CHAPTER 5

# Partial Differentiation and Its Applications



## 5.1 (1) FUNCTIONS OF TWO OR MORE VARIABLES

We often come across quantities which depend on two or more variables. For example, the area of a rectangle of length $x$ and breadth $y$ is given by $A = xy$. For a given pair of values of $x$ and $y$, $A$ has a definite value. Similarly, the volume of a parallelopiped ($= xyh$) depends on the three variables $x$(= length), $y$(= breadth) and $h$(=height).

**Def.** *A symbol $z$ which has a definite value for every pair of values of $x$ and $y$ is called a function of two independent variables $x$ and $y$ and we write $z = f(x, y)$ or $\phi(x, y)$.*

We may interpret $(x, y)$ as the coordinates of a point in the $XY$-plane and $z$ as the height of the surface $z = f(x, y)$. We have come across several examples of such surfaces in Chapter 4.

The set $R$ of points $(x, y)$ such that any two points $P_1$ and $P_2$ of $R$ can be so joined that any arc $P_1P_2$ wholly lies in $R$, is called as *region* in the $XY$-plane. A region is said to be a *closed region* if it includes all the points of its boundary, otherwise it is called an *open region.*

A set of points lying within a circle having centre at $(a, b)$ and radius $\delta > 0$, is said to be *neighbourhood* of $(a, b)$ in the circular region $R : (x - a)^2 + (y - b)^2 < \delta^2$.

When $z$ is a function of three or more variables $x, y, t, \ldots$, we represent the relation by writing $z = f(x, y, t, \ldots)$. For such functions, no geometrical representation is possible. However, the concepts of a region and neighbourhood can easily be extended to functions of three or more variables.

(2) **Limits.** *The function $f(x, y)$ is said to tend to the limit $l$ as $x \to a$ and $y \to b$ if and only if the limit $l$ is independent of the path followed by the point $(x, y)$ as $x \to a$ and $y \to b$ and we write*

$$\underset{\substack{x \to a \\ y \to b}}{Lt}\ f(x, y) = l$$

In terms of a circular neighbourhood, we have the following *definition of the limit :*

*The function $f(x, y)$ defined in a region $R$, is said to tend to the limit $l$ as $x \to a$ and $y \to b$ if and only if corresponding to a positive number $\varepsilon$, there exists another positive number $\delta$ such that $| f(x, y) - l | < \varepsilon$ for $0 < (x - a)^2 + (y - b)^2 < \delta^2$ for every point $(x, y)$ in $R$.*

(3) **Continuity.** *A function $f(x, y)$ is said to be continuous at the point $(a, b)$ if*

$$\underset{\substack{x \to a \\ y \to b}}{Lt}\ f(x, y) \text{ exists and } = f(a, b)$$

If a function is continuous at all points of a region, then it is said to be *continuous in that region*. A function which is not continuous at a point is said to be *discontinuous* at that point.

**Obs.** *Usually, the limit is the same irrespective of the path along which the point (x, y) approaches (a, b) and*

$$\underset{x\to a}{Lt}\left\{\underset{y\to b}{Lt} f(x,y)\right\}=\underset{y\to b}{Lt}\left\{\underset{x\to a}{Lt} f(x,y)\right\}$$

*But it is not always so, as the following examples show :*

$$\text{Lt}\left(\frac{x-y}{x+y}\right) \text{ as } (x,y)\to(0,0) \text{ along the line } y=mx$$

$$=\underset{x\to 0}{\text{Lt}}\frac{x-mx}{x+mx}=\frac{1-m}{1+m} \text{ which is different for lines with different slopes.}$$

Also $\underset{x\to 0}{\text{Lt}}\left[\underset{y\to 0}{\text{Lt}}\left(\frac{x-y}{x+y}\right)\right]=\underset{x\to 0}{\text{Lt}}\left(\frac{x}{x}\right)=1$, whereas $\underset{y\to 0}{\text{Lt}}\left[\underset{x\to 0}{\text{Lt}}\left(\frac{x-y}{x+y}\right)\right]=\underset{y\to 0}{\text{Lt}}\left(\frac{-y}{y}\right)=1.$

$\therefore$ As $(x, y)$ is made to approach $(0, 0)$ along different paths, $f(x, y)$ approaches different limits. Hence the two repeated limits are not equal and $f(x, y)$ is discontinuous at the origin.

Also the function is not defined at $(0, 0)$ since $f(x, y) = 0/0$ for $x = 0, y = 0$.

**(4)** *As in the case of functions of one variable, the following results hold :*

I. *If* $\underset{\substack{x\to a\\ y\to b}}{Lt} f(x,y)=l$ and $\underset{\substack{x\to a\\ y\to b}}{Lt} g(x,y)=m$,

*then* (*i*) *If* $\underset{\substack{x\to a\\ y\to b}}{Lt}[f(x,y)\pm g(x,y)]=l\pm m$ (*ii*) $\underset{\substack{x\to a\\ y\to b}}{Lt}[f(x,y)\,.\,g(x,y)]=l\,.\,m$

(*iii*) $\underset{\substack{x\to a\\ y\to b}}{Lt}[f(x,y)/g(x,y)]=l/m$ $(m\neq 0)$

II. *If* $f(x, y), g(x, y)$ *are continuous at* $(a, b)$ *then so also are the functions*

$f(x, y) \pm g(x, y), f(x, y) \,.\, g(x, y)$ and $f(x, y)/g(x, y)$

provided $g(x, y) \neq 0$ in the last case.

## PROBLEMS 5.1

Evaluate the following limits :

1. $\underset{\substack{x\to 1\\ y\to 2}}{\text{Lt}}\dfrac{2x^2y}{x^2+y^2+1}$ 2. $\underset{\substack{x\to 0\\ y\to 0}}{\text{Lt}}\dfrac{xy}{x^2+y^2}$ 3. $\underset{\substack{x\to \infty\\ y\to 2}}{\text{Lt}}\dfrac{xy+1}{x^2+2y^2}$ 4. $\underset{\substack{x\to 1\\ y\to 1}}{\text{Lt}}\dfrac{x(y-1)}{y(x-1)}$

5. If $f(x,y)=\dfrac{x-y}{2x+y}$, show that $\underset{x\to 0}{\text{Lt}}\left[\underset{y\to 0}{\text{Lt}} f(x,y)\right]\neq\underset{y\to 0}{\text{Lt}}\left[\underset{x\to 0}{\text{Lt}} f(x,y)\right]$

   Also show that the function is discountinuous at the origin.

6. Show that the function $f(x, y) = x^2 + 2y$, $(x, y) \neq (1, 2)$

   $3(x, y) = (1, 2)$ $= 0$

   is discontinuous at $(1, 2)$.

7. Investigate the continuity of the function

   $f(x, y) = xy/(x^2 + y^2)$, $(x, y) \neq (0, 0)$

   $= 0$ $(x, y) = (0, 0)$

   at the origin.

**Note.** In whatever follows, all the functions considered are continuous and their partial derivatives (as defined below) exist.

## 5.2 PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of two variables $x$ and $y$.

If we keep $y$ as constant and vary $x$ alone, then $z$ is a function of $x$ only. The derivative of $z$ with respect to $x$, treating $y$ as constant, is called the *partial derivative of z with respect to x* and is denoted by one of the symbols

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), D_x f. \qquad \text{Thus} \quad \frac{\partial z}{\partial x} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

Similarly, the derivative of $z$ with respect to $y$, keeping $x$ as constant, is called the *partial derivative of $z$ with respect to $y$* and is denoted by one of the symbols.

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), D_y f. \qquad \text{Thus} \quad \frac{\partial z}{\partial y} = \underset{\partial y \to 0}{\text{Lt}} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$$

Similarly, if $z$ is a function of three or more variables $x_1, x_2, x_3, \ldots$ *the partial derivative of $z$ with respect to $x_1$, is obtained by differentiating $z$ with respect to $x_1$, keeping all other variables constant* and is written as $\partial z/\partial x_1$.

In general $f_x$ and $f_y$ are also functions of $x$ and $y$ and so these can be differentiated further partially with respect to $x$ and $y$.

Thus $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } \frac{\partial^2 z}{\partial x^2} \text{ or } f_{xx}, \ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}{}^{*}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}, \text{ and } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}.$$

It can easily be verified that, in all ordinary cases,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

*Sometimes we use the following notation*

$$\frac{\partial z}{\partial x} = p, \ \frac{\partial z}{\partial y} = q, \ \frac{\partial^2 z}{\partial x^2} = r, \ \frac{\partial^2 z}{\partial x \partial y} = s, \ \frac{\partial^2 z}{\partial y^2} = t.$$

**Example 5.1.** *Find the first and second partial derivatives of $z = x^3 + y^3 - 3axy$.*

**Solution.** We have $z = x^3 + y^3 - 3axy$.

$$\therefore \quad \frac{\partial z}{\partial x} = 3x^2 + 0 - 3ay(1) = 3x^2 - 3ay, \text{ and } \frac{\partial z}{\partial y} = 0 + 3y^2 - 3ax(1) = 3y^2 - 3ax$$

Also $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3ay) = 6x, \ \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2 - 3ay) = -3a$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 - 3ax) = 6y, \text{ and } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(3y^2 - 3ax) = -3a.$$

We observe that $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

**Example 5.2.** *If $u = x^2 \tan^{-1} \dfrac{y}{x} - y^2 \tan^{-1} \dfrac{x}{y}$,*

*show that* $$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}.$$ *(Mumbai, 2008 S)*

*and* $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$ *(Madras, 2000)*

**Solution.** We have $$\frac{\partial u}{\partial y} = x^2 \cdot \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} - \left\{2y \cdot \tan^{-1}\frac{x}{y} + y^2 \cdot \frac{1}{1+(x/y)^2} \cdot \left(-\frac{x}{y}\right)\right\}$$

$$= \frac{x^3}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2} = x - 2y \tan^{-1}\frac{x}{y}.$$

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*It is important to note that in the subscript notation the subscripts are written in the same order in which we differentiate whereas in the '∂' notation the order is opposite.

$\therefore \quad \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left\{x - 2y \tan^{-1} \dfrac{x}{y}\right\} = 1 - 2y \cdot \dfrac{1}{1+(x/y)^2} \cdot \dfrac{1}{y} = 1 - \dfrac{2y^2}{x^2+y^2} = \dfrac{x^2-y^2}{x^2+y^2}.$

Similarly, $\dfrac{\partial u}{\partial x} = 2x \tan^{-1} y/x - y$

and $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left\{2x \tan^{-1} \dfrac{y}{x} - y\right\} = \dfrac{x^2-y^2}{x^2+y^2}$. Hence the result.

**Example 5.3.** *If $z = f(x + ct) + \phi(x - ct)$, prove that*

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

(*J.N.T.U., 2006 ; V.T.U., 2003 S*)

**Solution.** We have $\dfrac{\partial z}{\partial x} = f'(x+ct) \cdot \dfrac{\partial}{\partial x}(x+ct) + \phi'(x-ct)\dfrac{\partial}{\partial x}(x-ct) = f'(x+ct) + \phi'(x-ct)$

and $\dfrac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct)$ ...(*i*)

Again $\dfrac{\partial z}{\partial t} = f'(x+ct)\dfrac{\partial}{\partial t}(x+ct) + \phi'(x-ct)\dfrac{\partial}{\partial t}(x-ct) = cf'(x-ct) - c\phi'(x-ct)$

and $\dfrac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2\phi''(x-ct) = c^2[f''(x+ct) + \phi''(x-ct)]$ ...(*ii*)

From (*i*) and (*ii*), it follows that $\dfrac{\partial^2 z}{\partial t^2} = c^2 \dfrac{\partial^2 z}{\partial x^2}$.

**Obs.** This is an important partial differential equation, known as *wave equation* (§ 18.4).

**Example 5.4.** *If $\theta = t^n e^{-r^2/4t}$, what value of n will make $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$?*

(*Nagpur, 2009 ; Kurukshetra, 2006 ; U.P.T.U., 2006*)

**Solution.** We have $\dfrac{\partial \theta}{\partial r} = t^n \cdot e^{-r^2/4t} \cdot \left(\dfrac{-2r}{4t}\right) = -\dfrac{r}{2} t^{n-1} e^{-r^2/4t}$

$\therefore \quad r^2 \dfrac{\partial \theta}{\partial r} = -\dfrac{r^3}{2} t^{n-1} \cdot e^{-r^2/4t}$

and $\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = -\dfrac{3r^2}{2} t^{n-1} e^{-r^2/4t} - \dfrac{r^3}{2} t^{n-1} \cdot e^{-r^2/4t}\left(-\dfrac{2r}{4t}\right)$

$\therefore \quad \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \left(-\dfrac{3}{2} t^{n-1} + \dfrac{r^2}{4} t^{n-2}\right) e^{-r^2/4t}$

Also $\dfrac{\partial \theta}{\partial t} = nt^{n-1} \cdot e^{-r^2/4t} + t^n \cdot e^{-r^2/4t} \cdot \dfrac{r^2}{4t^2} = \left(nt^{n-1} + \dfrac{1}{4} r^2 t^{n-2}\right) e^{-r^2/4t}$

Since $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$,

$\therefore \quad \left(-\dfrac{3}{2} t^{n-1} + \dfrac{1}{4} r^2 t^{n-2}\right) e^{-r^2/4t} = \left(nt^{n-1} + \dfrac{1}{4} r^2 t^{n-2}\right) e^{-r^2/4t}$ or $\left(n + \dfrac{3}{2}\right) t^{n-1} e^{-r^2/4t} = 0.$

Hence $n = -3/2$.

**Example 5.5.** *If $v = (x^2 + y^2 + z^2)^{-1/2}$, prove that*

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0. \text{ (Laplace equation)*}$$

(*V.T.U., 2006 ; Osmania, 2003 S*)

*See footnote p. 18.

**Solution.** We have $\dfrac{\partial v}{\partial x} = -\dfrac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x = -x(x^2+y^2+z^2)^{-3/2}$

and
$$\frac{\partial^2 v}{\partial x^2} = -1[1 \cdot (x^2+y^2+z^2)^{-3/2} + x(-3/2)(x^2+y^2+z^2)^{-5/2} \cdot 2x]$$
$$= -(x^2+y^2+z^2)^{-5/2}[x^2+y^2+z^2-3x^2] = (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2)$$

Similarly,
$$\frac{\partial^2 v}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}(-x^2+2y^2-z^2) \text{ and } \frac{\partial^2 v}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}(-x^2-y^2+2z^2)$$

Hence
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = (x^2+y^2+z^2)^{-5/2} \cdot (0) = 0.$$

**Obs.** *A function v satisfying the Laplace equation is said to be a* **harmonic function.**

**Example 5.6.** *If $u = \log(x^3+y^3+z^3-3xyz)$, show that $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)^2 u = \dfrac{-9}{(x+y+z)^2}$.*

*(P.T.U., 2010 ; Anna, 2009 ; Bhopal, 2008 ; U.P.T.U., 2006)*

**Solution.** We have $\dfrac{\partial u}{\partial x} = \dfrac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$, $\dfrac{\partial u}{\partial y} = \dfrac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$, $\dfrac{\partial u}{\partial z} = \dfrac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$

$$\therefore \quad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$
$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{3}{x+y+z} \qquad \textit{(V.T.U., 2009)}$$

Now
$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$
$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) = \left(\frac{3}{x+y+z}\right)$$
$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}.$$

**Example 5.7.** *If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u} = 1$, prove that*
$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right) \qquad \textit{(U.P.T.U., 2003)}$$

**Solution.** We have $x^2(a^2+u)^{-1}+y^2(b^2+u)^{-1}+z^2(c^2+u)^{-1} = 1$ ...(*i*)

Differentiating (*i*) partially w.r.t. $x$, we get
$$2x(a^2+u)^{-1} - x^2(a^2+u)^{-2}\frac{\partial u}{\partial x} - y^2(b^2+u)^{-2}\frac{\partial u}{\partial y} - z^2(c^2+u)^{-2}\frac{\partial u}{\partial z} = 0$$

or
$$\frac{2x}{a^2+u} = \left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{y^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial x}$$

or
$$\frac{\partial u}{\partial x} = \frac{2x}{(a^2+u)v} \text{ where } v = \frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{y^2}{(c^2+u)^2}$$

Similarly differentiating (*i*) partially w.r.t. $y$, we get
$$\frac{2y}{b^2+u} = \left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial y} \text{ or } \frac{\partial u}{\partial y} = \frac{2y}{(b^2+u)v}$$

Similarly, differentiating (*i*) partially w.r.t. $z$, we get

$$\frac{2z}{(b^2+u)} = \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial z} \text{ or } \frac{\partial u}{\partial z} = \frac{2z}{(c^2+u)v}$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4}{v^2}\left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{y^2}{(c^2+u)^2}\right\} = \frac{4}{v} \quad ...(ii)$$

Also $$2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) = 2\left\{\frac{2x^2}{(a^2+u)v} + \frac{2y^2}{(b^2+u)v} + \frac{2z^2}{(c^2+u)v}\right\}$$

$$= \frac{4}{v}\left\{\frac{x^2}{(a^2+u)} + \frac{y^2}{(b^2+u)} + \frac{z^2}{(c^2+u)}\right\} = \frac{4}{v} \quad \text{[By (i)] ...(iii)}$$

Hence the equality of (*ii*) and (*iii*) proves the result.

**Example 5.8.** *If* $u = x^y$, *show that* $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$. *(Anna, 2009)*

**Solution.** We have $\dfrac{\partial u}{\partial y} = x^y \log_e x$ and $\dfrac{\partial^2 u}{\partial x \partial y} = yx^{y-1} \cdot \log x + x^y \cdot \dfrac{1}{x} = x^{y-1}(y \log x + 1)$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \quad ...(i)$$

Again $$\frac{\partial u}{\partial x} = yx^{y-1} \text{ and } \frac{\partial^2 u}{\partial y \partial x} = 1 \cdot x^{y-1} + y\left(\frac{1}{x}x^y \log x\right) = x^{y-1}(1 + y \log x)$$

$$\therefore \quad \frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \quad ...(ii)$$

From (*i*) and (*ii*) follows the required result.

## PROBLEMS 5.2

1. Evaluate $\partial z/\partial x$ and $\partial z/\partial y$, if
   (*i*) $z = x^2y - x \sin xy$ ; (*ii*) $z = \log(x^2 + y^2)$ ;
   (*iii*) $z = \tan^{-1}\{(x^2 + y^2)/(x + y)\}$ ; (*iv*) $x + y + z = \log z$.
2. If $z(x + y) = x^2 + y^2$, show that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$. *(V.T.U., 2003)*
3. If $z = e^{ax+by} f(ax - by)$, prove that $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz$. *(V.T.U., 2010)*
4. Given $u = e^{r\cos\theta}\cos(r \sin\theta)$, $v = e^{r\cos\theta}\sin(r\sin\theta)$ ; prove that $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.
5. If $z = \tan(y + ax) - (y - ax)^{3/2}$, show that $\dfrac{\partial^2 z}{\partial x^2} = a^2\dfrac{\partial^2 z}{\partial y^2}$. *(Mumbai, 2009)*
6. Verify that $f_{xy} = f_{yx}$, when $f$ is equal to (*i*) $\sin^{-1}(y/x)$ ; (*ii*) $\log x \tan^{-1}(x^2 + y^2)$.
7. If $f(x, y) = (1 - 2xy + y^2)^{-1/2}$, show that $\dfrac{\partial}{\partial x}\left[(1 - x^2)\dfrac{\partial f}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[y^2\dfrac{\partial f}{\partial y}\right] = 0$. *(Rohtak, 2006 S)*
8. Prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ if (*i*) $u = \tan^{-1}\left[\dfrac{2xy}{x^2 - y^2}\right]$, (*ii*) $u = \log(x^2 + y^2) + \tan^{-1}(y/x)$. *(Anna, 2009)*
9. If $v = \dfrac{1}{\sqrt{t}}e^{-x^2/4a^2t}$, prove that $\dfrac{\partial v}{\partial t} = a^2\dfrac{\partial^2 v}{\partial x^2}$.

10. The equation $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar without radiation, show that if $u = Ae^{-gx} \sin (nt - gx)$, where $A, g, n$ are positive constants then $g = \sqrt{(n/2\mu)}$.
11. Find the value of $n$ so that the equation $V = r^n (3 \cos^2 \theta - 1)$ satisfies the relation
$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial V}{\partial \theta}\right) = 0.$$
12. If $z = \log (e^x + e^y)$, show that $rt - s^2 = 0$ where $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, $t = \partial^2 z/\partial y^2$.
13. If $u = \frac{y}{z} + \frac{z}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.
14. Let $r^2 = x^2 + y^2 + z^2$ and $V = r^m$, prove that $V_{xx} + V_{yy} + V_{zz} = m(m+1)r^{m-2}$. (*Raipur, 2005*)
15. If $v = \log (x^2 + y^2 + z^2)$, prove that $(x^2 + y^2 + z^2)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = 2$.
16. If $v = x^y \cdot y^x$, prove that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v(x + y + \log v)$. (*Anna, 2005*)
17. If $x^x y^y z^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$. (*Bhopal, 2008*)
18. If $u = e^{xyz}$, find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$. (*Rajasthan, 2005 ; Osmania, 2003 S*)

## 5.3 WHICH VARIABLE IS TO BE TREATED AS CONSTANT

(1) Consider the equation $x = r \cos \theta, y = r \sin \theta$ ...(1)

To find $\partial r/\partial x$, we need a relation between $r$ and $x$. Such a relation will contain one more variable $\theta$ or $y$, for we can eliminate only one variable out of four from the relations (1). Thus the two possible relations are

$$r = x \sec \theta \quad ...(2) \qquad \text{and} \qquad r^2 = x^2 + y^2 \quad ...(3)$$

Now we can find $\partial r/\partial x$ either from (2) by treating $\theta$ as constant or from (3) by regarding $y$ as constant. And there is no reason to suppose that the two values of $\partial r/\partial x$ so found, are equal. *To avoid confusion as to which variable is regarded constant, we introduce the following :*

**Notation :** $(\partial r/\partial x)_\theta$ *means the partial derivative of $r$ with respect to $x$ keeping $\theta$ constant in* a relation expressing $r$ as a function of $x$ and $\theta$.

Thus from (2), $(\partial r/\partial x)_\theta = \sec \theta$.

When no indication is given regarding the variable to be kept constant, then according to convention $(\partial/\partial x)$ always means $(\partial/\partial x)_y$ and $\partial/\partial y$ means $(\partial/\partial y)_x$. Similarly, $\partial/\partial r$ means $(\partial/\partial r)_\theta$ and $\partial/\partial \theta$ means $(\partial/\partial \theta)_r$.

(2) In thermodynamics, we come across ten variables such as $p$ (pressure), $v$ (volume), $T$ (temperature), $W$ (work), $\phi$ (entropy) etc. Any one of these can be expressed as a function of other two variables *e.g.*, $T = f(p, v)$, $T = g(p, \phi)$

As we shall see, these respectively give rise to the following results :

$$dT = \frac{\partial T}{\partial p} dp + \frac{\partial T}{\partial v} dv \quad ...(i)$$

$$dT = \frac{\partial T}{\partial p} dp + \frac{\partial T}{\partial \phi} d\phi \quad ...(ii)$$

Now, $\partial T/\partial p$ appearing in (*i*), has been obtained from $T$ as function of $p$ and $v$, treating $v$ as constant, we write it as $(\partial T/\partial p)_v$.

Similarly, $\partial T/\partial p$ occurring in (*ii*), is written as $(\partial T/\partial p)_\phi$.

**Example 5.9.** *If $u = f(r)$ and $x = r \cos \theta$, $y = r \sin \theta$, prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$ (*S.V.T.U., 2008 ; Rajasthan, 2006 ; U.P.T.U., 2005*)

**Solution.** We have $\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2} = f''(r) \cdot \left(\frac{\partial r}{\partial x}\right)^2 + f'(r) \cdot \frac{\partial^2 r}{\partial x^2}$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \left(\frac{\partial r}{\partial y}\right)^2 + f'(r) \cdot \frac{\partial^2 r}{\partial y^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right]$$

Now to find $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}$ etc., we write $r = (x^2 + y^2)^{1/2}$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{r} \quad \text{and} \quad \frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x \cdot \partial r/\partial x}{r^2} = \frac{r - x^2/r}{r^2} = \frac{y^2}{r^3}.$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{x} \text{ and } \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}.$$

Substituting the values of $\partial r/\partial x$ etc. in (*i*), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] + f'(r)\left[\frac{y^2}{r^3} + \frac{x^2}{r^3}\right] = f''(r) + \frac{1}{r}f'(r).$$

**Example 5.10.** *If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and $y = e^{r\cos\theta}\sin(r\sin\theta)$, prove that $\frac{\partial x}{\partial \theta} = -r\frac{\partial y}{\partial r}, \frac{\partial y}{\partial \theta} = r\frac{\partial x}{\partial r}$.*

*Hence show that $\frac{\partial^2 x}{\partial \theta^2} + r\frac{\partial x}{\partial r} + r^2\frac{\partial^2 x}{\partial r^2} = 0.$*

**Solution.** We have $x = e^{r\cos\theta}\cos(r\sin\theta)$

$$\therefore \quad \frac{\partial x}{\partial \theta} = e^{r\cos\theta}(-r\sin\theta) \cdot \cos(r\sin\theta) + e^{r\cos\theta}[-\sin(r\sin\theta)] \cdot r\cos\theta$$
$$= -re^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$
$$= -re^{r\cos\theta}\sin(\theta + r\sin\theta) \quad ...(i)$$

and $$\frac{\partial x}{\partial r} = e^{r\cos\theta} \cdot \cos\theta \cdot \cos(r\sin\theta) - e^{r\cos\theta}\sin\theta\,(r\sin\theta)\sin\theta$$
$$= e^{r\cos\theta}\cos(\theta + r\sin\theta) \quad ...(ii)$$

Similarly, $y = e^{r\cos\theta}\sin(r\sin\theta)$ gives

$$\frac{\partial y}{\partial \theta} = re^{r\cos\theta}\cos(\theta + r\sin\theta) \quad ...(iii)$$

and $$\frac{\partial y}{\partial r} = e^{r\cos\theta}\sin(\theta + r\sin\theta) \quad ...(iv)$$

From (*i*) and (*iv*), $\frac{\partial x}{\partial \theta} = -r\frac{\partial y}{\partial r}$ ...(*v*)

From (*ii*) and (*iii*), $\frac{\partial y}{\partial \theta} = r\frac{\partial x}{\partial r}$ ...(*vi*)

From (*v*), $$\frac{\partial^2 x}{\partial \theta^2} = -r\frac{\partial^2 y}{\partial \theta \partial r} = -r\frac{\partial^2 y}{\partial r \partial \theta}$$

From (*vi*), $$\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta} \quad \text{which gives} \quad \frac{\partial^2 x}{\partial r^2} = -\frac{1}{r^2}\frac{\partial y}{\partial \theta} + \frac{1}{r}\frac{\partial^2 y}{\partial r \partial \theta}$$

$$\therefore \quad \frac{\partial^2 x}{\partial \theta^2} + r\frac{\partial x}{\partial r} + r^2\frac{\partial^2 x}{\partial r^2} = -r\frac{\partial^2 y}{\partial r \partial \theta} + \frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial \theta} + r\frac{\partial^2 y}{\partial r \partial \theta} = 0.$$

PROBLEMS 5.3

1. If $x = r\cos\theta$, $y = r\sin\theta$, show that (i) $\dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$ (ii) $\dfrac{1}{r}\dfrac{\partial x}{\partial \theta} = r\dfrac{\partial \theta}{\partial x}$, (iii) $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 = 1$. (Burdwan, 2003)

2. If $x^2 = au + bv$, $y^2 = au - bv$, prove that $\left(\dfrac{\partial u}{\partial x}\right)_y \cdot \left(\dfrac{\partial x}{\partial u}\right)_v = \dfrac{1}{2} = \left(\dfrac{\partial v}{\partial y}\right)_x \cdot \left(\dfrac{\partial y}{\partial v}\right)_u$.

3. If $u = lx + my$, $v = mx - ly$, show that $\left(\dfrac{\partial u}{\partial x}\right)_y \cdot \left(\dfrac{\partial x}{\partial u}\right)_v = \dfrac{l^2}{l^2 + m^2}$, $\left(\dfrac{\partial y}{\partial v}\right)_x \left(\dfrac{\partial v}{\partial y}\right)_u = \dfrac{l^2 + m^2}{l^2}$.

4. If $x = r\cos\theta$, $y = r\sin\theta$, prove that

(i) $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right]$ (ii) $\dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2} = 0$ $(x \neq 0, y \neq 0)$.

5. If $z = x\log(x + r) - r$ where $r^2 = x^2 + y^2$, prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{1}{x + y}$, $\dfrac{\partial^3 z}{\partial x^3} = -\dfrac{x}{r^3}$. (Mumbai, 2008)

6. If $u = f(r)$ where $r = \sqrt{(x^2 + y^2 + z^2)}$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = f''(r) + \dfrac{2}{r}f'(r)$.

## 5.4 (1) HOMOGENEOUS FUNCTIONS

An expression of the form $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_ny^n$ in which every term is of the $n$th degree, is called a homogeneous function of degree $n$. This can be rewritten as

$$x^n[a_0 + a_1(y/x) + a_2(y/x)^2 + \dots + a_n(y/x)^n].$$

Thus *any function $f(x, y)$ which can be expressed in the form $x^n\phi(y/x)$, is called a* **homogeneous function** of degree $n$ in $x$ and $y$.

For instance, $x^3\cos(y/x)$ is a homogeneous function of degree 3, in $x$ and $y$.

In general, a function $f(x, y, z, t, \dots)$ is said to be a homogeneous function of degree $n$ in $x, y, z, t, \dots$, if it can be expressed in the form $x^n\phi(y/x, z/x, t/x, \dots)$.

**(2) Euler's theorem on homogeneous functions*.** *If $u$ be a homogeneous function of degree $n$ in $x$ and $y$, then*

$$\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.}$$

Since $u$ is a homogeneous function of degree $n$ in $x$ and $y$, therefore,

$$u = x^n f(y/x)$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot y\left(-\frac{1}{x^2}\right) = nx^{n-1}f\left(\frac{y}{x}\right) - yx^{n-2}f'\left(\frac{y}{x}\right)$$

and $$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\frac{1}{x} = x^{n-1}f'\left(\frac{y}{x}\right). \text{ Hence } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) = nu.$$

In general, if $u$ be a homogeneous function of degree $n$ in $x, y, z, t, \dots$, then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + t\frac{\partial u}{\partial t} \dots = nu.$$

**Example 5.11.** *Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u\log u$ where $\log u = (x^3 + y^3)/(3x + 4y)$.*

**Solution.** Since $$z = \log u = \frac{x^3 + y^3}{3x + 4y} = x^2 \cdot \frac{1 + (y/x)^3}{3 + 4(y/x)},$$

* After an enormously creative Swiss mathematician *Leonhard Euler* (1707–1783). He studied under John *Bernoulli* and became a professor of mathematics in St. Petersburg, Russia. Even after becoming totally blind in 1771, he contributed to almost all branches of mathematics.

$\therefore$ $z$ is a homogeneous function of degree 2 in $x$ and $y$.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \quad ...(i)$$

But $$\frac{\partial z}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y}$$

Hence $(i)$ becomes

$$x \cdot \frac{1}{u}\frac{\partial u}{\partial x} + y \cdot \frac{1}{u}\frac{\partial u}{\partial y} = 2\log u \quad \text{or} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \log u.$$

**Example 5.12.** *If* $u = \sin^{-1}\dfrac{x+2y+3z}{x^8+y^8+z^8}$, *find the value of* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$. *(U.P.T.U., 2004)*

**Solution.** Here $u$ is not a homogeneous function. We therefore, write

$$\omega = \sin u = \frac{x+2y+3z}{x^8+y^8+z^8} = x^{-7} \cdot \frac{1+2(y/x)+3(z/x)}{1+(y/x)^8+(z/x)^8}$$

Thus $\omega$ is a homogeneous function of degree $-7$ in $x, y, z$. Hence by Euler's theorem

$$x\frac{\partial \omega}{\partial x} + y\frac{\partial \omega}{\partial y} + z\frac{\partial \omega}{\partial z} = (-7)\,\omega \quad ...(i)$$

But $$\frac{\partial \omega}{\partial x} = \cos u\,\frac{\partial u}{\partial x}, \frac{\partial \omega}{\partial y} = \cos u\,\frac{\partial u}{\partial y}, \frac{\partial \omega}{\partial z} = \cos u\,\frac{\partial u}{\partial z}$$

$\therefore$ $(i)$ becomes $$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = -7\sin u \quad \text{or} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -7\tan u.$$

**Example 5.13.** *If* $u = \dfrac{x^3y^3z^3}{x^3+y^3+z^3} + \log\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$, *find the value of* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$. *(Mumbai, 2009)*

**Solution.** Let $$v = \frac{x^3y^3z^3}{x^3+y^3+z^3} \text{ and } w = \log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right) \quad ...(i)$$

so that $u = v + w$

Since $v = x^6\,\dfrac{(y/x)^3(z/x)^3}{1+(y/x)^3+(z/x)^3}$, therefore $v$ is a homogeneous function of degree 6 in $x, y, z$.

Hence by Euler's theorem $$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 6v \quad ...(ii)$$

Since $w = \log\left\{\dfrac{\dfrac{y}{x}+\dfrac{y}{x}.\dfrac{z}{x}+\dfrac{z}{x}}{1+\left(\dfrac{y}{x}\right)^2+\left(\dfrac{z}{x}\right)^2}\right\}$ therefore $w$ is a homogeneous function of degree zero in $x, y, z$.

Hence by Euler's theorem $$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0 \quad ...(iii)$$

Addint $(ii)$ and $(iii)$, we obtain

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right) + z\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right) = 6v$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 6\,\frac{x^3y^3z^3}{x^3+y^3+z^3} \quad \text{[By }(i)]$$

**Example 5.14.** *If $z$ is a homogeneous function of degree $n$ in $x$ and $y$, show that*

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial z^2}=n(n-1)\,z.$$ *(Anna, 2009 ; V.T.U., 2007 ; U.P.T.U., 2006)*

**Solution.** By Euler's theorem, $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=nz$ ...(*i*)

Differentiating (*i*) partially w.r.t. $x$, we get $x\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial z}{\partial x}+y\dfrac{\partial^2 z}{\partial x\partial y}=n\dfrac{\partial z}{\partial x}$

*i.e.*, $$x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x}$$ ...(*ii*)

Again differentiating (*i*) partially w.r.t. $y$, we get $x\dfrac{\partial^2 z}{\partial y\partial x}+\dfrac{\partial z}{\partial y}+y\dfrac{\partial^2 z}{\partial y^2}=n\dfrac{\partial z}{\partial y}$

*i.e.*, $$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y}$$ ...(*iii*)

Multiplying (*ii*) by $x$ and (*iii*) by $y$ and adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=n(n-1)\,z.$$ [By (*i*)]

**Example 5.15.** *If $u=\sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{1}{2}\tan u.$*

*(Rajasthan, 2006 ; Calicut, 2005)*

*and* $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}.$$ *(P.T.U., 2006)*

**Solution.** Here $u$ is not a homogeneous function but $z=\sin u=\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of degree 1/2 in $x$ and $y$.

$\therefore$ By Euler's theorem, $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=\dfrac{1}{2}z$

or $$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\frac{1}{2}\sin u$$

Thus $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$$ ...(*i*)

Differentiating (*i*) w.r.t. $x$ partially, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}\quad\text{or}\quad x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial x}$$ ...(*ii*)

Again differentiating (*i*) w.r.t. $y$ partially, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}\quad\text{or}\quad x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial x}$$ ...(*iii*)

Multiplying (*ii*) by $x$ and (*iii*) by $y$ and adding, we obtain

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(\frac{1}{2}\tan u\right)$$ [By (*i*)]

$$=\frac{1}{4}\frac{\sin u}{\cos^3 u}-\frac{1}{2}\frac{\sin u}{\cos u}=-\frac{\sin u\,(2\cos^2 u-1)}{4\cos^3 u}$$

Hence $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}.$$

## PROBLEMS 5.4

1. Verify Euler's theorem, when (*i*) $f(x, y) = ax^2 + 2hxy + by^2$
   (*ii*) $f(x, y) = x^2 (x^2 - y^2)^3/(x^2 + y^2)^3$.
   (*iii*) $f(x, y) = 3x^2 yz + 5xy^2z + 4z^4$ (*J.N.T.U., 1999*)
2. If $u = \sin^{-1} \dfrac{x}{y} + \tan^{-1} \dfrac{y}{x}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$. (*Hazaribagh, 2009 ; Osmania, 2003 S*)
3. If $u = \sin^{-1} \dfrac{x^2 + y^2}{x + y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$ (*Bhopal, 2009 ; V.T.U., 2003*)
4. If $\sin u = \dfrac{x^2y^2}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3 \tan u$. (*Kottayam, 2005 ; V.T.U., 2003 S*)
5. If $u = \cos^{-1} \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2} \cot u$. (*V.T.U., 2004*)
6. Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u \log u$, where $u = e^{x^2 + y^2}$ (*P.T.U., 2010*)
7. If $z = f(y/x) + \sqrt{(x^2 + y^2)}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \sqrt{x^2 + y^2}$. (*Mumbai, 2008*)
8. If $u = \dfrac{x}{y + z} + \dfrac{y}{z + x} + \dfrac{z}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$. (*V.T.U., 2000 S*)
9. If $\sin u = \dfrac{x + 2y + 3z}{\sqrt{(x^8 + y^8 + z^8)}}$, show that $xu_x + yu_y + zu_z + 3 \tan u = 0$. (*S.V.T.U., 2009 ; U.T.U., 2009*)
10. If $z = x\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right)$, prove that $x^2 \dfrac{\partial^2 z}{\partial x^2} + 2xy \dfrac{\partial^2 z}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 0$. (*S.V.T.U., 2009 ; U.P.T.U., 2006*)
11. If $u = \tan^{-1} \dfrac{x^3 + y^3}{x + y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$. (*P.T.U., 2009 S*)
    and $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 2 \cos 3u \sin u$. (*Mumbai, 2009 ; Bhopal, 2008 ; S.V.T.U., 2007*)
12. Given $z = x^n f_1(y/x) + y^{-n} f_2(x/y)$, prove that $x^2 \dfrac{\partial^2 z}{\partial x^2} + 2xy \dfrac{\partial^2 z}{\partial x \partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = n^2 z$.
    (*Kurukshetra, 2009 S ; Rohtak, 2003*)
13. If $u = x^2 \tan^{-1} (y/x) - y^2 \tan^{-1} (x/y)$, evaluate $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2}$. (*U.T.U., 2009 ; Hissar, 2005 S*)
14. If $u = \tan^{-1} (y^2/x)$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = -\sin^2 u \,.\, \sin 2u$. (*Bhillai, 2005 ; P.T.U., 2005*)
15. If $u = \operatorname{cosec}^{-1} \left(\dfrac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}\right)^{1/2}$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{12}\left(\dfrac{13}{12} + \dfrac{\tan^2 u}{12}\right)$
    (*Mumbai, 2008 ; Rohtak, 2006 S*)

## 5.5 (1) TOTAL DERIVATIVE

If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$, then we can express $u$ as a function of $t$ alone by substituting the values of $x$ and $y$ in $f(x, y)$. Thus we can find the ordinary derivative $du/dt$ which is called the *total derivative* of $u$ to distinguish it from the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$.

Now to find $du/dt$ without actually substituting the values of $x$ and $y$ in $f(x, y)$, we establish the following **Chain rule :**

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \qquad ...(i)$$

***Proof.*** We have $u = f(x, y)$

Giving increment $\delta t$ to $t$, let the corresponding increments of $x, y$ and $u$ be $\delta x$, $\delta y$ and $\delta u$ respectively.

Then $\quad u + \delta u = f(x + \delta x, y + \delta y)$

Subtracting, $\quad \delta u = f(x + \delta x, y + \delta y) - f(x, y)$

$$= \{f(x + \delta x, y + \delta y) - f(x, y + \delta y)\} + \{f(x, y + \delta y) - f(x, y)\}$$

$$\therefore \quad \frac{\delta u}{\delta t} = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t}$$

Taking limits as $\delta t \to 0$, $\delta x$ and $\delta y$ also $\to 0$, we have

$$\frac{du}{dt} = \underset{\delta y \to 0}{\text{Lt}} \left[ \underset{\delta y \to 0}{\text{Lt}} \left\{ \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \right\} \right] \frac{dx}{dt} + \underset{\delta y \to 0}{\text{Lt}} \left\{ \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \right\} \frac{dy}{dt}$$

$$= \underset{\delta y \to 0}{\text{Lt}} \left\{ \frac{\partial f(x, y + \delta y)}{\delta y} \right\} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt}$$

[Supposing $\partial f(x, y)/\partial x$ to be a continuous function of $y$]

$$= \frac{\partial f(x, y)}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt} \text{ which is the desired formula.}$$

**Cor.** Taking $t = x$, (*i*) becomes, $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx}$ ...(*ii*)

**Obs.** If $u = f(x, y, z)$, where $x, y, z$ are all functions of a variable $t$, then **Chain rule** is

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt} \quad \text{...(iii)}$$

**(2) Differentiation of implicit functions.** If $f(x, y) = c$ be an implicit relation between $x$ and $y$ which defines as a differentiable function of $x$, then (*ii*) becomes

$$0 = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

This gives the *important formula* $\dfrac{\mathbf{dy}}{\mathbf{dx}} = -\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}} \Big/ \dfrac{\partial \mathbf{f}}{\partial \mathbf{y}}$ $\quad \left[\dfrac{\partial f}{\partial y} \neq 0\right]$

*for the first differential coefficient of an implicit function.*

**Example 5.16.** *Given $u = \sin(x/y)$, $x = e^t$ and $y = t^2$, find $du/dt$ as a function of $t$. Verify your result by direct substitution.*

**Solution.** We have $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dt} = \left(\cos \dfrac{x}{y}\right) \dfrac{1}{y} \cdot e^t + \left(\cos \dfrac{x}{y}\right)\left(-\dfrac{x}{y^2}\right) 2t$

$$= \cos(e^t/t^2) \cdot e^t/t^2 - 2\cos(e^t/t^2) \cdot e^t/t^3 = \{(t-2)/t^3\} e^t \cos(e^t/t^2)$$

Also $u = \sin(x/y) = \sin(e^t/t^2)$

$$\therefore \quad \frac{du}{dt} = \cos\left(\frac{e^t}{t^2}\right) \cdot \frac{t^2 e^t - e^t \cdot 2t}{t^4} = \frac{t-2}{t^3} e^t \cos\left(\frac{e^t}{t^2}\right) \text{ as before.}$$

**Example 5.17.** *If $x$ increases at the rate of 2 cm/sec at the instant when $x = 3$ cm. and $y = 1$ cm., at what rate must $y$ be changing in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing?*

**Solution.** Let $\quad u = 2xy - 3x^2y$, so that

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} = (2y - 6xy)\frac{dx}{dt} + (2x - 3x^2)\frac{dy}{dt} \quad \text{...(i)}$$

when $x = 3$ and $y = 1$, $dx/dt = 2$, and $u$ is neither increasing nor decreasing, *i.e.*, $du/dt = 0$.

$\therefore$ (*i*) becomes $\quad 0 = (2 - 6 \times 3)\, 2 + (2 \times 3 - 3 \times 9)\dfrac{dy}{dt}$

or $\quad \dfrac{dy}{dt} = -\dfrac{32}{21}$ cm/sec. Thus $y$ is decreasing at the rate of 32/21 cm/sec.

**Example 5.18.** *If $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$, find $du/dx$.* (V.T.U., 2009)

**Solution.** From $f(x, y) = x^3 + y^3 + 3xy - 1$, we have

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = -\frac{3x^2 + 3y}{3y^2 + 3x} = -\frac{x^2 + y}{y^2 + x} \quad ...(i)$$

Also $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = (1 \cdot \log xy + x \cdot 1/x) + (x/y) \cdot dy/dx.$$

Hence $$du/dx = 1 + \log xy - x(x^2 + y)/y(y^2 + x) \quad \text{[By } (i)]$$

**Example 5.19.** *If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$* (U.P.T.U., 2005)

**Solution.** Let $$v = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y} \quad \text{and} \quad w = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z} \quad ...(i)$$

so that $u = u(v, w)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial x} = \frac{\partial u}{\partial v}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial w}\left(-\frac{1}{x^2}\right) \quad \text{[Using } (i)]$$

or $$x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} \quad ...(ii)$$

Also $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial y} = \frac{\partial u}{\partial v}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial w}(0) \quad \text{[Using } (i)]$$

or $$y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \quad ...(iii)$$

Similarly $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\,\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\,\frac{\partial w}{\partial z} = \frac{\partial u}{\partial v}(0) + \frac{\partial u}{\partial w}\left(\frac{1}{z^2}\right) \quad \text{[Using } (i)]$$

or $$z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial w} \quad ...(iv)$$

Adding $(ii)$, $(iii)$ and $(iv)$, we have

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$$

**Example 5.20.** *Formula for the second differential coefficient of an implicit function. If $f(x, y) = 0$, show that*

$$\frac{d^2y}{dx^2} = -\frac{q^2r - 2pqs + p^2t}{q^3}$$ (Kurukshetra, 2006)

**Solution.** We have $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{p}{q} \quad ...(i)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}\left(\frac{p}{q}\right) = -\frac{q(dp/dx) - p(dq/dx)}{q^2} \quad ...(ii)$$

Using the notations : $r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial p}{\partial x}$, $s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial q}{\partial x}$, $t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial q}{\partial y}$,

we have $$\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} \cdot \frac{dy}{dx} = r + s\,(-p/q) = -\frac{qr - ps}{q}$$

and $$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} \cdot \frac{dy}{dx} = s + t\,(-p/q) = \frac{qs - pt}{q}$$

Substituting the values of $dp/dx$ and $dq/dx$ in (*ii*), we get

$$\frac{d^2y}{dx^2} = -\frac{1}{q^2}\left[q\left(\frac{qr-ps}{q}\right) - p\left(\frac{qs-pt}{q}\right)\right] = -\frac{q^2r - 2pqs + p^2t}{q^3}.$$

## PROBLEMS 5.5

1. If $z = u^2 + v^2$ and $u = at^2, v = 2at$, find $dz/dt$. (*P.T.U., 2005*)
2. If $u = \tan^{-1}(y/x)$ where $x = e^t - e^{-t}$, and $y = e^t + e^{-t}$, find $du/dt$. (*V.T.U., 2003*)
3. Find the value of $\dfrac{du}{dt}$ given $u = y^2 - 4ax, x = at^2, y = 2at$. (*Anna, 2009*)
4. At a given instant the sides of a rectangle are 4 ft. and 3 ft. respectively and they are increasing at the rate of 1.5 ft./sec. and 0.5 ft./sec. respectively, find the rate at which the area is increasing at that instant.
5. If $z = 2xy^2 - 3x^2y$ and if $x$ increases at the rate of 2 cm. per second and it passes through the value $x = 3$ cm., show that if $y$ is passing through the value $y = 1$ cm., $y$ must be decreasing at the rate of $2\frac{2}{15}$ cm. per second, in order that $z$ shall remain constant.
6. If $u = x^2 + y^2 + z^2$ and $x = e^{2t}, y = e^{2t}\cos 3t, z = e^{2t}\sin 3t$. Find $\dfrac{du}{dt}$ as a total derivative and verify the result by direct substitution.
7. If $\phi(cx - az, cy - bz) = 0$, show that $\dfrac{a\partial z}{\partial x} + \dfrac{b\partial z}{\partial y} = c$.
8. If $f(x, y) = 0, \phi(y, z) = 0$, show that $\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial \phi}{\partial z}\cdot\dfrac{dz}{dx} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial \phi}{\partial y}$.
9. If the curves $f(x, y) = 0$ and $\phi(y, z) = 0$ touch, show that at the point of contact, $\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial \phi}{\partial y} = \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial \varphi}{\partial \gamma}$.
10. If $f(x, y) = 0$, show that $\left(\dfrac{\partial f}{\partial y}\right)^3\dfrac{d^2y}{dx^2} = 2\left(\dfrac{\partial f}{\partial x}\right)\left(\dfrac{\partial f}{\partial y}\right)\left(\dfrac{\partial^2 f}{\partial x\partial y}\right) - \left(\dfrac{\partial f}{dy}\right)^2\dfrac{\partial^2 f}{\partial x^2} - \left(\dfrac{\partial f}{\partial x}\right)^2\left(\dfrac{\partial^2 f}{\partial y^2}\right)$.

## 5.6 CHANGE OF VARIABLES

If $\quad u = f(x, y)$ ...(1)

where $\quad x = \phi(s, t)$ and $y = \Psi(s, t)$ ...(2)

it is often necessary to change expressions involving $u, x, y, \partial u/\partial x, \partial u/\partial y$ etc. to expressions involving $u, s, t, \partial u/\partial s, \partial u/\partial t$ etc.

The necessary formulae for the change of variables are easily obtained. If $t$ is regarded as a constant, then $x, y, u$ will be functions of $s$ alone. Therefore, by (*i*) of page 208, we have

$$\frac{\partial \mathbf{u}}{\partial \mathbf{s}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\cdot\frac{\partial \mathbf{x}}{\partial \mathbf{s}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}}\cdot\frac{\partial \mathbf{y}}{\partial \mathbf{s}} \qquad ...(3)$$

where the ordinary derivatives have been replaced by the partial derivatives because $x, y$ are functions of two variables $s$ and $t$.

$\because$ Similarly, regarding $s$ as constant, we obtain $\dfrac{\partial \mathbf{u}}{\partial \mathbf{t}} = \dfrac{\partial \mathbf{u}}{\partial \mathbf{x}}\cdot\dfrac{\partial \mathbf{x}}{\partial \mathbf{t}} + \dfrac{\partial \mathbf{u}}{\partial \mathbf{y}}\cdot\dfrac{\partial \mathbf{y}}{\partial \mathbf{t}}$ ...(4)

On solving (3) and (4) as simultaneous equations in $\partial u/\partial x$ and $\partial u/\partial y$, we get their values in terms of $\partial u/\partial s$, $\partial u/\partial t, u, s, t$.

If instead of the equations (2), $s$ and $t$ are given in terms of $x$ and $y$, say: $s = \xi(x, y)$ and $t = \eta(x, y)$, ...(5)

then it is easier to use the formulae $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial s}{\partial x} + \dfrac{\partial u}{\partial t}\cdot\dfrac{\partial t}{\partial x}$ ...(6)

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y} \qquad ...(7)$$

The higher derivatives of $u$ can be found by repeated application of formulae (3) and (4) or of (6) and (7).

**Example 5.21.** *If $u = F(x - y, y - z, z - x)$, prove that*

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

*(V.T.U., 2010 ; U.T.U. 2009 ; U.P.T.U., 2003)*

**Solution.** Put $x - y = r$, $y - z = s$ and $z - x = t$, so that $u = f(r, s, t)$.

$$\therefore \quad \frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$=\frac{\partial u}{\partial r}.(1)+\frac{\partial x}{\partial s}.(0)+\frac{\partial u}{\partial t}.(-1)=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t} \qquad ...(i)$$

Similarly, $$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}=-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s} \qquad ...(ii)$$

and $$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}=-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t} \qquad ...(iii)$$

Adding $(i)$, $(ii)$ and $(iii)$, we get the required result.

**Example 5.22.** *If $z = f(x, y)$ and $x = e^u \cos v$, $y = e^u \sin v$, prove that $x\frac{\partial z}{\partial v}+y\frac{\partial z}{\partial u}=e^{2u}\frac{\partial z}{\partial y}$*

*and* $$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial u}{\partial v}\right)^2\right]$$

*(Mumbai, 2009)*

**Solution.** We have $\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$

$$=\frac{\partial z}{\partial x}(e^u\cos v)+\frac{\partial z}{\partial y}(e^u\sin v) \qquad ...(i)$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$=\frac{\partial z}{\partial x}(-e^u\sin v)+\frac{\partial x}{\partial y}(e^u\cos v) \qquad ...(ii)$$

$$\therefore \quad x\frac{\partial x}{\partial v}+y\frac{\partial z}{\partial u}=(e^u\cos v)\left[-e^u\sin v\frac{\partial z}{\partial x}+e^u\cos v\frac{\partial z}{\partial y}\right]+(e^u\sin v)\left[e^u\cos v\frac{\partial z}{\partial x}+e^u\sin v\frac{\partial z}{\partial y}\right]$$

$$=(e^{2u}\cos^2 v+e^{2u}\sin^2 v)\frac{\partial z}{\partial y}=e^{2u}\frac{\partial z}{\partial y}$$

Now squaring $(i)$ and $(ii)$ and adding, we get

$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2=e^{2u}\left(\cos v\frac{\partial z}{\partial x}+\sin v\frac{\partial z}{\partial y}\right)^2+e^{2u}\left(-\sin v\frac{\partial z}{\partial x}+\cos v\frac{\partial z}{\partial y}\right)^2$$

or $$e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]=\cos^2 v\left(\frac{\partial z}{\partial x}\right)^2+\sin^2 v\left(\frac{\partial z}{\partial y}\right)^2+2\sin v\cos v\,\frac{\delta z}{\delta z}\frac{\delta z}{\delta y}$$

$$+\sin^2 v\left(\frac{\partial z}{\partial x}\right)^2+\cos^2 v\left(\frac{\partial z}{\partial y}\right)^2-2\sin v\cos v\,\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$=(\cos^2 v+\sin^2 v)\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

Hence $$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right].$$

**Example 5.23.** *If $x + y = 2e^{\theta}\cos\phi$ and $x - y = 2ie^{\theta}\sin\phi$, show that*

$$\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial\phi^2}=4xy\frac{\partial^2 u}{\partial x\partial y}$$ *(Nagpur, 2009 ; U.P.T.U., 2002)*

**Solution.** We have $x = e^{\theta}(\cos\phi + i\sin\phi) = e^{\theta}\,.\,e^{i\phi}$ [See p. 205]

and $y = e^{\theta}(\cos\phi - i\sin\phi) = e^{\theta}\,.\,e^{-i\phi}$

Here $u$ is a composite function of $\theta$ and $\phi$.

$$\therefore \quad \frac{\partial u}{\partial\theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\theta}$$

$$=\frac{\partial u}{\partial x}\cdot(e^{\theta}\,.\,e^{i\phi})+\frac{\partial u}{\partial y}(e^{\theta}\,.\,e^{-i\phi})=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$

or $$\frac{\partial}{\partial\theta}=x\frac{\partial}{\partial u}+y\frac{\partial}{\partial y} \quad ...(i)$$

Also $$\frac{\partial u}{\partial\phi}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\phi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\phi}=\frac{\partial u}{\partial x}\cdot(e^{\theta}\,.\,ie^{i\phi})+\frac{\partial u}{\partial y}(e^{\theta}\,.-ie^{-i\phi})=ix\frac{\partial u}{\partial x}-iy\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial\phi}=ix\frac{\partial}{\partial x}-iy\frac{\partial}{\partial y} \quad ...(ii)$$

Using the operator $(i)$, we have

$$\frac{\partial^2 u}{\partial\theta^2}=\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial\theta}\right)=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)+x\frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial y}\right)+y\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x}\right)+y\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)$$

$$=x\left(x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\right)+xy\frac{\partial^2 u}{\partial x\partial y}+yx\frac{\partial^2 u}{\partial y\partial x}+y\left(y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\right)$$

$$=x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} \quad ...(iii)$$

Similarly using $(ii)$, $$\frac{\partial^2 u}{\partial\phi^2}=\frac{\partial}{\partial\phi}\left(\frac{\partial u}{\partial\phi}\right)=\left(ix\frac{\partial}{\partial x}-iy\frac{\partial}{\partial y}\right)\left(ix\frac{\partial u}{\partial x}-iy\frac{\partial u}{\partial y}\right)$$

$$=-x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}-y^2\frac{\partial^2 u}{\partial y^2}-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y} \quad ...(iv)$$

Adding $(iii)$ and $(iv)$, we get $$\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial\phi^2}=4xy\frac{\partial^2 u}{\partial x\partial y}$$

**Example 5.24.** *Transform the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ into polar coordinates.* *(P.T.U., 2010)*

**Solution.** We have $x = r\cos\theta$, $y = r\sin\theta$ and $r=\sqrt{(x^2+y^2)}$, $\theta=\tan^{-1}(y/x)$

$$\frac{\partial r}{\partial x}=\frac{x}{\sqrt{(x^2+y^2)}}=\cos\theta \text{ and } \frac{\partial\theta}{\partial x}=-\frac{y}{x^2+y^2}=-\frac{\sin\theta}{r}$$

Thus, $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}=\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}$$

*i.e.,* $$\frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\qquad\text{Similarly, }\frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$=\cos^2\theta\frac{\partial^2u}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2u}{\partial r\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}\qquad ...(i)$$

and $$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$=\sin^2\theta\frac{\partial^2u}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2u}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial r}\qquad ...(ii)$$

Adding (*i*) and (*ii*), we get $\dfrac{\partial^2u}{\partial x^2}+\dfrac{\partial^2u}{\partial y^2}=\dfrac{\partial^2u}{\partial r^2}+\dfrac{1}{r^2}\dfrac{\partial^2u}{\partial\theta^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}$.

Hence the transformed equation is $\dfrac{\partial^2u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2u}{\partial\theta^2}=0.$

## PROBLEMS 5.6

1. If $z=f(x,y)$ and $x=e^u+e^{-v}$, $y=e^{-u}-e^v$, prove that $\dfrac{\partial z}{\partial u}-\dfrac{\partial z}{\partial v}=x\dfrac{\partial z}{\partial x}-y\dfrac{\partial z}{\partial y}$. (*V.T.U., 2006*)
2. If $u=f(r,s)$, $r=x+at$, $s=y+bt$ and $x, y, t$ are independent variables, show that $\dfrac{\partial u}{\partial t}=a\dfrac{\partial u}{\partial x}+b\dfrac{\partial u}{\partial y}$.
3. If $\phi(z/x^3, y/x)=0$, prove that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=3z$. (*Mumbai, 2007*)
4. If $u=f(x,y)$ and $x=r\cos\theta$, $y=r\sin\theta$, prove that $\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2=\left(\dfrac{\partial u}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial\theta}\right)^2$.
(*V.T.U., 2010 ; Madras 2006 ; Rohtak, 2005*)
5. If $u=f(2x-3y, 3y-4z, 4z-2x)$, prove that $\dfrac{1}{2}\dfrac{\partial u}{\partial x}+\dfrac{1}{3}\dfrac{\partial u}{\partial y}+\dfrac{1}{4}\dfrac{\partial u}{\partial z}=0.$ (*U.P.T.U., 2006 ; Raipur, 2005*)
6. If $u=f(e^{y-z}, e^{z-x}, e^{x-y})$, prove that $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0$ (*Mumbai, 2008 S*)
7. If $u=f(r,s,t)$ and $r=x/y$, $s=y/z$, $t=z/x$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=0.$ (*Anna, 2009 ; Kurukshetra, 2006*)
8. If $x=u+v+w$, $y=vw+wu+uv$, $z=uvw$ and $F$ is a function of $x, y, z$, show that
$$u\frac{\partial F}{\partial u}+v\frac{\partial F}{\partial v}+w\frac{\partial F}{\partial w}=x\frac{\partial F}{\partial x}+2y\frac{\partial F}{\partial y}+3z\frac{\partial F}{\partial z}.$$
9. Given that $u(x,y,z)=f(x^2+y^2+z^2)$ where $x=r\cos\theta\cos\phi$, $y=r\cos\theta\sin\phi$ and $z=r\sin\theta$, find $\dfrac{\partial u}{\partial\theta}$ and $\dfrac{\partial u}{\partial\phi}$.
10. If the three thermodynamic variables $P, V, T$ are connected by a relation $f(P,V,T)=0$, show that
$$\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial T}{\partial V}\right)_P\left(\frac{\partial V}{\partial P}\right)_T=-1.$$
11. If by the substitution $u=x^2-y^2$, $v=2xy$, $f(x,y)=\theta(u,v)$, show that
$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=4(x^2+y^2)\left(\frac{\partial^2\theta}{\partial u^2}+\frac{\partial^2\theta}{\partial v^2}\right).$$ (*Anna, 2003*)
12. Transform $\dfrac{\partial^2 z}{\partial x^2}+2xy^2\dfrac{\partial z}{\partial x}+2(y-y^3)\dfrac{\partial z}{\partial y}+x^2y^2z=0$ by the substitution $x=uv$, $y=1/v$. Hence show that $z$ is the same function of $u$ and $v$ as of $x$ and $y$.

## 5.7 (1) JACOBIANS

If $u$ and $v$ are functions of two independent variables $x$ and $y$, then the determinant

$$\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$ is called the *Jacobian** of $u, v$ with respect to $x, y$

and is written as $\dfrac{\partial(u, v)}{\partial(x, y)}$ or $J\left(\dfrac{u, v}{x, y}\right)$.

Similarly the Jacobian of $u, v, w$ with respect to $x, y, z$ is

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

Likewise, we can define Jacobians of four or more variables. An important application of Jacobians is in connection with the change of variables in multiple integrals (§ 7.7).

**(2) Properties of Jacobians.** We give below two of the important properties of Jacobians. For simplicity, the properties are stated in terms of two variables only, but these are evidently true in general.

*I.* If $J = \partial(u, v)/\partial(x, y)$ and $J' = \partial(x, y)/\partial(u, v)$ then $JJ' = 1$.

Let $u = f(x, y)$ and $v = g(x, y)$.

Suppose, on solving for $x$ and $y$, we get $x = \phi(u, v)$ and $y = \psi(u, v)$.

Then
$$\left.\begin{aligned} \frac{\partial u}{\partial u} &= 1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u}, \\ \frac{\partial u}{\partial v} &= 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v}, \\ \frac{\partial v}{\partial u} &= 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u}, \\ \text{and} \quad \frac{\partial v}{\partial v} &= 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v}, \end{aligned}\right\} \quad ...(i)$$

$$\therefore \quad JJ' = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial y/\partial x & \partial v/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix}$$

(Interchanging rows and columns of the 2nd determinant).

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \qquad \text{[By virtue of } (i)]$$

*II.* **Chain rule for Jacobians.** *If $u$, $v$ are functions of $r$, $s$ and $r$, $s$ are functions of $x$, $y$, then*

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}.$$

$$\frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial r & \partial u/\partial s \\ \partial v/\partial r & \partial v/\partial s \end{vmatrix} \times \begin{vmatrix} \partial r/\partial x & \partial s/\partial x \\ \partial r/\partial y & \partial s/\partial y \end{vmatrix}$$

[Interchanging rows and columns of the 2nd det.]

$$= \begin{vmatrix} \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial x} & \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial y} \\ \dfrac{\partial v}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial v}{\partial s}\cdot\dfrac{\partial s}{\partial x} & \dfrac{\partial v}{\partial r}\cdot\dfrac{\partial r}{\partial y} + \dfrac{\partial v}{\partial s}\cdot\dfrac{\partial s}{\partial y} \end{vmatrix} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}.$$

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* Called after the German mathematician *Carl Gustav Jacob Jacobi* (1804–1851), who made significant contributions to mechanics, partial differential equations, astronomy, elliptic functions and the calculus of variations.

**Example 5.25.** *(i) In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, show that*

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r.$$ *(U.P.T.U., 2006 ; V.T.U., 2004 ; Andhra, 2000)*

*(ii) In cylindrical coordinates (Fig. 8.28), $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$, show that*

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

*(iii) In spherical polar coordidnates (Fig. 8.29), $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, show that*

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta.$$ *(Anna, 2009 ; Hazaribagh, 2009 ; Rohtak, 2003)*

**Solution.** *(i)* We have $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial\theta} = -r\sin\theta$ and $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial\theta} = -r\cos\theta$

$$\therefore \quad \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

*(ii)* We have $\frac{\partial x}{\partial\rho} = \cos\phi$, $\frac{\partial x}{\partial\phi} = -\rho\sin\phi$, $\frac{\partial x}{\partial z} = 0$,

$\frac{\partial y}{\partial\rho} = \sin\phi$, $\frac{\partial y}{\partial\phi} = \rho\cos\phi$, $\frac{\partial y}{\partial z} = 0$ and $\frac{\partial z}{\partial\rho} = 0$, $\frac{\partial z}{\partial\phi} = 0$, $\frac{\partial z}{\partial z} = 1$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \cos\phi & -\rho\sin\theta & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho.$$

*(iii)* We have $\frac{\partial x}{\partial r} = \sin\theta\cos\theta$, $\frac{\partial x}{\partial\theta} = r\cos\theta\cos\phi$, $\frac{\partial x}{\partial\phi} = -r\sin\theta\sin\phi$,

$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$, $\frac{\partial y}{\partial\theta} = r\cos\theta\sin\phi$, $\frac{\partial y}{\partial\phi} = r\sin\theta\cos\phi$,

and $\frac{\partial z}{\partial r} = \cos\theta$, $\frac{\partial z}{\partial\theta} = -r\sin\theta$, $\frac{\partial z}{\partial\phi} = 0$.

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

**Example 5.26.** *If $y_1 = \frac{x_2x_3}{x_1}$, $y_2 = \frac{x_3x_1}{x_2}$, $y_3 = \frac{x_1x_2}{x_3}$, show that the Jacobian of $y_1, y_2, y_3$ with respect to $x_1, x_2, x_3$ is 4.* *(U.P.T.U., 2006)*

**Solution.** We have $\frac{\partial y_1}{\partial x_1} = -\frac{x_2x_3}{x_1^2}$, $\frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}$, $\frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$

$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}$, $\frac{\partial y_2}{\partial x_2} = -\frac{x_3x_1}{x_2^2}$, $\frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$

and $\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}$, $\frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}$, $\frac{\partial y_3}{\partial x_3} = -\frac{x_1x_2}{x_3^2}$

$$\therefore \quad \frac{\partial(y_1y_2y_3)}{\partial(x_1x_2x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} \begin{vmatrix} \frac{-x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3x_1}{x_2^2} & \frac{x_1}{x_1} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix} = -\frac{x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1) = 0 + 2 + 2 = 4.$$

**Example 5.27.** *In $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$, evaluate $\partial(u, v, w)/\partial(x, y, z)$ at $(1, -1, 0)$.* (V.T.U., 2006)

**Solution.** $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

$\therefore$ At the point $(1, -1, 0)$ $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 4(-1+6) = 20.$$

**Example 5.28.** *If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial(u, v)}{\partial(r, \theta)}$.* (V.T.U., 2009 ; Madras, 2006)

**Solution.** We have $$\frac{\partial(u, v)}{\partial(r, \theta)} = \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(r, \theta)}$$

Since $u = x^2 - y^2$, $u = 2xy$

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2) \quad \text{...(ii)}$$

Since $x = r\cos\theta$, $y = r\sin\theta$,

$$\therefore \quad \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \quad \text{...(iii)}$$

Hence, $$\frac{\partial(u, v)}{\partial(r, \theta)} = 4(x^2 + y^2).\, r = 4(r^2\cos^2\theta + r^2\sin^2\theta).\, r = 4r^3 \quad \text{[Using (ii) \& (iii)]}$$

**(3) Jacobian of Implicit functions.** *If $u_1, u_2, u_3$ instead of being given explicitly in terms $x_1, x_2, x_3$ be connected with them equations such as*

$$f_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0,\ f_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0,\ f_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0, \text{ then}$$

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} \div \frac{\partial(f_1, f_2, f_3)}{\partial(u_1, u_2, u_3)}$$

**Obs.** This result can be easily generalised. It bears analogy to the result $\dfrac{dy}{dx} = -\dfrac{\partial f}{\partial x}\Big/\dfrac{\partial f}{\partial y}$, where $x, y$ are connected by the relation $f(x, y) = 0$.

**Example 5.29.** *If $u = x, y, z$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, find $\partial(x, y, z)/\partial(u, v, w)$.* (U.P.T.U., 2003)

**Solution.** Let $f_1 = u - x\,y\,z$, $f_2 = v - x^2 - y^2 - z^2$, $f_3 = w - x - y - z$.

We have $$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} \div \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} \quad ...(i)$$

Now, $$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z \\ \partial f_3/\partial x & \partial f_3/\partial y & \partial f_3/\partial z \end{vmatrix} = \begin{vmatrix} -yz & -xz & -xy \\ -2x & -2y & -2z \\ -1 & -1 & -1 \end{vmatrix}$$

$$= -2(x-y)(y-z)(z-x) \quad ...(ii)$$

and $$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad ...(iii)$$

Substituting values from (*ii*) and (*iii*) in (*i*), we get

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1) \times 1/[-2(x-y)(y-z)(z-x)] = 1/2(x-y)(y-z)(z-x).$$

**(4) Functional relationship.** *If $u_1, u_2, u_3$ be functions of $x_1, x_2, x_3$ then the necessary and sufficient condition for the existence of a functional relationship of the form $f(u_1, u_2, u_3) = 0$, is*

$$J\left(\frac{u_1, u_2, u_3}{x_1, x_2, x_3}\right) = 0.$$

**Example 5.30.** *If $u = x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}$, $v = \sin^{-1} x + \sin^{-1} y$, show that u, v are functionally related and find the relationship.* (*Kurukshetra, 2006*)

**Solution.** We have $$\frac{\partial u}{\partial x} = \sqrt{(1-y^2)} - \frac{xy}{\sqrt{(1-x^2)}}, \frac{\partial u}{\partial y} = \frac{-xy}{\sqrt{(1-y^2)}} + \sqrt{(1-x^2)}$$

and $$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{(1-x^2)}}, \frac{\partial v}{\partial y} = \frac{1}{\sqrt{(1-y^2)}}$$

$\therefore$ $$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sqrt{(1-y^2)} - \frac{xy}{\sqrt{(1-x^2)}}, & \sqrt{(1-x^2)} - \frac{xy}{\sqrt{(1-y^2)}} \\ \frac{1}{\sqrt{(1-x^2)}}, & \frac{1}{\sqrt{(1-y^2)}} \end{vmatrix}$$

$$= 1 - \frac{xy}{\sqrt{[(1-x^2)(1-y^2)]}} - 1 + \frac{xy}{\sqrt{[(1-x^2)(1-y^2)]}} = 0$$

Hence $u$ and $v$ are functionally related *i.e.*, they are not independent.

We have $$v = \sin^{-1} x + \sin^{-1} y = \sin^{-1} [x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}]$$

*i.e.*, $$u = \sin v$$

which is the required relationship between $u$ and $v$.

## PROBLEMS 5.7

1. If $x = r\cos\theta$, $y = r\sin\theta$, evaluate $\partial(r, \theta)/\partial(x, y)$, $\partial(x, y)/\partial(r, \theta)$ and prove that $[\partial(r, \theta)/\partial(x, y)] \cdot [\partial(x, y)/\partial(r, \theta)] = 1$. (*V.T.U., 2010*)
2. If $x = u(1-v)$, $y = uv$, prove that $JJ' = 1$. (*V.T.U., 2000 S*)
3. If $x = a\cosh\xi\cos\eta$, $y = a\sinh\xi\sin\eta$, show that $\partial(x, y)/\partial(\xi, \eta) = \frac{1}{2}a^2(\cosh 2\xi - \cos 2\eta)$ (*S.V.T.U., 2007*)
4. If $x = e^u \sec v$, $y = e^u \tan v$, find $J = \partial(u, v)/\partial(x, y)$, $J' = \partial(x, y)/\partial(u, v)$. Hence show $JJ' = 1$. (*V.T.U., 2007 S*)
5. If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ where $x = r\cos\theta$, $y = r\sin\theta$, show that $\frac{\partial(u, v)}{\partial(r, \theta)} = 6r^3 \sin 2\theta$.
6. If $u = x^2 + y^2 + z^2$, $v = xy + yz + zx$, $w = x + y + z$, find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$. (*U.T.U., 2009 ; V.T.U., 2008*)

7. If $F = xu + v - y$, $G = u^2 + vy + w$, $H = zu - v + vw$, compute $\partial(F, G, H)/\partial(u, v, w)$.
8. If $u = x + y + z$, $uv = y + z$, $uvw = z$, show that $\partial(x, y, z)/\partial(u, v, w) = u^2v$.
(*Kurukshetra, 2009; P.T.U., 2009 S; V.T.U., 2003*)
9. If $u^3 + v^3 = x + y$ and $u^2 + v^2 = x^3 + y^3$, show that $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{1}{2}\dfrac{y^2 - x^2}{uv(u - v)}$ (*U.P.T.U., 2006 MCA*)
10. If $u = \dfrac{x + y}{1 - xy}$ and $v = \tan^{-1}x + \tan^{-1}y$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$. Are $u$ and $v$ functionally related. If so, find this relationship.
(*Nagpur, 2008*)
11. If $u = 3x + 2y - z$, $v = x - 2y + z$ and $w = x(x + 2y - z)$, show that they are functionally related, and find the relation.
(*Nagpur, 2009*)

## 5.8 (1) GEOMETRICAL INTERPRETATION

If $P(x, y, z)$ be the coordinates of a point referred to axes $OX$, $OY$, $OZ$ then the equation $z = f(x, y)$ represents a surface. (Fig. 5.1)

Let a plane $y = b$ parallel to the $XZ$-plane pass through $P$ cutting the surface along the curve $APB$ given by

$$z = f(x, b).$$

As $y$ remains equal to $b$ and $x$ varies then $P$ moves along the curve $APB$ and $\partial z/\partial x$ is the ordinary derivative of $f(x, b)$ w.r.t. $x$.

Hence $\partial z/\partial x$ *at P is the tangent of the angle which the tangent at P to the section of the surface $z = f(x, y)$ by a plane through P parallel to the plane XOZ, makes with a line parallel to the x-axis.*

Similarly, $\partial z/\partial y$ *at P is the tangent of the angle which the tangent at P to the curve of intersection of the surface $z = f(x, y)$ and the plane $x = a$, makes with a line parallel to the y-axis.*

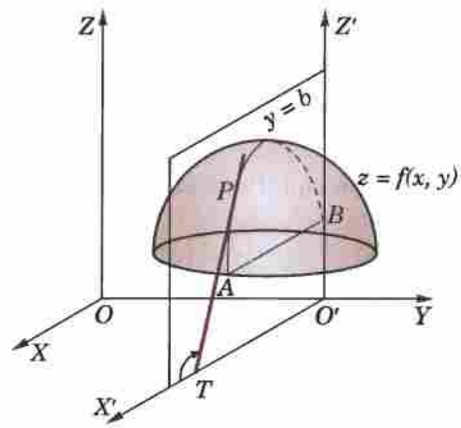


Fig. 5.1

**(2) Tangent plane and Normal to a surface.** Let $P(x, y, z)$ and $Q(x + \delta x, y + \delta y, z + \delta z)$ be two neighbouring points on the surface $F(x, y, z) = 0$. (Fig. 5.2) ...(*i*)

Let the arc $PQ$ be $\delta s$ and the chord $PQ$ be $\delta c$, so that (as for plane curves)

$$\underset{Q \to P}{\text{Lt}} (\delta s/\delta c) = 1.$$

The direction cosines of $PQ$ are $\dfrac{\delta x}{\delta c}, \dfrac{\delta y}{\delta c}, \dfrac{\delta z}{\delta c}$ *i.e.,* $\dfrac{\delta x}{\delta s}\cdot\dfrac{\delta s}{\delta c}, \dfrac{\delta y}{\delta s}\cdot\dfrac{\delta s}{\delta c}, \dfrac{\delta z}{\delta s}\cdot\dfrac{\delta s}{\delta c}$

When $\delta s \to 0$, $Q \to P$ and $PQ$ tends to tangent line $PT$. Then noting that the coordinates of any point on arc $PQ$ are functions of $s$ only, the direction cosines of $PT$ are

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \qquad ...(ii)$$

Differentiating (*i*) with respect to $s$, we obtain $\dfrac{\partial F}{\partial x}\cdot\dfrac{dx}{ds} + \dfrac{\partial F}{\partial y}\cdot\dfrac{dy}{ds} + \dfrac{\partial F}{\partial z}\cdot\dfrac{dz}{ds} = 0.$

This shows that the tangent line whose direction cosines are given by (*ii*), is perpendicular to the line having direction ratios

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \qquad ...(iii)$$

Since we can take different curves joining $Q$ to $P$, we get a number of tangent lines at $P$ and the line having direction ratios (*iii*) will be perpendicular to *all* these tangent lines at $P$. Thus all the tangent lines at $P$ lie in a plane through $P$ perpendicular to line (*iii*).

Hence *the equation of the tangent plane to (i) at the point P is*

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}}(\mathbf{X} - \mathbf{x}) + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(\mathbf{Y} - \mathbf{y}) + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}(\mathbf{Z} - \mathbf{z}) = 0$$

where $(X, Y, Z)$ are the current coordinates of any point on this tangent plane.

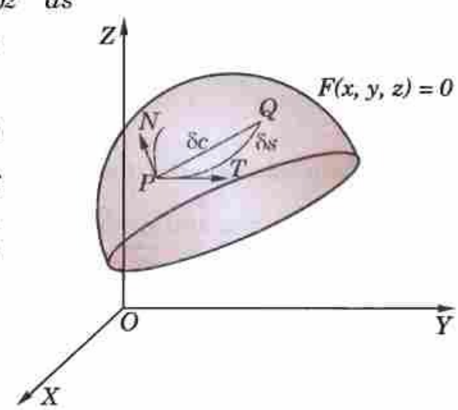


Fig. 5.2

Also *the equation of the normal to the surface at P (i.e., the line through P,* perpendicular to the tangent plane at *P*) is

$$\frac{\mathbf{X-x}}{\partial \mathbf{F}/\partial \mathbf{x}} = \frac{\mathbf{Y-y}}{\partial \mathbf{F}/\partial \mathbf{y}} = \frac{\mathbf{Z-z}}{\partial \mathbf{F}/\partial \mathbf{z}}.$$

**Example 5.31.** *Find the equations of the tangent plane and the normal to the surface $z^2 = 4(1 + x^2 + y^2)$ at (2, 2, 6).*

**Solution.** We have $F(x, y, z) = 4x^2 + 4y^2 - z^2 + 4$.

$\therefore$ $\partial F/\partial x = 8x$, $\partial F/\partial y = 8y$, $\partial F/\partial z = -2z$, and at the point (2, 2, 6)

$\partial F/\partial x = 16$, $\partial F/\partial y = 16$, $\partial F/\partial z = -12$

Hence the equation of the tangent plane at (2, 2, 6) is $16(X-2) + 16(Y-2) - 12(Z-6) = 0$

*i.e.,* $4X + 4Y - 3Z + 2 = 0$ ...(*i*)

Also the equation of the normal at (2, 2, 6) [being perpendicular to (*i*)] is

$$\frac{X-2}{4} = \frac{Y-2}{4} = \frac{Z-6}{-3}.$$

## PROBLEMS 5.8

Find the equations of the tangent plane and normal to each of the following surfaces at the given points :

1. $2x^2 + y^2 = 3 - 2z$ at $(2, 1, -3)$ (*Assam, 1998*)
2. $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$ (*Osmania, 2003 S*)
3. $xyz = a^2$ at $(x_1, y_1, z_1)$.
4. $2xz^2 - 3xy - 4x = 7$ at $(1, -1, 2)$.
5. Show the plane $3x + 12y - 6z - 17 = 0$ touches the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$. Find also the point of contact.
6. Show that the plane $ax + by + cz + d = 0$ touches the surface $px^2 + qy^2 + 2z = 0$, if $\frac{a^2}{p} + \frac{b^2}{q} + 2cd = 0$.
7. Find the equation of the normal to the surface $x^2 + y^2 + z^2 = a^2$. (*P.T.U., 2009 S*)

## 5.9 TAYLOR'S THEOREM FOR FUNCTIONS OF TWO VARIABLES

Considering $f(x + h, y + k)$ as a function of a single variable $x$, we have by Taylor's theorem*

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial f(x, y+k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x, y+k)}{\partial x^2} + \ldots \quad \ldots(i)$$

Now expanding $f(x, y + k)$ as a function of $y$ only,

$$f(x, y+k) = f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots$$

$\therefore$ (*i*) takes the form $f(x+h, y+k) = f(x, y) + k\dfrac{\partial f(x, y)}{\partial y} + \dfrac{k^2}{2!}\dfrac{\partial^2 f(x, y)}{\partial y^2} + \ldots$

$$+ h\frac{\partial}{\partial x}\left\{f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots\right\} + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x, y) + k\frac{\partial (x, y)}{\partial y} + \ldots\right\}$$

Hence, $$f(x+h, y+k) = f(x, y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots \quad \ldots(1)$$

In symbols we write it as $f(x+h, y+k) = f(x, y) + \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)f + \dfrac{1}{2!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^2 f + \ldots$

Taking $x = a$ and $y = b$, (1) becomes

$$f(a+h, b+k) = f(a, b) + [hf_x(a, b) + kf_y(a, b)] + \frac{1}{2!}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)] + \ldots$$

*See footnote on page 145.

Putting $a + h = x$ and $b + k = y$ so that $h = x - a$, $k = y - b$, we get

$$\mathbf{f(x, y) = f(a, b) + [(x - a)\, f_x\, (a, b) + (y - b)\, f_y\, (a, b)]}$$
$$\mathbf{+ \frac{1}{2!}\, [(x - a)^2\, f_{xx}\, (a, b) + 2(x - a)(y - b)\, f_{xy}\, (a, b) + (y - b)^2\, f_{yy}(a, b)] + \ldots} \qquad \ldots(2)$$

*This is Taylor's expansion of $f(x, y)$ in powers of $(x - a)$ and $(y - b)$. It is used to expand $f(x, y)$ in the neighbourhood of $(a, b)$.*

**Cor.** Putting $a = 0$, $b = 0$, in (2), we get

$$\mathbf{f(x, y) = f(0, 0) + [xf_x(0, 0) + yf_y\, (0, 0)] + \frac{1}{2!}\, [x^2\, f_{xx}\, (0, 0) + 2xy\, f_{xy}\, (0, 0) + y^2\, f_{yy}\, (0, 0)] + \ldots} \qquad \ldots(3)$$

This is Maclaurin's expansion of $f(x, y)$.

**Example 5.32.** *Expand $e^x \log (1 + y)$ in powers of x and y upto terms of third degree.*

*(V.T.U., 2010 ; P.T.U., 2009 ; J.N.T.U., 2006)*

**Solution.** Here

$$\begin{aligned}
f(x, y) &= e^x \log (1 + y) & \therefore \quad f(0, 0) &= 0\\
f_x(x, y) &= e^x \log (1 + y) & f_x(0, 0) &= 0\\
f_y(x, y) &= e^x\, \frac{1}{1 + y} & f_y(0, 0) &= 1\\
f_{xx}(x, y) &= e^x \log (1 + y) & f_{xx}(0, 0) &= 0\\
f_{xy}(x, y) &= e^x\, \frac{1}{1 + y} & f_{xy}(0, 0) &= 1\\
f_{yy}(x, y) &= - e^x (1 + y)^{-2} & f_{yy}(0, 0) &= - 1\\
f_{xxx}(x, y) &= e^x \log (1 + y) & f_{xxx}(0, 0) &= 0\\
f_{xxy}(x, y) &= e^x\, \frac{1}{1 + y} & f_{xxy}(0, 0) &= 1\\
f_{xyy}(x, y) &= - e^x (1 + y)^{-2} & f_{xyy}(0, 0) &= - 1\\
f_{yyy}(x, y) &= 2e^x (1 + y)^{-3} & f_{yyy}(0, 0) &= 2
\end{aligned}$$

Now Maclaurin's expansion of $f(x, y)$ gives

$$f(x, y) = f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2!}\, \{x^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0) + y^2\, f_{yy}(0, 0)\}$$
$$+ \frac{1}{3!}\, \{x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)\} + \ldots$$

$$\therefore \quad e^x \log (1 + y) = 0 + x(0) + y(1) + \frac{1}{2!}\, \{x^2\, (0) + 2xy(1) + y^2\, (-1)\}$$
$$+ \frac{1}{3!}\, \{x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)\} + \ldots$$
$$= y + xy - \frac{1}{2} y^2 + \frac{1}{2}\, (x^2y - xy^2) + \frac{1}{3} y^3 + \ldots$$

**Example 5.33.** *Expand $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ using Taylor's theorem.*

*(P.T.U., 2010 ; V.T.U., 2008 ; U.P.T.U., 2006 ; Anna, 2005)*

**Solution.** Taylor's expansion of $f(x, y)$ in powers of $(x - a)$ and $(x - b)$ is given by

$$f(x, y) = f(a, b) + [(x - a)\, f_x\, (a, b) + (y - b)\, f_y(a, b)] + \frac{1}{2!}\, [(x - a)^2\, f_{xx}\, (a, b)$$
$$+ 2\, (x - a)(y - b)\, f_{xy}(a, b) + (y - b)^2\, f_{yy}(a, b)] + \frac{1}{3!}\, [(x - a)^3\, f_{xxx}(a, b)$$
$$+ 3\, (x - a)^2\, (y - b)\, f_{xxy}(a, b) + 3\, (x - a)\, (y - b)^2\, f_{xyy}\, (a, b)$$
$$+ (y - b)^3\, f_{yyy}\, (a, b)] + \ldots \qquad \ldots(i)$$

Hence $a = 1$, $b = - 2$ and $f(x, y) = x^2y + 3y - 2$

$\therefore \quad f(1, -2) = -10, f_x = 2xy, f_x(1, -2) = -4; f_y = x^2 + 3, f_y(1, -2) = 4; f_{xx} = 2y,$
$f_{xx}(1, -2) = -4; f_{xy} = 2x, f_{xy}(1, -2) = 2; f_{yy} = 0, f_{yy}(1, -2) = 0; f_{xxx} = 0, f_{xxx} = (1, -2) = 0;$
$f_{xxy}(1, -2) = 2, f_{xyy}(1, -2) = 0, f_{yyy}(1, -2) = 0$

All partial derivatives of higher order vanish.

Substituting these in $(i)$, we get

$$x^2y + 3y - 2 = -10 + [(x-1)(-4) + (y+2)4] + \frac{1}{2}[(x-1)^2(-4) + 2(x-1)(y+2)(2)$$
$$+ (y+2)^2(0)] + \frac{1}{6}[(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3(0)]$$
$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2).$$

**Example 5.34.** *Expand $f(x, y) = \tan^{-1}(y/x)$ in powers of $(x-1)$ and $(y-1)$ upto third-degree terms. Hence compute $f(1.1, 0.9)$ approximately.* *(V.T.U., 2010 ; J.N.T.U., 2006 ; U.P.T.U., 2006)*

**Solution.** Here $a = 1$, $b = 1$ and $f(1, 1) = \tan^{-1}(1) = \pi/4$.

$$f_x = \frac{-y}{x^2+y^2} \qquad f_x(1,1) = -\frac{1}{2}; \qquad f_y = \frac{x}{x^2+y^2}, \qquad f_y(1,1) = \frac{1}{2}$$

$$f_{xx} = \frac{2xy}{(x^2+y^2)^2}, \qquad f_{xx}(1,1) = \frac{1}{2}; \qquad f_{xy} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad f_{xy}(1,1) = 0$$

$$f_{yy} = \frac{-2xy}{(x^2+y^2)^2}, \qquad f_{yy}(1,1) = -\frac{1}{2};$$

$$f_{xxx} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}, \qquad f_{xxx}(1,1) = -\frac{1}{2}; \qquad f_{xxy} = \frac{2x^3 - 6xy^2}{(x^3+y^2)^3}, \qquad f_{xxy}(1,1) = -\frac{1}{2}$$

$$f_{xyy} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}, \qquad f_{xyy}(1,1) = \frac{1}{2}; \qquad f_{yyy} = \frac{6xy^2 - 2x^3}{(x^2+y^2)^3}, \qquad f_{yyy}(1,1) = \frac{1}{2}$$

Taylor's expansion of $f(x, y)$ in powers of $(x-1)$ and $(y-1)$ is given by

$$f(x,y) = f(1,1) + \frac{1}{1!}\{(x-1)f_x(1,1) + (y-1)f_y(1,1)\} + \frac{1}{2!}\{(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)$$
$$f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1) + \frac{1}{3!}\{(x-1)^3 f_{xxx}(1,1) + 3(x-1)^2(y-1)f_{xxy}(1,1)$$
$$+ 3(x-1)(y-1)^2 f_{xyy}(1,1) + (y-1)^3 f_{yyy}(1,1)\} + \ldots$$

$$\therefore \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + \left\{(x-1)\left(-\frac{1}{2}\right) + (y-1)\frac{1}{2}\right\} + \frac{1}{2!}\left\{(x-1)^2\frac{1}{2} + 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right\}$$
$$+ \frac{1}{3!}\left\{(x-1)^3\left(-\frac{1}{2}\right) + 3(x-1)^2(y-1)\left(-\frac{1}{2}\right) + 3(x-1)(y-1)^2\frac{1}{2} + (y-1)^3\frac{1}{2}\right\} + \ldots$$
$$= \frac{\pi}{4} - \frac{1}{2}\{(x-1) - (y-1)\} + \frac{1}{4}\{(x-1)^2 - (y-1)^2\} - \frac{1}{12}\{(x-1)^3 + 3(x-1)^2(y-1)$$
$$- 3(x-1)(y-1)^2 - (y-1)^3\} + \ldots$$

Putting $x = 1.1$ and $y = 0.9$, we get

$$f(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}(0.2) + \frac{1}{4}(0) - \frac{1}{12}\{(0.1)^3 - 3(0.1)^3 - 3(0.1)^3 - (-0.1)^3\}$$
$$= 0.7854 - 0.1000 + 0.0003 = 0.6857.$$

## 5.10 (1) ERRORS AND APPROXIMATIONS

Let $f(x, y)$ be a continuous function of $x$ and $y$. If $\delta x$ and $\delta y$ be the increments of $x$ and $y$, then the new value of $f(x, y)$ will be $f(x + \delta x, y + \delta y)$. Hence

$$\delta f = f(x + \delta x, y + \delta y) - f(x, y).$$

Expanding $f(x + \delta x, y + \delta y)$ by Taylor's theorem and supposing $\delta x$, $\delta y$ to be so small that their products, squares and higher powers can be neglected, we get

$$\delta \mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \delta \mathbf{x} + \frac{\partial \mathbf{f}}{\partial \mathbf{y}} \delta \mathbf{y}, \text{ approximately.}$$

Similarly if $f$ be a function of several variables $x, y, z, t, \ldots$, then

$$\delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial z} \delta z + \frac{\partial f}{\partial t} \delta t + \ldots \text{ approximately.}$$

These formulae are very useful in correcting the effect of small errors in measured quantities.

**(2) Total Differential**

If $u$ is a function of two variables $x$ and $y$, the *total differential* of $u$ is defined as

$$\mathbf{du} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{dx} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \mathbf{dy} \qquad \ldots(1)$$

The differentials $dx$ and $dy$ are respectively the increments $\delta x$ and $\delta y$ in $x$ and $y$. If $x$ and $y$ are not independent variables but functions of another variable $t$ even then the formula (1) holds and we write $dx = \frac{dx}{dt} dt$ and $dy = \frac{dy}{dt} dt$. Similar definition can be given for a function of three or more variables.

**Example 5.35.** *The diameter and altitude of a can in the shape of a right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the values computed for the volume and the lateral surface.*

**Solution.** Let $x$ be the diameter and $y$ the height of the can. Then its volume $V = \frac{\pi}{4} x^2 y$

$$\therefore \qquad \delta V = \frac{\partial V}{\partial x} \delta x + \frac{\partial V}{\partial y} \delta y = \frac{\pi}{4} (2xy\delta x + x^2 \delta y)$$

When $x = 4$ cm., $y = 6$ cm. and $\delta x = \delta y = 0.1$ cm.

$$\therefore \qquad \delta V = \frac{\pi}{4}(2 \times 4 \times 6 \times 0.1 + 4^2 \times 0.1) = 1.6\pi \text{ cm}^3$$

Also its lateral surface $S = \pi xy$

$$\therefore \qquad \delta S = \pi(y\, \delta x + x\, \delta y)$$

When $x = 4$ cm., $y = 6$ cm. and $\delta x = \delta y = 0.1$ cm., we have $\delta S = \pi(6 \times 0.1 + 4 \times 0.1) = \pi \text{ cm}^2$.

**Example 5.36.** *The period of a simple pendulum is $T = 2\pi \sqrt{(l/g)}$, find the maximum error in T due to the possible error upto 1% in l and 2.5% in g.* (U.P.T.U., 2004)

**Solution.** We have $T = 2\pi \sqrt{(l/g)}$

or $$\log T = \log 2\pi + \frac{1}{2} \log l - \frac{1}{2} \log g$$

$$\therefore \qquad \frac{1}{T} \delta T = 0 + \frac{1}{2} \frac{1}{l} \delta l - \frac{1}{2} \frac{1}{g} \delta g$$

$$\frac{\delta T}{T} 100 = \frac{1}{2}\left(\frac{\delta l}{l} 100 - \frac{\delta g}{g} 100\right) = \frac{1}{2}(1 \pm 2.5) = 1.75 \text{ or } -0.75$$

Thus the maximum error in $T = 1.75\%$

**Example 5.37.** *A balloon is in the form of right circular cylinder of radius 1.5 m and length 4 m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and length by 0.05 m, find the percentage change in the volume of balloon.* (U.P.T.U., 2005)

**Solution.** Let the volume of the balloon (Fig. 5.3) be $V$, so that

$$V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$$

$\therefore$
$$\delta V = 2\pi r \delta r h + \pi r^2 \delta h + \frac{4}{3}\pi 3 r^2 \delta r$$

or
$$\frac{\delta V}{V} = \frac{\pi r[2h\delta r + r\delta h + 4r\delta r]}{\pi r^2 h + \frac{4}{3}\pi r^3}$$

$$= \frac{2(h+2r)\,\delta r + r\delta h}{rh + \frac{4}{3}r^2} = \frac{2(4+3)(.01) + 1.5(.05)}{1.5 \times 4 + \frac{4}{3}(1.5)^2}$$

$$= \frac{0.14 + 0.075}{6+3} = \frac{0.215}{9}$$

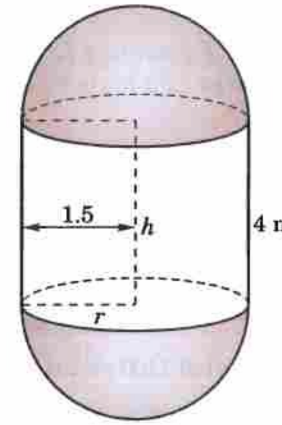


Fig. 5.3

Hence, the percentage change in $V = 100\ \dfrac{\delta V}{V} = \dfrac{21.5}{9} = 2.39\%$

**Example 5.38.** *In estimating the cost of a pile of bricks measured as 2 m × 15 m × 1.2 m, the tape is stretched 1% beyond the standard length. If the count is 450 bricks to 1 cu. m. and bricks cost ₹ 530 per 1000, find the approximate error in the cost.* (V.T.U., 2001)

**Solution.** Let $x$, $y$ and $z$ $m$ be the length, breadth and height of the pile so that its volume $V = xyz$

or
$$\log V = \log x + \log y + \log z \ \therefore\ \frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$$

Since
$$V = 2 \times 15 \times 1.2 = 36 \text{ m}^3, \text{ and } \frac{\delta x}{x} = \frac{\delta y}{y} = \frac{\delta z}{z} = \frac{1}{100}$$

$\therefore$
$$\delta V = 36\left(\frac{3}{100}\right) = 1.08 \text{ m}^3.$$

Number of bricks in $\delta V = 1.08 \times 450 = 486$

Thus error in the cost $= 486 \times \dfrac{530}{1000} =$ ₹ 257.58 which is a loss to the brick seller.

**Example 5.39.** *The height h and semi-vertical angle α of a cone are measured and from them A, the total area of the surface of the cone including the base is calculated. If h and α are in error by small quantities δh and δα respectively, find the corresponding error in the area. Show further that if α = π/6, an error of + 1% in h will be approximately compensated by an error of – 0.33 degrees in α.*

**Solution.** If $r$ be the base radius and $l$ the slant height of the cone, (Fig. 5.4), then total area

$$A = \text{area of base} + \text{area of curved surface}$$
$$= \pi r^2 + \pi r l = \pi r(r + l)$$
$$= \pi h \tan\alpha\,(h \tan\alpha + h \sec\alpha)$$
$$= \pi h^2 (\tan^2\alpha + \tan\alpha \sec\alpha)$$

$\therefore$
$$\delta A = \frac{\delta A}{\delta h}\,\delta h + \frac{\delta A}{\delta \alpha}\,\delta\alpha$$
$$= 2\pi h(\tan^2\alpha + \tan\alpha\sec\alpha)\delta h$$
$$+ \pi h^2 (2\tan\alpha\sec^2\alpha + \sec^3\alpha + \tan\alpha\sec\alpha\tan\alpha)\delta\alpha$$

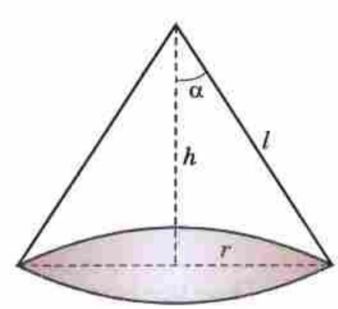


Fig. 5.4

which gives the error in the area $A$.

Putting $\delta h = h/100$ and $\alpha = \pi/6$, we get

$$\delta A = 2\pi h\left[\left(\frac{1}{\sqrt{3}}\right)^2 + \frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}\right]\frac{h}{100} + \pi h^2\left[2.\frac{1}{\sqrt{3}}\cdot\frac{4}{3} + \frac{8}{3\sqrt{3}} + \frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right]\delta\alpha$$

$$= \frac{2\pi h^2}{100} + 2\sqrt{3}\pi h^2\ \delta\alpha$$

The error in $h$ will be compensated by the error in $\alpha$, when

$$\delta A = 0 \text{ i.e., } \frac{2\pi h^2}{100} + 2\sqrt{3}\pi h^2\,\delta\alpha = 0$$

or $$\delta\alpha = -\frac{1}{100\sqrt{3}} \text{ radians} = -\frac{.01}{1.732} \times 57.3° = -0.33°.$$

**Example 5.40.** *Show that the approximate change in the angle A of a triangle ABC due to small changes δa, δb, δc in the sides a, b, c respectively, is given by*

$$\delta A = \frac{a}{2\Delta}\ (\delta a - \delta b \cos C - \delta c \cos B)$$

*where Δ is the area of the triangle. Verify that δA + δB + δC = 0.*

**Solution.** We know that $a^2 = b^2 + c^2 - 2bc \cos A$

so that $$2a\delta a = 2b\delta b + 2c\delta c - 2(c\delta b \cos A - b\delta c \cos A + bc \sin A\ \delta A)$$

$\therefore$ $$bc \sin A\ \delta A = a\delta a - (b - c \cos A)\ \delta b - (c - b \cos A)\ \delta c$$

or $$2\ \Delta\delta A = a\delta a - (c \cos A + a \cos C - c \cos A)\ \delta b - (a \cos B + b \cos A - b \cos A)\ \delta c$$

[$\because$ $b = c \cos A + a \cos C$ etc. ...(i)]

$$= a\delta a - a \cos C\ \delta b - a \cos B\ \delta c$$

or $$\delta A = \frac{a}{2\Delta}\ (\delta a - \delta b \cos C - \delta c \cos B)$$

By symmetry, we have

$$\delta B = \frac{b}{2\Delta}\ (\delta b - \delta c \cos A - \delta a \cos C)$$

$$\delta C = \frac{c}{2\Delta}\ (\delta c - \delta a \cos B - \delta b \cos A)$$

$\therefore$ $$\delta A + \delta B + \delta C = \frac{1}{2\Delta}\ (a - b \cos C - c \cos B)\ \delta a + (b - c \cos A - a \cos C)\ \delta b + (c - a \cos B - b \cos A)]$$

$$= \frac{1}{2\Delta}\ [(a - a)\ \delta a + (b - b)\ \delta b + (c - c)\ \delta c] = 0 \qquad \text{[By (i)]}$$

**Example 5.41.** *If the sides of a plane triangle ABC vary in such a way that its circumradius remains constant, prove that* $\dfrac{da}{\cos A} + \dfrac{db}{\cos B} + \dfrac{dc}{\cos C} = 0.$

**Solution.** The circumradius $R$ of $\Delta ABC$ is given by

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}$$

$\therefore$ $$a = 2R \sin A \qquad [\because\ R \text{ is constant}$$

Taking differentials, $da = 2R \cos A\ dA$ or $\dfrac{da}{\cos A} = 2RdA$

Similarly, $\dfrac{db}{\cos B} = 2RdB,\ \dfrac{dc}{\cos C} = 2RdC$

$\therefore$ $$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R\ (dA + dB + dC)$$

Now $A + B + C = \pi$, gives $dA + dB + dC = 0$ ...(i)

Thus $$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0 \qquad \text{[By (i)]}$$

## PROBLEMS 5.9

1. Expand the following functions as far as terms of third degree :
   (*i*) $\sin x \cos y$ (*V.T.U., 2009*) (*ii*) $e^x \sin y$ at $(-1, \pi/4)$ (*Anna, 2009*)
   (*iii*) $xy^2 + \cos xy$ about $(1, \pi/2)$. (*Hissar, 2005 S ; V.T.U., 2003*)
2. Expand $f(x, y) = x^y$ in powers of $(x-1)$ and $(y-1)$. (*U.T.U., 2009*)
3. If $f(x, y) = \tan^{-1} xy$, compute $f(0.9, -1.2)$ approximately.
4. If the kinetic energy $k = wv^2/2g$, find approximately the change in the kinetic energy as $w$ changes from 49 to 49.5 and $v$ changes from 1600 to 1590. (*V.T.U., 2006*)
5. Find the possible percentage error in computing the resistance $r$ from the formula $1/r = 1/r_1 + 1/r_2$, if $r_1, r_2$ are both in error by 2%.
6. The voltage $V$ across a resistor is measured with an error $h$, and the resistance $R$ is measured with an error $k$. Show that the error in calculating the power $W(V, R) = V^2/R$ generated in the resistor, is $VR^{-2}(2Rh - Vk)$. (*V.T.U., 2009*)
7. Find the percentage error in the area of an ellipse if one per cent error is made in measuring the major and minor axes. (*V.T.U., 2011*)
8. The time of oscillation of a simple pendulum is given by the equation $T = 2\pi\sqrt{(l/g)}$. In an experiment carried out to find the value of $g$, errors of 1.5% and 0.5% are possible in the values of $l$ and $T$ respectively. Show that the error in the calculated value of $g$ is 0.5%. (*Cochin, 2005*)
9. If $pv^2 = k$ and the relative errors in $p$ and $v$ are respectively 0.05 and 0.025, show that the error in $k$ is 10%. (*Mysore, 1999*)
10. If the H.P. required to propel a steamer varies as the cube of the velocity and square of the length. Prove that a 3% increase in velocity and 4% increase in length will require an increase of about 17% in H.P.
11. The range $R$ of a projectile which starts with a velocity $v$ at an elevation $\alpha$ is given by $R = (v^2 \sin 2\alpha)/g$. Find the percentage error in $R$ due to an error of 1% in $v$ and an error of $\frac{1}{2}$% in $\alpha$. (*Kurukshetra, 2009*)
12. In estimating the cost of a pile of bricks measured as 6 m × 50 m × 4 m, the tape is stretched 1% beyond the standard length. If the count is 12 bricks in 1 m³ and bricks cost ₹ 100 per 1000, find the approximate error in the cost. (*U.T.U., 2010 ; U.P.T.U., 2005*)
13. The deflection at the centre of a rod of length $l$ and diameter $d$ supported at its ends, loaded at the centre with a weight $w$ varies at $wl^3d^{-4}$. What is the increase in the deflection corresponding to $p$% increase in $w$, $q$% decrease in $l$ and $r$% increase in $d$ ?
14. The work that must be done to propel a ship of displacement $D$ for a distance $s$ in time $t$ is proportional to $(s^2D^{2/3}/t^2)$. Find approximately the increase of work necessary when the displacement is increased by 1%, the time is diminished by 1% and the distance diminished by 2%.
15. The indicated horse power $I$ of an engine is calculated from the formula $I = PLAN/33{,}000$, where $A = \pi d^2/4$. Assuming that error of $r$ per cent may have been made in measuring $P, L, N$ and $d$, find the greatest possible error in $I$.
16. The torsional rigidity of a length of wire is obtained from the formula $N = 8\pi\, Il/t^2r^4$. If $l$ is decreased by 2%, $r$ is increased by 2%, $t$ is increased by 1.5%, show that the value of $N$ is diminished by 13% approximately. (*V.T.U., 2003*)
17. If $x^2 + y^2 + z^2 - 2xyz = 1$, show that $\dfrac{dx}{\sqrt{(1-x^2)}} + \dfrac{dy}{\sqrt{(1-y^2)}} + \dfrac{dz}{\sqrt{(1-z^2)}} = 0$.
   [**Hint.** $2(x - yz)\,dx + 2(y - zx)\,dy + 2(z - xy)\,dz = 0$. Also $(x - yz)^2 = (1 - y^2)(1 - z^2)$, ...]

## 5.11 (1) MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

**Def.** *A function $f(x, y)$ is said to have a* **maximum** *or* **minimum** *at $x = a, y = b$, according as*

$$f(a + h, b + k) < \text{or} > f(a, b),$$

*for all positive or negative small values of h and k.*

In other words, if $\Delta = f(a + h, b + k) - f(a, b)$, *is of the same sign for all small values of h, k, and if this sign is negative, then $f(a, b)$ is a maximum. If this sign is positive, $f(a, b)$ is a minimum.*

Considering $z = f(x, y)$ as a surface, maximum value of $z$ occurs at the top of an elevation (*e.g.*, a dome) from which the surface descends in every direction and a minimum value occurs at the bottom of a depression (*e.g.*, a bowl) from which the surface ascends in every direction. Sometimes the maximum or minimum value may form a *ridge* such that the surface descends or ascends in all directions except that of the ridge. Besides these, we have such a point of the surface, where the tangent plane is horizontal and the surface looks like leather seat on the horse's back [Fig. 5.5 (*c*)] which falls for displacement in certain directions and rises for displacements in other directions. Such a point is called a **saddle point.**

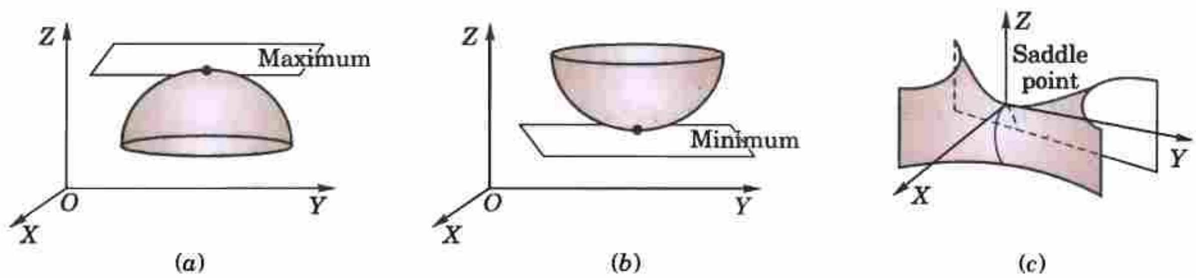


**Fig. 5.5**

**Note.** *A maximum or minimum value of a function is called its* **extreme value.**

**(2) Conditions for f(x, y) to be maximum or minimum**

Using Taylor's theorem page 235, we have $\Delta = f(a + h, b + k) - f(a, b)$

$$= \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{a,b} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots \qquad \ldots(i)$$

For small values of $h$ amd $k$, the second and higher order terms are still smaller and hence may be neglected. Thus

$$\text{sign of } \Delta = \text{sign of } [hf_x(a, b) + kf_y(a, b)].$$

Taking $h = 0$ we see that the right hand side changes sign when $k$ changes sign. Hence $f(x, y)$ cannot have a maximum or a minimum at $(a, b)$ unless $f_y(a, b) = 0$.

Similarly taking $k = 0$, we find that $f(x, y)$ cannot have a maximum or minimum at $(a, b)$ unless $f_x(a, b) = 0$.

Hence *the necessary conditions for $f(x, y)$ to have a maximum or minimum at $(a, b)$ are that*

$$\mathbf{f_x(a, b) = 0,\ f_y(a, b) = 0.}$$

If these conditions are satisfied, then for small value of $h$ and $k$, $(i)$ gives

$$\text{sign of } \Delta = \text{sign of } \left[\frac{1}{2!}(h^2r + 2hks + k^2t)\right] \text{ where } r = f_{xx}(a, b),\ s = f_{xy}(a, b) \text{ and } t = f_{yy}(a, b).$$

Now $\quad h^2r + 2hks + k^2t = \dfrac{1}{r}\left[(h^2r^2 + 2hkrs + k^2rt)\right] = \dfrac{1}{r}\left[(hr + ks)^2 + k^2(rt - s^2)\right]$

$$\text{Thus sign of } \Delta = \text{sign of } \frac{1}{2r}\left\{(hr + ks)^2 + k^2(rt - s^2)\right\} \qquad \ldots(ii)$$

In $(ii)$, $(hr + ks)^2$ is always positive and $k^2(rt - s^2)$ will be positive if $rt - s^2 > 0$. In this case, $\Delta$ will have the same sign as that of $r$ for all values of $h$ and $k$.

Hence if $rt - s^2 > 0$, then $f(x, y)$ *has a maximum or a minimum at $(a, b)$ according as $r <$ or $> 0$.*

If $rt - s^2 < 0$, then $\Delta$ will change with $h$ and $k$ and hence there is no maximum or minimum at $(a, b)$ *i.e.*, it is a *saddle point*.

If $rt - s^2 = 0$, further investigation is required to find whether there is a maximum or minimum at $(a, b)$ or not.

**Note. Stationary value.** *$f(a, b)$ is said to be a stationary value of $f(x, y)$, if $f_x(a, b) = 0$* and *$f_y(a, b) = 0$ i.e.* the function is stationary at $(a, b)$.

Thus every extreme value is a stationary value but the converse may not be true.

**(3) Working rule to find the maximum and minimum values of f(x, y)**

1. *Find $\partial f/\partial x$ and $\partial f/\partial y$ and equate each to zero. Solve these as simultaneous equations in $x$ and $y$.* Let $(a, b)$, *$(c, d)$, ... be the pairs of values.*
2. *Calculate the value of $r = \partial^2 f/\partial x^2$, $s = \partial^2 f/\partial x\partial y$, $t = \partial^2 f/\partial y^2$ for each pair of values.*

3. *(i) If $rt - s^2 > 0$ and $r < 0$ at $(a, b)$, $f(a, b)$ is a max. value.*
*(ii) If $rt - s^2 > 0$ and $r > 0$ at $(a, b)$, $f(a, b)$ is a min. value.*
*(iii) If $rt - s^2 < 0$ at $(a, b)$, $f(a, b)$ is not an extreme value, i.e., $(a, b)$ is a saddle point.*
*(iv) If $rt - s^2 = 0$ at $(a, b)$, the case is doubtful and needs further investigation.*

Similarly examine the other pairs of values one by one.

**Example 5.42.** *Examine the following function for extreme values :*

$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2. \qquad \text{(J.N.T.U., 2003)}$$

**Solution.** We have $f_x = 4x^3 - 4x + 4y$ ; $f_y = 4y^3 + 4x - 4y$

and $r = f_{xx} = 12x^2 - 4$, $s = f_{xy} = 4$, $t = f_{yy} = 12y^2 - 4$ ...(i)

Now $f_x = 0, f_y = 0$ give $x^3 - x + y = 0$, ...(i) $y^3 + x - y = 0$ ...(ii)

Adding these, we get $4(x^3 + y^3) = 0$ or $y = -x$.

Putting $y = -x$ in (i), we obtain $x^3 - 2x = 0$, *i.e.* $x = \sqrt{2}, -\sqrt{2}, 0$.

$\therefore$ Corresponding values of $y$ are $-\sqrt{2}, \sqrt{2}, 0$.

*At* $(\sqrt{2}, -\sqrt{2})$, $rt - s^2 = 20 \times 20 - 4^2 = +$ ve and $r$ is also + ve. Hence $f(\sqrt{2}, -\sqrt{2})$ is a minimum value.

*At* $(-\sqrt{2}, \sqrt{2})$ also both $rt - s^2$ and $r$ are + ve.

Hence $f(-\sqrt{2}, \sqrt{2})$, is also a minimum value.

*At* $(0, 0)$, $rt - s^2 = 0$ and, therefore, further investigation is needed.

Now $f(0, 0) = 0$ and for points along the $x$-axis, where $y = 0$, $f(x, y) = x^4 - 2x^2 = x^2(x^2 - 2)$, which is negative for points in the neighbourhood of the origin.

Again for points along the line $y = x$, $f(x, y) = 2x^4$ which is positive.

Thus in the neighbourhood of (0, 0) there are points where $f(x, y) < f(0, 0)$ and there are points where $f(x, y) > f(0, 0)$.

Hence $f(0, 0)$ is not an extreme value *i.e.*, it is a *saddle point*.

**Example 5.43.** *Discuss the maxima and minima of $f(x, y) = x^3y^2(1 - x - y)$.*

*(Anna, 2009 ; J.N.T.U., 2006 ; Bhopal, 2002)*

**Solution.** We have $f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ ; $f_y = 2x^3y - 2x^4y - 3x^3y^2$

and $r = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$ ; $s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$ ; $t = f_{yy} = 2x^3 - 2x^4 - 6x^3y$.

When $f_x = 0, f_y = 0$, we have $x^2y^2(3 - 4x - 3y) = 0$, $x^3y(2 - 2x - 3y) = 0$

Solving these, the stationary points are (1/2, 1/3), (0, 0).

Now $rt - s^2 = x^4y^2[12(1 - 2x - y)(1 - x - 3y) - (6 - 8x - 9y)^2]$

*At* $(1/2, 1/3)$, $rt - s^2 = \dfrac{1}{16} \cdot \dfrac{1}{9}\left[12\left(1 - 1 - \dfrac{1}{3}\right)\left(1 - \dfrac{1}{2} - 1\right) - (6 - 4 - 3)^2\right] = \dfrac{1}{14} > 0$

Also $r = 6\left(\dfrac{1}{2} \cdot \dfrac{1}{9} - \dfrac{2}{4} \cdot \dfrac{1}{9} - \dfrac{1}{2} \cdot \dfrac{1}{27}\right) = -\dfrac{1}{9} < 0$

Hence $f(x, y)$ has a maximum at (1/2, 1/3) and maximum value $= \dfrac{1}{8} \cdot \dfrac{1}{9}\left(1 - \dfrac{1}{2} - \dfrac{1}{3}\right) = \dfrac{1}{432}$.

*At* $(0, 0)$, $rt - s^2 = 0$ and therefore further investigation is needed.

For points along the line $y = x$, $f(x, y) = x^5(1 - 2x)$ which is positive for $x = 0.1$ and negative for $x = -0.1$ *i.e.*, in the neighbourhood of (0, 0) there are points where $f(x, y) > f(0, 0)$ and there are points where $f(x, y) < f(0, 0)$. Hence $f(0, 0)$ is not an extreme value.

**Example 5.44.** *In a plane triangle, find the maximum value of cos A cos B cos C.*

*(V.T.U., 2010 ; Nagpur, 2009 ; Anna, 2005 S)*

**Solution.** We have $A + B + C = \pi$ so that $C = \pi - (A + B)$.

$$\cos A \cos B \cos C = \cos A \cos B \cos [\pi - (A + B)]$$
$$= -\cos A \cos B \cos (A + B) = f(A, B), \text{ say.}$$

We get $\dfrac{\partial f}{\partial A} = \cos B\,[\sin A \cos(A+B) + \cos A \sin(A+B)]$

$= \cos B \sin(2A+B)$

and $\dfrac{\partial f}{\partial B} = \cos A \sin(A+2B)$

$\dfrac{\partial f}{\partial A} = 0,\ \dfrac{\partial f}{\partial B} = 0$ only when $A = B = \pi/3$.

Also $r = \dfrac{\partial^2 f}{\partial A^2} = 2\cos B \cos(2A+B),\ t = \dfrac{\partial^2 f}{\partial B^2} = 2\cos A \cos(A+2B)$

$s = \dfrac{\partial^2 f}{\partial A \partial B} = -\sin B \sin(2A+B) + \cos B \cos(2A+B) = \cos(2A+2B)$

When $A = B = \pi/3$, $r = -1$, $s = -1/2$, $t = -1$ so that $rt - s^2 = 3/4$.

These show that $f(A, B)$ is maximum for $A = B = \pi/3$.

Then $C = \pi - (A+B) = \pi/3$.

Hence $\cos A \cos B \cos C$ is maximum when each of the angles is $\pi/3$ *i.e.*, triangle is equilateral and its maximum value = 1/8.

## 5.12 LAGRANGE'S METHOD OF UNDERTERMINED MULTIPLIERS

Sometimes it is required to find the stationary values of a function of several variables which are not all independent but are connected by some given relations. Ordinarily, we try to convert the given function to the one, having least number of independent variables with the help of given relations. Then solve it by the above method. When such a procedure becomes impracticable, Lagrange's method* proves very convenient. Now we explain this method.

Let $u = f(x, y, z)$ ...(1)

be a function of three variables $x, y, z$ which are connected by the relation.

$\phi(x, y, z) = 0$ ...(2)

For $u$ to have stationary values, it is necessary that

$\partial u/\partial x = 0,\ \partial u/\partial y = 0,\ \partial u/\partial z = 0.$

$\therefore$ $\dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy + \dfrac{\partial u}{\partial z}dz = du = 0$ ...(3)

Also differentiating (2), we get $\dfrac{\partial \phi}{\partial x}dx + \dfrac{\partial \phi}{\partial y}dy + \dfrac{\partial \phi}{\partial z}dz = d\phi = 0$ ...(4)

Multiply (4) by a parameter $\lambda$ and add to (3). Then

$$\left(\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

This equation will be satisfied if $\dfrac{\partial u}{\partial x} + \lambda\dfrac{\partial \phi}{\partial x} = 0,\ \dfrac{\partial u}{\partial y} + \lambda\dfrac{\partial \phi}{\partial y} = 0,\ \dfrac{\partial u}{\partial z} + \lambda\dfrac{\partial \phi}{\partial z} = 0.$

These three equations together with (2) will determine the values of $x, y, z$ and $\lambda$ for which $u$ is stationary.

**Working rule : 1.** *Write* $F = f(x, y, z) + \lambda\phi(x, y, z)$

**2.** *Obtain the equations* $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial z} = 0.$

**3.** *Solve the above equations together with* $\phi(x, y, z) = 0$.

*The values of* $x, y, z$ *so obtained will give the stationary value of* $f(x, y, z)$.

**Obs.** Although the Lagrange's method is often very useful in application yet *the drawback is that we cannot determine the nature of the stationary point*. This can sometimes, be determined from physical considerations of the problem.

*See footnote page 142.

**Example 5.45.** *A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.* (Kurukshetra, 2006 ; P.T.U., 2006 ; U.P.T.U., 2005)

**Solution.** Let $x, y$ and $z$ ft. be the edges of the box and $S$ be its surface.

Then $$S = xy + 2yz + 2zx \qquad ...(i)$$

and $$xyz = 32 \qquad ...(ii)$$

Eliminating $z$ from $(i)$ with the help of $(ii)$, we get $S = xy + 2(y + x)\dfrac{32}{xy} = xy + 64\left(\dfrac{1}{x} + \dfrac{1}{y}\right)$

$\therefore$ $$\partial S/\partial x = y - 64/x^2 = 0 \quad \text{and} \quad \partial S/\partial y = x - 64/y^2 = 0.$$

Solving these, we get $x = y = 4$.

Now $r = \partial^2 S/\partial x^2 = 128/x^3,\ s = \partial^2 S/\partial x \partial y = 1,\ t = \partial^2 S/\partial y^2 = 128/y^3$.

At $x = y = 4,\ rt - s^2 = 2 \times 2 - 1 = +$ ve and $r$ is also + ve.

Hence $S$ is minimum for $x = y = 4$. Then from $(ii)$, $z = 2$.

*Otherwise* (*by Lagrange's method*) :

Write $$F = xy + 2yz + 2zx + \lambda(xyz - 32)$$

Then $$\frac{\partial F}{\partial x} = y + 2z + \lambda yz = 0 \qquad ...(iii)$$

$$\frac{\partial F}{\partial y} = x + 2z + \lambda zx = 0 \qquad ...(iv)$$

$$\frac{\partial F}{\partial z} = 2y + 2x + \lambda xy = 0 \qquad ...(v)$$

Multiplying $(iii)$ by $x$ and $(iv)$ by $y$ and subtracting, we get $2zx - 2zy = 0$ or $x = y$.

[The value $z = 0$ is neglected, as it will not satisfy $(ii)$]

Again multiplying $(iv)$ by $y$ and $(v)$ by $z$ and subtracting, we get $y = 2z$.

Hence the dimensions of the box are $x = y = 2z = 4$ ...$(vi)$

Now let us see what happens as $z$ increases from a small value to a large one. When $z$ is small, the box is flat with a large base showing that $S$ is large. As $z$ increases, the base of the box decreases rapidly and $S$ also decreases. After a certain stage, $S$ again starts increasing as $z$ increases. Thus $S$ must be a minimum at some intermediate stage which is given by $(vi)$. Hence $S$ is minimum when $x = y = 4$ ft and $z = 2$ ft.

**Example 5.46.** *Given $x + y + z = a$, find the maximum value of $x^m y^n z^p$.* (Anna, 2009)

**Solution.** Let $f(x, y, z) = x^m y^n z^p$ and $\phi(x, y, z) = x + y + z - a$.

Then $$F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z)$$
$$= x^m y^n z^p + \lambda(x + y + z - a).$$

For stationary values of $F$, $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0, \dfrac{\partial F}{\partial z} = 0$

$\therefore$ $$mx^{m-1} y^n z^p + \lambda = 0,\ nx^m y^{n-1} z^p + \lambda = 0,\ px^m y^n z^{p-1} + \lambda = 0$$

or $$-\lambda = mx^{n-1} y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$$

*i.e.* $$\frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m + n + p}{x + y + z} = \frac{m + n + p}{a} \qquad [\because\ x + y + z = a$$

$\therefore$ The maximum value of $f$ occurs when

$$x = am/(m + n + p),\ y = an/(m + n + p),\ z = ap/(m + n + p)$$

Hence the maximum value of $f(x, y, z) = \dfrac{a^{m+n+p} \cdot m^m \cdot n^n \cdot p^p}{(m + n + p)^{m+n+p}}$.

**Example 5.47.** *Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 4$.* (U.T.U., 2010)

**Solution.** Let $P(x, y, z)$ be any point on the sphere and $A(3, 4, 12)$ the given point so that

$$AP^2 = (x - 3)^2 + (y - 4)^2 + (z - 12)^2 = f(x, y, z), \text{ say} \qquad ...(i)$$

We have to find the maximum and minimum values of $f(x, y, z)$ subject to the condition

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 4 = 0 \quad ...(ii)$$

Let $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$

$$= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 4)$$

Then $\frac{\partial F}{\partial x} = 2(x-3) + 2\lambda x, \; \frac{\partial F}{\partial y} = 2(y-4) + 2\lambda y, \; \frac{\partial F}{\partial z} = 2(z-12) + 2\lambda z$

$\therefore$ $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$ give

$$x - 3 + \lambda x = 0, y - 4 + \lambda y = 0, z - 12 + \lambda z = 0 \quad ...(iii)$$

which give $\lambda = -\frac{x-3}{x} = -\frac{y-4}{y} = -\frac{z-12}{z}$

$$= \pm \frac{\sqrt{[(x-3)^2 + (y-4)^2 + (z-12)^2]}}{\sqrt{(x^2 + y^2 + z^2)}} = \pm \frac{\sqrt{f}}{1}$$

Substituting for $\lambda$ in $(iii)$, we get

$$x = \frac{3}{1+\lambda} = \frac{3}{1 \pm \sqrt{f}}, y = \frac{4}{1 \pm \sqrt{f}}, z = \frac{12}{1 \pm \sqrt{f}}$$

$\therefore$ $x^2 + y^2 + z^2 = \frac{9 + 16 + 144}{(1 \pm \sqrt{f})^2} = \frac{169}{(1 \pm \sqrt{f})^2}$

Using $(ii)$, $1 = \frac{169}{(1 \pm \sqrt{f})^2}$ or $1 \pm \sqrt{f} = \pm 13, \sqrt{f} = 12, 14.$

[We have left out the negative values of $\sqrt{f}$, because $\sqrt{f} = AP$ is + ve by $(i)$]

Hence maximum $AP = 14$ and minimum $AP = 12$.

**Example 5.48.** *Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.* *(Kurukshetra, 2006 ; U.P.T.U., 2004)*

**Solution.** Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular solid so that its volume

$$V = 8xyz \quad ...(i)$$

Let $R$ be the radius of the sphere so that $x^2 + y^2 + z^2 = R^2$ ...$(ii)$

Then $F(x, y, z) = 8xyz + \lambda(x^2 + y^2 + z^2 - R^2)$

and $\partial F/\partial x = 0, \partial F/\partial y = 0$ and $\partial F/\partial z = 0$ give

$$8yz + 2x\lambda = 0, 8zx + 2y\lambda = 0, 8xy + 2z\lambda = 0$$

or $2x^2\lambda = -8xyz = 2y^2\lambda = 2z^2\lambda$

Thus for a maximum volume $x = y = z$.

*i.e.*, the rectangular solid is a cube.

**Example 5.49.** *A tent on a square base of side x, has its sides vertical of height y and the top is a regular pyramid of height h. Find x and y in terms of h, if the canvas required for its construction is to be minimum for the tent to have a given capacity.*

**Solution.** Let $V$ be the volume enclosed by the tent and $S$ be its surface area (Fig. 5.6).

Then $V$ = cuboid $(ABCD, A'B'C'D')$ + pyramid $(K, A'B'C'D')$

$$= x^2y + \frac{1}{3}x^2h = x^2(y + h/3)$$

$$S = 4(ABGF) + 4\Delta\, KGH = 4xy + 4\frac{1}{2}(x \cdot KM)$$

$$= 4xy + x\sqrt{(x^2 + 4h^2)} \quad [\because \; KM = \sqrt{(KL^2 + LM^2)} = \sqrt{[h^2 + (x/2)^2]}]$$

For constant $V$, we have

$$\delta V = 2x(y + h/3)\,\delta x + x^2(\delta y) + \frac{x^2}{3}\,\delta h = 0$$

For minimum $S$, we have

$$\delta S = [4y + \sqrt{(x^2+4h^2)} + x\cdot\frac{1}{2}(x^2+4h^2)^{-1/2}\cdot 2x]\,\delta x$$
$$+ 4x\delta y + x\cdot\frac{1}{2}(x^2+4h^2)^{-1/2}\cdot 8h\delta h = 0$$

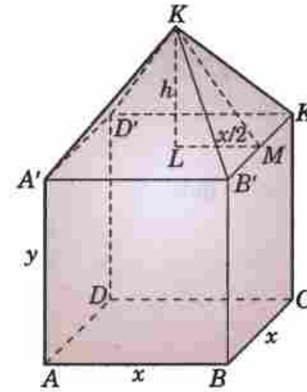

Fig. 5.6

By Lagrange's method,

$$[4y + \sqrt{(x^2+4h^2)} + x^2(x^2+4h^2)^{-1/2}] + \lambda\cdot 2x(y+h/3) = 0 \qquad ...(i)$$
$$4x + \lambda\cdot x^2 = 0 \qquad ...(ii)$$
$$4hx(x^2+4h^2)^{-1/2} + \lambda\cdot x^2/3 = 0 \qquad ...(iii)$$

$(ii)$ gives $\lambda = -4/x$. Then $(iii)$ becomes

$$4hx(x^2+4h^2)^{-1/2} - 4x/3 = 0 \quad \text{or} \quad x = \sqrt{5}\,h$$

Now putting $x = \sqrt{5}\,h$, $\lambda = -4/x$ in $(i)$, we get

$$4y + 3h + \frac{5}{3}h - \frac{4}{x}\cdot 2x(y+h/3) = 0 \quad \text{or} \quad 4y + \frac{14}{3}h - 8y - \frac{8h}{3} = 0, \quad i.e., \quad y = h/2.$$

**Example 5.50.** *If $u = a^3x^2 + b^3y^2 + c^3z^2$ where $x^{-1} + y^{-1} + z^{-1} = 1$, show that the stationary value of $u$ is given by $x = \Sigma a/a$, $y = \Sigma a/b$, $z = \Sigma a/c$.* *(Kerala, 2005)*

**Solution.** Let $u = f(x,y,z) = a^3x^2 + b^3y^2 + c^3z^2$

and $\phi(x,y,z) = x^{-1} + y^{-1} + z^{-1} - 1$

Let $F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$
$= a^3x^2 + b^3y^2 + c^3z^2 + \lambda(x^{-1} + y^{-1} + z^{-1} - 1)$

Then $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0$ and $\dfrac{\partial F}{\partial z} = 0$ give

$$2a^3x^2 - \lambda/x^2 = 0, \quad 2b^3y - \lambda/y^2 = 0, \quad 2c^3z - \lambda/z^2 = 0$$

or
$$2a^3x^3 = \lambda,\ 2b^3y^3 = \lambda,\ 2c^3z^3 = \lambda$$

which give $ax = by = cz = k$ (say) *i.e.*, $x = k/a$, $y = k/b$, $z = k/c$.

Substituting these in $x^{-1} + y^{-1} + z^{-1} = 1$, we get $k = a + b + c$

Hence the stationary value of $u$ is given by

$$x = \Sigma a/a, \quad y = \Sigma a/b \quad \text{and} \quad z = \Sigma a/c.$$

**Example 5.51.** *Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$ *(U.T.U., 2010 ; Anna, 2009 ; Madras, 2006)*

**Solution.** Let the edges of the parallelopiped be $2x$, $2y$ and $2z$ which are parallel to the axes. Then its volume $V = 8xyz$.

Now we have to find the maximum value of $V$ subject to the condition that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \qquad ...(i)$$

Write
$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

Then
$$\frac{\partial F}{\partial x} = 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \qquad ...(ii)$$

$$\frac{\partial F}{\partial y} = 8zx + \lambda\left(\frac{2y}{b^2}\right) = 0 \quad ...(iii) \qquad \frac{\partial F}{\partial z} = 8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 \quad ...(iv)$$

Equating the values of $\lambda$ from (*ii*) and (*iii*), we get $x^2/a^2 = y^2/b^2$

Similarly from (*iii*) and (*iv*), we obtain $y^2/b^2 = z^2/c^2 \quad \therefore \quad x^2/a^2 = y^2/b^2 = z^2/c^2$

Substituting these in (*i*), we get $x^2/a^2 = \frac{1}{3}$ *i.e.* $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}$

These give $x = a/\sqrt{3}$, $y = b/\sqrt{3}$, $z = c/\sqrt{3}$ ...(*v*)

When $x = 0$, the parallelopiped is just a rectangular sheet and as such its volume $V = 0$.

As $x$ increases, $V$ also increases continuously.

Thus $V$ must be greatest at the stage given by (*v*).

Hence the greatest volume $= \frac{8abc}{3\sqrt{3}}$.

## PROBLEMS 5.10

1. Find the maximum and minimum values of
   (*i*) $x^3 + y^3 - 3axy$ (*U.P.T.U., 2005*) (*ii*) $xy + a^3/x + a^3/y$.
   (*iii*) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ (*Mumbai, 2007*) (*iv*) $2(x^2 - y^2) - x^4 + y^4$ (*Osmania, 2003*)
   (*v*) $\sin x \sin y \sin (x + y)$.
2. If $xyz = 8$, find the values of $x, y$ for which $u = 5xyz/(x + 2y + 4z)$ is a maximum. (*S.V.T.U., 2007 ; Kurukshetra, 2005*)
3. Find the minimum value of $x^2 + y^2 + z^2$, given that
   (*i*) $xyz = a^3$ (*P.T.U., 2009 ; Osmania, 2003*) (*ii*) $ax + by + cz = p$. (*V.T.U., 2010 ; U.P.T.U., 2006*)
   (*iii*) $xy + yz + zx = 3a^2$ (*Anna, 2009*)
4. Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm. (*Madras, 2000 S*)
5. The sum of three numbers is constant. Prove that their product is maximum when they are equal.
6. Find the points on the surface $z^2 = xy + 1$ nearest to the origin. (*Burdwan, 2003 ; Andhra, 2000*)
7. Show that, if the perimeter of a triangle is constant, the triangle has maximum area when it is equilateral.
8. Find the maximum and minimum distances from the origin to the curve $5x^2 + 6xy + 5y^2 - 8 = 0$.
9. The temperature $T$ at any point $(x, y, z)$ in space is $T = 400\,xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$. (*V.T.U., 2009 ; Hissar, 2005 S*)
10. Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum. (*Bhillai, 2005*)
11. Find the stationary values of $u = x^2 + y^2 + z^2$ subject to $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. (*S.V.T.U., 2008*)

## 5.13 DIFFERENTIATION UNDER THE INTEGRAL SIGN

If a function $f(x, \alpha)$ of two variables $x$ and $\alpha$ (called a parameter), be integrated with respect to $x$ between the limits $a$ and $b$, then $\int_a^b f(x, \alpha)\, dx$ is a function of $\alpha$ : $F(\alpha)$, say. To find the derivative of $F(\alpha)$, when it exists, it is not always possible to first evaluate this integral and then to find the derivative. Such problems are solved by the following rules :

**(1) Leibnitz's rule***

*If* $f(x, \alpha)$ and $\frac{\partial f(x, \alpha)}{\partial \alpha}$ *be continuous functions of x and* $\alpha$, *then*

$$\frac{d}{d\alpha}\left[\int_a^b f(x, \alpha)\, dx\right] = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha}\, dx \quad \textit{where, a, b are constants independent of } \alpha.$$

*See foot note on p. 139.

Let $\displaystyle\int_a^b f(x, \alpha)dx = F(\alpha),$

then $$F(\alpha + \delta\alpha) - F(\alpha) = \int_a^b f(x, \alpha + \delta\alpha)dx - \int_a^b f(x, \alpha)dx = \int_a^b [f(x, \alpha + \delta\alpha) - f(x, \alpha)]dx$$

$$= \delta\alpha \int_a^b \frac{\partial f(x, \alpha + \theta\delta\alpha)}{\partial \alpha}dx, \ (0 < \theta < 1) \qquad \left\{\begin{array}{l}\because \ f(x, \alpha + h) - f(x, \alpha) = hf'(x, \alpha + \theta h) \\ \text{where } 0 < \theta < 1, \text{ by Mean Value Theorem}\end{array}\right.$$

Proceeding to limits as $\delta\alpha \to 0$, $\displaystyle\underset{\delta\alpha \to 0}{\text{Lt}} \frac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \int_a^b \frac{\partial f(x, \alpha + \theta \cdot 0)}{\partial \alpha}dx$

or $\displaystyle\frac{dF}{d\alpha} = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha}dx$ which is the desired result.

**Obs. 1.** Leibnitz's rule enables us to derive from the value of a simple definite integral, the value of another definite integral which it may otherwise be difficult, or even impossible, to evaluate.

**Obs. 2.** The rule for differentiation under the integral sign of an infinite integral is the same as for a definite integral.

**Example 5.52.** *Evaluate* $\displaystyle\int_0^1 \frac{x^\alpha - 1}{\log x}dx$, $\alpha \geq 0$. *(V.T.U., 2010)*

**Solution.** Let $\displaystyle F(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x}dx$ ...(*i*)

then $$F(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha}\left(\frac{x^\alpha - 1}{\log x}\right)dx = \int_0^1 \frac{x^\alpha \log x}{\log x}dx$$

$$= \int_0^1 x^\alpha \, dx = \left|\frac{x^{\alpha+1}}{\alpha + 1}\right|_0^1 = \frac{1}{1 + \alpha} \qquad \left[\because \ \frac{d}{dt}(n^t) = n^t \log n\right]$$

Now integrating both sides w.r.t. $\alpha$, $F(\alpha) = \log(1 + \alpha) + c$ ...(*ii*)

From (*i*), when $\alpha = 0$, $F(0) = 0$

$\therefore$ From (*ii*), $F(0) = \log(1) + c$, *i.e.*, $c = 0$. Hence (*ii*) gives, $F(\alpha) = \log(1 + \alpha)$.

**Example 5.53.** *Given* $\displaystyle\int_0^\pi \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{(a^2 - b^2)}}$ $(a > b)$,

*evaluate* $\displaystyle\int_0^\pi \frac{dx}{(a + b\cos x)^2}$ *and* $\displaystyle\int_0^\pi \frac{\cos x}{(a + b\cos x)^2}dx$ *(Madras, 2006)*

**Solution.** We have $\displaystyle\int_0^\pi \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{(a^2 - b^2)}}$ ...(*i*)

Differentiating both sides of (*i*) w.r.t. $a$,

$$\int_0^\pi \frac{\partial}{\partial a}\left(\frac{1}{a + b\cos x}\right)dx = \frac{\partial}{\partial a}\left\{\frac{\pi}{\sqrt{(a^2 - b^2)}}\right\}$$

*i.e.* $$\int_0^\pi \frac{-dx}{(a + b\cos x)^2} = \pi \cdot \left(-\frac{1}{2}\right)(a^2 - b^2)^{-3/2} \cdot 2a$$

$\therefore$ $$\int_0^\pi \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}}$$

Now differentiating both sides of (*i*) w.r.t. $b$,

$$\int_0^\pi -(a + b\cos x)^{-2} \cdot \cos x \, dx = \pi\left(-\frac{1}{2}\right)(a^2 - b^2)^{-3/2} \cdot (-2b)$$

$$\therefore \qquad \int_0^{\pi} \frac{\cos x}{(a + b\cos x)^2} dx = \frac{\pi b}{(a^2 - b^2)^{3/2}}.$$

**(2) Leibnitz's rule** *for variable limits of integration*

*If* $f(x, \alpha)$, $\dfrac{\partial f(x,\alpha)}{\partial \alpha}$ *be continuous functions of* $x$ *and* $\alpha$, *then*

$$\frac{d}{d\alpha}\left\{\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx\right\} = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha} dx + \frac{d\psi}{d\alpha} f[\psi(\alpha), \alpha] - \frac{d\phi}{d\alpha} f[\phi(\alpha), \alpha]$$

*provided* $\phi(\alpha)$ *and* $\psi(\alpha)$ *possesses continuous first order derivatives w.r.t.* $\alpha$.

Its proof is beyond the scope of this book.

**Example 5.54.** *Evaluate* $\displaystyle\int_0^{\alpha} \frac{\log(1+\alpha x)}{1+x^2} dx$ *and hence show that*

$$\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8}\log_e 2$$

(*Hissar, 2005 S*)

**Solution.** Let $\qquad F(\alpha) = \displaystyle\int_0^{\alpha} \frac{\log(1+\alpha x)}{1+x^2} dx \qquad$ ...(*i*)

Then by the above rule, $F'(\alpha) = \displaystyle\int_0^{\alpha} \frac{\partial}{\partial\alpha}\left(\frac{\log(1+\alpha x)}{1+x^2}\right) dx + \frac{d(\alpha)}{d\alpha}\cdot\frac{\log(1+\alpha^2)}{1+\alpha^2} - 0$

$$= \int_0^{\alpha} \frac{x}{(1+\alpha x)(1+x^2)} dx + \frac{\log(1+\alpha^2)}{1+\alpha^2} \qquad \text{...}(ii)$$

Breaking the integrand into partial fractions,

$$\int_0^{\alpha} \frac{x\,dx}{(1+\alpha x)(1+x^2)} = -\frac{\alpha}{1+\alpha^2}\int_0^{\alpha}\frac{dx}{1+\alpha x} + \frac{1}{2(1+\alpha^2)}\int_0^{\alpha}\frac{2x}{1+x^2}dx + \frac{\alpha}{1+\alpha^2}\int_0^{\alpha}\frac{dx}{1+x^2}$$

$$= -\frac{1}{1+\alpha^2}\Big|\log(1+\alpha x)\Big|_0^{\alpha} + \frac{1}{2(1+\alpha^2)}\times\Big|\log(1+x^2)\Big|_0^{\alpha} + \frac{\alpha}{1+\alpha^2}\Big|\tan^{-1}x\Big|_0^{\alpha}$$

$$= -\frac{\log(1+\alpha^2)}{1+\alpha^2} + \frac{\log(1+\alpha^2)}{2(1+\alpha^2)} + \frac{\alpha\tan^{-1}\alpha}{1+\alpha^2}$$

Substituting this value in (*ii*), $F'(\alpha) = \dfrac{\log(1+\alpha^2)}{2(1+\alpha^2)} + \dfrac{\alpha\tan^{-1}\alpha}{1+\alpha^2}$

Now integrating both sides w.r.t. $\alpha$,

$$F(\alpha) = \frac{1}{2}\int \log(1+\alpha^2)\cdot\frac{1}{1+\alpha^2}d\alpha + \int\frac{\alpha\tan^{-1}\alpha}{1+\alpha^2}d\alpha \qquad \text{[Integrating by parts]}$$

$$= \frac{1}{2}\left[\log(1+\alpha^2)\cdot\tan^{-1}\alpha - \int\frac{2\alpha}{1+\alpha^2}\cdot\tan^{-1}\alpha\,d\alpha\right] + \int\frac{\alpha\tan^{-1}\alpha}{1+\alpha^2}d\alpha + c$$

$$= \frac{1}{2}\log(1+\alpha^2)\cdot\tan^{-1}\alpha + c \qquad \text{...}(iii)$$

But from (*i*), when $\alpha = 0$, $F(0) = 0$.

$\therefore$ From (*iii*), $F(0) = 0 + c$, *i.e.*, $c = 0$. Hence (*iii*) gives, $F(\alpha) = \dfrac{1}{2}\log(1+\alpha^2)\tan^{-1}\alpha$

Putting $\alpha = 1$, we get $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}dx = F(1) = \frac{\pi}{8}\log_e 2.$

### PROBLEMS 5.11

1. Differentiating $\int_0^x \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$ under the integral sign, find the value of $\int_0^x \frac{dx}{(x^2+a^2)^2}$.

2. By successive differentiation of $\int_0^1 x^m\, dx = \frac{1}{m+1}$ w.r.t. $m$, evaluate $\int_0^1 x^m (\log x)^n\, dx$.

3. Evaluate $\int_0^\pi \log(1 + a\cos x)\, dx$, using the method of differentiation under the sign of integration.

4. Given that $\int_0^\pi \frac{dx}{a-\cos x} = \frac{\pi}{\sqrt{(a^2-1)}}$, evaluate $\int_0^\pi \frac{dx}{(a-\cos x)^2}$. (*V.T.U., 2009*)

Prove that :

5. $\int_0^\infty e^{-x}\cdot\frac{\sin ax}{x}\,dx = \tan^{-1} a.$ $\left[\textbf{Hint.} \text{ Use } \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$

6. $\int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx = \tan^{-1}\frac{1}{a}$. Hence show that $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$. (*Rohtak, 2003*)

7. $\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\,dx = \frac{\pi}{2}\log(1+a)$ where $a \ge 0$. (*V.T.U., 2010 ; S.V.T.U., 2009 ; Rohtak, 2006 S ; Anna, 2005 S*)

8. $\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\,dx = \log(1+a),\ (a > -1).$

9. $\int_0^{\pi/2} \log(\alpha^2\cos^2\theta + \beta^2\sin^2\theta)\, d\theta = \pi\log\frac{\alpha+\beta}{2}$ (*S.V.T.U., 2008*)

10. $\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}\,dx = \pi[\sqrt{(1+y)} - 1]$ (*S.V.T.U., 2008*)

11. $\int_0^\pi \frac{\log(1+\alpha\cos x)}{\cos x}\,dx = \pi\sin^{-1}\alpha.$ (*V.T.U., 2007*)

12. $\int_0^\infty e^{-x^2}\cos\alpha x\, dx = \frac{\sqrt{\pi}}{2}e^{-\alpha^2/4}$ (*Mumbai, 2009 S*)

13. $\frac{d}{da}\int_0^{a^2} \tan^{-1}\frac{x}{a}\,dx = 2a\tan^{-1} a - \frac{1}{2}\log(a^2+1)$. Verify your result by direct integration.

14. $\int_{\pi/2-\alpha}^{\pi/2} \sin\theta\cos^{-1}(\cos\alpha\cos\theta)\, d\theta = \frac{\pi}{2}(1-\cos\alpha).$ (*Burdwan, 2003*)

15. If $y = \int_0^x f(t)\sin[k(x-t)]dt$, prove that $y$ satisfies the differential equation $\frac{d^2y}{dx^2} + k^2y = k f(x)$.

## 5.14 OBJECTIVE TYPE OF QUESTIONS

### PROBLEMS 5.12

*Select the correct answer or fill up the blanks in each of the following problems :*

1. If $u = e^x(x\cos y - y\sin y)$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = ...$

2. If $x = uv,\ y = u/v$, then $\frac{\partial(x, y)}{\partial(u, v)}$ is

(*a*) $-2u/v$ (*b*) $-2v/u$ (*c*) 0 (*d*) 1. (*V.T.U., 2010*)

3. If $J_1 = \dfrac{\partial(u, v)}{\partial(x, y)}$ and $J_2 = \dfrac{\partial(x, y)}{\partial(u, v)}$, then $J_1 J_2 = ...$

4. If $u = f(y/x)$, then

(a) $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = 0$ (b) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$ (c) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$ (d) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$.

5. If $u = x^y$, then $\partial u/\partial x$ is

(a) 0 (b) $yx^{y-1}$ (c) $x^y \log x$.

6. If $x = r\cos\theta$, $y = r\sin\theta$, then

(a) $\dfrac{\partial x}{\partial r} = 1/\dfrac{\partial r}{\partial x}$ (b) $\dfrac{\partial x}{\partial r} = \dfrac{\partial r}{\partial x}$ (c) $\dfrac{\partial x}{\partial r} = 0$.

7. If $u = x^y$, then $\partial u/\partial y$ is

(a) $yx^{y-1}$ (b) 0 (c) $x^y \log x$.

8. If $u = x^3 + y^3$, then $\dfrac{\partial^2 u}{\partial x \partial y}$ is equal to

(a) $-3$ (b) 3 (c) 0 (d) $3x + 3y$ (*V.T.U., 2010 S*)

9. If $u = x^2 + 2xy + y^2 + x + y$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ is equal to

(a) $2u$ (b) $u$ (c) 0 (d) none of these.

10. If $u = \log\dfrac{x^2}{y}$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ is equal to

(a) $2u$ (b) $3u$ (c) $u$ (d) 1. (*V.T.U., 2010 S*)

11. If $x = r\cos\theta$, $y = r\sin\theta$, then $\dfrac{\partial(x, y)}{\partial(r, \theta)}$ is equal to

(a) 1 (b) $r$ (c) $1/r$ (d) 0. (*V.T.U., 2010 S*)

12. If $A = f_{xx}(a, b)$, $B = f_{xy}(a, b)$, $C = f_{yy}(a, b)$, then $f(x, y)$ will have a maximum at $(a, b)$ if

(a) $f_x = 0, f_y = 0,\ AC < B^2$ and $A < 0$ (b) $f_x = 0, f_y = 0,\ AC = B^2$ and $A > 0$
(c) $f_x = 0, f_y = 0,\ AC > B^2$ and $A > 0$ (d) $f_x = 0, f_y = 0,\ AC > B^2$ and $A < 0$.

13. If $z = \sin^{-1}\dfrac{\sqrt{x^2 + y^2}}{x + y}$, then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}$ is

(a) 0 (b) 1/2 (c) 1 (d) 2. (*Bhopal, 2008*)

14. If $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$, then $x\,\partial u/\partial x + y\,\partial u/\partial y$ equals

(a) $\sin^{-1}(x/y) + \tan^{-1}(y/x)$ (b) $2[\sin^{-1}(x/y) + \tan^{-1}(y/x)]$
(c) $3[\sin^{-1}(x/y) + \tan^{-1}(y/x)]$ (d) zero.

15. If an error of 1% is made in measuring its length and breadth, the percentage error in the area of a rectangle is

(a) 0.2% (b) 0.02% (c) 2% (d) 1%. (*V.T.U., 2010*)

16. $\dfrac{\partial(u, v)}{\partial(x, y)} \times \dfrac{\partial(x, y)}{\partial(u, v)}$ equals

(a) $-1$ (b) 1 (c) zero (d) none of these.

17. $\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$ is a homogeneous function of degree ......

18. If $z = \log(x^3 + y^3 - x^2y - xy^2)$, then $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}$ is equal to ......

19. If $r = \partial^2 f/\partial x^2$, $s = \partial^2 f/\partial x\partial y$ and $t = \partial^2 f/\partial y^2$, then the condition for the saddle point is ......

20. If $f(x, y) = \dfrac{1}{x^3} + \dfrac{1}{y^3} + \dfrac{1}{x^3 + y^3}$, then $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y}$ is

(a) 0 (b) $3f$ (c) 9 (d) $-3f$. (*V.T.U., 2009 S*)

21. If $u = x^4 + y^4 + 3x^2y^2$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = ......$

**22.** If $u$ and $v$ are functions of $r, s$ where $r, s$ are functions of $x, y$, then $\dfrac{\partial(u, v)}{\partial(r, s)} \cdot \dfrac{\partial(r, s)}{\partial(x, y)} = \ldots\ldots$

**23.** The necessary conditions for a function $f(x, y)$ to have an extreme at $(a, b)$ are ......

**24.** If $u = (x-y)^4 + (y-z)^4 + (z-x)^4$, then $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}$ is

(*a*) 1 (*b*) $u$ (*c*) $4u$ (*d*) 0. *(V.T.U., 2010)*

**25.** If $u$ is a composite function of $t$, defined by the relations $u = f(x, y); x = \phi(t), y = \psi(t)$, then total derivative $\dfrac{du}{dt} = \ldots\ldots$

**26.** If $u = \cos^{-1}(x/y) + \tan^{-1}(y/x)$, then $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$ is

(*a*) $u$ (*b*) $2u$ (*c*) 0 (*d*) 1. *(V.T.U., 2010)*

**27.** If $f(x, y, z) = 0$, then $\dfrac{\partial x}{\partial y} \cdot \dfrac{\partial y}{\partial z} \cdot \dfrac{\partial z}{\partial x} = \ldots\ldots$

**28.** If $u = f(x + ay) + g(x - ay)$, then $\dfrac{\partial^2 u}{\partial y^2}$ equals

(*a*) $\dfrac{\partial^2 u}{\partial x^2}$ (*b*) $a\dfrac{\partial^2 u}{\partial x^2}$ (*c*) $a^2\dfrac{\partial^2 u}{\partial x^2}$ (*d*) $\dfrac{\partial^2 u}{\partial x \partial y}$. *(V.T.U., 2010)*

**29.** If sum of three numbers is constant, then their product is a maximum when the numbers are ......

**30.** $y = \cosh(\lambda x)\cosh(-\lambda at)$ is a solution of $\dfrac{\partial^2 y}{\partial t^2} = a^2\dfrac{\partial^2 y}{\partial x^2}$. *(True or False)*