CHAPTER 32

Numerical Solution of Ordinary Differential Equations

32.1 INTRODUCTION

The methods of solution so far presented are applicable to a limited class of differential equations. Frequently differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realise that computing machines are now available which reduce numerical work considerably.

A number of numerical methods are available for the solution of first order differential equations of the form :

$$\frac{dy}{dx} = f(x, y), \text{ given } y(x_0) = y_0 \qquad ...(1)$$

These methods yield solutions either as a power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y. The methods of Picard and Taylor series belong to the former class of solutions whereas those of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class. In these later methods, the values of y are calculated in short steps for equal intervals of x and are therefore, termed as *step-by-step methods.*

Euler and Runge-Kutta methods are used for computing y over a limited range of x-values whereas Milne and Adams-Bashforth methods may be applied for finding y over a wider range of x-values. These later methods require starting values which are found by Picard's or Taylor series or Runge-Kutta methods.

The initial condition in (1) is specified at the point x_0. *Such problems in which all the initial conditions are given at the initial point only are called* ***initial value problems.*** *But there are problems involving second and higher order differential equations in which the conditions may be given at two or more points. These are known as* ***boundary value problems.*** In this chapter, we shall first explain methods for solving initial value problems and then give a method of solving boundary value problems.

32.2 PICARD'S METHOD*

Consider the first order equation $dy/dx = f(x, y)$ ...(1)

* Called after the French mathematician *Emile Picard* (1856—1941) who was professor in Paris since 1881 and is famous for his researches in the theory of functions.

It is required to find that particular solution of (1) which assumes the value y_0 when $x = x_0$. Integrating (1) between limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\,dx \quad \text{or} \quad y = y_0 + \int_{x_0}^{x} f(x, y)\,dx \qquad ...(2)$$

This is an integral equation equivalent to (1), for it contains the unknown y under the integral sign.

As a first approximation y_1 *to the solution*, we put $y = y_0$ in $f(x, y)$ and integrate (2), giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

For a second approximation y_2, we put $y = y_1$ in $f(x, y)$ and integrate (2), giving

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx.$$

Similarly, *a third approximation is* $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx.$

Continuing this process, a sequence of functions of x, *i.e.*, y_1, y_2, y_3 ... is obtained each giving a better approximation of the desired solution than the preceding one.

Obs. Picard's method is of considerable theoretical value, but can be applied only to a limited class of equations in which the successive integrations can be performed easily. The method can be extended to simultaneous equations and equations of higher order (See § 32.11 and 32.12).

Example 32.1. *Using Picard's process of successive approximation, obtain a solution upto the fifth approximation of the equation $dy/dx = y + x$, such that $y = 1$ when $x = 0$. Check your answer by finding the exact particular solution.*

Solution. (*a*) We have $y = 1 + \int_0^x (y + x)\,dx.$

First approximation. Put $y = 1$, in $y + x$, giving

$$y_1 = 1 + \int_0^x (1 + x)\,dx = 1 + x + x^2/2.$$

Second approximation. Put $y = 1 + x + x^2/2$ in $y + x$, giving

$$y_2 = 1 + \int_0^x (1 + 2x + x^2/2)\,dx = 1 + x + x^2 + x^3/6.$$

Third approximation. Put $y = 1 + x + x^2 + x^3/6$ in $y + x$, giving

$$y_3 = 1 + \int_0^x (1 + 2x + x^2 + x^3/6)\,dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

Fourth approximation. Put $y = y_3$ in $y + x$, giving

$$y_4 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}.$$

Fifth approximation. Put $y = y_4$ in $y + x$, giving

$$y_5 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720} \qquad ...(i)$$

(*b*) Given equation :

$$\frac{dy}{dx} - y = x \text{ is a Leibnitz's linear in } x.$$

Its I.F. being e^{-x}, the solution is

$$ye^{-x} = \int xe^{-x}\,dx + c = -xe^{-x} - \int (-e^{-x})\,dx + c = -xe^{-x} - e^{-x} + c \qquad \text{[Integrate by parts]}$$

$$\therefore \qquad y = ce^x - x - 1.$$

Since $y = 1$, when $x = 0$, $\therefore$ $c = 2$.

Thus the desired particular solution is $y = 2e^x - x - 1$ ...(*ii*)

Or using the series : $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dots \infty,$

we get $$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360} + \dots \infty \qquad \text{...}(iii)$$

Comparing (*i*) and (*iii*), it is clear that (*i*) approximates to the exact particular solution (*ii*) upto the term in x^5.

Obs. At $x = 1$, the fourth approximation $y_4 = 3.433$ and the fifth approximation $y_5 = 3.434$ whereas exact value is 3.44.

Example 32.2. *Find the value of y for x = 0.1 by Picard's method, given that*

$$\frac{dy}{dx} = \frac{y - x}{y + x}, \; y(0) = 1. \qquad \text{(P.T.U., 2002)}$$

Solution. We have $y = 1 + \displaystyle\int_0^x \frac{y - x}{y + x}\, dx$

First approximation. Put $y = 1$ in the integrand, giving

$$y_1 = 1 + \int_0^x \frac{1 - x}{1 + x}\, dx = 1 + \int_0^x \left(-1 + \frac{2}{1 + x} \right) dx$$

$$= 1 + \left[-x + 2 \log (1 + x) \right]_0^x = 1 - x + 2 \log (1 + x) \qquad \text{...}(i)$$

Second approximation. Put $y = 1 - x + 2 \log (1 + x)$ in the integrand, giving

$$y_2 = 1 + \int_0^x \frac{1 - x + 2 \log (1 + x) - x}{1 - x + 2 \log (1 + x) + x}\, dx = 1 + \int_0^x \left[1 - \frac{2x}{1 + 2 \log (1 + x)} \right] dx$$

which is very difficult to integrate.

Hence we use the first approximation and taking $x = 0.1$ in (*i*) we obtain

$$y(0.1) = 1 - (.1) + 2 \log 1.1 = 0.9828.$$

32.3 TAYLOR'S SERIES METHOD*

Consider the first order equation $dy/dx = f(x, y)$ ...(1)

Differentiating (1), we have

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} \qquad i.e., \quad y'' = f_x + f_y f' \qquad \text{...}(2)$$

Differentiating this successively, we can get y''', y^{iv} etc. Putting $x = x_0$ and $y = 0$, the values of $(y')_0$, $(y'')_0$, $(y''')_0$ can be obtained. Hence the Taylor's series

$$y(x) = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \dots \qquad \text{...}(3)$$

gives the values of y for every value of x for which (3) converges.

On finding the value y_1 for $x = x_1$ from (3), y', y'' can be evaluated at $x = x_1$ by means of (1), (2) etc. Then y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (3).

Example 32.3. *Find by Taylor's series method the value of y at x = 0.1 and x = [illegible] to five places of decimals from $dy/dx = x^2y - 1$, $y(0) = 1$.* (V.T.U., 2009 ; [illegible]htak, 2005)

Solution. Here $(y)_0 = 1$, $y' = x^2y - 1$, $(y')_0 = -1$

$\therefore$ Differentiating successively and substituting, we get

$$\begin{aligned} y'' &= 2xy + x^2y', & (y'')_0 &= 0 \\ y''' &= 2y + 4xy' + x^2y'', & (y''')_0 &= 2 \\ y^{iv} &= 6y' + 6xy'' + x^2y''', & (y^{iv})_0 &= -6 \text{ etc.} \end{aligned}$$

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*See footnote p. 145.

Putting these values in the Taylor's series,

$$y(x) = y_0 + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + ...,$$

we have $$y(x) = 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + ... = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

Hence $y(0.1) = 0.90033$ and $y(0.2) = 0.80227$.

Example 32.4. *Employ Taylor's method to obtain approximate value of y at $x = 0.2$ for the differential equation $dy/dx = 2y + 3e^x$, $y(0) = 0$. Compare the numerical solution obtained with the exact solution.* (V.T.U., 2009 ; P.T.U., 2003)

Solution. (*a*) We have $y' = 2y + 3e^x$ $\qquad y'(0) = 2y(0) + 3e^0 = 3.$

Differentiating successively and substituting $x = 0, y = 0$, we get

$$y'' = 2y' + 3e^x, \qquad y''(0) = 2y'(0) + 3 = 9$$
$$y''' = 2y'' + 3e^x, \qquad y'''(0) = 2y''(0) + 3 = 21$$
$$y^{iv} = 2y''' + 3e^x, \qquad y^{iv}(0) = 2y'''(0) + 3 = 45 \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + ...$$

$$= 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + ... = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + ...$$

Hence $\quad y(0.2) = 3(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 + 1.875(0.4)^4 + ... = 0.8110$ ...(*i*)

(*b*) Now $\dfrac{dy}{dx} - 2y = 3e^x$ is a Leibnitz's linear in x.

Its I.F. being e^{-2x}, the solution is

$$ye^{-2x} = \int 3e^x \cdot e^{-2x}\,dx + c = -3e^{-x} + c \quad \text{or} \quad y = -3e^x + ce^{2x}$$

Since $y = 0$ when $x = 0$, $\therefore$ $c = 3$.

Thus the exact solution is $y = 3(e^{2x} - e^x)$

When $x = 0.2$, $y = 3(e^{0.4} - e^{0.2}) = 0.8112$ ...(*ii*)

Comparing (*i*) and (*ii*), it is clear that (*i*) approximates to the exact value upto 3 decimal places.

Example 32.5. *Solve by Taylor's series method the equation $\dfrac{dy}{dx} = \log(xy)$ for $y(1.1)$ and $y(1.2)$, given $y(1) = 2$.* (Hazaribagh, 2009)

Solution. We have $y' = \log x + \log y$; $\quad y'(1) = \log 2$

Differentiating w.r.t. x and substituting $x = 1, y = 2$, we get

$$y'' = \frac{1}{x} + \frac{1}{y}y' \; ; \; y''(1) = 1 + \frac{1}{2}\log 2$$

$$y''' = -\frac{1}{x^2} + \frac{1}{y} + y'' + y'\left(-\frac{1}{y1}\right)y' \; ; \; y'''(1) = -1 + \frac{1}{2}\left(1 + \frac{1}{2}\log 2\right) - \frac{1}{4}(\log 2)^2$$

Substituting these values in the Taylor's series about $x = 1$, we have

$$y(x) = y(1) + (x-1)y'(1) + \frac{(x-1)^2}{2!}y''(1) + \frac{(x-1)^3}{3!}y'''(1) + ...$$

$$= 2 + (x-1)\log 2 + \frac{1}{2}(x-1)^2\left(1 + \frac{1}{2}\log 2\right) + \frac{1}{6}(x-1)^3\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right]$$

$$\therefore \quad y(1.1) = 2 + (0.1)\log 2 + \frac{(0.1)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(0.1)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right] = 2.036$$

$$y(1.2) = 2 + (0.2) \log 2 + \frac{(0.2)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(0.2)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right] = 2.081.$$

PROBLEMS 32.1

1. Using Picard's method, solve $dy/dx = -xy$ with $x_0 = 0, y_0 = 1$ upto third approximation. *(Mumbai, 2005)*
2. Employ Picard's method to obtain, correct to four places of decimal, solution of the differential equation $dy/dx = x^2 + y^2$ for $x = 0.4$, given that $y = 0$ when $x = 0$. *(J.N.T.U., 2009)*
3. Obtain Picard's second approximate solution of the initial value problem : $y' = x^2/(y^2 + 1), y(0) = 0$. *(Marathwada, 2008)*
4. Find an approximate value of y when $x = 0.1$, if $dy/dx = x - y^2$ and $y = 1$ at $x = 0$, using
   (*a*) Picard's method (*b*) Taylor's series. *(V.T.U., 2010 ; Madras, 2006)*
5. Solve $y' = x + y$ given $y(1) = 0$. Find $y(1.1)$ and $y(1.2)$ by Taylor's method. Compare the result with its exact value. *(J.N.T.U., 2008 ; Anna, 2005)*
6. Evaluate $y(0.1)$ correct to six places of decimals by Taylor's series method if $y(x)$ satisfies
   $y' = xy + 1, y(0) = 1.$
7. Solve $y' = 3x + y^2, y(0) = 1$ using Taylor's series method and computer $y(0.1)$. *(Mumbai, 2007)*
8. Using Taylor series method, find $y(0.1)$ correct to 3-decimal places given that
   $dy/dx = e^x - y^2, y(0) = 1.$

32.4 EULER'S METHOD*

Consider the equation $\dfrac{dy}{dx} = f(x, y)$ ...(1)

given that $y(x_0) = y_0$. Its curve of solution through $P(x_0, y_0)$ is shown dotted in Fig. 32.1. Now we have to find the ordinate of any other point Q on this curve.

Let us divide LM into n sub-intervals each of width h at $L_1, L_2, \ldots$ so that h is quite small. In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$, then

$$y_1 = L_1P_1 = LP + R_1P_1$$
$$= y_0 + PR_1 \tan\theta = y_0 + h\left(\frac{dy}{dx}\right)_P$$
$$= y_0 + h\, f(x_0, y_0)$$

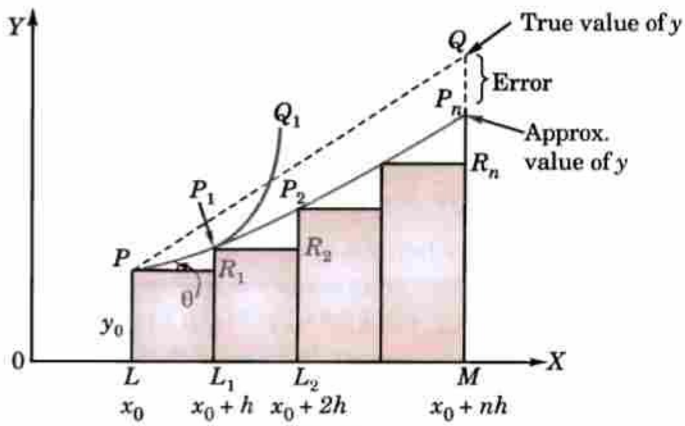

Fig. 32.1

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$. Then

$$y_2 = y_1 + h\, f(x_0 + h, y_1) \qquad \text{...(2)}$$

Repeating this process n times, we finally reach an approximation MP_n of MQ given by

$$y_n = y_{n-1} + h\, f(x_0 + \overline{n-1}\,h, y_{n-1})$$

This is *Euler's method* of finding an approximate solution of (1).

Obs. In Euler's method, we approximate the curve of solution by the tangent in each interval, *i.e.* by a sequence of short lines. Unless h is small, the error is bound to be quite significant. This sequence of lines may also deviate considerably from the curve of solution. Hence there is a modification of this method which is given in the next section.

Example 32.6. *Using Euler's method, find an approximate value of y corresponding to x = 1, given that dy/dx = x + y and y = 1 when x = 0.* *(Mumbai, 2005 ; Rohtak, 2003)*

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*See footnote p. 302.

Solution. We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows :

x	y	$x + y = dy/dx$	Old $y + 0.1\ (dy/dx)$ = new y
0.0	1.00	1.00	1.00 + 0.1(1.00) = 1.10
0.1	1.10	1.20	1.10 + 0.1(1.20) = 1.22
0.2	1.22	1.42	1.22 + 0.1(1.42) = 1.36
0.3	1.36	1.66	1.36 + 0.1(1.66) = 1.53
0.4	1.53	1.93	1.53 + 0.1(1.93) = 1.72
0.5	1.72	2.22	1.72 + 0.1(2.22) = 1.94
0.6	1.94	2.54	1.94 + 0.1(2.54) = 2.19
0.7	2.19	2.89	2.19 + 0.1(2.89) = 2.48
0.8	2.48	3.89	2.48 + 0.1(3.89) = 2.81
0.9	2.81	3.71	2.81 + 0.1(3.71) = 3.18
1.0	3.18		

Thus the required approximate value of $y = 3.18$.

Obs. In example 32.1, the true value of y from its exact solution at $x = 1$ is 3.44 whereas by Euler's method $y = 3.18$ and by Picard's method $y = 3.434$. In the above solution, had we chosen $n = 20$, the accuracy would have been considerably increased but at the expense of double the labour of computation. Euler's method is no doubt very simple but cannot be considered as one of the best.

Example 32.7. *Given* $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$ *with initial condition* $y = 1$ *at* $x = 0$ *; find* y *for* $x = 0.1$ *by Euler's method.*

(P.T.U., 2001)

Solution. We divide the interval (0, 0.1) into five steps *i.e.* we take $n = 5$ and $h = 0.02$. The various calculations are arranged as follows :

x	y	$(y - x)/(y + x) = dy/dx$	Old $y + 0.02\ (dy/dx)$ = new y
0.00	1.0000	1.0000	1.0000 + 0.02 (1.0000) = 1.0200
0.02	1.0200	0.9615	1.0200 + 0.02 (.9615) = 1.0392
0.04	1.0392	0.926	1.0392 + 0.02 (.926) = 1.0577
0.06	1.0577	0.893	1.0577 + 0.02 (.893) = 1.0756
0.08	1.0756	0.862	1.0756 + 0.02 (.862) = 1.0928
0.10	1.0928		

Hence the required approximate value of $y = 1.0928$.

32.5 MODIFIED EULER'S METHOD

In the Euler's method, the curve of solution in the interval LL_1 is approximated by the tangent at P (Fig. 32.1) such that at P_1, we have

$$y_1 = y_0 + h\, f(x_0, y_0) \qquad \text{...(1)}$$

Then the slope of the curve of solution through P_1 [*i.e.* $(dy/dx)_{P_1} = f(x_0 + h, y_1)$] is computed and the tangent at P_1 to P_1Q_1 is drawn meeting the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$.

Now we find a better approximation $y_1^{(1)}$ of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and P_1, *i.e.*

$$y_1^{(1)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_0 + h, y_1)] \qquad \text{...(2)}$$

As the slope of the tangent at P_1 is not known, we take y_1 as found in (1) by Euler's method and insert it on R.H.S. of (2) to obtain the first modified value $y_1^{(1)}$. The equation (1) is therefore, called the *predictor* while (2) serves as the *corrector of* y_1.

Again the corrector is applied and we find a still better value $y_1^{(2)}$ corresponding to L_1 as

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})]$$

We repeat this step, till two consecutive values of y agree. This is then taken as the starting point for the next interval L_1L_2.

Once y_1 is obtained to desired degree of accuracy, y corresponding to L_2 is found from the *predictor*

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

and a better approximation $y_2^{(1)}$ is obtained from the *corrector*

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)].$$

We repeat this step until y_2 becomes stationary. Then we proceed to calculate y_3 as above and so on.

This is the *modified Euler's method* which is a predictor-corrector method.

Example 32.8. *Using modified Euler's method, find an approximate value of y when x = 0.3, given that dy/dx = x + y and y = 1 when x = 0.* (Rohtak, 2005 ; Bhopal, 2002 S ; Delhi, 2002)

Solution. Taking $h = 0.1$, the various calculations are arranged as follows :

x	$x + y = y'$	*Mean slope*	*Old y + 0.1 (mean slope) = new y*
0.0	0 + 1	—	1.00 + 0.1 (1.00) = 1.10
0.1	.1 + 1.1	$\frac{1}{2}(1 + 1.2)$	1.00 + 0.1 (1.1) = 1.11
0.1	.1 + 1.11	$\frac{1}{2}(1 + 1.21)$	1.00 + 0.1 (1.105) = 1.1105
0.1	.1 + 1.1105	$\frac{1}{2}(1 + 1.2105)$	1.00 + 0.1 (1.1052) = 1.1105
0.1	1.2105	—	1.1105 + 0.1 (1.2105) = 1.2316
0.2	.2 + 1.2316	$\frac{1}{2}(1.2105 + 1.4316)$	1.1105 + 0.1 (1.3211) = 1.2426
0.2	.2 + 1.2426	$\frac{1}{2}(1.2105 + 1.4426)$	1.1105 + 0.1 (1.3266) = 1.2432
0.2	.2 + 1.2432	$\frac{1}{2}(1.2105 + 1.4432)$	1.1105 + 0.1 (1.3268) = 1.2432
0.2	1.4432	—	1.2432 + 0.1 (1.4432) = 1.3875
0.3	.3 + 1.3875	$\frac{1}{2}(1.4432 + 1.6875)$	1.2432 + 0.1 (1.5654) = 1.3997
0.3	.3 + 1.3997	$\frac{1}{2}(1.4432 + 1.6997)$	1.2432 + 0.1 (1.5715) = 1.4003
0.3	.3 + 1.4003	$\frac{1}{2}(1.4432 + 1.7003)$	1.2432 + 0.1 (1.5718) = 1.4004
0.3	.3 + 1.4004	$\frac{1}{2}(1.4432 + 1.7004)$	1.2432 + 0.1 (1.5718) = 1.4004

Hence $y(0.3) = 1.4004$ approximately.

Obs. In example 32.6, the approximate value of y for $x = 0.3$ would be 1.53 whereas by modified Euler's method the corresponding value is 1.4004 which is nearer its true value 1.3997, obtained from its exact solution $y = 2e^x - x - 1$ by putting $x = 0.3$.

Example 32.9. *Using modified Euler's method, find y(0.2) and y(0.4) given*

$$y' = y + e^x, \; y(0) = 0.$$

(J.N.T.U., 2009)

Solution. We have $y' = y + e^x = f(x, y)$; $x = 0, y = 0$ and $h = 0.2$

The various calculations are arranged as under :

To calculate $y(0.2)$:

x	$y + e^x = y'$	*Mean slope*	*Old y + h (mean slope) = new y*
0.0	1	—	$0 + 0.2\,(1) = 0.2$
0.2	$0.2 + e^{0.2} = 1.4214$	$\frac{1}{2}(1 + 1.4214) = 1.2107$	$0 + 0.2\,(1.2107) = 0.2421$
0.2	$0.2421 + e^{0.2} = 1.4635$	$\frac{1}{2}(1 + 1.4635) = 1.2317$	$0 + 0.2\,(1.2317) = 0.2463$
0.2	$0.2463 + e^{0.2} = 1.4677$	$\frac{1}{2}(1 + 1.4677) = 1.2338$	$0 + 0.2\,(1.2338) = 0.2468$
0.2	$0.2468 + e^{0.2} = 1.4682$	$\frac{1}{2}(1 + 1.4682) = 1.2341$	$0 + 0.2\,(1.2341) = 0.2468$

Since the last two values of y are equal, we take $y(0.2) = 0.2468$.

To calculate $y(0.4)$.

x	$y + e^x = y'$	*Mean slope*	*Old y + h (Mean slope) = new y*
0.2	$0.2468 + e^{0.2} = 1.4682$	—	$0.2468 + 0.2\,(1.4682) = 0.5404$
0.4	$0.5404 + e^{0.4} = 2.0322$	$\frac{1}{2}(1.4682 + 2.0322) = 1.7502$	$0.2468 + 0.2\,(1.7502) = 0.5968$
0.4	$0.5968 + e^{0.4} = 2.0887$	$\frac{1}{2}(1.4682 + 2.0887) = 1.7784$	$0.2468 + 0.2\,(1.7784) = 0.6025$
0.4	$0.6025 + e^{0.4} = 2.0943$	$\frac{1}{2}(1.4682 + 2.0943) = 1.78125$	$0.2468 + 0.2\,(1.78125) = 0.6030$
0.4	$0.6030 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949) = 1.7815$	$0.2468 + 0.2\,(1.7815) = 0.6031$
0.4	$0.6031 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949) = 1.7816$	$0.2468 + 0.2\,(1.7815) = 0.6031$

Since the last two value of y are equal, we take $y(0.4) = 0.6031$.

Hence $y(0.2) = 0.2468$ and $y(0.4) = 0.6031$ approximately.

Example 32.10. *Solve the following by Euler's modified method :*

$$\frac{dy}{dx} = \log(x + y),\ y(0) = 2.$$

at $x = 1.2$ and 1.4 with $h = 0.2$. *(Bhopal, 2009 ; U.P.T.U., 2007)*

Solution. The various calculations are arranged as follows :

x	$\log(x + y) = y'$	*Mean slope*	*Old y + 0.2 (mean slope) = new y*
0.0	$\log(0 + 2)$	—	$2 + 0.2\,(0.301) = 2.0602$
0.2	$\log(0.2 + 2.0602)$	$\frac{1}{2}(0.301 + 0.3541)$	$2 + 0.2\,(0.3276) = 2.0655$
0.2	$\log(0.2 + 2.0655)$	$\frac{1}{2}(0.301 + 0.3552)$	$2 + 0.2\,(0.3281) = 2.0656$
0.2	0.3552	—	$2.0656 + 0.2\,(0.3552) = 2.1366$
0.4	$\log(0.4 + 2.1366)$	$\frac{1}{2}(0.3552 + 0.4042)$	$2.0656 + 0.2\,(0.3797) = 2.1415$
0.4	$\log(0.4 + 2.1415)$	$\frac{1}{2}(0.3552 + 0.4051)$	$2.0656 + 0.2\,(0.3801) = 2.1416$

x	$\log(x+y) = y'$	Mean slope	Old y + 0.2 (mean slope) = new y
0.4	0.4051	—	2.1416 + 0.2 (0.4051) = 2.2226
0.6	log (0.6 + 2.2226)	$\frac{1}{2}(0.4051 + 0.4506)$	2.1416 + 0.2 (0.4279) = 2.2272
0.6	log (0.6 + 2.2272)	$\frac{1}{2}(0.4051 + 0.4514)$	2.1416 + 0.2 (0.4282) = 2.2272
0.6	0.4514	—	2.2272 + 0.2 (0.4514) = 2.3175
0.8	log (0.8 + 2.3175)	$\frac{1}{2}(0.4514 + 0.4938)$	2.2272 + 0.2 (0.4726) = 2.3217
0.8	log (0.8 + 2.3217)	$\frac{1}{2}(0.4514 + 0.4943)$	2.2272 + 0.2 (0.4727) = 2.3217
0.8	0.4943	—	2.3217 + 0.2 (0.4943) = 2.4206
1.0	log (1 + 2.4206)	$\frac{1}{2}(0.4943 + 0.5341)$	2.3217 + 0.2 (0.5142) = 2.4245
1.0	log (1 + 2.4245)	$\frac{1}{2}(0.4943 + 0.5346)$	2.3217 + 0.2 (0.5144) = 2.4245
1.0	0.5346	—	2.4245 + 0.2 (0.5346) = 2.5314
1.2	log (1.2 + 2.5314)	$\frac{1}{2}(0.5346 + 0.5719)$	2.4245 + 0.2 (0.5532) = 2.5351
1.2	log (1.2 + 2.5351)	$\frac{1}{2}(0.5346 + 0.5723)$	2.4245 + 0.2 (0.5534) = 2.5351
1.2	0.5723	—	2.5351 + 0.2 (0.5723) = 2.6496
1.4	log (1.4 + 2.6496)	$\frac{1}{2}(0.5723 + 0.6074)$	2.5351 + 0.2 (0.5898) = 2.6531
1.4	log (1.4 + 2.6531)	$\frac{1}{2}(0.5723 + 0.6078)$	2.5351 + 0.2 (0.5900) = 2.6531

Hence $y(1.2) = 2.5351$ and $y(1.4) = 2.6531$ approximately.

Example 32.11. *Using Euler's modified method, obtain a solution of the equation $dy/dx = x + |\sqrt{y}|$, with initial conditions $y = 1$ at $x = 0$, for the range $0 \le x \le 0.6$ in steps of 0.2.* (V.T.U., 2007)

Solution. The various calculations are arranged as follows :

x	$x + \|\sqrt{y}\| = y'$	Mean slope	Old y + .2 (mean slope) = new y
0.0	0 + 1 = 1	—	1 + 0.2 (1) = 1.2
0.2	$0.2 + \|\sqrt{(1.2)}\| = 1.2954$	$\frac{1}{2}(1 + 1.2954) = 1.1477$	1 + 0.2 (1.1477) = 1.2295
0.2	$0.2 + \|\sqrt{(1.2295)}\| = 1.3088$	$\frac{1}{2}(1 + 1.3088) = 1.1544$	1 + 0.2 (1.1544) = 1.2309
0.2	$0.2 + \|\sqrt{(1.2309)}\| = 1.3094$	$\frac{1}{2}(1 + 1.3094) = 1.1547$	1 + 0.2 (1.1547) = 1.2309
0.2	1.3094	—	1.2309 + 0.2 (1.3094) = 1.4927
0.4	$0.4 + \|\sqrt{(1.4927)}\| = 1.6218$	$\frac{1}{2}(1.3094 + 1.6218) = 1.4654$	1.2309 + 0.2 (1.4654) = 1.5240
0.4	$0.2 + \|\sqrt{(1.524)}\| = 1.6345$	$\frac{1}{2}(1.3094 + 1.6345) = 1.4718$	1.2309 + 0.2 (1.4718) = 1.5253
0.4	$0.4 + \|\sqrt{(1.5253)}\| = 1.6350$	$\frac{1}{2}(1.3094 + 1.6350) = 1.4721$	1.2309 + 0.2 (1.4721) = 1.5253

x	$x + \lvert\sqrt{y}\rvert = y'$	Mean slope	Old y + .2 (mean slope) = new y
0.4	1.6350	—	1.5253 + 0.2 (1.635) = 1.8523
0.6	$0.6 + \lvert\sqrt{(1.8523)}\rvert = 1.9610$	$\frac{1}{2}(1.635 + 1.961) = 1.798$	1.5253 + 0.2 (1.798) = 1.8849
0.6	$0.6 + \lvert\sqrt{(1.8849)}\rvert = 1.9729$	$\frac{1}{2}(1.635 + 1.9729) = 1.8040$	1.5253 + 0.2 (1.804) = 1.8861
0.6	$0.6 + \lvert\sqrt{(1.8861)}\rvert = 1.9734$	$\frac{1}{2}(1.635 + 1.9734) = 1.8042$	1.5253 + 0.2 (1.8042) = 1.8861

Hence $y(0.6) = 1.8861$ approximately.

PROBLEMS 32.2

1. Apply Euler's method to solve $y' = x + y$, $y(0) = 0$, choosing the step length = 0.2. (Carry out 6 steps). (*Kottayam, 2005*)
2. Using simple Euler's method solve for y at $x = 0.1$ from $dy/dx = x + y + xy$, $y(0) = 1$, taking step size $h = 0.025$.
3. Using Euler's method, find the approximate value of y when $dy/dx = x^2 + y^2$ and $y(0) = 1$ in five steps (*i.e.* $h = 0.2$). (*Mumbai, 2006*)
4. Solve $y' = 1 - y$, $y(0) = 0$ by modified Euler's method and obtain y at $x = 0.1, 0.2, 0.3$. (*Anna, 2005*)
5. Given $y' = x + \sin y$, $y(0) = 1$. Compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$ using Euler's modified method. (*J.N.T.U., 2007*)
6. Given that $dy/dx = x^2 + y$ and $y(0) = 1$. Find an approximate value of $y(0.1)$ taking $h = 0.05$ by modified Euler's method. (*V.T.U., 2010*)
7. Given $\frac{dy}{dx} = \frac{y - x}{y + x}$ with boundary conditions $y = 1$ when $x = 0$, find approximately y for $x = 0.1$, by Euler's modified method (5 steps). (*V.T.U., 2007*)
8. Given that $dy/dx = 2 + \sqrt{(xy)}$ and $y = 1$ when $x = 1$. Find approximate value of y at $x = 2$ in steps of 0.2, using Euler's modified method. (*Anna, 2004*)

32.6 RUNGE'S METHOD*

Consider the differential equation,

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \quad ...(1)$$

Clearly the slope of the curve through $P(x_0, y_0)$ is $f(x_0, y_0)$ (Fig. 32.2).

Integrate both sides of (1) from (x_0, y_0) to $(x_0 + h, y_0 + k)$, we have

$$\int_{y_0}^{y_0 + k} dy = \int_{x_0}^{x_0 + h} f(x, y)\, dx \quad ...(2)$$

To evaluate the integral on the right, we take N as the mid-point of LM and find the values of $f(x, y)$ (*i.e.* dy/dx) at the points $x_0, x_0 + h/2, x_0 + h$. For this purpose, we first determine the values of y at these points.

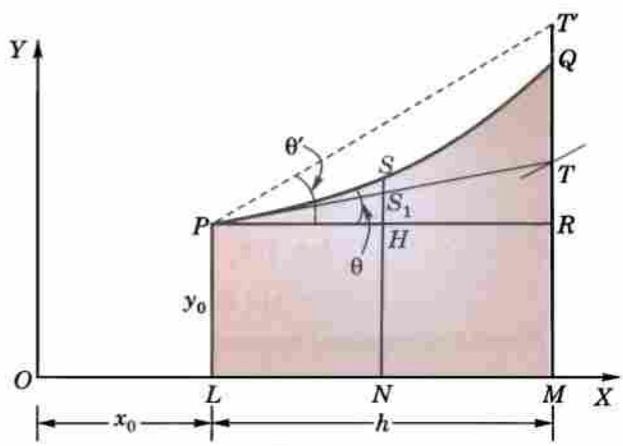

Fig. 32.2

Let the ordinate through N cut the curve PQ in S and the tangent PT in S_1. The value of y_s is given by the point S_1.

$$\therefore \quad y_s = NS = LP + HS_1 = y_0 + PH \tan\theta$$

* Called after the German mathematician *Carl Runge* (1856–1927) who was professor at Gottingen.

$$= y_0 + \frac{h}{2}(dy/dx)_P = y_0 + \frac{h}{2}f(x_0, y_0) \qquad ...(3)$$

Also $\qquad y_T = MT = LP + RT = y_0 + PR \tan\theta = y_0 + hf(x_0, y_0).$

Now the value of y_Q at $x_0 + h$ is given by the point T' where the line through P drawn with slope at $T(x_0 + h, y_T)$ meets MQ.

$\because$ Slope at $T = \tan\theta' = f(x_0 + h, y_T) = f[x_0 + h, y_0 + hf(x_0, y_0)]$

$$\therefore \qquad y_Q = MR + RT' = y_0 + PT \tan\theta' = y_0 + hf[x_0 + h, y_0 + hf(x_0, y_0)] \qquad ...(4)$$

Thus the value of $f(x, y)$ at $P = f(x_0, y_0)$,

the value of $f(x, y)$ at $S = f(x_0 + h/2, y_s)$

and the value of $f(x, y)$ at $Q = f(x_0 + h, y_Q)$

where y_S and y_Q are given by (3) and (4).

Hence from (2), we obtain

$$k = \int_{x_0}^{x_0+h} f(x, y)\,dx = \frac{h}{6}[f_P + 4f_S + f_Q] \qquad \text{[By Simpsons' rule (p. 1106)]}$$

$$= \frac{h}{6}[f(x_0, y_0) + 4f(x_0 + h/2, y_S) + f(x_0 + h, y_Q)] \qquad ...(5)$$

which gives a sufficiently accurate value of k and also of $y = y_0 + k$.

The repeated application of (5) gives the values of y for equispaced points.

Working rule *to solve (1) by Runge's method :*

Calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1\right)$$

$$k' = hf(x_0 + h, y_0 + k_1)$$

and $$k_3 = hf(x_0 + h, y_0 + k')$$

Finally compute, $\qquad k = \frac{1}{6}(k_1 + 4k_2 + k_3).$

(Note that k is the weighted mean of k_1, k_2 and k_3)

Example 32.12. *Apply Runge's method to find an approximate value of y when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.*

Solution. Here we have $x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

$\therefore \qquad k_1 = hf(x_0, y_0) = 0.2\,(1) \qquad = 0.200$

$k_2 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1\right) = 0.2\,f(0.1, 1.1) \qquad = 0.240$

$k' = hf(x_0 + h, y_0 + k_1) = 0.2\,f(0.2, 1.2) \qquad = 0.280$

and $\qquad k_3 = hf(x_0 + h, y_0 + k') = 0.2\,f(0.1, 1.28) \qquad = 0.296$

$\therefore \qquad k = \frac{1}{6}(k_1 + 4k_2 + k_3)$

$= \frac{1}{6}(0.200 + 0.960 + 0.296) \qquad = 0.2426$

Hence the required approximate value of y is 1.2426.

32.7 RUNGE-KUTTA METHOD*

The Taylor's series method of solving differential equations numerically is restricted by the labour involved in finding the higher order derivatives. However there is a class of methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives. These methods agree with Taylor's series solution upto the terms in h^r, where r differs from method to method and is called the *order of that method. Euler's method, Modified Euler's method and Runge's method are the Runge-Kutta methods of the first, second and third order respectively.*

* See footnote p. 1017. Named after *Wilhelm Kutta* (1867—1944).

The fourth-order Runge-Kutta method is most commonly used and is often referred to as 'Runge-Kutta method' only.

Working rule *for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from*

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \text{ is as follows :}$$

Calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \tfrac{1}{2}h,\ y + \tfrac{1}{2}k_2\right)$$

and $$k_4 = hf(x_0 + h, y_0 + k_3)$$

Finally compute $$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

which gives the required approximate value $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, k_3 and k_4)

Obs. One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

Example 32.13. *Apply Runge-Kutta fourth order method, to find an approximate value of y when x = 0.2, given that dy/dx = x + y and y = 1 when x = 0.* *(V.T.U., 2009 ; P.T.U., 2007 ; S.V.T.U., 2007)*

Solution. Here $x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

$\therefore$ $$k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2000$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.2400$$

$$k_3 = hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2\right) = 0.2 \times f(0.1, 1.12) = 0.2440$$

and $$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) = 0.2888$$

$\therefore$ $$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \tfrac{1}{6}(0.2000 + 0.4800 + 0.4880 + 0.2888) = \tfrac{1}{6} \times (1.4568) = 0.2468.$$

Hence the required approximate value of y is 1.2428.

Example 32.14. *Using Runge-Kutta method of fourth order, solve* $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ *with y(0) = 1 at x = 0.2, 0.4.* *(U.P.T.U., 2010 ; J.N.T.U., 2009 ; V.T.U., 2008)*

Solution. We have $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$

To find y(0.2) :

Here $x_0 = 0, y_0 = 1, h = 0.2$

$$k_1 = h\,f(x_0, y_0) = 0.2\,f(0, 1) = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right) = 0.2\,f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right) = 0.2\,f(0.1, 1.09836) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2\,f(0.2, 1.1967) = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891]$$

$$= 0.19599$$

Hence $$y(0.2) = y_0 + k = 1.196.$$

To find y(0.4) :

Here $x_1 = 0.2, y_1 = 1.196, h = 0.2$

$$k_1 = h\,f(x_1, y_1) \qquad = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2\,f(0.3, 1.2906) \qquad = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2\,f(0.3, 1.2858) \qquad = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2\,f(0.4, 1.3753) \qquad = 0.1688$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] \qquad = 0.1792$$

Hence $y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752.$

Example 32.15. *Apply Runge-Kutta method to find an approximate value of y for x = 0.2 in steps of 0.1, if $dy/dx = x + y^2$, given that y = 1, where x = 0.* *(V.T.U., 2009 ; Osmania, 2007 ; Madras, 2000)*

Solution. Here we take $h = 0.1$ and carry out the calculations in two steps.

Step I. $x_0 = 0, y_0 = 1, h = 0.1$

$\therefore$

$$k_1 = hf(x_0, y_0) = 0.1\,f(0, 1) \qquad = 0.1000$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1\right) = 0.1\,f(0.05, 1.1) \qquad = 0.1152$$

$$k_3 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2\right) = 0.1\,f(0.05, 1.1152) \qquad = 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1\,f(0.1, 1.1168) \qquad = 0.1347$$

$\therefore$

$$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \tfrac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347) \qquad = 0.1165$$

giving $y(0.1) = y_0 + k = 1.1165.$

Step II. $x_1 = x_0 + h = 0.1, y_1 = 1.1165, h = 0.1$

$\therefore$

$$k_1 = hf(x_1, y_1) = 0.1\,f(0.1, 1.1165) \qquad = 0.1347$$

$$k_2 = hf\left(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_1\right) = 0.1\,f(0.15, 1.1838) \qquad = 0.1551$$

$$k_3 = hf\left(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_2\right) = 0.1\,f(0.15, 1.194) \qquad = 0.1576$$

$$k_4 = hf(x_1 + h, y_2 + k_3) = 0.1\,f(0.2, 1.1576) \qquad = 0.1823$$

$\therefore$

$$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.1571$$

Hence $y(0.2) = y_1 + k = 1.2736.$

Example 32.16. *Using Runge-Kutta method of fourth order, solve for y at x = 1.2, 1.4 from* $\dfrac{dy}{dx} = \dfrac{2xy + e^x}{x^2 + xe^x}$ *given $x_0 = 1, y_0 = 0$.* *(Mumbai, 2008)*

Solution. We have $f(x, y) = \dfrac{2xy + e^x}{x^2 + xe^2}$

To find y(1.2) :

Here $x_0 = 1, y_0 = 0, h = 0.2$

$\therefore$

$$k_1 = h\,f(x_0, y_0) = 0.2\,\frac{0 + e}{1 + e} = 0.1462$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\left\{\frac{2(1 + 0.1)(0 + 0.073) + e^{1+0.1}}{(1 + 0.1)^2 + (1 + 0.1)\,e^{1+0.1}}\right\} = 0.1402$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2\left\{\frac{2(1+0.1)(0+0.07)+e^{1.1}}{(1+0.1)^2+(1+0.1)\,e^{1.1}}\right\} = 0.1399$$

$$k_4 = hf(x_0+k, y_0+k_3) = 0.2\left\{\frac{2(1.2)(0.1399)+e^{1.2}}{(1.2)^2+(1.2)\,e^{1.2}}\right\} = 0.1348$$

and $$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = \frac{1}{6}[0.1462+0.2804+0.2798+0.1348]$$

$$= 0.1402.$$

Hence $y(1.2) = y_0 + k = 0 + 0.1402 = 0.1402.$

To find y(1.4) :

Here $x_1 = 1.2, y_1 = 0.1402, h = 0.2$

$$k_1 = hf(x_1, y_1) = 0.2\,f(1.2, 0) = 0.1348$$
$$k_2 = hf(x_1 + h/2, y_1 + k_1/2) = 0.2\,f(1.3, 0.2076) = 0.1303$$
$$k_3 = hf(x_1 + h/2, y_1 + k_1/2) = 0.2\,f(1.3, 0.2053) = 0.1301$$
$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2\,f(1.3, 0.2703) = 0.1260$$

$\therefore$ $$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = \frac{1}{6}[0.1348+0.2606+0.2602+0.1260] = 0.1303$$

Hence $y(1.4) = y_1 + k = 0.1402 + 0.1303 = 0.2705.$

PROBLEMS 32.3

1. Use Runge's method to approximate y when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and $dy/dx = 3x + y^2$.
2. Using Runge-Kutta method of order 4, find $y(0.2)$ given that $dy/dx = 3x + \frac{1}{2}y$, $y(0) = 1$, taking $h = 0.1$. *(V.T.U., 2004)*
3. Using Runge-Kutta method of order 4, compute $y(.2)$ and $(.4)$ from $10\dfrac{dy}{dx} = x^2 + y^2$, $y(0) = 1$, taking $h = 0.1$. *(Rohtak, 2003 ; Bhopal, 2002)*
4. Use Runge-Kutta method to find y when $x = 1.2$ in steps of 0.1, given that :
   $dy/dx = x^2 + y^2$ and $y(1) = 1.5$. *(Mumbai, 2007)*
5. Find $y(0.1)$ and $y(0.2)$ using Runge-Kutta 4th order formula, given that $y' = x^2 - y$ and $y(0) = 1$. *(J.N.T.U., 2006)*
6. Using 4th order Runge-Kutta method, solve the following equation, taking each step of $h = 0.1$, given $y(0) = 3$, $dy/dx = (4x/y - xy)$. Calculate y for $x = 0.1$ and 0.2. *(Anna, 2007)*
7. Use fourth order Runge-Kutta method to find y at $x = 0.1$, given that $\dfrac{dy}{dx} = 3e^x + 2y$, $y(0) = 0$ and $h = 0.1$. *(V.T.U., 2006)*
8. Find by Runge-Kutta method an approximate value of y for $x = 0.8$, given that $y = 0.41$ when $x = 0.4$ and $dy/dx = \sqrt{(x+y)}$. *(S.V.T.U., 2007 S)*
9. Using Runge-Kutta method of order 4, find $y(0.2)$ for the equation $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$, $y(0) = 1$. Take $h = 0.2$. *(V.T.U., 2011 S)*
10. Given that $dy/dx = (y^2 - 2x)/(y^2 + x)$ and $y = 1$ at $x = 0$; find y for $x = 0.1, 0.2, 0.3, 0.4$ and 0.5. *(Delhi, 2002)*

32.8 PREDICTOR-CORRECTOR METHODS

If x_{i-1} and x_i be two consecutive mesh points, we have $x_i = x_{i-1} + h$. In the Euler's method (§ 32.4), we have

$$y_i = y_{i-1} + hf(x_0 + \overline{i-1}\,h, y_{i-1})\ ;\ i = 1, 2, 3, \ldots \quad \ldots(1)$$

The modified Euler's method (§ 32.5), gives

$$y_i = y_{i-1} + \frac{h}{2}\,[f(x_{i-1}, y_{i-1}) + f(x_i, y_i)] \qquad ...(2)$$

The value of y_i is first estimated by using (1), then this value is inserted on the right side of (2), giving a better approximation of y_i. This value of y_i is again substituted in (2) to find a still better approximation of y_i. This step is repeated till two consecutive values of y_i agree. *This technique of refining an initially crude estimate of y_i by means of a more accurate formula is known as* **predictor-corrector method.** The equation (1) is therefore called the *predictor* while (2) serves as a *corrector* of y_i.

In the methods so far explained, to solve a differential equation over an interval (x_i, x_{i+1}) only the value of y at the beginning of the interval was required. In the *predictor-corrector* methods, four prior values are required for finding the value of y at x_{i+1}. A predictor formula is used to predict the value of y at x_{i+1} and then a corrector formula is applied to improve this value.

We now describe two such methods, namely : Milne's method and Adams-Bashforth method.

32.9 MILNE'S METHOD

Given $dy/dx = f(x, y)$ and $y = y_0, x = x_0$; to find an approximate value of y for $x = x_0 + nh$ by Milne's method, we proceed as fo... :

The value $y_0 = y(x_0)$ being given, we compute

$$y_1 = y(x_0 + h),\ y_2 = y(x_0 + 2h),\ y_3 = y(x_0 + 3h),$$

by Picard's or Taylor's series method.

Next we calculate,

$$f_0 = f(x_0, y_0),\ f_1 = f(x_0 + h, y_1),\ f_2 = f(x_0 + 2h, y_2),\ f_3 = f(x_0 + 3h, y_3)$$

Then to find $y_4 = y(x_0 + 4h)$, we substitute Newton's forward interpolation formula

$$f(x, y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\,\Delta^2 f_0 + \frac{n(n-1)(n-2)}{6}\,\Delta^3 f_0 + ...$$

in the relation $y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x, y)\,dx$

$$\therefore \quad y_4 = y_0 + \int_{x_0}^{x_0+4h}\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + ...\right)dx \qquad [\text{Put } x = x_0 + nh,\ dx = h\,dn]$$

$$= y_0 + h\int_0^4\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + ...\right)dn$$

$$= y_0 + h\left(4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + ...\right)$$

Neglecting fourth and higher order differences and expressing Δf_0, $\Delta^2 f_0$ and $\Delta^3 f_0$ in terms of the function values, we get

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \text{ which is called a } \textit{predictor}.$$

Having found y_4, we obtain a first approximation to $f_4 = f(x_0 + 4h, y_4)$.

Then a better value of y_4 is found by Simpson's rule (p. 1106) as

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \text{ which is called a } \textit{corrector}.$$

Then an improved value of f_4 is computed and again the corrector is applied to find a still better value of y_4. We repeat this step until y_4 remains unchanged.

Once y_4 and f_4 are obtained to desired degree of accuracy, $y_5 = y(x_0 + 5h)$ is found from the *predictor* as

$$y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4)$$

and $f_5 = f(x_0 + 5h, y_5)$ is calculated. Then a better approximation to the value of y_5 is obtained from the *corrector* as

$$y_5^{(c)} = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5).$$

We repeat this step till y_5 becomes stationary and we, then proceed to calculate y_6 as before.

This is *Milne's predictor-corrector method.* To ensure greater accuracy, we must first improve the accuracy of the starting values and then sub-divide the intervals.

Example 32.17. *Apply Milne's method, to find a solution of the differential equation $y' = x - y^2$ in the range $0 \le x \le 1$ for the boundary conditions $y = 0$ at $x = 0$.* (V.T.U., 2009, Anna, 2005, Rohtak, 2005)

Solution. Using Picard's method, we have

$$y = y(0) + \int_0^x f(x, y)\, dx, \text{ where } f(x, y) = x - y^2.$$

To get the first approximation, we put $y = 0$ in $f(x, y)$,

giving $$y_1 = 0 + \int_0^x x\, dx = \frac{x^2}{2}$$

To find the second approximation, we put $y = x^2/2$ in $f(x, y)$,

giving $$y_2 = \int_0^x \left(x - \frac{x^4}{4}\right) dx = \frac{x^2}{2} - \frac{x^5}{20}$$

Similarly, the third approximation is

$$y_3 = \int_0^x \left[x - \left(\frac{x^2}{2} - \frac{x^5}{20}\right)^2\right] dx = \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400} \quad ...(i)$$

Now let us determine the starting values of the Milne's method from (i), by choosing $h = 0.2$.

$\therefore$ $x_0 = 0.0$, $y_0 = 0.0000$, $f_0 = 0.0000$
$x_1 = 0.2$, $y_1 = 0.020$, $f_1 = 0.1996$
$x_2 = 0.4$, $y_2 = 0.0795$, $f_2 = 0.3937$
$x_3 = 0.6$, $y_3 = 0.1762$, $f_3 = 0.5689$

Using the *predictor*, $$y_4^{(p)} = y_0 \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$x = 0.8$, $y_4^{(p)} = 0.3049$, $f_4 = 0.7070$

and the *corrector*, $$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4), \text{ yields}$$

$$y_4^{(c)} = 0.3046, \quad f_4 = 0.7072 \quad ...(ii)$$

Again using the *corrector*, $y_4^{(c)} = 0.3046$, which is same as in (ii)

Now using the *predictor*, $$y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4),$$

$x = 1.0$, $y_5^{(p)} = 0.4554$, $f_5 = 0.7926$

and the *corrector*, $$y_5^{(c)} = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5), \text{ gives}$$

$$y_5^{(c)} = 0.4555, \quad f_5 = 0.7925$$

Again using the *corrector*,

$y_5^{(c)} = 0.4555$, a value which is the same as before.

Hence, $y(1) = 0.4555$.

Example 32.18. *Given $y' = x(x^2 + y^2)\, e^{-x}$, $y(0) = 1$, find y at $x = 0.1, 0.2$ and 0.3 by Taylor's series method and compute $y(0.4)$ by Milne's method.* (Anna, 2007)

Solution. Given $y(0) = 1$ and $h = 0.1$

We have $y'(x) = x(x^2 + y^2)e^{-x}$; $\quad y'(0) = 0$

$$y''(x) = [(x^3 + xy^2)(-e^{-x}) + 3x^2 + y^2 + x(2y)y']e^{-x}$$

$$= e^{-x}[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy'] ; \quad y''(0) = 1$$

$$y'''(x) = -e^{-x}[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy' + 3x^2 + y^2 + 2xyy' - 6x - 2yy' - 2xy'^2 - 2xyy'] \quad y'''(0) = -2$$

Substitute these values in the Taylor's series,

$$y(x) = y(0) + \frac{x}{1!}y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \ldots$$

$$y(0.1) = 1 + (0.1)(0) + \frac{1}{2}(0.1)^2(1) + \frac{1}{6}(0.1)^3(-2) + \ldots$$

$$= 1 + 0.005 - 0.0003 = 1.0047 \quad i.e., \quad 1.005$$

Now taking $x = 0.1, y(0.1) = 1.005, h = 0.1$

$$y'(0.1) = 0.092, y''(0.1) = 0.849, y'''(0.1) = -1.247$$

Substituting these values in the Taylor's series about $x = 0.1$,

$$y(0.2) = y(0.1) + \frac{0.1}{1!}y'(0.1) + \frac{(0.1)^2}{2!}y''(0.1) + \frac{(0.1)^3}{3!}y'''(0.1) + \ldots$$

$$= 1.005 + (0.1)(0.092) + \frac{(0.1)^2}{2}(0.849) + \frac{(0.1)^3}{3}(-1.247) + \ldots$$

$$= 1.018$$

Now taking $x = 0.2, y(0.2) = 1.018, h = 0.1$

$$y'(0.2) = 0.176, y''(0.2) = 0.77, y'''(0.2) = 0.819$$

Substituting these values in the Taylor's series

$$y(0.3) = y(0.2) + \frac{0.1}{1!}y''(0.2) + \frac{(0.1)^2}{2!}y''(0.2) + \frac{(0.1)^3}{3!}y'''(0.2) + \ldots$$

$$= 1.018 + 0.0176 + 0.0039 + 0.0001 = 1.04$$

Thus the starting values of the Milne's method with $h = 0.1$ are

$x_0 = 0.0$	$y_0 = 1$	$f_0 = y'_0 = 0$
$x_1 = 0.1$	$y_1 = 1.005$	$f_1 = 0.092$
$x_2 = 0.2$	$y_2 = 1.018$	$f_2 = 0.176$
$x_3 = 0.3$	$y_3 = 1.04$	$f_3 = 0.26$

Using the predictor, $y_4^{(p)} = y_0 + \dfrac{4h}{3}(2f_1 - f_2 + 2f_3)$

$$= 1 + \frac{4(0.1)}{3}[2(0.092) - (0.176) + 2(0.26)] = 1.09$$

$\therefore$ $\quad x = 0.4 \quad y_4^{(p)} = 1.09 \quad f_4 = y'(0.4) = 0.362$

Using the corrector, $y_4^{(c)} = y_2 + \dfrac{h}{3}(f_2 + 4f_3 + f_4)$

$\therefore$ $$y_4^{(c)} = 0.018 + \frac{0.1}{3}(0.176 + 4(0.26) + 0.362) = 1.071$$

Hence $y(0.4) = 1.071.$

Example 32.19. *Using Runge-Kutta method of order 4, find y for x = 0.1, 0.2, 0.3 given that $dy/dx = xy + y^2$, $y(0) = 1$. Continue the solution at x = 0.4 using Milne's method.*

(V.T.U., 2008 ; S.V.T.U., 2007 ; Madras, 2006)

Solution. We have $f(x, y) = xy + y^2$.

To find y(0.1) :

Here $x_0 = 0, y_0 = 1, h = 0.1.$

$\therefore$

$$k_1 = h\,f(x_0, y_0) = (0.1)\,f(0.1) \qquad = 0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1)\,f(0.05, 1.05) \qquad = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1)\,f(0.05, 1.0577) \qquad = 0.1172$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)\,f(0.1, 1.1172) \qquad = 0.13598$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2348 + 0.13598) \qquad = 0.11687$$

Thus $y(0.1) = y_1 = y_0 + k = 1.1169$.

To find y(0.2) :

Here $x_1 = 0.1$, $y_1 = 1.1169$, $h = 0.1$.

$$k_1 = h\,f(x_1, y_1) = (0.1)\,f(0.1, 1.1169) \qquad = 0.1359$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1)\,f(0.15, 1.1848) \qquad = 0.1581$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1)\,f(0.15, 1.1959) \qquad = 0.1609$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)\,f(0.2, 1.2778) \qquad = 0.1888$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.1605$$

Thus $y(0.2) = y_2 = y_1 + k = 1.2773$.

To find y(0.3) :

Here $x_2 = 0.2$, $y_2 = 1.2773$, $h = 0.1$.

$$k_1 = hf(x_2, y_2) = (0.1)\,f(0.2, 1.2773) \qquad = 0.1887$$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1)\,f(0.25, 1.3716) \qquad = 0.2224$$

$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1)\,f(0.25, 1.3885) \qquad = 0.2275$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1)\,f(0.3, 1.5048) \qquad = 0.2716$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.2267$$

Thus $y(0.3) = y_3 = y_2 + k = 1.504$.

Now the starting values of the Milne's method are :

$x_0 = 0.0$	$y_0 = 1.0000$	$f_0 = 1.0000$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = 2.7132$

Using the *predictor*,

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$x_4 = 0.4 \qquad y_4^{(p)} = 1.8344 \qquad f_4 = 4.0988$$

and the *corrector*,

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \text{ yields}$$

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.098]$$

$$= 1.8386 \qquad f_4 = 4.1159$$

Again using the *corrector*,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}\,[1.8869 + 4\,(2.7132) + 4.1159]$$
$$= 1.8391 \qquad\qquad f_4 = 4.1182 \qquad ...(i)$$

Again using the *corrector*

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}\,[1.8869 + 4\,(2.7132) + 4.1182]$$
$$= 1.8392 \text{ which is same as } (i).$$

Hence $y\,(0.4) = 1.8392$.

PROBLEMS 32.4

1. Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 2$. The value of $y(0.2) = 2.073$, $y(0.4) = 2.452$, and $y(0.6) = 3.023$ are got by R.K. Method of 4th order. Find $y(0.8)$ by Milne's predictor-corrector method taking $h = 0.2$. *(Anna, 2004)*
2. Given $2\,dy/dx = (1 + x^2)y^2$ and $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$, evaluate $y(0.4)$ by Milne's predictor-corrector method. *(V.T.U., 2011 S ; Madras, 2003)*
3. From the data given below, find y at $x = 1.4$, using Milne's predictor-corrector formula :

$$\frac{dy}{dx} = x^2 + \frac{y}{2}$$

x :	1	1.1	1.2	1.3
y :	2	2.2156	2.4549	2.7514

(V.T.U., 2007)

4. Using Milne's method, find $y(4.5)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$, $y(4.4) = 1.0187$. *(Anna, 2007)*
5. If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.04$ and $y(0.3) = 2.09$; find $y(0.4)$ using Milne's predictor-corrector method. *(V.T.U., 2010)*
6. Using Runge-Kutta method, calculate $y(0.1)$, $y(0.2)$, and $y(0.3)$ given that $\frac{dy}{dx} - \frac{2xy}{1+x^2} = 1$, $y(0) = 0$. Taking these values as starting values, find $y(0.4)$ by Milne's method.

32.10 ADAMS-BASHFORTH METHOD

Given $\frac{dy}{dx} = f\,(x, y)$ and $y_0 = y(x_0)$, we compute

$$y_{-1} = y(x_0 - h),\ y_{-2} = y(x_0 - 2h),\ y_{-3} = y(x_0 - 3h)$$

by Taylor's series of **E**uler's method or Runge-Kutta method.

Next we calculate $f_{-1} = f\,(x_0 - h, y_{-1})$, $f_{-2} = f\,(x_0 - 2h, y_{-2})$, $f_{-3} = f\,(x_0 - 3h, y_{-3})$.

Then to find y_1, we substitute Newton's backward interpolation formula

$$f\,(x, y) = f_0 + n\,\nabla f_0 + \frac{n(n+1)}{2}\,\nabla^2 f_0 + \frac{n(n+1)\,(n+2)}{6}\,\nabla^3 f_0 + \cdots$$

in
$$y_1 = y_0 + \int_{x_0}^{x_0+h} f(x, y)\,dx \qquad ...(1)$$

$$\therefore \quad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\,\nabla^2 f_0 + \cdots\right) dx \qquad [\text{Put } x = x_0 + nh,\ dx = hdn]$$

$$= y_0 + h\int_0^1\left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\,\nabla^2 f_0 + \cdots\right) dn$$

$$= y_0 + h\left(f_0 + \frac{1}{2}\,\nabla f_0 + \frac{5}{12}\,\nabla^2 f_0 + \frac{3}{8}\,\nabla^3 f_0 + \cdots\right)$$

Neglecting fourth and higher order differences and expressing $\nabla f_0, \nabla^2 f_0$ and $\nabla^3 f_0$ in terms of function values, we get

$$y_1^{(p)} = y_0 + \frac{h}{24}\,(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}) \qquad \ldots(2)$$

This is called *Adams-Bashforth predictor formula.*

Having found y_1, we find $f_1 = f(x_0 + h_1, y_1)$.

Then to find a better value of y_1, we derive a *corrector formula* by substituting Newton's backward formula at f_1 *i.e.*,

$$f(x, y) = f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \frac{n(n+1)(n+2)}{6}\nabla^3 f_1 + \cdots \text{ in (1).}$$

$$\therefore \qquad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \cdots\right)dx \quad [\text{Put } x = x_1 + nh, dx = hdn]$$

$$= y_0 + \int_{-1}^{0}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \cdots\right)dn$$

$$= y_0 + h\left(f_1 - \frac{1}{2}\nabla f_1 - \frac{1}{12}\nabla^2 f_1 - \frac{1}{24}\nabla^3 f_1 - \cdots\right)$$

Neglecting fourth and higher order differences and expressing $\nabla f_1, \nabla^2 f_1$ and $\nabla^3 f_1$ in terms of function values, we obtain

$$y_1^{(c)} = y_0 + \frac{h}{24}\,(9f_1 + 19f_0 - 5f_{-1} + f_{-2}) \qquad \ldots(3)$$

which is called a *Adams-Moulton corrector formula.*

Then an improved value of f_1 is calculated and again the corrector (3) is applied to find a still better value of y_1. This step is repeated till y_1 remains unchanged and then proceed to calculate y_2 as above.

Obs. To apply both Milne and Adams-Bashforth methods, we require four starting values of y which are calculated by means of Picard's method or Taylor's series method or Euler's method or Runge-Kutta method. In practice, the Adams formulae (2) and (3) above together with fourth order Runge-Kutta formulae have been found to be most useful.

Example 32.20. *Given $\frac{dy}{dx} = x^2(1 + y)$ and $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$, evaluate $y(1.4)$ by Adams-Bashforth method.* *(V.T.U., 2010 ; J.N.T.U., 2009 ; Anna, 2004)*

Solution. Here $f(x, y) = x^2(1 + y)$.

Starting values of the Adams-Bashforth method with $h = 0.1$, are

$x = 1.0, y_{-3} = 1.000, f_{-3} = (1.0)^2(1 + 1.000) = 2.000$

$= 1.1, y_{-2} = 1.233, f_{-2} = 2.702$

$= 1.2, y_{-1} = 1.548, f_{-1} = 3.669$

$= 1.3, y_0 = 1.979, f_0 = 5.035$

Using th

$\frac{h}{\ }(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$

$73, f_1 = 7.004$

U:

$5f_{-1} + f_{-2})$

$19 \times 5.035 - 5 \times 3.669 + 2.702) = 2.575$

Hence 5.

Example 32.21. *If* $\frac{dy}{dx} = 2e^x y$, $y(0) = 0$, *find* $y(4)$ *using Adams predictor-corrector formula by calculating* $y(1)$, $y(2)$ *and* $y(3)$ *using Euler's modified formula.* (J.N.T.U., 2006)

Solution. We have $f(x, y) = 2e^x y$.

To find 0.1 :

x	$2e^x y = y'$	*Mean slope*	*Old y + h (Mean slope) = new y*
0.0	4	—	$2 + 0.1\,(4) = 2.4$
0.1	$2e^{0.1}\,(2.4) = 5.305$	$\frac{1}{2}(4 + 5.305) = 4.6524$	$2 + 0.1\,(4.6524) = 2.465$
0.1	$2e^{0.1}\,(2.465) = 5.449$	$\frac{1}{2}(4 + 5.449) = 4.7244$	$2 + 0.1\,(4.7244) = 2.472$
0.1	$2e^{0.1}\,(2.4724) = 5.465$	$\frac{1}{2}(4 + 5.465) = 4.7324$	$2 + 0.1\,(4.7324) = 2.473$
0.1	$2e^{0.1}\,(2.473) = 5.467$	$\frac{1}{2}(4 + 5.467) = 4.7333$	$2 + 0.1\,(4.7333) = 2.473$
0.1	5.467	—	$2 + 0.1\,(5.467) = 3.0199$
0.2	$2e^{0.2}\,(3.0199) = 7.377$	$\frac{1}{2}(5.467 + 7.377) = 6.422$	$2.473 + 0.1\,(6.422) = 3.1155$
0.2	7.611	$\frac{1}{2}(5.467 + 7.611) = 6.539$	$2.473 + 0.1\,(6.539) = 3.127$
0.2	7.639	$\frac{1}{2}(5.467 + 7.639) = 6.553$	$2.473 + 0.1\,(6.553) = 3.129$
0.2	7.643	$\frac{1}{2}(5.467 + 7.643) = 6.555$	$2.473 + 0.1\,(6.455) = 3.129$
0.2	7.643	—	$3.129 + 0.1\,(7.643) = 3.893$
0.3	$2e^{0.3}\,(3.893) = 10.51$	$\frac{1}{2}(7.643 + 10.51) = 9.076$	$3.129 + 0.1\,(9.076) = 4.036$
0.3	10.897	$\frac{1}{2}(7.643 + 10.897) = 9.266$	$3.129 + 0.1\,(9.2696) = 4.056$
0.3	10.949	$\frac{1}{2}(7.643 + 10.949) = 9.296$	$3.129 + 0.1\,(9.296) = 4.058$
0.3	10.956	$\frac{1}{2}(7.643 + 10.956) = 9.299$	$3.129 + 0.1\,(9.299) = 4.0586$

To find y(0.4) by Adam's method, the starting values with $h = 0.1$ are

$x = 0.0$	$y_{-3} = 2.4$	$f_{-3} = 4$
$x = 0.1$	$y_{-2} = 2.473$	$f_{-2} = 5.467$
$x = 0.2$	$y_{-1} = 3.129$	$f_{-1} = 7.643$
$x = 0.3$	$y_0 = 4.059$	$f_{-0} = 10.956$

Using the predictor formula

$$y_1^{(p)} = y_0 + \frac{h}{24}\,(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$= 4.059 + \frac{0.1}{24}\,(55 \times 10.957 - 59 \times 7.643 + 37 \times 5.467 - 9 \times 4)$$

$$= 5.383$$

Now $x = 0.4$ $\quad y_1 = 5.383 \quad f_1 = 2e^{0.4}\,(5.383) = 16.061$

Using the corrector formula,

$$y_1^{(c)} = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$= 4.0586 + \frac{0.1}{24}(9 \times 6.061 + 19 \times 10.956 - 5 \times 7.643 + 5.467) = 5.392$$

Hence $y(0.4) = 5.392$.

Example 32.22. *Solve the initial value problem $dy/dx = x - y^2$, $y(0) = 1$ to find $y(0.4)$ by Adam's method. Starting solutions required are to be obtained using Runge-Kutta method of order 4 using step value $h = 0.1$.* *(P.T.U., 2003)*

Solution. We have $f(x, y) = x - y^2$.

To find y (0.1) :

Here $x_0 = 0, y_0 = 1, h = 0.1$

$\therefore$

$$k_1 = hf(x_0, y_0) = (0.1) f(0, 1) = -0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1) f(0.05, 0.95) = -0.08525$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1) f(0.05, 0.9574) = -0.0867$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1) f(0.1, 0.9137) = -0.07341$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0883$$

Thus $y(0.1) = y_1 = y_0 + k = 1 - 0.0883 = 0.9117$

To find y(0.2) :

Here $x_1 = 0.1, y_1 = 0.9117, h = 0.1$.

$\therefore$

$$k_1 = hf(x_1, y_1) = (0.1) f(0.1, 0.9117) = -0.0731$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1) f(0.15, 0.8751) = -0.0616$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1) f(0.15, 0.8809) = -0.0626$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) f(0.2, 0.8491) = -0.0521$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0623$$

Thus $y(0.2) = y_2 = y_1 + k = 0.8494$.

To find y (0.3) :

Here $x_2 = 0.2, y_2 = 0.8494, y = 0.1$

$$k_1 = hf(x_2, y_2) = (0.1) f(0.2, 0.8494) = -0.0521$$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1) f(0.25, 0.8233) = -0.0428$$

$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1) f(0.25, 0.828) = -0.0436$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1) f(0.3, 0.8058) = -0.0349$$

$$k = \frac{1}{6}(k_1 + 2y_2 + 2k_3 + k_4) = -0.0438$$

Thus $y(0.3) = y_3 = y_2 + k = 0.8061$

Now the starting values of Adam's method with $h = 0.1$ are :

$x = 0.0$	$y_{-3} = 1.0000$	$f_{-3} = 0.0 - (1.0)^2$	$= -1.0000$
$x = 0.1$	$y_{-2} = 0.9117$	$f_{-2} = 0.1 - (0.9117)^2$	$= -1.7312$
$x = 0.2$	$y_{-1} = 0.8494$	$f_{-1} = 0.2 - (0.8494)^2$	$= -0.5215$
$x = 0.3$	$y_0 = 0.8061$	$f_0 = 0.3 - (0.8061)^2$	$= -0.3498$

Using the *predictor*,

$$y_1^{(p)} = y_0 + \frac{h}{24}\,(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$x = 0.4$
$$y_1^{(p)} = 0.8061 + \frac{0.1}{24}\,[55\,(-0.3498) - 59\,(-0.5215) + 37\,(-0.7312) - 9\,(-1)]$$
$$= 0.7789 \qquad\qquad f_1 = -0.2067$$

Using the *corrector*,

$$y_1^{(c)} = y_0 + \frac{h}{24}\,(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$y_1^{(c)} = 0.8061 + \frac{0.1}{24}\,[9\,(-0.2067) + 19\,(-0.3498) - 5\,(-0.5215) - 0.7312] = 0.7785$$

Hence $\quad y\,(0.4) = 0.7785.$

PROBLEMS 32.5

1. Using Adams-Bashforth method, obtain the solution of $dy/dx = x - y^2$ at $x = 0.8$, given the values

x :	0	0.2	0.4	0.6
y :	0	0.0200	0.0795	0.1762

(*Bhopal, 2002*)

2. Using Adams-Bashforth formulae, determine $y(0, 4)$ given the differential equation $dy/dx = \frac{1}{2}\,xy$ and the data

x :	0	0.1	0.2	0.3
y :	1	1.0025	1.0101	1.0228

3. Given $y' = x^2 - y$, $y(0) = 1$ and the starting values $y(0.1) = 0.90516$, $y(0.2) = 0.82127$, $y(0.3) = 0.74918$, evaluate $y(0, 4)$ using Adams-Bashforth method. (*S.V.T.U., 2007*)
4. Using Adams-Bashforth method, find $y(4.4)$ given $5xy' + y^2 = 2$, $y(4) = 1$, $y(4, 1) = 1.0049$, $y(4, 2) = 1.0097$ and $y(4.3) = 1.0143$.
5. Given the differential equation $dy/dx = x^2y + x^2$ and the data :

x :	1	1.1	1.2	1.3
y :	1	1.233	1.548488	1.978921

(*Indore, 2003 S*)

6. Using Adams-Bashforth method, evaluate $y\,(1.4)$, if y satisfies $dy/dx + y/x = 1/x^2$ and $y(1) = 1$, $y\,(1.1) = 0.996$, $y\,(1.2) = 0.986$, $y\,(1.3) = 0.972$. (*Madras, 2003*)

32.11 SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

The simultaneous differential equations of the type

$$\frac{dy}{dx} = f\,(x, y, z) \qquad ...(1)$$

and
$$\frac{dz}{dx} = \phi\,(x, y, z) \qquad ...(2)$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by the methods discussed in the preceding sections, especially by Picard's or Runge-Kutta methods.

(*i*) *Picard's method gives*

$$y_1 = y_0 + \int f\,(x, y_0, z_0)\,dx,\; z_1 = z_0 + \int \phi\,(x, y_0, z_0)\,dx$$

$$y_2 = y_0 + \int f\,(x, y_1, z_1)\,dx,\; z_2 = z_0 + \int \phi\,(x, y_1, z_1)\,dx$$

$$y_3 = y_0 + \int f\,(x, y_2, z_2)\,dx,\; z_3 = z_0 + \int \phi\,(x, y_2, z_2)\,dx$$

and so on.

(*ii*) *Taylor's series* method is used as follows :

If h be the step-size, $y_1 = y\,(x_0 + h)$ and $z_1 = z\,(x_0 + h)$. Then Taylor's algorithm for (1) and (2) gives

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \quad \ldots(3)$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} + z_0''' + \ldots \quad \ldots(4)$$

Differentiating (1) and (2) successively, we get y'', z'', etc. So the values $y_0', y_0'', y_0''' \ldots$ and $z_0', z_0'', z_0''' \ldots$ are known. Substituting these in (3) and (4), we obtain y_1, z_1 for the next step.

Similarly, we have the algorithms

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots \quad \ldots(5)$$

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!} z_1'' + \frac{h^3}{3!} z_1''' + \ldots \quad \ldots(6)$$

Since y_1 and z_1 are known, we can calculate $y_1', y_1'', \ldots$ and $z_1', z_1'' \ldots$. Substituting these in (5) and (6), we get y_2 and z_2.

Proceeding further, we can calculate the other values of y and z step by step.

(*iii*) *Runge-Kutta method* is applied as follows :

Starting at (x_0, y_0, z_0) and taking the step-sizes for x, y, z to be h, k, l respectively, the Runge-Kutta method gives,

$$k_1 = hf(x_0, y_0, z_0) \qquad l_1 = h\phi(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) \qquad l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) \qquad l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) \qquad l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

Hence $\quad y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad$ and $\quad z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

To compute y_2 and z_2, we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in the above formulae.

Example 32.23. *Using Picard's method find approximate values of y and z corresponding to x = 0.1, given that y(0) = 2, z(0) = 1 and dy/dx = x + z, dz/dx = x − y².*

Solution. Here $x_0 = 0, y_0 = 2, z_0 = 1$,

$$\frac{dy}{dx} = f(x, y, z) = x + z \; ; \qquad \text{and} \qquad \frac{dz}{dx} = \phi(x, y, z) = x - y^2$$

$$\therefore \qquad y = y_0 + \int_{x_0}^{x} f(x, y\, z)\, dx \qquad \text{and} \qquad z = z_0 + \int_{x_0}^{x} \phi(x, y\, z)\, dx.$$

First approximations $\quad y_1 = y_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\, dx = 2 + \int_0^x (x+1)\, dx = 2 + x + \frac{1}{2}x^2$

$$z_1 = z_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\, dx = 1 + \int_0^x (x - 4)\, dx = 1 - 4x + \frac{1}{2}x^2$$

Second approximations $\quad y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\, dx = 2 + \int_0^x \left(x + 1 - 4x + \frac{1}{2}x^2\right) dx$

$$= 2 + x - \frac{3}{2}x^2 + \frac{x^3}{6}$$

$$z_2 = z_0 + \int_{x_0}^{x} \phi(x, y_1, z_1)\, dx$$

$$= 1 + \int_{x_0}^{x} \left[x - \left(2 + x + \frac{1}{2}x^2\right)^2\right] dx = 1 - 4x + \frac{3}{2}x^2 - x^3 - \frac{x^4}{4} - \frac{x^5}{20}.$$

Third approximations $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2, z_2)\, dx$

$$= 2 + x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{4}x^4 - \frac{1}{20}x^5 - \frac{1}{120}x^6$$

$$z_3 = z_0 + \int_{x_0}^{x} \phi(x, y_2, z_2)\, dx$$

$$= 1 - 4x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{12}x^4 - \frac{31}{60}x^5 + \frac{1}{12}x^6 - \frac{1}{252}x^7$$

and so on.

When $x = 0.1$,

$$y_1 = 2.105,\ y_2 = 2.08517,\ y_3 = 2.08447$$

$$z_1 = 0.605,\ z_2 = 0.58397,\ z_3 = 0.58672.$$

Hence $y(0.1) = 2.0845,\ z(0.1) = 0.5867$

correct to four decimal places.

Example 32.24. *Solve the differential equations*

$$\frac{dy}{dx} = 1 + xz,\quad \frac{dz}{dx} = -xy \text{ for } x = 0.3,$$

using fourth order Runge-Kutta method. Initial values are $x = 0, y = 0, z = 1$.

Solution. Here $f(x, y, z) = 1 + xz,\ \phi(x, y, z) = -xy$

$$x_0 = 0,\ y_0 = 0,\ z_0 = 1. \text{ Let us take } h = 0.3.$$

$\therefore$ $k_1 = h f(x_0, y_0, z_0) = 0.3 f(0, 0, 1) = 0.3(1 + 0) = 0.3$

$$l_1 = h\phi(x_0, y_0, z_0) = 0.3(-0 \times 0) = 0$$

$$k_2 = h f\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1,\ z_0 + \frac{1}{2}l_1\right)$$

$$= (0.3) f(0.15, 0.15, 1) = 0.3(1 + 0.15) = 0.345$$

$$l_2 = h\phi\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1,\ z_0 + \frac{1}{2}l_1\right)$$

$$= 0.3[-(0.15)(0.15)] = -0.00675.$$

$$k_3 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$

$$= (0.3) f(0.15, 0.1725, 0.996625)$$

$$= 0.3[1 + 0.996625 \times 0.15] = 0.34485$$

$$l_3 = h\phi\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$

$$= 0.3[-(0.15)(0.1725)] = -0.007762$$

$$k_4 = h f(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$

$$= (0.3) f(0.3, 0.34485, 0.99224) = 0.3893$$

$$l_4 = h\phi(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$

$$= 0.3[-(0.3)(0.34485)] = -0.03104$$

Hence $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

i.e., $y(0.3) = 0 + \frac{1}{6}[0.3 + 2(0.345) + 2(0.34485) + 0.3893] = 0.34483$

and $z(x_0 + h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

i.e., $z(0.3) = 1 + \frac{1}{6}[0 + 2 + (-0.00675) + 2(-0.0077625) + (-0.03104)] = 0.98999$

32.12 SECOND ORDER DIFFERENTIAL EQUATIONS

Consider the second order differential equation $\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$

By writing $dy/dx = z$, it can be reduced to two first order simultaneous differential equations

$$\frac{dy}{dx} = z, \qquad \frac{dz}{dx} = f(x, y, z)$$

These equations can be solved as explained above.

Example 32.25. *Using Runge-Kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$ correct to 4 decimal places. Initial conditions are $x = 0$, $y = 1$, $y' = 0$.* (Delhi, 2002)

Solution. Let $dy/dx = z = f(x, y, z)$. Then $dz/dx = xz^2 - y^2 = \phi(x, y, z)$

We have $x_0 = 0, y_0 = 1, z_0 = 0, h = 0.2$.

Using k_1, k_2 ... for $f(x, y, z)$ and $l_1, l_2, ...$ for $\phi(x, y, z)$, Runge-Kutta formulae become

$$k_1 = hf(x_0, y_0, z_0) = 0.2\,(0) = 0$$

$$l_1 = h\phi(x_0, y_0, z_0) = 0.2(-1) = -0.2$$

$$k_2 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1, z_0 + \tfrac{1}{2}l_1) = 0.2\,(-0.1) = -0.02$$

$$l_2 = h\phi(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1, z_0 + \tfrac{1}{2}l_1) = 0.2(-0.999) = -0.1998$$

$$k_3 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2, z_0 + \tfrac{1}{2}l_2) = 0.2\,(-0.0999) = -0.02$$

$$l_3 = h\phi(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2, z_0 + \tfrac{1}{2}l_2) = 0.2(-0.9791) = -0.1958$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2\,(-0.1958) = -0.0392$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2(0.9527) = -0.1905$$

$$\therefore \quad k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0199$$

$$l = \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + 2l_4) = -0.1970$$

Hence at $x = 0.2$,

$$y = y_0 + k = 1 - 0.0199 = 0.9801$$

and $$y' = z = z_0 + l = 0 - 0.1970 = -0.1970.$$

Example 32.26. *Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$, obtain y for $x = 0(0.1)\,0.3$ by any method. Further, continue the solution by Milne's method to calculate $y(0.4)$.* (Anna, 2004; Madras, 2003 S)

Solution. Putting $y' = z$, the given equation reduces to the simultaneous equations

$$z' + xz + y = 0, \; y' = z \qquad ...(i)$$

We employ Taylor's series method to find y.

Differentiating the given equation n times, we get

$$y_{n+2} + xy_{n+1} + ny_n + y_n = 0$$

At $$x = 0, \; (y_{n+2})_0 = -(n+1)(y_n)_0$$

$\therefore$ $$y(0) = 1, \text{ gives } y_2(0) = -1, \; y_4(0) = 32, \; y_6(0) = -5 \times 3, ...$$

and $$y_1(0) = 0 \text{ yields } y_3(0) = y_5(0) = ... = 0.$$

Expanding $y(x)$ by Taylor's series, we have

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + ...$$

$$\therefore \quad y(x) = 1 - \frac{x^2}{2!} + \frac{3}{4!}x^4 - \frac{5 \times 3}{6!}x^6 + ... \qquad ...(ii)$$

and $$z(x) = y'(x) = -x + \frac{1}{2}x^3 - \frac{1}{8}x^5 = ... = -xy \qquad ...(iii)$$

From (ii), we have

$$y(0.1) = 1 - \frac{(0.1)^2}{2} + \frac{1}{8}(0.1)^4 - ... = 0.995$$

$$y(0.2) = 1 - \frac{(0.2)^2}{2} + \frac{(0.2)^4}{8} - \ldots = 0.9802$$

$$y(0.3) = 1 - \frac{(0.3)^2}{2} + \frac{(0.3)^4}{8} - \frac{(0.3)^6}{48} + \ldots = 0.956$$

From (*iii*), we have

$$z(0.1) = -0.0995, z(0.2) = -0.196, z(0.3) = -0.2863.$$

Also from (*i*), $z'(x) = -(xz + y) \quad \therefore \quad z'(0.1) = 0.985, z'(.2) = -0.941, z'(0.3) = -0.87.$

Applying Milne's predictor formula, first to z and then to y, we obtain

$$z(0.4) = z(0) + \frac{4}{3}(0.1)\{2z'(0.1) - z'(0.2) + 2z'(0.3)\}$$

$$= 0 + \left(\frac{0.4}{3}\right)\{-1.79 + 0.941 - 1.74\} = -0.3692$$

and

$$y(0.4) = y(0) + \frac{4}{3}(0.1)\{2y'(0.1) - y'(0.2) + 2y'(0.3)\}$$

$$= 0 + \left(\frac{0.4}{3}\right)\{-0.199 + 0.196 - 0.5736\} = 0.9231 \qquad [\because \quad y' = z]$$

Also $z'(0.4) = -\{x(0.4)z(0.4) + y(0.4)\} = \{0.4(-0.3692) + 0.9231\} = -0.7754.$

Now applying Milne's corrector formula, we get

$$z(0.4) = z(0.2) + \frac{h}{3}\{z'(0.2) + 4z'(0.3) + z'(0.4)\}$$

$$= -0.196 + \left(\frac{0.1}{3}\right)\{-0.941 - 3.48 - 0.7754\} = -0.3692$$

and

$$y(0.4) = y(0.2) + \frac{h}{3}\{y'(0.2) + 4y'(0.3) + y'(0.4)\}$$

$$= 0.9802 + \left(\frac{0.1}{3}\right)\{-0.196 - 1.1452 - 0.3692\} = 0.9232$$

Hence $\quad y(0.4) = 0.9232$ and $z(0.4) = -0.3692.$

PROBLEMS 32.6

1. Apply Picard's method to find the third approximation to the values of y and z, given that $dy/dx = z$, $dz/dx = x^3(y + z)$, given $y = 1$, $z = \frac{1}{2}$ when $x = 0$.
2. Solve the following differential equations using Taylor series method of the 4th order, for $x = 0.1$ and 0.2, $\frac{dy}{dx} = xz + 1, \frac{dz}{dy} = -xy$; $y(0) = 0$ and $z(0) = 1$.
3. Find $y(0.1)$, $z(0.1)$, $y(0.2)$ and $z(0.2)$ from the system of equations $y' = x + z$, $z' = x - y^2$ given $y(0) = 0$, $z(0) = 1$ using Runge-Kutta of 4th order. (*J.N.T.U., 2009*)
4. Using Picard's method, obtain the second approximation to the solution of
$$\frac{d^2y}{dx^2} = x^3\frac{dy}{dx} + x^3y \text{ so that } y(0) = 1, y'(0) = \frac{1}{2}.$$
5. Use Picard's method to approximate y when $x = 0.1$, given that $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$ and $y = 0.5$, $\frac{dy}{dx} = 0.1$, when $x = 0$.
6. Using Runge-Kutta method of order four, solve $y'' = y + xy'$, $y(0) = 1$, $y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$.
7. Consider the second order value problem $y'' - 2y' + 2y = e^{2t}\sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$. Using the fourth order Runge-Kutta method, find $y(0.2)$. (*Anna, 2003*)

8. The angular displacement θ of a simple pendulum is given by the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0$$

where $l = 98$ cm and $g = 980$ cm/sec^2. If $\theta = 0$ and $d\theta/dt = 4.472$ at $t = 0$, use Runge-Kutta method to find θ and $d\theta/dt$ when $t = 0.2$ sec.

32.13 BOUNDARY VALUE PROBLEMS

Such a problem requires the solution of a differential equation in a region R subject to the various conditions on the boundary of R. Practical applications give rise to many such problems. We shall discuss two-point linear boundary value problems of the following types :

(i) $\dfrac{d^2y}{dx^2} + \lambda(x)\dfrac{dy}{dx} + \mu(x)y = \gamma(x)$ with the conditions $y(x_0) = a$, $y(x_n) = b$.

(ii) $\dfrac{d^4y}{dx^4} + \lambda(x)y = \mu(x)$ with the conditions $y(x_0) = y'(x_0) = a$ and $y(x_n) = y'(x_n) = b$.

While there exist many numerical methods for solving such boundary value problems, the method of finite-differences is most commonly used. We shall explain this method in the next section.

32.14 FINITE-DIFFERENCE METHOD

In this method, the derivatives appearing in the differential equation and the boundary conditions are replaced by their finite-difference approximations and the resulting linear system of equations are solved by any standard procedure. These roots are the values of the required solution at the pivotal points.

The finite-difference approximations to the various derivatives are derived as under :

If $y(x)$ and its derivatives are single-valued continuous functions of x then by Taylor's expansion, we have

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^2}{3!}y'''(x) + \ldots \qquad \ldots(1)$$

and $$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - \frac{h^3}{3!}y'''(x) + \ldots \qquad \ldots(2)$$

Equation (1) gives $y'(x) = \dfrac{1}{h}[y(x+h) - y(x)] - \dfrac{h}{2}y''(x) - \ldots$

i.e., $$y'(x) = \frac{1}{h}[y(x+h) - y(x)] + O(h)$$

which is the *forward difference approximation of* $y'(x)$ with an error of the order h.

Similarly (2) gives $y'(x) = \dfrac{1}{h}[y(x) - y(x-h)] + O(h)$

which is the *backward difference approximation of* $y'(x)$ with an error of the order h.

Subtracting (2) from (1), we obtain

$$y'(x) = \frac{1}{2h}[y(x+h) - y(x-h)] + O(h^2)$$

which is the *central-difference approximation of* $y'(x)$ with an error of the order h^2. Clearly this central difference approximation to $y'(x)$ is better than the forward or backward difference approximations and hence should be preferred.

Adding (1) and (2), we get

$$y''(x) = \frac{1}{h^2}[y(x+h) - 2y(x) + y(x-h)] + O(h^2)$$

which is the central difference approximation of $y''(x)$. Similarly we can derive central difference approximations to higher derivatives.

Hence *the working expressions for the central difference approximations to the first four derivatives of* y_i *are as under :*

$$y'_i = \frac{1}{2h}(y_{i+1} - y_{i-1}) \qquad ...(3)$$

$$y''_i = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \qquad ...(4)$$

$$y'''_i = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}) \qquad ...(5)$$

$$y_i^{iv} = \frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i+1} + y_{i-2}) \qquad ...(6)$$

Obs. The accuracy of this method depends on the size of the sub-interval h and also on the order of approximation. As we reduce h, the accuracy improves but the number of equations to be solved also increases.

Example 32.27. *Solve the equation $y'' = x + y$ with the boundary conditions $y(0) = y(1) = 0$.* (Calicut, 1999)

Solution. We divide the interval (0, 1) into four sub-intervals so that $h = 1/4$ and the pivot points are $x_0 = 0, x_1 = 1/4, x_2 = 1/2, x_3 = 3/4$ and $x_4 = 1$.

The differential equation is approximated as

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] = x_i + y_i$$

or
$$16y_{i+1} - 33y_i + 16y_{i-1} = x_i, \; i = 1, 2, 3.$$

Using $y_0 = y_4 = 0$, we get the system of equations

$$16y_2 - 33y_1 = \frac{1}{4}$$

$$16y_3 - 33y_2 + 16y_1 = \frac{1}{2}$$

$$-33y_3 + 16y_2 = \frac{3}{4}$$

Their solution gives

$$y_1 = -0.03488, y_2 = -0.05632, y_3 = -0.05003.$$

Obs. The exact solution being $y(x) = \frac{\sinh x}{\sinh 1} - x$, the error at each nodal point is given in the table:

x	*Computed value y(x)*	*Exact value y(x)*	*Error*
0.25	− 0.03488	− 0.03505	0.00017
0.5	− 0.05632	− 0.05659	0.00027
0.75	− 0.05003	− 0.05028	0.00025

Example 32.28. *Determine values of y at the pivotal points of the interval (0, 1), if y satisfies the boundary value problem $y^{iv} + 81y = 81x^2$, $y(0) = y(1) = y''(0) = y''(1) = 0$. (Take $n = 3$).*

Solution. Here $h = 1/3$ and the pivotal points are $x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 1$. The corresponding y-values are $y_0(= 0), y_1, y_2, y_3 (= 0)$.

Replacing y^{iv} by its central difference approximation, the differential equation becomes

$$\frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + 81y_i = 81x_i^2$$

or
$$y_{i+2} - 4y_{i+1} + 7y_i - 4y_{i-1} + y_{i-2} = x_i^2, \; i = 1, 2$$

At $i = 1$, $\quad y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = 1/9$

At $i = 2$, $\quad y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 4/9$

Using $y_0 = y_3 = 0$, we get $\quad -4y_2 + 7y_2 + y_{-1} = 1/9 \qquad ...(i)$

$$y_4 + 7y_1 - 4y_1 = 4/9 \qquad ...(ii)$$

Regarding the conditions $y''_0 = y''_3 = 0$, we know that

$$x_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

At $i = 0$, $\quad y_0'' = 9(y_1 - 2y_0 + y_{-1})$

or $\quad y_{-1} = -y_1 \qquad [\because \; y_0 = y_0'' = 0] \quad ...(iii)$

At $i = 3$, $\quad y_3'' = 9(y_4 - 2y_3 + y_2)$

$y_4 = -y_2 \qquad [\because \; y_3 = y_3'' = 0] \quad ...(iv)$

Using (*iii*), the equation (*i*) becomes

$$-4y_2 + 6y_1 = 1/9 \qquad ...(v)$$

Using (*iv*), the equation (*ii*) reduces to

$$6y_2 - 4y_1 = 4/9 \qquad ...(vi)$$

Solving (*v*) and (*vi*), we obtain

$$y_1 = 11/90 \text{ and } y_2 = 7/45.$$

Hence $y(1/3) = 0.1222$ and $y(2/3) = 0.1556$.

Example 32.29. *The deflection of a beam is governed by the equation*

$$\frac{d^4y}{dx^4} + 81y = \phi(x)$$

where $\phi(x)$ is given by the table

[illegible]	:	1/3	2/3	1,
$\phi(x)$	:	81	162	243,

and boundary condition $y(0) = y'(0) = y''(1) = y'''(1) = 0$. Evaluate the deflection at the pivotal points of the beam using three sub-intervals.

Solution. Here $h = 1/3$ and the pivotal points are $x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 1$. The corresponding y-values are $y_0(= 0), y_1, y_2, y_3$.

The given differential equation is approximated to

$$\frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + 81y_i = \phi(x_i)$$

At $i = 1$, $\quad y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = 1 \qquad ...(i)$

At $i = 2$, $\quad y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 2 \qquad ...(ii)$

At $i = 3$, $\quad y_5 - 4y_4 + 7y_3 - 4y_2 + y_1 = 3 \qquad ...(iii)$

We have $\quad y_0 = 0 \qquad ...(iv)$

Since $\quad y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1})$

$\therefore$ for $i = 0$, $\quad 0 = y_0' = \frac{1}{2h}(y_1 - y_{-1})$ *i.e.* $y_{-1} = y_1 \qquad ...(v)$

Since $\quad y_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$

$\therefore$ for $i = 3$, $\quad 0 = y_3'' = \frac{1}{h^2}(y_4 - 2y_3 + y_2)$, *i.e.* $y_4 = 2y_3 - y_2 \qquad ...(vi)$

Also $\quad y_i''' = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} 2y_{i-1} - y_{i-2})$

$\therefore$ for $i = 3$, $\quad 0 = y_3''' = \frac{1}{2h^3}(y_5 - 2y_4 + 2y_2 - y_1)$

$$y_5 = 2y_4 - 2y + y_1 \qquad ...(vii)$$

Using (*iv*) and (*v*), the equation (*i*) reduces to

$$y_3 - 4y_2 + 8y_1 = 1 \qquad ...(viii)$$

Using (*iv*) and (*vi*), the equation (*ii*) becomes

$$-y_3 + 3y_2 - 2y_1 = 1 \qquad ...(ix)$$

Using (*vi*) and (*vii*), the equation (*iii*) reduces to

$$3y_3 - 4y_2 + 2y_1 = 3 \qquad ...(x)$$

Solving (*viii*), (*ix*) and (*x*), we get

$$y_1 = 8/13, y_2 = 22/13, y_3 = 37/13.$$

Hence $\quad y(1/3) = 0.6154, y(2/3) = 1.6923, y(1) = 2.8462.$

PROBLEMS 32.7

1. Solve the boundary value problem for $x = 0.5$:

$$\frac{d^2y}{dx^2} + y + 1 = 0, y(0) = y(1) = 0. \qquad \text{(Take } n = 4)$$

2. Find an approximate solution of the boundary value problem :

$$y'' + 8(\sin^2 \pi y)y = 0, 0 \le x \le 1, y(0) = y(1) = 1. \qquad \text{(Take } n = 4)$$

3. Solve the boundary value problem

$$xy'' + y = 0, y(1) = 1, y(2) = 2. \qquad \text{(Take } n = 4)$$

4. Solve the equation

$$y'' - 4y' + 4y = e^{3x}, \text{ with the conditions } y(0) = 0, y(1) = -2, \text{ taking } n = 4.$$

5. Solve the boundary value problem $y'' - 64y + 10 = 0$ with $y(0) = y(1) = 0$ by the finite difference method. Compute the value of $y(0.5)$ and compare with the true value.
6. Solve the boundary value problem

$$y'' + xy' + y = 3x^2 + 2, y(0) = 0, y(1) = 1.$$

7. The boundary value problem governing the deflection of a beam of length 3 metres is given by

$$\frac{d^4y}{dx^4} + 2y = \frac{1}{9}x^2 + \frac{2}{3}x + 4, y(0) = y'(0) = y(3) = y'(3) = 0.$$

The beam is built-in at the left end ($x = 0$) and simply supported at the right end ($x = 3$). Determine y at the pivotal points $x = 1$ and $x = 2$.

8. Solve the boundary value problem,

$$\frac{d^4y}{dx^4} + 81y = 729x^2, y(0) = y'(0) = y''(1) = y'''(1) = 0. \qquad \text{(Use } n = 3)$$

9. Solve the equation $y^{iv} - y''' + y = x^2$ subject to the boundary conditions

$$y(0) = y'(0) = 0 \text{ and } y(1) = 2, y'(1) = 0. \qquad \text{(Take } n = 5)$$

32.15 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 32.8

Select the correct answer or fill up the blanks in the following questions :

1. Which of the following is a step by step method :
   (*a*) Taylor' s (*b*) Adams-Bashforth (*c*) Picard's (*d*) None.
2. The finite difference scheme for the equation $2y'' + y = 5$ is ....
3. If $y'' = x + y, y(0) = 1$ and $y^{(1)}(x) = 1 + x + x^2/2$, then by Picard's method, the value of $y^{(2)}(x)$ is ....
4. The iterative formula of Euler's method for solving $y' = f(x, y)$ with $y(x_0) = y_0$, is ....
5. Taylor's series for solution of first order ordinary differential equations is ....
6. Using Runge-Kutta method of order four, the value of $y(0.1)$ for $y' = x - 2y, y(0) = 1$ taking $h = 0.1$ is
   (*a*) 0.813 (*b*) 0.825 (*c*) 0.0825 (*d*) none.
7. Given y_0, y_1, y_2, y_3, Milne's corrector formula to find y_4 for $dy/dx = f(x, y)$, is ....
8. The second order Runge-Kutta formula is ....
9. Adams-Bashforth predictor formula to solve $y' = f(x, y)$ given $y_0 = y(x_0)$ is ....
10. The multi-step methods available for solving ordinary differential equations are ....
11. To predict Adam's method atleast ...... values of y, prior to the desired value, are required.
12. Taylor's series solution of $y' = -xy, y(0) = 1$ upto x^4 is ....

13. Using modified Euler's method, the value of $y(0.1)$ for $\frac{dy}{dx} = x - y, y(0) = 1$ is ....
    (*a*) 0.809 (*b*) 0.909 (*c*) 0.0809 (*d*) none.
14. Milne's Predictor formula is ....
15. Adam's corrector formula is ....
16. Using Euler's method, $dy/dx = (y - 2x)/y, y(0) = 1$; gives $y(0.1) =$ ....
17. $\frac{d^2y}{dx^2} + y^2 \frac{dy}{dx} + y = 0$ is equivalent to a set of two first order differential equations ...... and ....
18. The formula for the 4th order Runge-Kutta method is ....
19. Taylor's series method will be useful to give some .... of Milne's method.
20. The name of two self-starting methods to solve $y' = f(x, y)$ given $y(x_0) = y_0$ are ......
21. In the derivation of fourth order Runge-Kutta formula, it is called fourth order because ......
22. If $y' = x, y(0) = 1$ then by Picard's method, the value of $y(1)$ is ......
    (*a*) 0.915 (*b*) 0.905 (*c*) 0.981 (*d*) none.
23. The finite difference scheme of the differential equation $y'' + 2y = 0$ is ......
24. If $y' = -y, y(0) = 1$, the Euler's method, the value of $y(1)$ is
    (*a*) 0.99 (*b*) 0.999 (*c*) 0.981 (*d*) none.
25. In Euler's method if h is small the method is too slow, if h is large, it gives inaccurate value. (True or False)
26. Runge-Kutta method is a self-starting method. (True or False)
27. Predictor-corrector methods are self-starting methods. (True or False)