CHAPTER 22

# Fourier Transforms



## 22.1 INTRODUCTION

In the previous chapter, the reader has already been acquainted with the use of Laplace transforms in the solution of ordinary differential equations. In this chapter, the well-known Fourier transforms will be introduced and their properties will be studied which will be used in the solution of partial differential equations. The choice of a particular transform to be employed for the solution of an equation depends on the boundary conditions of the problem and the ease with which the transform can be inverted. A Fourier transform when applied to a partial differential equation reduces the number of its independent variables by one.

The theory of integral transforms afford mathematical devices through which solutions of numerous boundary value problems of engineering can be obtained *e.g.*, conduction of heat, transverse vibrations of a string, transverse oscillations of an elastic beam, free and forced vibrations of a membrane, transmission lines etc. Some of these applications will be illustrated in the last section.

## 22.2 DEFINITION

*The integral transform of a function f(x) denoted by I[f(x)], is defined by*

$$\bar{f}(s) = \int_{x_1}^{x_2} f(x) K(s, x)\, dx$$

where $K(s, x)$ is called the *kernel* of the transform and is a known function of $s$ and $x$. The function $f(x)$ is called the *inverse transform* of $\bar{f}(s)$.

Three simple examples of a kernel are as follows :

(*i*) *When* $K(s, x) = e^{-sx}$, *it leads to the* **Laplace transform** *of* $f(x)$, *i.e.*,

$$\bar{f}(s) = \int_0^\infty f(x) e^{-sx}\, dx. \qquad \text{[Chap. 21]}$$

(*ii*) *When* $K(s, x) = e^{isx}$, *we have the* **Fourier transform** *of* $f(x)$, *i.e.*,

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

(*iii*) When $K(s, x) = x^{s-1}$, it gives the *Mellin transform* of $f(x)$ *i.e.*,

$$M(s) = \int_0^\infty f(x)x^{s-1}\,dx.$$

Other special transforms arise when the kernel is a sine or a cosine function or a Bessel's function. These lead to *Fourier sine* or *cosine transforms* and the *Hankel transform* respectively.

In order to introduce the *Fourier transforms*, we shall first derive the Fourier integral theorem.

## 22.3 (1) FOURIER INTEGRAL THEOREM

Consider a function $f(x)$ which satisfies the Dirichlet's conditions (Art. 10.3) in every interval $(-c, c)$ so that, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right) \qquad \text{...(1)}$$

where $\quad a_0 = \frac{1}{c}\int_{-c}^{c} f(t)\,dt,\ a_n = \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}\,dt$, and $b_n = \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}\,dt.$

Substituting the values of $a_0$, $a_n$ and $b_n$ in (1), it takes the form

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c} f(t)\cos\frac{n\pi(t-x)}{c}\,dt \qquad \text{...(2)}$$

If we assume that $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges, the first term on the right side of (2) approaches 0 as $c \to \infty$, since

$$\left|\frac{1}{2c}\int_{-c}^{c} f(t)\,dt\right| \le \frac{1}{2c}\int_{-\infty}^{\infty} |f(t)|\,dt$$

The second term on the right side of (2) tends to

$$\underset{c\to\infty}{\text{Lt}}\ \frac{1}{c}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\frac{n\pi(t-x)}{c}\,dt$$

$$= \underset{\delta\lambda\to 0}{\text{Lt}}\ \frac{1}{\pi}\sum_{n=1}^{\infty}\delta\lambda\int_{-\infty}^{\infty} f(t)\cos n\delta\lambda\,(t-x)\,dt, \text{ on writing } \pi/c = \delta\lambda$$

This is of the form $\underset{\delta\lambda\to 0}{\text{Lt}}\ \sum_{n=1}^{\infty} F(n\delta\lambda)$, *i.e.*, $\int_0^\infty F(\lambda)\,d\lambda$

Thus as $c \to \infty$, (2) becomes $f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda$ ...(3)

which is known as the **Fourier integral** of $f(x)$.

**Obs.** We have given a heuristic demonstration of the Fourier integral theorem which simply helps in deriving the result (3). It cannot however, be taken as a rigorous proof for that would, involve a proof of the convergence of the Fourier integral which is beyond the scope of this book. When $f(x)$ satisfies the above-mentioned conditions, equation (3) holds good at a point of continuity. If however, $x$ is point of discontinuity, we replace $f(x)$ by $\frac{1}{2}[f(x+0) + f(x-0)]$ as in the case of Fourier series.

**(2) Fourier sine and cosine integrals.** Expanding $\cos\lambda(t-x)$, (3) may be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty \cos\lambda x\int_{-\infty}^{\infty} f(t)\cos\lambda t\,dt\,d\lambda + \frac{1}{\pi}\int_0^\infty \sin\lambda x\int_{-\infty}^{\infty} f(t)\sin\lambda t\,dt\,d\lambda \qquad \text{...(4)}$$

If $f(x)$ is an odd function, $f(t)\cos\lambda t$ is also an odd function while $f(t)\sin\lambda t$ is even. Then the first term on the right side of (4) vanishes and, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\int_0^\infty f(t)\sin\lambda t\,dt\,d\lambda \qquad \text{...(5)}$$

which is known as the *Fourier sine integral*.

Similarly, if $f(x)$ is an even function, (4) takes the form

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t\, dt\, d\lambda \quad ...(6)$$

which is known as the *Fourier cosine integral*.

**Obs.** A function $f(x)$ defined in the interval $(0, \infty)$ is expressed either as a Fourier sine integral or as a Fourier cosine integral, merely looking upon it as an odd or even function in $(-\infty, \infty)$ on the lines of half-range Fourier series.

**(3) Complex form of Fourier integrals.** Equation (3) can be written as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t-x)dtd\lambda \quad ..(7)$$

because $\cos\lambda(t-x)$ is an even function of $\lambda$. Also since $\sin\lambda(t-x)$ is an odd function of $\lambda$, we have

$$0 = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\sin\lambda(t-x)dtd\lambda \quad ...(8)$$

Now multiply (8) by $i$ and add it to (7), so that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\, e^{i\lambda(t-x)}dtd\lambda \quad ...(9)$$

which is the *complex* form of the *Fourier integral*.

**(4) Fourier integral representation of a function**

Using (4), *a function F(x) may be represented by a Fourier integral as*

$$F(x) = \frac{1}{\pi}\int_0^\infty [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\, d\lambda$$

where $$A(\lambda) = \int_{-\infty}^\infty f(t)\cos\lambda t\, dt\ ;\ B(\lambda) = \int_{-\infty}^\infty f(t)\sin\lambda t\, dt \quad ...(10)$$

*If $f(x)$ is an odd function*, then [By (5)]

$$f(x) = \frac{1}{\pi}\int_0^\infty B(\lambda)\ \sin\lambda x\, d\lambda \text{ where } B(\lambda) = 2\int_0^\infty f(t)\sin\lambda t\, dt \quad ...(11)$$

*If $f(x)$ is an even function*, then [By (6)]

$$f(x) = \frac{1}{\pi}\int_0^\infty A(\lambda)\cos\lambda x\, d\lambda \text{ where } A(\lambda) = 2\int_0^\infty f(t)\cos\lambda t\, dt \quad ...(12)$$

**Example 22.1.** *Express $f(x) = 1$ for $0 \le x \le \pi$,*
*$= 0$ for $x > \pi$,*
*as a Fourier sine integral and hence evaluate*

$$\int_0^\infty \frac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)d\lambda.$$

*(Kottayam, 2005 ; J.N.T.U., 2004 S)*

**Solution.** The Fourier sine integral for $f(x) = \frac{2}{\pi}\int_0^\infty \sin(\lambda x)\, d\lambda \int_0^\infty f(t)\sin(\lambda t)\, dt$

$$= \frac{2}{\pi}\int_0^\infty \sin(\lambda x)\, d\lambda \int_0^\pi \sin(\lambda t)\, dt$$

$$= \frac{2}{\pi}\int_0^\infty \sin(\lambda x)\, d\lambda \left|\frac{-\cos(\lambda t)}{\lambda}\right|_0^\pi = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(\lambda\pi)}{\lambda}\sin(\lambda x)d\lambda$$

$$\therefore \quad \int_0^\infty \frac{1-\cos(\lambda\pi)}{\lambda}\sin(\lambda x)d\lambda = \frac{\pi}{2}f(x) = \begin{cases}\pi/2 & \text{for } 0 \le x < \pi \\ 0 & \text{for } x > \pi\end{cases}$$

At $x = \pi$, which is a point of discontinuity of $f(x)$, the value of the above integral

$$= \frac{\pi}{2}\left[\frac{f(\pi-0)+f(\pi+0)}{2}\right] = \frac{\pi}{2}\cdot\frac{1+0}{2} = \frac{\pi}{4}.$$

## 22.4 (1) FOURIER TRANSFORMS

Rewriting (9) of § 22.3 as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\, ds \int_{-\infty}^{\infty} f(t)e^{ist}\, dt,$$

it follows that if
$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \qquad ...(1)$$

then
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \qquad ...(2)$$

*The function F(s), defined by (1), is called the* **Fourier transform** of $f(x)$. *Also the function* $f(x)$, *as given by* (2), *is called the* **inverse Fourier transform** *of* $F(s)$. Sometimes, we call (2) as an *inversion formula* corresponding to (1).

**(2) Fourier sine and cosine transforms.** From (5) of § 22.3, it follows that if

$$F_s(s) = \int_0^{\infty} f(x)\sin sx\, dx \qquad ...(3)$$

then
$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(s)\sin sx\, ds \qquad ...(4)$$

*The function* $F_s(s)$, *as defined by* (3), *is known as the* **Fourier sine transform** *of* $f(x)$ *in* $0 < x < \infty$. *Also the function* $f(x)$, *as given by* (4) *is called the* **inverse Fourier sine transform** *of* $F_s(s)$.

Similarly, if follows from (6) of § 22.3 that if

$$F_c(s) = \int_0^{\infty} f(x)\cos sx\, dx \qquad ...(5)$$

then
$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(s)\cos sx\, ds \qquad ...(6)$$

*The function* $F_c(s)$ *as defined by* (5) *is known as the* **Fourier cosine transform** *of* $f(x)$ *in* $0 < x < \infty$. *Also the function* $f(x)$, *as given by* (6), *is called the* **inverse Fourier cosine transform** *of* $F_c(s)$.

(3) **Finite Fourier sine and cosine transforms.** These transforms are useful for such a boundary-value problem in which at least two of the boundaries are parallel and separated by a finite distance.

*The* **finite Fourier sine transform** *of* $f(x)$, *in* $0 < x < c$, *is defined as*

$$F_s(n) = \int_0^{c} f(x)\sin\frac{n\pi x}{c}\, dx \qquad ...(7)$$

where $n$ is an integer.

*The function* $f(x)$ *is then called the* **inverse finite Fourier sine transform** *of* $F_s(n)$ *which is given by*

$$f(x) = \frac{2}{c}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{c} \qquad ...(8)$$

*The* **finite Fourier cosine transform** *of* $f(x)$, *in* $0 < x < c$, *is defined as*

$$F_c(n) = \int_0^{c} f(x)\cos\frac{n\pi x}{c}\, dx \qquad ...(9)$$

where $n$ is an integer.

*The function* $f(x)$ *is then called the* **inverse finite Fourier cosine transform** *of* $F_c(n)$ *which is given by*

$$f(x) = \frac{1}{c}F_c(0) + \frac{2}{c}\sum_{n=1}^{\infty} F_c(n)\cos\frac{n\pi x}{c} \qquad ...(10)$$

**Obs.** The finite Fourier sine transform is useful for problems involving boundary conditions of heat distribution on two parallel boundaries, while the finite cosine transform is useful for problems in which the velocities normal to two parallel boundaries are among the boundary conditions.

## 22.5 PROPERTIES OF FOURIER TRANSFORMS

(1) **Linear property.** *If* $F(s)$ *and* $G(s)$ *are Fourier transforms of* $f(x)$ *and* $g(x)$ *respectively, then*

$$F[a\, f(x) + bg(x)] = a\, F(s) + bG(s)$$

*where a and b are constants.*

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$ and $G(s) = \int_{-\infty}^{\infty} e^{isx} g(x)\,dx$

$$\therefore \quad F[af(x) + bg(x)] = \int_{-\infty}^{\infty} e^{isx} [af(x) + bg(x)]\,dx = a\int_{-\infty}^{\infty} e^{isx} f(x)\,dx + b\int_{-\infty}^{\infty} e^{isx} g(x)dx$$
$$= aF(s) + bG(s)$$

**(2) Change of scale property.** *If F(s) is the complex Fourier transform of f(x), then*

$$F\{f(ax)\} = \frac{1}{a}F\left(\frac{s}{a}\right), a \neq 0$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx$ ...(*i*)

$$\therefore \quad F\{f(ax)\} = \int_{-\infty}^{\infty} e^{isx} f(ax)dx \qquad \left| \begin{array}{l}\text{Put } ax = t \\ \text{so that } dx = dt/a\end{array}\right.$$

$$= \int_{-\infty}^{\infty} e^{ist/a} f(t)dt/a = \frac{1}{a}\int_{-\infty}^{\infty} e^{i(s/a)t} f(t)dt = \frac{1}{a}F\left(\frac{s}{a}\right) \qquad \text{[By } (i)]$$

**Cor.** *If $F_s(s)$ and $F_c(s)$ are the Fourier sine and cosine transforms of f (x) respectively, then*

$$F_s\{f(ax)\} = \frac{1}{a}F_s\left(\frac{s}{a}\right) \quad \text{and} \quad F_c\{f(ax)\} = \frac{1}{a}F_s\left(\frac{s}{a}\right).$$

**(3) Shifting property.** *If F(s) is the complex Fourier transform of f (x), then*

$$F\{f(x-a)\} = e^{isa} F(s)$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx$ ...(*i*)

$$\therefore \quad F\{f(x-a)\} = \int_{-\infty}^{\infty} e^{isx} f(x-a)dx \qquad \left| \begin{array}{l}\text{Put } x - a = t \\ \text{so that } dx = dt\end{array}\right.$$

$$= \int_{-\infty}^{\infty} e^{is(t+a)} f(t)dt = e^{isa}\int_{-\infty}^{\infty} e^{ist} f(t)dt = e^{isa} F(s) \qquad \text{[By } (i)]$$

**(4) Modulation theorem.** *If F(s) is the complex Fourier transform of f (x), then*

$$F\{f(x)\cos ax\} = \frac{1}{2}[F(s+a) + F(s-a)]$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx$ ...(*i*)

$$\therefore \quad F\{f(x)\cos ax\} = \int_{-\infty}^{\infty} e^{isx} f(x)\cos ax\,dx = \int_{-\infty}^{\infty} e^{isx}\,.\,f(x)\,.\,\frac{e^{iax} + e^{-iax}}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{i(s+a)x} f(x)dx + \int_{-\infty}^{\infty} e^{i(s-a)x} f(x)dx\right] = \frac{1}{2}[F(s+a) + F(s-a)].$$

**Cor.** *If $F_s(s)$ and $F_c(s)$ are Fourier sine and cosine transforms of f (x) respectively, then*

(*i*) $F_s\{f(x)\cos ax\} = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$ (*Anna, 2008*)

(*ii*) $F_c\{f(x)\sin ax\} = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$

(*iii*) $F_s\{f(x)\sin ax\} = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

**Obs.** This theorem is of great importance in radio and television where the harmonic carrier wave is modulated by an envelope.

**Example 22.2.** *Find the Fourier transform of*

$$f(x) = \begin{cases} 1 \text{ for } |x| < 1 \\ 0 \text{ for } |x| > 1 \end{cases}.$$

*Hence evaluate* $\int_0^{\infty} \frac{\sin x}{x}\,dx.$ (*V.T.U., 2010 ; S.V.T.U., 2009 ; U.P.T.U., 2008*)

**Solution.** The Fourier transform of $f(x)$, *i.e.*,

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{isx}\,dx = \int_{-1}^{1}(1)\,e^{isx}\,dx = \left|\frac{e^{isx}}{is}\right|_{-1}^{1} = \frac{e^{is}-e^{-is}}{is}$$

Thus $\quad F\{f(x)\} = F(s) = 2\,\dfrac{\sin s}{s}$, $s \neq 0$. For $s = 0$, we have $F(s) = 2$.

Now by the inversion formula, we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds, \text{ or } \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin s}{s}\,e^{-isx}ds = \begin{cases}1 \text{ for } |x| < 1\\ 0 \text{ for } |x| > 1\end{cases}$$

Putting $x = 0$, we get

$$\int_{-\infty}^{\infty}\frac{\sin s}{s}\,ds = \pi \quad \therefore \quad \int_{0}^{\infty}\frac{\sin s}{s}\,ds = \frac{\pi}{2}, \text{ since the integrand is even.}$$

**Example 22.3.** *Find the Fourier transform of :*

$$f(x) = \begin{cases}1 - x^2, & |x| \le 1\\ 0 & , |x| > 1\end{cases}$$

*Hence evaluate* $\displaystyle\int_{0}^{\infty}\frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\,dx.$ *(V.T.U., 2011 S ; Anna, 2005 S ; Mumbai, 2005 S)*

**Solution.** $F\{f(x)\} = \displaystyle\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx = F(s)$, say

$$= \int_{-\infty}^{-1}(0)\,e^{isx}\,dx + \int_{-1}^{1}(1-x^2)\,e^{isx}\,dx + \int_{1}^{\infty}(0)\,e^{isx}\,dx = \left|(1-x^2)\frac{e^{isx}}{is} - (2x)\frac{e^{isx}}{(is)^2} + (-2)\frac{e^{isx}}{(is)^3}\right|_{-1}^{1}$$

$$= 2\left(\frac{e^{is}+e^{-is}}{-s^2}\right) - 2\left(\frac{e^{is}-e^{-is}}{-is^3}\right) = -\frac{4}{s^3}(s\cos s - \sin s)$$

Now by inversion formula, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(s)e^{-isx}\,ds$$

or

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{s^3}(s\cos s - \sin s)\,e^{-isx}\,ds = \begin{cases}1-x^2, & |x| \le 1\\ 0, & |x| > 1\end{cases}$$

Putting $x = 1/2$, we obtain

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{s^3}(s\cos s - \sin s)e^{-is/2}ds = \frac{3}{4}$$

or

$$\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\left(\cos\frac{s}{2} - i\sin\frac{s}{2}\right)ds = -\frac{3\pi}{8}$$

or

$$\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\,.\,\cos\frac{s}{2}\,ds = -\frac{3\pi}{8}$$

or

$$\int_{0}^{\infty}\frac{x\cos x - \sin x}{x^3}\,.\,\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}, \text{ since the integral is even.}$$

**Example 22.4.** (*a*) *Find the Fourier transform of* $e^{-a^2x^2}$, $a < 0$. *Hence deduce that* $e^{-x^2/2}$ *is self reciprocal in respect of Fourier transform.* (Madras, 2006 ; Kottayam, 2005)

(*b*) *Find Fourier transform of* (*i*) $e^{-2(x-3)^2}$ (*ii*) $e^{-x^2}\cos 3x$.

**Solution.** (*a*) $\quad F(e^{-a^2x^2}) = \displaystyle\int_{-\infty}^{\infty} e^{-a^2x^2}\,.\,e^{isx}\,dx = \int_{-\infty}^{\infty} e^{-a^2(x^2 - isx/a^2)}dx$

$$= \int_{-\infty}^{\infty} e^{-a^2(x - is/2a^2)^2}\,.\,e^{-s^2/4a^2}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-t^2} \cdot e^{-s^2/4a^2}\, dt/a \qquad \text{[Putting } a(x - is/2a^2) = t,\ dx = dt/a$$

$$= \frac{e^{-s^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-t^2}\, dt = \frac{e^{-s^2/4a^2}}{a} \sqrt{\pi} \qquad \left[\because \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}\right]$$

Hence $\qquad F(e^{-a^2x^2}) = \dfrac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$

Taking $a^2 = 1/2$, we have

$$F(e^{-x^2/2}) = \frac{\sqrt{\pi}}{(1/\sqrt{2})} e^{-s^2/2} = \sqrt{2\pi} e^{-s^2/2}$$

*i.e.*, Fourier transform of $e^{-x^2/2}$ is a constant times $e^{-s^2/2}$. Also the functions $e^{-x^2/2}$ and $e^{-s^2/2}$ are the same. Hence it follows that $e^{-x^2/2}$ *is self-reciprocal under the Fourier transform.*

(*b*) Since $e^{-2x^2} = e^{-(2x)^2/2} = f(2x)$ where $f(x) = e^{-x^2/2}$

$\therefore$ by change of scale property, $F[f(2x)] = \dfrac{1}{2} F(s/2)$

*i.e.*, $$F(e^{-2x^2}) = F[e^{-(2x)^2/2}] = \sqrt{2\pi} e^{-(s/2)^2/2} = \sqrt{2\pi} e^{-s^2/8}$$

By shifting property $F f(x-3) = e^{i3s} F(3)$

$$\therefore \quad F[e^{-2(x-3)^2}] = e^{3is} \sqrt{2\pi} e^{-s^2/8} = \sqrt{2\pi} e^{(3is - s^2/8)} \qquad \text{...}(i)$$

Also by modulation theorem,

$$F[f(x) \cos 2x] = \frac{1}{2} [F(s+a) + F(s-a)]$$

$$F(e^{-x^2} \cos 3x) = \frac{1}{2} \sqrt{2\pi} [e^{-(s+3)^2/2} + e^{-(s-3)^2/2}]. \qquad \text{...}(ii)$$

**Example 22.5.** *Find the Fourier cosine transform of* $e^{-x^2}$. *(V.T.U., 2010; Rajasthan, 2006)*

**Solution.** We have $F_c(e^{-x^2}) = \int_0^{\infty} e^{-x^2} \cos sx\, dx = I$ (say)

Differentiating under the integral sign w.r.t. $s$,

$$\frac{dI}{ds} = -\int_0^{\infty} x e^{-x^2} \sin sx\, dx = \frac{1}{2} \int_0^{\infty} (\sin sx)(-2xe^{-x^2}) dx$$

$$= \frac{1}{2} \left\{ \left| \sin sx \cdot e^{-x^2} \right|_0^{\infty} - s \int_0^{\infty} \cos sx \cdot e^{-x^2}\, dx \right\}$$

$$= -\frac{s}{2} \int_0^{\infty} e^{-x^2} \cos sx\, dx = -\frac{s}{2} I \quad \text{or} \quad \frac{dI}{I} = -\int \frac{s}{2} ds + \log c$$

or $$\log I = -\frac{s^2}{4} + \log c = \log (ce^{-s^2/4})$$

$$\therefore \quad I = ce^{-s^2/4} \quad \text{or} \quad \int_0^{\infty} e^{-x^2} \cos sx\, dx = ce^{-s^2/4}$$

Putting $s = 0$, $\qquad c = \int_0^{\infty} e^{-x^2}\, dx = \dfrac{\sqrt{\pi}}{2}$. *i.e.* $I = \dfrac{\sqrt{\pi}}{2} e^{-s^2/4}$.

Hence $\qquad F_c(e^{-x^2}) = \dfrac{\sqrt{\pi}}{2} e^{-s^2/4}$.

**Example 22.6.** *Find the Fourier sine transform of* $e^{-|x|}$.

*Hence show that* $\int_0^{\infty} \dfrac{x \sin mx}{1+x^2} dx = \dfrac{\pi e^{-m}}{2}$, $m > 0$. *(V.T.U., 2010 ; S.V.T.U., 2008 ; Kottayam, 2005)*

**Solution.** $x$ being positive in the interval $(0, \infty)$, $e^{-|x|} = e^{-x}$

$\therefore$ Fourier sine transform of $f(x) = e^{-|x|}$ is given by

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx = \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \left| \frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \right|_0^\infty = \frac{s}{1+s^2}$$

Using Inversion formula for Fourier sine transforms, we get

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s\{f(x)\} \sin sx \, dx \quad \text{or} \quad e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin sx \, ds$$

or changing $x$ to $m$, $$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{s \sin ms}{1+s^2} ds = \frac{2}{\pi} \int_0^\infty \frac{x \sin mx}{1+m^2} dx$$

Hence $$\int_0^\infty \frac{x \sin mx}{1+m^2} dx = \frac{\pi e^{-m}}{2}.$$

**Example 22.7.** *Find the Fourier cosine transform of* $f(x) = \begin{cases} x, \text{ for } 0 < x < 1 \\ 2 - x, \text{ for } 1 < x < 2 \\ 0, \text{ for } x > 2 \end{cases}$ *(J.N.T.U., 2006)*

**Solution.** Fourier cosine transform of $f(x)$ *i.e.*, $F_c[f(x)]$

$$= \int_0^\infty f_c(x) \cos sx \, dx = \int_0^1 x \cos sx \, dx + \int_1^2 (2-x) \cos sx \, dx + \int_2^\infty 0 \, . \, dx$$

$$= \left| x \frac{\sin sx}{s} - \left( \frac{-\cos sx}{s^2} \right) \right|_0^1 + \left| (2-x) \frac{\sin sx}{s} - (-1) \frac{-\cos sx}{s^2} \right|_1^2$$

$$= \left( \frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} \right) + \left( -\frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2} \right)$$

$$= \frac{2 \cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2}.$$

**Example 22.8.** *Find the Fourier sine transform of* $e^{-ax}/x$. *(V.T.U., 2010 S ; P.T.U., 2006 ; Rohtak, 2005)*

**Solution.** Let $f(x) = e^{-ax}/x$, then its Fourier sine transform

*i.e.* $$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx = \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx = F(s), \text{ say}$$

Differentiating both sides w.r.t. $s$, we get

$$\frac{d}{ds}\{F(s)\} = \int_0^\infty \frac{xe^{-ax} \cos sx}{x} dx = \int_0^\infty e^{-ax} \cos sx \, dx = \frac{a}{s^2 + a^2}$$

Integrating w.r.t. $s$, we obtain $F(s) = \int_0^\infty \frac{a}{s^2+a^2} ds = \tan^{-1} \frac{s}{a} + c$

But $F(s) = 0$, when $s = 0$; $\therefore$ $c = 0$. Hence $F(s) = \tan^{-1}(s/a)$.

**Example 22.9.** *Find the Fourier cosine transform of* $f(x) = 1/(1+x^2)$. *(V.T.U., 2011 S ; Anna, 2009)*
*Hence derive Fourier sine transform of* $\phi(x) = x/(1+x^2)$. *(V.T.U., 2009 S)*

**Solution.** $$F_c[f(x)] = \int_0^\infty \frac{\cos sx}{1+x^2} dx = I, \text{ say} \quad ...(i)$$

$\therefore$ $$\frac{dI}{ds} = \int_0^\infty \frac{-x \sin sx}{1+x^2} dx = -\int_0^\infty \frac{x^2 \sin sx}{x(1+x^2)} dx \quad ...(ii)$$

$$= -\int_0^\infty \frac{[(1+x^2)-1]\sin sx}{x(1+x^2)}dx = -\int_0^\infty \frac{\sin sx}{x}dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)}dx$$

or $$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}dx \quad ...(iii)$$

$\therefore$ $$\frac{d^2I}{ds^2} = \int_0^\infty \frac{x\cos sx}{x(1+x^2)}dx = I$$

or $$\frac{d^2I}{ds^2} - I = 0 \quad \text{or} \quad (D^2-1)I = 0, \text{ where } D = \frac{dI}{ds}$$

Its solution is $$I = c_1e^s + c_2e^{-s} \quad ...(iv)$$

$\therefore$ $$dI/ds = c_1e^s - c_2e^{-s} \quad ...(v)$$

When $s = 0$, (*i*) and (*iv*) give $c_1 + c_2 = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$

Also when $s = 0$, (*iii*) and (*v*) give $c_1 - c_2 = -\pi/2$.

Solving these, $c_1 = 0, c_2 = \pi/2$.

Thus from (*i*) and (*iv*), we have $F_c[f(x)] = I = (\pi/2)e^{-s}$

Now $$F_s[\phi(x)] = \int_0^\infty \frac{x\sin sx}{1+x^2}dx = -\frac{dI}{ds}, \text{ from } (ii)$$

$$= (\pi/2)e^{-s}, \text{ from } (v), \text{ with } c_1 = 0, c_2 = \pi/2.$$

**Example 22.10.** *Find the Fourier sine and cosine transform of $x^{n-1}$, $n > 0$.* (*Madras, 2006*)

**Solution.** We know that $$F_s(x^{n-1}) = \int_0^\infty x^{n-1}\sin sx\,dx \quad ...(i)$$

and $$F_c(x^{n-1}) = \int_0^\infty x^{n-1}\cos sx\,dx \quad ...(ii)$$

$\therefore$ $$F_c(x^{n-1}) + i\,F_s(x^{n-1}) = \int_0^\infty (\cos sx + i\sin sx)x^{n-1}dx$$

$$= \int_0^\infty e^{isx}x^{n-1}dx = \int_0^\infty e^{-t}\left(-\frac{t}{is}\right)^{n-1}\left(-\frac{dt}{is}\right) \quad [\text{Where } isx = -t]$$

$$= \left(-\frac{1}{(i)}\right)^n \int_0^\infty e^{-t}t^{n-1}dt = \frac{(i)^{2n}}{(i)^n s^n}\Gamma(n) = \frac{(i)^n}{s^n}\Gamma(n)$$

$$= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^n \Gamma(n)/s^n = \left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right)\Gamma(n)/s^n$$

Equating real and imaginary parts, we get

$$F_c(x^{n-1}) = \frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2} \quad \text{and} \quad F_s(x^{n-1}) = \frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2}.$$

**Example 22.11.** (*a*) *Show that* $F_s[x\,f(x)] = -\frac{d}{ds}\{F_c(s)\}$; $F_c[x\,f(x)] = \frac{d}{ds}\{F_s(s)\}$.

(*b*) *Find the Fourier sine and cosine transform of* $xe^{-ax}$ (*Madras, 2006*)

**Solution.** (*a*) $$\frac{d}{ds}\{F_c(s)\} = \frac{d}{ds}\left\{\int_0^\infty f(x)\cos sx\,dx\right\} = \int_0^\infty f(x)(-x\sin sx)dx$$

$$= -\int_0^\infty \{x\,f(x)\}\sin sx dx = -F_s[x\,f(x)] \quad ...(i)$$

$$\frac{d}{ds}\{F_s(s)\} = \frac{d}{ds}\left\{\int_0^\infty f(x)\sin sx\,dx\right\} = \int_0^\infty f(x)(x\cos sx)dx$$

$$= \int_0^\infty \{x\,f(x)\}\cos sx dx = F_c[x\,f(x)] \quad ...(ii)$$

(*b*) We have $$F_s(e^{-ax}) = \int_0^\infty e^{-ax} \sin sx\, dx = \frac{e^{-ax}}{a^2+s^2} \left| -a \sin sx - s \cos sx \right|_0^\infty$$

$$= \frac{s}{a^2+s^2} \qquad \text{...(iii)}$$

and $$F_c(e^{-ax}) = \int_0^\infty e^{-ax} \cos sx\, dx = \frac{e^{-ax}}{a^2+s^2} \left| -a \cos sx + s \sin sx \right|_0^\infty$$

$$= \frac{a}{a^2+s^2} \qquad \text{...(iv)}$$

Now $$F_s(xe^{-ax}) = -\frac{d}{ds}\{F_c(e^{-ax})\} \qquad \text{[by (i)}$$

$$= -\frac{d}{ds}\left(\frac{a}{a^2+s^2}\right) = \frac{2as}{(a^2+s^2)^2} \qquad \text{[by (iv)}$$

$$F_c(xe^{-ax}) = \frac{d}{ds}\{F_s(e^{-ax})\} \qquad \text{[by (ii)}$$

$$= \frac{d}{ds}\left(\frac{s}{a^2+s^2}\right) = \frac{(a^2+s^2) - s\,(2s)}{(a^2+s^2)^2} = \frac{a^2-s^2}{(a^2+s^2)^2}. \qquad \text{[by (iii)}$$

**Example 22.12.** *If the Fourier sine transform of* $f(x) = \dfrac{1-\cos n\pi}{n^2\pi^2}$ $(0 \le x \le \pi)$, *find* $f(x)$. *(Delhi, 2002)*

**Solution.** We have $f(x)$ = inverse finite Fourier sine transform of $F_s(n)$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} F_s(n) \sin\frac{n\pi x}{\pi} = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1-\cos n\pi}{n^2\pi^2}\right\} \sin nx$$

$$= \frac{2}{\pi^3}\sum_{n=1}^{\infty}\left\{\frac{1-\cos n\pi}{n^2}\right\} \sin nx.$$

**Example 22.13.** *Solve the integral equation**

$$\int_0^\infty f(\theta) \cos \alpha\theta\, d\theta = \begin{cases} 1-\alpha, & 0 \le a \le 1 \\ 0, & \alpha > 1 \end{cases}$$

*Hence evaluate* $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\, dt.$ *(V.T.U., 2011 S ; Kurukshetra, 2005)*

**Solution.** We have $$\int_0^\infty f(\theta) \cos \alpha\theta\, d\theta = F_c(\alpha)$$

$\therefore$ $$F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \le a \le 1 \\ 0, & \alpha > 1 \end{cases} \qquad \text{...(i)}$$

By the inversion formula, we have

$$f(\theta) = \frac{2}{\pi}\int_0^\infty F_c(\alpha) \cos \alpha\theta\, d\alpha = \frac{2}{\pi}\int_0^1 (1-\alpha) \cos \alpha\theta\, d\alpha \qquad \text{[Integrating by parts]}$$

$$= \frac{2}{\pi}\left[\left|(1-\alpha)\frac{\sin \alpha\theta}{\theta}\right|_0^1 - \int_0^1 (-1)\frac{\sin \alpha\theta}{\theta}\, d\alpha\right] = \frac{2}{\pi\theta}\left|-\frac{\cos \alpha\theta}{\theta}\right|_0^1 = \frac{2(1-\cos\theta)}{\pi\theta^2}$$

Now $$F_c(\alpha) = \int_0^\infty f(\theta) \cos \alpha\theta\, d\theta = \int_0^\infty \frac{2(1-\cos\theta)}{\pi\theta^2} \cos \alpha\theta\, d\theta \qquad \text{...(ii)}$$

* Refer to Chapter 26.

$\therefore$ From (*i*) and (*ii*), we have

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos\theta}{\theta^2}\cos\alpha\theta\, d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0 & , \alpha > 1 \end{cases}$$

Now letting $\alpha \to 0$, we get $\frac{2}{\pi}\int_0^\infty \frac{1-\cos\theta}{\theta^2}\, d\theta = 1$ (*V.T.U., 2008*)

or $$\int_0^\infty \frac{2\sin^2\theta/2}{\theta^2}\, d\theta = \pi/2$$ [Put $\theta/2 = t$, so that $d\theta = 2dt$]

$\therefore$ $$\int_0^\infty \frac{\sin^2 t}{t^2} = \pi/2.$$

## PROBLEMS 22.1

1. Express the function $f(x) = \begin{cases} 1 \text{ for } |x| \le 1 \\ 0 \text{ for } |x| > 1 \end{cases}$ as a Fourier integral.

   Hence evaluate $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\, d\lambda$. (*Kottayam, 2005*)

2. Find the Fourier integral representation for

   (*i*) $f(x) = \begin{cases} 1-x^2, \text{ for } |x| \le 1 \\ 0, \text{ for } |x| > 1 \end{cases}$ (*Mumbai, 2008*) (*ii*) $f(x) = \begin{cases} e^{ax}, \text{ for } x \le 0, a > 0 \\ e^{-ax}, \text{ for } x \ge 0\ a < 0 \end{cases}$

3. Using the Fourier integral representation, show that

   (*i*) $\int_0^\infty \frac{\omega\sin x\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2}e^{-x}\ (x > 0)$ (*ii*) $\int_0^\infty \frac{\cos\omega x}{1+\omega^2}\, d\omega = \frac{\pi}{2}e^{-x}\ \ (x \ge 0)$ (*U.P.T.U., 2008*)

   (*iii*) $\int_0^\infty \frac{\sin\omega\cos x\omega}{\omega}\, d\omega = \frac{\pi}{2}$ when $0 \le x < 1$. (*iv*) $\int_0^\infty \frac{\sin\pi\alpha\sin\alpha\theta}{1-\alpha^2}\, d\alpha = \begin{cases} \frac{1}{2}\pi\sin\theta, & 0 \le \theta \le \pi \\ 0, & \theta > \pi \end{cases}$

4. Find the Fourier transforms of

   (*i*) $f(x) = \begin{cases} 1, |x| < a \\ 0, |x| > a \end{cases}$ (*W.B.T.U., 2005 ; Madras, 2003 ; P.T.U., 2003*)

   Hence evaluate $\int_{-\infty}^\infty \frac{\sin ax}{x}\, dx$ (*Mumbai, 2009*)

   (*ii*) $f(x) = \begin{cases} x^2, |x| < a \\ 0, |x| > a \end{cases}$ (*S.V.T.U., 2008*)

5. Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2 \text{ for } |x| \le a \\ 0 \text{ for } |x| > a \end{cases}$ (*V.T.U., 2007*)

   Hence deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}$. (*Anna, 2009*)

6. Given $F(e^{-x^2}) = \sqrt{\pi}e^{-s^2/4}$, find the Fourier transform of

   (*i*) $e^{-x^2/3}$ (*ii*) $e^{-4(x-3)^2}$

7. Find the Fourier sine and cosine transforms of $f(x) = \begin{cases} 1, 0 \le x < 2 \\ 0, x \ge 2 \end{cases}$ (*V.T.U., 2008*)

8. Using the Fourier sine transform of $e^{-ax}$ $(a > 0)$, show that $\int_0^\infty \frac{x\sin kx}{a^2+x^2}\, dx = \frac{\pi}{2}e^{-ak}\ (k > 0)$.

   Hence obtain the Fourier sine transform of $x/(a^2+x^2)$. (*Rohtak, 2006 ; Madras, 2003 S*)

9. Find the Fourier cosine transform of $e^{-ax}$. (*Anna, 2009*)

   Hence evaluate $\int_0^\infty \frac{\cos\lambda x}{x^2+\alpha^2}\, dx$. (*V.T.U., 2003 S*)

10. If the Fourier sine transform of $f(x)$ is $e^{-as}/s$, find $f(x)$. Hence obtain the inverse Fourier sine transform of $1/s$. (*Mumbai, 2009*)

11. Find the Fourier cosine transform of $e^{-x^2}$ and hence evaluate Fourier sine transform of $xe^{-x^2}$.

12. Find the Fourier cosine transform of $e^{-a^2x^2}$ for any $a > 0$ and hence prove that $e^{-x^2/2}$ is self-reciprocal under Fourier cosine transform. *(Anna, 2009)*

13. Find the Fourier sine transform of (*i*) $\dfrac{1}{x(x^2+a^2)}$. *(Rohtak, 2006)*

(*ii*) $[e^{-ax}/x]$, $a > 0$ *(U.P.T.U., 2008)*

14. Obtain Fourier sine transform of

(*i*) $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$ *(Madras, 2000)* (*ii*) $f(x) = \begin{cases} 4x, & \text{for } 0 < x < 1 \\ 4 - x, & \text{for } 1 < x < 4 \\ 0, & \text{for } x > 4 \end{cases}$ *(V.T.U., 2006)*

15. Find the Fourier cosine transform of $(1 - x/\pi)^2$. *(P.T.U., 2006)*

16. Find the finite Fourier sine and cosine transforms of $f(x) = 2x$, $0 < x < 4$. *(V.T.U., 2011)*

17. Find the finite sine transform of $f(x) = \begin{cases} -x, & x < c \\ \pi - x, & x > c \end{cases}$ where $0 \le c \le \pi$.

18. Show that the inverse finite Fourier sine transform of $F_s(n) = \dfrac{1}{\pi}\left\{1 + \cos n\pi - 2\cos\dfrac{n\pi}{2}\right\}$ is

$$f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ -1, & \pi/2 < x < \pi \end{cases}$$ *(V.T.U., 2008)*

19. Solve the integral equation $\displaystyle\int_0^\infty f(x)\sin tx\, dx = \begin{cases} 1, & 0 \le t < 1, \\ 2, & 1 \le t < 2, \\ 0, & t \ge 2 \end{cases}$ *(Kottayam, 2005)*

20. Solve the integral equation $\displaystyle\int_0^\infty f(x)\cos \alpha x\, dx = e^{-\alpha}$. *(S.V.T.U., 2009 ; Rohtak, 2004)*

## 22.6 (1) CONVOLUTION

*The convolution of two functions f(x) and g(x) over the interval $(-\infty, \infty)$ is defined as*

$$f * g = \int_{-\infty}^{\infty} f(u)\, g(x-u)\,du = h(x).$$

(2) **Convolution theorem for Fourier transforms.** *The Fourier transform of the convolution of f(x) and g(x) is the product of their Fourier transforms, i.e.,*

$$F\{f(x) * g(x)\} = F\{f(x)\} \,.\, F\{g(x)\}$$

We have
$$F\{f(x) * g(x)\} = F\left\{\int_{-\infty}^{\infty} f(u)\, g(x-u)\,du\right\}$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\, g(x-u)\,du\right\} e^{isx}\,dx = \int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} g(x-u)\,.\,e^{isx}\,dx\right\}du$$

[Changing the order of integration]

$$= \int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} e^{is(x-u)}\,.\,g(x-u)\,d(x-u)\right\} e^{isu}\,du$$

$$= \int_{-\infty}^{\infty} e^{isu} f(u)\left\{\int_{-\infty}^{\infty} e^{ist}\, g(t)\,dt\right\} du \quad \text{where } x - u = t$$

$$= \int_{-\infty}^{\infty} e^{isu} f(u)\,du \,.\, F\{g(t)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx \,.\, F\{g(x)\} = F\{f(x)\}\,.\,F\{g(x)\}$$

## 22.7 PARSEVAL'S IDENTITY FOR FOURIER TRANSFORMS

*If the Fourier transforms of f(x) and g(x) are F(s) and G(s) respectively, then*

(*i*) $\displaystyle\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\overline{G}(s)\,ds = \int_{-\infty}^{\infty} f(x)\overline{g}(x)\,dx$ (*ii*) $\displaystyle\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$

*where bar implies the complex conjugate.*

(i) $\int_{-\infty}^{\infty} f(x)\,\overline{g}\,(dx) \int_{-\infty}^{\infty} f(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{G}(s)\,e^{isx}\,ds\right\}dx$ [Using the inversion formula for Fourier transform]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{G}(s)\left\{\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx\right\}ds$$ [Changing the order of integration]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{G}(s)F(s)ds,$$ by definition of F-transform.

(ii) Taking $g(x) = f(x)$, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,\overline{F}\,(s)ds = \int_{-\infty}^{\infty} f(x)\,\overline{f}(x)dx \text{ or } \frac{1}{2\pi}\int_{-\infty}^{\infty} |\,F(s)\,|^2\,ds = \int_{-\infty}^{\infty} |\,f(x)\,|^2\,dx$$

**Obs.** The following Parseval's identities for Fourier cosine and sine transforms can be proved as above :

(i) $\frac{2}{\pi}\int_0^{\infty} F_c(s)\,G_c(s)ds = \int_0^{\infty} f(x)\,g(x)dx$ (ii) $\frac{2}{\pi}\int_0^{\infty} F_s(s)\,G_s(s)ds = \int_0^{\infty} f(x)\,g(x)dx$

(iii) $\frac{2}{\pi}\int_0^{\infty} |F_c(s)|^2\,ds = \int_0^{\infty} |f(x)|^2\,dx$ (iv) $\frac{2}{\pi}\int_0^{\infty} |F_s(s)|^2\,ds = \int_0^{\infty} |f(x)|^2\,dx.$

**Example 22.14.** *Using Parseval's identities, prove that*

(i) $\int_0^{\infty}\frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$ *(S.V.T.U., 2009 ; [illegible] 2008)*

(ii) $\int_0^{\infty}\frac{t^2}{(t^2+1)^2}\,dt = \frac{\pi}{4}$ (iii) $\int_0^{\infty}\frac{\sin at}{t(a^2+t^2)}\,dt = \frac{\pi}{2}\cdot\frac{1-e^{-a^2}}{a^2}.$

**Solution.** (i) Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$. Then $F_c(s) = \frac{a}{a^2+s^2}, G_c(s) = \frac{b}{b^2+s^2}$

Now using Parseval's identity for Fourier cosine transforms, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty} F_c(s)G_c(s)ds = \int_0^{\infty} f(x)\,g(x)dx \qquad ...(1)$$

We have $$\frac{2}{\pi}\int_0^{\infty}\frac{ab}{(a^2+s^2)(b^2+s^2)}\,ds = \int_0^{\infty} e^{-(a+b)x}dx$$

or $$\frac{2ab}{\pi}\int_0^{\infty}\frac{ds}{(a^2+s^2)(b^2+s^2)} = \left|\frac{e^{-(a+b)x}}{-(a+b)}\right|_0^{\infty} = \frac{1}{a+b}$$

Thus $$\int_0^{\infty}\frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$

(ii) Let $f(x) = \frac{x}{x^2+1}$ so that $F_s[f(x)] = \frac{\pi}{2}e^{-s}$

Now using Parseval's identity for sine transform, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty}[F_s\{f(x)\}]^2\,ds = \int_0^{\infty}[f(x)]^2\,dx$$

or $$\int_0^{\infty}\left(\frac{x}{x^2+1}\right)^2 dx = \frac{2}{\pi}\int_0^{\infty}\left(\frac{\pi}{2}e^{-s}\right)^2 ds = \frac{\pi}{2}\left|\,e^{-2s}/-2\right|_0^{\infty} = \frac{\pi}{-4}(0-1) = \frac{\pi}{4}$$

Hence $$\int_0^{\infty}\frac{t^2}{(t^2+1)^2}\,dt = \frac{\pi}{4}$$

(iii) Let $f(x) = e^{-ax}$ and $g(x) = \begin{cases}1, & 0 < x < a\\ 0, & x > a\end{cases}$. Then $F_c(s) = \frac{a}{a^2+s^2}, G_c(s) = \frac{\sin as}{s}$

Now using (1) above, we have $\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\frac{a\sin as}{s(a^2+s^2)}\,ds = \int_0^a e^{-ax}\,.1\,dx = \frac{1-e^{-a^2}}{a}$

Thus $$\int_0^{\infty}\frac{\sin at}{t(a^2+t^2)}\,dt = \frac{\pi}{2a^2}(1-e^{-a^2}).$$

**Example 22.15.** *Find the Fourier transform of $f(x)$ given by* $f(x)=\begin{cases} a-|x|, & \text{for } |x|<a \\ 0 & \text{, for } |x|>a \end{cases}$.

*Hence show that* $\displaystyle\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ *and* $\displaystyle\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \pi/3.$ (Anna, 2008)

**Solution.** Fourier transform of $f(x)$ i.e. $F[f(x)] = \displaystyle\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx = \int_{-a}^{a}[a-|x|]e^{isx}\,dx$

$$= \int_{-a}^{a}[a-|x|](\cos x + i\sin sx)\,dx$$

$$= 2\int_0^a (a-x)\cos sx\,dx + 0 \qquad \left\{\because \begin{array}{l}[a-|x|]\cos x \text{ in an even function} \\ [a-|x|]\sin x \text{ is an odd function}\end{array}\right.$$

$$= 2\left|(a-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right|_0^a = 2\,\frac{1-\cos as}{s^2} = 4\,\frac{\sin^2 as/2}{s^2}$$

(*i*) By inversion formula,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,e^{-isx}\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4\sin^2 as/2}{s^2}\,e^{-isx}\,ds$$

To evaluate $\displaystyle\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$, put $x = 0$ and $a = 2$ so that

$$f(0) = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 s}{s^2}\,ds = \frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin s}{s}\right)^2 ds \qquad \left[\because \frac{\sin s}{s} \text{ is an even function}\right]$$

$$\therefore \quad \int_0^{\infty}\left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{4}f(0) = \frac{\pi}{2}. \qquad [\because f(0) = a = 2$$

(*ii*) Using Parseval's identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[F(s)]^2 ds = \int_{-\infty}^{\infty}|f(x)|^2\,dx$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{4\sin^2 as/2}{s^2}\right)^2 dx = \int_{-a}^{a}|[a-|x|]^2\,dx$$

$$\frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin as/2}{s}\right)^4 ds = 2\int_0^a (a-x)^2\,dx = 2\left|\frac{(a-x)^3}{-3}\right|_0^a = \frac{2}{3}a^3$$

Putting $t = as/2$ and $dt = ads/2$

$$\frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin t}{2t/a}\right)\frac{2}{a}\,dt = \frac{2}{3}a^3 \quad \text{or} \quad \frac{2a^3}{\pi}\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{2}{3}a^3$$

Hence $$\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}.$$

## PROBLEMS 22.2

1. Verify Convolution theorem for $f(x) = g(x) = e^{-x^2}$. (V.T.U., 2000 S)
2. Use Convolution theorem to find the inverse Fourier transform of $\frac{i}{(1+s^2)^2}$, given that $\frac{2}{(1+s^2)}$ is the Fourier transform of $e^{-|x|}$. (V.T.U., 2010 S)
3. Using Parseval's identity, show that

   (i) $\int_0^\infty \frac{dx}{(t^2+1)^2} = \frac{\pi}{4}$, (Hissar, 2007) (ii) $\int_0^\infty \frac{t^2}{(4+t^2)(9+t^2)}\, dt = \frac{\pi}{10}$. (Rohtak, 2003)
4. Find the Fourier transform of $f(x) = \begin{cases} 1 \text{ if } |x| < 1 \\ 0 \text{ if } |x| > 1 \end{cases}$.

   Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$. (Anna, 2009)
5. Evaluate $\int_0^\infty \left(\frac{1-\cos x}{x}\right)^2 dx$.

## 22.8 RELATION BETWEEN FOURIER AND LAPLACE TRANSFORMS

*If* $$f(t) = \begin{cases} e^{-xt}\, g(t), & t > 0 \\ 0 \quad , & t < 0 \end{cases} \qquad \text{...}(i)$$

*then* $$F\{f(t)\} = L\{g(t)\}.$$

We have $$F\{f(t)\} = \int_{-\infty}^{\infty} e^{ist} f(t)dt = \int_{-\infty}^{0} e^{ist} . 0 . dt + \int_0^\infty e^{ist} . e^{-xt}\, g(t)dt$$

$$= \int_0^\infty e^{(is-x)t}\, g(t)\, dt = \int_0^\infty e^{-pt}\, g(t)dt \quad \text{where } p = x - is$$

Hence *the Fourier transform of* $f(t)$ [defined by (*i*)] is the *Laplace transform of* $g(t)$.

## 22.9 FOURIER TRANSFORMS OF THE DERIVATIVES OF A FUNCTION

The Fourier transform of the function $u(x, t)$ is given by

$$F[u(x,t)] = \int_{-\infty}^{\infty} ue^{isx}\, dx$$

*Then the Fourier transform of* $\partial^2 u/\partial x^2$, *i.e.*

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx}\, dx = \left| e^{isx}\frac{\partial u}{\partial x} - is\, e^{isx} . u \right|_{-\infty}^{\infty} + (is)^2 \int_{-\infty}^{\infty} ue^{isx}\, dx,$$

on applying the general rule of integration by parts (p. 398). If $u$ and $\frac{\partial u}{\partial x}$ tend to zero as $x$ tends to $\pm\infty$, then

$$\mathbf{F}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -\,\mathbf{s}^2\,\mathbf{F}[\mathbf{u}] \qquad \text{...(1)}$$

Similarly in the case of Fourier sine and cosine transforms, we have

$$\mathbf{F}_s\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = \mathbf{s}(\mathbf{u})_{\mathbf{x}=0} - \mathbf{s}^2\mathbf{F}_s[\mathbf{u}] \qquad \text{...(2)}$$

and $$\mathbf{F}_c\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)_{\mathbf{x}=0} - \mathbf{s}^2\,\mathbf{F}_c[\mathbf{u}] \qquad \text{...(3)}$$

In general, the *Fourier transform of the nth derivative of* $f(x)$ *is given by*

$$\mathbf{F}\left[\frac{d^n f}{dx^n}\right] = (-is)^n\, \mathbf{F}[f(x)] \qquad ...(4)$$

provided the first $n-1$ derivatives vanish as $x \to \pm\infty$.

For $\quad F[f^n(x)] = \int_{-\infty}^{\infty} f^n(x)\, e^{isx}\, dx$

$$= \left| e^{isx} f^{n-1} - ise^{isx} f^{n-2} + (is)^2 e^{isx} f^{n-3} - ... \right|_{-\infty}^{\infty} + (-is)^n \int_{-\infty}^{\infty} f \,.\, e^{isx}\, dx$$

by the general rule of integration by parts, whence follows (4).

## 22.10 INVERSE LAPLACE TRANSFORMS BY METHOD OF RESIDUES

Let the Laplace transform of $f(x)$ be $\bar{f}(s)$ so that

$$\bar{f}(s) = \int_0^{\infty} f(t) e^{-st}\, dt \qquad ...(1)$$

Multiply both sides by $e^{xs}$ and integrate w.r.t. $s$ within the limits $a - ir$ and $a + ir$. Then

$$\int_{a-ir}^{a+ix} e^{xs}\, \bar{f}(s) ds = \int_{a-ir}^{a+ir} e^{xs} \int_0^{\infty} f(t) e^{-st}\, dt\, ds \qquad \text{[Put } s = a - iu]$$

$$= \int_r^{-r} e^{x(a-iu)} \int_0^{\infty} f(t) e^{-(a-iu)t}\, dt\, (-idu) = ie^{ax} \int_{-r}^{r} e^{-ixu} \int_0^{\infty} [e^{-at} f(t)] e^{iut}\, dt du$$

$$= ie^{ax} \int_{-r}^{r} e^{-ixu} \int_{-\infty}^{\infty} \phi(t)\, e^{iut}\, dt\, du$$

where $\phi(t) = \begin{cases} e^{-at} f(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$.

Proceeding to limits as $r \to \infty$, we get

$$\int_{a-i\infty}^{a+i\infty} e^{xs}\, \bar{f}(s)\, ds = ie^{ax} \,.\, 2\pi\phi(x), \text{ by (2) of § 22.4} = 2\pi i e^{ax} . e^{-ax} f(x) \text{ for } x > 0.$$

Hence $\qquad f(x) = \int_{a-i\infty}^{a+i\infty} e^{xs}\, \bar{f}(s) ds \ (x > 0) \qquad ...(2)$

which is called the *complex inversion formula*. It provides a direct means for obtaining the inverse Laplace transform of a given function.

The integration in (2) is performed along a line $LM$ parallel to the imaginary axis in the complex plane $z = x + iy$ such that all the singularities of $\bar{f}(s)$ lie to its left* (Fig. 22.1). Let us take a contour $C$ which is composed of the line $LM$ and the semi-circle $C'$ (*i.e.*, $MNL$). Then from (2)

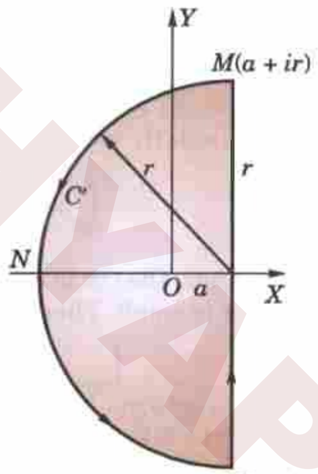


Fig. 22.1

$$\frac{1}{2\pi i} \int_{LM} e^{xs}\, \bar{f}(s)\, ds = \frac{1}{2\pi i} \int_C e^{xs}\, \bar{f}(s)\, ds - \frac{1}{2\pi i} \int_{C'} e^{xs}\, \bar{f}(s)\, dx$$

The integral over $C'$ tends to zero as $r \to \infty$ (under certain conditions†). Therefore,

$$\mathbf{f(x)} = \underset{r\to\infty}{\mathbf{Lt}} \frac{1}{2\pi i} \int_C \mathbf{e^{xs}\, \bar{f}(s)\, dx}$$

$$= \textbf{sum of the residues of } e^{xs}\bar{f}(s) \textbf{ at the poles of } f(s) \qquad ...(3)$$

[By §20.18]

* This has been so assumed simply to ensure the convergence of the integral (1).

† If positive constants $A$ and $k$ can be so found that $|\bar{f}(s)| < Ar^{-k}$ for every point on $C'$, then

$$\underset{r\to\infty}{\text{Lt}} \frac{1}{2\pi i} \int_{C'} e^{xs} \bar{f}(s)\, ds = 0. \qquad \textit{(Jordan's Lemma)}$$

**Example 22.16.** *Evaluate* $L^{-1}\left\{\dfrac{1}{(s-1)(s^2+1)}\right\}$ *by the method of residues.*

**Solution.** Since $\left|\dfrac{1}{(s-1)(s^2+1)}\right| \sim \left|\dfrac{1}{s^3}\right|$ for $|s| \to \infty$, therefore,

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \text{sum of Res}\left[\frac{e^{xs}}{(s-1)(s^2+1)}\right] \text{at the poles } s = 1, \pm i$$

Now $$(\text{Res})_{s=1} = \underset{s\to 1}{\text{Lt}}\left[\frac{(s-1)\,.\,e^{xs}}{(s-1)(s^2+1)}\right] = \frac{e^x}{2} \qquad \text{[By § 20.19 (1)]}$$

$$(\text{Res})_{s=i} = \underset{s\to i}{\text{Lt}}\left[\frac{(s-i)\,.\,e^{xs}}{(s-1)(s^2+1)}\right] = \frac{e^{ix}}{(i-1)(i-1)} = -\frac{1}{2}\cdot\frac{e^{ix}}{1+i}$$

Changing $i$ to $-i$, we get $(\text{Res})_{s=-i} = -\dfrac{1}{2}\cdot\dfrac{e^{ix}}{1-i}$

$$\therefore \qquad L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \frac{e^x}{2} - \frac{1}{2}\left(\frac{e^{ix}}{1+i} + \frac{e^{-ix}}{1-i}\right) = \frac{1}{2}(e^x - \sin x - \cos x).$$

**Example 22.17.** *Prove that* $L^{-1}\left(\dfrac{e^{-c\sqrt{s}}}{s}\right) = 1 - erf\left(\dfrac{c}{\sqrt{2x}}\right)$.

**Solution.** By the complex inversion formula,

$$L^{-1}\left(\frac{e^{-c\sqrt{s}}}{s}\right) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{xs}\,.\,\frac{e^{-c\sqrt{s}}}{s}\,ds.$$

Since $s = 0$ is a branch point of the integrand, we take a contour $LMNPQST$ as shown in Fig. 22.2, so that it doesn't include any singularity. Therefore, by Cauchy's theorem (§ 20.13), we have

$$\left\{\int_{LM} + \int_{MN} + \int_{NP} + \int_{PQS} + \int_{ST} + \int_{TL}\right\} \times e^{xs}\,\frac{e^{-c\sqrt{s}}}{s}\,ds = 0 \qquad \ldots(i)$$

If $ON = \rho$ and $OP = \varepsilon$, then along $NP$, $s = Re^{i\pi}$, therefore,

$$\int_{NP} = \int_{\rho}^{\varepsilon} e^{-xR}\,\frac{e^{-ic\sqrt{R}}}{R}\,dR$$

Similarly along $ST$, $s = Re^{-i\pi}$, therefore,

$$\int_{ST} = \int_{\varepsilon}^{\rho} e^{-xR}\,\frac{e^{c\sqrt{R}}}{R}\,dR$$

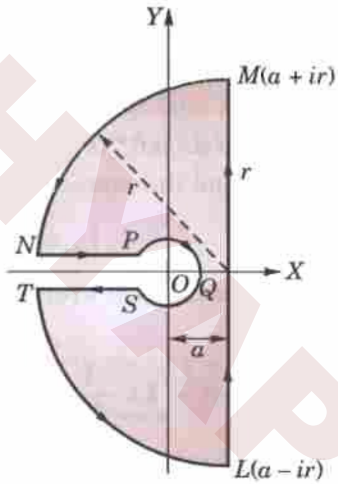

Fig. 22.2

Along the circle $PQS$, $s = \varepsilon\, e^{i\theta}$. Also $e^{xs}$ and $e^{-c\sqrt{\sigma}}$ are both approximately 1 since $\varepsilon$ is small. Therefore,

$$\int_{PQS} = \int_{\pi}^{-\pi} \frac{1}{\varepsilon e^{i\theta}}\,.\,\varepsilon e^{i\theta}\, i d\theta = -2\pi i \text{ approximately.}$$

For $c > 0$, $|e^{-c\sqrt{s}}/s| < |s|^{-1}$.

But $\int_{MN}$ and $\int_{TL}$ both tend to zero as $r \to \infty$

Thus $(i)$ takes the form

$$\int_{a-ir}^{a+ir} \frac{e^{xs-c\sqrt{s}}}{s}\,ds + \int_{\varepsilon}^{\rho} e^{-xR}\,\frac{e^{ic\sqrt{R}} - e^{-ic\sqrt{R}}}{R}\,dR - 2\pi i = 0$$

Taking limits as $\varepsilon \to 0$ and $\rho \to \infty$, we get

$$\int_{a-i\infty}^{a+i\infty} \frac{e^{xs-c\sqrt{s}}}{s}\,ds = 2\pi i - 2i\int_0^{\infty} e^{-xR}\,\frac{\sin c\sqrt{R}}{R}\,dR$$

or $$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{xs - c\sqrt{s}}}{s}\, ds = 1 - \frac{2}{\pi}\int_0^\infty e^{-t^2} \frac{\sin(ct/\sqrt{x})}{t}\, dt^*, \text{ where } R = t^2/x$$

$$= 1 - \frac{2}{\pi}\cdot\frac{\pi}{2}\, erf\left(\frac{C}{2\sqrt{x}}\right) \text{ whence follows the result.}$$

**PROBLEMS 22.3**

Using the method of residues, evaluate the inverse Laplace transform of each of the following :

1. $\dfrac{1}{(s+1)(s-2)^2}$.
2. $\dfrac{1}{(s-2)(s^2+1)}$.
3. $\dfrac{1}{s^2(s^2-a^2)}$.
4. $\dfrac{1}{(s-1)^2(s^2+1)}$.
5. $\dfrac{1}{(s^2+1)^2}$. (*V.T.U., 2008 S*)

## 22.11 APPLICATION OF TRANSFORMS TO BOUNDARY VALUE PROBLEMS

In one dimensional boundary value problems, the partial differential equation can easily be transformed into an ordinary differential equation by applying a suitable transform. The required solution is then obtained by solving this equation and inverting by means of the complex inversion formula or by any other method. In two dimensional problems, it is sometimes required to apply the transforms twice and the desired solution is obtained by double inversion.

(*i*) If in a problem $u(x, t)_{x=0}$ is given then we use infinite sine transform to remove $\partial u^2/\partial x^2$ from the differential equation.

In case $[\partial u(x, t)/\partial x]_{x=0}$ is given then we employ infinite cosine transform to remove $\partial^2 u/\partial x^2$.

(*ii*) If in a problem $u(0, t)$ and $u(l, t)$ are given, then we use finite sine transform to remove $\partial^2 u/\partial x^2$ from the differential equation.

In case $(\partial u/\partial x)_{x=0}$ and $(\partial u/\partial x)_{x=l}$ are given, then we employ finite cosine transform to remove $\partial^2 u/\partial x^2$.

The method of solution is best explained through the following examples.

### Heat conduction

**Example 22.18.** *Determine the distribution of temperature in the semi-infinite medium $x \geq 0$, when the end $x = 0$ is maintained at zero temperature and the initial distribution of temperature is $f(x)$.* (*Osmania, 2003*)

**Solution.** Let $u(x, t)$ be the temperature at any point $x$ and at any time $t$. We have to solve the heat-flow equation (§ 18.5)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \; (x > 0, t > 0) \qquad ...(i)$$

subject to the initial condition $u(x, 0) = f(x)$ ...(*ii*)

and the boundary condition $u(0, t) = 0$ ...(*iii*)

Taking Fourier sine transform of (1) and denoting $F_s[u(x, t)]$ by $\bar{u}_s$, we have

$$\frac{d\bar{u}_s}{dt} = c^2[su(0, t) - s^2\bar{u}_s] \qquad \text{[By (2) of § 22.9]}$$

---

* We know that $\int_0^\infty e^{-t^2}\cos 2mt\, dt = \frac{1}{2}\sqrt{\pi}e^{-m^2}$ [Example 20.44

Integrating both sides w.r.t. $m$ from 0 to $c/2\sqrt{x}$.

$$\int_0^\infty e^{-t^2}\left|\frac{\sin 2mt}{2t}\right|_0^{c/2\sqrt{x}} dt = \frac{1}{2}\sqrt{\pi}\int_0^{c/2\sqrt{x}} e^{-m^2}\, dm$$

or $$\int_0^\infty e^{-t^2}\frac{\sin(ct/\sqrt{x})}{t}\, dt = \frac{\pi}{2}\, erf\left(\frac{c}{2\sqrt{x}}\right). \qquad \text{[By § 7.18(1)]}$$

or $$\frac{d\bar{u}_s}{dt} + c^2s^2\bar{u}_s = 0 \qquad \text{[By (iii)] ...(iv)}$$

Also the Fourier sine transform of (*ii*) is $\bar{u}_s = \bar{f}(s)$ at $t = 0$. ...(*v*)

Solving (*iv*) and using (*v*), we get $\bar{u}_s = \bar{f}_s(s)e^{-c^2s^2t}$

Hence taking its inverse Fourier sine transform, we obtain

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \bar{f}_s(s)e^{-c^2s^2t} \sin xs\, ds.$$

**Example 22.19.** *Solve $\partial u/\partial t = 2\partial^2u/\partial x^2$, if $u(0, t) = 0$, $u(x, 0) = e^{-x}$ $(x > 0)$, $u(x, t)$ is bounded where $x > 0$, $t > 0$.* (*Rohtak, 2006*)

**Solution.** Given $\partial u/\partial t = 2\partial^2u/\partial x^2, x > 0, t > 0$ ...(*i*)

with boundary conditions : $u(0, t) = 0$, $u(x, t)$ is bounded ...(*ii*)

and initial condition $u(x, 0) = e^{-x}, x > 0$ ...(*iii*)

Since $u(0, t)$ is given, we take Fourier sine transform of both sides of (*i*) so that

$$\int_0^\infty \frac{\partial u}{\partial t}\sin px\, dx = 2\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin px\, dx$$

or $$\frac{d}{dt}\int_0^\infty u(x,t)\sin px\, dx = 2\left[\left|\frac{\partial u}{\partial x}\sin px\right|_0^\infty - \int_0^\infty \frac{\partial u}{\partial x}\,.\,p\cos px\, dx\right] \qquad \text{(Integrating by parts)}$$

or $$\frac{d\bar{u}_s}{dt} = -2p\int_0^\infty \frac{\partial u}{\partial x}\cos px\, dx, \text{ if } \frac{\partial u}{\partial x} \to \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty \text{ where } \bar{u}_s(p, t) = \int_0^\infty u(x, t)\sin px\, dx$$

$$= -2p\left[\left| u(x,t)\cos px\right|_0^\infty - \int_0^\infty u(x,t) - (-p\sin px)\, dx\right] \qquad \text{[Again integrating by parts]}$$

$$= -2p\left[0 - u(0,t) + p\int_0^\infty u(x,t)\sin px\, dx\right] \qquad [\because\ u(x,t) \to 0 \text{ as } x \to \infty \text{ by } (ii)]$$

$$= 2pu(0,t) - 2p^2\bar{u}_s$$

or $$\frac{d\bar{u}_s}{dt} = -2p^2\bar{u}_s \qquad \text{[By (ii)]}$$

Integrating $\int \frac{\partial\bar{u}_s}{\bar{u}_s} - \log c = -2p^2\int dt$ or $\log\bar{u}_s - \log c = -2p^2t$

$\therefore$ $$\bar{u}_s(p, t) = ce^{-2p^2t} \qquad ...(iv)$$

Taking Fourier sine transform of both sides of (*iii*), we get

$$\int_0^\infty u(x, 0)\sin px\, dx = \int_0^\infty e^{-x}\sin px\, dx$$

or $$\bar{u}_s(p, 0) = \left|\frac{e^{-x}}{1+p^2}(-\sin px - p\cos px)\right|_0^\infty = \frac{p}{1+p^2} \qquad ...(v)$$

Putting $t = 0$ in (*iv*) and using (*v*), we obtain $p/(1 + p^2) = c$

Thus (*iv*) becomes $\bar{u}_s(p, t) = \frac{p}{1+p^2}e^{-2p^2t}$

Now taking inverse Fourier sine transform, we get

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{pe^{-2p^2t}}{1+p^2}\sin px\, dp.$$

**Example 22.20.** *Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $(x > 0, t > 0)$ subject to the conditions*

*(i) $u = 0$, when $x = 0, t > 0$ (ii) $u = \begin{cases} 1, 0 < x < 1 \\ 0, x \le 1, \text{ when } t = 0\end{cases}$ (iii) $u(x, t)$ is bounded.* (*U.P.T.U., 2003 S*)

**Solution.** Since $u(0, t) = 0$, we take Fourier sine transform of both sides of the given equation, we get

$$\int_0^\infty \frac{\partial u}{\partial t} \sin sx\, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx\, dx$$

$$\frac{\partial}{\partial t} \int_0^\infty u \sin sx\, dx = -s^2\, \bar{u}(s) + s\, u(0) \qquad [\because u = 0, \text{ when } x = 0]$$

or $$\frac{\partial \bar{u}}{\partial t} = -s^2 \bar{u} \quad \text{or} \quad \frac{\partial \bar{u}}{\partial t} + s^2 \bar{u} = 0 \quad \text{or} \quad (D^2 + s^2)\bar{u} = 0 \text{ i.e., } D = \pm is$$

$\therefore$ Its solution is $\bar{u}(s, t) = e^{-s^2 t}$ ...(1)

Since $$\bar{u}(s, t) = \int_0^\infty u(x, t) \sin sx\, dx$$

$\therefore$ $$\bar{u}(s, 0) = \int_0^\infty u(x, 0) \sin sx\, dx = \int_0^1 1 . \sin sx\, dx \qquad [\text{By } (ii)]$$

$$= \frac{1 - \cos s}{s} \qquad ...(2)$$

From (1) and (2), $$c = \bar{u}(s, 0) = \frac{1 - \cos s}{s}$$

Thus (1) gives $$\bar{u}(s, t) = \frac{1 - \cos s}{s} e^{-s^2 t}$$

Now taking inverse Fourier sine transform, we get

$$u(x, t) = \int_0^\infty \frac{1 - \cos s}{s} e^{-s^2 t}\, ds$$

which is the desired solution.

**Example 22.21.** *Using finite Fourier transform, solve*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

*given $u(0, t) = 0$, $u(4, t) = 0$ and $u(x, 0) = 2x$ where $0 < x < 4$, $t > 0$.* *(Rajasthan, 2006)*

**Solution.** Since $u(0, t) = 0$, we take finite Fourier sine transform of both sides of the given equation

$$\int_0^4 \frac{\partial u}{\partial t} \sin \frac{n\pi}{4} x\, dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi}{4} x\, dx$$

or $$\frac{d}{dt}(\bar{u}_s) = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$= -\frac{n^2\pi^2}{16} \bar{u}_s + \frac{n\pi}{4} [u(0, t) - (-1)^n u(4, t)]$$

$$= -\frac{n^2\pi^2}{16} \bar{u}_s \qquad [\because u(0, t) = 0, u(4, t) = 0.]$$

or $$\frac{d\bar{u}_s}{\bar{u}_s} = -\frac{n^2\pi^2}{16}\, dt$$

Integrating both sides, $$\log \bar{u}_s = -\frac{n^2\pi^2}{16} t + c$$

or $$\bar{u}_s(x, 0) = \alpha e^{\frac{-n^2\pi^2 t}{16}} \qquad ...(i)$$

Putting $t = 0$, $$a = \bar{u}_s(x, 0) = \int_0^4 u(x, 0) \sin \frac{n\pi x}{4} dx \qquad [\because \quad u(x, 0) = 2x]$$

$$= \int_0^4 2x \sin \frac{n\pi x}{4} dx = -\frac{32}{n\pi} \cos n\pi$$

Thus (*i*) gives, $$\bar{u}_s(x, 0) = -\frac{32}{n\pi}\cos n\pi e^{-n^2\pi^2 t/16} = -\frac{32}{n\pi}(-1)^n e^{-n^2\pi^2 t/16}$$

Now taking inverse Fourier sine transform, we get

$$u(x, 0) = \frac{2}{4}\sum_{n=1}^{\infty}\frac{32}{n\pi}(-1)^{n+1} e^{-n^2\pi^2 t/16}\sin\left(\frac{n\pi x}{4}\right)$$

$$= 16\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\pi} e^{-n^2\pi^2 t/16}\sin\left(\frac{n\pi x}{4}\right).$$

**Example 22.22.** *If the initial temperature of an infinite bar is given by*

$$\theta(x) = \begin{cases}\theta_0 \text{ for } |x| < a\\ 0 \text{ for } |x| > a,\end{cases}$$

*determine the temperature at any point x and at any instant t.* (S.V.T.U., 2008 ; Rohtak, 2004)

**Solution.** To determine the temperature $\theta(x, t)$ at any point at any time, we have to solve the equation

$$\frac{\partial\theta}{\partial t} = c^2\frac{\partial^2\theta}{\partial x^2} \quad (t > 0) \qquad ...(i)$$

subject to the initial condition $\theta(x, 0) = \begin{cases}\theta_0 \text{ for } |x| < a\\ 0 \text{ for } |x| > a\end{cases}$ ...(*ii*)

Taking Fourier transform of (*i*) and denoting $F[\theta(x, t)]$ by $\bar{\theta}$, we find

$$\frac{d\bar{\theta}}{dt} = -c^2 s^2\bar{\theta} \qquad \text{[by (1) of § 22.9] ...(iii)}$$

Also the Fourier transform of (2) is

$$\bar{\theta}(s, 0) = \int_{-\infty}^{\infty}\theta(x, 0)e^{isx}\,dx = \int_{-a}^{a}\theta_0 e^{isx}\,dx = \theta_0\frac{e^{isa} - e^{-isa}}{is} = 2\theta_0\frac{\sin as}{s} \qquad ...(iv)$$

Solving (*iii*) and using (*iv*), we get $\bar{\theta} = \frac{2\theta_0\sin as}{s}e^{-c^2s^2t}$

Hence taking its inverse Fourier transform, we get

$$\theta(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\theta_0\sin as}{s}e^{-c^2s^2t}e^{-isx}\,ds = \frac{\theta_0}{\pi}\int_{-\infty}^{\infty}\frac{\sin as}{s}e^{-c^2s^2t}(\cos xs - i\sin xs)\,ds$$

$$= \frac{2\theta_0}{\pi}\int_0^{\infty}\frac{\sin as}{s}e^{-c^2s^2t}\cos xs\,ds \qquad \left\{\begin{array}{l}\text{The second integral vanishes as}\\ \text{its integrand is an odd function}\end{array}\right.$$

$$= \frac{\theta_0}{\pi}\int_0^{\infty}e^{-c^2s^2t}\frac{\sin(a+x)s + \sin(a-x)s}{s}\,ds$$

$$= \frac{\theta_0}{\pi}\int_0^{\infty}e^{-v^2}\left\{\sin\frac{(a+x)v}{c\sqrt{t}} + \sin\frac{(a-x)v}{c\sqrt{t}}\right\}\frac{dv}{v} \quad \text{where } v^2 = c^2s^2t$$

$$= \frac{\theta_0}{\pi}\left\{erf\,\frac{(a+x)}{2c\sqrt{t}} + erf\,\frac{(a-x)}{2c\sqrt{t}}\right\}.$$ [See footnote on p. 783]

**Example 22.23.** *A bar of length a is at zero temperature. At t = 0, the end x = a is suddenly raised to temperature $u_0$ and the end x = 0 is insulated. Find the temperature at any point x of the bar at any time t > 0, assuming that the surface of the bar is insulated.*

**Solution.** Here we have to solve the differential equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad (0 < x < a,\ t > 0) \qquad ...(i)$$

subject to the conditions

$u(x, 0) = 0$ ...(*ii*) ; $u_x(0, t) = 0$ ...(*iii*) and $u(a, t) = u_0$ (Rohtak, 2005) ...(*iv*)

The Laplace transform of $(i)$, if $L[u(x, t)] = \bar{u}(x, s)$, is

$$s\bar{u} - u(x, 0) = c^2 \frac{d^2\bar{u}}{dx^2}$$

Using $(ii)$, we get $$\frac{d^2\bar{u}}{dx^2} - \frac{s}{c^2}\bar{u} = 0 \qquad ...(v)$$

Similarly the Laplace transform of $(iii)$ and $(iv)$ are

$$\bar{u}_x(0, s) = 0 \qquad ...(vi)\,; \qquad \bar{u}(a, s) = \frac{u_0}{s} \qquad ...(vii)$$

Solving $(v)$, we have $\bar{u} = C_1 e^{x\sqrt{s}/c} + C_2 e^{-x\sqrt{s}/c}$

Using $(vi)$, we find $C_1 = C_2$ so that

$$\bar{u} = C_1 (e^{\sqrt{s}x/c} + e^{-\sqrt{s}x/c}) = 2C_1 \cosh(\sqrt{s}x/c)$$

Now using $(vii)$, we have $$\bar{u} = \frac{u_0 \cosh(\sqrt{s}x/c)}{s \cosh(\sqrt{s}a/c)}$$

By the inversion formula (3) § 22.10, we get

$$u(x, t) = \text{sum of the residues of } \left(\frac{e^{st} \cdot u_0 \cosh(\sqrt{s}x/c)}{s \cosh(\sqrt{s}a/c)}\right) \text{ at all the poles which occur at } s = 0$$

and $\cosh(\sqrt{s}a/c) = 0$ *i.e.*, at $s = 0$, $\sqrt{s}a/c = \left(n - \frac{1}{2}\right)\pi i$, $n = 0, \pm 1, \pm 2, \ldots$

or at $s = 0,\ s\,(= s_n) = -\dfrac{(2n-1)^2 c^2 \pi^2}{4a^2} = 0, 1, 2, \ldots$

Now $$(\text{Res})_{s=0} = \underset{s \to 0}{\text{Lt}} \left\{ s \cdot \frac{u_0 e^{st} \cosh(\sqrt{s}x/c)}{s \cosh(\sqrt{s}a/c)} \right\} = u_0$$

$$(\text{Res})_{s=s_n} = u_0 \underset{s \to s_n}{\text{Lt}} \left\{ (s - s_n) \cdot \frac{u_0 e^{st} \cosh(\sqrt{s}x/c)}{s \cosh(\sqrt{s}a/c)} \right\}$$

$$= u_0 \underset{s \to s_n}{\text{Lt}} \left\{ \frac{s - s_n}{\cosh(\sqrt{s}a/c)} \right\} \cdot \underset{s \to s_n}{\text{Lt}} \left\{ \frac{e^{st} \cosh(\sqrt{s}x/c)}{s} \right\} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= u_0 \underset{s \to s_n}{\text{Lt}} \frac{1}{\sinh(\sqrt{s}a/c) \cdot (a/2\sqrt{s}/c)} \cdot \underset{s \to s_n}{\text{Lt}} \left\{ \frac{e^{st} \cosh(\sqrt{s}x/c)}{s} \right\}$$

$$= \frac{4u_0(-1)^n}{(2n-1)\pi} e^{-(2n-1)^2 \pi^2 c^2 t/4a^2} \cos \frac{(2n-1)\pi x}{2a}$$

Thus we get $$u(x, t) = u_0 + \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-(2n-1)^2 \pi^2 c^2 t/4a^2} \cos \frac{(2n-1)\pi x}{2a}.$$

### Vibrations of a string

**Example 22.24.** *An infinite string is initially at rest and that the initial displacement is $f(x)$, $(-\infty < x < \infty)$. Determine the displacement $y(x, t)$ of the string.* (Rohtak, 2000)

**Solution.** The equation for the vibration of the string is

$$\partial^2 y/\partial t^2 = c^2 \partial^2 y/\partial x^2 \qquad ...(i)$$

and the initial conditions are

$$(\partial y/\partial t)_{t=0} = 0\,;\ y(x, 0) = f(x) \qquad ...(ii)$$

Multiplying $(i)$ by $e^{isx}$ and integrating w.r.t. $x$ from $-\infty$ to $\infty$, we get

$$\partial^2 Y/\partial t^2 = c^2(-s^2 Y) \qquad \text{provided } y \text{ and } \partial y/\partial x \to 0 \text{ as } x \to \infty$$

$\therefore$ a solution of $d^2Y/dt^2 + c^2s^2Y = 0$ is $\quad Y = A_1 \cos cst + A_2 \sin cst$ ...(*iii*)

Also Fourier transforms of (*ii*) are

$$\partial y/\partial t = 0 \quad \text{and} \quad Y = F(s) \text{ when } t = 0$$

Applying these to (*iii*), we get

$$A_2 = 0 \quad \text{and} \quad A_1 = F(s)$$

Thus $\qquad Y = F(s) \cos cst$

Now taking inverse Fourier transforms, we get

$$y(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s) \cos cst \, . \, e^{-isx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\frac{e^{icst} + e^{-icst}}{2} . \, e^{-isx}\, dx$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} [F(s)e^{-is(x-ct)} + F(s)e^{-is(x+ct)}]\, ds$$

$$= \frac{1}{2}[f(x-ct) + f(x+ct)] \qquad \left[\because \; f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds\right.$$

**Example 22.25.** *An infinitely long string having one end at $x = 0$, is initially at rest along the x-axis. The end $x = 0$ is given a transverse displacement $f(t)$, $t > 0$. Find the displacement of any point of the string at any time.*

**Solution.** Let $y(x, t)$ be the transverse displacement of any point $x$ of the string at any time $t$. Then we have to solve the wave equation (§ 18.4)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad (x > 0, t > 0) \qquad ...(i)$$

subject to the conditions $y(x, 0) = 0$, $y_t(x, 0) = 0$, $y(0, t) = f(t)$ and the displacement $y(x, t)$ is bounded.

The Laplace transform of (*i*), writing $L[y(x, t)] = \bar{y}(x, s)$ is

$$s^2\bar{y} - sy(x, 0) - \frac{\partial y(x, 0)}{\partial t} = c^2\frac{\partial^2 \bar{y}}{\partial x^2}$$

Using the first two conditions, we have

$$\frac{\partial^2 \bar{y}}{\partial x^2} = \left[\frac{s}{c}\right]^2 \bar{y} \qquad ...(ii)$$

Similarly the Laplace transforms of the third and fourth conditions are

$$\bar{y}(0, s) = \bar{f}(s) \quad \text{at} \quad x = 0 \qquad ...(iii) \qquad \text{and} \qquad \bar{y}(x, s) \text{ is bounded.} \qquad ...(iv)$$

Solving (*ii*), we get

$$\bar{y}(x, s) = C_1 e^{sx/c} + C_2 e^{-sx/c}$$

To satisfy condition (*iv*), we must have $C_1 = 0$

Using the condition (*iii*), we get $C_2 = \bar{f}(s)$.

$$\therefore \qquad \bar{y}(x, s) = \bar{f}(s)e^{-sx/c}$$

Using the complex inversion formula, we obtain

$$y = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{(t-x/c)s}\,\bar{f}(s)\, ds = f(t - x/c).$$

**Example 22.26.** *A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, where $\mu$ is a constant and then released. Find the displacement of any point $x$ of the string at any time $t > 0$.* (V.T.U., M.E., 2006)

**Solution.** We have to solve the wave equation $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2} \qquad (x > 0, t > 0)$

subject to the conditions $y(0, t) = 0, y(l, t) = 0$
and $y(x, 0) = \mu x(l - x), y_t(x, 0) = 0$

Now taking Laplace transform, writing $L[y(x, t)] = \bar{y}(x, s)$, we get

$$s^2\bar{y} - s\bar{y}(x, 0) - \frac{\partial y(x, 0)}{\partial t} = c^2 \frac{\partial^2 \bar{y}}{\partial x^2} \qquad ...(i)$$

where $$\bar{y}(0, s) = 0, \ \bar{y}(l, s) = 0 \qquad ...(ii)$$

$\therefore$ (i) reduces to $\dfrac{\partial^2 \bar{y}}{\partial x^2} - \left(\dfrac{s}{c}\right)^2 \bar{y} = -\dfrac{\mu s x(l - x)}{c^2}$

Its solution is $\bar{y}(x, s) = c_1 \cosh(sx/c) + c_2 \sinh(sx/c) + \dfrac{\mu x(l - x)}{s} - \dfrac{2c^2\mu}{s^3}$

Applying the conditions (ii), we get

$$c_1 = 2c^2\mu/s^2 \quad \text{and} \quad c_2 = \frac{2c^2\mu}{s^3}\left[\frac{1 - \cosh(sl/c)}{\sinh(sl/c)}\right] - \frac{2c^2\mu}{s^3} \tanh(s/2c)$$

Thus $$\bar{y}(x, s) = \frac{2c^2\mu}{s^3}\left[\frac{\cosh\{s(2x - l)/2c\}}{\cosh(sl/2c)}\right] + \frac{\mu x(l - x)}{s} - \frac{2c^2\mu}{s^3}$$

Now using the inversion formula (3) § 22.10, we get

$y(x, t) =$ sum of the residues of

$$2c^2\mu\left[e^{st}\frac{\cosh\{s(2x - l)/2c\}}{s^3\cosh(sl/2c)}\right] \text{ at all the poles } + \mu x(l - x) - c^2\mu t^2$$

Proceeding exactly as in Example 22.23, we have,

sum of the residues of $2c^2\mu\left[\dfrac{e^{st}\cosh\{s(2x - l)/2c\}}{s^3\cosh sl/2c}\right]$ at all the poles

$$= c^2\mu\left[t^2 + \left(\frac{2x - l}{2c}\right)^2 - \left(\frac{l}{2c}\right)^2\right]$$

$$- \frac{32c^2\mu}{\pi^3}\left(\frac{l}{2c}\right)^2 \sum_{n=1}^{\infty}\left[\frac{(-1)^n}{(2n - 1)^3}\cos\left\{\frac{(2n - 1)\pi(2x - l)}{2l}\right\}\cos\left\{\frac{(2n - 1)\pi ct}{l}\right\}\right]$$

$$= c^2\mu t^2 - \mu x(l - x) + \frac{8\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1}{(2n - 1)^3}\sin\frac{(2n - 1)\pi x}{l}\cos\frac{(2n - 1)\pi ct}{l}\right]$$

Hence $$y(x, t) = \frac{8\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1}{(2n - 1)^3}\sin\frac{(2n - 1)\pi x}{l}\cos\frac{(2n - 1)\pi ct}{l}\right].$$

### Transmission lines

**Example 22.27.** *A semi-infinite transmission line of negligible inductance and leakance per unit length has its voltage and current equal to zero. A constant voltage $v_0$ is applied at the sending end ($x = 0$) at $t = 0$. Find the voltage and current at any point ($x > 0$) and at any instant.*

**Solution.** Let $v(x, t)$ and $i(x, t)$ be the voltage and current at any point $x$ and at any time $t$. If $L = 0$ and $G = 0$, then the transmission line equations [(1) and (2) of § 18.10] become

$$\frac{\partial v}{\partial x} = -Ri, \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} \quad i.e., \quad \frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \qquad ...(i)$$

The boundary conditions are $v(0, t) = v_0$ and $i(x, t)$ is finite for all $x$ and $t$.
The initial conditions are $v(x, 0) = 0, i(x, 0) = 0$. $\qquad ...(ii)$

Laplace transforms of $(i)$, are

$$\frac{d^2\bar{v}}{dx^2} = RC(s\bar{v} - 0) \quad \text{or} \quad \frac{d^2\bar{v}}{dx^2} - RCs\bar{v} = 0 \qquad ...(iii)$$

Laplace transforms of the conditions in $(ii)$, are

$$\bar{v}(0, s) = \frac{v_0}{s} \text{ at } x = 0 \qquad ...(iv)$$

and $\bar{v}(x, s)$ remains finite as $x \to \infty$ ...$(v)$

$\therefore$ the solution of $(iii)$ is

$$\bar{v}(x, s) = C_1 e^{\sqrt{RCs}\, x} + C_2 e^{-\sqrt{RCs}\, x}$$

To satisfy condition $(v)$, we must have $C_1 = 0$.

Using the condition $(iv)$, we get $C_2 = v_0/s$

Thus $\bar{v}(x, s) = \frac{v_0}{s} e^{-\sqrt{RCs}\, x}$

Using the inversion formula, we obtain

$$v(x, t) = v_0 L^{-1}\left\{\frac{e^{-\sqrt{RC}\, x\, .\, \sqrt{s}}}{s}\right\} = v_0 \, erfc\left(x \frac{\sqrt{RC}}{2\sqrt{t}}\right) \qquad \text{[By Ex. 22.17]}$$

$$= v_0 \frac{x\sqrt{RC}}{2\sqrt{\pi}} \int_0^t u^{-3/2} e^{-(RC x^2/4u)} du$$

$\therefore$ since $i = -\frac{1}{R}\frac{\partial v}{\partial x}$, we obtain by differentiation,

$$i(x, t) = \frac{v_0 x}{2\sqrt{x}} \sqrt{\frac{C}{R}}\, t^{-3/2} e^{(-RC x^2/4t)}.$$

**Example 22.28.** *A transmission line of length $l$ has negligible inductance and leakance. A constant voltage $v_0$ is applied at the sending end ($x = 0$) and is open circuited at the far end. Assuming the initial voltage and current to be zero, determine the voltage and current.*

**Solution.** For a transmission line with $L = G = 0$, the voltage $v$ and current $i$ are given by the equations

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \text{ and } \frac{\partial v}{\partial x} + Ri = 0 \qquad ...(i)$$

The boundary conditions are (for $t > 0$)

$$v = v_0 \text{ at } x = 0 \text{ and } i = \frac{\partial v}{\partial x} = 0 \quad \text{at } x = l \qquad ...(ii)$$

The initial condition is $v = 0$ at $t = 0$ $(x > 0)$

Laplace transforms of $(i)$ and $(ii)$ are

$$\frac{\partial^2 \bar{v}}{\partial x^2} = RC(s\bar{v} - 0) \qquad ...(iii)$$

and $\bar{v} = v_0/s$ at $x = 0$, $\frac{\partial \bar{v}}{\partial x} = 0$ at $x = l$ ...$(iv)$

$\therefore$ the solution of $(iii)$ is

$$\bar{v} = c_1 \cosh \sqrt{(RCs)}\, x + c_2 \sinh \sqrt{RCs}\, x$$

Applying conditions $(iv)$, it gives

$$v_0/s = c_1,\ 0 = c_1 \sinh \sqrt{(RCs)}\, l + c_2 \cosh \sqrt{(RCs)}\, l$$

$$\therefore \quad \bar{v} = \frac{v_0}{s}\left[\cosh \sqrt{(RCs)}x - \frac{\sinh \sqrt{(RCs)}l}{\cosh \sqrt{(RCs)}l} \sinh \sqrt{(RCs)}x\right]$$

$$= \frac{v_0}{s}\frac{\cosh pq\sqrt{s}}{\cosh p\sqrt{s}} \qquad \text{where } p = \sqrt{(RC)}\,l \text{ and } q = (l-x)/l$$

By the inversion formula (3) § 22.10, we get

$$v(x, t) = \text{sum of the residues of } (e^{st}\bar{v}) \text{ at all poles of } e^{st}\bar{v}. \qquad ...(iv)$$

These poles are at $s = 0$ and $p\sqrt{s} = \pm i(2n-1)\pi/2 = \pm ipk$ (say)

Now $\quad Res\,(e^{st}\bar{v})_{s=0} = \underset{s\to 0}{\text{Lt}} \dfrac{se^{st}v_0 \cosh pq\sqrt{s}}{s\cosh p\sqrt{s}} = v_0$

and $\quad Res(e^{st}\bar{v})_{s=-k^2} = \underset{s\to -k^2}{\text{Lt}} \dfrac{(s+k^2)e^{st}v_0\cosh pq\sqrt{s}}{s\cosh p\sqrt{s}} \qquad \left(\dfrac{0}{0}\text{ form}\right)$

$$= \underset{s\to -k^2}{\text{Lt}} \frac{v_0 . e^{st}\cosh pq\sqrt{s} + (s+k^2)(...)}{\cosh p\sqrt{s} + s\sinh p\sqrt{s}\,.\,\frac{1}{2}ps^{-1/2}}$$

$$= \frac{v_0 e^{-k^2t}\cosh(ipqk) + 0}{0 + 1/2\,(ipk)\sinh(ipk)} = \frac{2v_0 e^{-k^2 t}\cos(pqk)}{-pk\sin pk}$$

Adding up all the residues, (*iv*) gives

$$v(x, t) = v_0 + \frac{4v_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-[(2n-1)^2\pi^2t/4RCl^2]}\cos[(2n-1)\pi(l-x)/2l]$$

$$\left[\because \quad pk = (2n-1)\pi/2,\ -\sin pk = (-1)^n,\ pqk = \tfrac{1}{2}(2n-1)\pi(l-x)/l,\ k^2 = (2n-1)^2\pi^2/4RCl^2\right]$$

Also $\qquad i = -\dfrac{1}{R}\dfrac{\partial v}{\partial x}. \qquad$ [By (*i*)]

## PROBLEMS 22.4

1. Solve the differential equation using Laplace transform method, $\dfrac{\partial y}{\partial t} = 3\dfrac{\partial^2 y}{\partial x^2}$ where $y(\pi/2, t) = 0$, $(\partial y/\partial x)_{x=0} = 0$ and $y(x, 0) = 30\cos 5x$. (*U.P.T.U., 2005*)
2. Using suitable transforms, solve the differential equation $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{\partial V}{\partial t}$, $0 \le x \le \pi$, $t \ge 0$. where $V(0, t) = 0 = V(\pi, t)$ and $V(x, 0) = V_0$ constant.
3. The initial temperature along the length of an infinite bar is given by $u(x, 0) = \begin{cases} 2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$. If the temperature $u(x, t)$ satisfies the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$, find the temperature at any point of the bar at any point $t$. (*Rohtak, 2006*)
4. Use the complex form of the Fourier transform to show that
$$V = \frac{1}{2\sqrt{(\pi t)}}\int_{-\infty}^{\infty}\bar{f}(u)e^{[-(x-u)^2/4t]}\,du$$
is the solution of the boundary value problem
$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2},\ -\infty < x < \infty,\ t > 0\ ;\ V = f(x) \text{ when } t = 0. \qquad (U.P.T.U., 2008)$$
5. A semi-infinite solid ($x > 0$) is initially at temperature zero. At time $t = 0$, a constant temperature $\theta_0 > 0$ is applied and maintained at the face $x = 0$. Show that the temperature at any point $x$ and at any time $t$, is given by $\theta(x, t) = \theta_0\,erfc(x/2c\sqrt{t})$.

6. A solid is initially at constant temperature $\theta_0$, while the ends $x = 0$ and $x = a$ are maintained at temperature zero. Determine the temperature at any point of the solid at any later time $t > 0$.
7. An infinite string is initially at rest along the $x$-axis. Its one end which is at $x = 0$, is given a periodic transverse displacement $a_0 \sin \omega t$, $t > 0$. Show that the displacement of any point of the string at any time is given by
$$y(x, t) = \begin{cases} a_0 \sin \omega(t - x/c), & t > x/c \\ 0, & t < x/c, \end{cases}$$
where $c$ is the wave velocity.
8. An infinite string has an initial transverse displacement $y(x, 0) = f(x)$, $-\infty < x < \infty$, and is initially at rest. Show that
$$y(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)].$$
9. A semi-infinite transmission line has negligible inductance and leakance per unit length. A voltage $v$ is applied at the sending end $(x = 0)$ which is given by
$$v(0, t) = \begin{cases} v_0, & 0 < t < \tau \\ 0, & t > \tau \end{cases}$$
Show that the voltage at any point $x > 0$ at any time $t > 0$ is given by
$$v(x, t) = v_0\, erfc\left[\frac{x}{2}\sqrt{(RC/t)}\right].$$

## 22.12 OBJECTIVE TYPE OF QUESTIONS

### PROBLEMS 22.5

*Fill in the blanks or choose the correct answer in each of the following problems :*

1. Fourier cosine transform of $f(t)$ is .........
2. Fourier sine transform of $1/x$ is .........
3. Convolution theorem for Fourier transforms states that .........
4. If Fourier transform of $f(x)$ is $F(s)$, then the inversion formula is .........
5. $F[x^n f(x)] =$ .........
6. If $F\{f(x)\} = F(s)$, then $F\{f(x - a)\} =$ .........
7. Fourier sine integral representation of a function $f(x)$ is given by .........
8. If $F_c[f(ax)] = k\, F_c(s/a)$, then $k =$ .........
9. Fourier transform of second derivative of $u(x, t)$ is .........
10. If $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$, then Fourier sine integral of $f(x)$ is .........
11. Fourier sine transform of $f'(x)$ in the interval $(0, l)$ is .........
12. If $F(\lambda)$ is the Fourier transform of $f(x)$, then the Fourier transform of $f(ax)$ is .........
13. Inverse finite Fourier sine transform of $F_s(p) = \dfrac{1 - \cos p\pi}{(p\pi)^2}$ for $p = 1, 2, 3, \ldots$ and $0 < x < \pi$ is .........
14. If Fourier transform of $f(x) = F(s)$, then Fourier Transform of $f(2x)$ is .........
15. Fourier cosine transform of $e^{-x}$ is .........
16. $f(x) = 1$, $0 < x < \infty$ cannot be represented by a Fourier integral. (True or False)
17. $\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_c(s)|^2\, dx.$ (True or False)
18. Fourier transform is a linear operation. (True or False)
19. $F_s[x f(x)] = -\dfrac{d}{ds} F_c(s).$ (True or False)
20. Kernel of Fourier transform is $e^{isx}$. (True or False)
21. Finite Fourier cosine transform of $f(x) = 1$ in $(0, \pi)$ is zero. (True or False)