CHAPTER 16

Series Solution of Differential Equations and Special Functions

1. Introduction. 2. Validity of series solution. 3. Series solution when $x = 0$ is an ordinary point. 4. Frobenius method. 5. Bessel's equation. 6. Recurrence formulae for $J_n(x)$. 7. Expansions for J_0 and J_1. 8. Value of $J_{1/2}$. 9. Generating function for $J_n(x)$. 10. Equations reducible to Bessel's equation. 11. Orthogonality of Bessel functions; Fourier-Bessel expansion of $f(x)$. 12. Ber and Bei functions. 13. Legendre's equation. 14. Rodrigue's formula, Legendre polynomials. 15. Generating function for $P_n(x)$. 16. Recurrence formulae for $P_n(x)$. 17. Orthogonality of Legendre polynomials, Fourier-Legendre expansion for $f(x)$. 18. Other special functions. 19. Strum-Liouville problem, Orthogonality of eigen functions. 20. Objective Type of Questions.

16.1 INTRODUCTION

Many differential equations arising from physical problems are linear but have variable coefficients and do not permit a general solution in terms of known functions. Such equations can be solved by numerical methods (Chapter 28), but in many cases it is easier to find a solution in the form of an infinite convergent series.

The series solution of certain differential equations give rise to special functions such as Bessel's function, Legendre's polynomial, Lagurre's polynomial, Hermite's polynomial, Chebyshev polynomials. Strum-Lioville problem based on the orthogonality of functions is also included which shows that Bessel's, Legendre's and other equations can be considered from a common point of view. These special functions have many applications in engineering.

16.2 VALIDITY OF SERIES SOLUTION OF THE EQUATION

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \qquad ...(i)$$

can be determined with the help of the following theorems :

Def. 1. If $P_0(a) \neq 0$, then $x = a$ is called and **ordinary point** of (i), otherwise a **singular point.**

2. *A singular point $x = a$ of (1) is called* **regular** *if, when (i) is put in the form*

$$\frac{d^2y}{dx^2} + \frac{Q_1(x)}{x-a}\frac{dy}{dx} + \frac{Q_2(x)}{(x-a)^2}y = 0,$$

$Q_1(x)$ and $Q_2(x)$ possess derivatives of all orders in the neighbourhood of a.

3. *A singular point which is not regular is called an* **irregular singular point.**

Theorem I. *When $x = a$ is an ordinary point of (i), its every solution can be expressed in the form*

$$y = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots \qquad ...(ii)$$

Theorem II. *When $x = a$ is a regular singularity of (i), at least one of the solutions can be expressed as*

$$y = (x-a)^m\,[a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots] \qquad ...(iii)$$

Theorem III. *The series (ii) and (iii) are convergent at every point within the circle of convergence at a.* A solution in series will be valid only if the series is convergent.

16.3 SERIES SOLUTION WHEN X = 0 IS AN ORDINARY POINT OF THE EQUATION

$$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2\, y = 0 \qquad ...(1)$$

where P's are polynomials in x and $P_0 \neq 0$ at $x = 0$.

(*i*) Assume its solution to be of the form $y = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$ ...(2)

(*ii*) Calculate dy/dx, d^2y/dx^2 from (2) and substitute the values of y, dy/dx, d^2y/dx^2 in (1).

(*iii*) Equate to zero the coefficients of the various powers of x and determine a_2, a_3, a_4 ... in terms of a_0, a_1. (The result obtained by equating to zero is the coefficient of x^n that is called the *recurrence relation*).

(*iv*) Substituting the values of a_2, a_3, a_4 ... in (2), we get the desired series solution having a_0, a_1 as its arbitrary constants.

Example 16.1. *Solve in series the equation* $\frac{d^2y}{dx^2} + xy = 0.$ *(V.T.U., 2010)*

Solution. Here $x = 0$ is an ordinary point since coefficient of $y'' \neq 0$ at $x = 0$.

Assume its solution is $y = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...$ ...(*i*)

Then $$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + ... + na_nx^{n-1} + ...$$

and $$\frac{d^2y}{dx^2} = 2 \,.\, 1a_2 + 3 \,.\, 2a_3x + 4 \,.\, 3a_4x^2 + ... + n(n-1)a_nx^{n-2} + ...$$

Substituting in the given differential equation

$$1 \,.\, 1a_2 + 3 \,.\, 2a_3x + 4 \,.\, 3a_4x^2 + ... + n(n-1)a_nx^{n-2} + ... + x(a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...) = 0$$

or $$2 \,.\, 1a_2 + (3 \,.\, 2a_3 + a_0)x + (4 \,.\, 3a_4 + a_1)x^2 + (5 \,.\, 4a_5 + a_2)x^3 + ... + [(n+2)(n+1)a_{n+2} + a_{n-1}]x^n + ... = 0.$$

Equating to zero the co-efficients of the various powers of x,

$a_2 = 0,$ [Coeff. of $x^0 = 0$]

$3 \,.\, 2a_3 + a_0 = 0$, *i.e.*, $a_3 = -\frac{a_0}{3!}$ [Coeff. of $x = 0$]

$4 \,.\, 3a_4 + a_1 = 0$, *i.e.*, $a_4 = -\frac{2a_1}{4!}$ [Coeff. of $x^2 = 0$]

$5 \,.\, 4a_5 + a_2 = 0$, *i.e.*, $a_5 = -\frac{a_2}{5 \,.\, 4} = 0$ and so on. [Coeff. of $x^3 = 0$]

In general, $(n+2)(n+1)a_{n+2} + a_{n-1} = 0$ [Coeff. of $x^n = 0$]

i.e., $$a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)} \qquad ..(ii)$$

which is the *recurrence relation.*

Putting $n = 4, 5, 6, ...$ in (*ii*) successively, $a_6 = -\frac{a_3}{6 \,.\, 5} = \frac{4a_0}{6!}$; $a_7 = -\frac{a_4}{7 \,.\, 6} = \frac{5 \,.\, 2a_1}{7!}$

$$a_8 = -\frac{a_5}{8 \,.\, 7} = 0;\ a_9 = -\frac{a_6}{9 \,.\, 8} = -\frac{7 \,.\, 4a_0}{9!} \text{ and so on.}$$

Substituting these values in (*i*), we get

$$y = a_0\left(1 - \frac{x^3}{3!} + \frac{1 \,.\, 4x^6}{6!} - \frac{1 \,.\, 4 \,.\, 7x^9}{9!} + ...\right) + a_1\left(x - \frac{2x^4}{4!} + \frac{2 \,.\, 5x^7}{7!} - ...\right)$$

which is the required solution.

Example 16.2. *Solve in series* $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 4y = 0.$ *(Bhopal, 2008 ; U.P.T.U., 2006)*

Solution. Here $x = 0$ is an ordinary point since coefficient of $y'' \neq 0$ at $x = 0$.

Assume the solution of the given equation to be

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots \qquad \ldots(i)$$

Then $$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots$$

and $$\frac{d^2y}{dx^2} = 2a_2 + 3.2a_3x + 4.3a_4x^2 + \ldots + n(n-1)a_nx^{n-2} + \ldots$$

Substituting in the given equation, we get

$$(1-x^2)[2a_2 + 3.2a_3x + 4.3a_4x^2 + \ldots + n(n-1)a_nx^{n-2} + \ldots]$$
$$- x[a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots] + 4[a_0 + a_1x + a_2x^2 + \ldots. + a_nx^n + \ldots] = 0$$

Equating to zero the coefficients of the various powers of x,

$$2a_2 + 4a_0 = 0 \quad i.e., \quad a_2 = -2a_0 \qquad [\text{coeff. of } x^0 = 0]$$

$$3.2a_3 - a_1 + 4a_1 = 0 \quad i.e., \quad a_3 = -\frac{1}{2}a_1 \qquad [\text{coeff. of } x^1 = 0]$$

$$4.3a_4 - 2a_2 - 2a_2 + 4a_2 = 0 \quad i.e., \quad a_4 = 0 \qquad [\text{coeff. of } x^2 = 0]$$

$$5.4a_5 - 3.2a_3 - 3a_3 + 4a_3 = 0 \qquad [\text{coeff. of } x^3 = 0]$$

i.e., $$20a_5 - 5a_3 = 0 \quad i.e., \quad a_5 = -\frac{a_1}{8} \text{ and so on.}$$

In general, $(n+2)(n+1)a_{n+2} - n(n-1)a_n - na_n + 4a_n = 0$

or $$a_{n+2} = \frac{n-2}{n+1}a_n \qquad \ldots(ii)$$

which is the *recurrence relation*

Putting $n = 4, 5, 6, 7\ldots$ in (ii) successively,

$$a_6 = 0; \quad a_7 = \frac{3}{6}a_5 = -\frac{3}{6}\frac{a_1}{8}; a_8 = 0; a_9 = -\frac{5.3}{8.6}.\frac{a_1}{8}\ldots$$

Substituting these values in (i), we get

$$y = a_0(1-2x^2) + a_1x\left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{3}{6}.\frac{x^6}{8} - \frac{5.3}{8.6}.\frac{x^8}{8} - \ldots\right).$$

PROBLEMS 16.1

Solve the following equations in series :

1. $\dfrac{d^2y}{dx^2} + y = 0$, given $y(0) = 0$. *(B.P.T.U., 2005 S)*
2. $\dfrac{d^2y}{dx^2} + x^2y = 0$.
3. $y'' + xy' + y = 0$. *(V.T.U., 2008)*
4. $(1-x^2)y'' + 2y = 0$, given $y(0) = 4$, $y'(0) = 5$. *(P.T.U., 2006)*
5. $(1+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$. *(S.V.T.U., 2008)*
6. $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = 0$. *(U.P.T.U., 2004)*

16.4 FROBENIUS* METHOD : Series solution when x = 0 is a regular sigularity of the equation

$$\mathbf{P_0\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = 0} \qquad \ldots(1)$$

*A German mathematician *F.G. Frobenius* (1849–1917) who is known for his contributions to the theory of matrices and groups.

(*i*) Assume the solution to be $y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots)$ ...(2)

(*ii*) Substitute from (2) for y, dy/dx, d^2y/dx^2 in (1) as before.

(*iii*) Equate to zero the coefficient of the lowest degree term in x. It gives a quadratic equation known as the *indicial equation*.

(*iv*) Equating to zero the coefficients of the other powers of x, find the values of $a_1, a_2, a_3,\ldots$ in terms of a_0.

The complete solution depends on the nature of roots of the indicial equation.

Case I. When roots of the indicial equation are distinct and do not differ by an integer, *the complete solution is*

$$y = c_1(y)_{m_1} + c_2(y)_{m_2}$$

where m_1, m_2 are the roots.

Example 16.3. *Solve in series the equation* $9x(1-x)\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 4y = 0.$

(*Madras, 2006 ; Roorkee, 2000*)

Solution. Here $x = 0$ is a singular point since coefficient of $y'' = 0$ at $x = 0$.

Substituting $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots$

$$\therefore \quad \frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots$$

and
$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots$$

in the given equation, we obtain

$$9x(1-x)\,[m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots]$$
$$-\,12[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots] + 4[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots] = 0.$$

The lowest power of x is x^{m-1}. Its coefficient equated to zero gives

$$a_0(9m(m-1) - 12m] = 0, \text{ i.e., } m(3m-7) = 0 \quad \text{as} \quad a_0 \neq 0.$$

Thus the roots of the *indicial equation* are $m = 0, 7/3$. *i.e., Roots are distinct and do not differ by an integer.*

The coefficient of x^m equated to zero gives $a_1\{9(m+1)m - 12(m+1)\} + a_0\{4 - 9m(m-1)\} = 0$

i.e., $$3a_1(3m-4)(m+1) - a_0(3m-4)(3m+1) = 0$$

i.e., $$3a_1(m+1) = a_0(3m+1).$$

Similarly $3a_2(m+2) = a_1(3m+4)$, $3a_3(m+3) = a_2(5m+7)$ and so on.

$$\therefore \quad a_1 = \frac{3m+1}{3(m+1)}a_0,\ a_2 = \frac{(3m+4)a_1}{3(m+2)} = \frac{(3m+4)(3m+1)}{3^2(m+2)(m+1)}a_0,\ a_3 = \frac{(3m+7)(3m+4)(3m+1)}{3^3(m+3)(m+2)(m+1)}a_0 \text{ etc.}$$

When $m = 0$, $a_1 = \dfrac{1}{3}a_0$, $a_2 = \dfrac{1\,.\,4}{3\,.\,6}a_0$, $a_3 = \dfrac{1\,.\,4\,.\,7}{3\,.\,6\,.\,9}a_0$ etc. giving the particular solution

$$y_1 = a_0\left[1 + \frac{1}{3}x + \frac{1\,.\,4}{3\,.\,6}x^2 + \frac{1\,.\,4\,.\,7}{3\,.\,6\,.\,9}x^3 + \ldots\right]$$

When $m = 7/3$, the particular solution is

$$y_2 = a_0x^{7/3}\left[1 + \frac{8}{10}x + \frac{8\,.\,11}{10\,.\,13}x^2 + \frac{8\,.\,11\,.\,14}{10\,.\,13\,.\,16}x^3 + \ldots\right]$$

Thus the complete solution is $y = c_1y_1 + c_2y_2$

i.e.,
$$y = C_1\left[1 + \frac{1}{3}x + \frac{1\,.\,4}{3\,.\,6}x^2 + \frac{1\,.\,4\,.\,7}{3\,.\,6\,.\,9}x^3 + \ldots\right]$$
$$+ C_2x^{7/3}\left[1 + \frac{8}{10}x + \frac{8\,.\,11}{10\,.\,13}x^2 + \frac{8\,.\,11\,.\,14}{10\,.\,13\,.\,16}x^3 + \ldots\right], \qquad \text{where } C_1 = c_1a_0,\ C_2 = c_2a_0.$$

Case II. When roots of the indicial equation are equal *the complete solution is*

$$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

where m_1, m_1 are the roots.

Example 16.4. *Solve in series the equation* $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0.$ (V.T.U., 2010 ; S.V.T.U., 2007)

Solution. Substituting $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots$ ...(*i*)

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots$$

and $$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots$$

in the given equation, we obtain

$$x[m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots]$$
$$+ [ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots]$$
$$+ x[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots] = 0.$$

The lowest power of x is x^{m-1}. Its coefficient equated to zero gives $a_0[m(m-1) + m] = 0$.

i.e., $$m^2 = 0 \text{ as } a_0 \neq 0. \quad \therefore \quad m = 0, 0.$$

The coefficients of $x^m, x^{m+1}, \ldots$ equated to zero give

$$a_1[(m+1)m + m + 1] = 0, \quad i.e., \quad a_1 = 0$$
$$a_2(m+2)^2 + a_0 = 0, a_3(m+3)^2 + a_1 = 0, a_4(m+4)^2 + a_2 = 0 \text{ and so on.}$$

Clearly $a_3 = a_5 = a_7 \ldots\ldots = 0.$

Also $$a_2 = -\frac{a_0}{(m+2)^2}, a_4 = -\frac{a_2}{(m+4)^2} = \frac{a_0}{(m+2)^2(m+4)^2} \text{ etc.}$$

$\therefore$ $$y = a_0x^m\left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} - \frac{x^6}{(m+2)^2(m+4)^2(m+6)^2} + \ldots\ldots\right] \quad \ldots(ii)$$

Putting $m = 0$, the first solution is

$$y_1 = a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \ldots\ldots\right] \quad \ldots(iii)$$

This gives only one solution instead of two. To get the second solution, differentiate (*ii*) partially w.r.t. m.

$$\frac{\partial y}{\partial m} = y \log x + a_0x^m\left\{\frac{x^2}{(m+2)^2}\frac{2}{m+2} - \frac{x^4}{(m+2)^2(m+4)^2}\left[\frac{2}{m+2} + \frac{2}{m+4}\right] + \ldots\right\}$$

$\therefore$ the second solution is $y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0}$

$$= y_1 \log x + a_0\left\{\frac{1}{2^2}x^2 - \frac{1}{2^2 . 4^2}\left(1 + \frac{1}{2}\right)x^4 + \frac{1}{2^2 . 4^2 . 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 - \ldots\right\} \quad \ldots(iv)$$

Hence the complete solution is $y = c_1y_1 + c_2y_2$. [From (*iii*) & (*iv*)]

i.e., $$y = (C_1 + C_2 \log x)\left[1 - \frac{1}{2^2}x^2 + \frac{1}{2^2 . 4^2}x^4 - \frac{1}{2^2 . 4^2 . 6^2}x^6 + \ldots\right]$$
$$+ C_2\left\{\frac{1}{2^2}x^2 - \frac{1}{2^2 . 4^2}\left(1 + \frac{1}{2}\right)x^4 + \frac{1}{2^2 . 4^2 . 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 - \ldots\right\}$$

where $C_1 = a_0c_1, C_2 = a_0c_2$.

Obs. The above differential equation is called *Bessel's equation of order zero*, y_1 is called *Bessel function of the first kind of order zero* and is denoted by $J_0(x)$. It is absolutely convergent for all values of x whether real or complex.

y_2 is called the *Bessel function of the second kind of order zero or the Neumann function* and is denoted by $Y_0(x)$.

Thus the complete solution of the *Bessel's equation of order zero is* $y = AJ_0(x) + BY_0(x)$.

Case III. When roots of indicial equation are distinct and differ by an integer, making a coefficient of y infinite.

Let m_1 and m_2 be the roots such that $m_1 < m_2$. If some of the coefficients of y series become infinite when $m = m_1$, we modify the form of y by replacing a_0 by $b_0\,(m - m_1)$. Then the complete solution is

$$y = C_1(y)_{m_2} + C_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

Obs. 1. Two independent solution can also be obtained by putting $m = m_1$ (lesser of the two roots) in the modified form of y and $\partial y/\partial m$.

Obs. 2. *If one of the coefficients* (say : a_1) *becomes* **indeterminate** *when $m = m_2$, the complete solution is given by putting $m = m_2$ in y which contains two arbitrary constants.*

Example 16.5. *Obtain the series solution of the equation*

$$x(1-x)\frac{d^2y}{dx^2} - (1+3x)\frac{dy}{dx} - y = 0.$$

Solution. Here $x = 0$ is a singular point, since coefficient of y'' is zero at $x = 0$.

$\therefore$ substituting $\quad y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots \qquad \ldots(i)$

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)\,a_2\,x^{m+1} + \ldots$$

and
$$\frac{d^2y}{dx^2} = m(m-1)\,a_0x^{m-2} + (m+1)ma_1\,x^{m-1} + (m+2)\,(m+1)\,a_2x^m + \ldots$$

in the given equation, we obtain

$$x(1-x)[m(m-1)\,a_0x^{m-2} + (m+1)ma_1\,x^{m-1} + (m+2)\,(m+1)a_2x^m + \ldots]$$
$$-(1+3x)[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots] - [a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots] = 0$$

Equating to zero the coefficients of the lowest power of x, we get $a_0[m(m-1) - m] = 0$, $(a_0 \neq 0)$,

i.e., $\quad m(m-2) = 0$, *i.e.* $m = 0, 2$ *i.e.*, the two roots are distinct and differ by an integer.

Equating to zero the coefficients of successive powers of x, we get

$$(m-1)a_1 = (m+1)a_0,\ ma_2 = (m+2)a_1,\ (m+1)a_3 = (m+3)\,a_2 \text{ and so on.}$$

i.e.,
$$a_1 = \frac{m+1}{m-1}a_0,\ a_2 = \frac{(m+1)(m+2)}{(m-1)m}a_0,\ a_3 = \frac{(m+1)(m+2)(m+3)}{(m-1)m(m+1)}a_0 \text{ etc.}$$

Thus (i) becomes

$$y = a_0x^m\left[1 + \frac{m+1}{m-1}x + \frac{(m+1)(m+2)}{(m-1)m}x^2 + \frac{(m+1)(m+2)(m+3)}{(m-1)m(m+1)}x^3 + \ldots\right] \qquad \ldots(ii)$$

Putting $m = 2$ (greater of the two roots) in (ii), the first solution is

$$y_1 = a_0x^2\left[1 + 3x + \frac{3.4}{2}x^2 + \frac{4.5}{2}x^3 + \ldots\right]$$

If we put $m = 0$ in (ii), the coefficients become infinite.

To obviate this difficulty, put $a_0 = b_0\,(m - 0)$ so that

$$y = b_0x^m\left[m + \frac{m\,(m+1)}{m-1}x + \frac{(m+1)(m+2)}{m-1}x^2 + \frac{(m+1)(m+2)(m+3)}{(m-1)(m+1)}x^3 + \ldots\right]$$

$$\therefore \quad \frac{\partial y}{\partial m} = b_0x^m\log x\left[m + \frac{m\,(m+1)\,x}{m-1} + \frac{(m+1)(m+2)}{m-1}x^2 + \frac{(m+1)(m+2)(m+3)}{(m-1)(m+1)}x^3 + \ldots\right]$$
$$+ b_0x^m\left[1 + \frac{m^2-2m-1}{(m-1)^2}x + \frac{m^2-m-5}{(m-1)^2}x^2 + \frac{m^2-2m-11}{(m-1)^2}x^3 + \ldots\right]$$

$\therefore$ the second solution is $y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0}$

$$= b_0 \log x\,[-1.2x^2 - 2.3x^3 - 3.4x^4 - \ldots] + b_0[1 - x - 5x^2 - 11x^3 - \ldots]$$

Hence the complete solution is $y = c_1y_1 + c_2y_2$

i.e.,
$$y = \frac{1}{2}\, c_1a_0[1.2x^2 + 2.3x^3 + 3.4x^4 + \ldots] - b_0c_2 \log x\,[1.2x^2 + 2.3x^3 + 3.4x^4 + \ldots]$$
$$- b_0c_2\,[-1 + x + 5x^2 + 11x^3 + \ldots]$$

i.e.,
$$y = (C_1 + C_2 \log x)\,(1.2x^2 + 2.3x^3 + 3.4x^4 + \ldots) + C_2\,(-1 + x + 5x^2 + 11x^3 + \ldots)$$

where $C_1 = \frac{1}{2}\, c_1a_0,\ C_2 = -b_0c_2$

Example 16.6. *Solve in series* $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 4)\,y = 0.$ *(Bhopal, 2008 S; Rajasthan, 2003)*

Solution. $x = 0$ is a singular point, since coeff. of y'' is zero at $x = 0$.

Substituting
$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots \qquad \ldots(i)$$
$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)\,a_1x^m + (m+2)\,a_2x^{m+1} + \ldots$$

and
$$\frac{d^2y}{dx^2} = m\,(m-1)\,a_0x^{m-2} + (m+1)\,ma_1x^{m-1} + (m+2)\,(m+1)\,a_2x^m + \ldots$$

in the given equation, we get
$$x^2\,[m\,(m-1)\,a_0x^{m-2} + (m+1)\,ma_1x^{m-1} + (m+2)\,(m+1)\,a_2x^m + \ldots]$$
$$+ x\,[ma_0x^{m-1} + (m+1)\,a_1x^m + (m+2)\,a_2x^{m+1} + \ldots]$$
$$+ (x^2 - 4)\,[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots] = 0$$

Equating to zero the coefficients of the lowest power of x.
$$a_0\,[m\,(m-1) + m - 4] = 0 \text{ so that } m = \pm 2.$$

i.e., the two roots are distinct and differ by an integer.

Now equating to zero the coefficiens of successive powers of x, we get
$$m(m+4)\,a_2 = -a_0,\ i.e.,\ a_2 = \frac{-1}{m\,(m+4)}\,a_0,\ a_3 = 0$$
$$a_4 = \frac{1}{(m+2)\,(m+6)}\cdot\frac{1}{m\,(m+4)}\,a_0,\ a_5 = a_7 = \ldots = 0.$$
$$a_6 = \frac{-a_0}{m\,(m+2)\,(m+4)^2\,(m+6)\,(m+8)} \text{ etc.}$$

Substituting these values in (i), we get
$$y = a_0x^m\left[1 - \frac{x^2}{m\,(m+4)} + \frac{x^4}{m\,(m+2)\,(m+4)\,(m+6)} - \frac{x^6}{m\,(m+2)\,(m+4)^2\,(m+6)\,(m+8)} + \ldots\right] \qquad \ldots(ii)$$

Putting $m = 2$ (greater of the two roots) in (ii), the first solution is
$$y_1 = a_0x^2\left\{1 - \frac{x^2}{2\cdot 6} + \frac{x^4}{2\cdot 4\cdot 6\cdot 8} - \frac{x^6}{2\cdot 4\cdot 6^2\cdot 8\cdot 10} + \ldots\right\}$$

If we put $m = -2$ in (ii), the coefficients become infinite. To obviate this difficulty, let $a_0 = b_0\,(m+2)$, so that
$$y = b_0x^m\left[(m+2)\left\{1 - \frac{x^2}{m\,(m+4)}\right\} + \frac{x^4}{m\,(m+4)\,(m+6)} - \frac{x^6}{m\,(m+4)^2\,(m+6)\,(m+8)} + \ldots\right]$$

$\therefore$ $$\frac{\partial y}{\partial m} = b_0 x^m \log x \left[(m+2)\left\{1 - \frac{x^2}{m(m+4)}\right\} + \frac{x^4}{m(m+4)(m+6)} - \ldots\right]$$
$$+ b_0 x^m \left[1 - \frac{(m+2)}{m(m+4)}\left\{\frac{1}{m+2} - \frac{1}{m} - \frac{1}{m+4}\right\}x^2 + \frac{1}{m(m+4)(m+6)}\left\{-\frac{1}{m} - \frac{1}{m+4} - \frac{1}{m+6}\right\}x^4 + \ldots\right]$$

The second solution is $y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=-2}$

$$= b_0 x^{-2} \log x \left[-\frac{x^4}{2^2 \cdot 4} + \frac{x^6}{2^3 . 4 \cdot 6} \cdots\right] + b_0 x^{-2}\left[1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \ldots\right]$$

Hence the complete solution $y = c_1 y_1 + c_2 y_2$

i.e., $$y = C_1 x^2 \left\{1 - \frac{x^2}{2 \cdot 6} + \frac{x^4}{2 \cdot 4 \cdot 6 \cdot 8} - \frac{x^6}{2 \cdot 4 \cdot 6^2 \cdot 8 \cdot 10} + \ldots\right\}$$
$$+ C_2\left[x^2 \log x \left\{-\frac{1}{2^2 \cdot 4} + \frac{x^4}{2^3 \cdot 4 \cdot 6} \cdots\right\} + x^{-2}\left\{1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \cdots\right\}\right]$$

where $C_1 = c_1 a_0$, $C_2 = c_2 b_0$.

Example 16.7. *Solve in series $xy'' + 2y' + xy = 0$.* (U.P.T.U., 2003)

Solution. Here $x = 0$ is a singular point since coefficient of $y'' = 0$ at $x = 0$.

$\therefore$ Substituting $y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots$ ...(*i*)

$$\frac{dy}{dx} = m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + (m+3) a_3 x^{m+2} + \ldots$$

and $$\frac{d^2y}{dx^2} = m(m-1) a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + \ldots.$$

in the given equation, we get

$$x\,[m(m-1) a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + \ldots]$$
$$+ 2\,[m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + (m+3) a_3 x^{m+2} + \ldots]$$
$$+ x\,[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots]$$

Equating to zero, the coefficients of the lowest power of x,

$$m(m-1) a_0 + 2 m a_0 = 0 \text{ so that } m = 0, -1.$$

i.e., the roots are distinct of and differ by an integer.

Equating to zero, the coefficient of x^m, we get

$$(m+1) m a_1 + 2(m+1) a_1 = 0 \; i.e. \; (m+1)(m+2) a_1 = 0$$

or $$(m+1) a_1 = 0 \qquad [\because \; m+2 \neq 0]$$

When $m = -1$, $a_1 = 0/0$ *i.e., indeterminate.*

Hence the complete solution will be given by putting $m = -1$ in y itself (containing two arbitrary constants a_0 and a_1).

Now equating to zero, the coefficients of successive powers of x, we get

$$(m+2)(m+3) a_2 + a_0 = 0 \qquad [\text{Coeff. of } x^{m+1} = 0]$$
$$(m+3)(m+4) a_3 + a_1 = 0 \qquad [\text{Coeff. of } x^{m+2} = 0]$$
$$(m+4)(m+5) a_4 + a_2 = 0 \qquad [\text{Coeff. of } x^{m+3} = 0]$$
$$(m+5)(m+6) a_5 + a_3 = 0 \text{ etc.} \qquad [\text{Coeff. of } x^{m+4} = 0]$$

i.e., $$a_2 = -\frac{a_0}{(m+2)(m+3)}, a_3 = \frac{-a_1}{(m+3)(m+4)}, a_4 = \frac{a_0}{(m+2)(m+3)(m+4)(m+5)},$$

$$a_5 = \frac{a_1}{(m+3)(m+4)(m+5)(m+6)} \text{ and so on.}$$

Substituting the values in (*i*), we get

$$y = x^m\left[a_0 + a_1x - \frac{a_0}{(m+2)(m+3)}x^2 - \frac{a_1}{(m+3)(m+4)}x^3 + \frac{a_0}{(m+2)(m+3)(m+4)(m+5)}x^4 + \frac{a_1}{(m+3)(m+4)(m+5)(m+6)}x^5 - \cdots\right.$$

Putting $m = -1$, the complete solution is

$$y = x^{-1}\left\{a_0\left(1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \cdots\right) + a_1\left(x - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdots\right)\right\}$$

$$= x^{-1}(a_0 \cos x + a_1 \sin x).$$

PROBLEMS 16.2

Solve the following equations in power series :

1. $4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$ (*P.T.U., 2005*)
2. $y'' + xy' + (x^2 + 2)y = 0.$ (*P.T.U., 2007*)
3. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0.$
4. $3x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} - y = 0.$ (*S.V.T.U., 2008*)
5. $x\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + 2y = 0.$ (*J.N.T.U., 2006*)
6. $2x^2y'' + xy' - (x+1)y = 0.$ (*U.P.T.U., 2005*)
7. $8x^2\frac{d^2y}{dx^2} + 10x\frac{dy}{dx} - (1+x)y = 0.$ (*P.T.U., 2009*)
8. $2x(1-x)\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + 3y = 0.$ (*U.P.T.U., 2004*)
9. $x(1-x)\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$
10. $(2x + x^3)\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6xy = 0.$ (*Bhopal, 2008*)

16.5 BESSEL'S EQUATION*

One of the most important differential equations in applied mathematics is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad \ldots(1)$$

which is known as *Bessel's equation of order n*. Its particular solutions are called *Bessel functions of order n*. Many physical problems involving vibrations or heat conduction in cylindrical regions give rise to this equation.

Substituting $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots$

(1) takes the form

$$a_0(m^2 - n^2)x^m + a_1\{(m+1)^2 - n^2\}x^{m+1} + \{a_2[(m+2)^2 - n^2] + a_0\}x^{m+2} + \ldots = 0.$$

Equating to zero the coefficient of x^m, we obtain the indicial equation $m^2 - n^2 = 0$ (as $a_0 \neq 0$) where $m = n$ or $-n$.

$$a_1 = a_3 = a_5 = a_7 = \ldots = 0$$

and $$a_2 = -\frac{a_0}{(m+2)^2 - n^2},\ a_4 = -\frac{a_2}{(m+4)^2 - n^2} \text{ etc.}$$

These give $$y = a_0x^m\left(1 - \frac{1}{(m+2)^2 - n^2}x^2 + \frac{1}{[(m+2)^2 - n^2][(m+4)^2 - n^2]}x^4 - \cdots\right)$$

* Named after the German mathematician and astronomer *Friederich Wilhelm Bessel* (1784 – 1846) whose paper on Bessel functions appeared in 1826. He studied Astronomy of his own and became director of Königsberg observatory.

For $m = n$, we get

$$y_1 = a_0 x^n \left\{1 - \frac{1}{4(n+1)} x^2 + \frac{1}{4^2 \cdot 2!(n+1)(n+2)} x^4 - \frac{1}{4^3 \cdot 3!(n+1)(n+2)(n+3)} x^6 + \cdots\right\} \quad ...(2)$$

and for $m = -n$, we have

$$y_2 = a_0 x^{-n} \left\{1 - \frac{1}{4(-n+1)} x^2 + \frac{1}{4^2 \cdot 2!(-n+1)(-n+2)} x^4 - \frac{1}{4^3 \cdot 3!(-n+1)(-n+2)(-n+3)} x^6 + \cdots\right\} \quad ...(3)$$

Case I. *When n is not integral or zero,* the complete solution of (1) is $y = c_1 y_1 + c_2 y_2$.

If we take $a_0 = 1/2^n\, \Gamma(n+1)$, then the solution given by (2) is called the *Bessel function of the first kind of order n* and is denoted by $J_n(x)$. Thus

$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{\frac{1}{\Gamma(n+1)} - \frac{1}{1!\,\Gamma(n+2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(n+3)}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma(n+4)}\left(\frac{x}{2}\right)^6 + \cdots\right\} \ (n > 0)$$

i.e.

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)} \quad ...(4)$$

and corresponding to (3), we have

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{r!\,\Gamma(-n+r+1)} \quad ...(5)$$

which is called the *Bessel function of the first kind of order – n.*

Hence complete solution of the Bessel's equation (1) may be expressed in the form.

$$y = AJ_n(x) + BJ_{-n}(x). \quad ...(6)$$

Case II. *When n is zero,* $y_1 = y_2$ and the complete solution of (1), which reduces to the *Bessel's equation of order zero,* is obtained as in Example 16.4.

Case III. *When n is integral,* y_2 fails to give a solution for positive values of n and y_1 fails to give a solution for negative values. Thus another independent integral of the Bessel's equation (1) is needed to form its general solution. We now proceed to find an independent solution of (1), when n is an integer.

Let $y = u(x) J_n(x)$ be a solution of (1). Substituting the values of y, $y' = u'J_n + uJ_n'$ and $y'' = u''J_n + 2u'J_n' + uJ_n''$ in (1), we obtain

$$x^2(u''J_n + 2u'J_n' + uJ_n'') + x(u'J_n + uJ_n') + (x^2 - n^2)uJ_n = 0$$

or

$$u\{x^2 J_n'' + xJ_n' + (x^2 - n^2)J_n\} + x^2 u'' J_n + 2x^2 u' J_n' + xu' J_n = 0. \quad ...(7)$$

Now since J_n is a solution of (1), therefore, $x^2 J_n'' + xJ_n' + (x^2 - n^2)J_n = 0$

$\therefore$ (7) reduces to $x^2 u'' J_n + 2x^2 u' J_n' + xu' J_n = 0$.

Dividing throughout by $x^2 u' J_n$, it becomes $\dfrac{u''}{u'} + 2\dfrac{J_n'}{J_n} + \dfrac{1}{x} = 0$

i.e.,

$$\frac{d}{dx}(\log u') + 2\frac{d}{dx}(\log J_n) + \frac{d}{dx}(\log x) = 0 \text{ or } \frac{d}{dx}\{\log(u' J_n^2 x)\} = 0.$$

Integrating, $\log(u' J_n^2 x) = \log B$, whence $xu' J_n^2 = B$.

$$\therefore \quad u' = \frac{B}{xJ_n^2} \text{ or } u = B\int \frac{dx}{xJ_n^2} + A.$$

Thus

$$y = AJ_n(x) + BJ_n(x)\int \frac{dx}{x[J_n(x)]^2}.$$

Hence the complete solution of the Bessel's equation (1) is

$$y = AJ_n(x) + BY_n(x) \quad ...(8) \quad (\text{V.T.U., 2006})$$

where

$$Y_n(x) = J_n(x)\int \frac{dx}{x[J_n(x)]^2} \quad ...(9)$$

$Y_n(x)$ is called the *Bessel function of the second kind of order n or Neumann function*.*

* Named after the German mathematician and physicist *Carl Neumann* (1832–1925) whose work on potential theory gave impetus for development of integral equations by *Volterra* of Rome, *Fredholm* of Stockholm and *Hilbert* of Gottingen.

Obs. Putting $k = -n + r$, *i.e.* $r = k + n$, and noting that $\Gamma(k+1) = k\,!$ where k is an integer, (5) may be written as

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}(x/2)^{2k+n}}{(k+n)!\,k!} = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k\,!\,\Gamma(k+n+1)}$$

Hence $J_{-n}(x) = (-1)^n J_n(x)$. ...(10) (*Bhopal, 2008 ; S.V.T.U., 2008 ; V.T.U., 2006*)

16.6 RECURRENCE FORMULAE FOR $J_n(x)$

The following recurrence formulae can easily be derived from the series expression for $J_n(x)$:

(1) $\dfrac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$.

(2) $\dfrac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$.

(3) $J_n(x) = \dfrac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$.

(4) $J_n'(x) = \dfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$.

(5) $J_n'(x) = \dfrac{n}{x} J_n(x) - J_{n+1}(x)$.

(6) $J_{n+1}(x) = \dfrac{2n}{x} J_n(x) - J_{n-1}(x)$.

These formulae are very useful in the solution of boundary value problems and in establishing the various properties of Bessel functions.

Proofs. **(1)** Multiplying (4) of page 551 by x^n, we have

$$x^n J_n(x) = x^n \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r\,!\,\Gamma(n+r+1)} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2(n+r)}}{2^{n+2r}\, r\,!\,\Gamma(n+r+1)}$$

$$\therefore \quad \frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty} \frac{(-1)^r\, 2(n+r) x^{2(n+r)-1}}{2^{n+2r} r\,!\,\Gamma(n+r+1)} = x^n \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n-1+2r}}{r\,!\,\Gamma(n-1+r+1)} = x^n J_{n-1}(x).$$

(*Bhopal, 2008 ; V.T.U., 2005 ; U.P.T.U., 2005*)

(2) Multiplying (4) of page 551 by x^{-n}, we have

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{n+2r} r!\,\Gamma(n+r+1)}$$

$$\therefore \quad \frac{d}{dx}[x^{-n} J_n(x)] = \sum_{r=0}^{\infty} \frac{(-1)^r 2r\, x^{2r-1}}{2^{n+2r} r\,!\,\Gamma(n+r+1)} = -x^{-n} \sum_{r=1}^{\infty} \frac{(-1)^{r-1} x^{n+1+2(r-1)}}{2^{n+1+2(r-1)}(r-1)\,!\,\Gamma(n+r+1)}$$

$$= -x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+1+2k}}{k\,!\,\Gamma(n+1+k+1)} = -x^{-n} J_{n+1}(x), \text{ where } k = r-1.$$

(*P.T.U., 2006 ; B.P.T.U., 2005*)

(3) From (1), we have $x^n J_n'(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$

or dividing by x^n, $\quad J_n'(x) + (n/x) J_n(x) = J_{n-1}(x)$ ...(*i*)

Similarly from (2), we get $x^{-n} J_n'(x) - nx^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$

or $\quad -J_n'(x) + \dfrac{n}{x} J_n(x) = J_{n+1}(x)$ ...(*ii*)

Adding (*i*) and (*ii*), we obtain $\dfrac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

i.e., $\quad J_n(x) = \dfrac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$ (*S.V.T.U., 2008 ; Anna, 2005 S*)

(4) Subtracting (*ii*) from (*i*), we get $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

i.e., $\quad J_n'(x) = \dfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$. (*S.V.T.U., 2007 ; P.T.U., 2005*)

(5) is another way of writing (*ii*). (*J.N.T.U., 2006 ; Anna, 2005*)

(6) is another way of writing (3). (*Madras, 2006 ; V.T.U., 2005*)

16.7 (1) EXPANSIONS FOR J_0 AND J_1

We have from (4) of page 551,

$$J_0(x) = 1 - \frac{1}{(1!)}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots \qquad ...(1)$$

and

$$J_1(x) = \frac{x}{2}\left[1 - \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,4!}\left(\frac{x}{2}\right)^6 + \cdots\right] \qquad \textit{(B.P.T.U., 2005)} ...(2)$$

Because of their special importance, the values of $J_0(x)$ and $J_1(x)$ are given in Appendix 2 : Table II to four decimal places at intervals of 0.1. With the help of these values, the graphs of $J_0(x)$ and $J_1(x)$ can be drawn as shown in Fig. 16.1, for $x > 0$. Their close resemblance to graphs of $\cos x$ and $\sin x$ is interesting.

Obs. The roots of the equation $J_0(x) = 0$ are useful in some physical problems. This equation has no complex roots but an infinite number of real roots. Its first four roots are $x = 2.4, 5.52, 8.65, 11.79$ approximately.

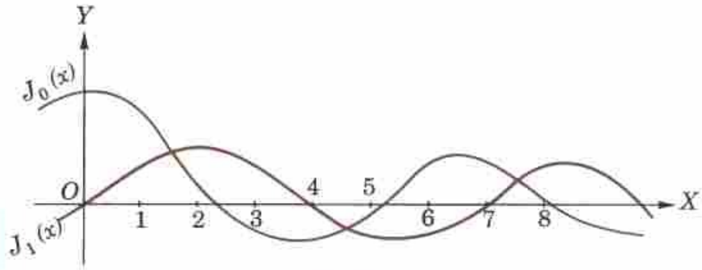

Fig. 16.1

16.8 VALUE OF $J_{1/2}$

We may think that $J_0(x)$ is the simplest of the J's but actually $J_{1/2}(x)$ is simpler, for it can be expressed in a finite form. Taking $n = \frac{1}{2}$ in (4) of page 551, we have

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2}\left\{\frac{1}{\Gamma\left(\frac{3}{2}\right)} - \frac{1}{1!\,\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma\left(\frac{7}{2}\right)}\left(\frac{x}{2}\right)^4 - \ldots\right\}$$

$$= \left(\frac{x}{2}\right)^{1/2}\left\{\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - \frac{1}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4 - \ldots\right\}$$

$$= \frac{\sqrt{x}}{\sqrt{2}\,\Gamma\left(\frac{1}{2}\right)}\left\{\frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!} - \ldots\right\}$$

Now multiplying the series by $x/2$ and outside by $2/x$, we get

$$J_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left\{\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right\} = \sqrt{\left(\frac{2}{\pi x}\right)}\sin x. \qquad ...(3) \quad \textit{(V.T.U., 2009 ; J.N.T.U., 2003)}$$

Similarly taking $n = \frac{1}{2}$ in (5) of page 551, it can be shown that

$$J_{-\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\cos x. \qquad ...(4) \quad \textit{(Anna, 2005 ; W.B.T.U., 2005 ; V.T.U., 2003)}$$

Example 16.8. *Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$.* *(Bhopal, 2008 S ; V.T.U., 2001)*

Solution. We know that

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right] \text{ i.e. } J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

Putting $n = 1, 2, 3, 4$ successively, $J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$ ...(i) $\quad J_3(x) = \frac{4}{x}J_2(x) - J_1(x)$ ...(ii)

$J_4(x) = \frac{6}{x}J_3(x) - J_2(x)$ ...(iii) $\quad J_5(x) = \frac{8}{x}J_4(x) - J_3(x)$ ...(iv)

Substituting the value of $J_2(x)$ in (*ii*), we have

$$J_3(x) = \frac{4}{x}\left\{\frac{2}{x}J_1(x) - J_0(x)\right\} - J_1(x) = \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x) \quad ...(v)$$

(*W.B.T.U., 2005 ; Madras, 2003*)

Now substituting the values of $J_3(x)$ from (*v*) and $J_2(x)$ from (*i*) in (*iii*), we get

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x) \quad ...(vi) \quad (V.T.U., 2003\ S)$$

Finally putting the values of $J_4(x)$ from (*vi*) and $J_3(x)$ from (*v*) in (*iv*), we obtain

$$J_5(x) = \left(\frac{384}{x^4} - \frac{72}{x^2} - 1\right)J_1(x) + \left(\frac{12}{x} - \frac{192}{x^3}\right)J_0(x).$$

Example 16.9. *Prove that* $J_{5/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left\{\frac{3 - x^2}{x^2}\sin x - \frac{3}{x}\cos x\right\}$. (*J.N.T.U., 2006*)

Solution. We know that $J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$ ...(*i*)

Putting $n = \frac{1}{2}$, we get $J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right)$

(*Bhopal, 2007 ; V.T.U., 2006*)

Again putting $n = \frac{3}{2}$ in (*i*), we get $J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x)$

$$= \frac{3}{x}\left[\sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right)\right] - \sqrt{\left(\frac{2}{\pi x}\right)}\sin x = \sqrt{\left(\frac{2}{\pi x}\right)}\left[\frac{3 - x^2}{x^2}\sin x - \frac{3}{x}\cos x\right]$$

which is the required result.

Example 16.10. *Prove that*

(*a*) $J_n''(x) = \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$. (*J.N.T.U., 2006*)

(*b*) $\frac{d}{dx}[xJ_n(x)J_{n+1}(x)] = x[J_n^2(x) - J_{n+1}^2(x)]$. (*V.T.U., 2006*)

Solution. (*a*) We know that $J_n'(x) = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$ ...(*i*)

Differentiating both sides, we get $J_n''(x) = \frac{1}{2}\{J_{n-1}'(x) - J_{n+1}'(x)\}$ ...(*ii*)

Changing n to $n - 1$ in (*i*), we obtain $J_{n-1}'(x) = \frac{1}{2}\{J_{n-2}(x) - J_n(x)\}$ ...(*iii*)

Changing n to $n + 1$ in (*i*), we have $J_{n+1}'(x) = \frac{1}{2}\{J_n(x) - J_{n+2}(x)\}$ ...(*iv*)

Substituting the values of $J_{n-1}'(x)$ and $J_{n+1}'(x)$ from (*iii*) and (*iv*) in (*ii*), we get

$$J_n'' = \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$$

(*b*) $\frac{d}{dx}[xJ_n(x)J_{n+1}(x)] = J_n(x)J_{n+1}(x) + x[J_n(x)J'_{n+1}(x) + J_n'(x)J_{n+1}(x)]$ ...(*i*)

From (5) of § 16.6, we have $J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$ ...(*ii*)

Changing n to $n + 1$ in (*i*) of page 499, we get $J'_{n+1}(x) = J_n(x) - \frac{n+1}{x}J_{n+1}(x)$ ...(*iii*)

Now substituting from (*iii*) and (*ii*) in (*i*), we get

$$\frac{d}{dx}[xJ_n(x)J_{n+1}(x)] = J_n(x)J_{n+1}(x) + x\left[J_n(x)\left\{J_n(x) - \frac{n+1}{x}J_{n+1}(x)\right\} + \left\{\frac{n}{x}J_n(x) - J_{n+1}(x)\right\}J_{n+1}(x)\right]$$

$$= x\{J_n^2(x) - J_{n+1}^2(x)\}.$$

Example 16.11. *Prove that :*

(*a*) $\int J_3(x)dx = c - J_2(x) - \frac{2}{x}J_1(x).$

(*b*) $\int xJ_0^2(x)dx = \frac{1}{2}x^2\{J_0^2(x) + J_1^2(x)\}.$ (*U.P.T.U., 2004 ; Osmania, 2002*)

Solution. (*a*) We know that $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$ [§ 16.6 (2)] ...(*i*)

or $\int x^{-n}J_{n+1}(x)dx = -x^{-n}J_n(x)$ ...(*ii*)

$\therefore$ $\int J_3(x)dx = \int x^2 \cdot x^{-2}J_3(x)dx + c$ [Integrate by parts]

$$= x^2 \cdot \int x^{-2}J_3(x)dx - \int 2x\left[\int x^{-2}J_3(x)dx\right]dx + c$$

$$= x^2[-x^{-2}J_2(x)] - \int 2x[-x^{-2}J_2(x)]\,dx + c \qquad \text{[By (ii) when } n = 2]$$

$$= c - J_2(x) + \int \frac{2}{x}J_2(x)\,dx = c - J_2(x) - \frac{2}{x}J_1(x) \qquad \text{[By (ii) when } n = 1]$$

(*b*) $\int xJ_0^2(x)dx = \int J_0^2(x)\cdot x\,dx$ [Integrate by parts]

$$= J_0^2(x)\cdot\frac{1}{2}x^2 - \int 2J_0(x)J_0'(x)\cdot\frac{1}{2}x^2\,dx$$

$$= \frac{1}{2}x^2J_0^2(x) + \int x^2J_0(x)J_1(x)\,dx \qquad \text{[By (i) when } n = 0]$$

$$= \frac{1}{2}x^2J_0^2(x) + \int xJ_1(x)\cdot\frac{d}{dx}[xJ_1(x)]\,dx \qquad \left[\because \frac{d}{dx}[xJ_1(x)] = xJ_0(x) \text{ by § 16.6 (1)}\right]$$

$$= \frac{1}{2}x^2J_0^2(x) + \frac{1}{2}[xJ_1(x)]^2 = \frac{1}{2}x^2[J_0^2(x) + J_1^2(x)].$$

16.9 GENERATING FUNCTION FOR $J_n(x)$

To prove that $e^{\frac{1}{2}x(t-1/t)} = \sum_{n=-\infty}^{\infty} t^nJ_n(x).$

We have $e^{\frac{1}{2}x(t-t^{-1})} = e^{xt/2} \times e^{-x/2t}$

$$= \left[1 + \left(\frac{xt}{2}\right) + \frac{1}{2!}\left(\frac{xt}{2}\right)^2 + \frac{1}{3!}\left(\frac{xt}{2}\right)^3 + \cdots\right] \times \left[1 - \left(\frac{x}{2t}\right) + \frac{1}{2!}\left(\frac{x}{2t}\right)^2 - \frac{1}{3!}\left(\frac{x}{2t}\right)^3 + \cdots\right]$$

The coefficient of t^n in this product

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \ldots = J_n(x).$$

As all the integral powers of t, both positive and negative occur, we have

$$e^{\frac{1}{2}x(t-t^{-1})} = J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \ldots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \ldots$$

$$= \sum_{n=-\infty}^{\infty} t^nJ_n(x)$$ (*V.T.U., 2007*)

This shows that Bessel functions of various orders can be derived as coefficients of different powers of t in the expansion of $e^{\frac{1}{2}x(t-1/t)}$. For this reason, it is known as the *generating function of Bessel functions*.

Example 16.12. *Show that*

(a) $J_n(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$, *n being an integer.* *(V.T.U., 2006)*

(b) $J_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos(x\cos\phi)\, d\phi$. *(Madras, 2006)*

(c) $J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \ldots = 1$. *(Kerala M. Tech, 2005 ; U.P.T.U., 2003; V.T.U., 2003 S)*

Solution. (*a*) We know that

$$e^{\frac{1}{2}x(t-1/t)} = J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \ldots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \ldots$$

Since $J_{-n}(x) = (-1)^n J_n(x)$

$$\therefore \quad e^{\frac{1}{2}x(t-1/t)} = J_0 + J_1(t - 1/t) + J_2(t^2 + 1/t^2) + J_3(t^3 - 1/t^3) + \ldots \qquad \ldots(i)$$

Now put $t = \cos\theta + i\sin\theta$

so that $\quad t^p = \cos p\theta + i\sin p\theta$ and $1/t^p = \cos p\theta - i\sin p\theta$

giving $\quad t^p + 1/t^p = 2\cos p\theta$ and $t^p - 1/t^p = 2i\sin p\theta$.

Substituting these in (*i*), we get

$$e^{ix\sin\theta} = J_0 + 2[J_2\cos 2\theta + J_4\cos 4\theta + \ldots] + 2i\,[J_1\sin\theta + J_3\sin 3\theta + \ldots] \qquad \ldots(ii)$$

Since $\quad e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta)$.

$\therefore$ equating the real and imaginary parts in (*ii*), we get

$$\cos(x\sin\theta) = J_0 + 2\,[J_2\cos 2\theta + J_4\cos 4\theta + \ldots] \qquad \ldots(iii)$$

$$\sin(x\sin\theta) = 2[J_1\sin\theta + J_3\sin 3\theta + \ldots] \qquad \ldots(iv)$$

which are known as *Jacobi series**. *(V.T.U., 2006)*

Now multiplying both sides of (*iii*) by $\cos n\theta$ and both sides of (*iv*) by $\sin n\theta$ and integrating each of the resulting expressions between 0 and π, we obtain

$$\frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos n\theta\, d\theta = \begin{cases} J_n(x), & n \text{ even or zero} \\ 0, & n \text{ odd} \end{cases}$$

and

$$\frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta)\sin n\theta\, d\theta = \begin{cases} 0 & n \text{ even} \\ J_n(x), & n \text{ odd} \end{cases}$$

Hence generally, if n is a positive integer,

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]\, d\theta = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta.$$

[This is Bessel's original definition of $J_n(x)$ given in 1824 while investigating Planetary motion.]

(*b*) Changing θ to $\frac{1}{2}\pi - \phi$ in (*iii*), we get

$$\cos(x\cos\phi) = J_0 + 2J_2\cos(\pi - 2\phi) + 2J_4\cos(2\pi - 4\phi) + \ldots$$
$$= J_0 - 2J_2\cos 2\phi + 2J_4\cos 4\phi - \ldots$$

Integrating both sides w.r.t. ϕ from 0 to π, we get

$$\int_0^{\pi} \cos(x\cos\phi)\, d\phi = \int_0^{\pi} \{J_0(x) - 2J_2(x)\cos 2\phi + 2J_4(x)\cos 4\phi - \cdots\}\, d\phi$$

$$= \left| J_0(x)\cdot\phi - 2J_2(x)\,.\,\frac{1}{2}\sin 2\phi + 2J_4(x)\cdot\frac{1}{4}\sin 4\phi - \cdots \right|_0^{\pi} = J_0(x)\,.\,\pi \text{ whence follows the result.}$$

* See footnot p. 215.

(c) Squaring (iii) and (iv) and integrating w.r.t. ϕ from 0 to π and noting that (m, n being integers),

$$\int_0^\pi \cos m\theta \cos n\theta \, d\theta = \int_0^\pi \sin m\theta \sin n\theta \, d\theta = 0, \qquad (m \neq n)$$

and $$\int_0^\pi \cos^2 n\theta \, d\theta = \int_0^\pi \sin^2 n\theta \, d\theta = \pi/2, \text{ we obtain}$$

$$[J_0(x)]^2 \pi + 4 [J_2(x)]^2 \frac{\pi}{2} + 4 [J_4(x)]^2 \frac{\pi}{2} + \dots = \int_0^\pi \cos^2 (x \sin \theta) \, d\theta$$

$$4 [J_1(x)]^2 \frac{\pi}{2} + 4 [J_3(x)]^2 \frac{\pi}{2} + \dots = \int_0^\pi \sin^2 (x \sin \theta) \, d\theta$$

Adding, $$\pi [J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots] = \int_0^\pi d\theta = \pi$$

Hence $$J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots = 1.$$

PROBLEMS 16.3

1. Compute $J_0(2)$, $J_1(1)$ correct to three decimal places.

2. Show that (i) $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^4}\right) J_0(x)$. (ii) $J_1(x) + J_3(x) = \frac{4}{x} J_2(x)$ (P.T.U., 2003)

3. Show that
   (i) $J_{-1/2}(x) = J_{1/2}(x) \cot x$. (S.V.T.U., 2008) (ii) $J'_{1/2}(x) J_{-1/2}(x) - J'_{-1/2}(x) J_{1/2}(x) = 2/\pi x$ (Delhi, 2002)
   (iii) $J_{-3/2}(x) = -\sqrt{\left(\frac{2}{\pi x}\right)\left(\sin x + \frac{\cos x}{x}\right)}$ (iv) $J_{-5/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)\left(\frac{3}{x} \sin x + \frac{3 - x^2}{x^2} \cos x\right)}$
   (V.T.U., 2000)

4. Prove that (i) $\frac{d}{dx} J_0(x) = -J_1(x)$. (ii) $\frac{d}{dx} [xJ_1(x)] = xJ_0(x)$.
   (iii) $\frac{d}{dx} [x^n J_n(ax)] = ax^n J_{n-1}(ax)$. (Madras, 2000 S) (iv) $J'_n(x) = -\frac{n}{2} J_n(x) + J_{n-1}(x)$ (P.T.U., 2009 S)

5. Show by the use of recurrence formula, that
   (i) $J_0''(x) = \frac{1}{2} [J_2(x) - J_0(x)]$ (ii) $J_1''(x) = J_1(x) - \frac{1}{x} J_2(x)$.
   (iii) $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$. (Osmania, 2003)

6. Prove that
   (i) $\frac{d}{dx} \{J_n^2(x)\} = \frac{x}{2n} [J_{n-1}^2(x) - J_{n+1}^2(x)]$ (S.V.T.U., 2008 ; Kerala M.E., 2005)
   (ii) $\frac{d}{dx} \{J_n^2(x) + J_{n+1}^2(x)\} = 2\left\{\frac{n}{2} J_n^2(x) - \frac{n+1}{x} J_{n+1}^2(x)\right\}$. (U.P.T.U., 2005 ; V.T.U., 2000 S)

7. Prove that (i) $\int_0^{\pi/2} \sqrt{(\pi x)} \, J_{1/2}(2x) \, dx = 1$. (P.T.U., 2005) (ii) $\int_0^r xJ_0(ax) dx = \frac{r}{a} J_1(ar)$.
   (iii) $\int x^2 J_1(x) dx = x^2 J_2(x)$. (P.T.U., 2007)

8. Prove that (i) $\int J_0(x) J_1(x) \, dx = -\frac{1}{2} [J_0(x)]^2$. (ii) $\int_0^\infty e^{-ax} J_0(bx) dx = \frac{1}{\sqrt{a^2 + b^2}}$.

9. Starting with the series of § 16.9, prove that
   $$2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)] \text{ and } xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x).$$

10. Establish the *Jacobi series*
    $$\cos (x \cos \theta) = J_0 - 2J_2 \cos 2\theta + 2J_4 \cos 4\theta - \dots$$
    $$\sin (x \cos \theta) = 2[J_1 \cos \theta - J_3 \cos 3\theta + J_5 \cos 5\theta - \dots]$$ (Madras, 2003 S)

11. Prove that (i) $\sin x = 2[J_1 - J_3 + J_5 - \dots]$ (Anna, 2005 S)
    (ii) $\cos x = J_0 - 2J_2 + 2J_4 - 2J_6 + \dots$ (Kerala M. Tech., 2005)
    (iii) $1 = J_0 + 2J_2 + 2J_4 + 2J_6 + \dots$

16.10 EQUATIONS REDUCIBLE TO BESSEL'S EQUATION

In many problems, we come across such differential equations which can easily be reduced to Bessel's equation and, therefore, can be solved by means of Bessel functions.

(1) *To reduce the differential equation* $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (k^2x^2 - n^2)\, y = 0$ *to Bessel form.*

Put $t = kx$, so that $\frac{dy}{dx} = k\frac{dy}{dt}$ and $\frac{d^2y}{dx^2} = k^2 \frac{d^2y}{dt^2}$.

Then (**1**) becomes $t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - n^2)y = 0$

$\therefore$ its solution is $y = c_1J_n(t) + c_2J_{-n}(t)$, n is non-integral,

or $\quad y = c_1J_n(t) + c_2Y_n(t)$, n is integral.

Hence the solution of (1) is

$y = c_1J_n(kx) + c_2J_{-n}(kx)$, n is non-integral

or $\quad y = c_1J_n(kx) + c_2Y_n(kx)$, n is integral.

(2) *To reduce the differential equation* $x\frac{d^2y}{dx^2} + a\frac{dy}{dx} + k^2xy = 0$ *to Bessel's equation,* *(Madras, 2006)*

put $\quad y = x^nz$,

so that $\quad \frac{dy}{dx} = x^n\frac{dz}{dx} + nx^{n-1}z$ and $\frac{d^2y}{dx^2} = x^n\frac{d^2z}{dx^2} + 2nx^{n-1}\frac{dz}{dx} + n(n-1)x^{n-2}z$

Then (**2**) takes the form $x^{n+1}\frac{d^2z}{dx^2} + (2n + a)x^n\frac{dz}{dx} + [k^2x^2 + n^2 + (a-1)n]x^{n-1}z = 0.$

Dividing throughout by x^{n-1} and putting $2n + a = 1$,

$$x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + (k^2x^2 - n^2)z = 0.$$

Its solution by (1) is $z = c_1J_n(kx) + c_2J_{-n}(kx)$, n is non-integral

or $\quad z = c_1J_n(kx) + c_2Y_n(kx)$, n is integral

Hence the solution of (2) is $y = x^n\,[c_1J_n(kx) + c_2J_{-n}(kx)]$, n is non-integral

or $\quad y = x^n\,[c_1J_n(kx) + c_2Y_n(kx)]$, n is integral, where $n = (1-a)/2$.

(3) *To reduce the differential equation* $x\frac{d^2y}{dx^2} + c\frac{dy}{dx} + k^2x^ry = 0$ *to Bessel form,* put $x = t^m$, *i.e.* $t = x^{1/m}$,

so that $\quad \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{m}t^{1-m}\frac{dy}{dt}$

and $\quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1}{m}t^{1-m}\frac{dy}{dt}\right)\cdot\frac{1}{m}t^{1-m}\ \frac{1}{m^2}t^{2-2m}\frac{d^2y}{dt^2} + \frac{1-m}{m^2}t^{1-2m}\frac{dy}{dt}$

Then (**3**) takes the form $\frac{1}{m^2}t^{2-m}\frac{d^2y}{dt^2} + \frac{1-m+cm}{m^2}t^{1-m}\frac{dy}{dt} + k^2t^{mr}y = 0$

or multiplying throughout by m^2/t^{1-m}, $t\frac{d^2y}{dt^2} + (1-m+cm)\frac{dy}{dt} + (km)^2t^{mr+m-1}y = 0.$

In order to reduce it to (2), we set $mr + m - 1 = 1$, *i.e.* $m = 2/(r+1)$

and $\quad a = 1 - m + cm = (r + 2c - 1)/(r + 1).$

Thus it reduces to $t\frac{d^2y}{dt^2} + a\frac{dy}{dt} + (km)^2ty = 0$ which is similar to (2).

Hence the solution of (3) is $y = x^{n/m}\{c_1J_n(k_mx^{1/m}) + c_2J_{-n}(kmx^{1/m})\}$, n is a fraction

or $\quad y = x^{n/m}\{c_1J_n(kmx^{1/m}) + c_2Y_n(kmx^{1/m})\}$, n is an integer

where $\quad n = \frac{1-a}{2} = \frac{1-c}{1+r}$ and $m = \frac{2}{1+r}$.

Example 16.13. *Solve the differential equations :*

(i) $y'' + \frac{y'}{x} + \left(8 - \frac{1}{x^2}\right) y = 0.$ *(ii)* $4y'' + 9xy = 0.$ *(iii)* $xy'' + y' + \frac{1}{4}y = 0.$ *(Anna, 2005)*

Solution. (*i*) Rewriting the given equation as $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (8x^2 - 1)y = 0,$

and comparing with (**1**) above, we see that $n = 1$ and $k = 2\sqrt{2}$.

$\therefore$ The solution of the given equation is $y = c_1J_n(kx) + c_2Y_n(kx)$

i.e., $$y = c_1J_1(2\sqrt{2}x) + c_2Y_1(2\sqrt{2}x).$$

(*ii*) Rewriting the given equation as $x\frac{d^2y}{dx^2} + \frac{9}{4}x^2y = 0$ ...(α)

and comparing with (**3**) above, we find that $c = 0$, $k = 3/2$ and $r = 2$.

$$\therefore \quad n = \frac{1-c}{1+r} = \frac{1}{3}, \quad m = \frac{2}{1+r} = \frac{2}{3} \quad \text{and} \quad \frac{n}{m} = \frac{1}{2}.$$

Hence the solution of (α) is $y = x^{n/m}\{c_1J_n(kmx^{1/m}) + c_2Y_{-n}(kmx^{1/m})\}$

$$y = \sqrt{x}\,[c_1J_{1/3}(x^{3/2}) + c_2J_{-1/3}(x^{3/2})].$$

(*iii*) Multiplying by x, the given equation becomes

$$x^2y'' + xy' + \frac{1}{4}xy = 0 \qquad \text{...}(\alpha)$$

Comparing with (**3**) above, we get $c = 1$, $k = 1/2$ & $r = 0$. $\therefore\ m = \frac{2}{1+r} = 2, n = \frac{1-c}{1+r} = 0\ \&\ \frac{n}{m} = 0$

Hence the solution of (α)

$$y = x^{n/m}\{c_1J_n(kmx^{1/m}) + c_2Y_n(kmx^{1/m})\} = x^0\left\{c_1J_0\left(\frac{1}{2}2x^{1/2}\right) + c_2Y_0\left(\frac{1}{2}.2x^{1/2}\right)\right\}$$

i.e., $$y = c_1J_0(\sqrt{x}) + c_2Y_0(\sqrt{x})$$

16.11 (1) ORTHOGONALITY OF BESSEL FUNCTIONS

We shall prove that

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = \begin{cases} 0, & \alpha \neq \beta \\ \frac{1}{2}[J_{n+1}(\alpha)]^2, & \alpha = \beta \end{cases}, \text{ where } \alpha, \beta \text{ are the roots of } J_n(x) = 0.$$

We know that the solution of the equation

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \qquad \text{...(1)}$$

and $$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \qquad \text{...(2)}$$

are $$u = J_n(\alpha x) \text{ and } v = J_n(\beta x) \text{ respectively.}$$

Multiplying (1) by v/x and (2) by u/x and subtracting, we get

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)xuv = 0$$

or $$\frac{d}{dx}\{x(u'v - uv')\} = (\beta^2 - \alpha^2)\,xuv.$$

Now integrating both sides from 0 to 1,

$$(\beta^2 - \alpha^2)\int_0^1 xuv\,dx = \left[x(u'v - uv')\right]_0^1 = (u'v - uv')_{x=1} \qquad \text{...(3)}$$

Since $$u = J_n(\alpha x),$$

$$\therefore \quad u' = \frac{d}{dx}[J_n(\alpha x)] = \frac{d}{d(\alpha x)}[J_n(\alpha x)].\frac{d(\alpha x)}{dx} = \alpha J_n'(\alpha x)$$

Similarly, $v = J_n(\beta x)$ and $v' = \beta J_n'(\beta x)$. Substituting these values in (3), we get

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{\alpha J_n'(\alpha)J_n(\beta) - \beta J_n(\alpha)J_n'(\beta)}{\beta^2 - \alpha^2} \quad ...(4)$$

If α and β are distinct roots of $J_n(x) = 0$, then $J_n(\alpha) = J_n(\beta) = 0$, and (4) reduces to

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = 0 \quad ...(5)$$

This is known as the *orthogonality relation of Bessel functions.*

When $\beta = \alpha$, the right side of (4) is of 0/0 form. Its value can be found by considering α as a root of $J_n(x) = 0$ and β as a variable approaching α. Then (4) gives

$$\underset{\beta\to\alpha}{\text{Lt}} \int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \underset{\beta\to\alpha}{\text{Lt}} \frac{\alpha J_n'(\alpha)J_n(\beta)}{\beta^2 - \alpha^2}$$

or by L'Hospital's rule, $\displaystyle\int_0^1 xJ_n^2(\alpha x)dx = \underset{\beta\to\alpha}{\text{Lt}} \frac{\alpha J_n'(\alpha)J_n'(\beta)}{2\beta} = \frac{1}{2}[J_n'(\alpha)]^2$

$$= \frac{1}{2}[J_{n+1}(\alpha)]^2 \quad ...(6) \quad \text{[By (5) of p. 552]}$$

Obs. If however, the interval be from 0 to 1, it can be shown that

$$\int_0^1 xJ_n^2(\alpha x)\,dx = \frac{1}{2}[J_n'(\alpha)]^2 \qquad \text{where } \alpha \text{ is the root of } J_n(x) = 0. \quad ...(7) \quad (V.T.U., 2006)$$

(2) Fourier-Bessel expansion. If $f(x)$ is a continuous function having finite number of oscillations in the interval $(0, a)$, then we can write

$$f(x) = c_1J_n(\alpha_1 x) + c_2J_n(\alpha_2 x) + ... + c_nJ_n(\alpha_n x) + ... \quad ...(8)$$

where α_1, α_2 ... are the positive roots of $J_n(x) = 0$.

To determine the coefficients c_n, multiply both sides of (8) by $xJ_n(\alpha_n x)$ and integrate from 0 to a. Then all integrals on the right of (1) vanish by (5), except the term in c_n. This gives

$$\int_0^a xf(x)J_n(\alpha_n x)dx = c_n\int_0^a xJ_n^2(\alpha_n x)dx = c_n\frac{a^2}{2}J_{n+1}^2(a\alpha_n) \quad \text{[By (7)]}$$

$$\therefore \qquad c_n = \frac{2}{a^2J_{n+1}^2(a\alpha_n)}\int_0^a xf(x)J_n(\alpha_n x)\,dx$$

Equation (8) is known as the *Fourier-Bessel expansion of f (x).*

Example 16.14. *If $\alpha_1, \alpha_2, ... \alpha_n$ are the positive roots of $J_0(x) = 0$, show that*

$$\frac{1}{2} = \sum_{n=1}^{\infty}[J_0(\alpha_n x)/\alpha_n J_1(\alpha_n)].$$

Solution. If $\quad f(x) = c_1J_n(\alpha_1 x) + c_2J_n(\alpha_2 x) + ... + c_rJ_n(\alpha_n x) + ... \quad ...(i)$

then
$$c_r = \frac{2}{a^2J_{n+1}^2(a\alpha_r)}\int_0^a xf(x)J_n(\alpha_n x)dx$$

Taking $f(x) = 1$, $a = 1$ and $n = 0$, we get

$$c_r = \frac{2}{J_1^2(\alpha_r)}\int_0^1 xJ_0(\alpha_r x)dx = \frac{2}{J_1^2(\alpha_r)}\left|\frac{xJ_1(\alpha_r x)}{\alpha_r}\right|_0^1 = \frac{2}{\alpha_r J_1(\alpha_r)}$$

From (*i*), $\qquad 1 = \displaystyle\sum_{r=1}^{\infty}\frac{2}{\alpha_r J_1(\alpha_r)}J_0(\alpha_r x) \quad$ or $\quad \frac{1}{2} = \sum_{n=1}^{\infty}\frac{J_0(\alpha_n x)}{\alpha_n J_1(\alpha_n)}.$

Example 16.15. *Expand $f(x) = x^2$ in the interval $0 < x < 2$ in terms of $J_2(\alpha_n x)$, where α_n are determined by $J_2(2\alpha_n) = 0$.*

Solution. Let the Fourier-Bessel expansion of $f(x)$ be $x^2 = \displaystyle\sum_{n=1}^{\infty} c_nJ_2(\alpha_n x)$.

Multiplying both sides by $xJ_2(\alpha_n x)$ and integrating w.r.t. x from 0 to 2, we get

$$\int_0^2 x^3 J_2(\alpha_n x)dx = c_n \int_0^2 xJ_2^2(\alpha_n x)dx = c_n \frac{(2)^2}{2} J_3^2(2\alpha_n) \qquad \text{[By (7)]}$$

or
$$\left| \frac{x^3 J_3(\alpha_n x)}{\alpha_n} \right|_0^2 = 2c_n J_3^2(2\alpha_n)$$

$\therefore$
$$c_n = \frac{4}{\alpha_n J_3(2\alpha_n)}$$

Hence
$$x^2 = 4\sum_{n=1}^{\infty} \frac{J_2(\alpha_n x)}{\alpha_n J_3(2\alpha_n)}.$$

16.12 BER AND BEI FUNCTIONS

Consider the differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - ixy = 0 \qquad \text{...(1)}$$

which occurs in certain problems of electrical engineering. This is equation (1) of §16.10 with $n = 0$ and $k^2 = -i$, so that its particular solution is

$$y = J_0(kx) = J_0[(-i)^{1/2} x] = J_0(i^{3/2} x)$$

Replacing $i^{3/2} x$ in the series for $J_0(x)$ [§16.8], we get

$$y = 1 - \frac{i^3 x^2}{2^2} + \frac{i^6 x^4}{(2\,!)^2 2^4} - \frac{i^9 x^6}{(3\,!)^2 2^6} + \frac{i^{12} x^8}{(4!)^2 2^8} - \ldots$$

$$= \left[1 - \frac{x^4}{2^2 \cdot 4^2} + \frac{x^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \ldots\right] + i\left[\frac{x^2}{2^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \frac{x^{10}}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2} - \ldots\right] \qquad \text{...(2)}$$

which is complex for x real. The series in the above brackets are taken to define *Bessel-real (or ber)* and *Bessel-imaginary (or bei) functions.*

Thus
$$ber\, x = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{x^{4m}}{2^2 \cdot 4^2 \cdot 6^2 \ldots (4m)^2} \qquad \text{...(3)}$$

and
$$bei\, x = -\sum_{m=1}^{\infty} (-1)^m \frac{x^{4m-2}}{2^2 \cdot 4^2 \cdot 6^2 \ldots (4m-2)^2} \qquad \text{...(4)}$$

so that $y = ber\, x + i\, bei\, x$ is a solution of (1).

Tables giving numerical values of $ber\, x$ and $bei\, x$ are also available.

Example 16.16. *Prove that* (i) $\frac{d}{dx}(x\, ber'\, x) = -x\, bei\, x$ (ii) $\frac{d}{dx}(x\, bei'\, x) = x\, ber\, x.$

Solution. We have $x\, ber' x = x\sum_{m=1}^{\infty} (-1)^m \frac{4mx^{4m-1}}{2^2 \cdot 4^2 \cdot 6^2 \ldots (4m)^2}$

$$= \sum_{m=1}^{\infty} (-1)^m \cdot \frac{x^{4m}}{2^2 \cdot 4^2 \cdot 6^2 \ldots (4m-2)^2\, 4m} = -\int_0^{\infty} x\, bei\, x\, dx$$

or
$$\frac{d}{dx}(x\, ber'\, x) = -x\, bei\, x$$

Again
$$\int_0^x x\, ber\, x\, dx = \frac{x^2}{2} + \sum_{m=1}^{\infty} (-1)^m \frac{x^{4m+2}}{2^2 \cdot 4^2 \cdot 6^2 \ldots (4m)^2 (4m+2)}$$

$$= -\sum_{m=1}^{\infty} (-1)^m \frac{x^{4m-2}}{2^2 \cdot 4^2 \cdot 6^2 \ldots (4m-4)^2 (4m-2)} = x\, bei'\, x \quad \text{or} \quad \frac{d}{dx}(x\, bei'\, x) = x\, ber\, x.$$

PROBLEMS 16.4

Obtain the solutions of the following differential equations in terms of Bessel functions :

1. $y'' + \frac{y'}{x} + \left(1 - \frac{1}{9x^2}\right) y = 0.$
2. $y'' + \frac{y'}{2} + \left(1 - \frac{1}{6.25x^2}\right) y = 0.$
3. $xy'' + \alpha y' + k^2 xy = 0.$ (V.T.U., 2010)
4. $x^2y'' - xy' + 4x^2y = 0.$
5. $xy'' + y = 0.$
6. Show that (*i*) $x^n J_n(x)$ is a solution of the equation $xy'' + (1-2n)y' + xy = 0.$ (V.T.U., 2001)
   (*ii*) $x^{-n}J_n(x)$ is a solution of the equation $xy'' + (1+2n)y' + xy = 0.$
7. Show that under the transformation $y = u/\sqrt{x}$, Bessel equation becomes

   $u'' + \left(1 + \frac{1-4n^2}{4x^2}\right)u = 0$. Hence find the solution of this equation.
8. By the use of substitution $y = u/\sqrt{x}$, show that the solution of the equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right)y = 0$ can be written in the form $y = c_1\frac{\sin x}{\sqrt{x}} + c_2\frac{\cos x}{\sqrt{x}}$.
9. Show that $\int_0^p x(ber^2 x + bei^2 x)\,dx = p(ber\, p\, bei'\, p - bei\, p\, ber'\, p).$
10. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the positive roots of $J_0(x) = 0$, prove that

$$x^2 = 2\sum_{n=1}^{\infty} \frac{\alpha_n^2 - 4}{\alpha_n^3 J_1(\alpha_n)} J_0(\alpha_n x).$$

11. Expand $f(x) = x^3$ in the interval $0 < x < 3$ in terms of functions $J_1(\alpha_n x)$ where α_n are determined by $J_1(3\alpha) = 0$.

16.13 LEGENDRE'S EQUATION*

Another differential equation of importance in Applied Mathematics, particularly in boundary value problems for spheres, is *Legendre's equation,*

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad ...(1)$$

Here n is a real number. But in most applications only integral values of n are required.

Substituting $\quad y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots\ (a_0 \neq 0),$

(1) takes the form

$$a_0(m)(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + \ldots + [a_{r+2}(m+r+2)(m+r+1)$$
$$-\{(m+r)(m+r+1) - n(n+1)\}a_r]x^{m+r} + \ldots = 0$$

Equating to zero the coefficient of the lowest power of x, *i.e.*, of x^{m-2}, we get

$$a_0 m(m-1) = 0,\ m = 0, 1 \qquad [\because\ a_0 \neq 0]$$

Equating to zero the coefficients of x^{m-1} and x^{m+r}, we get $a_1(m+1)m = 0 \qquad ...(2)$

$$a_{r+2}(m+r+2)(m+r+1) - \{(m+r)(m+r+1) - n(n+1)\}a_r = 0 \qquad ...(3)$$

When $m = 0$, (2) is satisfied and therefore, $a_1 \neq 0$. Then (3) gives, taking $r = 0, 1, 2, \ldots$ in turn,

$$a_2 = -\frac{n(n+1)}{2!}a_0, \quad a_3 = -\frac{(n-1)(n+2)}{3!}a_1$$

$$a_4 = \frac{-(n-2)(n+3)}{4.3}a_2 = \frac{n(n-2)(n+1)(n+3)}{4!}a_0$$

$$a_5 = -\frac{(n-3)(n+4)}{5.4}a_3 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!}a_1, \text{ etc.}$$

Hence for $m = 0$, there are two independent solutions of (1) :

$$y_1 = a_0\left\{1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \ldots\ldots\right\} \qquad ...(4)$$

*See footnote p. 493.

$$y_2 = a_1 \left\{ x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \ldots\ldots \right\} \qquad \ldots(5)$$

When $m = 1$, (2) shows that $a_1 = 0$. Therefore, (3) gives

$$a_3 = a_5 = a_7 = \ldots = 0$$

and
$$a_2 = -\frac{(n-1)(n+2)}{3!} a_0$$

$$a_4 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_0, \text{ etc.}$$

Thus for $m = 1$, we get the solution (5) again. Hence $y = y_1 + y_2$ is the general solution of (1).

If n is a positive even integer, the series (4) terminates at the term in x^n and y_1 becomes a polynomial. Similarly if n is an odd integer, (5) becomes a polynomial of degree n. Thus, whenever n is a positive integer, the general solution of (1) consists of a polynomial solution and an infinite series solution.

These polynomial solutions, with a_0 or a_1 so chosen that the value of the polynomial is 1 for $x = 1$, are called *Legendre polynomials* of order n and are denoted by $P_n(x)$. The infinite series solution with (a_0 or a_1 properly chosen) is called *Legendre function of the second kind* and is denoted by $Q_n(x)$. *(V.T.U., 2006)*

16.14 (1) RODRIGUE'S FORMULA*

We shall prove that $P_n(x) = \dfrac{1}{n!\,2^n} \dfrac{d^n}{dx^n}(x^2-1)^n$ ...(1)

Let $v = (x^2-1)^n$. Then $v_1 = \dfrac{dv}{dx} = 2nx(x^2-1)^{n-1}$

i.e., $(1-x^2)v_1 + 2nxv = 0$ ...(2)

Differentiating (2), $(n+1)$ times by Leibnitz's theorem

$$(1-x^2)v_{n+2} + (n+1)(-2x)v_{n+1} + \frac{1}{2!}(n+1)n(-2)v_n + 2n[xv_{n+1} + (n+1)v_n] = 0$$

or
$$(1-x^2)\frac{d^2(v_n)}{dx^2} - 2x\frac{d(v_n)}{dx} + n(n+1)v_n = 0$$

which is Legendre's equation and cv_n is its solution. Also its finite series solution is $P_n(x)$.

$\therefore$ $P_n(x) = cv_n = c\dfrac{d^n}{dx^n}(x^2-1)^n$ ...(3)

To determine the constant c, put $x = 1$ in (3). Then

$$1 = c\left[\frac{d^n}{dx^n}\{(x-1)^n(x+1)^n\}\right]_{x=1}$$

$= c[n!\,(x+1)^n]$

$+$ terms containing $(x-1)$ and its powers$]_{x=1}$

$= c\,.\,n!\,2^n$, *i.e.* $c = 1/n!\,2^n$.

Substituting this value of c in (3), we get (1), which is known as the *Rodrigue's formula*.

(V.T.U., 2008 ; Bhopal, 2007 ; U.P.T.U., 2004)

Obs. All roots of $P_n(x) = 0$ are real and lie between -1 and $+1$. *(Madras, 2003 S)*

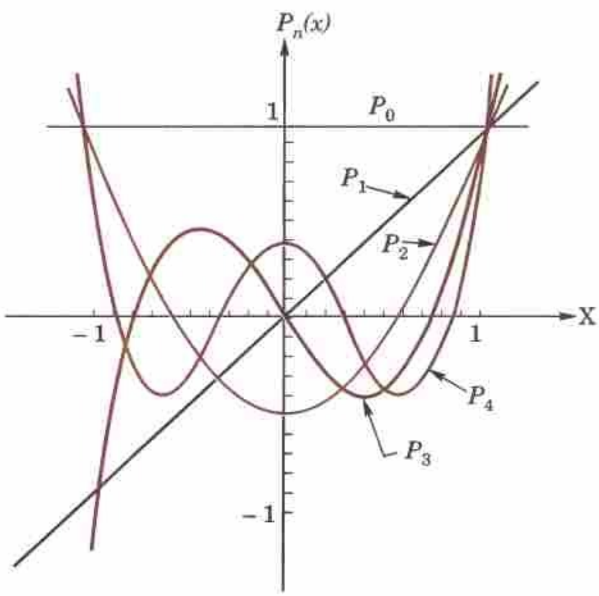

Fig. 16.2. Legendre polynomials.

(2) Legendre polynomials. Using (1), we get

$P_0(x) = 1,$ $\qquad P_1(x) = x,$

$P_2(x) = \dfrac{1}{2}(3x^2-1),$ $\qquad P_3(x) = \dfrac{1}{2}(5x^3-3x),$

* Named after the French mathematician and economist *Olinde Rodrigue* (1794–1851).

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x), \text{ etc.}$$ *(V.T.U., 2009)*

In general, we have $$P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r} \quad ...(4)$$

where $N = \frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

Let us derive (4) from (1).

By Binomial theorem, $(x^2-1)^n = \sum_{r=0}^{n} {}^nC_r (x^2)^{n-r}(-1)^r = \sum_{r=0}^{n} (-1)^r \frac{n!}{r!(n-r)!} x^{2n-2r}$

$\therefore$ by (1), $P_n = \frac{1}{n!\,2^n} \sum_{n=0}^{n} \frac{(-1)^r n!}{r!(n-r)!} \frac{d^n (x^{2n-2r})}{dx^n} = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r}$

This is same as (4), and the last term ($r = N$) is such that the power of x (*i.e.*, $n-2r$) for this term is either 0 or 1.

Example 16.17. *Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre polynomials.* *(V.T.U., 2010 ; S.V.T.U., 2007)*

Solution. Since $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$. $\therefore x^4 = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}$

$$\therefore \quad f(x) = \left[\frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}\right] + 3x^3 - x^2 + 5x - 2$$

$$= \frac{8}{35}P_4(x) + 3x^3 - \frac{1}{7}x^2 + 5x - \frac{73}{35} \qquad \left[\because x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x\,;\, x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}\right]$$

$$= \frac{8}{35}P_4(x) + 3\left[\frac{2}{5}P_3(x) + \frac{3}{5}x\right] - \frac{1}{7}\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + 5x - \frac{73}{35}$$

$$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}x - \frac{224}{105} \qquad [\because x = P_1(x), 1 = P_0(x)]$$

$$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}P_1 x - \frac{224}{105}P_0(x).$$

Example 16.18. *Show that for any function $f(x)$, for which the nth derivative is continuous,*

$$\int_{-1}^{1} f(x)\,P_n(x)dx = \frac{1}{2^n\,n!}\int_{-1}^{1}(1-x^2)^n f^n(x)\,dx.$$

Solution. Using Rodrigue's formula : $P_n(x) = \frac{1}{2^n\,n!}\frac{d^n}{dx^n}(x^2-1)^n$,

$$\int_{-1}^{1} f(x)\,P_n(x)\,dx = \frac{1}{2^n\,n!}\int_{-1}^{1} f(x)\frac{d^n(x^2-1)^n}{dx^n}\,dx \qquad \text{[Integrate by parts]}$$

$$= \frac{1}{2^n\,n!}\left[\left| f(x)\cdot\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\right|_{-1}^{1} - \int_{-1}^{1} f'(x)\cdot\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}dx\right]$$

$$= \frac{(-1)}{2^n\,n!}\int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

$$= \frac{(-1)^2}{2^n\,n!}\int_{-1}^{1} f''(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\,dx \qquad \text{[Again integrating by parts]}$$

$$= \frac{(-1)^n}{2^n\, n!} \int_{-1}^{1} f^n(x)\; (x^2-1)^n\, dx \qquad \text{[Integrating by parts } (n-2) \text{ times]}$$

$$= \frac{(-1)^{2n}}{2^n\, n!} \int_{-1}^{1} f^n(x)(1-x^2)^n\, dx = \frac{1}{2^n n!}\int_{-1}^{1} f^n\,(x)(1-x^2)^n\, dx$$

16.15 GENERATING FUNCTION FOR $P_n(x)$

To show that $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$.

Since
$$(1-z)^{-\frac{1}{2}} = 1 + \frac{1}{2}z + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}z^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}z^3 + \ldots$$

$$= 1 + \frac{2!}{(1!)^2\, 2^2}z + \frac{4!}{(2!)^2\, 2^4}z^2 + \frac{6!}{(3!)^2\, 2^6}z^3 + \ldots$$

$\therefore$
$$[1-t(2x-t)]^{-\frac{1}{2}} = 1 + \frac{2!}{(1!)^2\, 2^2}t(2x-t) + \frac{4!}{(2!)^2\, 2^4}t^2(2x-t)^2 + \ldots$$
$$+ \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}}t^{n-r}(2x-t)^{n-r} + \ldots + \frac{(2n)!}{(n!)^2\, 2^{2n}}t^n(2x-t)^n + \ldots \qquad \ldots(1)$$

The term in t^n from the term containing $t^{n-r}(2x-t)^{n-r}$

$$= \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}}\; t^{n-r}\,.\,{}^{n-r}C_r(-t)^r(2x)^{n-2r}$$

$$= \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}} \times \frac{(n-r)!}{r!\,(n-2r)!}(-1)^r t^n\,.\,(2x)^{n-2r} = \frac{(-1)^r\,(2n-2r)!}{2^n r!\,(n-r)!\,(n-2r)!}\,x^{n-2r}\,.\,t^n.$$

Collecting all terms in t^n which will occur in the term containing $t^n(2x-t)^n$ and the preceding terms, we see that terms in t^n.

$$= \sum_{r=0}^{N} \frac{(-1)^r\,(2n-2r)!}{2^n r!\,(n-r)!\,(n-2r)!}\,x^{n-2r}\,.\,t^n = P_n(x)t^n$$

where $N = \frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

Hence (1) may be written as $[1-t(2x-t)]^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$ ...(2)

(Kerala M.E., 2005 ; U.P.T.U., 2005)

This shows that $P_n(x)$ is the coefficient of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$. That is why, it is known as the *generating function of Legendre polynomials.*

Cor. 1. *$P_n(1) = 1$.* *(V.T.U., 2003 S ; Delhi, 2002)*

Taking $x = 1$ in (2), we have $(1-2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1)\, t^n$

i.e.,
$$\sum_{n=0}^{\infty} P_n(1)\, t^n = (1-t)^{-1} = 1 + t + t^2 + \ldots + t^n + \ldots$$

Equating coefficients of t^n, we get $P_n(1) = 1$.

Cor. 2. *$P_n(-1) = (-1)^n$.* *(B.P.T.U., 2005 S ; V.T.U., 2003)*

Taking $x = -1$ in (2), we have

$$\sum_{n=0}^{\infty} P_n(-1)\, t^n = (1+t)^{-1} = 1 - t + t^2 - \ldots + (-1)^n t^n + \ldots$$

Equating coefficients of t^n, we get the desired result.

Cor. 3. $P_n(0) = \begin{cases} (-1)^{n/2} \dfrac{1 \times 3 \times 5 \ldots (n-1)}{2 \times 4 \times 6 \times \ldots n}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd} \end{cases}$ (V.T.U., 2005)

Putting $x = 0$ in (2), we get $\sum_{n=0}^{\infty} P_n(0)\, t^n = (1 + t^2)^{-1/2}$

$$= 1 - \frac{1}{2}t^2 + \frac{1.3}{2.4} t^4 \ldots + (-1)^r \frac{1 \cdot 3 \cdot 5 \cdots (2r+1)}{2 \cdot 4 \cdot 6 \cdots 2r} t^{2r} + \ldots$$

Equating coefficient of t^{2m}, we get $P_{2m}(0) = (-1)^m \dfrac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2 \cdot 4 \cdot 6 \cdots 2m}$

Similarly equating coefficients of t^{2m+1}, we have $P_{2m+1}(0) = 0$.

Cor. 4. $P'_n(1) = \dfrac{1}{2} n(n+1)$ (U.P.T.U. 2003)

Since $P_n(x)$ is a solution of Legendre's eduation, $(1 - x^2) P''_n(x) - 2xP'_n(x) + n(n+1) P_n(x) = 0$

Putting $x = 1$, $-2P'_n(1) + n(n+1) P_n(1) = 0$ or $P'_n(1) = \dfrac{1}{2} n(n+1)$ $[\because\ P_n(1) = 1]$

16.16 RECURRENCE FORMULAE FOR $P_n(x)$

The following recurrence formulae can be easily derived from the generating function for $P_n(x)$:

(1) $(n+1) P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

(2) $nP_n(x) = xP'_n(x) - P'_{n-1}(x)$

(3) $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

(4) $P'_n(x) = xP'_{n-1}(x) + nP_{n-1}(x)$.

(5) $(1 - x^2)P'_n(x) = n\,[P_{n-1}(x) - xP_n(x)]$.

Proofs. **(1)** We know that $(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$ ...(*i*)

Differentiating partially w.r.t. t, we get

$$-\frac{1}{2}(1 - 2xt + t^2)^{-3/2}(-2x + 2t) = \Sigma nP_n(x)\, t^{n-1}$$

or $$(x - t)(1 - 2xt + t^2)^{-1/2} = (1 - 2xt + t^2)\, \Sigma nP_n(x)\, t^{n-1}$$

or $$(x - t)\, \Sigma P_n(x)\, t^n = (1 - 2xt + t^2)\, \Sigma nP_n(x)\, t^{n-1}$$

Equating coefficients of t^n from both sides, we get

$$xP_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2nxP_n(x) + (n-1) P_{n-1}(x)$$

whence follows the required result. (S.V.T.U., 2007 ; V.T.U., 2003)

(2) Differentiating (*i*) partially w.r.t. x,

$$-\frac{1}{2}(1 - 2xt + t^2)^{-3/2} \cdot (-2t) = \Sigma P_n{}'(x)t^n$$

i.e., $$t(1 - 2tx + t^2)^{-3/2} = \Sigma P_n{}'(x)\, t^n \qquad \text{...(ii)}$$

Again differentiating (*i*) partially w.r.t. t, we have

$$(x - t)(1 - 2tx + t^2)^{-3/2} = \Sigma nP_n(x)t^{n-1} \qquad \text{...(iii)}$$

Dividing (*iii*) by (*ii*), we get $\dfrac{x - t}{t} = \dfrac{\Sigma nP_n(x)t^{n-1}}{\Sigma P_n{}'(x)t^n}$

i.e., $$\Sigma nP_n(x)\, t^n = (x - t)\, \Sigma P_n{}'(x)\, t^n$$

Equating coefficients of t^n from both sides, we get (2). (J.N.T.U., 2006 ; U.P.T.U., 2006)

(3) Differentiating **(1)** w.r.t. x, we get

$$(n+1)P'_{n+1}(x) = (2n+1)P_n(x) + (2n+1)xP'_n(x) - nP'_{n-1}(x) \qquad \text{...(iv)}$$

Substituting for $xP'_n(x)$ from (2) in (*iv*), we obtain

$$(n+1) P'_{n+1}(x) = (2n+1) P_n(x) + (2n+1)\, [nP_n(x) + P'_{n-1}(x)] - nP'_{n-1}(x)$$

or $$(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$$ (Madras, 2006)

(4) Rewriting (*iv*) as

$$(n+1)P'_{n+1}(x) = (2n+1)P_n(x) + (n+1)xP_n'(x) + n[xP_n'(x) - P'_{n-1}(x)]$$

$$= (2n+1)P_n(x) + (n+1)xP_n'(x) + n^2P_n(x) \quad \text{[by (2)]}$$

$$= (n+1)xP_n'(x) + (n^2+2n+1)P_n(x)$$

or $$P'_{n+1}(x) = xP_n'(x) + (n+1)P_n(x)$$

Replacing n by $(n-1)$, we get (4).

(5) Rewriting (2) and (4) as

$$xP_n'(x) - P'_{n-1}(x) = nP_n(x) \quad ...(v)$$

and $$P_n'(x) - xP'_{n-1}(x) = nP_{n-1}(x) \quad ...(vi)$$

Multiplying (*v*) by x and substracting from (*vi*), we get

$$(1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)].$$

Example 16.19. *Prove that* $(2n+1)(1-x^2)P_n'(x) = n(n+1)[P_{n-1}(x) - P_{n+1}(x)]$.

Solution. We have the recurrence formula

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

or $$(\overline{n+1}+n)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

or $$(n+1)[xP_n(x) - P_{n+1}(x)] = n[P_{n-1}(x) - xP_n(x)]$$

$$= (1-x^2)P_n'(x) \quad [\because (1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)] \quad ...(i)$$

or $$xP_n(x) = P_{n+1}(x) + \frac{(1-x^2)P_n'(x)}{n+1} \quad ...(ii)$$

Also from (*i*) $$xP_n(x) = P_{n-1}(x) - \frac{(1-x^2)P_n'(x)}{n} \quad ...(iii)$$

From (*ii*) and (*iii*), $$P_{n-1}(x) - \frac{(1-x^2)P_n'(x)}{n} = P_{n+1}(x) + \frac{(1-x^2)P_n'(x)}{n+1}$$

or $$n(n+1)P_{n-1}(x) - (n+1)(1-x^2)P_n'(x) = n(n+1)P_{n+1}(x) + n(1-x^2)P_n'(x)$$

or $$(2n+1)(1-x^2)P_n'(x) = n(n+1)[P_{n-1}(x) - P_{n+1}(x)]$$

16.17 (1) ORTHOGONALITY OF LEGENDRE POLYNOMIALS

We shall prove that, $$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & m \neq n \\ \dfrac{2}{2n+1}, & m = n \end{cases}$$

We know that the solutions of

$$(1-x)^2u'' - 2xu' + m(m+1)u = 0 \quad ...(1)$$

and $$(1-x^2)v'' - 2xv' + n(n+1)v = 0 \quad ...(2)$$

are $P_m(x)$ and $P_n(x)$ respectively.

Multiplying (1) by v and (2) by u and subtracting, we get

$$(1-x^2)(u''v - uv'') - 2x(u'v - uv') + [m(m+1) - n(n+1)]uv = 0$$

or $$\frac{d}{dx}\{(1-x^2)(u'v - uv')\} + (m-n)(m+n+1)uv = 0.$$

Now integrating from -1 to 1, we get

$$(m-n)(m+n+1)\int_{-1}^{1} uv\,dx = \left|(1-x^2)(uv' - u'v)\right|_{-1}^{1} = 0.$$

Hence $$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0. \quad (m \neq n) \quad ...(3)$$

This is known as the *orthogonality property of Legendre polynomials.*

(S.V.T.U., 2008 ; Madras, 2006 ; V.T.U., 2006)

When $m = n$, we have from Rodrigue's formula,

$$(n\,!\,2^n)^2 \int_{-1}^{1} P_n^2(x)\,dx = \int_{-1}^{1} D^n(x^2-1)^n \cdot D^n(x^2-1)^n\,dx \qquad \text{[Integrate by parts]}$$

$$= \left| D^n(x^2-1)^n \cdot D^{n-1}(x^2-1)^n \right|_{-1}^{1} - \int_{-1}^{1} D^{n+1}(x^2-1)^n \,.\, D^{n-1}(x^2-1)^n\,dx$$

Since $D^{n-1}(x^2-1)^n$ has x^2-1 as a factor, the first term on the right vanishes for $x = \pm 1$. Thus

$$(n\,!\,2^n)^2 \int_{-1}^{1} P_n^2(x)\,dx = -\int_{-1}^{1} D^{n+1}(x^2-1)^n \,.\, D^{n-1}(x^2-1)^n\,dx$$

[Integrate by parts $(n-1)$ times]

$$= (-1)^n \int_{-1}^{1} D^{2n}(x^2-1)^n \,.\, (x^2-1)^n\,dx = (-1)^n \int_{-1}^{1} (2n)!\,(x^2-1)^n\,dx$$

$$= 2(2n)! \int_0^1 (1-x^2)^n\,dx \qquad \text{[Put } x = \sin\theta]$$

$$= 2(2n)! \int_0^{\pi/2} \cos^{2n+1}\theta\,d\theta = 2(2n)!\,\frac{2n(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 2\,.\,1}$$

$$= 2(2n)!\,[2n(2n-2)\ldots 4\,.\,2]^2/(2n+1)! = \frac{2}{2n+1}\,(2^n n\,!)^2$$

Hence $\displaystyle\int_{-1}^{1} P_n^2(x)dx = 2/(2n+1).$ ...(4) *(Bhopal, 2008 ; V.T.U., 2007 ; J.N.T.U., 2006)*

(2) Fourier-Legendre expansion of f(x). If $f(x)$ be a function defined from $x = -1$ to $x = 1$, we can write

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \qquad \text{...(5)}$$

To determine the coefficient c_n, multiply both sides by $P_n(x)$ and integrate from -1 to 1. Then (3) and (4) give

$$\int_{-1}^{1} f(x)P_n(x)dx = c_n \int_{-1}^{1} P_n^2(x)\,dx = \frac{2c_n}{2n+1} \quad \text{or} \quad c_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_n(x)\,dx$$

Equation (5) is known as *Fourier-Legendre expansion of f (x).*

Example 16.20. *Show that* $\displaystyle\int_{-1}^{1} xP_n(x)\,P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}.$

Solution. The recurrence formula (1) can be written as

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$

or
$$(2n-1)\,xP_{n-1} = nP_n + (n-1)P_{n-2} \qquad \text{[Changing } n \text{ to } n-1]$$

Multiplying by P_n, we get $\displaystyle xP_nP_{n-1} = \frac{1}{2n-1}\,[nP_n^2 + (n-1)P_nP_{n-2}]$

Integrating both sides w.r.t. x from $x = -1$ to $x = 1$, we get

$$\int_{-1}^{1} xP_nP_{n-1}dx = \frac{n}{2n-1}\int_{-1}^{1} P_n^2 dx + \frac{n-1}{2n-1}\int_{-1}^{1} P_nP_{n-2}\,dx$$

$$= \frac{n}{2n-1}\left(\frac{2}{2n+1}\right) + \frac{n-1}{2n-1}(0), \text{ by Orthogonal property}$$

Hence $\displaystyle\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}.$

Example 16.21. *Show that* $\displaystyle\int_{-1}^{1}(1-x^2)\,P_m'(x)\,dx, = \begin{cases} 0, & \text{when } m \neq n \\ \dfrac{2n(n+1)}{2n+1}, & \text{when } m = n \end{cases}$

(S.V.T.U., 2008 ; U.P.T.U., 2006)

Solution. Integrating by parts,

$$\int_{-1}^{1}(1-x^2)\ P'_m(x)\,P'_n(x)dx = \Big|(1-x^2)P'_m(x)\cdot P_n(x)\Big|_{-1}^{1} - \int_{-1}^{1}\cdot\frac{d}{dx}\left\{(1-x^2)\,P'_m(x)\right\}P_n\,dx$$

$$= -\int_{-1}^{1} P_n\left\{(1-x^2)\,P''_m(x) - 2x\,P'_m(x)\right\}dx \quad \text{...(i)}$$

Now $P_m(x)$ being a solution of Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + m(m+1)\,y = 0, \text{ we have}$$

$$(1-x^2)\,P''_m(x) - 2x\,P'_m(x) = -m(m+1)\,P_m(x)$$

Substituting this in (i), we get

$$\int_{-1}^{1}(1-x^2)\ P'_m(x)\,P'_n(x)\,dx = -\int_{-1}^{1} P_n\ \{-m(m+1)\,P_m(x)\}\,dx$$

$$= m(m+1)\int_{-1}^{1} P_m(x)\,P_n(x)dx \quad \text{...(ii)}$$

When $m \neq n$, $\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0$, by orthogonality property.

$$\therefore \quad \int_{-1}^{1}(1-x^2)\ P'_m(x)\,P'_n(x)\,dx = m(m+1)\,.\,0 = 0 \quad \text{[from (ii)]}$$

When $m = n$, $\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \dfrac{2}{2n+1}$, by orthogonality property.

$$\therefore \quad \int_{-1}^{1}(1-x^2)\ P'_m(x)\,P'_n(x)\,dx = n(n+1)\,.\,\frac{2}{2n+1} = \frac{2n(n+1)}{(2n+1)}.$$

Example 16.22. *Show that* $\int_{-1}^{1} x^2\ P_{n-1}P_{n+1}\,dx = \dfrac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$.

(J.N.T.U., 2006 ; Kerala M. Tech., 2005)

Solution. We have from the recurrence relation (1),

$$(2n+1)\,x\,P_n = (n+1)\,P_{n+1} + n\,P_{n-1}$$

$$\therefore \quad xP_{n-1} = \frac{1}{2n-1}\ \{nP_n + (n-1)\,P_{n-2}\}$$

and

$$xP_{n+1} = \frac{1}{2n+3}\ \{(n+2)\,P_{n+2} + (n+1)\,P_n\}P$$

$$\therefore \quad x^2P_{n-1}P_{n+1} = \frac{1}{(2n-1)(2n+3)}\{n(n+2)\,P_nP_{n+2} + n(n+1)P_n^{\,2}$$

$$+ (n-1)(n+2)P_{n-2}P_{n+2} + (n^2-1)P_n\,P_{n-2}\}$$

Integrating both sides from -1 to 1 and using orthogonality of Legendre polynomials, we get

$$\int_{-1}^{1} x^2P_{n-1}P_{n+1}\,dx = \frac{n(n+1)}{(2n-1)(2n+3)}\int_{-1}^{1} P_n^{\,2}\,dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}.$$

Example 16.23. *If* $f(x) = 0, \quad -1 < x \leq 0$

$\qquad = x, \quad 0 < x < 1,$

show that $f(x) = \dfrac{1}{4}P_0(x) + \dfrac{1}{2}P_1(x) + \dfrac{5}{16}\,P_2(x) - \dfrac{3}{32}\,P_4(x) + \ldots\ldots$ *(U.P.T.U., 2003)*

Solution. Let $\qquad f(x) = \displaystyle\sum_{n=0}^{\infty} c_n P_n(x)$

Then c_n is given by $c_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} f(x)\, P_n(x)\, dx$

$$= \left(n + \frac{1}{2}\right)\left[\int_{-1}^{0} 0 \,.\, P_n(x)\, dx + \int_0^1 xP_n(x)\, dx\right] = \left(n + \frac{1}{2}\right)\int_0^1 x\, P_n(x)\, dx$$

$\therefore$
$$c_0 = \frac{1}{2}\int_0^1 xP_0(x)\, dx = \frac{1}{2}\int_0^1 x\, dx = \frac{1}{4}$$

$$c_1 = \frac{3}{2}\int_0^1 xP_1(x)\, dx = \frac{3}{2}\int_0^1 x^2\, dx = \frac{1}{2}$$

$$c_2 = \frac{5}{2}\int_0^1 xP_2(x)\, dx = \frac{5}{2}\int_0^1 x \,.\, \frac{3x^2 - 1}{2}\, dx = \frac{5}{4}\left|\frac{3x^4}{4} - \frac{x^2}{2}\right|_0^1 = \frac{5}{16}$$

$$c_3 = \frac{7}{2}\int_0^1 xP_3(x)\, dx = \frac{7}{2}\int_0^1 x \,.\, \frac{5x^3 - 3x}{2}\, dx = \frac{7}{4}\left|5\frac{x^5}{5} - 3\frac{x^3}{3}\right|_0^1 = 0$$

$$c_4 = \frac{9}{2}\int_0^1 xP_4(x)\, dx = \frac{9}{2}\int_0^1 x \,.\, \frac{35x^4 - 30x^2 + 3}{8}\, dx$$

$$= \frac{9}{16}\left|35\frac{x^6}{6} - 35\frac{x^4}{4} + 3\frac{x^2}{2}\right|_0^1 = -\frac{3}{32} \text{ and so on.}$$

Hence
$$f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + \ldots .$$

PROBLEMS 16.5

1. Show that $P_n(-x) = (-1)^n P_n(x)$. *(Bhopal, 2008 ; V.T.U., 2003 S)*

2. Prove that (*i*) $P_{2n}'(0) = 0$ (*ii*) $P_{2n+1}'(0) = \frac{(-1)^n (2n+1)!}{2^{2n}(n!)^2}$. (*iii*) $P_n'(-1) = (-1)^n \frac{n(n+1)}{2}$ *(S.V.T.U., 2008)*

3. Express the following in terms of Legendre polynomials : (*i*) $5x^3 + x$
(*ii*) $x^3 + 2x^2 - x - 3$, *(Osmania, 2003)* (*iii*) $4x^3 + 6x^2 + 7x + 2$. *(S.V.T.U., 2008)*
(*iv*) $x^4 + 3x^3 - x^2 + 5x - 2$ *(Bhopal, 2008 ; Madras, 2006)*

4. Prove that (*i*) $(1 - x^2)\, P_n'(x) = (n + 1)\, [xP_n(x) - P_{n+1}(x)]$,
(*ii*) $P_n(x) = P'_{n+1}(x) - 2xP_n'(x) + P'_{n-1}(x)$. (*iii*) $P_n(x)\, P_{n+1/2}(x) = \frac{\sqrt{\pi}}{2^{2n+1}} P_{2m}(x)$ *(Anna, 2005 S)*

5. Prove that (*i*) $\int_{-1}^{1} [P_2(x)]^2\, dx = \frac{2}{5}$. *(P.T.U., 2002)* (*ii*) $\int_0^1 P_{2n}(x)\, dx = 0$.

6. Prove that $\int_{-1}^{1} P_n(x)(1 - 2hx + h^2)^{-1/2}\, dx = \frac{2h^n}{2n+1}$.

7. Show that $\int_{-1}^{1} (1 - x^2)[P_n'(x)]^2\, dx = \frac{2n(n+1)}{2n+1}$. *(U.P.T.U., 2006 ; Kerala M.E., 2005)*

8. Using Rodrigue's formula, show that $P_n(x)$ satisfies the differential equation
$$\frac{d}{dx}\left[(1 + x^2)\frac{d}{dx}[P_n(x)]\right] + n\,(n + 1)\, P_n(x) = 0.$$

9. Expand the following functions in terms of Legendre polynomials in the interval $-1 < x < 1$:
(*i*) $f(x) = x^3 + 2x^2 - x - 3$ *(V.T.U., 2008)* (*ii*) $f(x) = x^4 + x^3 + 2x^2 - x - 3$.

10. If $f(x) = 0, \quad -1 < x < 0$
$= 1, \quad 0 < x < 1$, show that $f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \ldots\ldots$

16.18 OTHER SPECIAL FUNCTIONS

The following special functions occur in numerous engineering problems. We state below their important properties which can be verified by similar methods :

(1) **Laguerre's polynomials***. These are the solutions of *Laguerre's differential equation*

$$xy'' + (1-x)y' + ny = 0 \qquad ...(1)$$

These polynomials $L_n(x)$, are given by the corresponding Rodrigue's formula

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}) \qquad ...(2)$$

In particular, $L_0(x) = 1$; $L_1(x) = 1 - x$, $L_2(x) = 2 - 4x + x^2$; $L_3(x) = 6 - 18x + 9x^2 - x^3$. *(Madras, 2006)*

Their *generating function* is given by

$$\frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n \qquad ...(3)$$

The *orthogonal property* for these polynomials is

$$\int_0^{\infty} e^{-x} L_m(x) L_n(x)\, dx = \begin{cases} 0, & m \neq n \\ (n!)^2, & m = n \end{cases} \qquad ...(4)$$

(2) **Hermite's polynomials†**. These are the solutions of Hermite's differential equation

$$y'' - 2xy' + 2ny = 0 \qquad ...(5)$$

These polynomials $H_n(x)$, are given by the Rodrigue's formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^{2n}}{dx^n}(e^{-x^2}) \qquad ...(6)$$

In particular, $H_0(x) = 1$; $H_1(x) = 2x$; $H_2(x) = 4x^2 - 2$; $H_3(x) = 8x^3 - 12x$. *(Madras, 2006)*

Their *generating function is* given by

$$e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \qquad ...(7) \quad \textit{(Madras, 2002 S)}$$

The *orthogonal property* of these polynomials is

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = \begin{cases} 0, & m \neq n \\ 2^n\, n!\sqrt{\pi}, & m = n \end{cases} \qquad ...(8)$$

(3) **Chebyshev polynomials****. These polynomials denoted by $T_n(x)$, are the solutions of the differential equation

$$(1 - x^2)y'' - xy' + n^2 y = 0 \qquad ...(9)$$

Their *generating function* is

$$\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x)\, t^n \qquad ...(10)$$

and

$$T_n(x) = \frac{n}{2} \sum_{r=0}^{N} (-1)^r \frac{(n-r-1)!}{r!(n-2r)!} (2x)^{n-2r} \qquad ...(11)$$

(J.N.T.U., 2006)

where $N = \frac{n}{2}$, if n is even and $N = \frac{1}{2}(n-1)$, if n is odd.

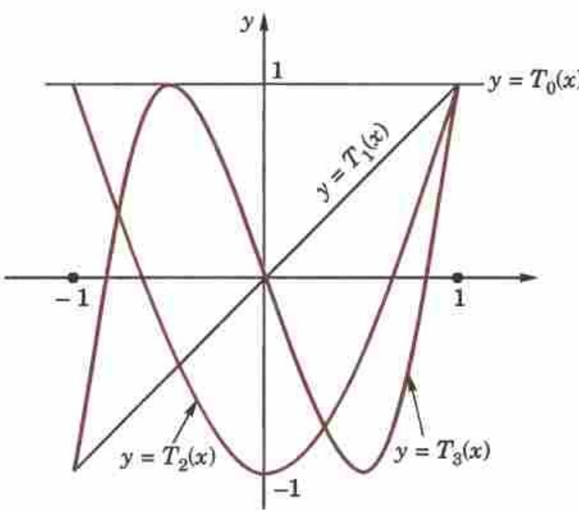

Fig. 16.3. Graphs of $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$.

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* Named after the French mathematician *Edmond Laguerre* (1834–86) who is known for his work in infinite series and geometry.

† See footnote p. 68.

** Named after the Russian mathematician *Pafnuti Chebyshev* (1821–1894) who is known for his work in the theory of numbers and approximation theory.

In particular, $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$. Also, we have the recurrence relation

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x) \qquad \text{...(12)} \quad (\textit{Bhopal, 2002})$$

which defines T_{n+1} in terms of T_n and T_{n-1}.

Their *orthogonal property* is

$$\int_{-1}^{1} (1-x^2)^{-1/2}\, T_m(x) T_n(x)\, dx = \begin{cases} 0, & m \neq n \\ \dfrac{\pi}{2}, & m = n \\ \pi, & m = n = 0 \end{cases} \qquad \text{...(13)}$$

Example 16.24. *Prove that* $\int_0^{\infty} e^{-x} L_m(x) L_n(x)\, dx = 0,\ m \neq n.$ *(Anna, 2006)*

Solution. Since $L_m(x)$ and $L_n(x)$ are the solutions of the Laguerre's differential equation (1).

$$\therefore \qquad xL_m'' + (1-x)L_m' + mL_m = 0 \qquad \text{...}(i)$$

$$xL_n'' + (1-x)L_n' + nL_n = 0 \qquad \text{...}(ii)$$

Multiplying (i) by L_n and (ii) by L_m and subtracting, we get

$$x(L_nL_m'' - L_mL_n'') + (1-x)(L_nL_m' - L_mL_n') = (n-m)L_mL_n$$

or

$$\frac{d}{dx}(L_nL_m' - L_mL_n') + \frac{1-x}{x}(L_nL_m' - L_mL_n') = \frac{(n-m)L_mL_n}{x}$$

This is Leibnitz's linear equation and its

$$\text{I.F.} = e^{\int \left(\frac{1}{x} - 1\right) dx} = e^{\log x - x} = xe^{-x}.$$

$$\therefore \quad \text{Its solution is } \left| (L_nL_m' - L_mL_n')\, xe^{-x} \right|_0^{\infty} = \int_0^{\infty} \frac{(n-m)L_mL_n}{x}\, xe^{-x}\, dx$$

or

$$\int_0^{\infty} e^{-x} L_mL_n\, dx = \left| \frac{(L_nL_m' - L_mL_n')xe^{-x}}{n-m} \right|_0^{\infty} = 0 \text{ which proves the result.}$$

Example 16.25. *Prove that* $H_n(x) = (-1)^n e^{x^2} \dfrac{d^{2n}}{dx^n}(e^{-x^2})$.

Solution. The generating function for $H_n(x)$ is $e^{2tx - t^2} = e^{x^2} \cdot e^{-(t-x)^2} = \sum_{n=0}^{\infty} H_n(x) \cdot \dfrac{t^n}{n!}$

$$\text{Then} \qquad \left[\frac{\partial^n}{\partial t^n}(e^{2tx - t^2})\right]_{t=0} = H_n(x) \qquad \text{...}(i)$$

$$\text{Also} \qquad \left[\frac{\partial^n}{\partial t^n}(e^{2tx - x^2})\right]_{t=0} = e^{x^2}\left[\frac{\partial^n}{\partial t^n}\{e^{-(t-x)^2}\}\right]_{t=0}$$

$$= e^{x^2}\left[\frac{\partial^n}{\partial(-x)^n}\{e^{-(t-x)^2}\}\right]_{t=0} = (-1)^n \frac{d^n}{dx^n}(e^{-x^2}) \qquad \text{...}(ii)$$

Equating (i) and (ii), we get the desired result.

PROBLEMS 16.6

1. Using the generating function (3) page 571, obtain the recurrence formula
$$L_{n+1}(x) = (2n+1-x)\, L_n(x) - n^2 L_{n-1}(x).$$
2. Show that (i) $nL_{n-1}(x) = nL'_{n-1}(x) - L'_n(x)$. (ii) $L'_n(x) = L'_{n-1}(x) - L_{n-1}(x)$. *(Anna, 2005)*
3. Show that (i) $H_{2n}(0) = (-1)^n \dfrac{2n!}{n!}$ $\qquad (ii)$ $H_{2n+1}(0) = 0$ *(Anna, 2005)*

4. Prove that (i) $H_n'(x) = 2n\, H_{n-1}(x)$ (ii) $\dfrac{d^m}{dx^m}[H_n(x)] = \dfrac{2^m \cdot n!}{(n-m)!} H_{n-m}(x),\ m < n.$
5. Using the generating function (7) page 515, obtain the recurrence formula $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$.
6. Prove that (i) $\int_{-\infty}^{\infty} e^{-x^2} H_2(x) H_3(x) = 0,$ (ii) $\int_{-\infty}^{\infty} e^{-x^2} \{H_2(x)\}^2 = 8\sqrt{\pi}.$ *(Madras, 2003)*
7. Express x^3 in terms of Chebyshev polynomials T_1 and T_3. *(U.P.T.U., 2009)*
8. Show that (i) $T_5 = 16x^5 - 20x^3 + 5x$. *(Bhopal, 2002)*
(ii) $(1 - x^2)T_n' = nT_{n-1}(x) - nxT_n(x)$. *(Osmania, 2003)*
9. Prove that $\dfrac{1-t^2}{1-2xt+t^2} = T_0(x) + 2\sum_{n=1}^{\infty} T_n(x)\, t^n$. *(J.N.T.U., 2006)*

16.19 (1) STRUM*-LIOUVILLE† PROBLEM

Legendre's equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ ...(i)
can be written as, $[(1 - x^2)y']' + \lambda y = 0$ $[\lambda = n(n + 1)]$

Bessel's equation $X^2 \dfrac{d^2y}{dx^2} + X\dfrac{dy}{dx} + (X^2 - n^2)y = 0$ can be transformed by putting $X = kx$ (so that $\dfrac{dy}{dx} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dX} = \dfrac{y'}{k}, \dfrac{d^2y}{dX^2} = \dfrac{y''}{k^2}$ to the form

$$x^2y'' + xy' + (k^2x^2 - n^2)y = 0$$

or $$(xy'' + y') + (\lambda x - n^2/x)y = 0 \qquad [\lambda = k^2]$$

or $$(xy')' + (\lambda x - n^2/x)y = 0 \qquad \text{...(ii)}$$

Both the equations (i) and (ii) are of the form

$$[r(x)\,y']' + [\lambda p(x) + q(x)]y = 0 \qquad \text{...(1)}$$

which is known as the *Strum-Liouville equation*. Similarly Laguerre's, Hermite's equations etc. can also be reduced to (1). Thus all the above equations of engineering utility can be considered with a common approach by means of Strum-Liouville's equation.

Eqution (1) considered on some interval $a \le x \le b$, satisfying the conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0,\ \beta_1 y(b) + \beta_2 y'(b) = 0 \qquad \text{...(2)}$$

with the real constants : α_1, α_2 not both zero and β_1, β_2 not both zero. The conditions (2) at the end points are called *boundary condtions.*

A differential equation together with the boundary conditions, is called a **boundary value problem.** *Equation (1) together with boundary conditions (2) is called a* **Strum-Liouville problem.**

Obviously $y = 0$ is a solution of the problem for any value of the parameter λ which is a trivial solution and as such is of no practical utility. Any other solution of (1) satisfying (2) is called an *eigen function* of the problem and the corresponding value of λ is called an eign value of the problem.

A special case. Taking $r = p = 1$ and $q = 0$ in (1), we get

$$y'' + \lambda y = 0 \qquad \text{...(3)}$$

Also if $\alpha_1 = \beta_1 = 1$ and $\alpha_2 = \beta_2 = 0$, then the boundary conditions (2) become

$$y(a) = 0, \quad y(b) = 0 \qquad \text{...(4)}$$

Thus (3) and (4) constitute the *simplest form of Strum-Liouville problem.*

(2) Orthogonality. Of the various properties of eigen functions of Strum-Liouville problem the orthogonality is of special importance.

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* Named after the Swiss mathematician *J.C.F. Strum* (1803–1855) who later became Poisson's successor at Sorbonne university, Paris.

† Named after the French professor *Joseph Liouville* (1809–1882) who is known for his important contributions to complex analysis, special functions, number theory and differential geometry.

Def. *Two functions $y_m(x)$ and $y_n(x)$ defined on some interval $a \le x \le b$, are said to be orthogonal on this interval w.r.t. the weight function $p(x) > 0$, if*

$$\int_a^b p(x)\, y_m(x) y_n(x)\, dx = 0 \text{ for } m \neq n.$$

The **norm** *of y_m, denoted by $||\, y_m \,||$, is defined to be the non-negative square root of* $\int_a^b p(x)\,[y_m(x)]^2\, dx$. *Thus*

$$||\, y_m \,|| = \sqrt{\left\{\int_a^b p(x)\,[y_m(x)]^2\, dx\right\}}$$

The functions which are orthogonal on $a \le x \le b$ and have norm equal to 1, are called **orthonormal** on this interval.

(3) Orthogonality of eigen functions.

Theorem. *If (i) the functions p, q, r and r' in the Strum-Liouville equation (1) be continuous in $a \le x \le b$;*

(ii) $y_m(x)$, $y_n(x)$ be two eigen functions of the Strum-Liouville problem corresponding to eigen values λ_m and *λ_n respectively ;*

then $y_m(x)$ and $y_n(x)$ $(m \neq n)$ are orthogonal on that interval w.r.t. the weight function $p(x)$.

Proof. Since y_m and y_n satisfy (1) above

$$(ry'_m)' + (\lambda_m p + q)\, y_m = 0$$
$$(ry'_n)' + (\lambda_n p + q)\, y_n = 0$$

Multiplying the first equation by y_n and the second by $-y_m$ and adding, we get

$$(\lambda_m - \lambda_n)\, p y_m y_n = y_m (ry'_n) - y_n (ry'_m)$$
$$= \frac{d}{dx}[(ry'_n)\, y_m - (ry'_m) y_n], \text{ after differentiation.}$$

Now integrating both sides w.r.t. x from a to b, we obtain

$$(\lambda_m - \lambda_n) \int_a^b p y_m y_n\, dx = \left[(ry'_n) y_m - (ry'_m) y_n\right]_a^b$$
$$= r(b)\,[y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b)] - r(a)\,[y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a)] \quad \text{...(A)}$$

The R.H.S. will vanish if the boundary conditions are of one of the following forms :

I. $y(a) = y(b) = 0$; II. $y'(a) = y'(b) = 0$; III. $\alpha_1 y(a) + \alpha_2 y'(a) = 0$, $\beta_1 y(b) + \beta_2 y'(b) = 0$ where either α_1 and α_2 is not zero and either β_1 or β_2 is not zero.

Thus in each case (A) reduces to $\int_a^b p y_m y_n\, dx = 0 \quad (m \neq n)$

which shows that the eigen functions y_m and y_n are orthogonal on $a \le x \le b$ w.r.t. the weight function $p(x) = 0$.

Obs. The third form of the boundary conditions in fact contains the first two forms as special cases.

Cor. 1. Orthogonality of Legendre polynomials has already been established directly in § 16.17. But if follows at once from the above theorem.

We have already seen in para (1) that Legendre's equation is Strum-Liouville equation

$$[(1 - x^2)\, y']' + \lambda y = 0 \qquad [\lambda = n(n + 1)]$$

with $r(x) = 1 - x^2$, $p(x) = 1$ and $q(x) = 0$.

Since $y(-1) = y(1) = 0$ and for $n = 0, 1, 2, \ldots, \lambda = 0, 1.2, 2.3, \ldots$, the Legendre polynomials are the solutions of the problem *i.e.*, these are the eigen functions. Thus it follows by the above theorem, that they are orthogonal on $-1 \le x \le 1$.

Cor. 2. Orthogonality of Bessel functions has also been established directly in § 16.11. But it can easily be seen to follow from the above theorem.

In para (1), we transformed the Bessel's equation

$$X^2 \frac{d^2 J_n}{dx^2} + X \frac{dJ_n}{dx} + (X^2 - n^2)\, J_n(x) = 0$$

into $[xJ'_n(kx)]' + (k^2 x - n^2/x)\, J_n(kx) = 0$ which is Strum-Liouville equation with $r(x) = x$, $p(x) = x$, $q(x) = -n^2/x$ and $\lambda = k^2$. Since $r(0) = 0$, it follows from the above theorem that those solutions of $J_n(kx)$ which are zero at $x = 0$ form an orthogonal set on $0 \le x \le R$ with weight function $p(x) = x$.

Example 16.26. *For the Strum-Liouville problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(l) = 0$, find the eigen functions and show that they are orthogonal.*

Solution. For $\lambda = -\gamma^2$, the general solution of the equation is $y(x) = c_1 e^{\gamma x} + c_2 e^{-\gamma x}$

The above boundary conditions give $c_1 = c_2 = 0$ and $y = 0$ which is not an eigen function.

For $\lambda = \gamma^2$, the general solution is $y(x) = A \cos \gamma x + B \sin \gamma x$

The first boundary condition gives $y(0) = A = 0$ and the second boundary condition gives $y(l) = B \sin \gamma l = 0$, $\gamma = 0, \pm \pi/l, \pm 2\pi/l, \ldots$ Thus the eigen values are $\lambda = 0, \pi^2/l^2, 4\pi^2/l^2, \ldots$ and taking $B = 1$, the corresponding eigen functions are

$$y_n(x) = \sin(n\pi x/l) \qquad n = 0, 1, 2, \ldots$$

From the above theorem, it follows that the said eigen functions are orthogonal on the interval $0 \le x \le l$.

Obs. This problem concerns an elastic string stretched between fixed points $x = 0$ and $x = l$ and allowed to vibrate. Here $y(x)$ is the space function of the deflection $u(x, t)$ of the string where t is the time. (See § 18.4).

PROBLEMS 16.7

Find the eigen functions of each of the following *Strum-Liouville problems* and verify their orthogonality :

1. $y'' + \lambda y = 0$, $y(0) = 0$, $y(\pi) = 0$.
2. $y'' + \lambda y = 0$, $y(0) = 0$, $y'(l) = 0$.
3. $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(\pi) = 0$.
4. $y'' + \lambda y = 0$, $y(\pi) = y(-\pi)$, $y'(\pi) = y'(-\pi)$.
5. $(xy')' + \lambda x^{-1} y = 0$, $y(1) = 0$, $y'(e) = 0$.

Transform each of the following equations to the *Strum-Liouville equations* indicating the weight function :

6. *Laguerre's equation* : $xy'' + (1 - x)y' + ny = 0$.
7. Hermite's equation : $y'' - 2xy' + 2ny = 0$.

16.20 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 16.8

Fill up the blanks or choose the correct answer in the following problems :

1. In terms of Legendre polynomials $2 - 3x + 4x^2$ is ......
2. $J_{-1/2} =$ ....
3. $\int_{-1}^{1} P_n^2(x)\, dx =$ ....
4. $P_{2n+1}(0) =$ ....
5. $\int_{-1}^{1} x^m P_n(x)\, dx =$ ......(m being an integer $< n$)
6. The recurrence relation connecting $J_n(x)$ to $J_{n-1}(x)$ and $J_{n+1}(x)$ is ....
7. Orthogonality relation for Bessel functions is ....
8. Bessel's equation of order zero is ....
9. $J_{1/2} =$ ....
10. $\dfrac{d}{dx}[x^n J_n(x)] =$ ....
11. Value of $P_2(x)$ is ....
12. $\int_{-1}^{1} P_3(x) P_4(x)\, dx =$ ....
13. $P_n(-1) = (-1)^n$ (True or False)
14. Rodrigue's formula for $P_n(x)$ is ....
15. $\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = 0$, if ....
16. Expansion of $5x^3 + x$ interms of Legendre polynomials is ....
17. Generating function of $P_n(x)$ is ....
18. $\dfrac{d}{dx}[J_0(x)] =$ ....
19. Bessel equation of order 4 is $x^2y'' + xy' + (x^2 - 4)y = 0$. (True or False)
20. $\dfrac{d}{dx}[x^2 J_2(x)] = x^2 J_1(x)$. (True or False)
21. Legendre's polynomial of first degree $= x$. (True or False)

22. If α is a root of $P_n(x) = 0$, then $P_{n+1}(\alpha)$ and $P_{n-1}(\alpha)$ are of opposite signs. (True or False)
23. $x = 0$ is a regular singular point of $2x^2y'' + 3xy' + (x^2 - 4)y = 0$. (True or False)
24. $\cos x = 2J_1 - 2J_3 + 2J_5 - \ldots\ldots$ (True or False)
25. If J_0 and J_1 are Bessel functions, then $J_1'(x)$ is given by

    (*a*) $-J_0$ (*b*) $J_0(x) - 1/x\, J_1(x)$ (*c*) $J_0(x) + \frac{1}{x} J_1(x)$.

26. If $J_n(x)$ is the Bessel function of first kind, then $\int_0^{\pi} [J_{-2}(x) - J_2(x)]\, dx =$

    (*a*) 2 (*b*) -2 (*c*) 0 (*d*) 1.

27. If $J_{n+1}(x) = \frac{2}{x} J_n(x) - J_0(x)$, then n is

    (*a*) 0 (*b*) 2 (*c*) -1 (*d*) none of these.

28. The series $x - \frac{x^3}{2^2(1!)^2} + \frac{x^5}{2^4(2!)^2} - \frac{x^7}{2^6(3!)^2} + \ldots\ldots \infty$ equals

    (*a*) $J_{1/2}(x)$ (*b*) $J_0(x)$ (*c*) $xJ_0(x)$ (*d*) $xJ_{1/2}(x)$.

29. If $\int_{-1}^{1} P_n(x)\, dx = 2$, then n is

    (*a*) 0 (*b*) 1 (*c*) -1 (*d*) none of these.

30. The value of $\int_{-1}^{1} (2x+1)P_3(x)\, dx$ where $P_3(x)$ is the third degree Legendre polynomial, is

    (*a*) 1 (*b*) -1 (*c*) 2 (*d*) 0.

31. The value of the integral $\int_{-1}^{1} x^3 P_3(x)\, dx$, where $P_3(x)$ is a Legendre polynomial of degree 3, is

    (*a*) 0 (*b*) $\frac{2}{35}$ (*c*) $\frac{4}{35}$ (*d*) $\frac{11}{35}$.

32. The polynomial $2x^2 + x + 3$ in terms of Legendre polynomials is

    (*a*) $\frac{1}{3}(4P_2 - 3P_1 + 11P_0)$ (*b*) $\frac{1}{3}(4P_2 + 3P_1 - 11P_0)$

    (*c*) $\frac{1}{3}(4P_2 + 3P_1 + 11P_0)$ (*d*) $\frac{1}{3}(4P_2 - 3P_1 - 11P_0)$.

33. If $P_n(x)$ be the Legendre polynomial, then $P_n'(-x)$ is equal to

    (*a*) $(-1)^n P_n(x)$ (*b*) $(-1)^n P_n'(x)$ (*c*) $(-1)^{n+1} P_n'(x)$ (*d*) $P_n''(x)$.

34. Legendre polynomial $P_5(x) = \lambda(63x^5 - 70x^3 + 15x)$ where λ is equal to

    (*a*) 1/2 (*b*) 1/5 (*c*) 1/8 (*d*) 1/10.

35. $\int_{-1}^{1} (1+x) P_n(x)\, dx$, $(n > 1)$, is equal to

    (*a*) $\frac{1}{2n+1}$ (*b*) $\frac{2}{2n+1}$ (*c*) $\frac{n}{2n+1}$ (*d*) 0.

36. The singular points of the differential equation $x^3(x-1)y'' + 2(x-1)y' + y - 0$ are .... (*P.T.U.*, 2009)