CHAPTER 20

Calculus of Complex Functions

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20.1 INTRODUCTION

In the previous chapter, we have dealt with some elementary complex functions—the exponential, logarithmic, circular and hyperbolic functions, evaluated at specific complex values. These functions are useful in the study of fluid mechanics, thermodynamics and electric fields. It, therefore, seems desirable to study the calculus of such functions.

20.2 (1) LIMIT OF A COMPLEX FUNCTION

A *function $w = f(z)$ is said to tend to* **limit** *l as z approaches a point z_0, if for every real $\in$, we can find a positive real δ such that*

$$| f(z) - l | < \in \quad \text{for} \quad | z - z_0 | < \delta$$

i.e., for every $z \neq z_0$ in the δ–disc (dotted) of z-plane, $f(z)$ has a value lying in the $\in$-disc of w-plane (Fig. 20.1). In symbols, we write $\underset{z \to z_0}{\text{Lt}} f(z) = l$.

This definition of limit though similar to that in ordinary calculus, is quite different for in real calculus x approaches x_0 only along the line whereas here *z approaches z_0 from any direction in the z-plane.*

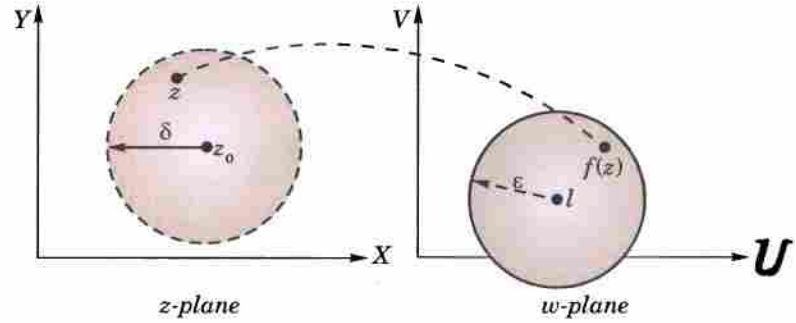

Fig. 20.1

(2) Continuity of f(z). *A function $w = f(z)$ is said to be* **continuous** *at $z = z_0$, if*

$$\underset{z \to z_0}{Lt} f(z) = f(z_0).$$

Further $f(z)$ is said to be continuous in any region R of the z-plane, if it is continuous at every point of that region.

Also if $w = f(z) = u(x, y) + iv(x, y)$ is continuous at $z = z_0$, then $u(x, y)$ and $v(x, y)$ are also continuous at $z = z_0$, *i.e.*, at $x = x_0$ and $y = y_0$. Conversely if $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0), then $f(z)$ will be continuous at $z = z_0$. [cf. § 5.1 (3)].

20.3 (1) DERIVATIVE OF f(z)

Let $w = f(z)$ be a single-valued function of the variable $z = x + iy$. Then *the derivative of $w = f(z)$ is defined to be*

$$\frac{dw}{dz} = f'(z) = \underset{\delta z \to 0}{Lt} \frac{f(z + \delta z) - f(z)}{\delta z},$$

provided the limit exists and has the same value for all the different ways in which δz approaches zero.

Suppose $P(z)$ is fixed and $Q(z + \delta z)$ is a neighbouring point (Fig. 20.2). The point Q may approach P along any straight or curved path in the given region, *i.e.*, δz may tend to zero in any manner and dw/dz may not exist. It, therefore, becomes a fundamental problem to determine the necessary and sufficient conditions for dw/dz to exist. The fact is settled by the following theorem.

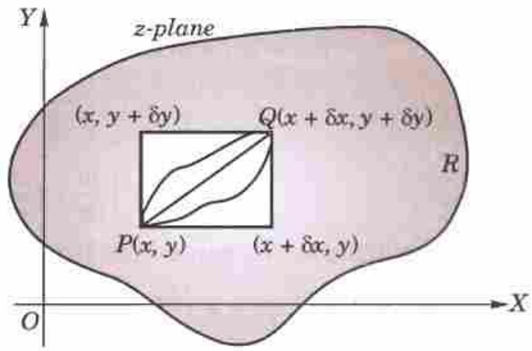

Fig. 20.2

(2) Theorem. *The necessary and sufficient conditions for the derivative of the function $w = u(x, y) + iv(x, y) = f(z)$ to exist for all values of z in a region R, are*

(*i*) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ *are continuous functions of x and y in R;*

(*ii*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

The relations (ii) are known as **Cauchy-Riemann* equations** or briefly *C-R* equations.

(*a*) *Condition is necessary.*

If $f(z)$ possesses a unique derivative at $P(z)$, then

$$f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z}$$

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\text{Lt}} \frac{\{u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)\} - \{u(x, y) + iv(x, y)\}}{\delta x + i\delta y}$$

Since δz can approach zero in any manner, we can first assume δz to be wholly real and then wholly imaginary. When δz is wholly real, then $\delta y = 0$ and $\delta z = \delta x$.

$$\therefore \quad f'(z) = \underset{\delta x \to 0}{\text{Lt}} \left(\frac{u(x + \delta x, y) - u(x, y)}{\delta x} + i\, \frac{v(x + \delta x, y) - v(x, y)}{\delta x} \right) = \frac{\partial u}{\partial x} + i\, \frac{\partial v}{\partial x} \quad ...(1)$$

When δz is wholly imaginary, then $\delta x = 0$ and $\delta z = i\delta y$.

$$\therefore \quad f'(z) = \underset{\delta y \to 0}{\text{Lt}} \left(\frac{u(x, y + \delta y) - u(x, y)}{i\delta y} + i\, \frac{v(x, y + \delta y) - v(x, y)}{i\delta y} \right) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\delta y} - i\, \frac{\partial u}{\partial y} \quad ...(2)$$

Now the existence of $f'(z)$ requires the equality of (1) and (2).

* Named after *Cauchy* (p. 144) and the German mathematician *Bernhard Riemann* (1826—1866) who along with *Weierstrass* (p. 390) laid the foundations of complex analysis. Riemann introduced the concept of integration and made basic contributions to number theory and mathematical analysis. He developed the Riemannian geometry which formed the mathematical base for Einstein's relativity theory.

$$\therefore \qquad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

On equating the real and imaginary parts from both sides, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad ...(3)$$

Thus the necessary conditions for the existence of the derivative of f (z) is that the C-R equations should be satisfied. (V.T.U., 2011 S)

(*b*) *Condition is sufficient.* Suppose $f(z)$ is a single-valued function possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of the region and the *C-R* equations (3) are satisfied.

Then by Taylor's theorem for functions of two variables (p. 220)

$$f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)$$

$$= u(x, y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + ... + i\left[v(x, y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + ...\right]$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\partial x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

[Omitting terms beyond the first powers of δx and δy]

or $$f(z + \delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\partial x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y.$$

Now using the *C-R* equation (3), replace $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $-\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x}$ respectively.

Then $$f(z + \delta z) - f(z) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right]\delta y = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right]i\delta y$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](\delta x + i\delta y) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta z$$

$$\therefore \qquad f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{or} \quad \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

which by (1) or (2) proves the sufficiency of conditions.

20.4 ANALYTIC FUNCTIONS

A function f (z) which is single-valued and possesses a unique derivative with respect to z at all points of a region R, is called an **analytic function** *of z in that region. An analytic function is also called a regular function* or an *holomorphic function.*

A function which is analytic everywhere in the complex plane, is known as an ***entire function.*** As derivative of a polynomial exists at every point, a polynomial of any degree is an *entire function.*

A point at which an analytic function ceases to possess a derivative is called a **singular point** *of the function.*

Thus if u and v are real single-valued functions of x and y such that $\delta u/\delta x$, $\delta u/\delta y$, $\delta v/\delta x$, $\delta v/\delta y$ are continuous throughout a region R, then the Cauchy-Riemann equations

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{y}} \text{ and } \frac{\partial \mathbf{u}}{\partial \mathbf{y}} = -\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \qquad ...(1)$$

are both necessary and sufficient conditions for the function $f(z) = u + iv$ to be analytic in R. The derivative of $f(z)$ is then, given by (1) of p. 664 or (2) of p. 665.

The real and imaginary parts of an analytic function are called *conjugate functions.* The relation between two conjugate functions is given by *C-R* equation (1).

Example 20.1. *If w = log z, find dw / dz and determine where w is non-analytic.*

(U.P.T.U., 2005 ; J.N.T.U., 2005)

Solution. We have $w = u + iv = \log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1} y/x$ [By (**2**), p. 665]

so that $u = \frac{1}{2}\log(x^2 + y^2),\ v = \tan^{-1} y/x.$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x}.$$

Since the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous except at (0, 0). Hence w is analytic everywhere except at $z = 0$.

$$\therefore \quad \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{x + iy} = \frac{1}{z}\ (z \neq 0).$$

Obs. The definition of the derivative of a function of complex variable is identical in form to that of the derivative of a function of real variable. Hence the rules of differentiation for complex functions are the same as those of real calculus. *Thus* **if, a complex function is once known to be analytic, it can be differentiated just in the ordinary way.**

Example 20.2. *If $f(z)$ is an analytic function with constant modulus, show that $f(z)$ is constant.* (*U.P.T.U., 2008 ; Mumbai, 2005 S ; Madras 2003 ; Bhopal, 2002 S*)

Solution. If $f(z) = u + iv$ is an analytic function, then

$$|f(z)| = \sqrt{(u^2 + v^2)} \text{ is constant} = c \text{ (say) or } u^2 + v^2 = c^2 \quad ...(i)$$

Differentiating (*i*) partially w.r.t. x and y, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0\ ; \qquad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

or
$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad ...(ii) \qquad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad ...(iii)$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y};\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ by C-R equations,

$$\therefore \quad (iii) \text{ becomes } -u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \quad ...(iv)$$

Squaring and adding (*ii*) and (*iv*), we obtain

$$u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 = 0$$

or
$$(u^2 + v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0 \text{ or } \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \quad [\because\ u^2 + v^2 = c^2 \neq 0] \quad ...(v)$$

Now
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\therefore \quad |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \quad \text{[By } (v)]$$

or $f'(z) = 0$. or $f(z) =$ constant.

Example 20.3. *Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin even though C.R. equations are satisfied thereof.* (*A.M.I.E.T.E., 2005 S ; Osmania, 2003*)

Solution. If $f(z) = \sqrt{|xy|} = u(x, y) + iv(x, y)$, then $u(x, y) = \sqrt{|xy|}$, $v(x, y) = 0$

At the origin, we have

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e., C.R. equations are satisfied at the origin.

However $$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z - 0} = \underset{z \to 0}{\text{Lt}} \frac{\sqrt{|xy|} - 0}{x + iy}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|mx^2|} - 0}{x(1 + im)}, \text{ when } z \to 0 \text{ along the line } y = mx$$

$$= \frac{\sqrt{|m|}}{1 + im} \text{ which is not unique.}$$

$\therefore$ $f'(0)$ does not exist. Hence $f(z)$ is not analytic at the origin.

Example 20.4. *Prove that the function $f(z)$ defined by*

$$f(z) = \frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2} (z \neq 0), f(0) = 0$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

(S.V.T.U., 2009 ; V.T.U., 2001)

Solution. $$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2} = \underset{y \to 0}{\text{Lt}} \frac{-y^3(1 - i)}{y^2} = \underset{y \to 0}{\text{Lt}} [-y(1 - i)] = 0$$

$$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{y \to 0 \\ x \to 0}}{\text{Lt}} \frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2} = \underset{x \to 0}{\text{Lt}} \frac{x^3(1 + i)}{x^2} = \underset{x \to 0}{\text{Lt}} [x(1 + i)] = 0$$

Also $f(0) = 0$ (given).

Thus $\underset{z \to 0}{\text{Lt}} f(z) = f(0)$ when $x \to 0$ first and then $y \to 0$ and also vice-versa. Now let both x and y tend to zero simultaneously along the path $y = mx$. Then

$$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{y \to mx \\ x \to 0}}{\text{Lt}} \frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^3(1 + i) - m^3x^3(1 - i)}{(1 + m^2)x^2} = \underset{x \to 0}{\text{Lt}} \frac{x[1 + i - m^3(1 - i)]}{1 + m^2} = 0$$

Hence $\underset{z \to 0}{\text{Lt}} f(z) = f(0)$, in whatever manner $z \to 0$. $\therefore$ $f(z)$ is continuous at the origin.

Now $$f(z) = \frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2} = u(x, y) + iv(x, y).$$

Also $u(0, 0) = 0$, and $v(0, 0) = 0$ $\quad [\because \ f(0) = 0]$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)_{0,0} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{0,0} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{-y}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{0,0} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

and $$\left(\frac{\partial v}{\partial y}\right)_{0,0} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{y}{y} = 1.$$

Hence at (0, 0), $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Thus the *C-R* equations are satisfied at the origin.

But $$f'(0) = \underset{z\to 0}{\text{Lt}} \frac{f(z)-f(0)}{z} = \underset{z\to 0}{\text{Lt}} \frac{f(z)}{z} = \underset{z\to 0}{\text{Lt}} \frac{x^3-y^3+i(x^3+y^3)}{(x^2+y^2)(x+iy)}.$$

If $z \to 0$ along the path $y = mx$, then $f'(0) = \dfrac{1-m^3+i(1+m^3)}{(1+m^2)(1+im)}$

which assumes different values as m varies. So $f'(z)$ is not unique at (0, 0) *i.e.*, $f'(0)$ does not exist. Thus $f(z)$ is not analytic at the origin even though it is continuous and satisfies the *C-R* equations thereat.

Example 20.5. *Show that polar form of Cauchy-Riemann equations are*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$ (U.P.T.U., 2008 ; V.T.U., 2006)

Deduce that $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0.$ (Bhopal, 2009 ; Kurukshetra, 2005)

Solution. If (r, θ) be the coordinates of a point whose cartesian coordinates are (x, y), then $z = x + iy = re^{i\theta}$.

$$\therefore \qquad u + iv = f(z) = f(re^{i\theta})$$

where u and v are now expressed in terms of r and θ.

Differentiating it partially w.r.t. r and θ, we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \,.\, e^{i\theta}$$

and $$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \,.\, ire^{i\theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ...(i) \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad ...(ii)$$

Differentiating (*i*) partially w.r.t. r, we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} \quad ...(iii)$$

Differentiating (*ii*) partially w.r.t. θ, we have

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial r \partial \theta} \quad ...(iv)$$

Thus using (*i*), (*ii*) and (*iv*)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} + \frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) + \frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial r \partial \theta}\right) = 0 \qquad \left[\because \frac{\partial^2 v}{\partial \theta \partial r} = \frac{\partial^2 v}{\partial r \partial \theta}\right]$$

20.5 (1) HARMONIC FUNCTIONS

If $f(z) = u + iv$ be an analytic function in some region of the z-plane, then the Cauchy-Riemann equations are satisfied.

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ...(1) \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad ...(2)$$

Differentiating (1) with respect to x and (2) with respect to y, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad ...(3) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}. \quad ...(4)$$

Adding (3) and (4) and assuming that $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y \partial x}$, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad ...(5)$$

Similarly, by differentiating (1) with respect to y and (2) with respect to x and subtracting, we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \quad ...(6)$$

Thus both the functions u and v satisfy the Laplace's equation in two variables. For this reason, they are known as **harmonic functions** and their theory is called **potential theory.** *(Rohtak, 2005)*

(2) Orthogonal system. Consider the two families of curves

$$u(x, y) = c_1 \quad ...(7) \qquad \text{and} \qquad v(x, y) = c_2 \quad ...(8)$$

Differentiating (7), we get $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx} = 0$

or $$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial y}{\partial v/\partial x} = m_1 \text{ (say)} \qquad \text{[By (1) and (2)]}$$

Similarly (8) gives $$\frac{dy}{dx} = -\frac{\partial v/\partial x}{\partial v/\partial y} = m_2 \text{ (say)}$$

$\therefore$ $m_1 m_2 = -1$, *i.e.*, (7) and (8) form an orthogonal system.

Hence *every analytic function $f(z) = u + iv$ defines two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$, which form an orthogonal system.* *(U.P.T.U., 2009)*

20.6 APPLICATIONS TO FLOW PROBLEMS

As the real and imaginary parts of an analytic function are the solutions of the Laplace's equation in two variables, the conjugate functions provide solutions to a number of field and flow problems.

As an illustration, consider the irrotational motion of an incompressible fluid in two dimensions. Assuming the flow to be in planes parallel to the xy-plane, the velocity $\mathbf{V}$ of a fluid particle can be expressed as

$$\mathbf{V} = v_x \mathbf{I} + v_y \mathbf{J} \quad ...(1)$$

Since the motion is irrotational, therefore, by § 6.18 (1), there exist a scalar function $\phi(x, y)$ such that

$$\mathbf{V} = \nabla\phi(x, y) = \frac{\partial \phi}{\partial x}\mathbf{I} + \frac{\partial \phi}{\partial y}\mathbf{J} \quad ...(2)$$

[The function $\phi(x, y)$ is called the *velocity potential* and the curves $\phi(x, y) = c$ are known as *equipotential lines*.]

Thus from (1) and (2), $v_x = \dfrac{\partial \phi}{\partial x}$ and $v_y = \dfrac{\partial \phi}{\partial y}$ ...(3)

Also the fluid being incompressible div $\mathbf{V} = 0$ [by § 8.7 (1)] *i.e.*, $\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0$.

Substituting the values of v_x and v_y from (3), we get $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$

which shows that the velocity potential ϕ is *harmonic*. It follows that there must exist a conjugate harmonic function $\psi(x, y)$ such that $w(z) = \phi(x, y) + i\psi(x, y)$ ...(4)
is analytic.

Also the slope at any point of the curve $\psi(x, y) = c'$ is given by

$$\frac{dy}{dx} = -\frac{\partial \psi/\partial x}{\partial \psi/\partial y} = \frac{\partial \phi/\partial y}{\partial \phi/\partial x} \qquad \text{[By } C\text{-}R \text{ equations]}$$

$$= v_y/v_x \qquad \text{[By (3)]}$$

This shows that the velocity of the fluid particle is along the tangent to the curve $\psi(x, y) = c'$, *i.e.* the particle moves along this curve. Such curves are known as *stream lines* and $\psi(x, y)$ is called the *stream function*. Also the equipotential lines $\phi(x, y) = c$ and the stream lines $\psi(x, y) = c'$ cut orthogonally.

From (4), $\dfrac{\partial w}{dz} = \dfrac{\partial \phi}{\partial x} + i\dfrac{\partial \psi}{\partial x} = \dfrac{\partial \phi}{\partial x} - i\dfrac{\partial \phi}{\partial y}$ [By *C-R* equations]

$= v_x - iv_y$ [By (3)]

∴ The magnitude of the fluid velocity $= \sqrt{(v_x^2 + v_y^2)} = |\, dw/dz \,|$.

Thus the flow pattern is fully represented by the function $w(z)$ which is known as the **complex potential.**

Similarly the complex potential $w(z)$ can be taken to represent any other type of 2-dimensional steady flow. In electrostatics and gravitational fields, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are *equipotential lines* and *lines of force*. In heat flow problems, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are known as *isothermals* and *heat flow lines* respectively.

Given $\phi(x, y)$, we can find $\psi(x, y)$ and *vice-versa*.

Example 20.6. *If $w = \phi + i\psi$ respresents the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$, determine the function ϕ.* (V.T.U., 2011 ; Mumbai, 2008 ; Bhopal, 2002 S)

Solution. It is readily verified that ψ satisfies the Laplace's equation.

∴ ϕ and ψ must satisfy the Cauchy-Riemann equations :

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad ...(i) \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad ...(ii)$$

∴ by (*i*), $$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial y}\left[x^2 - y^2 + \frac{x}{x^2 + y^2}\right] = -2y - \frac{2xy}{(x^2 + y^2)^2}$$

Integrating w.r.t. x, we get $\phi = -2xy + \dfrac{y}{x^2 + y^2} + \eta(y)$ where $\eta(y)$ is an arbitrary function of y.

∴ (*ii*) gives $-2x + \dfrac{x^2 - y^2}{(x^2 + y^2)^2} + \eta'(y) = -2x + \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$

whence $\eta'(y) = 0$, *i.e.*, $\eta(y) = c$, an arbitrary constant.

Thus $$\phi = -2xy + \frac{y}{x^2 + y^2} + c$$

Otherwise (Milne-Thomson's method*) :

We have $$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x} = \left[-2y - \frac{2xy}{(x^2 + y^2)^2}\right] + i\left[2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}\right]$$

By Milne-Thomson's method, we express dw/dz in terms of z, on replacing x by z and y by 0.

∴ $$\frac{dw}{dz} = i\left(2z - \frac{1}{z^2}\right)$$

Integrating w.r.t. z, we get $w = i\,(z^2 + 1/z) + A$ where A is a complex constant.

---

* Since $z = x + iy$ and $\bar{z} = x - iy$, we have

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z + \bar{z})$$

∴ $$f(z) = \phi(x, y) + i\psi(x, y) \quad ...(1)$$

$$= \phi\left[\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right] + i\psi\left[\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right]$$

Now considering this as a formal identity in the two independent variables z, $\bar{z}$ and putting $\bar{z} = z$, we get

$$f(z) = \phi(z, 0) + i\psi(z, 0) \quad ...(2)$$

∴ (2) is the same as (1), if we replace x by z and y by 0.

Thus **to express any function in terms of z, replace x by z and y by 0.** *This provides an elegant method of finding $f(z)$ when its real part or the imaginary part is given. It is due to Milne-Thomson.*

Hence $$\phi = R\left[i\left(z^2 + \frac{1}{z}\right) + A\right] = -2xy + \frac{y}{x^2+y^2} + c.$$

Example 20.7. *Find the analytic function, whose real part is $\sin 2x/(\cosh 2y - \cos 2x)$.* *(J.N.T.U., 2005 ; Anna, 2003)*

Solution. Let $f(z) = u + iv$, where $u = \sin 2x/(\cosh 2y - \cos 2x)$

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[By } C\text{-}R \text{ equations]}$$

$$= \frac{(\cosh 2y - \cos 2x)\, 2\cos 2x - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2} - i\,\frac{\sin 2x(-2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} + i\,\frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

By Milne-Thomson's method, we express $f'(z)$ in terms of z by putting $x = z$ and $y = 0$.

$$\therefore \quad f'(z) = \frac{2\cos 2z - 2}{(1-\cos 2z)^2} + i(0) = \frac{-2}{1-\cos 2z} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

Integrating w.r.t. z, we get $f(z) = \cot z + ic$, taking the constant of integration as imaginary since u does not contain any constant.

Example 20.8. *Determine the analytic function $f(z) = u + iv$, if $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$ and $f(\pi/2) = 0$.* *(A.M.I.E.T.E., 2005 ; Osmania, 2003)*

Solution. We have $\quad u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$

$$\therefore \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + 1 - e^{-y}\sin x}{2(\cos x - \cosh y)^2} \qquad ...(i)$$

and
$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{(\cos x - \cosh y)\, e^{-y} + (\cos x + \sin x - e^{-y})\sinh y}{2(\cos x - \cosh y)^2}$$

or
$$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = \frac{(\sin x + \cos x)\sinh y + e^{-y}(\cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2} \qquad ...(ii)$$

Subtracting (*ii*) from (*i*), we get

$$2\frac{\partial u}{\partial x} = \frac{(\sin x - \cos x)\cosh y - (\sin x + \cos x)\sinh y + 1 - e^{-y}(\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Adding (*i*) and (*ii*), we have

$$-2\frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + (\sin x + \cos x)\sinh y + 1 + e^{-y}(-\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Thus $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1-\cos z}{2(1-\cos z)^2} \qquad \text{[Putting } x = z \text{ and } y = 0]$$

$$= \frac{1}{2(1-\cos z)} = \frac{1}{4\sin^2 z/2} = \frac{1}{4}\operatorname{cosec}^2\frac{z}{2} \quad \text{or} \quad f(z) = -\frac{1}{2}\cot\frac{z}{2} + c$$

Since $f(\pi/2) = 0$, $\quad 0 = -\dfrac{1}{2}\cot \pi/4 + c$, whence $c = \dfrac{1}{2}$

Hence $$f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right).$$

Example 20.9. *Find the conjugate harmonic of $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$. Show that v is harmonic.* *(Marathwada, 2008)*

Solution. Let $f(z) = u + v$. Using C-R equations in polar coordinates (Ex. 20.5),

$$r \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = -2r^2 \sin 2\theta + r \sin \theta \quad \text{...(i)}$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r \cos 2\theta - \cos \theta \quad \text{...(ii)}$$

$\therefore$ (*i*) gives, $$\frac{\partial u}{\partial r} = -2r \sin 2\theta + \sin \theta$$

Integrating w.r.t., r

$$u = -r^2 \sin 2\theta + r \sin \theta + \phi(\theta)$$ where $\phi(\theta)$ is an arbitrary function.

$\therefore$ $$\frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta) \quad \text{...(iii)}$$

From (*ii*) and (*iii*), we get

$$-2r^2 \cos 2\theta + r \cos \theta = \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta)$$

$\therefore$ $\phi'(\theta) = 0$ or $\phi(\theta) = c$

Thus $u = -r^2 \sin 2\theta + r \sin \theta + c$ is the conjugate harmonic of v.

Now v will be harmonic if it satisfies the Laplace equation $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r} + \dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0$

From (*i*), $\dfrac{\partial^2 v}{\partial \theta^2} = -4r^2 \cos 2\theta + r \cos \theta$. From (*ii*), $\dfrac{\partial^2 v}{\partial r^2} = 2 \cos 2\theta$

$$\therefore \quad \frac{\partial t^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 2 \cos 2\theta + \frac{1}{r}(2r \cos 2\theta - \cos \theta) + \frac{1}{r^2}(-4r^2 \cos 2\theta + r \cos \theta)$$

$$= 4 \cos 2\theta - \frac{1}{r}\cos \theta - 4 \cos 2\theta + \frac{1}{r}\cos \theta = 0$$

Hence v is harmonic.

Example 20.10. (*a*) *Find the orthogonal trajectories of the family of curves*

$$x^4 + y^4 - 6x^2y^2 = \text{constant}$$

(*b*) *Show that the curves $r^n = \alpha \sec n\theta$ and $r^n = \beta \operatorname{cosec} n\theta$ cut orthogonally.* *(Mumbai, 2005 ; J.N.T.U., 2003)*

Solution. (*a*) Take $u(x, y) = x^4 + y^4 - 6x^2y^2$. Then the family of curves $v(x, y)$ = constant will be the required trajectories if $f(z) = u + iv$ is analytic.

Now $$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2, \quad \frac{\partial u}{\partial y} = 4y^3 - 12x^2y$$

$\therefore$ $$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 4x^3 - 12xy^2$$

Integrating, $$v = 4x^3y - 4xy^3 + c(x)$$

Differentiating partially w.r.t. x

$$12x^2y - 4y^3 + \frac{dc(x)}{dx} = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -4y^3 + 12x^2y$$

$\therefore$ $$\frac{dc(x)}{dx} = 0 \quad \text{or} \quad c = \text{constant}$$

Thus the required orthogonal trajectories are v = constant or $x^3y - xy^3$ = constant.

(*b*) Writing $u(r, \theta) = r^n \cos n\theta = \alpha$ and $v(r, \theta) = r^n \sin n\theta = \beta$,

we have $u(r, \theta) + iv(r, \theta) = \alpha + i\beta = r^n (\cos n\theta + i \sin n\theta) = r^n . e^{in\theta} = (re^{i\theta})^n = z^n$

This is an analytic function.

Thus $f(z) = u + iv$, gives the curves $u = \alpha$ and $v = \beta$ which cut orthogonally.

Example 20.11. *Two concentric circular cylinders of radii r_1, r_2 $(r_1 < r_2)$ are kept at potentials ϕ_1 and ϕ_2 respectively. Using complex function $w = a \log z + c$, prove that the capacitance per unit length of the capacitor formed by them is $2\pi\lambda / \log (r_2/r_1)$ where λ is the dielectric constant of the medium.*

Solution. We have $\quad \phi + i\psi = a \log (re^{i\theta}) + c \quad$ where $z = x + iy = re^{i\theta}$

$\therefore \quad \phi = a \log r + c, \quad$ and $\quad \psi = a\theta$

so that $\quad \phi_1 = a \log r_1 + c, \quad \phi_2 = a \log r_2 + c$

Thus the potential difference $= \phi_2 - \phi_1 = a (\log r_2 - \log r_1)$

Also the total charge (or flux) $= \int_0^{2\pi} d\psi = \int_0^{2\pi} a\, d\theta = 2\pi a.$

The capacitance being the charge required to maintain a unit potential difference ; the capacitance without dielectric

$$= \frac{\text{charge}}{\text{potential difference}} = \frac{2\pi a}{a(\log r_2 - \log r_1)} = \frac{2\pi}{\log(r_2/r_1)}.$$

A medium of dielectric constant λ increases the potential difference to λ times that in vacuum for the same charge. Thus the capacitance with dielectric $= 2\pi\lambda/\log (r_2/r_1)$.

Example 20.12. *If $f(z)$ is a regular function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) | f(z)|^2 = 4 \,| f'(z) |^2. \qquad \textit{(J.N.T.U., 2006 ; Kottayam, 2005)}$$

or

$$\nabla^2 \,| f(z) |^2 = 4 \,| f'(z) |^2 \qquad \textit{(Madras, 2006)}$$

Solution. Let $f(z) = u(x, y) + iv(x, y)$ so that $| f(z) |^2 = u^2 + v^2 = \phi(x, y)$, (say).

$$\therefore \quad \frac{\partial \phi}{\partial x} = 2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} \text{ and } \frac{\partial^2 \phi}{\partial x^2} = 2\left\{u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v \frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right\}$$

Similarly,
$$\frac{\partial^2 \phi}{\partial y^2} = 2\left\{u \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v \frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right\}$$

Adding, we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left\{u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right\} + 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} \quad ...(i)$$

Since u, v have to satisfy Cauchy-Riemann equations and the Laplace's equation.

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2 ; \left(\frac{\partial u}{\partial y}\right)^2 = \left(-\frac{\partial v}{\partial x}\right)^2 \text{ and } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}.$$

Thus (i) takes the form $\quad \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 4\left\{\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2\right\}$

Hence $\quad \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) | f(z)|^2 = 4 \,| f'(z) |^2 \quad$ or $\quad \nabla^2 \,| f(z) |^2 = 4 \,| f'(z) |^2.$

PROBLEMS 20.1

1. If $f(z) = \begin{cases} x^3 y(y - ix)/(x^6 + y^2), & z \neq 0 \\ 0 & , z = 0 \end{cases}$ prove that $|f(z) - f(0)|/z \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ along the curve $y = ax^3$.

2. Show that (a) $f(z) = xy + iy$ is everywhere continuous but is not analytic. (Osmania, 2003 S)
(b) $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane. (J.N.T.U., 2003)

3. If $f(z) = u + iv$ is analytic, then show that $|f'(z)|^2 = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$. (Mumbai, 2007)

4. Find the constants a, b, c, d and e if $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$ is analytic. (Mumbai, 2008)

5. Show that z^n is analytic. Hence find its derivative. (V.T.U., 2010 S)

6. Determine which of the following functions are analytic :
(i) $2xy + i(x^2 - y^2)$ (ii) $(x - iy)/(x^2 + y^2)$ (iii) $\cosh z$.

7. (a) Determine p such that the function $f(z) = \dfrac{1}{2}\log_e(x^2 + y^2) + i\tan^{-1}(px/y)$ be an analytic function. (Mumbai, 2007 ; J.N.T.U., 2003)
(b) Show that $u = \dfrac{1}{2}\log(x^2 + y^2)$ is harmonic and find its harmonic conjugate function. (U.P.T.U., 2010)

8. Show that each of the following functions is not analytic at any point :
(i) $\bar{z}$ (J.N.T.U., 2003) (ii) $|z|^2$.

9. Show that $u + iv = (x - iy)/(x - iy + a)$ where $a \neq 0$, is not an analytic function of $z = x + iy$ whereas $u - iv$ is such a function.

10. Show that $f(z) = \begin{cases} xy^2(x + iy) \div (x^2 + y^4), & z \neq 0 \\ 0, & z = 0 \end{cases}$
is not analytic at $z = 0$, although C-R equations are satisfied at the origin. (J.N.T.U., 2003)

11. Verify if $f(z) = \dfrac{xy^2(x + iy)}{x^2 + y^4}$, $z \neq 0$; $f(0) = 0$ is analytic or not. (U.P.T.U., 2008)

12. Examine the nature of the function $f(z) = \dfrac{x^2y^5(x + iy)}{x^4 + y^{10}}$, $z \neq 0$; $f(0) = 0$. (Rohtak, 2004)

13. For the function $f(z)$ defined by $f(z)^2 = (\bar{z})^2/z$, $z \neq 0$; $f(0) = 0$, show that the C-R equations are satisfied at (0, 0), but $f(z)$ is not differentiable at (0, 0). (P.T.U., 2010)

14. Determine the analytic function whose real part is
(i) $x^3 - 3xy^2 + 3x^2 - 3y^2$ (Bhopal, 2009) (ii) $\cos x \cosh y$ (Rohtak, 2004)
(iii) $y/(x^2 + y^2)$ (iv) $y + e^x \cos y$ (S.V.T.U., 2008 ; V.T.U., 2006)
(v) $e^{-x}(x \sin y - y \cos y)$ (U.P.T.U., 2008)
(vi) $e^{2x}(x \cos 2y - y \sin 2y)$ (V.T.U., 2008 S ; Mumbai, 2005 ; Kottayam, 2005)
(vii) $x \sin x \cosh y - y \cos x \sinh y$ (V.T.U., 2006)
(viii) $e^x[(x^2 - y^2)\cos y - 2xy \sin y]$. (V.T.U., 2010 S ; Rohtak, 2005)

15. Find the regular function whose imaginary part is
(i) $(x - y)/(x^2 + y^2)$ (ii) $-\sin x \sinh y$ (iii) $e^x \sin y$
(iv) $e^{-x}(x \sin y - y \cos y)$ (v) $e^{-x}(x \cos y + y \sin y)$ (U.P.T.U., 2009) (vi) $\dfrac{2 \sin x \sin y}{\cos 2x + \cosh 2y}$. (Mumbai, 2006)

16. Find the analytic function $z = u + iv$, if
(i) $u - v = (x - y)(x^2 + 4xy + y^2)$ (Mumbai, 2008 ; V.T.U., 2007 ; W.B.T.U., 2005)
(ii) $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2 \cos x - e^y - e^{-y}}$ when $f\left(\dfrac{\pi}{2}\right) = 0$ (Mumbai, 2007)
(iii) $u + v = \dfrac{2 \sin 2x}{e^{2y} - e^{-2y} - 2 \cos 2x}$. (P.T.U., 2002)

17. An electrostatic field in the xy-plane is given by the potential function $\phi = 3x^2y - y^3$, find the stream function.

18. If the potential function is $\log(x^2 + y^2)$, find the flux function and the complex potential function.

19. Prove that $u = x^2 - y^2$ and $v = -\dfrac{y}{x^2 + y^2}$ are harmonic functions of (x, y) but are not harmonic conjugates. (U.P.T.U., 2004 S)

20. Show that the function $u = e^{-2xy} \sin(x^2 - y^2)$ is harmonic. Find the conjugate function v and express $u + iv$ as an analytic function of z. *(Bhopal, 2007)*
21. For $w = \exp(z^2)$, find u and v, and prove that the curves $u(x, y) = c_1$ and $v(x, y) = c_2$ where c_1 and c_2 are constants, cut orthogonally. *(J.N.T.U., 2003)*
22. Find the orthogonal trajectories of the family of curves
    (*i*) $x^3y - xy^3 = c$ *(Mumbai, 2007)* (*ii*) $e^x \cos y - xy = c$ *(Mumbai, 2008)* (*iii*) $r^2 \cos 2\theta = c$.
23. In a two dimensional fluid flow, the stream function ψ is given, find the velocity potential ϕ :
    (*i*) $\psi = -y/(x^2 + y^2)$ (*ii*) $\psi = \tan^{-1}(y/x)$.
24. Find the analytic function $f(z) = u + iv$, given
    (*i*) $u = a(1 + \cos\theta)$ (*ii*) $v = (r - 1/r) \sin\theta$, $r \neq 0$.
25. If $f(z)$ is an analytic function of z, show that
$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2.$$
(U.P.T.U., 2009 ; V.T.U., 2008 S ; P.T.U., 2005)
26. If $f(z)$ is an analytic function of z, prove that
    (*i*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log |f(z)| = 0$ *(Madras, 2000 S)* (*ii*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |R f(z)|^2 = 2 |f'(z)|^2$
    (*iii*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^p = p^2 |f'(z)|^2 |f(z)|^{p-2}$. *(Kerala, 2005)*
27. Prove that $\psi = \log [(x - 1)^2 + (y - 2)^2]$ is harmonic in every region which does not include the point (1, 2). Find a function ϕ such that $\phi + i\psi$ is an analytic function of the complex variable $z = x + iy$. Express $\phi + i\psi$ as a function of z.

20.7 GEOMETRICAL REPRESENTATION OF w = f(z)

To find the geometrical representation of a function of a complex variable, it requires a departure from the usual practice of cartesian plotting, where we associate a curve to a real function $y = f(x)$.

In the complex domain, the function $w = f(z)$

i.e., $$u + iv = f(x + iy) \qquad ...(1)$$

involves four real variables x, y, u, v. Hence a four dimensional region is required to plot (1) in the cartesian fashion. As it is not possible to have 4-dimensional graph papers, we make use of two complex planes, one for the variable $z = x + iy$, and the other for the variable $w = u + iv$. If the point z describes some curve C in the z-plane, the point w will move along a corresponding curve C' in the w-plane, since to each point (x, y), there corresponds a point (u, v) (Fig. 20.3). We then, say that *a curve C in the z-plane is mapped into the corresponding curve C′ in the w-plane by the function w = f(z) which defines a* **mapping** *or* **transformation** *of the z-plane into the w-plane.*

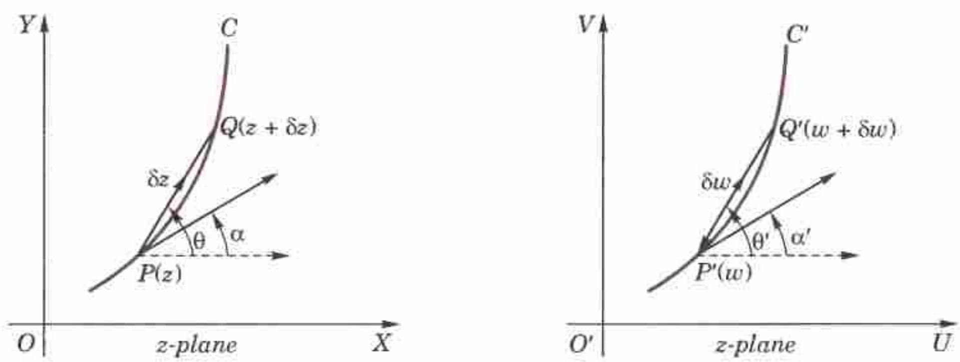

Fig. 20.3

20.8 SOME STANDARD TRANSFORMATIONS

(1) Translation. $w = z + c$, where c is a complex constant.

If $z = x + iy$, $c = c_1 + ic_2$ and $w = u + iv$, then the transformation becomes $u + iv = x + iy + c_1 + ic_2$ whence $u = x + c_1$ and $v = y + c_2$, *i.e.* the point $P(x, y)$ in the z-plane is mapped onto the point $P'(x + c_1, y + c_2)$ in the

w-plane. Every point in the z-plane is mapped onto w-plane in the same way. Thus if the w-plane is superposed on the z-plane, figure is shifted through a distance given by the vector c. Accordingly, this transformation maps a figure in the z-plane into a figure in the w-plane of the same shape and size.

In particular, this transformation changes circles into circles.

(2) Magnification and rotation. w = cz, where c is a complex constant.

If $$c = \rho e^{i\alpha},\ z = re^{i\theta} \text{ and } w = Re^{i\phi}, \text{ then}$$

$$Re^{i\phi} = \rho e^{i\alpha} \cdot re^{i\theta} = \rho r e^{i(\theta+\alpha)}$$

whence $R = \rho r$ and $\phi = \theta + \alpha$, *i.e.* the point $P(r, \theta)$ in the z-plane is mapped onto the point $P'(\rho r, \theta + \alpha)$ in the w-plane. Hence the transformation consists of magnification (or contraction) of the radius vector of P by $\rho = |c|$ and its rotation through an $\angle\alpha = \text{amp}(c)$. Accordingly any figure in the z-plane is transformed into a geometrically similar figure in the w-plane. In particular, this transformation maps circles into circles.

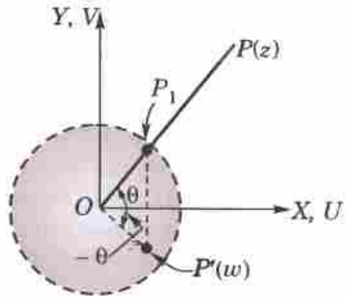

Fig. 20.4

(3) Inversion and reflection. w = 1/z.

Here it is convenient to think the w-plane as superposed on z-plane (Fig. 20.4).

If $z = re^{i\theta}$ and $w = Re^{i\phi}$, then $Re^{i\phi} = \dfrac{1}{r}e^{-i\theta}$

whence $R = 1/r$ and $\phi = -\theta$. Thus, if P be (r, θ) and P_1 be $(1/r, \theta)$, *i.e.* P_1 is the inverse* of P w.r.t. the unit circle with centre O, then the reflection P' of P_1 in the real axis represents $w = 1/z$.

Hence this transformation is an inversion of z w.r.t. the unit circle $|z| = 1$ followed by reflection of the inverse into the real axis.

Obs. 1. Clearly the function $w = 1/z$ maps the interior of the unit circle $|z| = 1$ onto the exterior of the unit circle $|w| = 1$ and the exterior of $|z| = 1$ onto the interior of $|w| = 1$. In particular, the origin $z = 0$ corresponds to the improper point $w = \infty$, called the *point at infinity* and the image of the improper point $z = \infty$ is the origin $w = 0$.

2. *This transformation maps a circle onto a circle or to a straight line if the former goes through the origin.*

To prove this, we write $z = 1/w$ as $x + iy = \dfrac{1}{u+iv} = \dfrac{u-iv}{u^2+v^2}$

so that $$x = \frac{u}{u^2+v^2} \text{ and } y = \frac{-v}{u^2+v^2}. \quad ...(1)$$

Now the general equation of any circle in the z-plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ...(2)$$

which on substituting from (1), becomes $\dfrac{u^2}{(u^2+v^2)^2} + \dfrac{v^2}{(u^2+v^2)^2} + 2g\dfrac{u}{u^2+v^2} + 2f\dfrac{-v}{u^2+v^2} + c = 0$

or $$c(u^2 + v^2) + 2gu - 2fv + 1 = 0 \quad ...(3)$$

This is the equation of a circle in the w-plane. If $c = 0$, the circle (2) passes through the origin and its image, *i.e.*, (3) reduces to a straight line. Hence the result.

Regarding a straight line as the limiting form of a circle with infinite radius, we conclude that the *transformation $w = 1/z$ always maps a circle into a circle.*

(4) Bilinear transformation. The transformation

$$w = \frac{az+b}{cz+d} \quad ...(1)$$

where a, b, c and d are complex constants and $ad - bc \neq 0$ is known as the **bilinear transformation.**** The condition $ad - bc \neq 0$ ensures that $dw/dz \neq 0$, *i.e.*, the transformation is conformal. If $ad - bc = 0$ every point of the z-plane is a *critical point.*

The inverse mapping of (1) is

$$z = \frac{-dw+b}{cw-a} \quad ...(2)$$

which is also a bilinear transformation.

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* The inverse of a point A w.r.t. a circle with centre O and radius k is defined as the point B on the line OA such that $OA.OB = k^2$.

** First studied by *Möbius* (p. 337). Hence, sometimes called *Möbius transformation*

Obs. 1. From (1), we see that each point in the z-plane except $z = -d/c$, corresponds a unique point in the w-plane. Similarly, (2) shows that each point in the w-plane except $w = a/c$, maps into a unique point in the z-plane. Including the images of the two exceptional points as the infinite points in the two planes, it follows that *there is one to one correspondence between all points in the two planes.*

Obs. 2. Invariant points of bilinear transformation. If z maps into itself in the w-plane (*i.e.*, $w = z$), then (1) gives

$$z = \frac{az+b}{cz+d} \quad \text{or} \quad cz^2 + (d-a)z - b = 0$$

The roots of this equation (say : z_1, z_2) *are defined as the invariant or fixed points of the bilinear transformation (1).*

If however, the two roots are equal, the bilinear transformation is said to be *parabolic.*

Obs. 3. Dividing the numerator and denominator of the right side of (1) by one of the four constants, it is clear that (1) has only three essential arbitrary constants. Hence *three conditions are required to determine a bilinear transformation.* For instance, three distinct points z_1, z_2, z_3 can be mapped into any three specified points w_1, w_2, w_3.

Two important properties :

I. A bilinear transformation maps circles into circles.

By actual division, (1) can be written as $w = \dfrac{a}{c} + \dfrac{bc-ad}{c^2} \cdot \dfrac{1}{z + d/c}$

which is a combination of the transformations

$$w_1 = z + d/c,\ w_2 = 1/w_1,\ w_3 = \frac{bc-ad}{c^2} w_2,\ w = \frac{a}{c} + w_3 .$$

By these transformations, we successively pass from z-plane to w_1-plane, from w_1-plane to w_2-plane, from w_2-plane to w_3-plane and finally from w_3-plane to w-plane. Now each of these transformations is one or other of the standard transformations $w = z + c$, $w = cz$, $w = 1/z$ and under each of these a circle always maps onto a circle. Hence the bilinear transformation maps circles into circles.

II. A bilinear transformation preserves cross-ratio† of four points.

Let the points z_1, z_2, z_3, z_4 of the z-plane map onto the points w_1, w_2, w_3, w_4 of the w-plane respectively under the bilinear transformation (1). If these points are finite, then from (1), we have

$$w_j - w_k = \frac{az_j + b}{cz_j + d} - \frac{az_k + b}{cz_k + d} = \frac{ad - bc}{(cz_j + d)(cz_k + d)}(z_j - z_k).$$

Using this relation for $j, k = 1, 2, 3, 4$, we get $\dfrac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$

Thus the cross-ratio of four points is invariant under bilinear transformation.

This property is very useful in finding a bilinear transformation. If one of the points, say : $z_1 \to \infty$, the quotient of those two differences which contain z_1, is replaced by 1 *i.e.*,

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{z_3 - z_4}{z_3 - z_2}.$$

Example 20.13. *Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$.*

Hence find (a) the image of $|z| < 1$, *(Mumbai, 2006 ; Delhi, 2002)*

(b) the invariant points of this transformation. *(U.P.T.U., 2008 ; V.T.U., 2000)*

Solution. Let the points $z_1 = 1, z_2 = i, z_3 = -1$ and $z_4 = z$ map onto the points $w_1 = i, w_2 = 0, w_3 = -i$ and $w_4 = w$. Since the cross-ratio remains unchanged under a bilinear transformation.

$$\therefore \quad \frac{(1-i)(-1-z)}{(1-z)(-1-i)} = \frac{(i-0)(-i-w)}{(i-w)(-i-0)} \quad \text{or} \quad \frac{w+i}{w-i} = \frac{(z+1)(1-i)}{(z-1)(1+i)}$$

By componendo dividendo, we get $\dfrac{2w}{2i} = \dfrac{(z+1)(1-i) + (z-1)(1+i)}{(z+1)(1-i) - (z-1)(1+i)}$

† **Def.** *If t_1, t_2, t_3, t_4 be any four numbers, then* $\dfrac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_4)(t_3 - t_2)}$ *is said to be their cross-ratio and is denoted* (t_1, t_2, t_3, t_4).

$\therefore \qquad w = \dfrac{1+iz}{1-iz}$ ...(*i*)

which is the required bilinear transformation.

(*a*) Rewriting (*i*) as $\quad z = i\,\dfrac{1-w}{1+w}$

$\therefore \qquad \left|\dfrac{i(1-w)}{1+w}\right| = |z| < 1 \quad \text{or} \quad |i|\,|1-w| < |1+w|$

or $\qquad |1-u-iv| < |1+u+iv|$ $\qquad [\because \; |i| = 1]$

or $\qquad (1-u)^2 + v^2 < (1+u)^2 + v^2$ which reduces to $u > 0$.

Hence the interior of the circle $x^2 + y^2 = 1$ in the z-plane is mapped onto the entire half of the w-plane to the right of the imaginary axis.

(*b*) To find the invariant points of the transformation, we put $w = z$ in (*i*).

$\therefore \qquad z = \dfrac{1+iz}{1-iz} \quad \text{or} \quad iz^2 + (i-1)z + 1 = 0$

or $\qquad z = \dfrac{1-i \pm \sqrt{[(i-1)^2 - 4i]}}{2i} = -\tfrac{1}{2}\{1 + i \pm \sqrt{(6i)}\}$

which are the required invariant points.

Example 20.14. *Show that* $w = \dfrac{i-z}{i+z}$ *maps the real axis of z-plane into the circle* $|w| = 1$ *and the half plane* $y > 0$ *into the interior of the unit circle* $|w| = 1$ *in the w-plane.* (*Mumbai, 2007*)

Solution. Since $\qquad w = (i-z)/(i+z)$,

$\therefore \qquad |w| = 1$ becomes $(i-z)/(i+z)\,| = 1 \quad \text{or} \quad |i-z| = |i+z|$

i.e., $\qquad |i-x-iy| = |i+x+iy| \quad \text{or} \quad |-x+i(1-y)| = |x+i(1+y)|$

$\therefore \qquad \sqrt{[x^2 + (1-y)^2} = \sqrt{(x^2 + (1+y)^2]}$ or $(1-y)^2 = (1+y)^2$

$\therefore \qquad 4y = 0 \quad \text{or} \quad y = 0$ which is the real axis.

Hence the real axis of the z-plane is mapped to the circle $|w| = 1$

Now for the interior of the circle $|w| = 1$

$\qquad |w| < 1 \quad i.e., \quad |i-z| < |i+z| \quad \text{or} \quad (1-y)^2 < (1+y)^2$

$\therefore \qquad -4y < 0 \quad i.e., \quad y > 0$

Hence the half plane $y > 0$ is mapped into the interior of the circle $|w| = 1$.

PROBLEMS 20.2

1. Find the invariant points of the transformation $w = (z-1)/(z+1)$. (*Madras, 2003*)
2. Find the transformation which maps the points $-1, i, 1$ of the z-plane onto $1, i, -1$ of the w-plane respectively. Also find its *invariant points*. (*V.T.U., 2011*)
3. Find the bilinear transformation which maps $1, i, -1$ to $2, i, -2$ respectively. Find the fixed and critical points of the transformation. (*S.V.T.U., 2008 ; Mumbai, 2007 ; V.T.U., 2006*)
4. Determine the bilinear transformation that maps the points $1-2i$, $2+i$, $2+3i$ respectively into $2+2i$, $1+3i$, 4. (*J.N.T.U., 2003 ; Coimbatore, 1999*)
5. Find the bilinear transformation which maps
   (*i*) the points $z = 1, i, -1$ into the points $w = 0, 1, \infty$ (*V.T.U., 2008 ; Mumbai, 2007*)
   (*ii*) the points $z = 0, 1, i$ into the points $w = 1+i, -i, 2-i$ (*V.T.U., 2010 S*)
   (*iii*) $R(z) > 0$ into interior of unit circle so that $z = \infty, i, 0$ map into $w = -1, -i, 1$.
6. Under the transformation $w = \dfrac{z-1}{z+1}$, show that the map of the straight line $x = y$ is a circle and find its centre and radius. (*Marathwada, 2008*)

7. Show that the bilinear transformation $w = (2z + 3)/(z - 4)$ maps the circle $x^2 + y^2 - 4x = 0$ into the line $4u + 3 = 0$. (*Mumbai, 2007 ; J.N.T.U., 2003 ; Bhopal, 2002*)
8. Show that the condition for transformation $w = (az + b)/(cz + d)$ to make the circle $|w| = 1$ correspond to a straight line in the z-plane is $|a| = |c|$.
9. Show that the transformation $w = i(1 - z)/(1 + z)$ maps the circle $|z| = 1$ into the real axis of the w-plane and the interior of the circle $|z| < 1$ into the upper half of the w-plane. (*Osmania, 2003 S ; V.T.U., 2001*)
10. If z_0 is the upper half of the z-plane, show that the bilinear transformation $w = e^{i\alpha}\left(\dfrac{z - z_0}{z - \bar{z}_0}\right)$ maps the upper half of the z-plane into the interior of the unit circle at the origin in the w-plane.

20.9 (1) CONFORMAL TRANSFORMATION

Suppose two curves C, C_1 in the z-plane intersect at the point P and the corresponding curves C' and C_1' in the w-plane intersect at P' (Fig. 20.5). *If the angle of intersection of the curves at P is the same as the angle of intersection of the curves at P′ in magnitude and sense, then the transformation is said to be* **conformal.**

(2) Theorem. *The transformation effected by an analytic function $w = f(z)$ is conformal at every point of the z-plane where $f'(z) \neq 0$.*

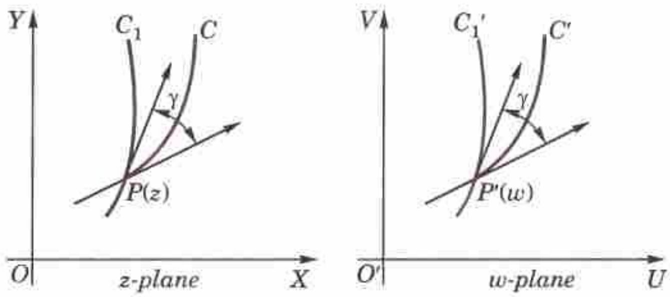

Fig. 20.5

Let $P(z)$ be a point in the region R of the z-plane and $P'(w)$ the corresponding point in the region R' of the w-plane (Fig. 20.3). Suppose z moves on a curve C and w moves on the corresponding curve C'. Let $Q(z + \delta z)$ be a neighbouring point on C and $Q'(w + \delta w)$ be the corresponding point on C' so that $\vec{PQ} = \delta z$ and $\vec{P'Q'} = \delta w$. Then δz is a complex number whose modulus r is the length PQ and amplitude θ is the angle which PQ makes with the x-axis.

$$\therefore \qquad \delta z = re^{i\theta}$$

Similarly, if the modulus and amplitude of δw be r' and θ', then $\delta w = r'e^{i\theta'}$.

Hence $$\frac{\delta w}{\delta z} = \frac{r'}{r} e^{i(\theta' - \theta)}$$

Now if the tangent at P to the curve C makes an $\angle\alpha$ with the x-axis and the tangent at P' to C' makes an $\angle\alpha'$ with the u-axis, then as $\delta z \to 0$, $\theta \to \alpha$ and $\theta' \to \alpha'$.

$$\therefore \qquad f'(z) = \frac{dw}{dz} = \left(\text{Lt}\,\frac{r'}{r}\right) . e^{i(\alpha' - \alpha)} \qquad ...(1)$$

If ρ is the modulus and ϕ the amplitude of the function $f(z)$ which is supposed to be non-zero, then

$$f'(z) = \rho e^{i\phi} \qquad ...(2)$$

$\therefore$ from (1) and (2), we have $$\rho = \text{Lt}\,\frac{r'}{r} \qquad ...(3)$$

and $$\phi = \alpha' - \alpha. \qquad ...(4)$$

Now let C_1 be another curve through P in the z-plane and C_1' the corresponding curve through P' in the w-plane. If the tangent at P to C_1 makes an $\angle\beta$ with the x-axis and tangent at P' to C_1' makes an $\angle\beta'$ with the u-axis, then as in (4),

$$\psi = \beta' - \beta \qquad ...(5)$$

Equating (4) and (5), $\qquad \alpha' - \alpha = \beta' - \beta \quad$ or $\quad \beta - \alpha = \beta' - \alpha' = \gamma$ (Fig. 20.5)

Thus the angle between the curves before and after the mapping is preserved in magnitude and direction. Hence the mapping by the analytic function $w = f(z)$ is conformal at each point where $f'(z) \neq 0$.

Obs. 1. *A point at which $f'(z) = 0$ is called a* **critical point** *of the transformation.*

Obs. 2. The relation (4), *i.e.*, $\alpha' = \alpha + \phi$ shows that the tangent at P to the curve C is rotated through an $\angle\phi$ = amp $\{f'(z)\}$ under the given transformation.

Obs. 3. The relation (3) shows that in the transformation, elements of arc passing through P in any direction are changed in the ratio $\rho : 1$, where $\rho = |f'(z)|$, *i.e., an infinitesimal length in the z-plane is magnified by the factor* $|f'(z)|$. Consequently *the infinitesimal areas are magnified by the factor* $|f'(z)|^2$ *in a conformal transformation.*

If $w = f(z)$ is analytic then u and v must satisfy C-R equations.

$$\therefore \quad J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{-\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2$$

Hence *in a conformal transformation, infinitesimal areas are magnified by the factor* $J\left(\frac{u,v}{x,y}\right)$.

Also the condition of a conformal mapping is $J\left(\frac{u,v}{x,y}\right) \neq 0$.

Obs. 4. *The angle preserving property of the conformal transformation has many important physical applications.* For instance, consider the flow of an incompressible fluid in a plane with velocity potential $\phi(x, y)$ and stream function $\psi(x, y)$. We know that ϕ and ψ are real and imaginary parts of some analytic function $w = f(z)$. As ϕ = constant and ψ = constant represent a system of orthogonal curves; these are transformed by the function $w = f(z)$ into a set of orthogonal lines in the w-plane and *vice-versa*.

Thus, *the conjugate functions ϕ and ψ when subjected to conformal transformation remain conjugate functions, i.e., the solutions of Laplace's equation remain solutions of the Laplace's equation after the transformation.* This is the main reason for the great importance of the conformal transformation in applications.

20.10 SPECIAL CONFORMAL TRANSFORMATIONS

(**1**) *Transformation* $w = z^2$.

We have $\quad u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$.

$\therefore \quad u = x^2 - y^2$ and $v = 2xy \quad$ ...(1)

If u is constant (say, a), then $x^2 - y^2 = a$ which is a rectangular hyperbola. Similarly, if v is constant (say, b), then $xy = b/2$ which also represents a rectangular hyperbola.

Hence a pair of lines $u = a$, $v = b$ parallel to the axes in the w-plane, map into pair of orthogonal rectangular hyperbolae in the z-plane as shown in Fig. 12.7 (p. 455).

Again, if x is constant (say, c), then $y = v/2c$ and $y^2 = c^2 - u$. Elimination of y from these equations gives $v^2 = 4c^2(c^2 - u)$, which represents a parabola. Similarly, if y is a constant (say, d), then elimination of x from the equations (1) gives $v^2 = 4d^2(d^2 + u)$ which is also a parabola.

Hence the pair of lines $x = c$ and $y = d$ parallel to the axes in the z-plane map into orthogonal parabolas in the w-plane as shown in Fig. 20.6.

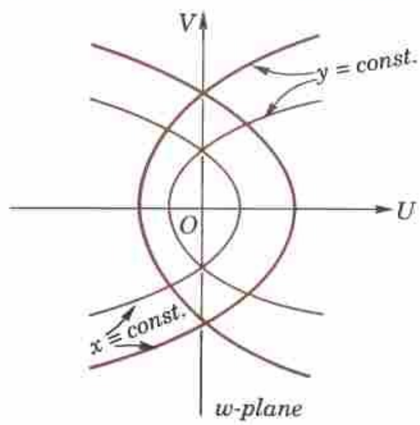

Fig. 20.6

Also since $\dfrac{dw}{dz} = 2z = 0$ for $z = 0$, therefore, it is a *critical point* of the mapping.

Taking $z = re^{i\theta}$ and $w = Re^{i\phi}$ then in polar form $w = z^2$ becomes $Re^{i\phi} = r^2e^{2i\theta}$.

This shows that upper half of the z-plane $0 < \theta < \pi$ transforms into the entire w-plane $0 \leq \phi < 2\pi$. The same is true of the lower half. (P.T.U., 2003)

Obs. 1. Taking the axes to represent two walls, a single quadrant could be used to represent fluid flow at a corner wall. This transformation can also represent the electrostatic field in the vicinity of a corner conductor.

Obs. 2. *For the tansformation* $w = z^n$, n being a positive integer, we have $dw/dz = 0$ at $z = 0$.

Also $\quad Re^{i\phi} = (re^{i\theta})^n = r^ne^{in\theta}$

$\therefore \quad R = r^n$ and $\phi = n\theta$, when $0 < \theta < \pi/n$, correspondingly $0 < \phi < \pi$.

Hence $w = z^n$ gives a conformal mapping of the z-plane everywhere except at the origin and that is fans out a sector of z-plane of central angle π/n to cover the upper half of the w-plane.

(2) *Joukowski's transformation** $w = z + 1/z$.

Since $\dfrac{dw}{dz} = \dfrac{(z+1)(z-1)}{z^2}$, the mapping is conformal except at the points $z = 1$ and $z = -1$ which correspond to the points $w = 2$ and $w = -2$ of the w-plane.

---

* Named after the Russian mathematician *Nikolai Jegorovich Joukowshi* (1847-1921).

Changing to polar coordinates,

$$w = u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$\therefore$ $$u = (r + 1/r)\cos\theta \text{ and } v = (r - 1/r)\sin\theta$$

Elimination of θ gives $$\frac{u^2}{(r+1/r)^2} + \frac{v^2}{(r-1/r)^2} = 1 \qquad ...(1)$$

while the elimination of r gives $$\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1 \qquad ...(2)$$

From (1), it follows that the circles r = constant of z-plane transform into a family of ellipses of the w-plane (Fig. 20.7). These ellipses are confocal for $(r + 1/r)^2 - (r - 1/r)^2 = 4$, *i.e.*, a constant.

In particular, the unit circle ($r = 1$) in the z-plane flattens out to become the segment $u = -2$ to $u = 2$ of the real axis in w-plane. Thus the exterior of the unit circle in the z-plane maps into the entire w-plane.

(*A.M.I.E.T.E., 2005 S*)

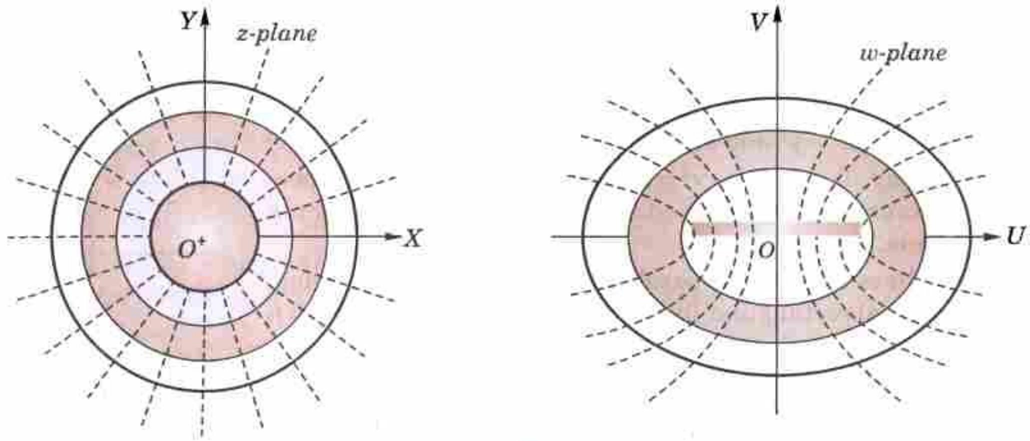

Fig. 20.7

From (2), it is clear that the radial lines θ = constant of the z-plane transform into a family of hyperbolae which are also confocal (Fig. 20.7).

Obs. 1. $v = \left(r - \frac{1}{r}\right)\sin\theta = 0$ gives $r = \pm 1$ or $\theta = 0, \pi$, *i.e.*, this streamline consists of the unit circle $r = 1$ and the x-axis ($\theta = 0$ to $\theta = \pi$). For large z, the flow is nearly uniform and parallel to the x-axis. This can be interpreted as a flow around a circular cylinder of unit radius having two **stagnation points*** at $A(z = 1)$ and $B(z = -1)$. (Fig. 20.8)

[$\because$ $dw/dz = 0$ at $z = \pm 1$]

Obs. 2. This transformation is also used to map the exterior of the profile of an aeroplane wing on the exterior of a nearly circular region. These airfoils are known as *Joukowski airfoils*.

(3) *Transformation* $w = e^z$.

Writing $z = x + iy$ and $w = \rho e^{i\phi}$, we have $\rho e^{i\phi} = e^{x+iy} = e^x \cdot e^{iy}$

whence $\rho = e^x$ ...(1) and $\phi = y$ ...(2)

From (1), it is clear that the lines parallel to y-axis (x = const.) map into circles (ρ = const.) in the w-plane, their radii being less than or greater than 1 according as x is less than or greater than 0 (Fig. 20.9). (*V.T.U., 2011*)

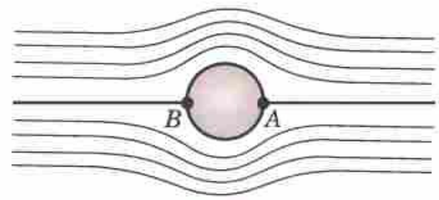

Fig. 20.8

Similarly, it follows from (2) that the lines parallel to the x-axis (y = const.) map into the radial lines (ϕ = const.) of the w-plane. Thus any horizontal strip of height 2π in the z-plane will cover once the entire w-plane.

* *Stagnation points* are those at which the fluid velocity is zero.

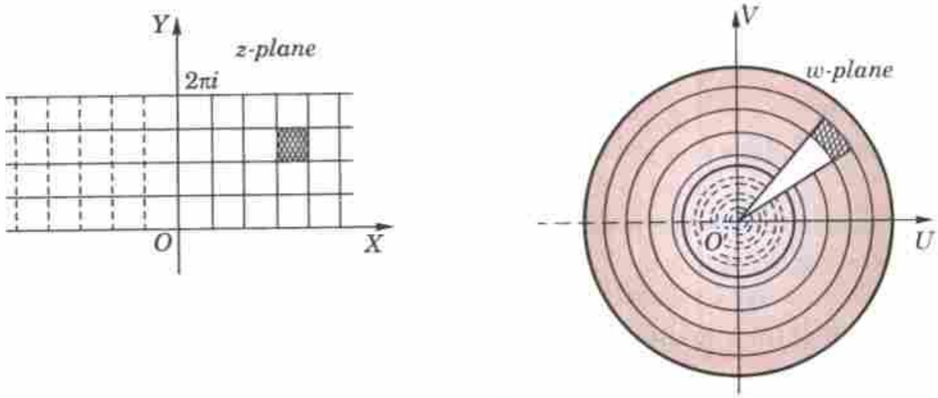

Fig. 20.9

The rectangular region $a_1 \le x \le a_2$, $b_1 \le y \le b_2$ in the z-plane (shown shaded) transforms into the region $e^{a_1} \le \rho \le e^{a_2}$, $b_1 \le \phi \le b_2$ in the w-plane bounded by circles and rays (shown shaded).

(P.T.U., 2005 ; Kerala, 2005)

Obs. This transformation can be used to obtain the circulation of a liquid around a cylindrical obstacle, the electrostatic field due to a charged circular cylinder etc.

(4) *Transformation $w = \cosh z$.*

We have $\quad u + iv = \cosh (x + iy) = \cosh x \cos y + i \sinh x \sin y \quad$ [By (2) (*ii*), p. 662]

so that $\quad u = \cosh x \cos y$ and $v = \sinh x \sin y$.

Elimination of x from these equations gives

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1 \qquad ...(1)$$

while elimination of y gives

$$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 z} = 1 \qquad ...(2)$$

(1) shows that the lines parallel to x-axis (*i.e.*, y = const.) in the z-plane map into hyperbolae in the w-plane.

(2) shows that the lines parallel to the y-axis (*i.e.*, x = const.) in the z-plane map into ellipse in the w-plane (Fig. 20.10). The rectangular region $a_1, \le x \le a_2$, $b_1 \le y \le b_2$ in the z-plane (shown shaded) transforms into the shaded region in the w-plane bounded by the corresponding hyperbolae and ellipses. *(Kerala M. Tech., 2005)*

Obs. This transformation can be used.

(*i*) to obtain the circulation of liquid around an elliptic cylinder;

(*ii*) to determine the electrostatic field due to a charged cylinder;

(*iii*) to determine the potential between two confocal elliptic (or hyperbolic) cylinders.

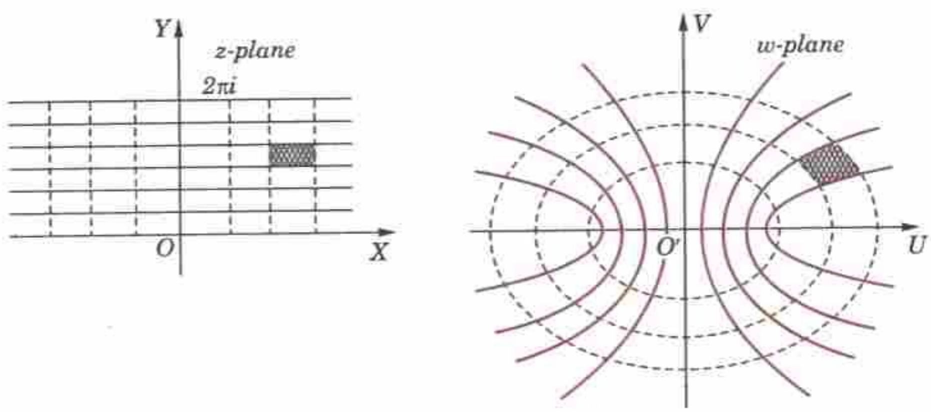

Fig. 20.10

Example 20.15. *Show that under the transformation $w = (z - i)/(z + i)$, real axis in the z-plane is mapped into the circle $|w| = 1$. Which portion of the z-plane corresponds to the interior of the circle?* (J.N.T.U., 2003)

Solution. We have $|w| = \left|\frac{z-i}{z+i}\right| = \frac{|z-i|}{|z+i|} = \frac{|x+i(y-1)|}{|x+i(y+1)|}$

$$= \sqrt{\{x^2 + (y-1)^2\}}/\sqrt{\{x^2 + (y+1)^2\}}$$

Now the real axis in z-plane *i.e.*, $y = 0$, transforms into

$$|w| = \sqrt{(x^2 + 1)}/\sqrt{\{x^2 + 1)} = 1.$$

Hence the real axis in the z-plane is mapped into the circle $|w| = 1$.

The interior of the circle, *i.e.*, $|w| < 1$, gives

$$\{x^2 + (y-1)^2\}/\{x^2 + (y+1)^2\} < 1 \text{ i.e., } -4y < 0 \text{ or } y > 0.$$

Thus the upper half of the z-plane corresponds to the interior of the circle $|w| = 1$.

PROBLEMS 20.3

1. Determine the region of the w-plane into which the following regions are mapped by the transformation $w = z^2$:
   (*i*) first quadrant of z-plane (*J.N.T.U., 2000*)
   (*ii*) region bounded by $x = 1$, $y = 1$, $x + y = 1$ (*Kottayam, 2005 ; V.T.U., 2000 S*)
   (*iii*) the region $1 \le x \le 2$ and $1 \le y \le 2$ (*Osmania, 2003 ; V.T.U., 2000*)
   (*iv*) circle $|z - 1| = 2$.
2. Find the transformation which maps the triangular region $0 \le \arg z \le \pi/3$ into the unit circle $w \le 1$.
3. Discuss the transformation $w = \sqrt{z}$. Is it conformal at the origin? (*Delhi, 2002*)
4. Under the transformation $w = 1/z$, find the image of
   (*a*) the circle $|z - 2i| = 2$ (*Bhopal, 2009 ; Kerala M.Tech., 2005*)
   (*b*) the straight line $y - x + 1 = 0$ (*P.T.U., 2007*)
   (*c*) the hyperbola $x^2 - y^2 = 1$. (*Mumbai, 2005 ; J.N.T.U., 2005*)
5. Show that under the transformation $w = 1/z$, (*a*) circle $x^2 + y^2 - 6x = 0$ is transformed into a straight line in the w-plane.
   (*b*) the circle $(x - 3)^2 + y^2 = 2$ is transformed into a circle with centre $(3/7, 0)$ and radius $\sqrt{2}/17$. (*Mumbai, 2007*)
6. Show that the transformation $w = 1/z$ transforms all circles and straight lines into the circles and straight lines in the w-plane. Which circles in the z-plane become straight lines in the w-plane, and which straight lines are transformed into other sti aight lines ? (*Anna, 2003*)
7. Show that the transformation $w = z + 1/z$ converts the straight line $\arg z = \alpha$ ($|\alpha| < \alpha/2$) into a branch of hyperbola of eccentricity $\sec \alpha$. (*Mumbai, 2005 S*)
8. Show that the transformation $w = z + (a^2 - b^2)/4z$ transforms the circle of radius $\frac{1}{2}(a + b)$, centre at the origin, in the z-plane into ellipse of semi-axes a, b in the w-plane.
9. Show that the transformation $w = z + a^2/z$ transforms circles with origin at the cent`re in the z-plane into co-axial concentric, confocal ellipses in the w-plane. (*Kurukshetra, 2005 ; J.N.T.U., 2005*)
10. Show that the function $w = A(z + a^2/z)$ may be used to represent the flow of a perfect incompressible fluid past a circular cylinder. Also find the stagnation points.
11. Show that by the relation $u + iv = \cos(x + iy)$, the infinite strip bounded by $x = c$, $x = d$, where c and d lie between 0 and $\pi/2$, is mapped into the region between the two branches of the hyperbola lying in $u > 0$. (*Osmania, 2002*)
12. Prove that the transformation $w = \sin z$, maps the families of lines x = constant and y = constant into two families of confocal central conics. (*J.N.T.U., 2003*)
13. Discuss the transformation $w = e^z$, and show that it transforms the region between the real azis and a line parallel to real axis at $y = \pi$, into the upper half of the w-plane.
14. Discuss fully the transformation $w = c \cosh z$, where c is a real number. What physical problem can we study with the help of this transformation ?

20.11 SCHWARZ-CHRISTOFFEL TRANSFORMATION

This transformation maps the interior of a polygon of the w-plane into the upper half of the z-plane and the boundary of the polygon into the real axis. The formula of this transformation is

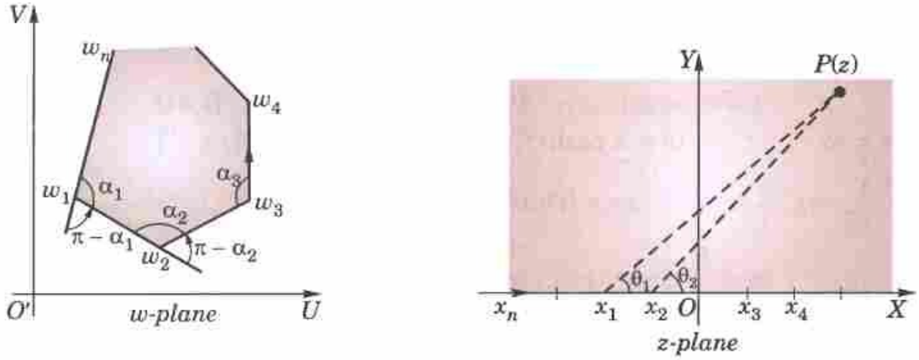

Fig. 20.11

$$\frac{dw}{dz} = A(z - x_1)^{\frac{\alpha_1}{\pi}-1}(z - x_2)^{\frac{\alpha_2}{\pi}-1} \ldots (z - x_n)^{\frac{\alpha_n}{\pi}-1} \quad \ldots(1)$$

or
$$w = A\int (z - x_1)^{\frac{\alpha_1}{\pi}-1}(z - x_2)^{\frac{\alpha_2}{\pi}-1} \ldots (z - x_n)^{\frac{\alpha_n}{\pi}-1}\, dz + B \quad \ldots(2)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the interior angles of the polygon having vertices $w_1, w_2, \ldots, w_n$ which map into the points $x_1, x_2, \ldots, x_n$ on the real-axis of the z-plane (Fig. 20.11). Also A and B are complex constants which determines the size and position of the polygon.

Proof. We have from (1),

$$\text{amp}\left(\frac{dw}{dz}\right) = \text{amp}\,[(A) + \left(\frac{\alpha_1}{\pi}-1\right)\text{amp}\,(z-x_1) + \left(\frac{\alpha_2}{\pi}-1\right)\text{amp}\,(z-x_2)$$
$$\ldots. + \left(\frac{\alpha_n}{\pi}-1\right)\text{amp}\,(z-x_n) \quad \ldots(3)$$

As z moves along the real axis from the left towards x_1, suppose that w moves along the side $w_n w_1$ of the polygon towards w_1. As z crosses x_1 from left to right, $\theta_1 = \text{amp}\,(z - x_1)$ changes from π to 0 while all other terms of (3) remain unaffected. Hence only $\left(\frac{\alpha_1}{\pi}-1\right)$ amp $(z - x_1)$ decreases by $\left(\frac{\alpha_1}{\pi}-1\right)\pi = \alpha_1 - \pi$, *i.e.* increases by $\pi - \alpha_1$ in the anti-clockwise direction. In other words, amp (dw/dz) increases by $\pi - \alpha_1$. Thus the direction of w_1 turns through the angle $\pi - \alpha_1$ and w now moves along the side w_1w_2 of the polygon.

Similarly when z passes through x_2, $\theta_1 = \text{amp}\,(z - x_1)$ and $\theta_2 = \text{amp}\,(z - x_2)$ change from π to 0 while all other terms remain unchanged. Hence the side w_1w_2 turns through the angle $\pi - \alpha_2$. Proceeding in this way, we see that as z moves along x-axis, w traces the polygon $w_1\, w_2\, w_3 \ldots w_n$ and conversely.

Example 20.16. *Find the transformation which maps the semi-infinite strip in the w-plane (Fig. 20.12) into the upper half of the z-plane* *(V.T.U., M.E. 2006 ; Osmania, 2003)*

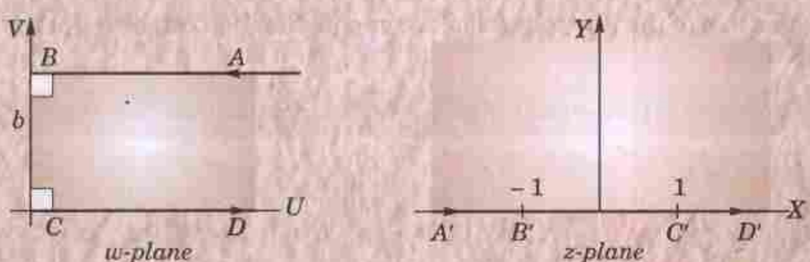

Fig. 20.12

Solution. Consider $ABCD$ as the limiting case of a triangle with two vertices B and C and the third vertex A or D at infinity. Let the vertices B and C map into the points B' $(-1)$and C' (1) of the z-plane. Since the interior angles at B and C are $\pi/2$, we have by the Schwarz-Christoffel transformation,

$$\frac{dw}{dz} = A(z+1)^{\frac{\pi/2}{\pi}-1}(z-1)^{\frac{\pi/2}{\pi}-1} = A/\sqrt{(z^2-1)}$$

$\therefore \qquad w = A \int \frac{dz}{\sqrt{(z^2 - 1)}} + B = A \cosh^{-1} z + B$

When $\qquad z = 1, w = 0. \quad \therefore \quad 0 = A \cosh^{-1}(1) + B$, *i.e.*, $B = 0$.

When $z = -1, w = ib. \quad \therefore \quad ib = A \cosh^{-1}(-1) + 0$, *i.e.*, $\cosh (ib/A) = -1$

or $\qquad \cos \frac{b}{A} = -1 = \cos \pi$. Thus $A = \frac{b}{\pi}$.

Hence $\qquad w = \frac{b}{\pi} \cosh^{-1} z$ or $z = \cosh \frac{\pi w}{b}$.

PROBLEMS 20.4

1. Find the transformation which maps the semi-infinite strip of width π bounded by the lines $v = 0$, $v = \pi$ and $u = 0$ into the upper half of the z plane.
2. Show how you will use Schwarz-Christoffel transformation to map the semi-infinite strip enclosed by the real axis and the lines $u = \pm 1$ of the w-plane into the upper half of the z-plane.
3. Find the mapping function which maps semi-infinite strip in the z-plane $-\pi/2 \le x \le \pi/2$, $y \ge 0$ into half w-plane for which $v \ge 0$, such that the points $(-\pi/2, 0)$, $(\pi/2, 0)$ in the π-plane are mapped into the points $(-1, 0)$, $(1, 0)$ respectively in w-plane.
4. Find the transformation which will map the interior of the infinite strip bounded by the lines $v = 0$, $v = \pi$ onto the upper half of the z-plane.

20.12 COMPLEX INTEGRATION

We have already discussed the concept of the line integral as applied to vector fields in § 8.11. Now we shall consider the line integral of a complex function.

Fig. 20.13

Consider a continuous function $f(z)$ of the complex variable $z = x + iy$ defined at all points of a curve C having end points A and B. Divide C into n parts at the points

$$A = P_0(z_0), P_1(z_1), \ldots, P_i(z_i), \ldots P_n(z_n) = B.$$

Let $\delta z_i = z_i - z_{i-1}$ and ζ_i be any point on the arc $P_{i-1} P_i$. The limit of the sum $\sum_{i=1}^{n} f(\zeta_i)\, \delta z_i$ as $n \to \infty$ *in such a way that the length of the chord* δz_i *approaches zero, is called the* **line integral of f (z)** *taken along the path C, i.e.,*

$$\int_C f(z)\, dz.$$

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that $dz = dx + idy$,

$$\int_C f(z)\, dz = \int_C (udx - vdy) + i \int_C (vdx + udy)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Obs. *The value of the integral is independent of the path of integration when the integrand is analytic.*

Example 20.17. *Prove that*

(*i*) $\int_C \frac{dz}{z - a} = 2\pi i.$ $\qquad$ (*ii*) $\int_C (z - a)^n\, dz = 0$ [*n, any integer* $\neq -1$]

where C is the circle $|z - a| = r$. (*U.P.T.U., 2003*)

Solution. The parametric equation of C is $z - a = re^{i\theta}$, where θ varies from 0 to 2π as z describes C once in the positive (anti-clockwise) sense. (Fig. 20.14)

(*i*) $$\int_C \frac{dz}{z - a} = \int_0^{2\pi} \frac{1}{re^{i\theta}} \cdot ire^{i\theta}\, d\theta \qquad [\because dz = ire^{i\theta}\, d\theta]$$

$$= i \int_0^{2\pi} d\theta = 2\pi i$$

(*ii*) $\int_C (z-a)^n \, dz = \int_0^{2\pi} r^n e^{ni\theta} \,.\, ire^{i\theta} \, d\theta$

$$= ir^{n+1} \int_0^{2\pi} e^{(n+1)\theta i} \, d\theta = \frac{r^{n+1}}{n+1} \left| e^{(n+1)\theta i} \right|_0^{2\pi}, \text{ provided } n \neq -1$$

$$= \frac{r^{n+1}}{n+1} [e^{2(n+1)\pi i} - 1] = 0 \qquad [\because \quad e^{2(n+1)\pi i} = 1]$$

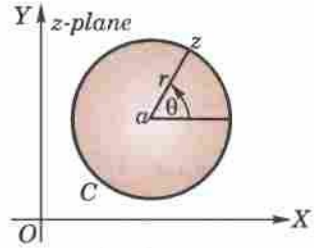

Fig. 20.14

Example 20.18. *Evaluate* $\int_0^{2+i} (\bar{z})^2 \, dz$, *along (i) the line* $y = x/2$, *(Bhopal, 2007 ; U.P.T.U., 2002)*
(ii) the real axis to 2 and then vertically to $2 + i$. *(S.V.T.U., 2009 ; P.T.U., 2008 S ; Mumbai, 2006)*

Solution. (*i*) Along the line *OA*, $x = 2y$, $z = (2+i)y$, $\bar{z} = (2-i)y$ and $dz = (2+i)\,dy$ (Fig. 20.15)

$$\therefore \qquad I = \int_0^{2+i} (\bar{z})^2 \, dz = \int_0^1 (2-i)^2 y^2 \,.\, (2+i)\, dy$$

$$= 5(2-i) \left| \frac{y^3}{3} \right|_0^1 = \frac{5}{3}(2-i)$$

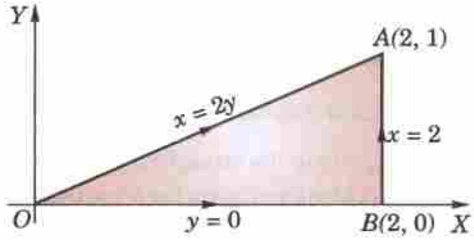

Fig. 20.15

(*ii*) $$I = \int_{OB} (\bar{z})^2 \, dz + \int_{BA} (\bar{z})^2 \, dz.$$

Now along *OB*, $z = x$, $\bar{z} = x$, $dz = dx$;
and along *BA*, $z = 2 + iy$, $\bar{z} = 2 - iy$, $dz = idy$

$$\therefore \qquad I = \int_0^2 x^2 dx + \int_0^1 (2-iy)^2 \,.\, idy = \left| \frac{x^3}{3} \right|_0^2 + \int_0^1 [\,4y + (4-y^2)i\,]\, dy$$

$$= \frac{8}{3} + 4 \,.\, \frac{1}{2} + \left(4.1 - \frac{1}{3}\right) i = \frac{1}{3}(14 + 11i).$$

Example 20.19. *Evaluate* $\int_C (z^2 + 3z + 2)\, dz$ *where C is the arc of the cycloid* $x = a\,(\theta + \sin\theta)$, $y = a\,(1 - \cos\theta)$ *between the points* $(0, 0)$ *and* $(\pi a, 2a)$. *(Rohtak, 2004)*

Solution. $f(z) = z^2 + 3z + 2$ is analytic in the *z*-plane being a polynomial. As such, the line integral of $f(z)$ between *O* and *A* is independent of the path (Fig. 20.16). We therefore, take the path from *O* to *L* and *L* to *A* so that

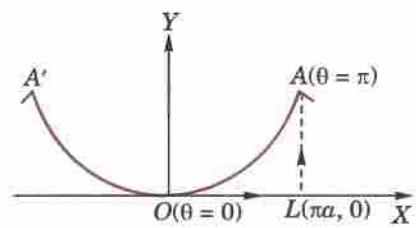

Fig. 20.16

$$\int_C f(z)\, dz = \int_{OL} f(z)\, dz + \int_{LA} f(z)\, dz \qquad ...(i)$$

$$\therefore \qquad \int_{OL} f(z)\, dz = \int_0^{\pi a} (x^2 + 3x + 2)\, dx$$

[$\because$ along *OL*, $y = 0$, $x = 0$ at *O*, $x = \pi a$ at *L*

$$= \left| \frac{x^3}{3} + \frac{3x^2}{2} + 2x \right|_0^{\pi a} = \frac{\pi a}{6}(2\pi^2 a^2 + 9\pi a + 12) \qquad ...(ii)$$

and $$\int_{LA} f(z)\, dz = \int_0^{2a} [(\pi a + iy)^2 + 3\,(\pi a + iy) + 2]\, idy$$

[$\because$ along *LA*, $x = \pi a$, $z = \pi a + iy$, $dz = idy$ and *y* varies from 0 (at *L*) to $2a$ (at *A*)

$$= L \left| \frac{(\pi a + iy)^3}{3i} + 3\,\frac{(\pi a + iy)^2}{2i} + 2y \right|_0^{2\pi} = \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 4ia \qquad ...(iii)$$

$\therefore$ substituting from (*ii*) and (*iii*) in (*i*), we get

$$\int_C f(z)\, dz = \frac{\pi a}{6}(2\pi^2 a^2 + 9\pi a + 12) + \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 4ia$$

PROBLEMS 20.5

1. Evaluate $\int_0^{1+i} (x^2 - iy)\,dz$ along the paths (a) $y = x$ and (b) $y = x^2$. (U.P.T.U., 2010)
2. Evaluate $\int_{1-i}^{2+i} (2x + iy + 1)\,dz$, along the two paths: (U.P.T.U., 2010)
   (i) $x = t + 1, y = 2t^2 - 1$ (ii) the straight line joining $1 - i$ and $2 + i$. (U.P.T.U., 2006)
3. Evaluate $\int_{1-i}^{2+3i} (z^2 + z)\,dz$ along the line joining the points $(1, -1)$ and $(2, 3)$. (V.T.U., 2004)
4. Show that for every path between the limits, $\int_{-2}^{-2+i} (2 + z)^2\,dz = -i/3$. (Delhi, 2002)
5. Show that $\oint_C (z + 1)\,dz = 0$, where C is the boundary of the square whose vertices are at the points $z = 0$, $z = 1$, $z = 1 + i$ and $z = i$. (Rohtak, 2006)
6. Evaluate $\int_C |z|\,dz$, where C is the contour
   (i) straight line from $z = -i$ to $z = i$.
   (ii) left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$.
   (iii) circle given by $|z + 1| = 1$ described in the clockwise sense.
7. Find the value of $\int_0^{1+i} (x - y + ix^2)\,dz$
   (i) along the straight line from $z = 0$ to $z = 1 + i$
   (ii) along real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$. (U.P.T.U., 2003)
8. Prove that $\int dz/z = -\pi i$ or πi, according as C is the semi-circular arc $|z| = 1$ above or below the real axis. (Rohtak, 2005)
9. Evaluate $\int_C (z - z^2)\,dz$, where C is the upper half of the circle $|z| = 1$.
   What is the value of this integral if C is the lower half of the above circle ?

20.13 CAUCHY'S THEOREM

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a closed curve C, then $\oint_C f(z)\,dz = 0$.

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that $dz = dx + idy$

$$\oint_C f(z)dz = \oint_C (udx - vdy) = i \oint_C (vdx + udy) \qquad ...(1)$$

Since $f'(z)$ is continuous, therefore, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the region D enclosed by C. Hence the Green's theorem (p. 376) can be applied to (1), giving

$$\oint_C f(z)dz = -\iint_D \left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right] dxdy + i \iint_D \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dxdy \qquad ...(2)$$

Now $f(z)$ being analytic, u and v necessarily satisfy the Cauchy-Riemann equations and thus the integrands of the two double integrals in (2) vanish identically.

Hence $$\oint_C f(z)dz = 0.$$

Obs. 1. The Cauchy-Riemann equations are precisely the conditions for the two real integrals in (1) to be independent of the path. *Hence the line integral of a function $f(z)$ which is analytic in the region D, is independent of the path joining any two points of D.*

Obs. 2. Extension of Cauchy's theorem. *If $f(z)$ is analytic in the region D between two simple closed curves C and C_1, then* $\oint_C f(z)dz = \oint_{C_1} f(z)dz$.

To prove this, we need to introduce the cross-cut AB. Then $\oint f(z)dz = 0$ where the path is as indicated by arrows in Fig. 20.17, *i.e.*, along AB—along C_1 in clockwise sense and along BA—along C in anti-clockwise sense.

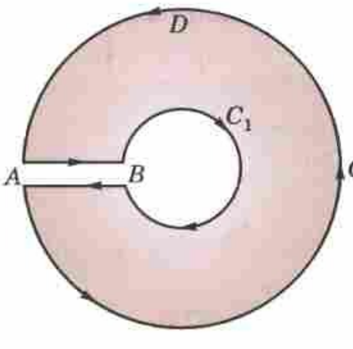

Fig. 20.17(*a*)

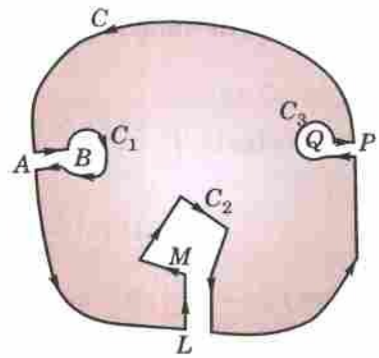

Fig. 20.17(*b*)

i.e., $$\int_{AB} f(z)dz + \int_{C_1} f(z)dz + \int_{AB} f(z)dz + \int_C f(z)dz = 0$$

But, since the integrals along AB and along BA cancel, it follows that

$$\int_C f(z)dz + \int_{C_1} f(z)dz = 0$$

Reversing the direction of the integral around C_1 and transposing, we get

$$\int_C f(z)dz + \int_{C_1} f(z)dz$$ each integration being taken in the anti-clockwise sense.

If C_1, C_2, C_3, ... be any number of closed curves within C (Fig. 20.17(b)), then

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz + \ldots$$

20.14 CAUCHY'S INTEGRAL FORMULA

If f (z) is analytic within and on a closed curve and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i} = \oint_C \frac{f(z)dz}{z-a}$$

Consider the function $f(z)/(z-a)$ which is analytic at all points within C except at $z = a$. With the point a as centre and radius r, draw a small circle C_1 lying entirely within C.

Now $f(z)/(z-a)$ being analytic in the region enclosed by C and C_1, we have by Cauchy's theorem,

$$\oint_C \frac{f(z)}{z-a}dz = \oint_{C_1} \frac{f(z)}{z-a}dz \qquad \left\{\begin{array}{l}\text{For any point on } C_1, \\ z-a = re^{i\theta} \text{ and } dz = ire^{i\theta}\,d\theta\end{array}\right.$$

$$= \oint_C \frac{f(a+re^{i\theta})}{re^{i\theta}}.\,ire^{i\theta}\,d\theta = i\oint_{C_1} f(a+re^{i\theta})d\theta \qquad \ldots(1)$$

In the limiting form, as the circle C_1 shrinks to the point a, *i.e.*, as $r \to 0$, the integral (1) will approach to

$$\oint_C f(a)d\theta = if(a)\int_0^{2\pi} d\theta = 2\pi i f(a)\oint f(z)dz. \text{ Thus } \oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

i.e., $$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \qquad \ldots(2)$$

which is the desired *Cauchy's integral formula*. (*V.T.U., 2011 S*)

Cor. Differentiating both sides of (2) w.r.t. a,

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right]dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}dz \qquad \ldots(3)$$

Similarly, $$f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\oint_C dz \qquad \ldots(4)$$

and in general, $$f^{n}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz. \quad ...(5)$$

Thus it follows from the results (2) to (5) that *if a function $f(z)$ is known to be analytic on the simple closed curve C then the values of the function and all its derivatives can be found at any point of C.* Incidently, we have established a remarkable fact that **an analytic function possesses derivatives of all orders and these are themselves all analytic.**

Example 20.20. *Evaluate* $\int_C \frac{z^2-z+1}{z-1}\,dx$, *where C is the circle*

(*i*) $|z| = 1$, (*ii*) $|z| = \frac{1}{2}$. (*S.V.T.U., 2007*)

Solution. (*i*) Here $f(z) = z^2 - z + 1$ and $a = 1$.

Since $f(z)$ is analytic within and on circle $C : |z| = 1$ and $a = 1$ lies on C.

$\therefore$ by Cauchy's integral formula $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a} = f(a) = 1$ *i.e.*, $\int_C \frac{z^2-z+1}{z-1}\,dz = 2\pi i.$

(*ii*) In this case, $a = 1$ lies outside the circle $C : |z| = 1/2$. So $(z^2 - z + 1)/(z - 1)$ is analytic everywhere within C.

$\therefore$ by Cauchy's theorem $\int_C \frac{z^2-z+1}{z-1}\,dz = 0.$

Example 20.21. *Evaluate, using Cauchy's integral formula:*

(*i*) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$ *where C is the circle* $|z| = 3$ (*U.P.T.U., 2010*)

(*ii*) $\oint_C \frac{\cos \pi z}{z^2-1}\,dz$ *around a rectangle with vertices* $2 \pm i, -2 \pm 1$

(*iii*) $\oint_C \frac{e^{tz}}{z^2+1}\,dz$ *where C is the circle* $|z| = 3$. (*U.P.T.U., 2009*)

Solution. (*i*) $f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic within the circle $|z| = 3$ and the two singular points $z = 1$ and $z = 2$ lie inside this circle.

$$\therefore \quad \oint_C \frac{f(z)dz}{(z-1)(z-2)} = \oint_C (\sin \pi z^2 + \cos \pi z^2) = \left(\frac{1}{z-2} - \frac{1}{z-1}\right)$$

$$= \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-2}\,dz - \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-1}\,dz$$

$$= 2\pi i\,[\sin \pi(2)^2 + \cos \pi(2)^2] - 2\pi i[\sin \pi(1)^2 + \cos \pi(1)^2]$$

[By Cauchy's integral formula]

$$= 2\pi i\,(0 + 1) - 2\pi i\,(0 - 1) = 4\pi i$$

(*ii*) $f(z) = \cos \pi z$ is analytic in the region bounded by the given rectangle and the two singular points $z = 1$ and $z = -1$ lie inside this rectangle. (Fig. 20.18)

$$\therefore \quad \oint_C \frac{\cos \pi z}{z^2-1}\,dz = \frac{1}{2}\oint_C \left(\frac{1}{z-1} - \frac{1}{z+1}\right)\cos \pi z\,dz$$

$$= \frac{1}{2}\oint_C \frac{\cos \pi z}{z-1}\,dz - \oint_C \frac{\cos \pi z}{z+1}\,dz$$

$$= \frac{1}{2}\{2\pi i \cos \pi(1)\} - \frac{1}{2}\{2\pi i \cos \pi(-1)\} = 0.$$

[By Cauchy's integral formula]

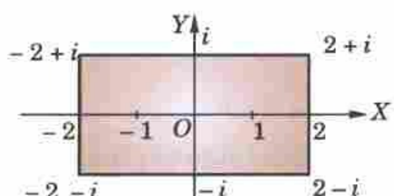

Fig. 20.18

(*iii*) $f(z) = e^{tz}$ is analytic within the circle $|z| = 3$.

The singular points are given by $z^2 + 1 = 0$ *i.e.*, $z = i$ and $z = -i$ which lie within this circle.

$$\therefore \quad \oint_C \frac{e^{tz}}{z^2+1} dz = \oint_C \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right) e^{tz} dz = \frac{1}{2i}\left\{\oint_C \frac{e^{tz}}{z-i} dz - \oint_C \frac{e^{tz}}{z+i} dz\right\}$$

$$= \frac{1}{2i}\{2\pi i e^{t(i)} - 2\pi i e^{t(-i)}\} \qquad \text{[By Cauchy's integral formula]}$$

$$= 2\pi i \left(\frac{e^{it} - e^{-it}}{2i}\right) = 2\pi i \sin t.$$

Example 20.22. *Evaluate*

(*i*) $\oint_C \frac{\sin^2 z}{(z - \pi/6)^3} dz$, *where C is the circle* $|z| = 1$ *(Rohtak, 2005)*

(*ii*) $\oint_C \frac{e^{2z}}{(z+i)^4} dz$, *where C is the circle* $|z| = 3$ *(U.P.T.U., 2008)*

(*iii*) $\oint_C \frac{e^z}{(z^2 + \pi^2)^2} dz$, *where C is* $|z| = 4$. *(U.P.T.U., 2008; J.N.T.U., 2000)*

Solution. (*i*) $f(z) = \sin^2 z$ is analytic inside the circle C: $|z| = 1$ and the point $a = \pi/6$ (= 0.5 approx.) lies within C.

$\therefore$ by Cauchy's integral formula $f''(a) = \dfrac{2!}{2\pi i}\oint_C \dfrac{f(z)}{(z-a)^3} dz$,

we get
$$\oint_C \frac{\sin^2 z}{(z-\pi/6)^3} dz = \pi i \left[\frac{d^2}{dz^2}(\sin^2 z)\right]_{z=\pi/6}$$

$$= \pi i (2\cos 2z)_{z=\pi/6} = 2\pi i \cos \pi/3 = \pi i.$$

(*ii*) $f(z) = e^{2z}$ is analytic within the circle C: $|z| = 3$. Also $z = -1$ lies inside C.

$\therefore$ By Cauchy's integral formula: $f'''(a) = \dfrac{3!}{2\pi i}\displaystyle\int_C \dfrac{f(z)dz}{(z-a)^4}$

we get
$$\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{6}\left|\frac{d^3(e^{2z})}{dz^3}\right|_{z=-1} = \frac{\pi i}{3}[8e^{2z}]_{z=-1} = \frac{8\pi i}{3} e^{-2}$$

(*iii*) $\dfrac{e^z}{(z^2+\pi^2)^2} = \dfrac{e^z}{(z+\pi i)^2 (z-\pi i)^2}$ is not analytic at $z = \pm \pi i$.

However both $z = \pm \pi i$ lie within the circle $|z| = 4$.

Now
$$\frac{1}{(z+\pi i)^2 (z-\pi i)^2} = \frac{A}{z+\pi i} + \frac{B}{(z+\pi i)^2} + \frac{C}{z-\pi i} + \frac{D}{(z-\pi i)^2}$$

where $A = 7/2\pi^3 i$, $C = -7/2\pi^3 i$, $B = D = -1/4\pi^2$

$$\therefore \quad \int_C \frac{e^z}{(z^2+\pi^2)^2} dz = \frac{7}{2\pi^3 i}\left\{\int_C \frac{e^z}{z+\pi i} dz - \int_C \frac{e^z}{z-\pi i} dz\right\} - \frac{1}{4\pi^2}\left\{\int_C \frac{e^z}{(z+\pi i)^2} dz + \int \frac{e^z}{(z-\pi i)^2} dz\right\}$$

$$= \frac{7}{2\pi^3 i}[2\pi i f(-\pi i) - 2\pi i f(\pi i)] - \frac{1}{4\pi^2}[2\pi i f'(-\pi i) + 2\pi i f'(\pi i)] \qquad \text{where } f(z) = e^z.$$

$$= \frac{7}{\pi^2}(e^{-\pi i} - e^{\pi i}) - \frac{i}{2\pi}(e^{-\pi i} + e^{\pi i}) = -\frac{14i}{\pi^2}\left(\frac{e^{\pi i} - e^{-\pi i}}{2i}\right) - \frac{i}{\pi}\left(\frac{e^{\pi i} + e^{-\pi i}}{2}\right) \qquad [\S\ 19.9]$$

$$= -\frac{14i}{\pi^2}\sin \pi - \frac{i}{\pi}\cos \pi = \frac{i}{\pi}.$$

Example 20.23. *Verify Cauchy's theorem by integrating e^{iz} along the boundary of the triangle with the verticles at the points $1 + i$, $-1 + i$ and $-1 - i$.* (U.P.T.U., 2006)

Solution. The boundary of the given triangle consists of three lines AB, BC, CA. (Fig. 29.19).

$$\oint_{ABC} e^{iz} dz = \int_{AB} e^{iz} dz + \int_{BC} e^{iz} dz + \int_{CA} e^{iz} dz$$

Now
$$\int_{AB} e^{iz} dz = \int_{1}^{-1} e^{i(x+i)} dx \quad \left| \begin{array}{l} \because \text{Along } AB : y = 1 \\ \therefore z = x + i \text{ and } dz = dx \end{array} \right.$$

$$= \int_{1}^{-1} e^{(ix-1)} dx = \left| \frac{e^{ix-1}}{i} \right|_{1}^{-1} = \frac{e^{-i-1} - e^{i-1}}{i}$$

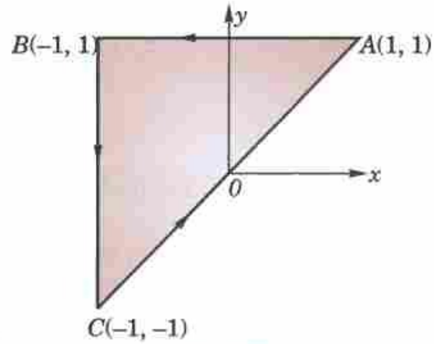

Fig. 20.19

$$\int_{BC} e^{iz} dz = \int_{1}^{-1} e^{i(-1+1iy)} \, idy$$

$$\left| \begin{array}{l} \because \text{Along } BC : x = -1 \\ \therefore z = -1 + iy, \; dz = idy \end{array} \right.$$

$$= i \int_{1}^{-1} e^{-i-y} dy = i \left| \frac{e^{-i-y}}{-1} \right|_{1}^{-1} = \frac{e^{-i+1} - e^{-i-1}}{i}$$

$$\int_{CA} e^{iz} dz = \int_{-1}^{1} e^{i(1+i)x} (1+i) dx \qquad \left| \begin{array}{l} \because \text{Along } CA : y = x \\ \therefore z = (1 + i, x, \; dz = (1 + i) dx \end{array} \right.$$

$$= (1+i) \, \frac{e^{(i-1)} - e^{-(i-1)}}{i(1+i)} = \frac{e^{i-1} - e^{-i+1}}{i}$$

Thus from (*i*) $$\oint_{ABC} e^{iz} \, dz = \frac{e^{-i-1} - e^{i-1}}{i} + \frac{e^{-i+1} - e^{-i-1}}{i} + \frac{e^{i-1} - e^{-i+1}}{i} = 0 \qquad ...(ii)$$

Also since $f(x) = e^{iz}$ is analytic everywhere,

$\therefore$ by Cauchy's theorem $\oint f(z) = 0$ ...(*iii*)

Hence from (*ii*) and (*iii*), $\oint_{ABC}$ Cauchy's theorem is verified.

Example 20.24. *If* $F(\zeta) = \oint_C \dfrac{4z^2 + z + 5}{z - \zeta} dz$, *where C is the ellipse* $(x/2)^2 + (y/3)^2 = 1$, *find the value of* (*a*) $F(3.5)$; (*b*) $F(i)$, $F''(-1)$ *and* $F''(-i)$. (Bhopal, 2009 ; Marathwada, 2008 ; Mumbai, 2006)

Solution. (*a*)
$$F(3.5) = \oint_C \frac{z^3 + z + 1}{z^2 - 7z + 2} dz$$

Since $\zeta = 3.5$ is the only singular point of $(4z^2 + z + 5)/(z - 3.5)$ and it lies outside the ellipse C, therefore, $(4z^2 + z + 5)/(z - 3.5)$ is analytic everywhere within C.

Hence by Cauchy's theorem,

$$\oint_C \frac{4z^2 + z + 5}{z - 3.5} dz = 0, \text{ i.e., } F(3.5) = 0.$$

(*b*) Since $f(z) = 4z^2 + z + 5$ is analytic within C and $\zeta = i, -1$ and $-i$ all lie within C, therefore, by Cauchy's integral formula

$$f(\zeta) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - \zeta} dz$$

i.e., $$\oint_C \frac{4z^2+z+5}{z-\zeta}dz = 2\pi i(4\zeta^2+\zeta+5)$$

i.e., $$F(\zeta) = 2\pi i(4\zeta^2+\zeta+5)$$

$\therefore$ $$F'(\zeta) = 2\pi i\,(8\zeta+1) \text{ and } F''(\zeta) = 16\pi i$$

Thus $$F(i) = 2\pi i(-4+i+5) = 2\pi(i-1)$$

$$F'(-1) = 2\pi i[8(-1)+1] = -14\pi i \text{ and } F''(-i) = 16\pi i.$$

20.15 (1) CONVERSE OF CAUCHY'S THEOREM: MORERA'S THEOREM*

If $f(z)$ is continuous in a region D and $\oint_C f(z)dz = 0$ around every simple closed curve C in D, then $f(z)$ is analytic in D.

Since $\oint f(z)dz = 0$, then the line integral of $f(z)$ from a fixed point z_0 to a variable point z must be independent of the path and hence must be a function of z only. Thus

$$\int_{z_0}^{z} f(z)dz = \phi(z), \text{ (say)},$$

Let $\phi(z) = U + iV$ and $f(z) = u + iv$

Then $$U + iV = \int_{(x_0,y_0)}^{(x,y)} (u+iv)(dx+idy) = \int_{(x_0,y_0)}^{(x,y)} (udx - vdy) + i\int_{(x_0,y_0)}^{(x,y)} (vdx+udy)$$

$\therefore$ $$U = \int_{(x_0,y_0)}^{(x,y)} (udx - vdy),\ V = \int_{(x_0,y_0)}^{(x,y)} (vdx+udy)$$

Differentiating under the integral sign,

$$\frac{\partial U}{\partial x} = u,\ \frac{\partial U}{\partial y} = -v,\ \frac{\partial V}{\partial x} = v,\ \frac{\partial V}{\partial y} = u \quad \therefore \quad \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y},\ \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

Thus U and V satisfy C-R equations.

Also, since $f(z)$ is given to be continuous, u and v and therefore, $\partial U/\partial x$, $\partial U/\partial y$, $\partial V/\partial x$, $\partial V/\partial y$, are also continuous.

$\therefore$ $\phi(z)$ is an analytic function and

$$\phi'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv = f(z).$$

Thus, $f(z)$ is the derivative of an analytic function $\phi(z)$. Hence $f(z)$ is analytic by § 20.14 Cor.

(2) Cauchy's inequality†. *If $f(z)$ is analytic within and on the circle C: $|z-a| = r$, then*

$$|f^n(a)| \le \frac{Mn!}{r^n} \quad \text{...(I)}$$

where M is the maximum value of $|f(z)|$ on C.

From (5) of § 20.14, we get

$$|f^n(a)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)dz}{(z-a)^{n+1}}\right|$$

$$\le \frac{n!}{2\pi}\cdot\frac{M}{r^{n+1}}\oint_C |z| \qquad [\because\ |f(z)| < M]$$

$$= \frac{n!\,M}{2\pi r^{n+1}}\oint_C ds = \frac{Mn!}{2\pi r^{n+1}}2\pi r = \frac{Mn!}{r^n} \qquad \textit{(U.P.T.U., 2005)}$$

(3) Liouville's theorem$. *If $f(z)$ is analytic and bounded for all z in the entire complex plane, then $f(z)$ is a constant.* *(U.P.T.U., 2008)*

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* Named after the Italian mathematician, *Giacinto Morera* (1856–1909) who worked in Turin.

† See footnote p. 144

$ See footnote p. 573.

Taking $n = 1$ and replacing a by z, (I) gives

$$| f'(z) | \leq M/r$$

As $r \to \infty$, it gives $f'(z) = 0$ *i.e.*, $f(z)$ is constant for all z.

(4) Poisson's integral formulae. *If $f(z)$ is analytic within and on the circle C: $|z| = \rho$ and $z = re^{i\theta}$ is any point within C, then*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} f(re^{i\phi})d\phi$$

By Cauchy's integral formula, $f(z) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(w)}{w - z}\, dw$ ...(1)

As the inverse of the point x w.r.t. C lies outside C and is given by $\rho^2/\bar{z}$. [See footnote p. 685]

$\therefore$ by Cauchy's theorem,

$$0 = \frac{1}{2\pi i}\int \frac{f(w)}{w - \rho^2/\bar{z}}\, dw \quad ...(2)$$

Subtracting (2) from (1), $f(z) = \dfrac{1}{2\pi i}\displaystyle\int \left(\frac{1}{w - z} - \frac{1}{w - \rho^2/\bar{z}}\right) f(w)\, dw$

$$= \frac{1}{2\pi i}\oint_C \frac{z\bar{z} - \rho^2}{\bar{z}w^2 - (z\bar{z} + \rho^2)w + z\rho^2)} f(w)\, dw \quad ...(3)$$

Taking $w = \rho e^{i\phi}$ and noting that $\bar{z} = re^{-i\theta}$, (3) gives

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(r^2 - \rho^2) f(\rho e^{i\phi}) . \rho i e^{i\phi}\, d\phi}{re^{-i\theta} . \rho^2 e^{2i\phi} - (r^2 + \rho^2)\rho e^{i\phi} + re^{i\theta}\rho^2}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(\rho^2 - r^2) f(\rho e^{i\phi})d\phi}{\rho^2 + r^2 - r\rho[e^{i(\theta - \phi)} + e^{-i(\theta - \phi)}]} = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\rho^2 - r^2) f(\rho e^{i\phi})d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \quad ...(4)$$

This is called *Poisson's integral formula** for a circle. It expresses the values of a harmonic function within a circle in terms of its values on the boundary.

Writing $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ and $f(\rho e^{i\phi}) = u(\rho, \phi) + iv(\rho, \phi)$ in (4) and equating real and imaginary parts from both sides, we get the formulae:

$$u(r, \theta) = \int_0^{2\pi} \frac{(\rho^2 - r^2)u(\rho, \phi)d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \quad ...(5)$$

and

$$v(r, \theta) = \int_0^{2\pi} \frac{(\rho^2 - r^2)v(\rho, \phi)d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \quad ...(6)$$

PROBLEMS 20.6

1. Evaluate $\oint_C (z - a)^{-1} dz$, where C is a simple closed curve and the point $z = a$ is (*i*) outside C, (*ii*) inside C.
2. Evaluate $\oint_C \dfrac{dz}{(z - a)^n}$, $n = 2, 3, 4, \ldots$, where C is a closed curve containing the point $z = a$.
3. Evaluate (*i*) $\oint_C \dfrac{e^z}{z^2 + 1} dz$, where C is the circle $|z| = 1/2$. (*P.T.U., 2010*)

   (*ii*) $\oint_C \dfrac{e^{3iz}}{(z + \pi)^3} dz$, where C is the circle $|z - \pi| = 3$. (*U.P.T.U., 2007*)

* Named after the French mathematician *Simeon Denis Poisson* (1781–1840) who was a professor in Paris and made contributions to partial differential equations, potential theory and probability.

4. Use Cauchy's integral formula to calculate:

   (i) $\oint_C \frac{3z\;5}{z^2+2z}dz$, where C is $|z| = 1$. (*P.T.U., 2005 S*) (ii) $\oint_C \frac{z^2+1}{z(2z+1)}dz$, where C is $|z| = 1$.

   (iii) $\oint_C \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)} dz$ where C is $|z| = 4$. (*U.P.T.U., 2008*)

5. Evaluate (a) $\oint_C \frac{z^3-2z+1}{(z-i)^2}$ where C is $|z| = 2$.

   (b) $\oint_C \frac{e^{-z}}{(z-1)(z-2)^2}dz$ where C is $|z| = 3$. (*Rohtak, 2003*)

6. Evaluate, using Cauchy's integral formulae:

   (i) $\oint_C \frac{z}{z^2-3z+2}dz$, where C is $|z-2| = \frac{1}{2}$. (*U.P.T.U., 2009 ; Hissar, 2007 ; Madras, 2000*)

   (ii) $\oint_C \frac{e^z dz}{(z+1)^2}$, where C is $|z-1| = 3$. (*Bhopal, 2009*)

   (iii) $\oint_C \frac{\log z}{(z-1)^3}dz$ where C is $|z-1| = \frac{1}{2}$. (*J.N.T.U., 2003*)

7. Evaluate $f(2)$ and $f(3)$ where $f(a) = \oint_C \frac{2z^2-z-2}{z-a}dz$ and C is the circle $|z| = 2.5$.

8. If $\phi(\zeta) = \oint_C \frac{3z^2+7z+1}{z-\zeta}dz$, where C is the circle $|z| = 2$ find the values of

   (i) $\phi(3)$, (ii) $\phi'(1-i)$ (iii) $\phi''(1-i)$. (*Mumbai, 2006*)

9. Evaluate $\oint_C \frac{z^3+z+1}{z^2-7z+2}dz$, where C is the ellipse $4x^2 + 9y^2 = 1$. (*Rohtak, 2006*)

10. Verify Cauchy's theorem for the integral of z^3 taken over the boundary of the (i) rectangle with vertices $-1, 1, 1+i, -1+i$; (ii) triangle with vertices (1, 2), (1, 4), (3, 2). (*V.T.U., 2003*)

20.16 (1) SERIES OF COMPLEX TERMS

Let $$(a_1 + ib_1) + (a_2 + ib_2) + \ldots + (a_n + ib_n) + \ldots \quad \ldots(1)$$

be an infinite series of complex terms ; a's and b's being real numbers. *If the series Σa_n and Σb_n converge to the sums A and B, then series (1) is said to* **converge** *to the sum A + iB.* Also if (1) is a convergent series, then $\underset{n\to\infty}{\text{Lt}} (a_n + ib_n) = 0$.

The series (1) is said to be **absolutely convergent** *if the series*

$$|a_1 + ib_1| + |a_2 + ib_2| + \ldots + |a_n + ib_n| + \ldots$$

is convergent. Since $|a_n|$ and $|b_n|$ are both $\leq |a_n + ib_n|$, it follows *that an absolutely convergent series is convergent.*

Next let the series of functions $u_1(z) + u_2(z) + \ldots + u_n(z) + \ldots\ldots$ ...(2)

converge to the sum $S(z)$ and $S_n(z)$ be the sum of its first n terms. Then *the series (2) is said to be* **uniformly convergent** *in a region R, if corresponding to any positive number ε, there exists a positive number N, depending on ε, but not on z, such that for every z in R.*

$$|S(z) - S_n(z)| < \varepsilon \text{ for } n > N. \quad [\text{cf. Def. p. 389}]$$

As in the case of real series (p. 390) **Weirstrass's M-test** holds for series of complex terms. So *the series (2) is uniformly convergent in a region R if there is a convergent series of positive constants ΣM_n such that $|u_n(z)| \leq M_n$ for all z in R.*

Also a *uniformly convergent series of continuous complex functions is itself continuous and can be integrated term by term.*

Obs. *If a power series $\Sigma a_n z^n$ converges for $z = z_1$, then it converges absolutely for $|z| < |z_1|$.*

Since $\Sigma a_n z_1^n$ converges, therefore, $\underset{n\to\infty}{\text{Lt}}\ a_1 z_1^n = 0$ and so we can find a number k such that $|a_n z_1^n| < k$ for all n. Then $\Sigma a_n z^n = \Sigma |a_n z_1^n| \cdot |z/z_1|^n < \Sigma k t^n$ where $t = |z/z_1|$.

But the series Σt^n converges for $t < 1$. Hence the series $\Sigma a_n z^n$ converges absolutely for $|z| < |z_1|$, *i.e.*, if a circle with centre at the origin and radius $|z|$ be drawn, then the given series converges absolutely at all points inside the circle.

Such a circle $|z| = R$ within which series $\Sigma a_n z^n$ converges, is called the *circle of convergence and R is called the radius of convergence*.

A power series is uniformly convergent in any region which lies entirely within its circle of convergence.

(2) Taylor's series*. *If $f(z)$ is analytic inside a circle C with centre at a, then for z inside C,*

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots + \frac{f^n(a)}{n!}(z-a)^n + \ldots \qquad \ldots(i)$$

Proof. Let z be any point inside C. Draw a circle C_1 with centre at a enclosing z (Fig. 20.20). Let t be a point on C_1. We have

$$\frac{1}{t-z} = \frac{1}{t-a-(z-a)} = \frac{1}{t-a}\left(1 - \frac{z-a}{t-a}\right)^{-1}$$

$$= \frac{1}{t-a}\left[1 + \frac{z-a}{t-a} + \left(\frac{z-a}{t-a}\right) + \ldots + \left(\frac{z-a}{t-a}\right)^n + \ldots\right] \qquad \ldots(ii)$$

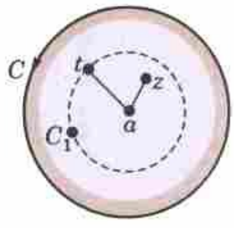

Fig. 20.20

As $|z-a| < |t-a|$, *i.e.* $|(z-a)/(t-a)| < 1$, this series converges uniformly. So, multiplying both sides of (ii) by $f(t)$, we can integrate over C_1.

$$\therefore \quad \oint_{C_1} \frac{f(t)}{t-z}dz = \oint_{C_1} \frac{f(t)}{t-a}dz + (z-a)\oint_{C_1}\frac{f(t)}{(t-a)^2}dt + \ldots + (z-a)^n \cdot \oint_{C_1}\frac{f(t)}{(t-a)^{n+1}}dt + \ldots \qquad \ldots(iii)$$

Since $f(t)$ is analytic on and inside C_1, therefore, applying the formulae (2) to (5) of p. 697-698 (iii), we get (i) which is known as *Taylor's series*.

Obs. Another remarkable fact is that *complex analytic functions can always be represented by power series of the form* (i).

(3) Laurent's series†. *If $f(z)$ is analytic in the ring-shaped region R bounded by two concentric circles C and C_1 of radii r and r_1 $(r > r_1)$ and with centre at a, then for all z in R*

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots$$

where $a_n = \dfrac{1}{2\pi i}\oint \dfrac{f(t)}{(t-a)^{n+1}}\,dt$,

Γ being any curve in R, encircling C_1 (as in Fig. 20.21).

Proof. Introduce cross-out AB, then $f(z)$ is analytic in the region D bounded by AB, C_1 described clockwise, BA and C described anti-clockwise (see Fig. 20.17). Then if z be any point in D, we have

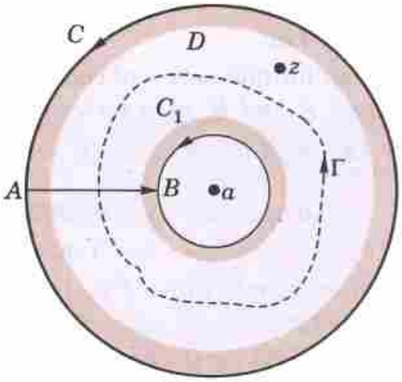

Fig. 20.21

$$f(z) = \frac{1}{2\pi i}\left[\int_{AB}\frac{f(t)}{t-z}dt + \oint_{C_1}\frac{f(t)}{t-z}dt + \int_{BA}\frac{f(t)}{t-z}dt + \oint_{C_1}\frac{f(t)}{t-z}dt\right]$$

$$= \frac{1}{2\pi i}\left[\oint_C \frac{f(t)}{t-z}dt - \oint_{C_1}\frac{f(t)}{t-z}dt\right] \qquad \ldots(i)$$

where both C and C_1 are described anti-clockwise in (i) and integrals along AB and BA cancel (Fig. 20.21).

For the first integral in (i), expanding $1/(t-z)$ as in § 20.16 (2), we get

$$\frac{1}{2\pi i}\oint_C \frac{f(t)}{t-z}dt = \sum_{n=1}^{\infty}\frac{(z-a)^n}{2\pi i}\oint_C \frac{f(t)}{(t-a)^{n+1}}dt$$

* See footnote p. 145.

† Named after the French engineer and mathematician *Pierre Alphonse Laurent* (1813–1854) who published this theorem in 1843.

$$= \Sigma a_n (z-a)^n \text{ where } \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(t)}{(t-a)^{n+1}} dt \quad ...(ii)$$

For the second integral in (i), let t lie on C_1. Then we write

$$\frac{1}{t-z} = \frac{1}{(t-a)-(z-a)} = -\frac{1}{z-a}\left(1 - \frac{t-a}{z-a}\right)^{-1}$$

$$= -\frac{1}{z-a}\left[1 + \frac{t-a}{z-a} + \left(\frac{t-a}{z-a}\right)^2 + ... + \left(\frac{t-a}{z-a}\right)^{n-1} + ...\right]$$

As $|t-a| < |z-a|$, *i.e.*, $|(t-a)/(z-a)| < 1$, this series converges uniformly. So multiplying both sides by $f(t)$ and integrating over C_1, we get

$$-\frac{1}{2\pi i} \oint_C \frac{f(t)}{t-z} dt = \sum_{n=1}^{\infty} \frac{1}{(z-a)^n} \cdot \frac{1}{2\pi i} \oint_C (t-a)^{n-1} f(t) dt = \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} \quad ...(iii)$$

where $$a_{-n} = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{(t-a)^{-n+1}} dt$$

Substituting from (ii) and (iii) in (i), we get

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}. \quad ...(iv)$$

Now $f(t)/(t-a)^{n+1}$ being analytic in the region between C and Γ, we can take the integral giving a_n over Γ. Similarly we can take the integral giving a_{-n} over Γ. Hence (iv) can be written as

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n \text{ where } a_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(t)}{(t-a)^{n+1}} dt$$

which is known as *Laurent's series.*

Obs. 1. *As $f(z)$ is not given to be analytic inside Γ,* $a_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(t)}{(t-a)^{n+1}} dt \neq \frac{f^n(a)}{n!}$

However, *if $f(z)$ is analytic inside* Γ, then $a_{-n} = 0$; $a_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(t)}{(t-a)^{n+1}} dt = \frac{f^n(a)}{n!}$

and *Laurent's series reduces to Taylor's series.*

Obs. 2. *To obtain Taylor's or Laurent's series, simply expand f(z) by binomial theorem* instead of finding a_n by complex integration which is quite complicated.

Obs. 3. *Laurent series of a given analytic function f(z) in its annulus of convergence is unique.* There may be different Laurent series of $f(z)$ in two annuli with the same centre.

Example 20.25. *Show that the series $z(1-z) + z^2(1-z) + z^3(1-z) + ... \infty$ converges for $|z| < 1$. Determine whether it converges absolutely or not.*

Solution. Let the sum of the first n terms of the series be s_n, so that

$$s_n = z - z^2 + z^2 - z^3 + z^3 - z^4 + ... + z^n - z^{n+1} = z - z^{n+1}$$

For $|z| < 1$, $z^{n+1} \to 0$ as $n \to \infty$.

$\therefore$ $\underset{n\to\infty}{\text{Lt}} (s_n) = z$, *i.e.*, the given series converges for $|z| < 1$.

$$|s_n(z)| = |z(1-z)| + |z^2(1-z)| + ... + |z^n(1-z)|$$
$$= |1-z| \{|z| + |z|^2 + |z|^3 + ... + |z|^n\}$$

For $|z| < 1$, $\underset{n\to\infty}{\text{Lt}} |s_n(z)| = |1-z| \frac{|z|}{1-|z|}$ [G.P.]

Hence the given series converges absolutely.

Example 20.26. *Expand sin z in a Taylor's series about z = 0 and determine the region of convergence.* (P.T.U., 2009 S)

Solution. Given $f(z) = \sin z, f'(z) = \cos \theta, f''(z) = -\sin z, f'''(z) = -\cos z, \ldots$

$\therefore$ $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = 1$

By Taylor's series about $z = 0$, we have

$$f(z) = f(0) + \frac{(z-0)}{1!}f'(0) + \frac{(z-0)^2}{2!}f''(0) + \frac{(z-0)^3}{3!}f'''(0) + \ldots$$

i.e.,
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots + (-1)^{n-1}\frac{z^{2n}-1}{(2n-1)!} + \ldots$$

Hence
$$\sin z = \sum_{n=1}^{\infty} a_n (z-0)^{2n-1} \text{ where } a_n = \frac{(-1)^{n-1}}{(2n-1)!}$$

Since
$$\rho = \operatorname*{Lt}_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \operatorname*{Lt}_{n\to\infty}\left|\frac{(2n-1)!}{(2n+1)!}\right| = 0$$

Thus the radius of convergence of $f(z) = 1/\rho = \infty$

i.e., the region of convergence of $f(z)$ is all reals.

Example 20.27. *Find Taylor's expansion of*

(*i*) $f(z) = \dfrac{1}{(z+1)^2}$ *about the point* $z = -i$. (V.T.U., 2009 S)

(*ii*) $f(z) = \dfrac{2z^3+1}{z^2+z}$ *about the point* $z = i$. (P.T.U., 2003)

Solution. (*i*) To expand $f(z)$ about $z = -i$, *i.e.*, in powers of $z + i$, put $z + i = t$. Then

$$f(z) = \frac{1}{(t-i+1)^2} = (1-i)^{-2}\,[1 + t/(1-i)]^{-2} = \frac{i}{2}\left[1 - \frac{2t}{1-i} + \frac{3t^2}{(1-i)^2} - \frac{4t^3}{(1-i)^3} + \ldots\right]$$

[Expanding by Binomial theorem]

$$= \frac{i}{2}\left[1 + \sum_{n=1}^{\infty}(-1)^n\frac{(n+1)(z+i)^n}{(1-i)^n}\right]$$

(*ii*)
$$f(z) = \frac{2z^3+1}{z(z+1)} = 2z - 2 + \frac{2z+1}{z(z+1)} = (2i-2) + 2(z-i) + \frac{1}{z} + \frac{1}{z+1} \quad \ldots(i)$$

[By partial fractions]

To expand $1/z$ and $1/(z + 1)$ about $z = i$, put $z - i = t$, so that

$$\frac{1}{z} = \frac{1}{(t+i)} = \frac{1}{i}\left(1 + \frac{t}{i}\right)^{-1}$$ [Expanding by Binomial theorem]

$$= \frac{1}{i}\left[1 - \frac{t}{i} + \frac{t^2}{i^2} - \frac{t^3}{i^3} + \frac{t^4}{i^4} - \ldots\infty\right] = \frac{1}{i} + \frac{t}{1} + \frac{t^2}{i^3} - \frac{t^3}{i^4} + \frac{t^4}{i^5} - \ldots\infty$$

$$= -i + (z-i) + \sum_{n=2}^{\infty}(-1)^n\frac{(z-i)^n}{i^{n+1}} \quad \ldots(ii)$$

and
$$\frac{1}{z+1} = \frac{1}{t+i+1} = \frac{1}{1+i}\left(1 + \frac{t}{1+i}\right)^{-1}$$ [Expanding by Binomial theorem]

$$= \frac{1}{1+i}\left[1 - \frac{t}{1+i} + \frac{t^2}{(1+t)^2} - \frac{t^3}{(1+i)^3} + \frac{t^4}{(1+i)^4} - \ldots\infty\right]$$

$$= \frac{1-i}{2} - \frac{t}{2i} + \left[\frac{t^2}{(1+i)^3} - \frac{t^3}{(1+i)^4} + \frac{t^4}{(1+i)^5} - \ldots\infty\right] = \frac{1}{2} - \frac{i}{2} - \frac{z-i}{2i} + \sum_{n=2}^{\infty}(-1)^n\frac{(z-i)^n}{(1+i)^{n+1}} \quad \ldots(iii)$$

Substituting from (*ii*) and (*iii*) in (*i*), we get

$$f(z) = \left(2i - 2 - i + \frac{1}{2} - \frac{i}{2}\right) + \left(2 + 1 - \frac{1}{2i}\right)(z - i) + \sum_{n=2}^{\infty} (-1)^n \left\{\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right\}(z-i)^n$$

$$= \left(\frac{i}{2} - \frac{3}{2}\right) + \left(3 + \frac{i}{2}\right)(z - i) + \sum_{n=2}^{\infty} (-1)^n \left\{\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right\}(z-i)^n.$$

Example 20.28. *Expand $f(z) = 1/[(z-1)(z-2)]$ in the region:*
(*a*) $|z| < 1$, (*b*) $1 < |z| < 2$, (*c*) $|z| > 2$, (*d*) $0 < |z-1| < 1$.
(U.P.T.U., 2010 ; V.T.U., 2010 ; Bhopal, 2009)

Solution. (*a*) By partial fractions $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$ ...(*i*)

$$= -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} + (1-z)^{-1} \quad \text{...(ii)}$$

For $|z| < 1$, both $|z/2|$ and $|z|$ are less than 1. Hence (*ii*) gives on expansion

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right) + (1 + z + z^2 + z^3 + \ldots)$$

$$= \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \ldots \text{ which is a } \textit{Taylor's series.}$$

(*b*) *For* $1 < |z| < 2$, we write (*i*) as

$$f(z) = -\frac{1}{2}\frac{1}{(1 - z/2)} - \frac{1}{z(1 - z^{-1})} \quad \text{...(iii)}$$

and notice that both $|z/2|$ and $|z^{-1}|$ are less than 1. Hence (*iii*) gives on expansion

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right) - \frac{1}{z}(1 + z^{-1} + z^{-2} + z^{-3} + \ldots)$$

$$= \ldots - z^{-4} - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \ldots$$

which is a *Laurent's series.*

(*c*) *For* $|z| > 2$, we write (*i*) as

$$f(z) = \frac{1}{z(1 - 2z^{-1})} - \frac{1}{z(1 - z^{-1})}$$

$$= z^{-1}(1 + 2z^{-1} + 4z^{-2} + 8z^{-3} + \ldots) - z^{-1}(1 + z^{-1} + z^{-2} + z^{-3} + \ldots)$$

$$= \ldots + 7z^{-4} + 3z^{-3} + z^{-2} + \ldots$$

(*d*) *For* $0 < |z-1| < 1$, we write (*i*) as

$$f(z) = \frac{1}{(z-1)-1} - \frac{1}{z-1}$$

$$= -(z-1)^{-1} - [1 - (z-1)]^{-1}$$

$$= -(z-1)^{-1} - [1 + (z-1) + (z-1)^2 + (z-1)^3 + \ldots].$$

Example 20.29. *Find the Laurents' expansion of $f(z) = \dfrac{7z-2}{(z+1)z(z-2)}$ in the region $1 < z + 1 < 3$.*
(S.V.T.U., 2009 ; Anna, 2003 ; V.T.U., 2003)

Solution. Writing $z + 1 = u$, we have

$$f(z) = \frac{7(u-1)-2}{u(u-1)(u-1-2)} = \frac{7u-9}{u(u-1)(u-3)}$$

$$= -\frac{3}{u} + \frac{1}{u-1} + \frac{2}{u-3} \quad \text{(splitting into partial fraction)}$$

$$= -\frac{3}{u} + \frac{1}{u(1-1/u)} - \frac{2}{3(1-u/3)} = -\frac{3}{u} + \frac{1}{u}\left(1 - \frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1 - \frac{u}{3}\right)^{-1}$$

Since $1 < u < 3$ or $1/u < 1$ and $u/3 < 1$, expanding by Binomial theorem,

$$f(z) = \frac{-3}{u} + \frac{1}{u}\left(1 + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \ldots \infty\right) - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \ldots \infty\right)$$

$$= -\frac{2}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \ldots \infty - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \ldots \infty\right)$$

Hence $$f(z) = -\frac{2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \ldots \infty - \frac{2}{3}\left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{3^2} + \frac{(z+1)^3}{3^3} + \ldots \infty\right]$$

which is valid in the region $1 < z + 1 < 3$.

20.17 (1) ZEROS OF AN ANALYTIC FUNCTION

Def. *A zero of an analytic function f(z) is that value of z for which f(z) = 0.*

If $f(z)$ is analytic in the neighbourhood of a point $z = a$, then by Taylor's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_n(z-a)^n + \ldots \qquad \text{where } a_n = \frac{f^n(a)}{n!}.$$

If $a_0 = a_1 = a_2 = \ldots = a_{m-1} = 0$ but $a_m \neq 0$, then $f(z)$ is said to have a *zero of order m* at $z = a$.

When $m = 1$, the zero is said to be *simple*. In the neighbourhood of zero ($z = a$) of order m,

$$f(z) = a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \ldots \infty$$
$$= (z-a)^m \phi(z) \text{ where } \phi(z) = a_m + a_{m+1}(z-a) + \ldots$$

Then $\phi(z)$ is analytic and non-zero in the neighbourhood of $z = a$.

(2) Singularities of an analytic function

We have already defined a *singular point of a function as the point at which the function ceases to be analytic.*

(*i*) **Isolated singularity.** *If $z = a$ is a singularity of $f(z)$ such that $f(z)$ is analytic at each point in its neighbourhood* (*i.e.*, there exists a circle with centre a which has no other singularity), *then $z = a$ is called an* **isolated singularity.**

In such a case, $f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots \qquad \ldots(1)$$

For example, $f(z) = \cot(\pi/z)$ is not analytic where $\tan(\pi/z) = 0$ *i.e.* at the points $\pi/z = 4\pi$ or $z = 1/n$ ($n = 1, 2, 3, \ldots$).

Thus $z = 1, 1/2, 1/3, \ldots$ are all *isolated singularities* as there is no other singularity in their neighbourhood.

But when n is large, $z = 0$ is such a singularity that there are infinite number of other singularities in its neighbourhood. Thus $z = 0$ is the *non-isolated singularity* of $f(z)$.

(*ii*) **Removable singularity.** If all the negative powers of $(z - a)$ in (1) are zero, then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$.

Here the singularity can be removed by defining $f(z)$ at $z = a$ in such a way that it becomes analytic at $z = a$. Such a singularity is called a *removable singularity.*

Thus *if* $\underset{z\to a}{\text{Lt}}\, f(z)$ *exists finitely, then $z = a$ is a removable singularity.*

(*iii*) **Poles.** *If all the negative powers of $(z - a)$ in (i) after the nth are missing, then the singularity at $z = a$ is called a* **pole of order n.**

A pole of first order is called a **simple pole.**

(*iv*) **Essential singularity.** *If the number of negative powers of $(z - a)$ in (1) is infinite, then $z = a$ is called an essential singularity.* In this case, $\underset{z\to a}{\text{Lt}}\, f(z)$ does not exist.

Example 20.30. *Find the nature and location of singularities of the following functions:*

(*i*) $\dfrac{z - \sin z}{z^2}$ (*ii*) $(z + 1)\sin\dfrac{1}{z-2}$ (*iii*) $\dfrac{1}{\cos z - \sin z}$.

Solution. (*i*) Here $z = 0$ is a singularity.

Also $$\frac{z-\sin z}{z^2} = \frac{1}{z^2}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\right)\right\} = \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} - \ldots$$

Since there are no negative powers of z in the expansion, $z = 0$ *is a removable singularity.*

(*ii*) $$(z+1)\sin\frac{1}{z-2} = (t+2+1)\sin\frac{1}{t} \qquad \text{where } t = z-2$$

$$= (t+3)\left\{\frac{1}{t} - \frac{1}{3!t^3} + \frac{1}{5!t^5} - \ldots\right\} = \left(1 - \frac{1}{3!t^2} + \frac{1}{5!t^4} - \ldots\right) + \left(\frac{3}{t} - \frac{1}{2t^3} + \frac{3}{5!t^5} - \ldots\right)$$

$$= 1 + \frac{3}{t} - \frac{1}{6t^2} - \frac{1}{2t^3} + \frac{1}{120t^4} - \ldots = 1 + \frac{3}{z-2} - \frac{1}{6(z-2)^2} - \frac{1}{2(z-2)^3} + \ldots$$

Since there are infinite number of terms in the negative powers of $(z-2)$, $z = 2$ *is an essential singularity.*

(*iii*) Poles of $f(z) = \dfrac{1}{\cos z - \sin z}$ are given by equating the denominator to zero, *i.e.*, by $\cos z - \sin z = 0$ or $\tan z = 1$ or $z = \pi/4$. Clearly $z = \pi/4$ is a *simple pole of* $f(z)$.

Example 20.31. *What type of singularity have the following functions :*

(*i*) $\dfrac{1}{1-e^z}$ (*ii*) $\dfrac{e^{2z}}{(z-1)^4}$ (*iii*) $\dfrac{e^{1/z}}{z^2}$. *(U.P.T.U., 2009)*

Solution. (*i*) Poles of $f(z) = 1/(1-e^z)$ are found by equating to zero $1 - e^z = 0$ or $e^z = 1 = e^{2n\pi i}$

$\therefore$ $$z = 2n\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$$

Clearly $f(z)$ *has a simple pole at* $z = 2\pi i$.

(*ii*) $$\frac{e^{2z}}{(z-1)^4} = \frac{e^{2(t+1)}}{t^4} = \frac{e^2}{t^4}\cdot e^{2t} \qquad \text{where } t = z-1$$

$$= \frac{e^2}{t^4}\left\{1 + \frac{2t}{1!} + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \ldots\right\} = e^2\left\{\frac{1}{t^4} + \frac{2}{t^3} + \frac{2}{t^2} + \frac{4}{3t} + \frac{2}{3} + \frac{4t}{15} + \ldots\right\}$$

$$= e^2\left\{\frac{1}{(z-1)^4} + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{4}{3(z-1)} + \frac{2}{3} + \frac{4}{15}(z-1) + \ldots\right\}$$

Since there are finite (4) number of terms containing negative powers of $(z-1)$,

$\therefore$ $z = 1$ *is a pole of 4th order.*

(*iii*) $$f(z) = \frac{e^{1/z}}{z^2} = \frac{1}{z^2}\left\{1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots\right\} = z^{-2} + z^{-3} + \frac{z^{-4}}{2} + \ldots \infty$$

Since there are infinite number of terms in the negative powers of z, therefore $f(z)$ has *an essential singularity* at $z = 0$.

PROBLEMS 20.7

1. Obtain the expansion of $(z-1)/z^2$ in a Taylor's series in powers of $(z-1)$ and determine the region of convergence.
2. Find the first three times of the Taylor's series expansion of $f(z) = 1/(z^2+4)$ about $z = -i$. Also find the region of convergence. *(U.P.T.U., 2006)*
3. Expand in Taylor's series (*i*) $(z-1)/(z+1)$ about the point $z = 1$. *(Andhra, 2000)*

   (*ii*) $\cos z$ about the point $z = \pi/2$. *(Marathwada, 2008)* (*iii*) $\dfrac{1}{z^2-z-6}$ about (*a*) $z = -1$ (*b*) $z = 1$ *(P.T.U., 2009)*
4. Expand the following functions in Laurent's series :

   (*i*) $f(z) = \dfrac{1}{z-z^2}$ for $1 < |z+1| < 2$. *(Madras, 2006)*

(ii) $f(z) = \dfrac{1}{(z-1)(z+3)}$ for $1 < |z| < 3$. (J.N.T.U., 2006)

(iii) $f(z) = z/[(z-1)(z-3)]$ for $|z-1| < 2$. (V.T.U., 2007)

5. Find the Laurent's expansion of (i) $\dfrac{e^z}{(z-1)^2}$, about $z = 1$. (Rohtak, 2006)

(ii) $e^{2z}/(z-1)^3$ about the singularity $z = 1$.

6. Expand the following functions in Laurent series.

(i) $(z-1)/z^2$ for $|z-1| > 1$ (ii) $\dfrac{1-\cos z}{z^3}$, about $z = 0$. (Rohtak, 2004)

7. Find the Laurent's series expansion of

(i) $\dfrac{z^2-1}{z^2+5z+6}$ about $z = 0$ in the region $2 < |z| < 3$ (V.T.U., 2011 S ; Osmania, 2003)

(ii) $\dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$

(iii) $\dfrac{7z^2-9z-18}{z^3-9z}$ in the region (a) $|z| > 3$ (b) $0 < |z-3| < 3$. (V.T.U., 2010 S)

8. Find the Laurent's expansion of $1/[(z^2+1)(z^2+2)]$ for (a) $0 < |z| < 1$; (b) $1 < |z| < \sqrt{2}$; (c) $|z| > 2$.

Find the nature and location of the singularities of the following functions : (P.T.U., 2005)

9. $\dfrac{1}{z(2-z)}$. 10. $\sin(1/z)$. (U.P.T.U., 2009) 11. $\tan\left(\dfrac{1}{z}\right)$. (P.T.U., 2006)

12. $\dfrac{z^2-1}{(z-1)^3}$. (Osmania, 2003) 13. $\dfrac{e^z}{(z-1)^4}$. 14. $\dfrac{\cot \pi z}{(z-a)^2}$. (U.P.T.U., 2008)

20.18 (1) RESIDUES

The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity is called the **residue** *of $f(z)$ at that point.* Thus in the Laurent's series expansion of $f(z)$ around $z = a$ *i.e.*, $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots$, the residue of $f(z)$ at $z = a$ is a_{-1}.

$$\therefore \qquad Res\, f(a) = \frac{1}{2\pi i}\oint_C f(z)$$

i.e.,

$$\oint_C f(z)\,dz = 2\pi i \text{ Res } f(a). \qquad \ldots(1)$$

(2) Residue Theorem

If $f(z)$ is analytic in a closed curve C except at a finite number of singular points within C, then $\oint_C f(z)\,dz = 2\pi i \times$ *(sum of the residues at the singular points within C).*

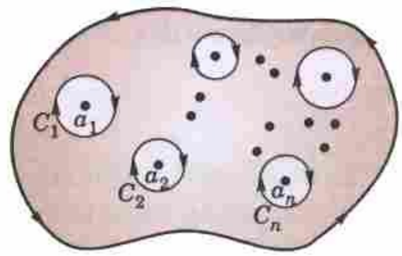

Fig. 20.22

Let us surround each of the singular points $a_1, a_2, \ldots, a_n$ by a small circle such that it encloses no other singular point (Fig. 20.22). Then these circles $C_1, C_2, \ldots, C_n$ together with C, form a multiply connected region in which $f(z)$ is analytic.

$\therefore$ applying Cauchy's theorem, we have

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \ldots + \oint_{C_n} f(z)dz \qquad \text{[by (1)}$$

$$= 2\pi i\,[\text{Res } f(a_1) + \text{Res } f(a_2) + \ldots + \text{Res } f(a_n)] \text{ which is the desired result.}$$

20.19 CALCULATION OF RESIDUES

(1) *If $f(z)$ has a simple pole at $z = a$, then*

$$\textbf{Res f(a)} = \underset{z \to a}{\textbf{Lt}} \, [(\mathbf{z - a})\mathbf{f(z)}]. \qquad \ldots(1)$$

Laurent's series in this case is

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 \ldots + c_{-1}(z-a)^{-1}$$

Multiplying throughout by $z - a$, we have

$$(z-a)f(z) = c_0(z-a) + c_1(z-a)^2 + \ldots + c_{-1}.$$

Taking limits as $z \to a$, we get

$$\underset{z \to a}{\text{Lt}} \, [(z-a)f(z)] = c_{-1} = \text{Res } f(a).$$

(2) *Another formula for Res $f(a)$:*

Let $f(z) = \phi(z)/\psi(z)$, where $\psi(z) = (z-a)F(z)$, $F(a) \neq 0$.

Then
$$\underset{z \to a}{\text{Lt}} \, [(z-a)\phi(z)/\psi(z)]$$
$$= \underset{z \to a}{\text{Lt}} \frac{(z-a)[\phi(a) + (z-a)\phi'(a) + \ldots]}{\psi(a) + (z-a)\psi'(a) + \ldots}$$
$$= \underset{z \to a}{\text{Lt}} \frac{\phi(a) + (z-a)\phi'(a) + \ldots}{\psi'(a) + (z-a)\psi''(a) + \ldots}, \quad \text{since } \psi(a) = 0$$

Thus
$$\textbf{Res f(a)} = \frac{\boldsymbol{\phi}\mathbf{(a)}}{\boldsymbol{\psi}'\mathbf{(a)}}.$$

(3) *If $f(z)$ has a pole of order n at $z = a$, then*

$$\textbf{Res f(a)} = \frac{\mathbf{1}}{\mathbf{(n-1)!}}\left\{\frac{\mathbf{d^{n-1}}}{\mathbf{dz^{n-1}}}[(\mathbf{z-a})^{\mathbf{n}}\mathbf{f(z)}]\right\}_{\mathbf{z=a}}$$

Here
$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots + c_{-1}(z-a)^{-1} + \ldots + c_{-n}(z-a)^{-n}.$$

Multiplying throughout by $(z-a)^n$, we get

$$(z-a)^n f(z) = c_0(z-a)^n + c_1(z-a)^{n+1} + c_2(z-a)^{n+2} + \ldots + c_{-1}(z-a)^{n-1} + c_{-2}(z-a)^{n-2} + \ldots + c_{-n}.$$

Differentiating both sides w.r.t. z, $n - 1$ times and putting $z = a$, we get

$$\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a} = (n-1)!\, c_{-1} \text{ whence follows the result.}$$

Obs. In many cases, the residue of a pole $(z = a)$ can be found, by putting $z = a + t$ in $f(z)$ and expanding it in powers of t where $|t|$ is quite small.

Example 20.32. *Find the sum of the residues of $f(z) = \dfrac{\sin z}{z \cos z}$ at its poles inside the circle $|z| = 2$.*

(Rohtak, 2004)

Solution. $f(z)$ has simple poles at $z = 0, \pm \pi/2, \pm 3\pi/2, - \ldots$

Only the poles $z = 0$ and $z = \pm \pi/2$ lies inside $|z| = 2$.

$$\therefore \quad \text{Res } f(0) = \underset{z \to 0}{\text{Lt}} \, [z \,.\, f(z)] = \underset{z \to 0}{\text{Lt}} \left(\frac{\sin z}{\cos z}\right) = 0.$$

$$\text{Res } f(\pi/2) = \underset{z \to \pi/2}{\text{Lt}} \left[\left(z - \frac{\pi}{2}\right) f(z)\right] = \underset{z \to \pi/2}{\text{Lt}} \left\{\frac{(z - \pi/2)\sin z}{z \cos z}\right\}$$

$$= \underset{z \to \pi/2}{\text{Lt}} \frac{(z - \pi/2)\cos z + \sin z}{\cos z - z \sin z} \qquad \left[\text{Being } \frac{0}{0} \text{ form}\right]$$

$$= \frac{1}{-\pi/2} = -\frac{2}{\pi}$$

and $$\text{Res } f(-\pi/2) = \underset{z\to-\pi/2}{\text{Lt}} \left\{\frac{(z+\pi/2)\sin z}{z\cos z}\right\} = \underset{z\to-\pi/2}{\text{Lt}} \frac{(z+\pi/2)\cos z + \sin z}{\cos z - z\sin z} = \frac{-1}{-\pi/2} = \frac{2}{\pi}$$

Hence sum of residues $= 0 - \frac{2}{\pi} + \frac{2}{\pi} = 0$.

Example 20.33. *Determine the poles of the function*

$f(z) = z^2/(z-1)^2(z+2)$ and the residue at each pole. (S.V.T.U., 2008 ; J.N.T.U., 2005)

Hence evaluate $\oint_C f(z)dz$, where C is the circle $|z| = 2.5$.

Solution. Since $\underset{z\to-2}{\text{Lt}} \{(z+2)f(z)\} = \underset{z\to-2}{\text{Lt}} \frac{z^2}{(z-1)^2} = \frac{4}{9}$.

which is finite and non-zero, the function has a simple pole at $z = -2$ and Res $f(-2) = 4/9$.

Also since $\underset{z\to1}{\text{Lt}} [(z-1)^2 f(z)]$ is finite and non-zero, $f(z)$ has a pole of order two at $z = 1$.

$$\therefore \quad \text{Res } f(1) = \frac{1}{1!}\left[\frac{d}{dz}\{(z-1)^2 f(z)\}\right]_{z=1} = \left[\frac{d}{dz}\left(\frac{z^2}{z+2}\right)\right]_{z=1} = \left[\frac{z^2+4z}{(z+2)^2}\right]_{z=1} = \frac{5}{9}.$$

[*Otherwise writing* $z = 1 + t$,

$$f(z) = \frac{(1+t)^2}{t^2(3+t)} = \frac{1}{3t^2}(1+t)^2(1+t/3)^{-1} = \frac{1}{3t^2}(1+t)^2\left(1 - \frac{t}{3} + \frac{t^2}{9} - \ldots\right)$$

$$= \frac{1}{3t^2}\left(1 + \frac{5}{3}t + \frac{4}{9}t^2 - \ldots\right) = \frac{1}{3t^2} + \frac{5}{9t} + \frac{4}{27} - \ldots \qquad \ldots(i)$$

$$\therefore \quad \text{Res } f(1) = \text{coefficient of } \frac{1}{t} \text{ in } (i) = \frac{5}{9}.]$$

Clearly $f(z)$ is analytic on $|z| = 2.5$ and at all points inside except the poles $z = -2$ and $z = 1$. Hence by residue theorem

$$\oint_C f(z)dz = 2\pi i\,[\text{Res } f(-2) + \text{Res } f(1)] = 2\pi i\left[\frac{4}{9} + \frac{5}{9}\right] = 2\pi i.$$

Example 20.34. *Find the residue of $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at its poles and hence evaluate $\oint_C f(z)dz$ where C is the circle $|z| = 2.5$.* (U.P.T.U., 2003)

Solution. The poles of $f(z)$ are given by $(z-1)^4(z-2)(z-3) = 0$.

$\therefore$ $z = 1$ is a pole of order 4, while $z = 2$ and $z = 3$ are simple poles.

$$\text{Res } f(1) = \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-1)^4 \cdot \frac{z^3}{(z-1)^4(z-2)(z-3)}\right\}_{z=1} = \frac{1}{6}\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}_{z=1}$$

$\therefore$ $z = 1$ is a pole of order 4, while $z = 2$ and $z = 3$ are simple poles.

$$\text{Res } f(1) = \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-1)^4 \cdot \frac{z^3}{(z-1)^4(z-2)(z-3)}\right\}_{z=1} = \frac{1}{6}\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}_{z=1}$$

$$= \frac{1}{6}\frac{d^3}{dz^3}\left[z + 5 - \frac{8}{z-2} + \frac{27}{z-3}\right] = \frac{1}{6}\left[-8 \cdot \frac{(-1)^3\,3!}{(z-2)^4} + \frac{27 \cdot (-1)^3\,3!}{(z-2)^4}\right]_{z=1}$$

$$= -\left[-8 + \frac{27}{16}\right] = \frac{101}{16}.$$

$$\text{Res } f(2) = \underset{z \to 2}{\text{Lt}} \left\{ (z-2) \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)} \right\} = \underset{z \to 2}{\text{Lt}} \left\{ \frac{z^3}{(z-1)^4 (z-3)} \right\} = \frac{8}{(1)^4(-1)} = -8$$

$$\text{Res } f(3) = \underset{z \to 3}{\text{Lt}} \left\{ (z-3) \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)} \right\} = \frac{27}{(2)^4 \cdot 1} = \frac{27}{16}$$

Now $\oint_C f(z)dz = 2\pi i \, [\text{Res } f(1) + \text{Res } f(2)]$ [$\because$ Pole $z = 3$ is outside C]

$$= 2\pi i \left(\frac{101}{16} - 8 \right) = \frac{-27\pi i}{8}.$$

Example 20.35. *Evaluate*

$$\oint_C \frac{z-3}{z^2+2z+5} dz, \text{ where } C \text{ is the circle}$$

(i) $|z| = 1$, *(ii)* $|z + 1 - i| = 2$, *(iii)* $|z + 1 + i| = 2$. *(J.N.T.U., 2003)*

Solution. The poles of $f(z) = \dfrac{z-3}{z^2+2z+5}$ are given by $z^2 + 2z + 5 = 0$

i.e., by $$z = \frac{-2 \pm \sqrt{(4-20)}}{2} = -1 \pm 2i.$$

(i) Both the poles $z = -1 + 2i$ and $z = -1 - 2i$ lie outside the circle $|z| = 1$. Therefore, $f(z)$ is analytic everywhere within C.

Hence by Cauchy's theorem, $\displaystyle\oint_C \frac{z-3}{z^2+2z+5} dz = 0$.

(ii) Here only one pole $z = -1 + 2i$ lies inside the circle $C : |z + 1 - i| = 2$. Therefore, $f(z)$ is analytic within C except at this pole.

$$\therefore \quad \text{Res } f(-1+2i) = \underset{z \to -1+2i}{\text{Lt}} [\{z - (-1+2i)\} f(z)] = \underset{z \to -1+2i}{\text{Lt}} \frac{(z+1-2i)(z-3)}{z^2+2z+5}$$

$$= \underset{z \to -1+2i}{\text{Lt}} \frac{z-3}{z+1+2i} = \frac{-4+2i}{4i} = i + 1/2.$$

Hence by residue theorem $\displaystyle\oint_C f(z)dz = 2\pi i \text{ Res } f(-1+2i) = 2\pi i(i + 1/2) = \pi(i-2)$.

(iii) Here only the pole $z = -1 - 2i$ lies inside the circle $C : |z + 1 + i| = 2$. Therefore, $f(z)$ is analytic within C except at this pole.

$$\therefore \quad \text{Res } f(-1-2i) = \underset{z \to -1-2i}{\text{Lt}} \frac{(z+1+2i)(z-3)}{z^2+2z+5}$$

$$= \underset{z \to -1-2i}{\text{Lt}} \frac{z-3}{z+1-2i} = \frac{-4-2i}{-4i} = \tfrac{1}{2} - i$$

Hence by residue theorem, $\displaystyle\oint_C f(z)dz = 2\pi i \text{ Res } f(-1-2i) = 2\pi i \left(\tfrac{1}{2} - i\right) = \pi(2+i)$.

Example 20.36. *Evaluate* $\displaystyle\oint_C \frac{e^z}{\cos \pi z} dz$, *where C is the unit circle* $|z| = 1$. *(Rohtak, 2006)*

Solution. $f(z) = e^x/\cos \pi z$ has simple poles at $z = \pm 1/2, \pm 3/2, \pm 5/2, \ldots$

Out of these only the poles at $z = 1/2$ and $z = -1/2$ lie inside the given circle $|z| = 1$.

$$\therefore \quad \text{Res } f(1/2) = \underset{z \to 1/2}{\text{Lt}} \left[\left(z - \frac{1}{2} \right) f(z) \right] = \underset{z \to 1/2}{\text{Lt}} \left\{ \frac{\left(z - \frac{1}{2} \right) e^z}{\cos \pi z} \right\} \qquad \left[\frac{0}{0} \text{ form} \right]$$

$$= \underset{z \to 1/2}{\text{Lt}} \frac{e^z + \left(z - \frac{1}{2}\right)e^z}{-\pi \sin \pi z} = \frac{e^{1/2}}{-\pi}$$

and $$\text{Res } f(-1/2) = \underset{z \to -1/2}{\text{Lt}} \left\{ \frac{\left(z + \frac{1}{2}\right)e^z}{\cos \pi z} \right\} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \underset{z \to -1/2}{\text{Lt}} \frac{e^z + \left(z + \frac{1}{2}\right)e^z}{-\pi \sin \pi z} = \frac{e^{-1/2}}{\pi}$$

Hence $$\oint_C \frac{e^z}{\cos \pi z} dz = 2\pi i \left(\text{Res } f\left(\frac{1}{2}\right) + \text{Res } f\left(-\frac{1}{2}\right)\right\}$$

$$= 2\pi i \left(-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi}\right) = -4i \left(\frac{e^{1/2} - e^{-1/2}}{2}\right) = -4i \sinh \frac{1}{2}.$$

Example 20.37. *Evaluate* $\oint_C \tan z \, dz$ *where C is the circle* $|z| = 2$. *(V.T.U., 2010 S)*

Solution. The poles of $f(z) = \sin z/\cos z$ are given by $\cos z = 0$ *i.e.* $z = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$
Of these poles, $z = \pi/2$, and $-\pi/2$ only are within the given circle.

$$\therefore \quad \text{Res } f(\pi/2) = \underset{z \to \pi/2}{\text{Lt}} \frac{\sin z}{\frac{d}{dz}(\cos z)} = \underset{z \to \pi/2}{\text{Lt}} \left(\frac{\sin z}{-\sin z}\right) = -1 \qquad \text{[By § 20.19 (2)]}$$

Similarly $$\text{Res } f(-\pi/2) = \underset{z \to -\pi/2}{\text{Lt}} \frac{\sin z}{\frac{d}{dz}(\cos z)} = -1.$$

Hence by residue theorem,

$$\oint_C f(z) dz = 2\pi i \{\text{Res } f(\pi/2) + \text{Res } f(-\pi/2)\} = 2\pi i(-1-1) = -4\pi i.$$

Example 20.38. *Evaluate* $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz$, *where C is the circle* $|z| = 3$.
(V.T.U., 2010 ; Anna, 2003 S ; U.P.T.U., 2002)

Solution. $$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}$$

is analytic within the circle $|z| = 3$ excepting the poles $z = 1$ and $z = 2$.

Since $z = 1$ is a pole of order 2.

$$\therefore \quad \text{Res } f(1) = \frac{1}{1!}\left[\frac{d}{dz}\{(z-1)^2 f(z)\}\right]_{z=1} = \left[\frac{d}{dz}\left(\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)}\right)\right]_{z=1}$$

$$= \left[\frac{(z-2)(2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2) - (\sin \pi z^2 + \cos \pi z^2)}{(z-2)^2}\right]_{z=1}$$

$$= (-1)(-2\pi) - (-1) = 2\pi + 1$$

Also $$\text{Res } f(2) = \underset{z \to 2}{\text{Lt}} [(z-2) f(z)] = \underset{z \to 2}{\text{Lt}} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = 1$$

Hence by residue theorem,

$$\oint_C f(z)\, dz = 2\pi i [\text{Res } f(1) + \text{Res } f(2)] = 2\pi i(2\pi + 1 + 1) = 4\pi(\pi + 1)i.$$

PROBLEMS 20.8

1. Expand $f(z) = 1/[z^2(z-i)]$ as a Laurent's series about i and hence find the residue thereat.
2. Find the residue of (*i*) $ze^z/(z-1)^3$ at its pole. (*J.N.T.U., 2003*)
   (*ii*) $z^2/(z^2+a^2)$ at $z = ai$. (*P.T.U., 2009 S*)
3. Determine the poles of the following functions and the residue at each pole :
   (*i*) $\dfrac{z^2+1}{z^2-2z}$ (*ii*) $\dfrac{z^2-2z}{(z+1)^2(z^2+1)}$ (*J.N.T.U., 2005*) (*iii*) $\dfrac{2z+4}{(z+1)(z^2+1)}$ (*J.N.T.U., 2006*)
4. Find the residues of the following functions at each pole.
   (*i*) $(1-e^{2z})/z^4$ (*ii*) $ze^{iz}/(z^2+1)$ (*P.T.U., 2010*) (*iii*) $\cot z$.
5. $\oint_C \dfrac{z^2+4}{(z-2)(z+3)}\,dz$, where C is (*i*) $|z+1| = 2$ (*ii*) $|z-2| = 2$. (*Mumbai, 2006*)
6. Evaluate the following integrals :
   (*i*) $\oint_C \dfrac{e^{2z}\,dz}{(z+2)(z+4)(z+7)}$ for C as circle $|z| = 3$. (*V.T.U., 2009*)
   (*ii*) $\oint_C \dfrac{4z^2-4z+1}{(z-2)(4+z^2)}\,dz$, $C : |z| = 1$
   (*iii*) $\oint_C \dfrac{3z^2+z+1}{(z^2-1)(z+3)}\,dz$, $C : |z| = 2$. (*U.P.T.U., 2004*)
7. Evaluate
   (*i*) $\int_C \dfrac{2z+1}{(2z-1)^2}\,dz$, where C is $|z| = 1$ (*ii*) $\oint_C \dfrac{z+4}{z^2+2z+5}\,dz$, where C is $|z+1-i| = 2$
   (*iii*) $\int_C \dfrac{z^2-2z}{(z+1)^2(z^2+4)}\,dz$, where C is the circle $|z| = 10$. (*U.P.T.U., 2009*)
8. Evaluate :
   (*i*) $\oint_C \dfrac{z\,dz}{(z-1)(z-2)^2}$, $C : |z-2| = \frac{1}{2}$. (*Madras, 2006*)
   (*ii*) $\oint_C \dfrac{3z^2+2}{(z-1)(z^2+9)}\,dz$, $C : |z-2| = 2$. (*Rohtak, 2005*)
   (*iii*) $\oint_C \dfrac{dz}{(z^2+4)^2}$, $C : |z-i| = 2$. (*Hissar, 2007 ; Anna, 2003 S ; Osmania, 2003*)
9. Evaluate :
   (*i*) $\oint_C \dfrac{e^{-z}}{z^2}\,dz$, $C : |z| = 1$. (*ii*) $\oint_C z^2 e^{1/z}$, $C : |z| = 1$.
   (*iii*) $\oint_C \dfrac{e^z\,dz}{z^2+4}$, $C : |z-i| = 2$. (*V.T.U., 2006*) (*iv*) $\oint_C \dfrac{e^{2z}\,dz}{(z+1)^4}$, $C : |z| = 2$.
10. Evaluate the following integrals : (*i*) $\oint_C \dfrac{\sin^6 z}{(z-\pi/6)^3}\,dz$, $C : |z| = 1$
    (*ii*) $\oint_C \dfrac{z\sec z}{(1-z)^2}\,dz$, $C : |z| = 3$ (*iii*) $\oint_C \dfrac{z\cos z}{(z-\pi/2)^3}\,dz$, $C : |z-1| = 1$. (*V.T.U., 2007*)
11. Evaluate $\oint_C \dfrac{dz}{\sinh 2z}$ where C is the circle $|z| = 2$. (*Marathwada, 2008*)
12. Obtain Laurent's expansion for the function $f(z) = 1/z^2 \sinh z$ and evaluate
    $\oint_C \dfrac{z}{z^2\sinh z}$, where C is the circle $|z-1| = 2$. (*J.N.T.U., 2005*)

20.20 EVALUATION OF REAL DEFINITE INTEGRALS

Many important definite integrals can be evaluated by applying the Residue theorem to properly chosen integrals. The contours chosen will consist of straight lines and circular arcs.

(*a*) **Integration around the unit circle.** An integral of the type $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$, where the integrand is a rational function of $\sin\theta$ and $\cos\theta$ can be evaluated by writing $e^{i\theta} = z$.

Since $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, then integral takes the form $\int_C f(z)\, dz$, where $f(z)$ is a rational function of z and C is a unit circle $|z| = 1$.

Hence *the integral is equal to $2\pi i$ times the sum of the residues at those poles of $f(z)$ which are within C.*

Example 20.39. *Show that*

$$\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi a^2}{1 - a^2},\ (a^2 < 1).$$

(Bhopal, 2009 ; Rohtak, 2003)

Solution. Putting $z = e^{i\theta}$, $d\theta = dz/iz$, $\cos\theta = \frac{1}{2}(z + 1/z)$ and $\cos 2\theta = \frac{1}{2}(e^{2i\theta} + e^{-2i\theta}) = \frac{1}{2}(z^2 + 1/z^2)$

$\therefore$ the given integral

$$I = \int_C \frac{\frac{1}{2}(z^2 + 1/z^2)}{1 - a(z + 1/z) + a^2} \cdot \frac{dz}{iz} = \frac{1}{2i}\int_C \frac{(z^4 + 1)\, dz}{z^2(z - az^2 - a + a^2 z)}$$

$$= \frac{1}{2i}\int_C \frac{(z^4 + 1)\, dz}{z^2(z - a)(1 - az)} = \int_C f(z)\, dz \qquad \text{where } C \text{ is the unit circle } |z| = 1.$$

Now $f(z)$ has simple poles at $z = a$, $1/a$ and the second order pole at $z = 0$, of which the poles at $z = 0$ and $z = a$ lie within the unit circle.

$$\therefore \qquad \text{Res}\, f(a) = \underset{z \to a}{\text{Lt}}\, [(z - a)\, f(z)] = \frac{1}{2i}\underset{z \to a}{\text{Lt}}\left[\frac{z^4 + 1}{z^2(1 - az)}\right] = \frac{a^4 + 1}{2ia^2(1 - a^2)}$$

and

$$\text{Res}\, f(0) = \underset{z \to 0}{\text{Lt}}\, \frac{d}{dz}[z^2 f(z)] = \frac{1}{2i}\underset{z \to 0}{\text{Lt}}\, \frac{d}{dz}\left[\frac{z^4 + 1}{z - az^2 - a + a^2 z}\right]$$

$$= \frac{1}{2i}\underset{z \to 0}{\text{Lt}}\, \frac{(z - az^2 - a + a^2 z)(4z^3) - (z^4 + 1)(1 - 2az + a^2)}{(z - az^2 - a + a^2 z)^2} = -\frac{1 + a^2}{2ia^2}$$

Hence

$$I = 2\pi i\, [\text{Res}\, f(a) + \text{Res}\, f(0)] = 2\pi i\left[\frac{a^4 + 1}{2ia^2(1 - a^2)} - \frac{1 + a^2}{2ia^2}\right] = \frac{2\pi a^2}{1 - a^2}.$$

Example 20.40. *By integrating around a unit circle, evaluate* $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta$.

(S.V.T.U, 2009 ; U.P.T.U., 2009 ; Madras, 2003)

Solution. Putting $z = e^{i\theta}$, $d\theta = dz/iz$, $\cos\theta = \frac{1}{2}(z + 1/z)$

and

$$\cos 3\theta = \frac{1}{2}(e^{3i\theta} + e^{-3i\theta}) = \frac{1}{2}(z^3 + 1/z^3).$$

$\therefore$ the given integral

$$I = \int_C \frac{\frac{1}{2}(z^3 + 1/z^3)}{5 - 2(z + 1/z)} \cdot \frac{dz}{iz}$$

$$= -\frac{1}{2i}\int_C \frac{z^6 + 1}{z^3(2z^2 - 5z + 2)}\, dz = -\frac{1}{2i}\int_C \frac{(z^6 + 1)\, dz}{z^3(2z - 1)(z - 2)}$$

$$= -\frac{1}{2i}\int_C f(z)\,dz, \quad \text{where } C \text{ is the unit circle } |z| = 1.$$

Now $f(z)$ has a pole of order 3 at $z = 0$ and simple poles at $z = \frac{1}{2}$ and $z = 2$. Of these only $z = 0$ and $z = 1/2$ lie within the unit circle.

$$\therefore \quad \text{Res } f(1/2) = \underset{z\to 1/2}{\text{Lt}} \frac{(z-1/2)(z^6+1)}{(2z-1)(z-2)} = \underset{z\to 1/2}{\text{Lt}} \left\{\frac{z^6+1}{2z^3(z-2)}\right\} = -\frac{65}{24}$$

$$\text{Res } f(0) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-0)^n f(z)]\right\}_{z=0} \quad \text{where } n = 3$$

$$= \frac{1}{2}\left\{\frac{d^2}{dz^2}\left(\frac{z^6+1}{2z^2-5z+2}\right)\right\}_{z=0} = \frac{d}{dz}\left[\frac{(2z^2-5z+2)\,6z^5-(z^6+1)(4z-5)}{2(2z^2-5z+2)^2}\right] \text{ at } z = 0$$

$$= \left\{\frac{d}{dz}\left[\frac{8z^7-25z^6+12z^5-4z+5}{2(2z^2-5z+2)^2}\right]\right\}_{z=0}$$

$$= \left[\frac{(2z^2-5z+2)^2(56z^6-150z^5+60z^4-4)-(8z^7-25z^6+12z^5-4z+5)2(2z^2-5z+2)(4z-5)}{2(2z^2-5z+2)^4}\right]_{z=0}$$

$$= \frac{4(-4)-5(-20)}{2\times 16} = \frac{84}{32} = \frac{21}{8}$$

Hence
$$I = \frac{-1}{2i}\{2\pi i\,[\text{Res } f(1/2) + \text{Res } f(0)]\} = -\pi\left[-\frac{65}{24}+\frac{21}{8}\right] = -\pi\left(-\frac{1}{12}\right) = \frac{\pi}{12}.$$

(*b*) **Integration around a small semi-circle.** To evaluate $\int_{-\infty}^{\infty} f(x)\,dx$, we consider $\int_C f(z)$, where C is the contour consisting of the semi-circle $C_R : |z| = R$, together with the diameter that closes it.

Supposing that $f(z)$ has no singular point on the real axis, we have, by the Residue theorem,

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \text{Res } f(a).$$

Finally making R tend to ∞, we find the value of $\int_{-\infty}^{\infty} f(x)\,dx$, provided $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$.

Example 20.41. *Evaluate* $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$. *(U.P.T.U., 2008)*

Solution. Consider $\int_C \frac{z^2 dz}{(z^2+1)(z^2+4)} = \int_C f(z)\,dz$ ...(*i*)

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R as shown in Fig. 20.23.

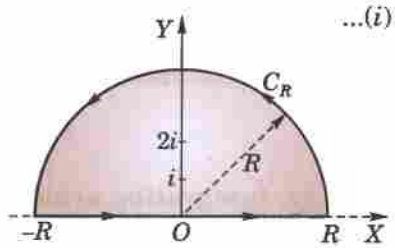

Fig. 20.23

The integrand has simple poles at $z = \pm i$, $z = \pm 2i$ of which $z = i, 2i$ only lie inside C.

$\therefore$ by the Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Res } f(i) + \text{Res } f(2i)]$$

$$= 2\pi i\,[\underset{z\to i}{\text{Lt}}\,(z-i)\,f(z) + \underset{z\to 2i}{\text{Lt}}\,(z-2i)\,f(z)]$$

$$= 2\pi i\left[\frac{i^2}{2i(i^2+4)} + \frac{4i^2}{(4i^2+1)(4i)}\right] = 2\pi i\left(\frac{i}{6}-\frac{i}{3}\right) = \frac{\pi}{3} \quad ...(ii)$$

Also $$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz \qquad ...(iii)$$

Now let $R \to \infty$, so as to show that the second integral in (*iii*) vanishes. For any point on C_R as $|z| \to \infty$

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{(1+z^{-2})(1+4z^{-2})}$$

decreases as $1/z^2$ and tends to zero whereas the length of C_R increases with z.

Consequently, $\underset{|z|\to\infty}{\text{Lt}} \int_{C_R} f(z)\,dz = 0.$

Hence from (*i*), (*ii*) and (*iii*), we get $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}$.

Example 20.42. *Evaluate* $\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx$. (*U.P.T.U., 2006 ; Delhi, 2002*)

Solution. Consider $\int_C \frac{e^{iaz}}{z^2+1}\,dz = \int_C f(z)\,dz$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R as shown in Fig. 20.23.

The integrand has simple poles at $z = i$ amd $z = -i$, of which $z = i$ only lies inside C.

$\therefore$ by Residue theorem, $\int_C f(z)\,dz = 2\pi i \,\text{Res}\, f(i) = 2\pi i \underset{z\to i}{\text{Lt}} [(z-i)\,f(z)]$

$$= 2\pi i \underset{z\to i}{\text{Lt}} \frac{(z-i)\,e^{iaz}}{z^2+1} = 2\pi i \underset{z\to i}{\text{Lt}} \frac{e^{iaz}}{z+i} = 2\pi i\,\frac{e^{-a}}{2i} = \pi e^{-a} \qquad ...(i)$$

Also $$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz \qquad ...(ii)$$

Now $\quad |z| = R$ on C_R and $|z^2+1| \ge R^2 - 1$.

Also $\quad |e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax} \cdot e^{-ay}| = e^{-ay} < 1 \qquad [\because\ y > 0]$

$$\therefore \quad \left|\frac{e^{iaz}}{z^2+1}\right| = |e^{iaz}| \cdot \frac{1}{|z^2+1|} < 1 \cdot \frac{1}{R^2-1}$$

Thus $$\int_{C_R} f(z)\,dz = \left|\int_{C_R} \frac{e^{iaz}}{z^2+1}\,dz\right| < \int_{C_R} \frac{1}{R^2-1}\,|dz| < \frac{\pi R}{R^2-1} \text{ which} \to \text{to } 0 \text{ as } R \to \infty. \qquad ...(iii)$$

Hence from (*i*), (*ii*) and (*iii*), we get

$$\pi e^{-a} = \int_{-\infty}^{\infty} f(x)\,dx + 0 \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a}$$

Equating real parts from both sides, we obtain

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}$$

Since $\cos ax/(x^2+1)$ is an even function of x, we have

$$2\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a} \quad \text{or} \quad \int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2} e^{-a}.$$

(*c*) Integration around rectangular contours

Example 20.43. *Evaluate* $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}\,dx$.

Solution. Consider $\int_C \frac{e^{az}}{e^z+1}\,dz = \int_C f(z)\,dz$ where C is the rectangle $ABCD$ with vertices at $(R, 0)$, $(R, 2\pi)$, $(-R, 2\pi)$ and $(-R, 0)$, R being positive (Fig. 20.24).

$f(z)$ has finite poles given by

$$e^z = -1 = e^{(2n+1)\pi i}$$

or $\quad z = (2n+1)\pi i$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

The only pole inside the rectangle is $z = \pi i$.

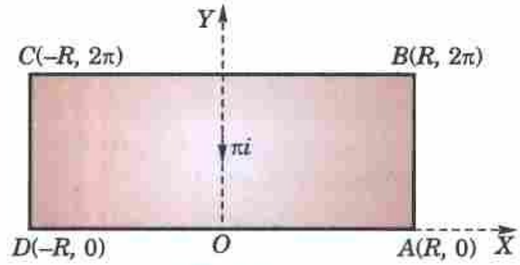

Fig. 20.24

$\therefore$ by Residue theorem,

$$\int_C f(z)dz = 2\pi i \text{ Res } f(\pi i)$$

$$= 2\pi i \left[e^{az} \Big/ \frac{d}{dz}(e^z + 1) \right]_{z=\pi i}$$

$$= 2\pi i e^{a\pi i}/e^{\pi i} = -2\pi i e^{a\pi i} \qquad [\because \; e^{\pi i} = -1] \qquad \ldots(i)$$

Also $\quad \displaystyle\int_C f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz$

$$= \int_0^{2\pi} f(R+iy)idy + \int_R^{-R} f(x+2\pi i)dx + \int_{2\pi}^0 f(-R+iy)idy + \int_{-R}^R f(x)dx$$

[$\because$ $z = R + iy$ along AB, $z = x + 2\pi i$ along BC, $z = -R + iy$ along CD and $z = x$ along DA.]

or
$$\int_C f(z)dz = i\int_0^{2\pi} \frac{e^{a(R+iy)}}{e^{R+iy}+1}dy - \int_{-R}^R \frac{e^{a(x+2\pi i)}}{e^{x+2\pi i}+1}dx - i\int_0^{2\pi} \frac{e^{a(-R+iy)}}{e^{-R+iy}+1}dy + \int_{-R}^R \frac{e^{ax}}{e^x+1}dx \qquad \ldots(ii)$$

Now for any two complex numbers z_1, z_2

$$|z_1| \geq |z_2|, \text{ we have } |z_1 + z_2| \geq |z_1| - |z_2|$$

so that $\quad |e^{R+iy} + 1| \geq e^R - 1$. Also $|e^{a(R+iy)}| = e^{aR}$

$\therefore$ for the integrand of first integral in (ii), we have

$$\left| \frac{e^{a(R+iy)}}{e^{R+iy}+1} \right| \leq \frac{e^{aR}}{e^R - 1} \text{ which} \to 0 \text{ as } R \to \infty. \qquad [\because \; a > 1]$$

Similarly, for the integrand of the third integral in (ii), we get

$$\left| \frac{e^{a(-R+iy)}}{e^{-R+iy}+1} \right| \leq \frac{e^{-aR}}{1 - e^{-R}} \text{ which also} \to 0 \text{ as } R \to \infty. \qquad [\because \; a < 0]$$

Hence as $R \to \infty$, since the first and third integrals in (ii) approach zero, we get

$$\int_C f(z)dz = -e^{2a\pi i}\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx + \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = \left(1 - e^{2a\pi i}\right)\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx \qquad \ldots(iii)$$

Thus from (i) and (iii), we obtain $(1 - e^{2a\pi i}) \displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = -2\pi i e^{a\pi i}$

$\therefore$ equating real parts, we get $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = \frac{\pi}{\sin a\pi}$.

Example 20.44. *Show that* $\displaystyle\int_0^{\infty} e^{-x^2} \cos 2mx \, dx = \frac{1}{2}\sqrt{\pi e^{-m^2}}$.

Solution. Integrate $f(z) = e^{-z^2}$ along the rectangle ABCDA having vertices $A(-l)$, $B(l)$, $C(l + im)$, $D(-l + im)$ (Fig. 20.25). $f(z)$ has no poles inside this contour. As such

$$\int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz = 0 \qquad \ldots(i)$$

On $AB : z = x$, on $BC : z = l + iy$, on $CD : z = x + im$ and on $DA : z = -l + iy$.

Therefore, (i) becomes

$$\int_{-l}^{l} e^{-x^2} dx + \int_0^m e^{-(l+iy)^2} idy + \int_l^{-l} e^{-(x+im)^2} dx + \int_m^0 e^{-(-l+iy)^2} dy = 0$$

or
$$\int_{-l}^{l} e^{-x^2} dx - \int_{-l}^{l} e^{-x^2 - 2imx + m^2} dx + \int_0^m e^{-l^2 - 2ily + y^2} . idy$$

$$- \int_0^m e^{-l^2 + 2ily + y^2} . idy = 0 \qquad \ldots(ii)$$

Now let $l \to \infty$. Then the last two integrals

$$= ie^{-l^2} \int_0^m e^{y^2} (e^{-2ily} - e^{2ily})\, dy = 2e^{-l^2} \int_0^m e^{y^2} \sin 2ly\, dy \to 0$$

[$\because$ As $l \to \infty$, $e^{-l^2} \to 0$ and $\sin 2ly$ is finite]

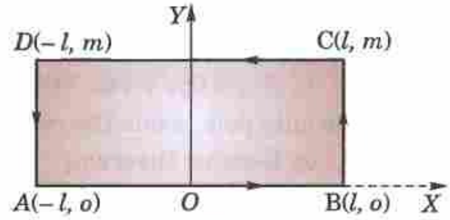

Fig. 20.25

Hence (*ii*) reduces to

$$\int_{-\infty}^{\infty} e^{-x^2} dx - e^{m^2} \int_{-\infty}^{\infty} e^{-x^2} (\cos 2mx - i \sin 2mx) dx = 0$$

Equating real parts, we get

$$e^{m^2} \int_{-\infty}^{\infty} e^{-x^2} \cos 2mx\, dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \qquad \text{[See p. 289]}$$

or
$$\int_0^{\infty} e^{-x^2} \cos 2mx\, dx = \frac{1}{2}\sqrt{\pi e^{-m^2}}$$

(*d*) **Indenting the contours having poles on the real axis.** So far we have considered such cases in which there is no pole on the real axis. When the integrand has a simple pole on the real axis, we delete it from the region by indenting the contour (*i.e.*, by drawing a small semi-circle having the pole for the centre). The method will be clear from the following example.

Example 20.45. *Evaluate* $\int_0^{\infty} \frac{\sin mx}{x} dx$, *when* $m > 0$. (*U.P.T.U., 2007*)

Solution. Consider the integral $\int_C \frac{e^{miz}}{z} dz = \int_C f(z) dz$ where C consists of

(*i*) the real axis from r to R,

(*ii*) the upper half of the circle $C_R : | z | = R$,

(*iii*) the real axis $-R$ to $-r$,

(*iv*) the upper half of the circle $C_r : | z | = r$ (Fig. 20.26).

Since $f(z)$ has no singularity inside C (its only singular point being a simple pole at $z = 0$ which has been deleted by drawing C_r), we have by Cauchy's theorem :

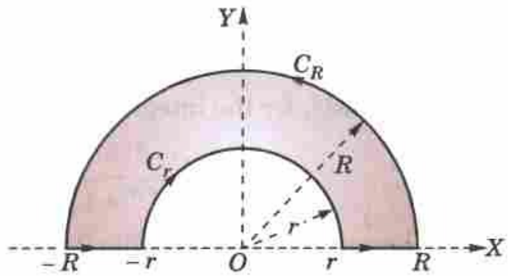

Fig. 20.26

$$\int_r^R f(x) dx + \int_{C_R} f(z) dz + \int_{-R}^{-r} f(x) dx + \int_{C_r} f(z) dz = 0 \qquad ...(i)$$

Now
$$\int_{C_R} f(z) dz = \int_0^{\pi} \frac{e^{imR(\cos\theta + i \sin\theta)}}{Re^{i\theta}} . Rie^{i\theta} d\theta \qquad [\because \quad z = Re^{i\theta}]$$

$$= i \int_0^{\pi} e^{imR(\cos\theta + i\sin\theta)} d\theta$$

Since $| e^{imR(\cos\theta + i\sin\theta)} | = | e^{-mR\sin\theta + imR\cos\theta} | = e^{-mR\sin\theta}$

$$\therefore \quad \left| \int_{C_R} f(z) dz \right| \le \int_0^{\pi} e^{-mR\sin\theta} d\theta = 2\int_0^{\pi/2} e^{-mR\sin\theta} d\theta$$

$$= 2\int_0^{\pi/2} e^{-2mR\theta/\pi} d\theta \qquad [\because \quad \text{for } 0 \le \theta \le \pi/2,\ \sin\theta/\theta \ge 2/\pi]$$

$$= \frac{\pi}{mR}\left(1 - e^{-mR}\right) \text{ which} \to 0 \text{ as } R \to \infty,$$

Also
$$\int_{C_r} f(z) dz = i\int_{\pi}^{0} e^{imr(\cos\theta + i\sin\theta)} d\theta \to i\int_{\pi}^{0} d\theta \ \ i.e., -i\pi \text{ as } r \to 0.$$

Hence as $r \to 0$ and $R \to \infty$, we get from (*i*) $\int_0^{\infty} f(x) dx + 0 + \int_{-\infty}^{0} f(x) dx - i\pi = 0$

or
$$\int_{-\infty}^{\infty} f(x) dx = i\pi \ \ i.e., \int_{-\infty}^{\infty} \frac{e^{imx}}{x} dx = i\pi \qquad ...(ii)$$

Equating imaginary parts from both sides,

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = \pi. \text{ Hence } \int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}.$$

Obs. Equating real parts from both sides of (*ii*), we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x} dx = 0.$$

Example 20.46. *Show that* $\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin px}$, $0 < p < 1$.

Solution. Integrate $f(z) = \frac{z^{p-1}}{1+z}$ along the contour consisting of the circles α and γ of radii a and R and the lines AB and FG along x-axis (Fig. 20.27). There is a simple pole at $z = -1$ which is within the contour.

$$\therefore \quad \text{Res}\, f(-1) = \underset{z \to -1}{\text{Lt}} (1+z) \cdot \frac{z^{p-1}}{1+z} = \underset{z \to -1}{\text{Lt}} z^{p-1} = (-1)^{p-1} = e^{i\pi(p-1)}$$

Thus $$\int_{AB} f(z)dz + \int_\gamma f(z)dz + \int_{FG} f(z)dz + \int_\alpha f(z)dz = 2\pi i\, e^{i\pi(p-1)} \quad ...(i)$$

On $AB : z = x$ and on $FG : z = xe^{2\pi i}$

$$\therefore \quad \int_{AB} f(z)dz + \int_{FG} f(z)dz = \int_a^R \frac{x^{p-1}}{1+x} dx + \int_R^a \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}} dx\, e^{2\pi i}$$

$$= \int_a^R \frac{x^{p-1}}{1+x} [1 - e^{2\pi i(p-1)}]\, dx$$

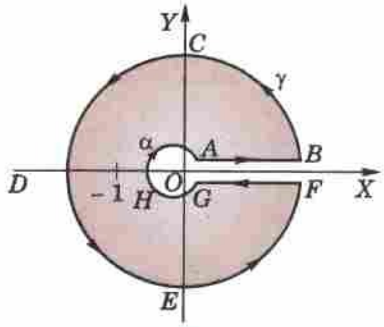

Fig. 20.27

On the circle $\gamma : z = Re^{i\theta}$. So

$$\int_\gamma f(z)dz = \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1}}{1+Re^{i\theta}} Re^{i\theta}\, i\, d\theta$$

For large R, the integrand is of the order $\frac{R^{p-1} \cdot R}{1+R}$ *i.e.* R^{p-1} which tends to zero as $R \to \infty$. $(\because \; p < 1)$

Hence $$\int_\gamma f(z)dz \to 0 \text{ as } R \to \infty$$

On the circle $\alpha : z = ae^{i\theta}$. So

$$\int_\alpha f(z)dz = \int_{2\pi}^0 \frac{(ae^{i\theta})^{p-1}}{1+ae^{i\theta}} ae^{i\theta}\, i d\theta$$

For small α, the integrand is of the order a^p which tends to zero as $a \to 0$. $(\because \; p > 0)$

Thus on taking limits as $a \to 0$ and $R \to \infty$, (*i*) gives

$$\int_0^\infty \frac{x^{p-1}}{1+x} \{1 - e^{2\pi i(p-1)}\} dx = 2\pi i\, e^{i\pi(p-1)}$$

or $$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{2\pi i\, e^{i\pi(p-1)}}{1 - e^{2\pi i(p-1)}} = \frac{2\pi i\, e^{ip\pi}(-1)}{1 - e^{2ip\pi}\,(1)} = \frac{2i \cdot \pi}{e^{ip\pi} - e^{-ip\pi}} = \frac{\pi}{\sin p\pi}.$$

Example 20.47. *Prove that* $\int_0^\infty \sin x^2\, dx = \int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)}$. *(Osmania, 2003)*

Solution. Consider $\int_C e^{-z^2} dz$ where C consists of the real axis from O to A, part of circle AB of radius R and the line $\theta = \frac{\pi}{4}$. (Fig. 20.28).

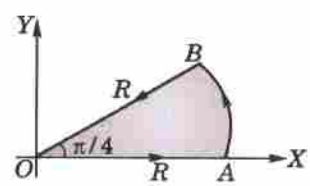

Fig. 20.28

e^{-z^2} has no singularity within C.

$$\therefore \quad \int_{OA} e^{-z^2} dz + \int_{AB} e^{-z^2} dz + \int_{BO} e^{-z^2} dz = 0 \quad ...(i)$$

On $OA : z = x$, $\therefore \int_{OA} e^{-z^2}\, dz = \int_0^R e^{-x^2} dx \to \sqrt{\pi/2}$ as $R \to \infty$ [See p. 289]

On $AB : z = Re^{i\theta}$,

$$\therefore \quad \int_{AB} e^{-z^2}\, dz = \int_0^{\pi/4} e^{-R^2 (\cos 2\theta + i \sin 2\theta)} . Re^{i\theta} . id\theta \to 0 \quad \text{as} \quad R \to \infty.$$

[$\because$ integrand $\to 0$ as $R \to \infty$]

On $BO : z = re^{i\pi/4}$ and $z^2 = r^2 e^{i\pi/2} = ir^2$

$$\therefore \quad \int_{BO} e^{-z^2}\, dz = \int_R^0 e^{-ir^2} . e^{i\pi/4} dr = -\int_0^R e^{-ix^2} \frac{1+i}{\sqrt{2}} dx$$

$$\to -\int_0^\infty (\cos x^2 - i \sin x^2) \frac{1+i}{\sqrt{2}} dx \qquad \text{when } R \to \infty$$

Substituting these in (i), we get

$$\frac{1}{2}\sqrt{\pi} + 0 - \int_0^\infty (\cos x^2 - i \sin x^2) \left(\frac{1+i}{\sqrt{2}}\right) dx = 0$$

Equating real and imaginary parts, we obtain

$$\int_0^\infty (\cos x^2 + \sin x^2) dx = \frac{1}{2}\sqrt{(2\pi)} \quad \text{and} \quad \int_0^\infty (\cos x^2 - \sin x^2) dx = 0$$

Hence $\int_0^\infty \sin x^2\, dx = \int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)}$.

PROBLEMS 20.9

Apply the calculus of residues, to prove that

1. $\int_0^{2\pi} \frac{d\theta}{1 - 2p \sin\theta + p^2} = \frac{2\pi}{1-p^2} (0 < p < 1)$. *(Hissar, 2007 ; Mumbai, 2006 ; Kerala, 2005)*
2. $\int_0^{2\pi} \frac{d\theta}{1 - 2r \cos\theta + r^2} = \frac{\pi}{1-r^2}$. *(J.N.T.U., 2006 ; Madras, 2006 ; Anna, 2003)*
3. $\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = \frac{2\pi}{\sqrt{(a^2-1)}} (a > 1)$. *(P.T.U., 2010)*
4. $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta = \frac{\pi}{6}$. *(U.P.T.U., 2010)*
5. $\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2} [a - \sqrt{(a^2 - b^2)}], (0 < b < a)$. *(J.N.T.U., 2003)*
6. $\int_0^{2\pi} \frac{a d\theta}{a^2 + \sin^2\theta} = \frac{2\pi}{\sqrt{(1+a^2)}}, (a > 0)$. *(S.V.T.U., 2009)*
7. $\int_0^{2\pi} \frac{d\theta}{(5 - 3\cos\theta)^2} = \frac{5\pi}{32}$.
8. $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b} (a, b > 0)$. *(P.T.U., 2007 ; Mumbai, 2006 ; Anna, 2003)*
9. $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$. *(A.M.I.E.T.E., 2003 ; Delhi, 2002)*
10. $\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$. *(J.N.T.U., 2006)*
11. $\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$. *(Madras, 2006 ; Kerala, 2005)*
12. $\int_0^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}$. *(Kerala, 2005)*
13. $\int_0^{\infty} \frac{x^2}{x^6+1} dx = \frac{\pi}{6}$. *(Rohtak, 2006)*
14. $\int_{-\infty}^{\infty} \frac{\cos mx}{e^x + e^{-x}} dx = \frac{\pi}{2} \operatorname{sech} \frac{m\pi}{2}$.
15. $\int_0^{\infty} \frac{1 - \cos x}{x^2} dx = \pi/2$. *(P.T.U., 2005)*
16. $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$. *(Kerala, 2005)*
17. $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} dx = \frac{-\pi \sin 2}{e}$.
18. $\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) (a > b > 0)$.
19. $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{\pi}{2} e^{-a}$.

20.12. OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 20.10

Select the correct answer or fill up the blanks in each of the following questions :

1. The only function that is analytic from the following is
(*i*) $f(z) = \sin z$ (*ii*) $f(z) = \bar{z}$ (*iii*) $f(z) = \text{Im}(z)$ (*iv*) $R(iz)$.

2. If $f(z) = u(x, y) + iv(x, y)$ is analytic, then $f'(z) =$
(*i*) $\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}$ (*ii*) $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (*iii*) $\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}$.

3. If $2x - x^2 + ay^2$ is to be harmonic, then a should be
(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 0.

4. The analytic function which maps the angular region $0 \le \theta \le \pi/4$ onto the upper half plane is
(*i*) z^2 (*ii*) $4z$ (*iii*) z^4 (*iv*) 2θ.

5. An angular domain in the complex plane is defined by $0 < \text{amp}(z) < \pi/4$. The mapping which maps this region onto the left half plane is
(*i*) $w = z^4$ (*ii*) $w = iz^4$ (*iii*) $w = -z^4$ (*iv*) $w = -iz^4$.

6. The mapping $w = z^2 - 2z - 3$ is
(*i*) conformal within $|z| = 1$ (*ii*) not conformal at $z = 1$
(*iii*) not conformal at $z = -1$ and $z = 3$ (*iv*) conformal everywhere.

7. If $z = re^{i\theta}$, then the image of θ = constant under the mapping $w(z) = Re^{i\phi} = iz^3$ is
(*i*) $\phi = 3\theta$ (*ii*) $\phi = 3\theta + \pi/2$ (*iii*) $\phi = 3\theta - \pi/2$ (*iv*) $\phi = \theta^3$.

8. The fixed points of the mapping $w = (5z + 4)/(z + 5)$ are
(*i*) 2, 2 (*ii*) 2, – 2 (*iii*) – 2, – 2 (*iv*) – 4/5, 5.

9. The value of $\int_C (4x^3\,dx + 3y^2z^2\,dy + 2y^3 z\,dz)$ where C is any path joining $A(-1, 1, 0)$ to $B(1, 2, 1)$ is
(*i*) 0 (*ii*) 1 (*iii*) 8 (*iv*) – 8.

10. The value of $\int_C \frac{3z^2 + 7z + 1}{z + 1}\,dz$ where C is $|z| = 1/2$ is
(*i*) $2\pi i$ (*ii*) 0 (*iii*) πi (*iv*) $\pi i/2$.

11. The value of $\int_C \frac{3z + 4}{z(2z + 1)}\,dz$ where C is the circle $|z| = 1$ is
(*i*) $2\pi i$ (*ii*) $3\pi i$ (*iii*) 4 (*iv*) – 4.

12. The residue of a function can be found if the pole is an isolated singularity :
(*i*) True (*ii*) False (*iii*) Partially false (*iv*) none of these.

13. The value of $\int_C \frac{z\,dz}{\sin z}$ where $C : |z| = 4$ is
(*i*) $2\pi i$ (*ii*) 0 (*iii*) $-2\pi i$ (*iv*) $4\pi i$.

14. The value of $\int_C \tanh z\,dz$, where $C : |z| = 3$, is
(*i*) 0 (*ii*) πi (*iii*) $2\pi i$ (*iv*) $4\pi i$.

15. The harmonic conjugate of the function $u(x, y) = 2x(1 - y)$ is ...... (*U.P.T.U., 2009*)

16. Harmonic conjugate of $x^3 - 3xy^2$ is ....

17. The curves $u(x, y) = c$ and $v(x, y) = c'$ are orthogonal if ......

18. The value of $\int_0^{1+i} z^2\,dz$ along the line $x = y$ is ......

19. Residue of $\frac{\cos z}{z}$ at $z = 0$ is ......

20. The critical point of the transformation $w^2 = (z - a)(z - b)$ is ......

21. Image of $|z + 1| = 1$ under the mapping $w = 1/z$ is......

22. The poles of $f(z) = (z^3 - 1)/(z^3 + 1)$ are $z =$ ......

23. $w = \log z$ is analytic everywhere except at $z =$ ......

24. If $f(z) = -\frac{1}{z - 1} - 2[1 + (z - 1) + (z - 1)^2 + \ldots\ldots]$, then the residue of $f(z)$ at $z = 1$ is ......

25. If $|z| < 1$ then Taylor's series expansion of $\log(1 + z)$ about $z = 0$ is ......

26. The value of $\int_C \frac{4z^2+z+5}{z-4}dz$ where C is $9x^2+4y^2=36$, is ......
27. The value of $\int z^4 e^{1/z}\,dz$, where C is $|z|=1$, is
(i) πi (ii) $\pi i/12$ (iii) $\pi i/60$ (iv) $-\pi i/60$.
28. If $f(z)$ has a pole of order three at $z=a$ Res $[f(a)]=$ ......
29. The value of $\int_C \frac{e^z\,dz}{(z-3)^2}$, C being $|z|=2$, is ......
30. The CR equations for $f(z)=u(x,y)+iv(x,y)$ to be analytic are ......
31. If $f(z)$ is analytic in a simply connected domain D and C is any simple closed path then $\int_C f(z)\,dz=$ ......
32. The harmonic conjugate of $e^x \cos y$ is ......
33. The value of $\oint_C \cos z\,dz$ where C is the circle $|z|=1$, is ......
34. The singularity of $f(z)=z/(z-2)^3$ is ......
35. The function $f(z)=\bar{z}$ is analytic at ......
36. C-R equations for a function to be analytic, in polar form, are ......
37. If C is the circle $|z-a|=r$, $\int_C (z-a)^n\,dz$ [n, any integer $\neq -1$] $=$ ......
38. A simply connected region is that ......
39. A holomorphic function is that ......
40. The poles of the function $f(z)=\frac{z^2}{(z-1)^2(z+2)}$ are at $z=$ ......
41. The cross-ratio of four points z_1, z_2, z_3, z_4 is ......
42. The value of $\int_C |z|\,dz$, where C is the contour represented by the straight line from $z=-i$ to $z=i$, is ......
43. Taylor's series expansion of $\left(\frac{1}{z-2}-\frac{1}{z-1}\right)$ in the region $|z|<1$, is ......
44. The invariant points of the transformation $w=(1+z)/(1-z)$ are $z=$ ......
45. The residue at $z=0$ of $\frac{1+e^z}{z\cos z+\sin z}$ is ......
46. The transformation $w=Cz$ consists of ......
47. The residue of $f(z)$ at a pole is ......
48. The value of $\int_C \frac{1}{z-1}dz$, C being $|z|=2$, is ......
49. If C is $|z|=1/2$, $\int_C \frac{z^2-z+1}{z-1}dz=$ ......
50. Singular points of $\frac{\cos \pi z}{(z-1)(z-2)}$ are ......
51. Taylor series expansion of $\frac{1}{z-2}$ in $|z|<1$ is ......
52. $\underset{z\to\infty}{\text{Lt}} \frac{iz^3+iz-1}{(2z+3)(z-1)^2}=$ .... (P.T.U., 2007)
53. The poles of $\frac{(z-1)^2}{z(z-2)^2}$ are at $z=$ ......
54. Cauchy's integral theorem states that ......
55. The critical points of the transformation $w=z+1/z$ are ......
56. $\int_c \frac{dz}{2z-3}$, where $|z|=1$, is ....
57. The zeroes and singularities of $\frac{z^2+1}{1-z^2}$ are ......
58. Residue of $\tan z$ at $z=\pi/2$ is ......
59. Singularity of $e^{z^{-1}}$ at $z=0$ is of the type ....
60. Res $(e^{1/z})_{z=0}=$ ......
61. Taylor's series expansion of $\sin z$ about $z=\pi/4$ is ......
62. Image of $|z|=2$ under $w=z+3+2i$ is ....
63. The poles of $\cot z$ are ....
64. If a is simple pole, then Res $[\phi(z)/\psi(z)]_{z=a}=$ ....
65. Bilinear transformation always transforms circles into ....
66. If $f(z)$ and $\overline{f(z)}$ are analytic functions, then $f(z)$ is constant. (True or False) (Mumbai, 2006)
67. The function $u(x,y)=2xy+3xy^2-2y^3$ is a harmonic functions. (True or False) (P.T.U., 2009 S)
68. The function $e^x \cos y$ is harmonic. (True or False)

69. $\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$, if $z = a$ is a point within C. (True or False)

70. The transformation affected by an analytic function $w = f(z)$ is conformal at every point of the z-plane where $f'(z) = 0$. (True or False)

71. The function $\bar{z}$ is not analytic at any point. (True or False)

72. Under the transformation $w = 1/z$, circle $x^2 + y^2 - 6x = 0$ transforms into a straight line in the w-plane. (True or False)

73. If $w = f(z)$ is analytic, then $\frac{dw}{dz} = -i \frac{\partial w}{\partial y}$. (True or False)

74. An analytic function with constant imaginary part is constant. (True or False)

75. If $u + iv$ is analytic, then $v - iu$ is also analytic. (True or False)

76. $f(z) = I_m(z)$ is not analytic. (True or False)

77. The cross-ratio of four points is not invariant under bilinear transformation. (True or False)

78. $z = 0$ is not a critical point of the mapping $w = z^2$. (True or False)

79. $f(z) = \text{Re}\,(z^2)$ is analytic. (True or False)

80. An analytic function with constant modulus is constant. (True or False)

81. The function $|\bar{z}|^2$ is not analytic at any point. (True or False)

82. If $f(z) = z^2$, then the family of curves $x^2 - y^2 = C_1$, and $xy = C_2$ are orthogonal. (True or False)