CHAPTER 23

Z-Transforms

23.1 INTRODUCTION

The development of communication branch is based on discrete analysis. Z-transform plays the same role in discrete analysis as Laplace transform in continuous systems. As such, Z-transform has many properties similar to those of the Laplace transform (§ 21.2). The main difference is that the Z-transform operates not on functions of continuous arguments but on sequences of the discrete integer-valued arguments, *i.e.* $n = 0, \pm 1, \pm 2,$ ...... The analogy of Laplace transform to Z-transform can be carried further. For every operational rule of Laplace transforms, there is a corresponding operational rule of Z-transforms and for every application of the Laplace transform, there is a corresponding application of Z-transform. A discrete system is expressible as a difference equation (§ 30.2) and its solutions are found using Z-transforms.

23.2 DEFINITION

If the function u_n is defined for discrete values ($n = 0, 1, 2, \ldots$) and $u_n = 0$ for $n < 0$, then its Z-transform is defined to be

$$Z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n} \text{ whenever the infinite series converges.} \quad \ldots(i)$$

The inverse Z-transform is written as $Z^{-1}[U(z)] = u_n$.

If we insert a particular complex number z into the power series (i), the resulting value of $Z(u_n)$ will be a complex number. Thus the Z-transform $U(z)$ is a complex valued function of a complex variable z.

23.3 SOME STANDARD Z-TRANSFORMS

The direct application of the definition gives the following results :

(1) $Z(a^n) = \dfrac{z}{z-a}$ **(2)** $Z(n^p) = -z\dfrac{d}{dz}Z(n^{p-1})$, *p being a +ve integer.*

Proof. **(1)** By definition, $Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}$

$$= 1 + (a/z) + (a/z)^2 + (a/z)^3 + \ldots = \frac{1}{1-(a/z)} = \frac{z}{z-a}$$ (*Kottayam, 2005*)

(2) $$Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n} = z \sum_{n=0}^{\infty} n^{p-1} . n . z^{-(n+1)} \quad \ldots(i)$$

Changing p to $p-1$, we get $Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} . z^{-n}$

Differentiating it w.r.t. z,

$$\frac{d}{dz}[Z(n^{p-1})] = \sum_{n=0}^{\infty} n^{p-1} . (-n) z^{-(n+1)} \quad \ldots(ii)$$

Substituting (*ii*) in (*i*), we obtain $Z(n^p) = -z\frac{d}{dz}[Z(n^{p-1})]$

which is the desired *recurrence formula*.

In particular, we have the following formulae :

(3) $Z(1) = \frac{z}{z-1}$ [Taking $a = 1$ in (1)]

(4) $Z(n) = \frac{z}{(z-1)^2}$ [Taking $p = 1$ in (2)]

(5) $Z(n^2) = \frac{z^2+z}{(z-1)^3}$ (*V.T.U., 2006*)

(6) $Z(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}$

(7) $Z(n^4) = \frac{z^4+11z^3+11z^2+z}{(z-1)^5}$.

23.4 LINEARITY PROPERTY

If a, b, c be any constants and u_n, v_n, w_n be any discrete functions, then

$$Z(au_n + bv_n - cw_n) = aZ(u_n) + bZ(v_n) - cZ(w_n)$$

Proof. By definition, $Z(au_n + bv_n - cw_n) = \sum_{n=0}^{\infty} (au_n + bv_n - cw_n)z^{-n}$

$$= a\sum_{n=0}^{\infty} u_n z^{-n} + b\sum_{n=0}^{\infty} v_n z^{-n} - c\sum_{n=0}^{\infty} w_n z^{-n}$$

$$= aZ(u_n) + bZ(v_n) - cZ(w_n).$$

23.5 DAMPING RULE

If $Z(u_n) = U(z)$, then $Z(a^{-n} u_n) = U(az)$

Proof. By definition, $Z(a^{-n} u_n) = \sum_{n=0}^{\infty} a^{-n} u_n . z^{-n} = \sum_{n=0}^{\infty} u_n . (az)^{-n} = U(az)$. (*Madras, 2006*)

Cor. $Z(a^n u_n) = U(z/a)$

Obs. The geometric factor a^{-n} when $|a| < 1$, damps the function u_n, hence the name *damping rule*.

23.6 SOME STANDARD RESULTS

The application of the damping rule leads to the following standard results :

(1) $Z(na^n) = \frac{az}{(z-a)^2}$

(2) $Z(n^2a^n) = \frac{az^2+a^2z}{(z-a)^3}$

(3) $Z(\cos n\theta) = \dfrac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$ (4) $Z(\sin n\theta) = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

(5) $Z(a^n \cos n\theta) = \dfrac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$ (6) $Z(a^n \sin n\theta) = \dfrac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$.

Proofs. (1) We know that $Z(n) = \dfrac{z}{(z-1)^2}$. Applying damping rule, we have

$$Z(na^n) = U(a^{-1}z) = \frac{a^{-1}z}{(a^{-1}z - 1)^2} = \frac{az}{(z-a)^2}.$$ *(Madras, 2000 S)*

(2) We know that $Z(n^2) = \dfrac{z^2 + z}{(z-1)^3}$. Applying damping rule, we have

$$Z(n^2a^n) = U(a^{-1}z) = \frac{(a^{-1}z)^2 + a^{-1}z}{(a^{-1}z - 1)^3} = \frac{a(z^2 + az)}{(z-a)^3}.$$

(3) and (4) We know that $Z(1) = \dfrac{z}{z-1}$. Applying damping rule, we have

$$Z(e^{-in\theta}) = Z(e^{-i\theta})^n \,.\, 1 = \frac{ze^{i\theta}}{ze^{i\theta} - 1} = \frac{z}{z - e^{-i\theta}} = \frac{z(z - e^{i\theta})}{(z - e^{-i\theta})(z - e^{i\theta})}$$

$$= \frac{z(z - \cos\theta) - iz\sin\theta}{z^2 - z(e^{i\theta} + e^{-i\theta}) + 1} = \frac{z(z - \cos\theta) - iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating real and imaginary parts, we get (3) and (4). *(V.T.U., 2010 S; Anna, 2009)*

(5) We know that $Z(\cos n\theta) = \dfrac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$. By damping rule, we have

$$Z(a^n \cos n\theta) = \frac{a^{-1}z\,(a^{-1}z - \cos\theta)}{(a^{-1}z)^2 - 2(a^{-1}z)\cos\theta + 1} = \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$$ *(V.T.U., 2006)*

Similarly using (4) above, we get (6).

Example 23.1. *Find the Z-transform of the following :*
(i) $3n - 4\sin n\pi/4 + 5a$ *(ii) $(n+1)^2$* *(V.T.U., 2010)*
(iii) $\sin(3n+5)$. *(V.T.U., 2009 S ; Kottayam, 2005)*

Solution. (*i*) $Z\left(3n - 4\sin\dfrac{n\pi}{4} + 5a\right) = 3Z(n) - 4Z\left(\sin\dfrac{n\pi}{4}\right) + 5a\,Z(1)$ [By Linearty property[

$$= 3\,.\,\frac{z}{(z-1)^2} - 4\,.\,\frac{z\sin n\pi/4}{z^2 - 2z\cos\pi/4 + 1} + 5a\,.\,\frac{z}{z-1}$$ [Using formulae for $Z(1)$, $Z(n)$, $Z(\sin n\theta)$]

$$= \frac{(3-5a)z + 5az^2}{(z-1)^2} - \frac{2\sqrt{2}z}{z^2 - \sqrt{2}z + 1}$$

(*ii*) $Z(n+1)^2 = Z(n^2 + 2n + 1) = Z(n^2) + 2Z(n) + Z(1)$

$$= \frac{z^2 + z}{(z-1)^3} + 2\frac{z}{(z-1)^2} + \frac{z}{z-1} = \frac{z^2(2z+1)}{(z-1)^3}$$

(*iii*) $Z[\sin(3n+5)] = Z(\sin 3n\cos 5 + \cos 3n\sin 5)$

$= \cos 5.Z(\sin 3n) + \sin 5.Z(\cos 3n)$ (using formulae for $Z(\sin n\theta)$, $Z(\cos n\theta)$

$$= \cos 5\,.\,\frac{z\sin 3}{z^2 - 2z\cos 3 + 1} + \sin 5\,.\,\frac{z(z - \cos 3)}{z^2 - 2z\cos 3 + 1} = z\,.\,\frac{(z\sin 5 - \sin 2)}{z^2 - 2z\cos 3 + 1}.$$

Example 23.2. *Find the Z-transforms of the following*
(*i*) e^{an} (*ii*) ne^{an} (*iii*) n^2e^{an}.

Solution. (*i*) Let $u_n = 1$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$. By damping rule $Z(k^{-n}\, u_n) = U(kz)$,

$$\therefore \quad Z(e^{an}) = Z(k^{-n}\,.\,1) = U(kz) = \frac{kz}{kz-1} \qquad \left[\because \; U(z) = Z(1) = \frac{z}{z-1}\right]$$

$$= \frac{z}{z-1/k} = \frac{z}{z-e^a}$$

(*ii*) Let $u_n = n$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$

By damping rule, $Z(e^{an}\,.\,n) = Z(k^{-n}\,.\,n) = U(k\,z)$ where $U(z) = Z(n) = \dfrac{z}{(z-1)^2}$

$$\frac{kz}{(kz-1)^2} = \frac{z}{k(z-1/k)^2} = \frac{e^a z}{(z-e^a)^2}$$

(*iii*) Let $u_n = n^2$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$

By damping rule,

$$Z(e^{an}\,.\,n^2) = Z(k^{-n}\,.\,n^2) = U(kz) \quad \text{where} \quad U(z) = Z(n^2) = \frac{z^2+z}{(z-1)^3}$$

$$= \frac{(kz)^2+kz}{(kz-1)^3} = \frac{z(z+1/k)}{(z-1/(k)^3} = \frac{ze^a(z+e^a)}{(z-e^a)^3}.$$

Example 23.3. *Find the Z-transform of* (*i*) *cosh nθ* (V.T.U., 2011) (*ii*) a^n *cosh nθ.*

Solution. (*i*)

$$Z(\cosh n\theta) = Z\left(\frac{e^{n\theta}+e^{-n\theta}}{2}\right)$$

$$= \frac{1}{2}\left[Z\left\{(e^{-\theta})^{-n}\,.\,1\right\} + Z\left\{(e^{\theta})^{-n}\,.\,1\right\}\right]$$

Apply damping rule to both terms, taking $u_n = 1$.

$$Z(\cosh n\theta) = \frac{1}{2}\left[\frac{ze^{-\theta}}{ze^{-\theta}-1} + \frac{ze^{\theta}}{ze^{\theta}-1}\right] \qquad \left[\because \; z(1) = \frac{z}{z-1}\right]$$

$$= \frac{1}{2}\left\{\frac{2z^2 - z(e^{\theta}+e^{-\theta})}{z^2 - z(e^{\theta}+e^{-\theta})+1}\right\} = \frac{z^2 - z\cosh\theta}{z^2 - 2z\cosh\theta + 1}$$

(*ii*)

$$Z(a^n \cosh n\theta) = Z[(a^{-1})^{-n}\,.\,\cosh n\theta] \qquad \text{[Apply damping rule using (i)]}$$

$$= \frac{(a^{-1}z)^2 - (a^{-1}\,z)\cosh\theta}{(a^{-1}\,z)^2 - 2(a^{-1}\,z)\cosh\theta + 1} = \frac{z(z-a\cosh\theta)}{z^2 - 2az\cosh\theta + a^2}.$$

Example 23.4. *Find the Z-transforms of*
(*i*) $e^t \sin 2t$ (Madras, 2003) (*ii*) $c^k \cos k\alpha$, $(k \geq 0)$ (U.P.T.U., 2004 S)

Solution. (*i*) We know that $Z(\sin 2t) = \dfrac{z\sin 2}{z^2 - 2z\cos 2 + 1}$ ...(A)

$$\therefore \quad Z(e^t \sin 2t) = Z[(e^{-1})^{-t}\,.\,\sin 2t] \qquad \text{[Apply damping rule, using (A)]}$$

$$= \frac{(e^{-1}\,z)\sin 2}{(e^{-1}z)^2 - 2(e^{-1}\,z)\cos 2 + 1} = \frac{ez\sin 2}{z^2 - 2ez\cos 2 + e^2}.$$

(*ii*) We know that $Z(\cos k\alpha) = \dfrac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$ ...(B)

$$\therefore \quad Z(c^k \cos k\alpha) = Z[(c^{-1})^{-k}\,.\,\cos k\alpha] \qquad \text{[Apply damping rule, using (B)]}$$

$$= \frac{(c^{-1}\,z)\,[c^{-1}\,z - \cos\alpha]}{(c^{-1}\,z)^2 - 2(c^{-1}\,z)\cos\alpha + 1} = \frac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}.$$

Example 23.5. *Find the Z-transforms of*

(*i*) $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$ (*V.T.U., 2011 S*) (*ii*) $\cosh\left(\frac{n\pi}{2}+\theta\right)$. (*U.P.T.U., 2008*)

Solution. (*i*) $Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right] = Z\left(\cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right)$

$$= \cos\frac{\pi}{4}\,.\,Z\left(\cos\frac{n\pi}{2}\right) - \sin\frac{\pi}{2}\,.\,Z\left(\sin\frac{n\pi}{2}\right) \quad \text{[Using formulae for } Z(\sin n\alpha) \text{ and } Z(\cos n\alpha)\text{]}$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{z(z-\cos\pi/2)}{z^2-2z\cos\pi/2+1} - \frac{z\sin\pi/2}{z^2-2z\cos\pi/2+1}\right\} = \frac{1}{\sqrt{2}}\left(\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right) = \frac{z(z-1)}{\sqrt{2}(z^2+1)}$$

(*ii*)
$$Z\left[\cosh\left(\frac{n\pi}{2}+\theta\right)\right] = Z\left[\frac{e^{n\pi/2+\theta}+e^{-(n\pi/2+\theta)}}{2}\right] = \frac{1}{2}\left[e^{\theta}\,Z(e^{n\pi/2}) + e^{-\theta}\,Z(e^{-n\pi/2})\right]$$

Since, $Z(a^n) = \frac{z}{z-a}$, $\therefore\ Z(e^{n\pi/2}) = Z(e^{\pi/2})^n = \frac{z}{z-e^{\pi/2}}$, $Z(e^{-n\pi/2}) = \frac{z}{z-e^{-\pi/2}}$

Thus
$$Z\left[\cosh\left(\frac{n\pi}{2}+\theta\right)\right] = \frac{1}{2}\left\{e^{\theta}\,.\,\frac{z}{z-e^{\pi/2}} + e^{-\theta}\,.\,\frac{z}{z-e^{-\pi/2}}\right\}$$

$$= \frac{z}{2}\left\{\frac{z(e^{\theta}+e^{-\theta}) - [e^{(\pi/2-\theta)}+e^{-(\pi/2-\theta)}]}{z^2-z(e^{\pi/2}+e^{-\pi/2})+1}\right\} = \frac{z^2\cosh\theta - z\cosh\left(\frac{\pi}{2}-\theta\right)}{z^2-2z\cosh\left(\frac{\pi}{2}\right)+1}.$$

Example 23.6. *Find the Z-transform of*

(*i*) ${}^nC_p$ $(0 \le p \le n)$ (*ii*) ${}^{n+p}C_p$.

Solution. (*i*)
$$Z({}^nC_p) = \sum_{p=0}^{n}\left({}^nC_p z^{-p}\right) = 1 + {}^nC_1 z^{-1} + {}^nC_2 z^{-2} + \ldots + {}^nC_n z^{-n} = (1+z^{-1})^n$$

(*ii*)
$$Z({}^{n+p}C_n) = \sum_{p=0}^{n} {}^{n+p}C_p z^{-p}$$
$$= 1 + {}^{n+1}C_1 z^{-1} + {}^{n+2}C_2 z^{-2} + {}^{n+3}C_3 z^{-3} + \ldots \infty$$
$$= 1 + (n+1)z^{-1} + \frac{(n+2)(n+1)}{2!}z^{-2} + \frac{(n+3)(n+2)(n+1)}{3!}z^{-3} + \ldots \infty$$
$$= 1 + (-n-1)(-z^{-1}) + \frac{(-n-1)(-n-2)}{2!}(-z^{-1})^2 + \frac{(-n-1)(-n-2)(-n-3)}{3!}(-z^{-1})^3 + \ldots \infty$$
$$= (1-z^{-1})^{-n-1}.$$

Example 23.7. *Find the Z-transform of*

(*i*) *unit impulse sequence* $\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \ne 0 \end{cases}$ (*ii*) *unit step sequence* $u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \ge 0 \end{cases}$.

Solution. (*i*) $Z[\delta(n)] = \sum_{n=0}^{\infty}\delta(n)z^{-n} = 1 + 0 + 0 + \ldots = 1$

(*ii*) $Z[u(n)] = \sum_{n=0}^{\infty}u(n)z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \ldots = \frac{1}{1-z^{-1}} = \frac{z}{z-1}.$

23.7 (1) SHIFTING U_N TO THE RIGHT

If $Z(u_n) = U(z)$, *then* $Z(u_{n-k}) = z^{-k}U(z)$ $(k > 0)$

Proof. By definition,

$$Z(u_{n-k}) = \sum_{n=0}^{\infty} u_{n-k}z^{-n} = u_0 z^{-k} + u_1 z^{-(k+1)} + \dots = z^{-k}\sum_{n=0}^{\infty} u_n z^{-n} = z^{-k}U(z)$$

Obs. This rule will be very useful in applications to difference equations.

(2) **Shifting u_n to the left.** If $Z(u_n) = U(z)$, then

$$Z(u_{n+k}) = z^k[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1}z^{-(k-1)}]$$

Proof. $$Z(u_{n+k}) = \sum_{n=0}^{\infty} u_{n+k}z^{-n} = z^k \sum_{n=0}^{\infty} u_{n+k}z^{-(n+k)}$$

$$= z^k\left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n}\right]$$

Hence $Z(u_{n+k}) = z^k[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1}z^{-(k-1)}]$ (*J.N.T.U., 2002*)

In particular, we have the following standard results :

(1) $Z(u_{n+1}) = z[U(z) - u_0]$; (2) $Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$

(3) $Z(u_{n+3}) = z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$.

Example 23.8. *Show that* $Z\left(\frac{1}{n!}\right) = e^{1/z}$.

Hence evaluate $Z[1/(n+1)!]$ *and* $Z[1/(n+2)!]$. (*Madras, 2006*)

Solution. We have $$Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}z^{-n} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \dots = e^{1/z}.$$

Shifting $(1/n!)$ one unit to the left gives

$$Z\left[\frac{1}{(n+1)!}\right] = z\left[Z\left(\frac{1}{n!}\right) - 1\right] = z(e^{-1/z} - 1)$$

Similarly shifting $(1/n!)$ two units to the left gives

$$Z\left[\frac{1}{(n+2)!}\right] = z^2(e^{1/z} - 1 - z^{-1}).$$

23.8 MULTIPLICATION BY n

If $Z(u_n) = u(z)$, then $Z(nu_n) = -z\dfrac{dU(z)}{dz}$

Proof. $$Z(nu_n) = \sum_{n=0}^{\infty} n \,.\, u_n z^{-n} = -z\sum_{n=0}^{\infty} u_n(-n)z^{-n-1} = -z\sum_{n=0}^{\infty} u_n \frac{d}{dz}(z^{-n}).$$

$$= -z\sum_{n=0}^{\infty}\frac{d}{dz}(u_n z^{-n}) = -z\frac{d}{dz}\left(\sum_{n=0}^{\infty} u_n z^{-n}\right) = -z\frac{d}{dz}U(z).$$

Obs. We have, $Z(n^2 u_n) = \left(-z\dfrac{d}{dz}\right)^2 u(z)$ (*Madras, 2006*)

In general, $Z(n^m u_n) = \left(-z\dfrac{d}{dz}\right)^m u(z)$.

Example 23.9. *Find the Z-transform of (i) $n \sin n\theta$ (ii) $n^2 e^{n\theta}$.*

Solution. (*i*) We know that $Z(nu_n) = -z\dfrac{dU(z)}{dz}$ and $Z(\sin n\theta) = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

$$\therefore \quad Z(n\sin n\theta) = -z\frac{d}{dz}[Z(\sin n\theta)] = -z\frac{d}{dz}\left(\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}\right)$$

$$= -z\frac{\sin\theta - z^2\sin\theta}{(z^2 - 2z\cos\theta + 1)^2} = \frac{z(z^2-1)\sin\theta}{(z^2 - 2z\cos\theta + 1)^2}$$

(*ii*) We know that $\quad Z(e^{n\theta}) = \dfrac{z}{z - e^{\theta}}$

$$\therefore \quad Z(n^2 e^{n\theta}) = \left(-z\frac{d}{dz}\right)^2 (Ze^{n\theta}) = \left(-z\frac{d}{dz}\right)\left[-z\frac{d}{dz}\left(\frac{z}{z-e^{\theta}}\right)\right]$$

$$= \left(-z\frac{d}{dz}\right)\left\{-z\frac{(z-e^{\theta})(1) - z(1)}{(z-e^{\theta})^2}\right\} = -z\frac{d}{dz}\left\{\frac{ze^{\theta}}{(z-e^{\theta})^2}\right\}$$

$$= -ze^{\theta}\left\{\frac{(z-e^{\theta})^2(1) - z[2(z-e^{\theta})]}{(z-e^{\theta})^4}\right\} = -ze^{\theta}\frac{z - e^{\theta} - 2z}{(z-e^{\theta})^3} = \frac{z(z+e^{\theta})e^{\theta}}{(z-e^{\theta})^3}.$$

23.9 TWO BASIC THEOREMS

In applications, we often need the values of u_n for $n = 0$ or as $n \to \infty$ without requiring complete knowledge of u_n. We can find this as the behaviour of u_n for small values of n is related to the behaviour of $U(z)$ as $z \to \infty$ and *vice-versa*. The precise relationship is given by the following *initial and final value theorems :*

(1) Initial value theorem. *If $Z(u_n) = U(z)$, then $u_0 = \underset{z\to\infty}{Lt}\, U(z)$*

Proof. We know that $U(z) = Z(u_n) = u_0 + u_1 z^{-1} + u_2 z^{-2} + \ldots$

Taking limits as $z \to \infty$, we get $\underset{z\to\infty}{\text{Lt}}\,[U(z)] = u_0$, as required.

Similarly additional initial values can be found successively, giving :

$$u_1 = \underset{z\to\infty}{\text{Lt}}\left\{z[U(z) - u_0]\right\};\ u_2 = \underset{z\to\infty}{\text{Lt}}\left\{z^2[U(z) - u_0 - u_1 z^{-1}]\right\} \text{ and so on.}$$

(2) Final value theorem. *If $Z(u_n) = U(z)$, then*

$$\underset{n\to\infty}{Lt}\,(u_n) = \underset{z\to 1}{Lt}\,(z-1)\,U(z)$$

Proof. By definition, $Z(u_{n+1} - u_n) = \displaystyle\sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$

or $$Z(u_{n+1}) - Z(u_n) = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

or $$z[U(z) - u_0] - U(z) = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

or $$U(z)(z-1) - u_0 z = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

Taking limits of both sides as $z \to 1$, we get

$$\underset{z\to 1}{\text{Lt}}\,[(z-1)U(z)] - u_0 = \sum_{n=0}^{\infty}(u_{n+1} - u_n) = \underset{n\to\infty}{\text{Lt}}\,[(u_1 - u_0) + (u_2 - u_1) + \ldots + (u_{n+1} - u_n)]$$

$$= \underset{n \to \infty}{\text{Lt}} \; [u_{n+1}] - u_0 = u_\infty - u_0$$

Hence $$u_\infty = \underset{z \to 1}{\text{Lt}} \; [(z-1)\,U(z)].$$ *(Anna, 2005 S)*

Example 23.10. *If* $U(z) = \dfrac{2z^2 + 5z + 14}{(z-1)^4}$, *evaluate* u_2 *and* u_3.

Solution. Writing $$U(z) = \frac{1}{z^2} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4}$$

By initial value theorem, $u_0 = \underset{z \to \infty}{\text{Lt}} \; U(z) = 0$

Similarly, $u_1 = \underset{z \to \infty}{\text{Lt}} \; \{z\,[U(z) - u_0]\} = 0$

Now $u_2 = \underset{z \to \infty}{\text{Lt}} \; \{z^2\,[U(z) - u_0 - u_1 z^{-1}]\} = 2 - 0 - 0 = 2$

and $$u_3 = \underset{z \to \infty}{\text{Lt}} \; z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}] = \underset{z \to \infty}{\text{Lt}} \; z^3[U(z) - 0 - 0 - 2z^{-2}]$$

$$= \underset{z \to \infty}{\text{Lt}} \; z^3 \left[\frac{2z^2 + 5z + 14}{(z-1)^4} - \frac{2}{z^2}\right] = \underset{z \to \infty}{\text{Lt}} \; z^3 \left\{\frac{13z^3 + 2z^2 + 8z - 2}{z^2(z-1)^4}\right\} = 13.$$

PROBLEMS 23.1

1. Find the Z-transforms of the following sequences :
   (*i*) $\dfrac{a^n}{n!}$ $(n \geq 0)$ *(S.V.T.U., 2009)* (*ii*) $\dfrac{1}{(n+1)!}$ (*iii*) $(\cos\theta + i\sin\theta)^n$.
2. Using the linearity property, find the Z- transforms of the following functions :
   (*i*) $2n + 5\sin n\pi/4 - 3a^4$ (*ii*) $\dfrac{1}{2}(n-1)(n+2)$ *(S.V.T.U., 2007)*
   (*iii*) $(n+1)(n+2)$ *(Anna, 2008)* (*iv*) $(2n-1)^2$ *(V.T.U., 2011 S)*
3. Show that (*i*) $Z(\sinh n\theta) = \dfrac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}$ *(V.T.U., 2011)* (*ii*) $Z(a^n \sinh n\theta) = \dfrac{az\sinh\theta}{z^2 - 2az\cosh\theta + a^2}$.
4. Show that (*i*) $Z(e^{-an}\cos n\theta) = \dfrac{ze^a(ze^a - \cos\theta)}{z^2e^{2a} - 2ze^a\cos\theta + 1}$; (*ii*) $Z(e^{-an}\sin n\theta) = \dfrac{ze^a\sin\theta}{z^2e^{2a} - 2ze^a\cos\theta + 1}$
   Also evaluate $Z(e^{3n}\sin 2n)$. *(S.V.T.U., 2007)*
5. Using $Z(n^2) = \dfrac{z^2 + z}{(z-1)^3}$, show that $Z\,(n+1)^2 = \dfrac{z^3 + z^2}{(z-1)^3}$.
6. Find the Z-transforms of (*i*) $\sin(n+1)\theta$, (*ii*) $\cos\left(\dfrac{k\pi}{8} + \alpha\right)$. *(Marathwada, 2008)*
7. Find the Z-transform of $\cos n\theta$ and hence find $Z\,(n\cos n\theta)$. *(Anna, 2009)*
8. Find the Z-transform of $\cos(n\pi/2)$ and $a^n\cos(n\pi/2)$. *(Anna, 2008 S)*
9. Find the Z-transforms of the following
   (*i*) e^{-an} (*ii*) e^{-2n} *(V.T.U., 2010 S)* (*iii*) $e^{-an}\,n^2$.
10. Show that (*i*) $Z[\delta(n+1)] = 1/z$ (*ii*) $(1/2)^n\,u(n) = \dfrac{2z}{2z-1}$.
11. Show that $Z({}^{n+p}C_p) = (1 - 1/z)^{-(p+1)}$. Using the damping rule, deduce that
    $$Z({}^{n+p}C_p\,a^n) = (1 + a/z)^{-(p+1)}.$$
12. If $Z(u_n) = \dfrac{z}{z-1} + \dfrac{z}{z^2+1}$, find the Z-transform of u_{n+2}. *(S.V.T.U., 2009)*
13. If $U(z) = \dfrac{2z^2 + 3z + 12}{(z-1)^4}$, find the value of u_2 and u_3.

14. Given that $Z(u_n) = \dfrac{2z^2+3z+4}{(z-3)^3}$, $|z| > 3$, show that $u_1 = 2$, $u_2 = 21$, $u_3 = 139$.

15. Show that (i) $Z\left(\dfrac{1}{n}\right) = z \log \dfrac{z}{z-1}$. *(Madras, 2003 S)* (ii) $Z\left\{\dfrac{1}{n(n+1)}\right\}$. *(Anna, 2005 S)*

16. Using $Z(n) = \dfrac{z}{(z-1)^2}$, show that $Z(n \cos n\theta) = \dfrac{(z^3+z)\cos\theta - 2z^2}{(z^2 - 2z\cos\theta + 1)^2}$.

23.10 SOME USEFUL Z-TRANSFORMS

Sr. No.	*Sequence u_n $(n \geq 0)$*	*Z-transform $U(z) = Z(u_n)$*
1.	k	$kz/(z-1)$
2.	$-k$	$kz/(z+1)$
3.	n	$z/(z-1)^2$
4.	n^2	$(z^2+z)/(z-1)^3$
5.	n^p	$-z\, d/dz\, [Z(n^{p-1})]$, p + ve integer.
6.	$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$	1
7.	$u(n) = \begin{Bmatrix} 0 & n < 0 \\ 1 & n \geq 0 \end{Bmatrix}$	$z/(z-1)$
8.	a^n	$z/(z-a)$
9.	na^n	$az/(z-a)^2$
10.	n^2a^n	$(az^2+a^2z)/(z-a)^3$
11.	$\sin n\theta$	$\dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$
12.	$\cos n\theta$	$\dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$
13.	$a^n \sin n\theta$	$\dfrac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$
14.	$a^n \cos n\theta$	$\dfrac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$
15.	$\sinh n\theta$	$\dfrac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}$
16.	$\cosh n\theta$	$\dfrac{z(z-\cosh\theta)}{z^2 - 2z\cosh\theta + 1}$
17.	$a^n \sinh n\theta$	$\dfrac{az\sinh\theta}{z^2 - 2az\cosh\theta + a^2}$
18.	$a^n \cosh n\theta$	$\dfrac{z(z - a\cosh\theta)}{z^2 - 2az\cosh\theta + a^2}$
19.	$a^n u_n$	$U(z/a)$
20.	u_{n+1}	$z[U(z) - u_0]$
	u_{n+2}	$z^2[U(z) - u_0 - u_1 z^{-1}]$
	u_{n+3}	$z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$
21.	u_{n-k}	$z^{-k}U(z)$
22.	nu_n	$-zd/dz[U(z)]$
23.	u_0	$\underset{z\to\infty}{\text{Lt}}\ U(z)$
24.	$\underset{n\to\infty}{\text{Lt}}\ (u_n)$	$\underset{z\to 1}{\text{Lt}}\ [(z-1)\,U(z)]$

23.11 SOME USEFUL INVERSE Z-TRANSFORMS

Sr. No.	$U(z)$	Inverse Z-transform $u_n = z^{-1}[U(z)]$
1.	$\frac{1}{z-a}$	a^{n-1}
2.	$\frac{1}{z+a}$	$(-a)^{n-1}$
3.	$\frac{1}{(z-a)^2}$	$(n-1)a^{n-2}$
4.	$\frac{1}{(z-a)^3}$	$\frac{1}{2}(n-1)(n-2)a^{n-3}$
5.	$\frac{z}{z-a}$	a^n
6.	$\frac{z}{z+a}$	$(-a)^n$
7.	$\frac{z^2}{(z-a)^2}$	$(n+1)a^n$
8.	$\frac{z^3}{(z-a)^3}$	$\frac{1}{2!}(n+1)(n+2)a^n u(n)$

23.12 CONVOLUTION THEOREM

If $Z^{-1}[U(z)] = u_n$ and $Z^{-1}[V(z)] = v_n$, then

$$Z^{-1}[U(z) \cdot V(z)] = \sum_{m=0}^{n} u_m \cdot v_{n-m} = u_n * v_n$$

where the symbol $*$ denotes the convolution operation.

Proof. We have $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$, $V(z) = \sum_{n=0}^{\infty} v_n z^{-n}$

$$\therefore \quad U(z)\,V(z) = (u_0 + u_1 z^{-1} + u_2 z^{-2} + \ldots + u_n z^{-n} + \ldots \infty) \times (v_0 + v_1 z^{-1} + v_2 z^{-2} + \ldots + v_n z^{-n} + \ldots \infty)$$

$$= \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \ldots + u_n v_0) z^{-n} = Z(u_0 v_n + u_1 v_{n-1} + \ldots + u_n v_0)$$

whence follows the desired result.

Obs. The convolution theorem plays an important role in the solution of difference equations and in probability problems involving sums of two independent random variables.

Example 23.11. *Use convolution theorem to evaluate* $Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$.

Solution. We know that $Z^{-1}\left\{\frac{z}{z-a}\right\} = a^n$ and $Z^{-1}\left\{\frac{z}{z-b}\right\} = b^n$

$$\therefore \quad Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = Z^{-1}\left\{\frac{z}{z-a} \cdot \frac{z}{z-b}\right\} = a^n * b^n$$

$$= \sum_{m=0}^{n} a^m \cdot b^{n-m} = b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m \text{ which is a G.P.}$$

$$= b^n \cdot \frac{(a/b)^{n+1} - 1}{a/b - 1} = \frac{a^{n+1} - b^{n+1}}{a - b}.$$

23.13 CONVERGENCE OF Z-TRANSFORMS

Z-transform operation is performed on a sequence u_n which may exist in the range of integers $-\infty < n < \infty$, and we write

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \qquad ...(1)$$

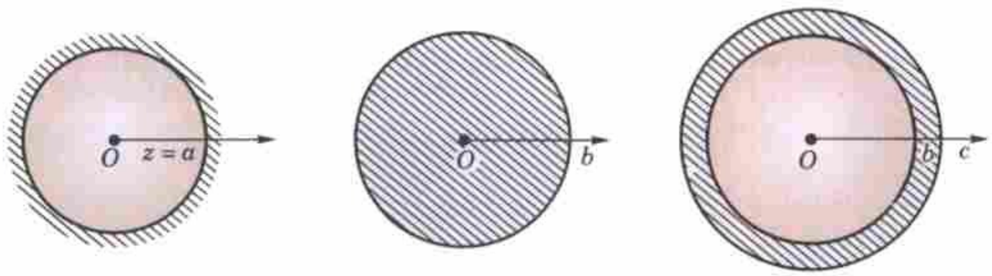
where u_n represents a number in the sequence for n = an integer. *The region of the z-plane in which (1) converges absolutely is known as the region of convergence (ROC) of U(z).*

We have so far discussed *one-sided Z-transform only for which $n \geq 0$*. Here the sequence is always *right-sided* and the *ROC* is always outside a prescribed circle say $|z| > |a|$ Fig. 23.2 (*i*)]. For a left-handed sequence, the ROC is always inside any prescribed contour $|z| < |b|$. [Fig. 23.2 (*ii*)].

23.14 TWO-SIDED Z-TRANSFORM OF u_n IS DEFINED BY

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \qquad ...(2)$$

In this case, the sequence is two-sided and the region of convergence for (2) is the *annular region* $|b| < |z| < |c|$ [Fig. 23.2 (*iii*)]. The inner circle bounds the terms in negative powers of z and the outer circle bounds the terms in positive powers of z. *The shaded annulus of convergence is necessary for the two sided sequence and its Z-transform to exist.*



$|z| > |a|$ (*i*) $\qquad$ $|z| < |b|$ (*ii*) $\qquad$ $|b| < |z| < |c|$ (*iii*)

Fig. 23.1

Example 23.12. *Find the Z-transfrom and region of convergence of*

(*a*) $u(n) = \begin{cases} 4^n & \text{for } n < 0 \\ 2^n & \text{for } n \geq 0 \end{cases}$ $\qquad$ (*b*) $u(n) = {}^nc_k,\ n \geq k.$

Solution. By definition $\quad Z[u(n)] = \sum_{-\infty}^{\infty} u(n) Z^{-n} = \sum_{-\infty}^{-1} 4^n z^{-n} + \sum_{n=0}^{\infty} 2^n z^{-n}$

Putting $-n = m$ in the first series, we get

$$Z[u(n)] = \sum_{1}^{\infty} 4^{-m} z^m + \sum_{n=0}^{\infty} 2^n z^{-n}$$

$$\left\{\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + ...\right\} + \left\{1 + \frac{2}{z} + \frac{2^2}{z^2} + ...\right\}$$

$$= \frac{z}{4}\left\{1 + \left(\frac{z}{4}\right) + \left(\frac{z}{4}\right)^2 + ...\right\} + \left\{1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + ...\right\} \qquad ...(i)$$

$$= \frac{z}{4} \cdot \frac{1}{1-(z/4)} + \frac{1}{1-(2/z)} = \frac{z}{4-z} + \frac{z}{z-2} = \frac{2z}{(4-z)(z-2)}$$

Now the two series in (i) being G.P. will be convergent if $| z/4 | < 1$ and $| 2/z | < 1$ *i.e.*, if $| z | < 4$ and $2 < | z |$. *i.e.* $2 < z < 4$.

Hence $Z[u(n)]$ is convergent if z lies between the annulus as shown shaded in Fig. 23.3. Hence ROC is $2 < z < 4$.

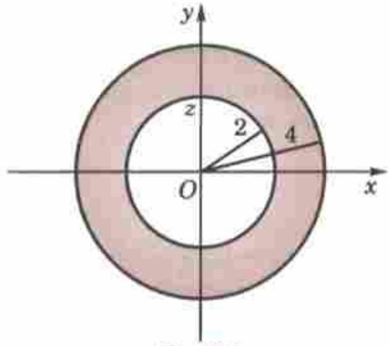


Fig. 23.2

(b) By definition, $Z[u(n)] = \sum_{-\infty}^{\infty} {}^nC_k z^{-n} = \sum_{n=k}^{\infty} {}^nC_k 2^n z^{-n}$

To find the sum of this series, we replace n by $k + r$

$$\therefore \quad Z[u(n)] = \sum_{r=0}^{\infty} {}^{k+r}C_k z^{-(k+r)} = z^{-k} \sum_{r=0}^{\infty} {}^{k+r}C_r z^{-r} \qquad [\because \ {}^kC_r = {}^kC_{k-r}]$$

$$= z^{-k}[1 + {}^{k+1}C_1 z^{-1} + {}^{k+1}C_2 z^{-2} + \ldots]$$

$$= z^{-k}(1 - 1/z)^{-(k+1)}$$

This series is convergence for $| 1/z | < 1$ *i.e.*, for $| z | > 1$.

Hence ROC is $| z | > 1$.

Example 23.13. *Find the Z-transform and the radius of convergence of*

(a) $f(n) = 2^n, n < 0$ (b) $f(n) = 5^n/n!, n \geq 0$. (*Mumbai, 2009*)

Solution. (a) Assuming that $f(n) = 0$ for $n \geq 0$ we have

$$Z[f(n)] = \sum_{-\infty}^{\infty} f(n)z^{-n} = \sum_{n=-\infty}^{-1} 2^n z^{-n} = \sum_{m=1}^{\infty} 2^{-m} z^m \quad \text{where} \quad m = -n$$

$$= \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots\infty = \frac{z}{2}\{1 + (z/2) + (z/2)^2 + \ldots \infty\}$$

$$= \frac{z}{2} \cdot \frac{1}{1-(z/2)} = \frac{z}{2-z}$$

This series being a G.P. is convergent if $| z/2 | < 1$ *i.e.*, $| z | < 2$.

Hence ROC is $| z | < 2$.

(b) By definition, $Z[u(n)] = \sum_{n=0}^{\infty} \frac{5^n}{n!} \cdot z^{-n} = \sum_{0}^{\infty} \frac{(5/z)^n}{n!} = 1 + \left(\frac{5}{z}\right) + \frac{1}{2!}\left(\frac{5}{z}\right)^2 + \frac{1}{3!}\left(\frac{5}{z}\right)^3 + \ldots \infty$

$$= e^{5/z}$$

The above series is convergent for all values of z.

Hence ROC is the entire z-plane.

PROBLEMS 23.2

Find the Z-transform and its ROC in each of the following sequences :

1. $u(n) = 4^n, n \geq 0$.
2. $u(n) = 2^n, n < 0$.
3. $u(n) = 4^n$, for $n < 0$ and $= 3^n$ for $n \geq 0$.
4. $u(n) = n5^n, n \geq 0$.
5. $u(n) = 2^n/n, n > 1$.
6. $u(n) = 3^n/n!, n \geq 0$.
7. $u(n) = e^{an}, n \geq 0$.

23.15 EVALUATION OF INVERSE Z-TRANSFORMS

We can obtain the inverse Z-transforms using any of the following three methods :

I. **Power series method.** This is the simplest of all the methods of finding the inverse Z-transform. If $U(z)$ is expressed as the ratio of two polynomials which cannot be factorized, we simply divide the numerator by the denominator and take the inverse Z-transform of each term in the quotient.

Example 23.14. *Find the inverse Z-transform of log (z / z + 1) by power series method.*

Solution. Putting $z = 1/t$, $U(z) = \log\left(\frac{1/y}{1/y+1}\right) = -\log(1+y) = -y + \frac{1}{2}y^2 - \frac{1}{3}y^3 + \ldots$

$$= -z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{3}z^{-3} + \ldots$$

Thus $$u_n = \begin{cases} 0 & \text{for } n = 0 \\ (-1)^n/n & \text{otherwise} \end{cases}.$$

Example 23.15. *Find the inverse Z-transform of $z/(z+1)^2$ by division method.*

Solution. $$U(z) = \frac{z}{z^2+2z+1} = z^{-1} - \frac{2+z^{-1}}{z^2+2z+1}, \text{ by actual division}$$

$$= z^{-1} - 2z^{-2} + \frac{3z^{-1}+2z^{-2}}{z^2+2z+1} = z^{-1} - 2z^{-2} + 3z^{-3} - \frac{4z^{-2}+3z^{-3}}{z^2+2z+1}$$

Continuing this process of division, we get an infinite series *i.e.*,

$$U(z) = \sum_{n=0}^{\infty} (-1)^{n-1} nz^{-n}$$

Thus $u_n = (-1)^{n-1} n.$

II. **Partial fractions method.** This method is similar to that of finding the inverse Laplace transforms using partial fractions. The method consists of decomposing $U(z)/z$ into partial fractions, multiplying the resulting expansion by z and then inverting the same.

Example 23.16. *Find the inverse Z-transforms of*

(*i*) $\frac{2z^2+3z}{(z+2)(z-4)}$ (*V.T.U., 2008 S ; S.V.T.U., 2007*) (*ii*) $\frac{z^3-20z}{(z-2)^3(z-4)}$. (*V.T.U., 2011*)

Solution. (*i*) We write $U(z) = \frac{2z^2+3z}{(z+2)(z-4)}$ as $\frac{U(z)}{z} = \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4}$

where $A = 1/6$ and $B = 11/6$

$$\therefore \quad U(z) = \frac{1}{6}\frac{z}{z+2} + \frac{11}{6}\frac{z}{z-4}$$

On inversion, we have

$$u_n = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n \qquad \text{[Using § 23.10 (9)]}$$

(*ii*) We write $$U(z) = \frac{z^3-20z}{(z-2)^3(z-4)}$$

as $$\frac{U(z)}{z} = \frac{z^2-20}{(z-2)^3(z-4)} = \frac{A+Bz+Cz^2}{(z-2)^3} + \frac{D}{z-4}$$

Readily we get $D = 1/2$. Multiplying throughout by $(z-2)^3(z-4)$, we get

$$z^2 - 20 = (A + Bz + Cz^2)(z-4) + D(z-2)^3.$$

Putting $z = 0, 1, -1$ successively and solving the resulting simultaneous equations, we get $A = 6$, $B = 0$, $C = 1/2$.

Thus $$U(z) = \frac{1}{2}\cdot\frac{12z+z^3}{(z-2)^3} - \frac{z}{z-4} = \frac{1}{2}\frac{z(z-2)^2+4z^2+8z}{(z-2)^3} - \frac{z}{z-4}$$

$$= \frac{1}{2}\left\{\frac{z}{z-2} + 2\frac{2z^2+4z}{(z-2)^3}\right\} - \frac{z}{z-4}$$

On inversion, we get $u_n = \frac{1}{2}(2^n + 2 \cdot n^2 2^n) - 4^n$ [Using § 23.10 (9) & (11)]

$= 2^{n-1} + n^2 2^n - 4^n.$

Example 23.17. *Find the inverse Z-transform of*

$2(z^2 - 5z + 6.5)/[(z-2)(z-3)^2]$, *for* $2 < |z| < 3$.

Solution. Splitting into partial fractions, we obtain

$$U(z) = \frac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{(z-3)^2} \qquad \text{where } A = B = C = 1$$

$$\therefore \quad U(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

$$= \frac{1}{2}\left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1 - \frac{z}{3}\right)^{-2} \qquad \text{so that } 2/z < 1 \text{ and } z/3 < 1$$

$$= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \ldots\right) - \frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \ldots\right) + \frac{1}{9}\left(1 + \frac{2z}{3} + \frac{3z^2}{9} + \frac{4z^3}{27} + \ldots\right)$$

where $2 < |z| < 3$.

$$= \left(\frac{1}{2} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \ldots\right) - \left(\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \frac{z^3}{3^4} + \ldots\right) + \left(\frac{1}{3^2} + \frac{2z}{3^3} + \frac{3z^2}{3^4} + \frac{4z^3}{3^5} + \ldots\right)$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} z^n + \sum_{n=0}^{\infty} (n+1)\left(\frac{1}{3}\right)^{n+2} z^n$$

On inversion, we get $u_n = 2^{n-1}, n \geq 1$ and $u_n = -(n+2)3^{n-2}, n \leq 0$.

III. **Inversion integral method.** *The inverse Z-transform of U(z) is given by the formula*

$$u_n = \frac{1}{2\pi i}\int_c U(z) z^{n-1}\, dz$$

= *sum of residues of* $U(z)\, z^{n-1}$ *at the poles of U(z) which are inside the contour C drawn according to the ROC given.*

The following examples will illustrate the application of this formula :

Example 23.18. *Using the inversion integral method, find the inverse Z-transform of*

$$\frac{z}{(z-1)(z-2)}$$

(V.T.U., 2010 S)

Solution. Let $U(z) = \dfrac{z}{(z-1)(z-2)}$. Its poles are at $z = 1$ and $z = 2$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i}\int_C U(z) z^{n-1}\, dz,$$

where C is a circle large enough to enclose both the poles of $U(z)$.

= sum of residues of $U(z)\, z^{n-1}$ at $z = 1$ and $z = 2$.

Now $$Res\, [U(z)\, z^{n-1}]_{z=1} = \underset{z \to 1}{\text{Lt}} \left\{(z-1) \cdot \frac{z^n}{(z-1)(z-2)}\right\} = -1$$

and $$Res\, [U(z)\, z^{n-1}]_{z=2} = \underset{z \to 2}{\text{Lt}} \left\{(z-2) \cdot \frac{z^n}{(z-1)(z-2)}\right\} = 2^n$$

Thus the required inverse Z-transform $u_n = 2^n - 1, n = 0, 1, 2, \ldots$

Example 23.19. *Find the inverse Z-transform of $2z/[(z-1)(z^2+1)]$.* (*Madras, 2000 S*)

Solution. Let $U(z) = \dfrac{2z}{(z-1)(z+i)(z-i)}$. It has three poles at $z = 1, z = \pm i$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i}\int_C U(z) \,.\, z^{n-1}\, dz \text{, where } C \text{ is a circle large enough to enclose the poles of } U(z).$$

$= \text{sum of residues of } U(z) \,.\, z^{n-1}$ at $z = 1, z = \pm i$.

Now $$Res\,[U(z)\, z^{n-1}]_{z=1} = \underset{z\to 1}{\text{Lt}} \left\{(z-1)\frac{2z^n}{(z-1)(z^2+1)}\right\} = 1$$

$$Res\,[U(z)\, z^{n-1}]_{z=i} = \underset{z\to i}{\text{Lt}} \left\{(z-i)\frac{2z^n}{(z-1)(z+i)(z-i)}\right\} = \frac{-(i)^n}{1+i}$$

$$Res\,[U(z)\, z^{n-1}]_{z=-i} = \underset{z\to -i}{\text{Lt}} \left\{(z+i)\frac{2z^n}{(z-1)(z+i)(z-i)}\right\} = \frac{(-i)^n}{i-1}$$

Hence $$u_n = 1 - \frac{(i)^n}{1+i} - \frac{(-i)^n}{1-i}.$$

PROBLEMS 23.3

Using convolution theorem, evaluate the inverse Z-transforms of the following :

1. $\dfrac{z^2}{(z-1)(z-3)}$.
2. $\left(\dfrac{z}{z-a}\right)^2$ (*Madras, 2003*)
3. $\left(\dfrac{z}{z-1}\right)^3$.
4. Show that (*a*) $\dfrac{1}{n!} * \dfrac{1}{n!} = \dfrac{2^n}{n!}$ (*b*) $Z^{-1}\left(\dfrac{z^2}{(z+a)(z+b)}\right) = \dfrac{(-1)}{b-a}(b^{n+1} - a^{n+1})$. (*Anna, 2009*)

Find the inverse Z-transforms of the following :

5. $\dfrac{4z}{z-a}$, $|z| > |a|$. (*Kottayam, 2005*)
6. $\dfrac{5z}{(2-z)(3z-1)}$. (*Madras, 1999*)
7. $\dfrac{z}{(z-1)^2}$.
8. $\dfrac{18z^2}{(2z-1)(4z+1)}$. (*S.V.T.U., 2009*)
9. $\dfrac{8z-z^3}{(4-z)^3}$.
10. $\dfrac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$. (*Anna, 2005 S*)
11. $\dfrac{4z^2-2z}{z^3-5z^2+8z-4}$. (*V.T.U., 2011 S*)
12. $\dfrac{z^3+3z}{(z-1)^2(z^2+1)}$. (*Anna, 2009*)
13. $\dfrac{(1-e^{at})z}{(z-1)(z-e^{-at})}$.
14. Obtain $Z^{-1}\{1/[(z-2)(z-3)]\}$ for (*i*) $|z| < 2$; (*ii*) $2 < |z| < 3$; (*iii*) $|z| > 3$. (*Marathwada, 2008*)
15. Evaluate $Z^{-1}\{(z-5)^{-3}\}$ for $|z| > 5$. (*Mumbai, 2009*)

Using inversion integral, find the inverse Z-transform of the following functions :

16. $\dfrac{z+3}{(z+1)(z-2)}$.
17. $\dfrac{(2z-1)z}{2(z-1)(z+0.5)}$.
18. $\dfrac{1}{z(z-1)(z+0.5)}$. (*S.V.T.U., 2008*)
19. $\dfrac{z^2+z}{(z-1)(z^2+1)}$. (*Madras, 2003*)
20. $\dfrac{2z(z^2-1)}{(z^2+1)^2}$.

23.16 (1) APPLICATION TO DIFFERENCE EQUATIONS

Just as the Laplace transforms method is quite effective for solving linear differential equations (§ 21.15), the Z-transforms are quite useful for solving linear difference equations.

The performance of discrete systems is expressed by suitable difference equations. Also Z-transform plays an important role in the analysis and representation of discrete-time systems. To determine the frequency response of such systems, the solution of difference equations is required for which Z-transform method proves useful.

(2) **Working procedure** *to solve a linear difference equation with constant coefficients by Z-transforms :*

1. *Take the Z-transform of both sides of the difference equations using the formulae of § 26.16 and the given conditions.*

2. *Transpose all terms without $U(z)$ to the right.*

3. *Divide by the coefficient of $U(z)$, getting $U(z)$ as a function of z.*

4. *Express this function in terms of the Z-transforms of known functions and take the inverse Z-transform of both sides. This gives u_n as a function of n which is the desired solution.*

Example 23.20. *Using the Z-transform, solve*

$$u_{n+2} + 4u_{n+1} + 3u_n = 3^n \text{ with } u_0 = 0,\ u_1 = 1.$$ *(U.P.T.U., 2003)*

Solution. If $\quad Z(u_n) = U(z)$, then $Z(u_{n+1}) = z[U(z) - u_0]$,

$$Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$$

Also $\quad Z(2^n) = z/(z-2)$

$\therefore$ taking the Z-transforms of both sides, we get

$$z^2[U(z) - u_0 - u_1 z^{-1}] + 4z\,[U(z) - u_0] + 3U(z) = z/(z-3)$$

Using the given conditions, it reduces to

$$U(z)\,(z^2 + 4z + 3) = z + z/(z-3)$$

$$\therefore \quad \frac{U(z)}{z} = \frac{1}{(z+1)(z+3)} + \frac{1}{(z-3)(z+1)(z+3)} = \frac{3}{8}\frac{1}{z+1} + \frac{1}{24}\frac{1}{z-3} - \frac{5}{12}\frac{1}{z+3},$$

on breaking into partial fractions.

$$U(z) = \frac{3}{8}\frac{z}{z+1} + \frac{1}{24}\frac{z}{z-3} - \frac{5}{12}\frac{z}{z+3}$$

On inversion, we obtain

$$u_n = \frac{3}{8}Z^{-1}\left(\frac{z}{z+1}\right) + \frac{1}{24}Z^{-1}\left(\frac{z}{z-3}\right) - \frac{5}{12}Z^{-1}\left(\frac{z}{z+3}\right) = \frac{3}{8}(-1)^n + \frac{1}{24}3^n - \frac{5}{12}(-3)^n.$$

Example 23.21. *Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0$, using Z-transforms.*
(V.T.U., 2011 ; Anna, 2009 ; S.V.T.U., 2009)

Solution. If $Z(y_n) = Y(z)$, then $Z(y_{n+1}) = z\,[(Y(z) - y_0]$, $Z(y_{n+2}) = z^2[Y(z) - y_0 - y_1 z^{-1}]$

Also $\quad Z(2^n) = z/(z-2)$.

Taking Z-transforms of both sides, we get

$$z^2[Y(z) - y_0 - y_1 z^{-1}] + 6z[Y(z) - y_0] + 9Y(z) = z/(z-2)$$

Since $\;y_0 = 0$, and $y_1 = 0$, we have $Y(z)\,(z^2 + 6z + 9) = z/(z-2)$

or $$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{1}{25}\left[\frac{1}{z-2} - \frac{1}{z+3} - \frac{5}{(z+3)^2}\right],$$ on splitting into partial fractions.

or $$Y(z) = \frac{1}{25}\left[\frac{z}{z-2} - \frac{z}{z+3} - 5\frac{z}{(z+3)^2}\right]$$

On taking inverse Z-transform of both sides, we obtain

$$y_n = \frac{1}{25}\left[Z^{-1}\left(\frac{z}{z-2}\right) - Z^{-1}\left(\frac{z}{z+3}\right) + \frac{5}{3}Z^{-1}\left(-\frac{3z}{(z+3)^2}\right)\right]$$

$$= \frac{1}{25}\{2^n - (-3)^n + \frac{5}{3}n(-3)^n\} \qquad \left[\because\ Z^{-1}\left\{\frac{az}{(z-a)^2}\right\} = na^n\right]$$

Example 23.22. *Find the response of the system $y_{n+2} - 5y_{n+1} + 6y_n = u_n$, with $y_0 = 0, y_1 = 1$ and $u_n = 1$ for $n = 0, 1, 2, 3, ...$ by Z-transform method.* (V.T.U., 2010)

Solution. Taking Z-transform of both sides of the given equation, we get

$$z^2(Y(z) - y_0 - y_1 z^{-1}) - 5z(Y(z) - y_0) + 6Y(z) = \frac{z}{z-1}$$

Substituting the values $y_0 = 0, y_1 = 1$, it reduces to

$$(z^2 - 5z + 6)\, Y(z) = \frac{z}{z-1} + z = \frac{z^2}{z-1}$$

or
$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-2)(z-3)}$$

$$= \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3} \qquad \text{where } A = \frac{1}{2}, B = -2, C = \frac{3}{2}$$

so that
$$Y(z) = \frac{1}{2}\frac{z}{z-1} - 2\frac{z}{z-2} + \frac{3}{2}\frac{z}{z-3}$$

On inversion, we obtain
$$y_n = \frac{1}{2} - 2(2)^n + \frac{3}{2}(3)^n$$

Obs. The initial values given in the problem automatically appear in the generated sequence.

Example 23.23. *Solve the difference equation $y_n + \frac{1}{4}y_{n-1} = u_n + \frac{1}{3}u_{n-1}$ where u_n is a unit step sequence.*

Solution. Taking Z-transform of both sides of the given equation, we get

$$Y(z) + \frac{1}{4}z^{-1}\, Y(z) = 1 + \frac{1}{3}z^{-1}$$

or
$$Y(z) = \left(1 + \frac{1}{3}z^{-1}\right)\Big/\left(1 + \frac{1}{4}z^{-1}\right) = \left(z + \frac{1}{3}\right)\Big/\left(z + \frac{1}{4}\right)$$

There being only one simple pole at $z = -1/4$, consider the contour $|z| > 1/4$.

$$\therefore \quad Res\,[Y(z)\, z^{n-1}]_{z=-1/4} = \underset{z \to -1/4}{\text{Lt}} \left\{\left(z + \frac{1}{4}\right) . \left(z + \frac{1}{3}\right) z^{n-1} \Big/ \left(z + \frac{1}{4}\right)\right\}$$

$$= \underset{z \to -1/4}{\text{Lt}} \left(z + \frac{1}{3}\right) z^{n-1} = \left(-\frac{1}{4} + \frac{1}{3}\right)\left(-\frac{1}{4}\right)^{n-1} = \frac{1}{12} . \left(-\frac{1}{4}\right)^{n-1}$$

Hence by inversion integral method, we have

$$y_n = \frac{1}{12}\left(-\frac{1}{4}\right)^{n-1}.$$

Example 23.24. *Using the Z-transform, solve $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.* (S.V.T.U., 2007)

Solution. Given equation is $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.
Taking the Z-transforms of both sides, we get

$$z^2[U(z) - u_0 - u_1 z^{-1}] - 2z[U(z) - u_0] + U(z) = 3 . \frac{z}{(z-1)^2} + 5 . \frac{z}{z-1}$$

or
$$U(z)\,(z^2 - 2z + 1) = \frac{5z^2 - 2z}{(z-1)^2} + u_0(z^2 - 2z) + u_1 z$$

or
$$U(z) = \frac{5z^2 - 2z}{(z-1)^4} + u_0\frac{z^2 - 2z}{(z-1)^2} + u_1\frac{z}{(z-1)^2}$$

On inversion, we obtain

$$u_n = Z^{-1}\left\{\frac{5z^2 - 2z}{(z-1)^4}\right\} + u_0 Z^{-1}\left\{\frac{z^2 - 2z}{(z-1)^2}\right\} + u_1 Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} \qquad ...(i)$$

Noting that $Z(1) = \dfrac{z}{z-1}$, $Z(n) = \dfrac{z}{(z-1)^2}$

$$Z(n^2) = \frac{z^2 + z}{(z-1)^3},\ Z(n^3) = \frac{z^3 + 4z^2 + z}{(z-1)^4}$$

We write $\dfrac{5z^2 - 2z}{(z-1)^4} \equiv A\,\dfrac{z^3 + 4z^2 + z}{(z-1)^4} + B\,\dfrac{z^2 + z}{(z-1)^3} + C\,\dfrac{z}{(z-1)^2} + D\,\dfrac{z}{z-1}$

Equating coefficients of like powers of z, we find

$$A = \frac{1}{2},\ B = 1,\ C = -\frac{3}{2},\ D = 0$$

$$\therefore \qquad Z^{-1}\left\{\frac{5z^2 - 2z}{(z-1)^4}\right\} = \frac{1}{2}n^3 + n^2 - \frac{3}{2}n = \frac{1}{2}n(n-1)(n+3)$$

Also $\quad Z^{-1}\left\{\dfrac{z^2 - 2z}{(z-1)^2}\right\} = Z^{-1}\left\{\dfrac{z}{z-1}\right\} - Z^{-1}\left\{\dfrac{z}{(z-1)^2}\right\} = 1 - n$

and $\quad Z^{-1}\left\{\dfrac{z}{(z-1)^2}\right\} = n.$

Substituting these values in (i) above, we get

$$u_n = \frac{1}{2}n(n-1)(n+3) + u_0(1-n) + u_1 n$$

$$= \frac{1}{2}n(n-1)(n+3) + c_0 + c_1 n. \qquad \text{where } c_0 = u_0,\ c_1 = u_1 - u_0$$

Example 23.25. *Using residue method, solve* $y_k + \dfrac{1}{9}y_{k-2} = \dfrac{1}{3^k}\cos\dfrac{k\pi}{2},\ k \geq 0.$

Solution. Taking Z-transform of both sides of the given equation, we get

$$Z\left\{y_k + \frac{1}{9}y_{k-2}\right\} = Z\left\{\frac{1}{3^k}\cos\frac{k\pi}{2}\right\}$$

or $\quad Y(z) + \dfrac{1}{9}z^{-2}\,Y(z) = \dfrac{z^2}{z^2 + 1/9}$ or $\left(1 + \dfrac{1}{9}z^{-2}\right)Y(z) = \dfrac{z^2}{z^2 + \dfrac{1}{9}}$

or $\quad Y(z) = \dfrac{z^2}{\left(1 + \dfrac{1}{9}z^{-2}\right)\left(z^2 + \dfrac{1}{9}\right)} = \dfrac{z^4}{\left(z^2 + \dfrac{1}{9}\right)^2}$

There are two poles of second order at $z = i/3$ and $z = -i/3$.

$$\therefore \qquad \text{Residue at } (z = i/3) = \left[\frac{d}{dz}\left\{\left(\frac{z-i}{3}\right)^2 \frac{z^{k-1}\,z^4}{(z^2 + 1/9)^2}\right\}\right]$$

$$= \left[\frac{d}{dz}\left\{\frac{z^{k+3}}{(z + i/3)^2}\right\}\right]_{z=i/3} = \left[\frac{(z + i/3)^2\,(k+3)z^{k+2} - z^{(k+3)}\,.\,2(z + i/3)}{(z + i/3)^4}\right]_{z=i/3}$$

$$= \left[\frac{(z + i/3)\,(k+3)z^{k+2} - 2z^{k+3}}{(z + i/3)^3}\right]_{z=i/3} = \left(\frac{3}{2i}\right)^3\left[(2k+6)\left(\frac{i}{3}\right)^{k+3} - 2\left(\frac{i}{3}\right)^{k+3}\right]$$

$$= \frac{1}{8}(2k+4)\left(\frac{i}{3}\right)^k = \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^k = \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2}\right) \quad ...(i)$$

Changing i to $-i$ in (i), we have

$$\text{Residue at } (z = -i/3) = \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{k\pi}{2} - i\sin\frac{k\pi}{2}\right) \quad ...(ii)$$

Adding (i) and (ii), we obtain $y_k = \frac{1}{2}(k+2)\left(\frac{1}{3}\right)^k \cos\frac{k\pi}{2}$.

PROBLEMS 23.4

Solve the following difference equations using Z-transforms (1 – 8) :

1. $6y_{k+2} - y_{k+1} - y_k = 0$, given that $y(0) = y(1) = 1$. *(Kottayam, 2005)*
2. $y(n+2) + 2y(n+1) + y(n) = 0$, given that $y(0) = y(1) = 0$. *(V.T.U., 2008 S)*
3. $y_{n+2} - 4y_n = 0$ given that $y_0 = 0, y_1 = 2$. *(U.P.T.U., 2008)*
4. $f(n) + 3f(n-1) - 4f(n-2) = 0, n \geq 2$, given that $f(0) = 3, f(1) = -2$. *(Madras, 2003 S)*
5. $y_{(n+3)} - 3y(n+1) + 2y(n) = 0$, given that $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$. *(Anna, 2005 S)*
6. $y_{n+2} - 5y_{n+1} + 6y_n = 36$, given that $y(0) = y(1) = 0$. *(Anna, 2009)*
7. $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$.
8. $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$.
9. $y_{n+1} + \frac{1}{4}y_n = \left(\frac{1}{4}\right)^n \quad (n \geq 0), y_0 = 0$.
10. $u_{x+2} + u_x = 5(2^x)$ given that $u_0 = 1, u_1 = 0$. *(Marathwada, 2008)*
11. $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0, y_1 = 1$. *(Madras, 2006)*
12. $u_{k+2} - 2u_{k+1} + u_k = 2^k$ with $y_0 = 2, y_1 = 1$.
13. $y_{n+2} - 6y_{n+1} + 8y_n = 2^n + 6n$.
14. $y_k + \frac{1}{25}y_{k-2} = \left(\frac{1}{5}\right)^k \cos\frac{k\pi}{2}, \quad (k \geq 0)$.
15. Find the response of the system given by $y_n + 3y_{(n-1)} = u_n$ where u_n is a unit step sequence and $y_{(-1)} = 1$.
16. Find the impulse response of a system described by $y_{(n+1)} + 2y_{(n)} = \delta_n$; $y_0 = 0$.

23.1 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 23.5

Choose the correct answer or fill up the blanks in each of the following problems :

1. $Z(1) = ...$
2. If u_n is defined for $n = 0, 1, 2, ...$ only, then $Z(u_n) = ...$
3. Z-transform of $n = ...$ *(Anna, 2009)*
4. $Z(na^n) = ...$
5. $Z(\sin n\theta) = ...$
6. Z-transform of $(1/n!)$ is
7. $Z(n^2) = ...$
8. Linear property of Z-transform states that...
9. $Z^{-1}\left(\frac{1}{z-2}\right) = ...$
10. $Z^{-1}\left\{\frac{z}{(z+1)^2}\right\} = ...$
11. Initial value theorem on Z-transform states that ....
12. Z-transform is linear. (True or False)
13. If $Z(u_n) = u(z)$, then $\underset{n\to\infty}{\text{Lt}}(u_n) = \underset{z\to\infty}{\text{Lt}}(z-1)\,u(z)$. (True or False)
14. Z-transform of the sequence $\{2^k\}, k \geq 0$ is $z/(z-2)$. (True or False)
15. Z-transform of $\{a^k/k!\}, k \geq 0 = e^{a/z}$. (True or False)
16. Z-transform of $\{^nC_r\}, (0 \leq r \leq n)$ is $(1+z)^n$. (True or False)
17. Z-transform of unit impulse sequence $\delta(n) = \begin{cases} 1, & n < 0 \\ 0, & n \geq 0 \end{cases}$, is $z/z - 1$. (True or False)
18. Z-transform of unit step sequence $u(n) = \begin{cases} 0, & n < 0, \\ 1, & n \geq 0, \end{cases}$ is 1. (True or False)