the free surface. *(P.T.U., 2003)*
16. A uniform semi-circular is lamina immersed in a fluid with its plane vertical and its boundary diameter on the free surface. If the density at any point of the fluid varies as the depth of the point below the free surface, find the position of the centre of pressure of the lamina.

7.12 (1) MOMENT OF INERTIA

If a particle of mass m of a body be at a distance r from a given line, then mr^2 is called the *moment of inertia of the particle about the given line* and the sum of similar expressions taken for all the particles of the body, *i.e.*, Σmr^2 is called the *moment of inertia of the body about the given line* (Fig. 7.39).

If M be the total mass of the body and we write its moment of inertia $= Mk^2$, then k is called the *radius of gyration* of the body about the axis.

(2) M.I. of plane lamina. Consider the elementary mass $\rho\delta x\delta y$ at the point $P(x, y)$ of a plane area A so that its M.I. about x-axis $= \rho\delta x\delta y\, y^2$.

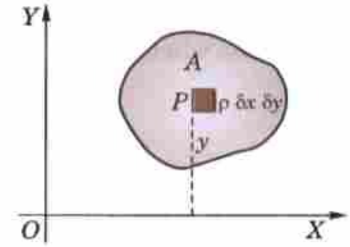

Fig. 7.39

$\therefore$ M.I. of the lamina about x-axis, *i.e.* $I_x = \iint_A \rho\, y^2\, dxdy$.

Similarly, M.I. of the lamina about y-axis' *i.e.*, $I_y = \iint_A \rho\, x^2\, dxdy$.

Also M.I. of the lamina about an axis perpendicular to the xy-plane, *i.e.*,

$$I_z = \iint_A \rho(x^2 + y^2)\, dxdy.$$

(3) M.I. of a solid. Consider an elementary mass $\rho\delta x\delta y\delta z$ enclosing a point $P(x, y, z)$ of a solid of volume V.

Distance of P from the x-axis $= \sqrt{(y^2 + z^2)}$.

$\therefore$ M.I. of this element about the x-axis $= \rho\, \delta x\delta y\delta z\, (y^2 + z^2)$.

Thus *M.I. of this solid about x-axis, i.e.,* $I_x = \iiint_V \rho(y^2 + z^2)\, dxdydz.$

Similarly, *its M.I. about y-axis, i.e.,* $I_y = \iiint_V \rho(z^2 + x^2)\, dxdydz$

and M.I. *about z-axis, i.e.,* $I_z = \iiint_V \rho(x^2 + y^2)\, dxdydz.$

(4) Sometimes we require the moment of inertia of a body about axes other than the principal axes. The following theorems prove useful for this purpose :

I. Theorem of perpendicular axis. *If the moment of inertia of a lamina about two perpendicular axes OX, OY in its plane are I_x and I_y, then the moment of inertia of the lamina about an axis OZ, perpendicular to it is given by $I_z = I_x + I_y$.*

Its proof follows from the relations giving I_x, I_y and I_z for a plane lamina [(2) above].

II. Steiner's theorem*. *If the moment of inertia of a body of mass M about an axis through its centre of gravity is I, then I′, moment of inertia about a parallel axis at a distance d from the first axis, is given by $I' = I + Md^2$.*

Its proof will be found in any text book on Dynamics of a Rigid Body.

Example 7.37. *Find the M.I. of the area bounded by the curve $r^2 = a^2 \cos 2\theta$ about its axis.*

Solution. Given curve is symmetrical about the pole and for half of the loop in the first quadrant θ varies from 0 to $\pi/4$ (Fig. 7.40).

Elementary area at $P(r, \theta) = r\, d\theta dr$.

If ρ be the surface density, then elementary mass

$$= \rho r\, d\theta dr \qquad ...(i)$$

$$\therefore \text{ its total mass } M = 4 \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \rho r\, dr d\theta$$

$$= 2\rho a^2 \int_0^{\pi/4} \cos 2\theta\, d\theta = \rho a^2 \qquad ...(ii)$$

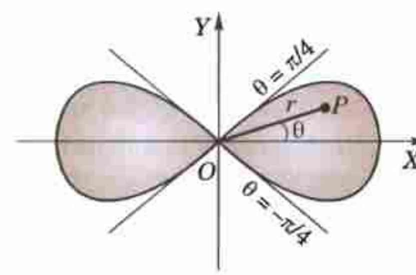

Fig. 7.40

Now M.I. of the elementary mass (i) about the x-axis.

$$= \rho r\, d\theta dr \cdot y^2 = \rho r d\theta\, dr\, (r \sin\theta)^2 = \rho r^3 \sin^2\theta\, dr d\theta$$

Hence the M.I. of the whole area

$$= 4 \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3 \sin^2\theta\, dr d\theta = 4\rho \int_0^{\pi/4} \sin^2\theta \cdot \left| r^4/4 \right|_0^{a\sqrt{(\cos 2\theta)}} d\theta$$

$$= \rho a^2 \int_0^{\pi/4} \cos^2 2\theta \cdot \sin^2\theta\, d\theta = \rho a^4 \int_0^{\pi/2} \cos^2\phi \cdot \sin^2\frac{\phi}{2} \cdot \frac{d\phi}{2} \qquad [\text{Put } 2\theta = \phi,\ d\theta = d\phi/2]$$

$$= \frac{\rho a^4}{4} \int_0^{\pi/2} (\cos^2\phi - \cos^3\phi)\, d\phi = \frac{\rho a^4}{48}(3\pi - 8) = \frac{Ma^2}{48}(3\pi - 8). \qquad [\text{By } (ii)]$$

*Named after a Swiss geometrer *Jacob Steiner* (1796–1863) who was a professor at Berlin University.

Example 7.38. *Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being 5 metres and 4 metres.*

Solution. Let ρ be the density of the given hollow sphere. Then the M.I. about the diameter, *i.e.*, x-axis is

$I_x = \iiint_V \rho(y^2 + z^2)\, dx\, dy\, dz$

Changing to polar spherical coordinates, we get

$$I_x = \int_0^{2\pi} \int_0^{\pi} \int_4^5 \rho\{(r \sin\theta \sin\phi)^2 + (r\cos\theta)^2\}\; r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \rho\left\{\int_0^{2\pi} \sin^2\phi\, d\phi \cdot \int_0^{\pi} \sin^3\theta\, d\theta \left|\frac{r^5}{5}\right|_4^5 + \int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta \cdot \left|\frac{r^5}{5}\right|_4^5\right\}$$

$$= \frac{8\pi\rho}{15}(5^5 - 4^5) = 1120.5 \text{ m.}$$

Example 7.39. *A solid body of density ρ is in the shape of the solid formed by revolution of the centroid $r = a(1 + \cos\theta)$ about the initial line. Show that its moment of inertia about a straight line through the pole perpendicular to the initial line is $\frac{352}{105}\pi\rho a^5$.* *(U.P.T.U., 2001)*

Solution. An elementary area $rd\theta dr$, when revolved about OX generates a circular ring of radius $LP = r\sin\theta$ (Fig. 7.41).

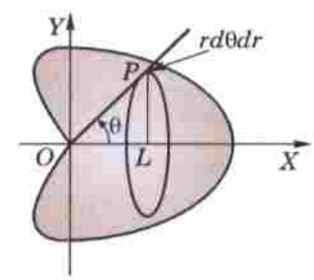

Fig. 7.41

M.I. of this ring about a diameter parallel to OY

$$= (2\pi r \sin\theta)(rd\theta dr)\rho \cdot \frac{(r\sin\theta)^2}{2}.$$

[∵ M.I. of a ring about a diameter $= Ma^2/2$.]

Now using Steiner's theorem, we have M.I. of the ring about OY = M.I. of the ring about a diameter LP parallel to OY + Mass of the ring $(OL)^2$ $(r\cos\theta)^2$

$$= 2\pi\rho r^4 \sin^3\theta\, d\theta dr + 2\pi r \sin\theta\,(rd\theta dr)\,(r\cos\theta)^2$$

Hence M.I. of the solid generated by revolution about OY

$$= \pi\rho \int_0^{\pi} \int_0^{r=a(1+\cos\theta)} (r^4 \sin^3\theta + 2r^4 \sin\theta\cos^2\theta)\, d\theta dr$$

$$= \pi\rho \int_0^{\pi} (\sin^3\theta + 2\sin\theta\cos^2\theta)\, d\theta \int_0^{a(1+\cos\theta)} r^4\, dr$$

$$= \frac{\pi\rho a^5}{5} \int_0^{\pi} \sin\theta\,(1 + \cos^2\theta)(1+\cos\theta)^5\, d\theta \qquad [\text{Put } \theta = 2\phi]$$

$$= \frac{\pi\rho a^5}{5} \int_0^{\pi/2} \sin 2\phi\,(1 + \cos^2 2\phi)(1+\cos 2\phi)^5\, 2d\phi$$

$$= \frac{\pi\rho a^5}{5} \int_0^{\pi/2} 2\sin\phi\cos\phi\,\{1 + (2\cos^2\phi - 1)^2\}\,(2\cos^2\phi)^5\, 2d\phi$$

$$= \frac{256\,\pi\rho a^5}{5} \int_0^{\pi/2} (\cos^{11}\phi - 2\cos^{13}\phi + 2\cos^{15}\phi)\sin\phi\, d\phi$$

$$= \frac{256\,\pi\rho a^5}{5}\left| -\frac{\cos^{12}\phi}{12} + \frac{2\cos^{14}\phi}{14} - \frac{2\cos^{16}\phi}{16}\right|_0^{\pi/2} = \frac{352\,\pi\rho a^5}{105}.$$

Example 7.40. *A hemisphere of radius R has a cylindrical hole of radius a drilled through it, the axis of the hole being along the radius normal to the plane face of the hemisphere. Find its radius of gyration about a diameter of this face.*

Solution. M.I. of the given solid about x-axis

$$= \iiint \rho(y^2 + z^2)\, dx\, dy\, dz$$

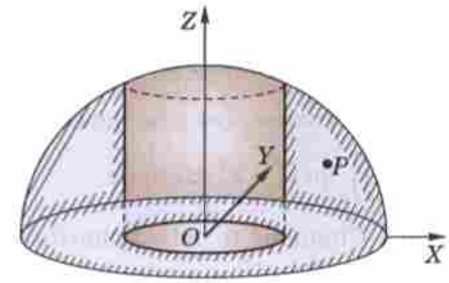

Fig. 7.42

The limits of integration for z are from 0 to $z = \sqrt{(R^2 - x^2 - y^2)}$ found from the equation of the sphere $x^2 + y^2 + z^2 = R^2$. The limits for x and y are to be such as to cover the shaded area A in the xy-plane between the concentric circles of radii a and R (Fig. 7.42).

Thus the required M.I. about x-axis

$$= \rho \iint_A \int_0^{\sqrt{(R^2 - x^2 - y^2)}} (y^2 + z^2)\, dz\, dx\, dy$$

$$= \rho \iint_A \left| y^2 z + z^3/3 \right|_0^{\sqrt{(R^2 - x^2 - y^2)}} dx\, dy = \rho \iint_A \left[y^2 (R^2 - x^2 - y^2)^{1/2} + \frac{1}{3}(R^2 - x^2 - y^2)^{3/2} \right] dx\, dy.$$

Now changing to polar coordinates, we have $x = r\cos\theta$, $y = r\sin\theta$ and $dxdy = rd\theta dr$.

Also to cover the area A, r varies from a to R and θ varies from 0 to 2π.

Hence the required M.I. about x-axis

$$= \rho \int_a^R \int_0^{2\pi} \left[r^2 \sin^2\theta \cdot (R^2 - r^2)^{1/2} + \frac{1}{3}(R^2 - r^2)^{3/2} \right] r d\theta dr$$

$$= \rho \int_a^R \int_0^{2\pi} \left[\frac{1}{2} r^2 (1 - \cos 2\theta) + \frac{1}{3}(R^2 - r^2) \right] d\theta \cdot r(R^2 - r^2)^{1/2}\, dr$$

$$= \rho \int_a^R \left| \frac{r^2}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + \frac{1}{3}(R^2 - r^2)\,\theta \right|_0^{2\pi} \cdot r(R^2 - r^2)^{1/2}\, dr$$

$$= \rho \int_a^R 2\pi \left(\frac{r^2}{2} + \frac{R^2 - r^2}{3} \right) . r(R^2 - r^2)^{1/2}\, dr$$

$$= \frac{\pi\rho}{3} \int_a^R (2R^2 + r^2)(R^2 - r^2)^{1/2} \cdot r dr \qquad \text{[Put } r^2 = t \text{ and } rdr = dt/2]$$

$$= \frac{\pi\rho}{6} \int_{a^2}^{R^2} (2R^2 + t)(R^2 - t)^{1/2}\, dt \qquad \text{[Integrate by parts]}$$

$$= \frac{\pi\rho}{9} \left[(2R^2 + a^2)(R^2 - a^2)^{3/2} + \frac{2}{5}(R^2 - a^2)^{5/2} \right] = \frac{2\pi\rho}{3}(R^2 - a^2)^{3/2} \times \frac{1}{10}(4R^2 + a^2)$$

$$\left[\because \text{ Mass} = \rho \int_0^{2\pi} \int_a^R \int_0^{\sqrt{(R^2 - r^2)}} dz \cdot rdr \cdot d\theta = \frac{2\pi\rho}{3}(R^2 - a^2)^{3/2} \right]$$

Hence, the radius of gyration $= [(4R^2 + a^2)/10]^{1/2}$.

7.13 (1) PRODUCT OF INERTIA

If a particle of mass m of a body be at distances x and y from two given perpendicular lines, then $\sum mxy$ is called the *product of inertia* of the body about the given lines.

Consider an elementary mass $\delta x \delta y \delta z$ enclosing the point $P(x, y, z)$ of solid of volume V. Then the product of inertia (P.I.) of this element about the axes of x and $y = \rho \delta x \delta y \delta z\, xy$.

$\therefore$ P.I. of the solid about x and y-axes, *i.e.*, $P_{xy} = \iiint_V \rho\, xy\, dxdydz$

Similarly, $P_{yz} = \iiint_V \rho\, yz\, dxdydz$ and $P_{zx} = \iiint_V \rho\, zx\, dxdydz$.

In particular, for a plane lamina of surface density ρ and covering a region A in the xy-plane,

$$P_{xy} = \iint_A \rho\, xy\, dxdy \text{ whereas } P_{yz} = P_{zx} = 0. \qquad [\because \; z = 0]$$

(2) Principal axes. *The principal axes* of a lamina at a given point are that pair of axes in its plane through the given point, about which the product of inertia of the lamina vanishes.

Let $P(x, y)$ be a point of the plane area A referred to rectangular axes OX, OY. Let (x', y') be the coordinates of P referred to another pair of rectangular axes OX', OY' in the same plane and inclined at an angle θ to the first pair (Fig. 7.43).

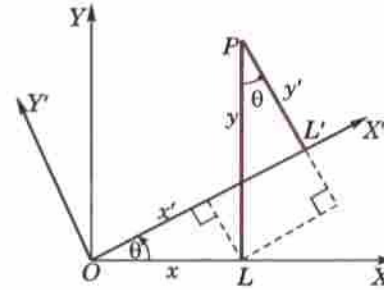

Fig. 7.43

Then $$x' = x\cos\theta + y\sin\theta$$
$$y' = y\cos\theta - x\sin\theta$$

If I_x, I_y be the moments of inetria of the area A about OX and OY and P_{xy} be its product of inertia about these axes, then

$$I_x = \iint_A \rho y^2\, dA,\; I_y = \iint_A \rho x^2\, dA,\; P_{xy} = \iint \rho\, xy\, dA.$$

$\therefore$ the product of inertia P'_{xy} about OX' and OY' is given by

$$P'_{xy} = \int_A \rho\, x'y'\, dA = \iint_A \rho(x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\, dA$$

$$= \sin\theta\cos\theta \iint_A \rho(y^2 - x^2)\, dA + (\cos^2\theta - \sin^2\theta)\iint_A \rho xy\, dA$$

$$= 1/2 \sin 2\theta \cdot (I_x - I_y) + \cos 2\theta\, P_{xy}.$$

Now OX', OY' will be the principal axes of the area A if P'_{xy} vanishes.

i.e., If $\quad 1/2 \sin 2\theta\,(I_x - I_y) + \cos 2\theta\, P_{xy} = 0$

i.e., If $\quad \mathbf{\tan 2\theta = 2P_{xy}/(I_y - I_x)}$.

This gives two values of θ differing by $\pi/2$.

Example 7.41. *Show that the principal axes at the node of a half-loop of the lemniscate $r^2 = a^2 \cos 2\theta$ are inclined to the initial line at angles*

$$\frac{1}{2}\tan^{-1}\frac{1}{2} \text{ and } \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\frac{1}{2}.$$

Solution. Let the element of mass at $P(r, \theta)$ be $\rho r d\theta dr$.

Then $$I_x = \rho\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} r^2\sin^2\theta \cdot r d\theta dr$$ [See Fig. 7.40]

$$= \frac{\rho a^4}{4}\int_0^{\pi/4}\sin^2\theta\cos^2 2\theta\, d\theta = \frac{\rho a^4}{16}\left(\frac{\pi}{4} - \frac{2}{3}\right)$$

$$I_y = \rho\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} r^2\cos^2\theta \cdot r\, d\theta dr = \frac{\rho a^4}{16}\left(\frac{\pi}{4} + \frac{2}{3}\right)$$

and $$P_{xy} = \rho\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} r^2 \sin\theta\cos\theta \cdot r d\theta dr = \frac{\rho a^4}{48}.$$

Hence the required direction of the principal axes at O are given by

$$\tan 2\theta = \frac{2P_{xy}}{I_y - I_x} = \frac{\rho a^4/24}{(\rho a^4/16)\times(4/3)} = \frac{1}{2}$$

or by $$\theta = \frac{1}{2}\tan^{-1}\frac{1}{2} \text{ and } \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\frac{1}{2}.$$

PROBLEMS 7.7

1. Using double integrals, find the moment of inertia about the x-axis of the area enclosed by the lines $x = 0, y = 0, (x/a) + (y/b) = 1$. (*P.T.U., 2005*)
2. Find the moment of inertia of a circular plate about a tangent.
3. Find the moment of inertia of the area $y = \sin x$ from $x = 0$ to $x = 2\pi$ about OX.

4. Find the moment of inertia of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$ of mass M about the x-axis, if the density at a point is proportional to xy.
5. Find the moment of inertia about the initial line of the cardioid $r = a(1 + \cos\theta)$.
6. Find the moment of inertia of a uniform spherical ball of mass M and radius R about a diameter.
7. Find the moment of inertia of a solid right circular cylinder about (*i*) its axis (*P.T.U., 2006*)
   (*ii*) a diameter of the base.
8. Find the M.I. of a solid right circular cone having base-radius r and height h, about (*i*) its axis, (*ii*) an axis through the vertex and perpendicular to its axis, (*iii*) a diameter of its base.
9. Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being 51 metres and 49 metres.
10. Find the moment of inertia about z-axis of a homogeneous tetrahedron bounded by the planes $x = 0, y = 0, z = x + y$ and $z = 1$.
11. Find the moment of inertia of an octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, about the x-axis.
12. Find the product of inertia of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$, about the coordinate axes.
13. Show that the principal axes at the origin of the triangle enclosed by $x = 0, y = 0, (x/a) + (y/b) = 1$ are inclined to the x-axis at angles α and $\alpha + \pi/2$, where $\alpha = \frac{1}{2}\tan^{-1}[ab/(a^2 - b^2)]$ (*U.P.T.U., 2002*)
14. The lengths AB and AD of the sides of a rectangle $ABCD$ are $2a$ and $2b$. Show that the inclination to AB of one of the principal axes at A is $\frac{1}{2}\tan^{-1}\left\{\frac{3ab}{2(a^2 - b^2)}\right\}$.

7.14 BETA FUNCTION

The beta function is defined as

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \begin{cases} m > 0 \\ n > 0 \end{cases} \qquad ...(1)$$

Putting $x = 1 - y$ in (1), we get $\beta(m, n) = -\int_1^0 (1-y)^{m-1}\, y^{n-1}\,dy$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = \beta(n, m)$$

Thus $\beta(m, n) = \beta(n, m)$ ...(2)

Putting $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$, (1) becomes

$$\beta(m, n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \qquad ...(3)$$

which is another form of $\beta(m, n)$.

This function is also *Euler's integral of the first kind**.

7.15 (1) GAMMA FUNCTION

The gamma function is defined as

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx \ (n > 0) \qquad ...(i)$$

This integral is also known as *Euler's integral of the second kinds.* It defines a function of n for positive values of n.

*After an enormously creative Swiss mathematician *Leonhard Euler (1707–1783).* He studied under *John Bernoulli* and became a professor of mathematics in St. Petersburg, Russia. Even after becoming totally blind in 1771, he contributed to almost all branches of mathematics.

In particular, $\Gamma(1) = \int_0^\infty e^{-x}\, dx = \left| -e^{-x} \right|_0^\infty = 1.$ ...(*ii*)

(2) Reduction formula for Γ(n).

Since $\Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx$ [Integrating by parts] $= \left| -x^n e^{-x} \right|_0^\infty + n \int_0^\infty e^{-x} x^{n-1} dx$

$\therefore$ $\Gamma(n+1) = n\Gamma(n)$ ...(*iii*)

which is the reduction formula for $\Gamma(n)$. From this formula, it is clear that if $\Gamma(n)$ is known throughout a unit interval say : $1 < n \le 2$, then the values of $\Gamma(n)$ throughout the next unit interval $2 < n \le 3$ are found, from which the values of $\Gamma(n)$ for $3 < n \le 4$ are determined and so on. In this way, *the values of $\Gamma(n)$ for all positive values of $n > 1$ may be found by successive application of (iii).*

Also using (*iii*) in the form

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \quad \text{...(iv)}$$

We can define $\Gamma(n)$ for values of n for which the definition (1) fails. It gives the value of $\Gamma(n)$ for $0 < n \le 1$ in terms of the values of $\Gamma(n)$ for $1 < n \le 2$. Thus we can define $\Gamma(n)$ for all $n < 0$ provided its value for $1 < n \le 2$ is known. Also if $-1 < n < 0$, (4) gives $\Gamma(n)$ in terms of its values for $0 < n < 1$. Then we may find, $\Gamma(n)$ for $-2 < n < -1$ and so on.

Thus (i) and (iv) together give a complete definition of Γ(n) for all values of n except when n is zero or a negative integer and its graph is as shown in Fig.7.44. The values of *Γ(n)* for *1 < n ≤ 2* are given in (Table I-Appendix 2) from which the values of *Γ(n)* for values of n outside the interval $1 < n \le 2$ ($n \ne 0, -1, -2, -3$ .....) may be found.

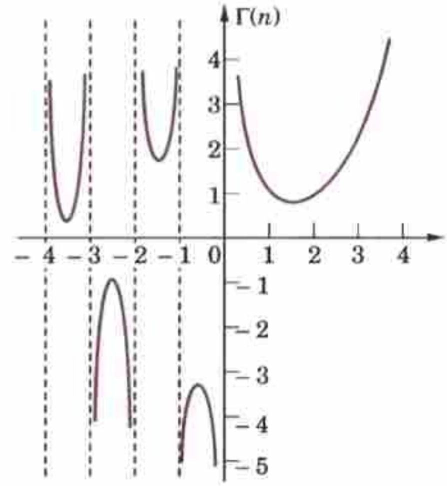

Fig. 7.44

(3) Value of Γ(n) in terms of factorial.

Using $\Gamma(n+1) = n\Gamma(n)$ successively, we get

$\Gamma(2) = 1 \times \Gamma(1) = 1\,!$

$\Gamma(3) = 2 \times \Gamma(2) = 2 \times 1 = 2\,!$

$\Gamma(4) = 3 \times \Gamma(3) = 3 \times 2\,! = 3\,!$

....

In general $\Gamma(n+1) = n\,!$ provided n is a positive integer ...(*v*)

Taking $n = 0$, it defines $0\,! = \Gamma(1) = 1$.

(4) Value of $\Gamma\left(\frac{1}{2}\right)$**.** We have

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-x} x^{-1/2}\, dx \qquad \text{[Put } x = y^2 \text{ so that } dx = 2y\, dy]$$

$$= 2\int_0^\infty e^{-y^2}\, dy \text{ which is also } = 2\int_0^\infty e^{-x^2}\, dx$$

$$\therefore \quad \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = 4\int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r\, dr\, d\theta$$

$$= 4 \cdot \frac{\pi}{2} \int_0^\infty e^{-r^2}\, r dr = 2\pi\left[\left(-\frac{1}{2}\right)e^{-r^2}\right]_0^\infty = \pi$$

whence $\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} = 1.772$ ...(*vi*) (*V.T.U., 2006*)

7.16 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

We have $\Gamma(m) = \int_0^\infty e^{-t}\, t^{m-1}$ [Put $t = x^2$ so that $dt = 2x\, dx$

$$= 2\int_0^\infty e^{-x^2} x^{2m-1}\,dx \qquad ...(2)$$

Similarly, $\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\,dy$

$\therefore$ $\Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1}\,dx \int_0^\infty e^{-y^2} y^{2n-1}\,dy$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\,dx\,dy \qquad ...(3)$$ [$\because$ the limits of integration are constant.]

Now change to polar coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r d\theta dr$. To cover the region in (3) which is the entire first quadrant, r varies from 0 to ∞ and θ from 0 to $\pi/2$. Thus (3) becomes

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta dr$$

$$= \left[2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta\right] \times \left[2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\,dr\right] \qquad ...(4)$$

But by (2), $2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\,dr = \Gamma(m+n)$

and by (3) of § 7.14, $2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta = \beta(m, n)$.

Thus (4) gives $\Gamma(m)\Gamma(n) = \beta(m, n)\,\Gamma(m+n)$ *(U.T.U., 2010 ; Bhopal, 2009 ; V.T.U., 2008 S)*
whence follows (1) which is extremely useful for evaluating definite integrals in terms of gamma functions.

Cor. *Rule to evaluate* $\int_0^{\pi/2} \sin^p x \cos^q x\, dx$.

$$\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \qquad \text{[By (3) of § 7.14]}$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \qquad ...(5)$$

In particular, when $q = 0$, and $p = n$, we have

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}$$

Similarly, $$\int_0^{\pi/2} \cos^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \qquad ...(6)$$

Example 7.42. *Show that*

(a) $\Gamma(n) = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy \ (n > 0)$. *(J.N.T.U., 2003 ; Madras, 2003 S)*

(b) $\beta(p, q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\, dy$ *(V.T.U., 2003 ; Gauhati, 1999)*

$= \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}}\, dx$. *(V.T.U., 2008 ; Osmania, 2003 ; Rohtak, 2003)*

Solution. (*a*) $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \ (n > 0)$

$$= \int_1^0 \left(\log \frac{1}{y}\right)^{n-1} y\left(-\frac{1}{y}dy\right) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy.$$

| Put $y = e^{-x}$ *i.e.*, $x = \log(1/y)$ so that $dx = -(1/y)\, dy$

(*b*) $$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx$$

$$= \int_\infty^0 \frac{1}{(1+y)^{p-1}} \left(\frac{y}{1+y}\right)^{q-1} \frac{-1}{(1+y)^2}\, dy$$

| Put $x = \dfrac{1}{1+y}$ *i.e.*, $y = \dfrac{1}{x} - 1$ so that $dx = \dfrac{-1}{(1+y)^2}\, dy$

$$= \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\, dy = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}\, dy + \int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\, dy$$

Now substituting $y = 1/z$ in the second integral, we get

$$\int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\, dy = \int_1^0 \frac{1}{z^{q-1}} \cdot \frac{1}{(1+1/z)^{p+q}} \left(-\frac{1}{z^2}\right) dz = \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\, dz.$$

Hence, $$\beta(p, q) = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}\, dy + \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\, dz = \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}}\, dx.$$

Example 7.43. *Express the following integrals in terms of gamma functions :*

(*a*) $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$ (*b*) $\displaystyle\int_0^{\pi/2} \sqrt{(\tan\theta)}\, d\theta.$ (*Madras, 2006*)

(*c*) $\displaystyle\int_0^\infty \frac{x^c}{c^x}\, dx$ (*U.P.T.U., 2006*) (*d*) $\displaystyle\int_0^\infty a^{-bx^2}\, dx.$

(*e*) $\displaystyle\int_0^1 x^5 [\log(1/x)]^3\, dx$ (*Madras, 2000*)

Solution. (*a*) $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$

| Put $x^2 = \sin\theta$, *i.e.*, $x = \sin^{1/2}\theta$ so that $dx = 1/2 \sin^{-1/2}\theta \cos\theta\, d\theta$

$$= \int_0^{\pi/2} \frac{1}{2} \cdot \frac{\sin^{-1/2}\theta \cdot \cos\theta\, d\theta}{\sqrt{(1-\sin^2\theta)}} = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta = \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\dfrac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\dfrac{-\frac{1}{2}+2}{2}\right)} = \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

(*b*) $$\int_0^{\pi/2} \sqrt{(\tan\theta)}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta$$

$$= \frac{\Gamma\left(\dfrac{\frac{1}{2}+1}{2}\right)\Gamma\left(\dfrac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\dfrac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

(*c*) $$\int_0^\infty \frac{x^c}{c^x}\, dx = \int_0^\infty \frac{x^c}{e^{x\log c}}\, dx$$ $[\because\ c^x = e^{\log c^x} = e^{x\log c}]$

$$= \int_0^\infty e^{-x\log c}\, x^c\, dx$$ [Put $x \log c = t$ so that $dx = dt/\log c$]

$$= \int_0^\infty e^{-t}\left(\frac{t}{\log c}\right)^c \frac{dt}{\log c} = \frac{1}{(\log c)^{c+1}} \int_0^\infty t^c\, e^{-t}\, dt = \Gamma\,(c+1)/(\log c)^{c+1}$$

(d) $\int_0^\infty a^{-bx^2}\, dx = \int_0^\infty e^{-bx^2 \log a}\, dx$ $\left[\begin{array}{l}\text{Put } (b \log a)\, x^2 = t \\ \text{so that } dx = dt/2\sqrt{(b \log a)}\end{array}\right.$

$$= \frac{1}{2\sqrt{(b \log a)}} \int_0^\infty e^{-t}\, t^{-1/2}\, dt = \frac{\Gamma\left(\frac{1}{2}\right)}{2\sqrt{(b \log a)}} = \frac{\sqrt{\pi}}{2\sqrt{(b \log a)}}$$

(e) $\int_0^1 x^4\, [\log(1/x)]^3\, dx = \frac{1}{625}\int_0^\infty e^{-t}\, .\, t^3\, dt$ $\left[\begin{array}{l}\text{Put } x = e^{-t/5} \text{ so that } \log(1/x) = t/5 \\ dx = -\frac{1}{5} e^{-t/5}\, dt\end{array}\right]$

$$= \frac{\Gamma(4)}{625} = \frac{6}{625}.$$

Example 7.44. *Evaluate* $\int_0^\infty e^{-ax}\, x^{m-1}\, \sin bx\, dx$ *in terms of Gamma function.* *(U.P.T.U., 2003)*

Solution. We have $\Gamma(m) = \int_0^\infty e^{-x}\, x^{m-1}\, dx$ [Put $x = ay$, $dx = a dy$]

$$= \int_0^\infty e^{-ay}\, a^m\, y^{m-1}\, dy \quad \text{or} \quad \int_0^\infty e^{-ay}\, y^{m-1}\, dy = \Gamma\,(m)/a^m. \qquad ...(i)$$

Then $I = \int_0^\infty e^{-ax}\, x^{m-1} \sin bx\, dx = \int_0^\infty e^{-ax}\, x^{m-1}$ (Imaginary part of e^{ibx}) dx

$= \text{I.P. of } \int_0^\infty e^{-(a-ib)x}\, x^{m-1}\, dx$

$= \text{I.P. of } \{\Gamma(m)/(a-ib)^m$ [By (i)]

$= \text{I.P. of } \{\Gamma(m)/(r^m (\cos\theta - i \sin\theta)^m]$ where $a = r\cos\theta$, $b = r \sin\theta$

$= \text{I.P. of } \Gamma(m)/(r^m (\cos m\theta - i \sin m\theta)]$ (Using Demoivere's theorem §19.5)

$$= \text{I.P. of } \left\{\frac{\Gamma(m)\,.\,(\cos m\theta + i \sin m\theta)}{r^m\, (\cos m\theta + i \sin m\theta)\, (\cos m\theta - i \sin m\theta)}\right\}$$

$= \frac{\Gamma(m)}{r^m} \sin m\theta$ where $r\ \sqrt{(a^2+b^2)}$, $\theta = \tan^{-1} b/a$.

Example 7.45. *Prove that* $\int_0^1 \frac{x^2 dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\pi}{4\sqrt{2}}$

Solution. $\int_0^1 \frac{x^2 dx}{\sqrt{(1-x^4)}} \times \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{2\sqrt{(\sin\theta)}}\, d\theta$ $\left[\text{Putting } x^2 = \sin\theta,\ dx = \frac{\cos\theta\, d\theta}{2\sqrt{(\sin\theta)}}\right.$

$$= \frac{1}{2}\int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta = \frac{1}{4}\beta\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{1}{4}\frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(5/4)} = \frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(1/4)}$$

$\int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{2\sqrt{(\tan\theta)}\sec\theta}$ $\left[\text{Putting } x^2 = \tan\theta,\ dx = \frac{\sec^2\theta\, d\theta}{2\sqrt{(\tan\theta)}}\right.$

$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{d\theta}{\sqrt{(\sin 2\theta)}} = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2}\phi\, d\phi$ $\left[\text{Putting } 2\theta = \phi,\ d\theta = \frac{1}{2} d\phi\right.$

$$= \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4\sqrt{2}}\frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

$$\therefore \qquad \int_0^1 \frac{x^2\,dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \frac{\pi}{4\sqrt{2}}.$$

Example 7.46. *Prove that (i)* $\beta(m, 1/2) = 2^{2m-1}\beta(m, m)$ *(V.T.U., 2004)*

$$(ii)\ \Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m) \qquad \textit{(Duplication Formula)}$$

(V.T.U., 2010 ; Kerala, M.E., 2005 ; Madras, 2003 S)

Solution. (*i*) We know that $\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$ ...(1)

Putting $n = \frac{1}{2}$, we have $\beta\left(m, \frac{1}{2}\right) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\,d\theta$ ...(2)

Again putting $n = m$ in (*i*), we get $\beta(m, m) = 2\int_0^{\pi/2}(\sin\theta\cos\theta)^{2m-1}\,d\theta$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2}\sin^{2m-1}2\theta\,d\theta$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi}\sin^{2m-1}\phi\,d\phi, \text{ putting } 2\theta = \phi$$

$$= \frac{1}{2^{2m-1}}\cdot 2\int_0^{\pi/2}\sin^{2m-1}\phi\,d\phi$$

or $$2^{2m-1}\beta(m, m) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\,d\theta = \beta\left(m, \frac{1}{2}\right)$$ [by (2)

(*ii*) Rewriting the above result in terms of Γ functions, we get

$$2^{2m-1}\frac{\Gamma(m)\Gamma(m)}{\Gamma(m+m)} = \frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

or $$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}.$$

Example 7.47. *Prove that*

(*a*) $\iint_D x^{l-1}y^{m-1}\,dx\,dy = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}h^{l+m}$ *where D is the domain* $x \ge 0$, $y \ge 0$ *and* $x + y \le h$.

(U.P.T.U., 2005)

(*b*) $\iiint_V x^{l-1}y^{n-1}z^{n-1}\,dxdydz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$

where V is the region $x \ge 0$, $y \ge 0$, $z \ge 0$ *and* $x + y + z \le 1$. *This important result is known as* **Dirichlet's integral*.**

Solution. (*a*) Putting $x/h = X$ and $y/h = Y$, we see that the given integral

$$= \iint_{D'}(hX)^{l-1}(hY)^{m-1}h^2\ dXdY \text{ where } D' \text{ is the domain } X \ge 0, Y \ge 0 \text{ and } X + Y \le 1.$$

$$= h^{l+m}\int_0^1\int_0^{1-X}X^{l-1}Y^{m-1}\,dY\,dX = h^{l+m}\int_0^1 X^{l-1}\left|\frac{Y^m}{m}\right|_0^{1-X}dX$$

$$= \frac{h^{l+m}}{m}\int_0^1 X^{l-1}(1-X)^m\,dX = \frac{h^{l+m}}{m}\beta(l, m+1) = \frac{h^{l+m}}{m}\cdot\frac{\Gamma(l)\,\Gamma(m+1)}{\Gamma(l+m+1)}$$

*Named after a German mathematician *Peter Gustav Lejeune Dirichlet* (1805–1859) who studied under Cauchy and succeeded Gauss at Gottingen. He is known for his contributions to Fourier series and number theory.

$$= h^{l+m} \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)} \qquad \ldots(i)\ [\because\ \Gamma(m+1)/m = \Gamma(m)]$$

(*b*) Taking $y + z \le 1 - x (= h$: say), the triple integral

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{l-1}\, y^{m-1}\, z^{n-1}\, dz dy dx$$

$$= \int_0^1 x^{l-1} \left[\int_0^h \int_0^{h-y} y^{m-1}\, z^{n-1}\, dz dy \right] dx = \int_0^1 x^{l-1} \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} h^{m+n}\, dx \qquad \ldots \text{[By } (i)]$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \int_0^1 x^{l-1} (1-x)^{m+n}\, dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)}\, \beta(l, m+n+1)$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(l)\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)} = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}.$$

Example 7.48. *Evaluate the integral* $\iiint x^{l-1} y^{m-1} z^{n-1}\, dxdydz$ *where x, y, z are all positive with condition,* $(x/a)^p + (y/b)^q + (z/c)^r \le 1$. *(U.P.T.U., 2005 S)*

Solution. Put $(x/a)^p = u$, *i.e.*, $x = au^{1/p}$ so that $dx = \dfrac{a}{p} u^{1/p-1}\, du$

$(y/b)^q = v$, *i.e.*, $y = bv^{1/q}$ so that $dy = \dfrac{b}{q} v^{1/q-1}\, dv$

and $(z/c)^r = w$, *i.e.*, $z = cw^{1/r}$ so that $dz = \dfrac{c}{r} w^{1/r-1}\, dw$

Then $\iiint x^{l-1}\, y^{m-1}\, z^{n-1}\, dxdydz$

$$= \iiint (au^{1/p})^{l-1} (bv^{1/q})^{m-1} (cw^{1/r})^{n-1} \left(\frac{a}{p}\right) u^{1/p-1} \left(\frac{b}{q}\right) v^{1/q-1} \left(\frac{c}{r}\right) w^{1/r-1}\, dudvdw$$

$$= \frac{a^l\, b^m\, c^n}{pqr} \iiint u^{l/p-1}\, v^{m/q-1}\, w^{n/r-1}\, du\, dv\, dw \text{ where } u + v + w \le 1.$$

$$= \frac{a^l\, b^m\, c^n}{pqr} \frac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma(l/p + m/q + n/r + 1)} \qquad \text{[By Dirichlet's integral]}$$

Example 7.49. *The plane* $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ *meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron OABC. Also find its mass if the density at any point is kxyz.* *(U.P.T.U., 2004)*

Solution. Put $x/a = u$, $y/b = v$, $z/c = w$ then the tetrahedron *OABC* has $u \ge 0$, $v \ge 0$, $w \ge 0$ and $u + v + w \le 1$.

$\therefore$ volume of this tetrahedron $= \iiint_D dx\, dy\, dz$

$$= \iiint_{D'} abc\, du\, dv\, dw \qquad \left[\begin{array}{l} \because dx = adu,\ dy = bdv,\ dz = cdw \\ \text{for } D' = u \ge 0, v \ge 0, w \ge 0\ \&\ u + v + w \le 1. \end{array}\right.$$

$$= abc \iiint_{D'} u^{1-1}\, v^{1-1}\, w^{1-1}\, du\, dv\, dw$$

$$= abc \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{6} \qquad \text{[By Dirichlet's integral]}$$

$$\text{Mass} = \iiint kxyz dx\, dy\, dz = \iiint k\,(au)\,(bv)\,(cw)\, abc\, du\, dv\, dw$$

$$= ka^2b^2c^2 \iiint u^{2-1}\, v^{2-1}\, w^{2-1}\, du\, dv\, dw$$

$$= ka^2b^2c^2 \frac{\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(2+2+2+1)} ka^2b^2c^2 \cdot \frac{1}{6!} = \frac{k}{720}\, a^2b^2c^2.$$

PROBLEMS 7.8

1. Compute :
   (*i*) $\Gamma(3.5)$ (*Assam, 1998*) (*ii*) $\Gamma(4.5)$
   (*iii*) $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ (*S.V.T.U., 2009*) (*iv*) $\beta(2.5, 1.5)$ (*v*) $\beta\left(\frac{9}{2}, \frac{7}{2}\right)$. (*Andhra, 2000*)
2. Express the following integrals in terms of gamma functions :
   (*i*) $\int_0^\infty e^{-x^2} dx$ (*ii*) $\int_0^\infty x^{p-1} e^{-kx} dx (k > 0)$ (*Delhi, 2002 ; V.T.U., 2000*)
   (*iii*) $\int_0^\infty \sqrt{x}\, e^{-x^3} dx$ (*J.N.T.U., 2003*) (*iv*) $\int_0^\infty \frac{dx}{x^{p+1}.(x-1)^q} (-p < q < 1)$
3. Show that :
   (*i*) $\int_0^\infty \frac{x^4}{4^x} dx = \frac{\Gamma(5)}{(\log 4)^5}$ (*Marathwada, 2008*)
   (*ii*) $\int_0^{\pi/2} \sqrt{(\cot\theta)}\, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ (*Osmania, 2003 S ; V.T.U., 2001*)
   (*iii*) $\int_0^{\pi/2} [\sqrt{(\tan\theta)} + \sqrt{(\sec\theta)}]\, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left\{\Gamma\left(\frac{3}{4}\right) + \sqrt{\pi}/\Gamma\left(\frac{3}{4}\right)\right\}$ (*J.N.T.U., 2000*)
   (*iv*) $\int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin\theta)}} = \pi$. (*V.T.U., 2007*)
4. Given $\int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}$, show that $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$. (*S.V.T.U., 2008*)

   Hence evaluate $\int_0^\infty \frac{dy}{1+y^4}$. (*V.T.U., 2006 ; J.N.T.U., 2005*)
5. Prove that :
   (*i*) $\int_0^1 \frac{x\,dx}{\sqrt{(1-x^5)}} = \frac{1}{5}\beta\left(\frac{2}{5}, \frac{1}{2}\right)$ (*Raipur, 2006*) (*ii*) $\int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$. (*V.T.U., 2003*)
   (*iii*) $\int_0^1 x^3 (1-\sqrt{x})^5 dx = 2\beta(8, 6)$. (*Marathwada, 2008 ; J.N.T.U., 2006*)
6. Show that (*i*) $\int_a^b (x-a)^{m-1}(b-x)^{n-1} dx = (b-a)^{m+n-1}\beta(m, n)$ (*P.T.U., 2010 ; Mumbai, 2005*)
   (*ii*) $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \frac{1}{a^n . b^m}\beta(m, n)$ (*Nagpur, 2009*) (*iii*) $\int_0^\infty \frac{x^{10} - x^{18}}{(1+x)^{30}} dx = 0$ (*Mumbai, 2005*)
   (*iv*) $\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2} dx = \frac{1}{2^{9/2}}\beta\left(\frac{7}{4}, \frac{1}{4}\right)$ (*Mumbai, 2007*)
7. Prove that $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$, where n is a positive integer and $m > -1$. (*S.V.T.U., 2006*)

   Hence evaluate $\int_0^1 x (\log x)^3 dx$. (*Nagpur, 2009*)
8. Show that $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p}$, where $p > 0, q > 0$. (*Rohtak, 2006 S*)
9. Express $\int_0^1 x^m (1-x^n)^p dx$ in terms of gamma functions (*Marathwada, 2008*)

   Hence evaluate : (*i*) $\int_0^1 x(1-x^3)^{10} dx$. (*Bhopal, 2008*) (*ii*) $\int_0^1 \frac{dx}{\sqrt{(1-x^n)}}$ (*Anna, 2005*)

10. Prove that $\int_0^{\infty} \frac{dx}{(e^x + e^{-x})^n} = \frac{1}{4}\beta\left(\frac{n}{2}, \frac{n}{2}\right)$ and hence evaluate $\int_0^{\infty} \text{sech}^8 x\, dx$.

11. Prove that $\beta\left(n + \frac{1}{2}, n + \frac{1}{2}\right) = \frac{\Gamma(n + 1/2)\sqrt{\pi}}{2^{2n}\,\Gamma(n+1)}$. Hence show that $2^n\,\Gamma(n + 1/2) = 1 \cdot 3 \cdot 5 \ldots (2n - 1)\sqrt{\pi}$

(*Mumbai, 2007*)

12. Prove that :

(*i*) $\frac{\beta(m+1, n)}{m} = \frac{\beta(m, n+1)}{n} = \frac{\beta(m, n)}{m+n}$

(*ii*) $\beta(n, n) = \frac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\,\Gamma\left(n + \frac{1}{2}\right)}$

(*iii*) $\Gamma\left(n + \frac{1}{2}\right) = \frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}\,.\,\Gamma(n+1)}$

(*iv*) $\beta(m + 1) + \beta(m, n + 1) = \beta(m, n)$

(*Bhopal, 2008 ; J.N.T.U., 2006 ; Madras, 2003*)

13. Show that $\iint x^{m-1}\, y^{n-1}\, dx\, dy$ over the positive quadrant of the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \text{ is } \frac{a^m b^n}{2n}\,\beta\left(\frac{m}{2}, \frac{n}{2} + 1\right).$$

14. Show that the area in the first quadrant enclosed by the curve $(x/a)^{\alpha} + (y/b)^{\beta} = 1$, $\alpha > 0$, $\beta > 0$, is given by

$$\frac{ab}{\alpha + \beta}\,\frac{\Gamma(1/\alpha)\,\Gamma(1/\beta)}{\Gamma(1/\alpha + 1/\beta)}.$$

15. Find the mass of an octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, the density at any point being $\rho = kxyz$.

(*U.P.T.U., 2002*)

7.17 (1) ELLIPTIC INTEGRALS

In Applied Mathematics, we often come across integrals of the form $\int_0^1 e^{-x^2}\, dx$ or $\int_0^1 \sin x^2\, dx$ which cannot be evaluated by any of the standard methods of integration. In such cases, we may find the value to any desired degree of accuracy by expanding their integrands as power series. An important class of such integrals is the *elliptic integrals.*

Def. The integral $F(k, \phi) = \int_0^{\phi} \frac{dx}{\sqrt{(1 - k^2 \sin^2 x)}}\ (k^2 < 1)$ ...(*i*)

which is a function of the two variables k and ϕ, is called the *elliptic integral of the first kind with modulus k* and *amplitude* ϕ.

The integral $E(k, \phi) = \int_0^{\phi} \sqrt{(1 - k^2 \sin^2 x)}\, dx\ (k^2 < 1)$ ...(*ii*)

is called the *elliptic integral of the second kind* with *modulus k and amplitude* ϕ.

The name *elliptic integral* arose from its original application in finding the length of an elliptic arc (Fig.7.45), For instance, consider the ellipse

$$x = a\cos\phi, \qquad y = b\sin\phi, \qquad (a < b)$$

Then length of its arc

$$AP = \int_0^{\phi} \sqrt{\left[\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right]}\, d\phi = \int_0^{\phi} \sqrt{[(-a\sin\phi)^2 + (b\cos\phi)^2]}\, d\phi$$

$$= \int_0^{\phi} \sqrt{[(b^2 + (a^2 - b^2)\sin^2\phi]}\, d\phi = b\int_0^{\phi} \sqrt{\left[1 - \left(1 - \frac{a^2}{b^2}\right)\sin^2\phi\right]}\, d\phi$$

$$= bE(k, \phi) \text{ for } k^2 = 1 - a^2/b^2 \le 1.$$

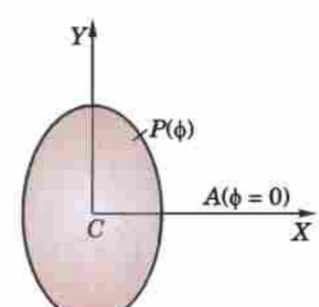

Fig. 7.45

Also the perimeter of the ellipse

$$= 4b \int_0^{\pi/2} \sqrt{(1 - k^2 \sin^2 \phi)}\ d\phi = 4bE(k, \pi/2).$$

This particular integral with upper limit $\phi = \pi/2$ is called the *complete elliptic integral of the second kind and is denoted by $E(k)$.*

Thus $$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)\, d\phi\ (k^2 < 1) \quad ...(iii)$$

Similarly, the *complete elliptic integral of first kind is*

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}} (k^2 < 1) \quad ...(iv)$$

To evaluate it, we expand the integral in the form

$$(1 - k^2 \sin^2 \phi)^{-1/2} = 1 + \frac{k^2}{2} \sin^2 \phi + \frac{3k^4}{4}\ \sin^4 \phi + ...$$

This series can be shown to be uniformly convergent for all k, and may, therefore, be integrated term by term [See § 9.19-II]. Then we have

$$K(k) = \int_0^{\pi/2} \left(1 + \frac{k^2}{2} \sin^2 \phi + \frac{3k^4}{8} \sin^4 \phi + \frac{5k^6}{16} \sin^6 \phi + ...\right) d\phi$$

$$= \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6.}\right)^2 k^6 + ...\right] \quad ...(v)$$

This series may be used to compute K for various values of k. In particular, if $k = \sin 10°$; we have

$$K = \frac{\pi}{2}(1 + 0.00754 + 0.00012 + ...) = 1.5828 \quad ...(vi)$$

In this way tables of the elliptic integrals are constructed. Values of $F(k, \phi)$ and $E(k, \phi)$ are readily available for $0 \le \phi \le \pi/2$, $0 < k < 1$. *(See Peirce's short tables).*

Example 7.50. *Express* $\int_0^{\pi/6} \frac{dx}{\sqrt{(\sin x)}}$ *in terms of elliptic integral.*

Solution. Put $\cos x = \cos^2 \phi$ and $dx = \dfrac{2 \cos \phi\, d\phi}{\sqrt{(1 + \cos^2 \phi)}}$

Then $$I = \int_0^{\pi/2} \frac{2 \cos^2 \phi}{\sqrt{(1 + \cos^2 \phi)}} d\phi = 2 \int_0^{\pi/2} \frac{(1 + \cos^2 \phi) - 1}{\sqrt{(1 + \cos^2 \phi)}}\ d\phi$$

$$= 2\left\{\int_0^{\pi/2} \sqrt{(1 + \cos^2 \phi)}\ d\phi - \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 + \cos^2 \phi)}}\right\} = 2\left\{\int_0^{\pi/2} \sqrt{(2 - \sin^2 \phi)}\ d\phi - \int_0^{\pi/2} \frac{d\phi}{\sqrt{(2 - \sin^2 \phi)}}\right\}$$

$$= 2\sqrt{2} \int_0^{\pi/2} \sqrt{(1 - 1/2 \sin^2 \phi)}\ d\phi - \sqrt{2} \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - 1/2 \sin^2 \phi)}} = 2\sqrt{2}\, E\left(\frac{1}{\sqrt{2}}\right) - \sqrt{2}\, K\left(\frac{1}{\sqrt{2}}\right)$$

(2) **Elliptic functions.** By putting $\sin x = t$ and $\sin \phi = z$, (i) becomes

$$u = \int_0^{z} \frac{dt}{\sqrt{[(1 - t^2)(1 - k^2 t^2)]}} (k^2 < 1) \quad ...(vii)$$

This is known as *Jacobi's form of the elliptic integral of first kind** whereas (i) is the *Legendre's form*†.

If $k = 0$, (vii) gives $u = \sin^{-1} z$. By analogy, we denote (vii) $sn^{-1} z$ for a fixed non-zero value of k. This leads to the functions $sn\ u = z = \sin \phi$ and $cn\ u = \cos \phi$
which are called the *Jacobi's elliptic functions.*

* See footnote p. 215.

† A French mathematician *Adrien Marie Legendre* (1752–1833) who made important contributions to number theory, special functions, calculus of variations and elliptic integrals.

The elliptic functions $sn\ u$ and $cn\ u$ are periodic with a period depending on k and an amplitude equal to unity. These behave somewhat like $\sin u$ and $\cos u$. For instance

$$sn(0) = 0, cn(1) = 1 \quad \text{and} \quad sn(-u) = -sn(u), cn(-u) = cn(u).$$

Example 7.51. *Show that* $\int_0^{a/2} \frac{dx}{\sqrt{(2ax - x^2)}\sqrt{(a^2 - x^2)}} = \frac{2}{3a} K\left(\frac{1}{3}\right)$.

Solution. Putting $x = \frac{a}{2}(1 - \sin\theta), dx = -\frac{a}{2}\cos\theta\, d\theta$,

$$2ax - x^2 = \frac{a^2}{4}(1 - \sin\theta)(3 + \sin\theta) \text{ and } a^2 - x^2 = \frac{a^2}{4}(1 + \sin\theta)(3 - \sin\theta)$$

Also when $x = 0, \theta = \pi/2$; when $x = a/2, \theta = 0$.

Thus the given integral

$$= \frac{4}{a^2}\int_{\pi/2}^{0} \frac{-(a/2)\cos\theta\, d\theta}{\sqrt{[(1 - \sin^2\theta)(2 - \sin^2\theta)]}} = \frac{2}{3a}\int_0^{\pi/2} \frac{d\theta}{\sqrt{[(1 - (1/3)^2 \sin^2\theta]}} = \frac{2}{3a} K\left(\frac{1}{3}\right).$$

7.18 (1) ERROR FUNCTION OR PROBABILITY INTEGRAL

The error function or the probability integral is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt.$$

This integral arises in the solution of certain partial differential equations of applied mathematics and occupies an important position in the probability theory.

The complementary error function erfc (x) is defined as $erfc(x) = 1 - erf(x)$.

(2) Properties : (*i*) $erf(-x) = -erf(x)$; (*ii*) $erf(0) = 0$

$$(iii)\ erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2} dt = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1$$ [By (*iii*), p. 289]

This proves that the total area under the *Normal* or *Gaussian error function curve* is unity – § 26.16.

PROBLEMS 7.9

1. By means of the substitution $k \sin x = \sin z$, show that

   (*i*) $\int_0^\pi \frac{dx}{\sqrt{(1 - k^2 \sin^2 x)}} = \frac{1}{k} F\left(\frac{1}{k}, \phi'\right)$;

   (*ii*) $\int_0^\phi \sqrt{(1 - k^2\sin^2 x)}\, dx = \left(\frac{1}{k} - k\right) F\left(\frac{1}{k}, \phi'\right) + kE\left(\frac{1}{k}, \phi'\right)$

   where $k > 1$ and $\phi' = \sin^{-1}(k \sin\phi)$.

   Express the following integrals in terms of elliptic integrals :

2. $\int_0^{\pi/2} \frac{dx}{\sqrt{(1 + 3\sin^2 x)}}$. (*Kerala, M.E., 2005*)
3. $\int_0^{\pi/2} \frac{dx}{\sqrt{(2 - \cos x)}}$.
4. $\int_0^{\pi/2} \sqrt{(\cos x)}\, dx$.
5. Expand $erf(x)$ in ascending powers of x. Hence evaluate $erf(0)$. (*P.T.U., 2009 S*)
6. Compute (*i*) $erf(0.3)$, (*ii*) $erf(0.5)$, correct to three decimal places.
7. Show that (*i*) $erf(x) + erf(-x) = 0$ (*ii*) $erfc(x) + erfc(-x) = 2$
8. Prove that

   (*i*) $\frac{d}{dx}[erf(ax)] = \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$ (*Osmania, 2003*) (*ii*) $\frac{d}{dx}[erfc(ax)] = -\frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$.

9. Prove that $\int_0^\infty e^{-x^2 - 2ax}\, dx = \frac{\sqrt{\pi}}{2} e^{a^2}[1 - erf(0)]$

7.19 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 7.10

Fill up the blanks or choose the correct answer from the following problems :

1. $\int_0^2 \int_0^x (x+y)\, dx\, dy = \ldots\ldots$

2. $\int_0^1 \int_0^{1-x} dx\, dy \ldots\ldots$

3. $\int_0^1 e^{-x^2}\, dx = \ldots\ldots$

4. $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \ldots\ldots$ (*V.T.U., 2010*)

5. $\Gamma(3.5) = \ldots\ldots$

6. The surface area of the sphere $x^2 + y^2 + z^2 + 2x - 4y + 8z - 2 = 0$ is ....

7. $\int_0^2 \int_1^3 \int_1^2 xy^2 z\, dz dy dx = \ldots\ldots$

8. If $u = x + y$ and $v = x - 2y$, then the area-element $dxdy$ is replaced by .... $dudv$.

9. In terms of Beta function $\int_0^{\pi/2} \sin^7 \theta \sqrt{\cos \theta}\, d\theta = \ldots\ldots$

10. The value of $\beta(2, 1) + \beta(1, 2)$ is ....

11. $\int_0^1 \int_1^2 xy\, dy\, dx = \ldots\ldots$

12. Volume bounded by $x \geq 0, y \geq 0, z \geq 0$ and $x^2 + y^2 + z^2 = 1$ as a triple integral integral.

13. Value of $\int_0^1 \int_0^{x^2} xe^y\, dy\, dx$ is equal to

    (*a*) $e/2$ (*b*) $e - 1$ (*c*) $1 - e$ (*d*) $e/2 - 1$. (*Bhopal, 2008*)

14. $\iint x^2 y^3\, dxdy$ over the rectangle $0 \leq x \leq 1$ and $0 \leq y \leq 3$ is ....

15. $\int_0^{\pi} \int_0^{a \sin \theta} r\, dr\, d\theta = \ldots\ldots$

16. $\int_{x=0}^{x=3} \int_{y=0}^{y=1/x} ye^{xy}\, dxdy = \ldots\ldots$

17. $\int_0^{\pi/2} \int_0^{\infty} \frac{r\, dr\, d\theta}{(r^2 + a^2)} = \ldots\ldots$

18. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dx\, dy = \ldots\ldots$

19. To change cartesian coordinates (x, y, z) to spherical polar coordinate (r, θ, ϕ); $dx\, dy\, dz$ is replaced by ....

20. $\int_0^2 \int_0^{x^2} e^{y/x}\, dydx = \ldots\ldots$

21. $\iint (x + y)^2\, dxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is ....

22. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\, dy}{1 + x^2 + y^2} = \ldots\ldots$

23. $\iint xy(x + y)\, dx\, dy$ over the area between $y + x^2$ and $y = x$, is ....

24. Value of $\int_0^1 \int_x^{\sqrt{x}} xy\, dx\, dy$ is

    (*a*) zero (*b*) $-1/24$ (*c*) $1/24$ (*d*) 24. (*V.T.U., 2010*)

25. $\iint dx\, dy$ over the area bounded by $x = 0, y = 0, x^2 + y^2 = 1$ and $5y = 3$, is ....

26. $\iint_R y\, dx\, dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$, is ....

27. $\iint (x^2 + y^2)\, dx\, dy$ in the positive quadrant for which $x + y \leq 1$, is ....

28. Area between the parabolas $y^2 = 4x$ and $x^2 = 4y$ is ....

29. Changing the order of integration in $\int_{-a}^{a} \int_0^{\sqrt{a^2 - y^2}} f(x, y)\, dx\, dy = \ldots\ldots$

30. $\Gamma(1/4)\Gamma(3/4) = \ldots\ldots$ (*V.T.U., 2011*)

31. $\beta(5/2, 7/2) = \ldots\ldots$

32. $\int_0^{\infty} \int_0^{x} xe^{-x^2/y}\, dydx = \ldots\ldots$

33. On changing to polar coordinates $\int_0^{2a} \int_0^{\sqrt{(2ax - x^2)}} dx\, dy$ becomes ....

34. A square lamina is immersed in the liquid with one vertex in the surface and the diagonal of length vertical. Its centre of pressure is at a depth ....
35. The centroid of the area enclosed by the parabola $y^2 = 4x$, x-axis and its latus-rectum is ....
36. The moment of inertia of a uniform spherical ball of mass 10 gm and radius 2 cm about a diameter is ....
37. M.I. of a solid right circular cone (base-radius r and height h) about its axis is ....

38. $erf_c(-x) - erf(x) =$ ....

39. $\int_0^1 \frac{x-1}{\log x}\, dx =$ ....

40. $\Gamma\left(\frac{3}{2}\right) =$ ....

41. Value of $\int_0^a \int_0^b \int_0^c x^2 y^2 z^2 \, dx\, dy\, dz$ is

(a) $\frac{abc}{3}$ (b) $\frac{a^2b^2c^2}{27}$ (c) $\frac{a^3b^3c^3}{27}$ (d) $\frac{a^2b^2c^2}{9}$.

42. The integral $\int_0^1 \int_0^{\sqrt{(1-x^2)}} (x+y)\, dy\, dx$ after changing the order of integration.

(a) $\int_0^2 \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$ (b) $\int_0^1 \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$

(c) $\int_0^1 \int_0^{\sqrt{(1+y^2)}} (x+y)\, dx\, dy$ (d) $\int_0^{-1} \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$. (V.T.U., 2011)