CHAPTER 9

Infinite Series

9.1 INTRODUCTION

Infinite series occur so frequently in all types of problems that the necessity of studying their convergence or divergence is very important. Unless a series employed in an investigation is convergent, it may lead to absurd conclusions. Hence it is essential that the students of engineering begin by acquiring an intelligent grasp of this subject.

9.2 SEQUENCES

(1) An ordered set of real numbers, $a_1, a_2, a_3, \ldots, a_n$ is called a *sequence* and is denoted by (a_n). If the number of terms is unlimited, then the sequence is said to be an *infinite sequence* and a_n is its *general term*.

For instance (*i*) $1, 3, 5, 7, \ldots, (2n-1), \ldots$, (*ii*) $1, 1/2, 1/3, \ldots, 1/n, \ldots$,

(*iii*) $1, -1, 1, -1, \ldots, (-1)^{n-1}, \ldots$ are infinite sequences.

(2) Limit. A sequence is said to tend to a limit l, if for every $\varepsilon > 0$, a value N of n can be found such that $|a_n - l| < \varepsilon$ for $n \geq N$.

We then write $\underset{n\to\infty}{\text{Lt}} (a_n) = l$ or simply $(a_n) \to l$ as $n \to \infty$.

(3) Convergence. *If a sequence (a_n) has a finite limit, it is called a* **convergent sequence.** If (a_n) is not convergent, it is said to be **divergent.**

In the above examples, (*ii*) is convergent, while (*i*) and (*iii*) are divergent.

(4) Bounded sequence. A sequence (a_n) is said to be bounded, if there exists a number k such that $a_n < k$ for every n.

(5) Monotonic sequence. The sequence (a_n) is said to increase steadily or to decrease steadily according as $a_{n+1} \geq a_n$ or $a_{n+1} \leq a_n$, for all values of n. Both increasing and decreasing sequences are called *monotonic sequences.*

A monotonic sequence always tends to a limit, finite or infinite. Thus, *a sequence which is monotonic and bounded is* **convergent.**

(6) Convergence, Divergence and Oscillation. If $\underset{n\to\infty}{\text{Lt}} (a_n) = l$ is finite and unique then the sequence is said to be *convergent*.

If $\underset{n \to \infty}{\text{Lt}} (a_n)$ is infinite ($\pm \infty$), the sequence is said to be *divergent*.

If $\underset{n \to \infty}{\text{Lt}} (a_n)$ is not unique, then (a_n) is said to be *oscillatory*.

Example 9.1. *Examine the following sequences for convergence :*

(*i*) $a_n = \dfrac{n^2 - 2n}{3n^2 + n}$. (*ii*) $a_n = 2^n$ (*iii*) $a_n = 3 + (-1)^n$.

Solution. (*i*) $\underset{n \to \infty}{\text{Lt}} \left(\dfrac{n^2 - 2n}{3n^2 + n} \right) = \underset{n \to \infty}{\text{Lt}} \dfrac{1 - 2/n}{3 + 1/n} = 1/3$ which is finite and unique. Hence the sequence (a_n) is convergent.

(*ii*) $\underset{n \to \infty}{\text{Lt}} (2^n) = \infty$. Hence the sequence (a_n) is divergent.

(*iii*) $\underset{n \to \infty}{\text{Lt}} [3 + (-1)^n] = 3 + 1 = 4$ when n is even

$= 3 - 1 = 2$, when n is odd

i.e., this sequence doesn't have a unique limit. Hence it oscillates.

PROBLEMS 9.1

Examine the convergence of the following sequences :

1. $a_n = \dfrac{3n - 1}{1 + 2n}$
2. $a_n = 1 + 2/n$
3. $a_n = [n + (-1)^n]^{-1}$
4. $a_n = \sin n$
5. $a_n = 1/2n$
6. $a_n = 1 + (-1)^n/n$
7. $\left(\dfrac{n}{n-1} \right)^2$
8. $a_n = 2n$.

9.3 SERIES

(1) Def. *If $u_1, u_2, u_3, \ldots, u_n, \ldots$ be an infinite sequence of real numbers, then*

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$$

is called an infinite series. An infinite series is denoted by Σu_n and the sum of its first n terms is denoted by s_n.

(2) Convergence, divergence and oscillation of a series.

Consider the infinite series $\Sigma u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$ and let the sum of the first n terms be $s_n = u_1 + u_2 + u_3 + \ldots + u_n$

Clearly, s_n is a function of n and as n increases indefinitely three possibilities arise :

(*i*) If s_n tends to a finite limit as $n \to \infty$, the series Σu_n is said to be *convergent*.

(*ii*) If s_n tends to $\pm \infty$ as $n \to \infty$, the series Σu_n is said to be *divergent*.

(*iii*) If s_n does not tend to a unique limit as $n \to \infty$, then the series Σu_n is said to be *oscillatory* or *non-convergent*.

Example 9.2. *Examine for convergence the series (i) $1 + 2 + 3 + \ldots + n + \ldots \infty$.*
(ii) $5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \ldots \infty$

Solution. (*i*) Here $s_n = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

$\therefore$ $\underset{n \to \infty}{\text{Lt}} s_n = \dfrac{1}{2} \underset{n \to \infty}{\text{Lt}} n(n+1) \to \infty$. Hence this series is *divergent*.

(*ii*) Here $s_n = 5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \ldots n$ terms

$= 0, 5$ or 1 according as the number of terms is $3m$, $3m + 1$, $3m + 2$.

Clearly in this case, s_n does not tend to a unique limit. Hence the series is *oscillatory*.

Examples 9.3. Geometric series. *Show that the series $1 + r + r^2 + r^3 + \ldots \infty$ (i) converges if $|r| < 1$, (ii) diverges if $r \geq 1$, and (iii) oscillates if $r \leq -1$.*

Solution. Let $s_n = 1 + r + r^2 + \ldots + r^{n-1}$

Case I. When $|r| < 1$, $\underset{n\to\infty}{\text{Lt}}\ r^n = 0$.

Also $$s_n = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r} \text{ so that } \underset{n\to\infty}{\text{Lt}}\ s_n = \frac{1}{1-r}$$

$\therefore$ the series is convergent.

Case II. (*i*) When $r > 1$, $\underset{n\to\infty}{\text{Lt}}\ r^n \to \infty$.

Also $$s_n = \frac{r^n-1}{r-1} = \frac{r^n}{r-1} - \frac{1}{r-1} \text{ so that } \underset{n\to\infty}{\text{Lt}}\ s_n \to \infty$$

$\therefore$ the series is divergent.

(*ii*) When $r = 1$, then $s_n = 1 + 1 + 1 \ldots\ldots + 1 = n$

and $\underset{n\to\infty}{\text{Lt}}\ s_n \to \infty$ $\quad\therefore$ The series is divergent.

Case III. (*i*) When $r = -1$, then the series becomes $1 - 1 + 1 - 1 + 1 - 1 \ldots$ which is an oscillatory series.

(*ii*) When $r < -1$, let $r = -\rho$ so that $\rho > 1$. Then $r^n = (-1)^n \rho^n$

and $$s_n = \frac{1-r^n}{1-r} = \frac{1-(-1)^n\rho^n}{1+\rho} \text{ as } \underset{n\to\infty}{\text{Lt}}\ \rho^n \to \infty.$$

$\therefore$ $\underset{n\to\infty}{\text{Lt}}\ s_n \to -\infty$ or $+\infty$ according as n is even or odd. Hence the series oscillates.

PROBLEMS 9.2

Examine the following series for convergence :

1. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \infty$.
2. $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} - \ldots \infty$.
3. $6 - 10 + 4 + 6 - 10 + 4 + 6 - 10 + 4 + \ldots \infty$.
4. $\frac{1}{1\cdot 2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots \infty$. *(V.T.U., 2006)*
5. A ball is dropped from a height h metres. Each time the ball hits the ground, it rebounds a distance r times the distance fallen where $0 < r < 1$. If $h = 3$ metres and $r = 2/3$, find the total distance travelled by the ball.

9.4 GENERAL PROPERTIES OF SERIES

The truth of the following properties is self-evident and these may be regarded as axioms :

1. *The convergence or divergence of an infinite series remains unaffected by the addition or removal of a finite number of its terms* ; for the sum of these terms being the finite quantity does not on addition or removal alter the nature of its sum.

2. *If a series in which all the terms are positive is convergent, the series remains convergent even when some or all of its terms are negative* ; for the sum is clearly the greatest when all the terms are positive.

3. *The convergence or divergence of an infinite series remains unaffected by multiplying each term by a finite number.*

9.5 SERIES OF POSITIVE TERMS

1. *An infinite series in which all the terms after some particular terms are positive, is a positive term series.* *e.g.*, $-7 - 5 - 2 + 2 + 7 + 13 + 20 + \ldots$ is a positive term series as all its terms after the third are positive.

2. *A series of positive terms either converges or diverges to $+\infty$* ; for the sum of its first n terms, omitting the negative terms, tends to either a finite limit or $+\infty$.

3. **Necessary condition for convergence.** *If a positive term series Σu_n is convergent, then* $\underset{n \to \infty}{Lt}\, u_n = 0$.

(P.T.U., 2009)

Let $s_n = u_1 + u_2 + u_3 + \ldots + u_n$. Since Σu_n is given to be convergent.

$\therefore$ $\underset{n \to \infty}{\text{Lt}}\, s_n =$ a finite quantity k (say). Also $\underset{n \to \infty}{\text{Lt}}\, s_{n-1} = k$

But $u_n = s_n - s_{n-1}$ $\therefore$ $\underset{n \to \infty}{\text{Lt}}\, u_n = \underset{n \to \infty}{\text{Lt}}\, (s_n - s_{n-1}) = 0.$

Hence the result.

Obs. 1. *It is important to note that the converse of this result is not true.*

Consider, for instance, the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots \infty$

Since the term go on descending,

$$\therefore \quad s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} \quad \text{i.e.,} \quad \sqrt{n}$$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}}\, s_n = \underset{n \to \infty}{\text{Lt}}\, \sqrt{n} \to \infty$$

Thus the series is divergent even though $\underset{n \to \infty}{\text{Lt}}\, u_n = \underset{n \to \infty}{\text{Lt}}\, \frac{1}{\sqrt{n}} = 0$

Hence $\underset{n \to \infty}{Lt}\, u_n = 0$ *is a necessary but not sufficient condition for convergence of* Σn_n.

Obs. The above result leads to a simple test for divergence :

If $\underset{n \to \infty}{Lt}\, u_n \neq 0$, *the series* Σu_n *must be divergent.*

9.6 COMPARISON TESTS

I. If two positive term series Σu_n and Σv_n be such that

(i) Σv_n converges, *(ii) $u_n \le v_n$ for all values of n, then Σu_n also converges.*

Proof. Since Σv_n is convergent,

$$\therefore \quad \underset{n \to \infty}{\text{Lt}}\, (v_1 + v_2 + v_3 + \ldots + v_n) = \text{a finite quantity } k \text{ (say)}$$

Also since $u_1 \le v_1, u_2 \le v_2, \ldots, u_n \le v_n$

$\therefore$ Adding, $u_1 + u_2 + \ldots + u_n \le v + v_2 + \ldots + v_n$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}}\, (u_1 + u_2 + \ldots + u_n) \le \underset{n \to \infty}{\text{Lt}}\, (v_1 + v_2 + \ldots + v_n) = k.$$

Hence the series Σu_n also converges.

Obs. If, however, the relation $u_n \le v_n$ holds for values of n greater than a fixed number m, then the first m terms of both the series can be ignored without affecting their convergence or divergence.

II. If two positive term series Σu_n and Σv_n be such that :

(i) Σv_n diverges, *(ii) $u_n \ge v_n$ for all values of n, then Σu_n also diverges.*

Its proof is similar to that of Test *I*.

III. **Limit form**

If two positive term series Σu_n and Σv_n be such that

$$\underset{n \to \infty}{Lt}\, \frac{u_n}{v_n} = \textit{finite quantity } (\neq 0), \textit{ then } \Sigma u_n \textit{ and } \Sigma v_n \textit{ converge or diverge together.}$$

Proof. Since $\underset{n \to \infty}{\text{Lt}}\, \frac{u_n}{v_n} = l$, a finite number $(\neq 0)$

By definition of a limit, there exists a positive number ε, however small, such that

$$\left| \frac{u_n}{v_n} - l \right| < \varepsilon \qquad \text{for } n \ge m$$

or $$-\varepsilon < \frac{u_n}{v_n} - l < \varepsilon \qquad \text{for } n \ge m$$

or $$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \qquad \text{for } n \ge m$$

Omitting the first m terms of both the series, we have

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \qquad \text{for all } n \qquad ...(1)$$

Case I. When Σv_n is convergent, then

$$\underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + ... + v_n) = k, \text{ a finite number} \qquad ...(2)$$

Also from (1), $\frac{u_n}{v_n} < l + \varepsilon$, *i.e.*, $u_n < (l + \varepsilon)v_n$ for all n.

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} (u_1 + u_2 + ... + u_n) < (l + \varepsilon) \underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + ... + v_n) = (l + \varepsilon)k \qquad \text{[By (2)]}$$

Hence Σu_n is also convergent.

Case II. When Σv_n is divergent, then

$$\underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + ... + v_n) \to \infty \qquad ...(3)$$

Also from (1) $l - \varepsilon < \frac{u_n}{v_n}$ or $u_n > (l - \varepsilon)v_n$ for all n

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} (u_1 + u_2 + ... + u_n) > (l - \varepsilon) \underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + ... + v_n) \to \infty \qquad \text{[By (3)]}$$

Hence Σu_n is also divergent.

9.7 INTEGRAL TEST

A positive term series $f(1) + f(2) + ... + f(n) + ...$, where $f(n)$ decreases as n increases, converges or diverges according as the integral

$$\int_1^\infty f(x)\, dx \qquad \textit{...(1) is finite or infinite.}$$

The area under the curve $y = f(x)$, between any two ordinates lies between the set of inscribed and escribed rectangles formed by ordinates at $x = 1, 2, 3, ...$ as in Fig. 9.1. Then

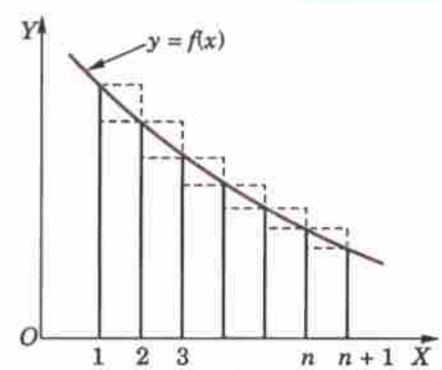

Fig. 9.1

$$f(1) + f(2) + ... + f(n) \ge \int_1^{n+1} f(x)\, dx \ge f(2) + f(3) + ... + f(n+1)$$

or $$s_n \ge \int_1^{n+1} f(x)\, dx \ge s_{n+1} - f(1)$$

Taking limits as $n \to \infty$, we find from the second inequality that $\text{Lt}\, s_{n+1} \le \int_1^\infty f(x)\, dx + f(1)$.

Hence if integral (1) is finite, so is $\text{Lt}\, s_{n+1}$. Similarly, from the first inequality, we see that if the integral (1) is infinite, so is $\text{Lt}\, s_n$. But the given series either converges or diverges to $+\infty$, *i.e.*, $\text{Lt}\, s_n$ is either finite or infinite as $n \to \infty$.

Hence the result follows.

Example 9.4. Test for Comparison. *Show that the p-series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + ... \infty$$

(i) converges for $p > 1$ (ii) diverges for $p \le 1$. *(P.T.U., 2009 ; V.T.U., 2006 ; Rohtak, 2003)*

Solution. By the above test, this series will converge or diverge according as $\int_1^\infty \frac{dx}{x^p}$ is finite or infinite.

If $p \neq 1$, $$\int_1^\infty \frac{dx}{x^p} = \underset{m \to \infty}{\text{Lt}} \int_1^m \frac{dx}{x^p} = \underset{m \to \infty}{\text{Lt}} \left(\frac{m^{1-p} - 1}{1-p} \right)$$

$$= \frac{1}{p-1}, \text{ i.e. finite} \quad \text{for } p > 1$$

$$\to \infty \quad \text{for } p < 1$$

If $p = 1$, $\int_1^\infty \frac{dx}{x} = \int_1^\infty \log x \to \infty$, this proves the result.

Obs. Application of comparison tests. Of all the above tests the 'limit form' is the most useful. To apply this comparison test to a given series Σu_n, the auxiliary series Σv_n must be so chosen that Lt (u_n/v_n) is non-zero and finite. To do this, we take v_n equal to that term of u_n which is of the highest degree in $1/n$ and the convergence or divergence of v_n is known with the help of the above series.

Example 9.5. *Test for convergence the series*

(*i*) $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots \infty$ (*P.T.U., 2009*)

(*ii*) $\frac{1}{4.7.10} + \frac{4}{7.10.13} + \frac{9}{10.13.16} + \ldots \infty$ (*V.T.U., 2010*)

(*iii*) $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \ldots \infty$

Solution. (*i*) We have $$u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{1}{n^2} \frac{2 - 1/n}{(1+1/n)(1+2/n)}$$

Take $v_n = 1/n^2$; then

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \frac{2 - 1/n}{(1+1/n)(1+2/n)} = \frac{2-0}{(1+0)(1+0)}$$

$= 2$, which is finite and non-zero

$\therefore$ both Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma 1/n^2$ is known to be convergent.

Hence Σu_n is also convergent.

(*ii*) Here $$u_n = \frac{n^2}{(3n+1)(3n+4)(3n+7)} = \frac{1}{n\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)}$$

Taking $v_n = \frac{1}{n}$, we find that

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \frac{1}{\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)} = \frac{1}{27} \neq 0$$

Now since Σv_n is divergent, therefore Σu_n is also divergent.

(*iii*) Here $$u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{1}{n+1} \cdot \left(\frac{n}{n+1}\right)^n, \text{ ignoring the first term.}$$

Taking $v_n = 1/n$, we have

$$\underset{n \to \infty}{\text{Lt}} \left(\frac{u_n}{v_n}\right) = \underset{n \to \infty}{\text{Lt}} \frac{n}{n+1} \cdot \underset{n \to \infty}{\text{Lt}} \left(\frac{n}{n+1}\right)^n$$

$$= \underset{n \to \infty}{\text{Lt}} \left(\frac{1}{1+1/n}\right) \cdot \underset{n \to \infty}{\text{Lt}} \frac{1}{(1+1/n)^n} = 1 \cdot \frac{1}{e} \neq 0$$

Now since Σv_n is divergent, therefore Σu_n is also divergent.

Example 9.6. *Test the convergence of the series :*

(i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{(n+1)}}$ *(V.T.U., 2008)* *(ii)* $\sum_{n=1}^{\infty} \frac{1}{x^n+x^{-n}}$ *(iii)* $\sum_{n=1}^{\infty} \sqrt{\frac{3^n-1}{2^n+1}}$ *(V.T.U., 2000 S)*

Solution. (*i*) We have $u_n = \frac{\sqrt{(n+1)}-\sqrt{n}}{[\sqrt{(n+1)}+\sqrt{n}][\sqrt{(n+1)}-\sqrt{n}]} = \sqrt{(n+1)}-\sqrt{n}$

$= \sqrt{n}\,[(1+1/n)^{1/2}-1]$ **(Expanding by Binomial Theorem)**

$$= \sqrt{n}\left\{\left(1+\frac{1}{2n}-\frac{1}{8n^2}+\ldots\right)-1\right\} = \sqrt{n}\left(\frac{1}{2n}-\frac{1}{8n^2}+\ldots\right) = \frac{1}{\sqrt{n}}\left(\frac{1}{2}-\frac{1}{8n}+\ldots\right)$$

Taking $v_n = 1/\sqrt{n}$, we have

$$\underset{u\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{2}-\frac{1}{8n}+\ldots\right) = \frac{1}{2}, \text{ which is finite and non-zero.}$$

$\therefore$ both Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma 1/\sqrt{n}$ is known to be divergent. Hence Σu_n is also divergent.

(*ii*) *When $x < 1$*, comparing the given series Σu_n with $\Sigma v_n = \Sigma x^n$,

we get $$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{x^n+x^{-n}}\cdot\frac{1}{x^n}\right) = \underset{n\to\infty}{\text{Lt}} \frac{1}{x^{2n}+1} = 1 \qquad [\because\ x^{2n}\to 0 \text{ as } n\to\infty]$$

But Σv_n is convergent, so Σu_n is also convergent.

When $x > 1$, comparing Σu_n with $\Sigma w_n = \Sigma x^{-n}$, we get

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{w_n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{x^n+x^{-n}}\cdot x^n\right) = \underset{n\to\infty}{\text{Lt}} \frac{1}{1+x^{-2n}} = 1. \qquad [\because\ x^{-2n}\to 0 \text{ as } n\to\infty]$$

But Σw_n is convergent, so Σu_n is also convergent.

When $x = 1$, $\Sigma u_n = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots\infty$ which is divergent.

Hence, Σu_n converges for $x < 1$ and $x > 1$ but diverges for $x = 1$.

(*iii*) Here $$u_n = \sqrt{\frac{3^n-1}{2^n+1}} = \left(\frac{3}{2}\right)^{n/2} \sqrt{\left(\frac{1-1/3^n}{1+1/2^n}\right)}$$

Taking $$v_n = \left(\frac{3}{2}\right)^{n/2}, \text{ we get}$$

$$\underset{n\to\infty}{\text{Lt}} \left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{\text{Lt}} \sqrt{\left(\frac{1-1/3^n}{1+1/2^n}\right)} = 1 \neq 0$$

Also since $\Sigma v_n = r^n$ where $r = \sqrt{3/2}$ is a geometric series having $r > 1$, is divergent.

$\therefore$ Σu_n is also divergent.

Example 9.7. *Determine the nature of the series :*

(i) $\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\ldots\infty$ *(ii)* $\sum \frac{1}{n}\sin\frac{1}{n}$

(iii) $\sum_{1}^{\infty} \frac{(\log n)^2}{n^{3/2}}$ *(iv)* $\sum_{2}^{\infty} \frac{1}{n(\log n)\,p}\ (p > 0)$ *(P.T.U., 2010)*

Solution. (*i*) We have $u_n = \frac{\sqrt{(n+1)}-1}{(n+2)^3-1} = \frac{\sqrt{n}\,[(1+1/n)-1/\sqrt{n}]}{n^3[(1+2/n)^3-1/n^3]}$

Taking $v_n = \frac{1}{n^{5/2}}$, we find that

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \frac{[\sqrt{(1+1/n)} - 1/\sqrt{n}]}{[(1+2/n)^3 - 1/n^3]} = 1 \neq 0$$

Since Σv_n is convergent, therefore Σu_n is also convergent.

(*ii*) Here $$u_n = \frac{1}{n} \sin \frac{1}{n} = \frac{1}{n}\left[\frac{1}{n} - \frac{1}{3!\,n^3} + \frac{1}{5!\,n^5} - \dots\right] = \frac{1}{n^2}\left[1 - \frac{1}{3!\,n^2} + \frac{1}{5!\,n^4} - \dots\right]$$

Taking $v_n = \frac{1}{n^2}$, we have

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \left[1 - \frac{1}{3!\,n^2} + \frac{1}{5!\,n^4} \dots\right] = 1 \neq 0$$

Since Σv_n is convergent, therefore Σu_n is also convergent.

(*iii*) We have $\underset{n \to \infty}{\text{Lt}} \frac{(\log n)^2}{n^{1/4}} = 0$, *i.e.*, $\frac{(\log n)^2}{n^{1/4}} < 1$ or $(\log n)^2 < n^{1/4}$

$$\therefore \quad u_n = \frac{(\log n)^2}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$$

Since $\Sigma 1/n^{5/4}$ converges by p-series. $(\because \quad p = 5/4 > 1)$

Hence by comparison test, Σu_n also converges.

(*iv*) Let $$f(n) = \frac{1}{n(\log n)^p} \text{ so that } f(x) = \frac{(\log x)^{-p}}{x}$$

$$\therefore \quad f'(x) = \frac{-p}{x}(\log x)^{-p-1} \cdot \frac{1}{x} + (\log x)^{-p} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}\left\{\frac{p}{(\log x)^{p+1}} + \frac{1}{(\log x)^p}\right\} < 0$$

i.e., $f(x)$ is a decreasing function.

Also $$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{dx}{x(\log x)^p} = \left|\frac{(\log x)^{-p+1}}{-p+1}\right|_2^\infty$$

If $p > 1$, then $p - 1 = k(\text{say}) > 0$

$$\therefore \quad \int_2^\infty f(x)\,dx = \left|\frac{(\log x)^{-k}}{-k}\right|_2^\infty = \frac{1}{k}\,[0 + (\log 2)^{-k}] \text{ which is finite}$$

Thus by integral test, *the given series converges for p > 1.*

If $p < 1$, then $1 - p > 0$ and $(\log x)^{1-p} \to \infty$ as $x \to \infty$.

$$\therefore \quad \int_2^\infty f(x)\,dx \to \infty.$$

Thus *the given series diverges for p < 1.*

If $p = 1$, then $$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{dx}{x \log x} = |\log(\log x)|_2^\infty \to \infty$$

Thus *the given series diverges for* $p = 1$.

PROBLEMS 9.3

Test the following series for convergence :

1. $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} - \dots \infty$ (*J.N.T.U., 2000*)
2. $\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \dots \infty$
3. $\frac{1}{1.2} + \frac{2}{3.4} + \frac{3}{5.6} + \dots \infty$ (*Cochin, 2001*)
4. $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \dots \infty$ (*P.T.U., 2009*)

5. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\ldots\infty$

6. $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots\infty$

7. $\frac{1}{1^2}+\frac{1+2}{1^2+2^2}+\frac{1+2+3}{1^2+2^2+3^2}+\ldots\infty$

8. $\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\ldots\infty$ *(V.T.U., 2009 S)*

9. $\frac{3}{1}+\frac{4}{8}+\frac{5}{27}+\frac{6}{64}+\ldots\infty$

10. $\sum\frac{\sqrt{n}}{n^2+1}$ *(Osmania, 2000 S)*

11. $\sum_{n=0}^{\infty}\frac{2n^3+5}{4n^5+1}$

12. $\sum\frac{(n+1)(n+2)}{n^2\sqrt{n}}$ *(J.N.T.U., 2006 S)*

13. $\sum_{n=1}^{\infty}[\sqrt{(n^2+1)}-n]$ *(V.T.U., 2010 ; P.T.U., 2009)*

14. $\sum[\sqrt[3]{(n^3+1)}-n]$ *(P.T.U., 2007 ; Rohtak 2003)*

15. $\sum[\sqrt{(n^4+1)}-\sqrt{(n^4-1)}]$

16. $\sum\frac{1}{\sqrt{n}}\sin\frac{1}{n}$

17. $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n\log n}}$

18. $\sum_{n=1}^{\infty}\frac{\sqrt{(n+1)}-1}{(n+2)^3-1}$ *(J.N.T.U., 2003)*

9.8 COMPARISON OF RATIOS

If Σu_n and Σv_n be two positive term series, then Σu_n converges if (i) Σv_n converges, and (ii) from and after some particular term,

$$\frac{u_{n+1}}{u_n}<\frac{u_{n+1}}{v_n}$$

Let the two series beginning from the particular term be $u_1+u_2+u_3+\ldots$ and $v_1+v_2+v_3+\ldots$

If $\frac{u_2}{u_1}<\frac{v_2}{v_1},\frac{u_3}{u_2}<\frac{v_3}{v_2},\ldots$

then $$u_1+u_2+u_3+\ldots=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\ldots\right)$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_2}{u_1}\cdot\frac{u_3}{u_2}+\ldots\right)<u_1\left(1+\frac{v_2}{v_1}+\frac{v_2}{v_1}\cdot\frac{v_3}{v_2}+\ldots\right)<\frac{u_1}{v_1}(v_1+v_2+v_3+\ldots).$$

Hence, if Σv_n converges, Σu_n also converges.

Obs. A more convenient form of the above test to apply is as follows :

Σu_n converges if (i) Σv_n converges and (ii) from and after a particular term $\frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}}$.

Similarly, Σu_n diverges, if (i) Σv_n diverges and (ii) from and after a particular term $\frac{u_n}{u_{n+1}}<\frac{v_n}{v_{n+1}}$.

9.9 D'ALEMBERT'S RATIO TEST*

In a positive term series Σu_n, if

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\lambda,\text{ then the series converges for }\lambda<1\text{ and diverges for }\lambda>1.$$

Case I. *When* $\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\lambda<1.$

*Called after the French mathematician *Jean le-Rond d'Alembert* (1717–1783), who also made important contributions to mechanics.

By definition of a limit, we can find a positive number $r\ (< 1)$ such that $\frac{u_{n+1}}{u_n} < r$ for all $n > m$

Leaving out the first m terms, let the series be $u_1 + u_2 + u_3 + \ldots$

so that $\frac{u_2}{u_1} < r, \frac{u_3}{u_2} < r, \frac{u_4}{u_3} < r, \ldots$ and so on. Then $u_1 + u_2 + u_3 + \ldots \infty$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots \infty\right) < u_1(1 + r + r^2 + r^3 + \ldots \infty)$$

$$= \frac{u_1}{1-r}, \text{ which is finite quantity. Hence } \Sigma u_n \text{ is convergent.} \qquad [\because\ r < 1]$$

Case II. *When* $\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \lambda > 1$

By definition of limit, we can find m, such that $\frac{u_{n+1}}{u_n} \geq 1$ for all $n \geq m$.

Leaving out the first m terms, let the series be

$$u_1 + u_2 + u_3 + \ldots \text{ so that } \frac{u_2}{u_1} \geq 1,\quad \frac{u_3}{u_2} \geq 1,\quad \frac{u_4}{u_3} \geq 1 \text{ and so on.}$$

$$\therefore \quad u_1 + u_2 + u_3 + u_4 + \ldots + u_n = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots\right)$$

$$\geq u_1(1 + 1 + 1 + \ldots \text{ to } n \text{ terms}) = nu_1$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ (u_1 + u_2 + \ldots + u_n)$$

$$\geq \underset{n\to\infty}{\text{Lt}}\ (nu_1), \text{ which tends to infinity. Hence } \Sigma u_n \text{ is divergent.}$$

Obs. 1. *Ratio test fails when $\lambda = 1$. Consider, for instance, the series $\Sigma u_n = \Sigma 1/n^p$.*

Here $$\lambda = \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}}\left[\frac{1}{(n+1)^p}\cdot\frac{n^p}{1}\right] = \underset{n\to\infty}{\text{Lt}} \frac{1}{(1+1/n)^p} = 1.$$

Then for all values of p, $\lambda = 1$; whereas $\Sigma 1/n^p$ converges for $p > 1$ and diverges for $p < 1$.

Hence $\lambda = 1$ both for convergence and divergence of Σu_n, which is absurd.

Obs. 2. *It is important to note that this test makes no reference to the magnitude of u_{n+1}/u_n but concerns only with the limit of this ratio.*

For instance in the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$, the ratio $\frac{u_{n+1}}{u_n} = \frac{n}{n+1} < 1$ for all finite values of n, but tends to unity as $n \to \infty$. Hence the Ratio test fails although this series is divergent.

Practical form of Ratio test. Taking reciprocals, the ratio test can be stated as follows :

In the positive term series Σu_n, if $\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = k$, then the series converges for $k > 1$ and diverges for $k < 1$ but fails for $k = 1$.

Example. 9.8. *Test for convergence the series*

(*i*) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots \infty.$ *(P.T.U., 2005 ; V.T.U., 2003 ; I.S.M., 2001)*

(*ii*) $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \ldots + \frac{2^n - 2}{2^n + 1}x^{n-1} + \ldots (x > 0).$ *(P.T.U., 2009 ; V.T.U., 2004)*

Solution. (*i*) We have $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$ and $u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{(n+1)}}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{x^{2n-2}}{(n+1)\sqrt{n}} \cdot \frac{(n+2)\sqrt{(n+1)}}{x^{2n}}$$

$$= \underset{n\to\infty}{\text{Lt}} \left[\frac{n+2}{n+1}\left(\frac{n+1}{n}\right)^{1/2}\right] x^{-2} = \underset{n\to\infty}{\text{Lt}} \left[\frac{1+2/n}{1+1/n} \cdot \sqrt{(1+1/n)}\right] x^{-2} = x^{-2}.$$

Hence Σu_n converges if $x^{-2} > 1$, *i.e.*, for $x^2 < 1$ and diverges for $x^2 > 1$.

If $x^2 = 1$, then, $u_n = \dfrac{1}{(n+1)\sqrt{n}} = \dfrac{1}{n^{3/2}} \cdot \dfrac{1}{1+1/n}$

Taking $v_n = \dfrac{1}{n^{3/2}}$, we get $\underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \dfrac{1}{1+1/n} = 1$, a finite quantity.

$\therefore$ Both Σu_n and Σv_n converge or diverge together. But $\Sigma v_n = \sum \dfrac{1}{n^{3/2}}$ is a convergent series.

$\therefore$ Σu_n is also convergent. Hence the given series converges if $x^2 \le 1$ and diverges if $x^2 > 1$.

(*ii*) Here $$\frac{u_n}{u_{n+1}} = \frac{2^n - 2}{2^n + 1} x^{n-1} \cdot \frac{2^{n+1}+1}{2^{n+1}-2} \frac{1}{x^n} = \frac{1-\frac{2}{2^n}}{1+\frac{1}{2^n}} \cdot \frac{2+\frac{1}{2^n}}{2-\frac{2}{2^n}} \frac{1}{x}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1-0}{1+0} \cdot \frac{2+0}{2-0} \frac{1}{x} = \frac{1}{x}$$

Thus by Ratio test, Σu_n converges for $x^{-1} > 1$ *i.e.*, for $x < 1$ diverges for $x > 1$. But it fails for $x = 1$.

When $x = 1$, $$\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}} \frac{2^n - 2}{2^n + 1} = \underset{n\to\infty}{\text{Lt}} \frac{1-\frac{2}{2^n}}{1+\frac{1}{2^n}} = 1 \neq 0$$

$\therefore$ Σu_n diverges for $x = 1$. Hence the given series converges for $x < 1$ and diverges for $x \ge 1$.

Example 9.9. *Discuss the convergence of the series*

(*i*) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n^n)^2}$ (*P.T.U., 2010*) (*ii*) $1 + \dfrac{2!}{2^2} + \dfrac{3!}{3^3} + \dfrac{4!}{4^4} + \ldots \infty$ (*V.T.U., 2008 S*)

Solution. (*i*) We have $u_n = \dfrac{n!}{(n^n)^2}$ and $u_{n+1} = \dfrac{(n+1)!}{[(n+1)^{n+1}]^2}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{n!}{n^{2n}} \times \frac{(n+1)^{2(n+1)}}{(n+1)!}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{(n+1)^{2n+1}}{n^{2n}} = \underset{n\to\infty}{\text{Lt}} \left(\frac{n+1}{n}\right)^{2n} . (n+1)$$

$$= \underset{n\to\infty}{\text{Lt}} \left[\left(1+\frac{1}{n}\right)^n\right]^2 . (n+1) = e . \underset{n\to\infty}{\text{Lt}} (n+1) \to \infty$$

Hence the given series is convergent.

(*ii*) Given series is $\Sigma u_n = \displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$. Here $\dfrac{u_n}{u_{n+1}} = \dfrac{n!}{n^n} \cdot \dfrac{(n+1)^{n+1}}{(n+1)!} = \dfrac{(n+1)^n}{n^n} = \left(1+\dfrac{1}{n}\right)^n$

$\therefore$ $\underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \left(1+\dfrac{1}{n}\right)^n = e$, which is > 1. Hence the given series is convergent.

Example 9.10. *Examine the convergence of the series :*

(*i*) $\frac{x}{1+x}+\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}+\ldots\infty$

(*ii*) $1+\frac{a+1}{b+1}+\frac{(a+1)(2a+1)}{(b+1)(2b+1)}+\frac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)}+\ldots\infty$

Solution. (*i*) Here $u_n=\frac{x^n}{1+x^n}$ and $u_{n+1}=\frac{x^{n+1}}{1+x^{n+1}}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\left(\frac{x^n}{x^{n+1}}\cdot\frac{1+x^{n+1}}{1+x^n}\right)=\underset{n\to\infty}{\text{Lt}}\left(\frac{1+x^{n+1}}{x+x^{n+1}}\right)$$

$$=\frac{1}{x}, \text{ if } x<1. \qquad [\because \; x^{n+1}\to 0 \text{ as } n\to\infty]$$

Also
$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\left(\frac{1+1/x^{n+1}}{1+x/x^{n+1}}\right)=1 \text{ if } x>1.$$

$\therefore$ by *Ratio test*, Σu_n converges for $x<1$ and fails for $x\geq 1$.

When $x=1$, $\Sigma u_n=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots+\infty$, which is divergent.

Hence the given series converges for $x<1$ and diverges for $x\geq 1$.

(*ii*) Neglecting the first term, we have

$$u_{n+1}=u_n\cdot\frac{n_{a+1}}{n_{b+1}}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\frac{n_{b+1}}{n_{a+1}}=\underset{n\to\infty}{\text{Lt}}\frac{b+1/n}{a+1/n}=\frac{b}{a}.$$

By *Ratio test*, Σu_n converges for $b/a>1$ or $a<b$, and diverges for $a>b$.

When $a=b$, the series becomes $1+1+1+\ldots\infty$, which is divergent.

Hence the given series converges for $0<a<b$ and diverges for $0<b\leq a$.

PROBLEMS 9.4

Test for convergence the following series :

1. $x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\infty.$
2. $\sqrt{\frac{1}{2}}x+\sqrt{\frac{2}{3}}x^2+\sqrt{\frac{3}{4}}x^3+\ldots\infty$
3. $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\ldots+\frac{x^n}{n^2+1}\ldots\infty$
4. $\sum_{n=2}^{\infty}\frac{x^n}{n(n-1)(n-2)}$ (*J.N.T.U., 2006*)
5. $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\ldots\infty$ (*Kurukshetra, 2005*)
6. $\sum_{n=1}^{\infty}\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right)$ (*Rohtak, 2005*)
7. $\sum_{n=1}^{\infty}\frac{n!\,3^n}{n^n}$ (*Kerala, 2005*)
8. $\sum_{n=1}^{\infty}\frac{n^3+a}{2^n+a}$
9. $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{\sqrt{(n^2+1)}}x^n$ (*P.T.U., 2006*)
10. $\sum_{1}^{\infty}\frac{n^3-n+1}{n!}$ (*Madras, 2000*)
11. $\frac{2}{3.4}+\frac{2.4}{3.5.6}+\frac{2.4.6}{3.5.7.8}+\ldots$ (*V.T.U., 2010*)
12. $\left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\ldots$

13. $1+\frac{1^2 \cdot 2^2}{1 \cdot 3 \cdot 5}+\frac{1^2 \cdot 2^2 \cdot 3^2}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}+\ldots \infty$ (Delhi, 2002)

14. $\frac{4}{18}+\frac{4 \cdot 12}{18 \cdot 27}+\frac{4 \cdot 12 \cdot 20}{18 \cdot 27 \cdot 36}+\ldots \infty$ (Madras, 2000)

15. $\frac{1}{1^p}+\frac{x}{3^p}+\frac{x^2}{5^p}+\ldots+\frac{x^{n-1}}{(2n-1)^p}+\ldots \infty$ (J.N.T.U., 2006)

16. $\sum_{n=1}^{\infty} \frac{3 \cdot 6 \cdot 9 \ldots 3n}{4 \cdot 7 \cdot 10 \ldots (3n+1)} \cdot \frac{5^n}{3n+2}$ (V.T.U., 2004)

17. $1+\frac{1+\alpha}{1+\beta}+\frac{(1+\alpha)(1+2\alpha)}{(1+\beta)(1+2\beta)}+\frac{(1+\alpha)(1+2\alpha)(1+3\alpha)}{(1+\beta)(1+2\beta)(1+3\beta)}+\ldots$

9.10 FURTHER TESTS OF CONVERGENCE

When the Ratio test fails, we apply the following tests :

(1) Raabe's test*. *In the positive term series* Σu_n, if $\underset{n \to \infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right)=k,$

then the series converges for $k > 1$ and diverges for $k < 1$, but the test fails for $k = 1$.

When $k > 1$, choose a number p such that $k > p > 1$, and compare Σu_n with the series $\sum \frac{1}{n^p}$ which is convergent since $p > 1$.

$\therefore$ Σu_n will converge, if from and after some term,

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} \text{ or } \left(1+\frac{1}{n}\right)^p \qquad \text{or if, } \quad \frac{u_n}{u_{n+1}} > 1+\frac{p}{n}+\frac{p(p-1)}{2n^2}+\ldots$$

or if, $$n\left(\frac{u_n}{u_{n+1}}-1\right) > p+\frac{p(p-1)}{2n}+\ldots \quad \text{or if, } \quad \underset{n \to \infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right) > \underset{n \to \infty}{\text{Lt}}\left[p+\frac{p(p-1)}{2n}+\ldots\right]$$

i.e., if $k > p$, which is true. Hence Σu_n is convergent.

The other case when $k < 1$ can be proved similarly.

(2) Logarithmic test. *In the positive term series* Σu_n *if* $\underset{n \to \infty}{Lt}\left(n \log \frac{u_n}{u_{n+1}}\right)=k,$

then the series converges for $k > 1$, and diverges for $k < 1$, but the test fails for $k = 1$.

Its proof is similar to that of Raabe's test.

Obs. 1. *Logarithmic test is a substitute for Raabe's test and should be applied when* **either** *n occurs as an exponent in* u_n/u_{n+1}, **or** *evaluation of* $\underset{n \to \infty}{Lt}$ *becomes easier on taking logarithm of* u_n/u_{n+1}.

Obs. 2. If u_n/u_{n+1} does not involve n as an exponent or a logarithm, the series Σu_n diverges.

Example 9.11. *Test for convergence the series*

(*i*) $\sum \frac{4 \cdot 7 \ldots (3n+1)}{1 \cdot 2 \ldots n} x^n$ (V.T.U., 2009 ; P.T.U., 2006 S) (*ii*) $\sum \frac{(n!)^2}{(2n)!} x^{2n}$.

Solution. (*i*) Here $\frac{u_n}{u_{n+1}}=\frac{4 \cdot 7 \ldots (3n+1)}{1 \cdot 2 \ldots n} x^n \div \frac{4 \cdot 7 \ldots (3n+4)}{1 \cdot 2 \ldots (n+1)} x^{n+1}=\frac{n+1}{3n+4} \cdot \frac{1}{x}=\left[\frac{1+1/n}{3+4/n}\right]\frac{1}{x}$

$$\therefore \qquad \underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}}=\frac{1}{3x}.$$

*Called after the Swiss mathematician *Joseph Ludwig Raabe* (1801–1859).

Thus by *Ratio test*, the series converges for $\frac{1}{3x} > 1$, *i.e.*, for $x < \frac{1}{3}$ and diverges for $x > \frac{1}{3}$. But it fails for $x = \frac{1}{3}$. $\therefore$ Let us try the *Raabe's test*.

Now $$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)\left(1+\frac{4}{3n}\right)^{-1}$$ [*Expand by Binomial Theorem*]

$$= \left(1+\frac{1}{n}\right)\left(1-\frac{4}{3n}+\frac{16}{9n^2}-\ldots\right) = 1-\frac{1}{3n}+\frac{4}{9n^2}+\ldots$$

$\therefore$ $$n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{3}+\frac{4}{9n}+\ldots \quad \therefore \quad \underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{3} \text{ which} < 1.$$

Thus by *Raabe's test,* the series diverges.

Hence the given series converges for $x < \frac{1}{3}$ and diverges for $x \geq \frac{1}{3}$.

(*ii*) Here $$\frac{u_n}{u_{n+1}} = \left(\frac{n!}{(n+1)!}\right)^2 \frac{[2(n+1)]!}{(2n)!}\cdot\frac{x^{2n}}{x^{2(n+1)}} = \frac{(2n+1)(2n+2)}{(n+1)^2}\frac{1}{x^2} = \frac{2(2n+1)}{n+1}\frac{1}{x^2}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\frac{2(2+1/n)}{1+1/n}\cdot\frac{1}{x^2} = \frac{4}{x^2}$$

Thus by *Ratio Test,* the series converges for $x^2 < 4$ and diverges for $x^2 > 4$. But fails for $x^2 = 4$.

When $x^2 = 4$, $$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{2n+1}{2n+2}-1\right) = -\frac{n}{2n+2}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{2} < 1$$

Thus by *Raabe's test*, the series diverges.

Hence the given series converges for $x^2 < 4$ and diverges for $x^2 \geq 4$.

Example 9.12. *Discuss the convergence of the series*

$$x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\frac{5^5x^5}{5!}+\ldots\infty$$ (*P.T.U., 2008 ; Cochin, 2005 ; Rohtak, 2003*)

Solution. Here $$\frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n!} \div \frac{(n+1)^{n+1}x^{n+1}}{(n+1)!} = \frac{n^n}{(n+1)^n\, x} = \frac{1}{(1+1/n)^n}\cdot\frac{1}{x}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}} = \frac{1}{ex}.$$

Thus by Ratio test, the series converges for $x < 1/e$ and diverges for $x > 1/e$. But it fails for $x = 1/e$. Let us try the *log-test*.

Now $$\frac{u_n}{u_{n+1}} = \frac{e}{(1+1/n)^n}$$

$\therefore$ $$\log\frac{u_n}{u_{n+1}} = \log_e e - n\log\left(1+\frac{1}{n}\right) = 1-n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right) = \frac{1}{2n}-\frac{1}{3n^2}+\ldots$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\left(n\log\frac{u_n}{u_{n+1}}\right) = \frac{1}{2},$$ which < 1. Thus by the *log-test*, the series diverges.

Hence the given series converges for $x < 1/e$ and diverges for $x \geq 1/e$.

Example 9.13. *Discuss the convergence of the* ***hypergeometric series***

$$1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\ldots\infty.$$ (*Kurukshetra, 2005*)

Solution. Neglecting the first term, we have

$$u_{n+1} = u_n \frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)} x$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} \cdot \frac{1}{x} = \underset{n\to\infty}{\text{Lt}} \frac{(1+1/n)(1+\gamma/n)}{(1+\alpha/n)(1+\beta/n)} \cdot \frac{1}{x} = \frac{1}{x}$$

$\therefore$ by *Ratio test*, the series converges for $1/x > 1$, *i.e.*, for $x < 1$, and diverges for $x > 1$. But it fails for $x = 1$.

$\therefore$ let us try the *Raabe's test.*

$$\underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}} - 1\right) = \underset{n\to\infty}{\text{Lt}}\, n\left\{\frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)} - 1\right\} = \underset{n\to\infty}{\text{Lt}}\, n\left\{\frac{n(1+\gamma-\alpha-\beta)+\gamma-\alpha\beta}{n^2+n(\alpha+\beta)+\alpha\beta}\right\}$$

$$= \underset{n\to\infty}{\text{Lt}} \left\{\frac{(1+\gamma-\alpha-\beta)+(\gamma-\alpha\beta)\frac{1}{n}}{1+(\alpha+\beta)\frac{1}{n}+\alpha\beta\cdot\frac{1}{n^2}}\right\} = 1+\gamma-\alpha-\beta$$

Thus the series converges for $1+\gamma-\alpha-\beta > 1$, *i.e.*, for $\gamma > \alpha+\beta$ and diverges for $\gamma < \alpha+\beta$. But it fails for $\gamma = \alpha+\beta$. Since u_n/u_{n+1} does not involve n as an exponent or a logarithm, the series Σu_n diverges for $\gamma = \alpha+\beta$.

Hence the series converges for $x < 1$ and diverges for $x > 1$. When $x = 1$, the series converges for $\gamma > \alpha+\beta$ and diverges for $\gamma \le \alpha+\beta$.

PROBLEMS 9.5

Test the following series for convergence :

1. $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \dots \infty \; (x > 0)$ *(Mumbai, 2009)*
2. $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \frac{x^4}{4.5} + \dots \infty$ *(V.T.U., 2008 ; J.N.T.U., 2003)*
3. $1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \dots \infty \; (x > 0)$ *(Raipur, 2005)*
4. $1 + \frac{2}{3}x + \frac{2.3}{3.5}x^2 + \frac{2.3.4}{3.5.7}x^3 + \dots \infty$ *(V.T.U., 2009 S)*
5. $1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \frac{4!}{5^4}x^4 + \dots \infty$
6. $1 + \frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + \frac{3.6.9.12}{7.10.13.16}x^4 + \dots \infty$
7. $\frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \dots \infty. \; (x > 0)$ *(V.T.U., 2007 ; Raipur, 2005)*
8. $1 + \frac{1}{2}\cdot\frac{x^2}{4} + \frac{1.3.5}{2.4.6}\cdot\frac{x^4}{8} + \frac{1.3.5.7.9}{2.4.6.8.10}\cdot\frac{x^6}{12} + \dots \infty$ *(Rohtak, 2006 S ; Roorkee, 2000)*
9. $1 + \frac{(1!)^2}{2!}x^2 + \frac{(2!)^2}{4!}x^4 + \frac{(3!)^2}{6!}x^6 + \dots \infty \; (x > 0)$
10. $\frac{1^2}{4^2} + \frac{1^2.5^2}{4^2.8^2} + \frac{1^2.5^2.9^2}{4^2.8^2.12^2} + \dots \infty$
11. $\frac{a+x}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + \dots \infty$
12. $x^2 (\log 2)^q + x^3 (\log 3)^q + x^4 (\log 4)^q + \dots \infty$
13. $\frac{1}{1^2} + \frac{1+2}{1^2+2^2} + \frac{1+2+3}{1^2+2^2+3^2} + \dots$ *(V.T.U., 2000)*
14. $1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)}x^2 + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3 + \dots \infty \; (a, b > 0, x > 0).$

9.11 CAUCHY'S ROOT TEST*

In a positive series Σu_n, if $\underset{n \to \infty}{\text{Lt}} (u_n)^{1/n} = \lambda$,

then the series converges for $\lambda < 1$, and diverges for $\lambda > 1$.

Case I. When $\underset{n \to \infty}{\text{Lt}} (u_n)^{1/n} = \lambda < 1$.

By definition of a limit, we can find a positive number r $(\lambda < r < 1)$ such that

$$(u_n)^{1/n} < r \text{ for all } n > m, \quad \text{or} \quad u_n < r^n \text{ for all } n > m.$$

Since $r < 1$, the geometric series Σr^n is convergent. Hence, by comparison test, Σu_n is also convergent.

Case II. When $\text{Lt}\,(u_n)^{1/n} = \lambda > 1$.

By definition of a limit, we can find a number m, such that

$$(u_n)^{1/n} > 1 \text{ for all } n > m, \quad \text{or} \quad u_n > 1 \text{ for all } n > m.$$

Omitting the first m terms, let the series be $u_1 + u_2 + u_3 + \ldots$ so that $u_1 > 1, u_2 > 1, u_3 > 1$ and so on.

$$\therefore \quad u_1 + u_2 + u_3 + \ldots + u_n > n \quad \text{and} \quad \underset{n \to \infty}{\text{Lt}} (u_1 + u_2 + \ldots + u_n) \to \infty$$

Hence the series Σu_n is divergent.

Obs. *Cauchy's root test fails when $\lambda = 1$.*

Example 9.14. *Test for convergence the series*

(i) $\Sigma \dfrac{n^3}{3^n}$ *(ii)* $\Sigma (\log n)^{-2n}$ *(iii)* $\Sigma (1 + 1/\sqrt{n})^{-n^{3/2}}$ *(P.T.U., 2009 ; Kurukshetra, 2005)*

Solution. (*i*) We have $u_n = n^3/3^n$.

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} (u_n)^{1/n} = \underset{n \to \infty}{\text{Lt}} \left(\frac{n^{3/n}}{3}\right) = \underset{n \to \infty}{\text{Lt}} \frac{(n^{1/n})^3}{3} = \frac{1}{3} \ (< 1) \qquad \left[\because \underset{n \to \infty}{\text{Lt}} n^{1/n} = 1\right]$$

Hence the given series converges by Cauchy's root test.

(*ii*) Here $u_n = (\log n)^{-2n}$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} (u_n)^{1/n} = \underset{n \to \infty}{\text{Lt}} (\log n)^{-2} = 0 \ (< 1) \qquad \left[\because \underset{n \to \infty}{\text{Lt}} \log n = 0\right]$$

Hence, by Cauchy's root test, the given series converges.

(*iii*) Here $u_n = (1 + 1/\sqrt{n})^{-n^{3/2}}$

$$\therefore \quad (u_n)^{1/n} = \left[\frac{1}{(1 + 1/\sqrt{n})^{n^{3/2}}}\right]^{1/n} = \frac{1}{(1 + 1/\sqrt{n})^{\sqrt{n}}}$$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} (u_n)^{1/n} = \underset{n \to \infty}{\text{Lt}} \frac{1}{(1 + 1/\sqrt{n})^{\sqrt{n}}} = \frac{1}{e}$$, which is < 1. Hence the given series is convergent.

Example 9.15. *Discuss the nature of the following series :*

(i) $\dfrac{1}{2} + \dfrac{2}{3}x + \left(\dfrac{3}{4}\right)^2 x^2 + \left(\dfrac{4}{5}\right)^3 x^3 + \ldots \infty \ (x > 0)$ *(J.N.T.U., 2006)*

(ii) $\Sigma \dfrac{(n+1)^n x^n}{n^{n+1}}$

(iii) $\left(\dfrac{2^2}{1^2} - \dfrac{2}{1}\right)^{-1} + \left(\dfrac{3^3}{2^3} - \dfrac{3}{2}\right)^{-2} + \left(\dfrac{4^4}{3^4} - \dfrac{4}{3}\right)^{-3} + \ldots \infty$ *(V.T.U., 2006)*

*See footnote p. 144.

Solution. (*i*) After leaving the first term, we find that $u_n = \left(\frac{n+1}{n+2}\right)^n x^n$, so that

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{1+1/n}{1+2/n}\right) x = x$$

$\therefore$ By Cauchy's root test, the given series converges for $x < 1$ and diverges for $x > 1$.

When $x = 1$, $\quad u_n = \left(\frac{n+1}{n+2}\right)^n = \frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}\left(1+\frac{1}{n+1}\right)$

$\therefore \quad \underset{n\to\infty}{\text{Lt}} u_n = \frac{1}{e} \neq 0$. Since u_n does not tend to zero, Σu_n is divergent.

Thus the given series converges for $x < 1$ and diverges for $x \geq 1$.

(*ii*) Here $\quad (u_n^{1/n}) = \frac{n+1}{n^{1+1/n}} x$

$\therefore \quad \underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \frac{n+1}{n} \cdot \frac{1}{n^{1/n}} x = \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)\left(\frac{1}{n^{1/n}}\right) x = x \qquad \left[\because \underset{n\to\infty}{\text{Lt}} n/n = 1\right]$

$\therefore$ The given series converges for $x < 1$ and diverges for $x > 1$.

When $x = 1$, $\quad u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{1}{n}\left(1+\frac{1}{n}\right)^n$

Taking $\quad v_n = \frac{1}{n}, \ \underset{n\to\infty}{\text{Lt}} \left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e \neq 0$ and finite.

$\therefore$ By comparison test both Σu_n and Σv_n behave alike.

But $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent ($\because \ p = 1$). $\therefore \ \Sigma u_n$ also diverges. Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

(*iii*) Here $\quad u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-n}$

$$\therefore \quad (u_n)^{1/n} = \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^n - 1\right]^{-1} = \left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^n - 1\right]^{-1}$$

$\therefore \quad \underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = 1 \cdot (e-1)^{-1} = \frac{1}{e-1} 9 < 1 \qquad [\because \ e > 1]$

Thus the given series converges.

PROBLEMS 9.6

Discuss the convergence of the following series :

1. $\sum \frac{1}{n^n}$
2. $\sum \frac{1}{(\log n)^n}$ (*P.T.U., 2005*)
3. $\sum \left(\frac{n}{n+1}\right)^{n^2}$ (*P.T.U., 2010*)
4. $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \ldots + (x > 0)$
5. $\sum \left(\frac{n+2}{n+3}\right)^n x^n$
6. $\sum \frac{[(2n+1)x]^n}{n^{n+1}}, x > 0$
7. $\frac{3}{4}x + \left(\frac{4}{5}\right)^2 x^2 + \left(\frac{5}{6}\right)^3 x^3 + \ldots - \infty \ (x > 0)$ (*V.T.U., 2007*)

9.12 ALTERNATING SERIES

(1) Def. *A series in which the terms are alternately positive or negative is called an alternating series.*

(2) Leibnitz's series. *An alternating series* $u_1 - u_2 + u_3 - u_4 + \ldots$ *converges if* (*i*) *each term is numerically less than its preceding term, and* (*ii*) $\underset{n\to\infty}{\text{Lt}}\, u_n = 0$.

If $\underset{n\to\infty}{\text{Lt}}\, u_n \neq 0$, *the given series is oscillatory.*

The given series is $u_1 - u_2 + u_3 - u_4 + \ldots$

Suppose $u_1 > u_2 > u_3 > u_4 \ldots > u_{n+1} \ldots$ ...(1)

and $\underset{n\to\infty}{\text{Lt}}\, u_n = 0$ ...(2)

Consider the sum of $2n$ terms. It can be written as

$$s_{2n} = (u_1 - u_2) + (u_3 - u_4) + \ldots + (u_{2n-1} - u_{2n}) \qquad \ldots(3)$$

or as

$$s_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) \ldots - u_{2n} \qquad \ldots(4)$$

By virtue of (1), the expressions within the brackets in (3) and (4) are all positive.

$\therefore$ It follows from (3) that s_{2n} is positive and increases with n.

Also from (4), we note that s_{2n} always remains less than u_1.

Hence s_{2n} must tend to a finite limit.

Moreover $\underset{n\to\infty}{\text{Lt}}\, s_{2n+1} = \underset{n\to\infty}{\text{Lt}}\,(s_{2n} + u_{2n+1}) = \underset{n\to\infty}{\text{Lt}}\, s_{2n} + 0$ [by (2)

Thus $\underset{n\to\infty}{\text{Lt}}\, s_n$ tends to the same finite limit whether n is even or odd.

Hence the given series is convergent.

When $\underset{n\to\infty}{\text{Lt}}\, u_n \neq 0$, $\underset{n\to\infty}{\text{Lt}}\, s_{2n} \neq \underset{n\to\infty}{\text{Lt}}\, s_{2n+1}$. $\therefore$ The given series is oscillatory.

Example 9.16. *Discuss the convergence of the series*

(*i*) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$ (*ii*) $\frac{5}{2} - \frac{7}{4} + \frac{9}{6} - \frac{11}{8} + \ldots$

(*iii*) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots$ (*P.T.U., 2010*)

Solution. (*i*) The terms of the given series are alternately positive and negative ; each term is numerically less than its preceding term $\left[\because\ u_n - u_{n-1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}} < 0\right]$

Also $\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}}\,(1/\sqrt{n}) = 0$. Hence by Leibnitz's rule, the given series is convergent.

(*ii*) The terms of the given series are alternately positive and negative and

$$u_n - u_{n-1} = \frac{2n+3}{2n} - \frac{2n+1}{2n-2} = \frac{-6}{4n(n-1)} < 0 \text{ for } n > 1.$$

i.e., $u_n < u_{n-1}$ for $n > 1$. Also $\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}}\, \frac{2n+3}{2n} = 1 \neq 0$

Hence by Leibnitz's rule, the given series is oscillatory.

(*iii*) The terms of the given series are alternately positive and negative.

Also $n + 2 > n + 1$, *i.e.*, $\log(n+2) > \log(n+1)$

i.e., $\frac{1}{\log(n+2)} < \frac{1}{\log(n+1)}$, *i.e.*, $u_{n+1} < u_n$.

and $\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}}\, \frac{1}{\log(n+1)} = 0$

Hence the given series is convergent.

Example 9.17. *Examine the character of the series*

(i) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2n-1}$. *(ii)* $\sum_{n=2}^{\infty} \frac{(-1)^{n-1} x^n}{n(n-1)}, 0 < x < 1.$

Solution. (*i*) The terms of the given series are alternately positive and negative ; each term is numerically less than its preceding term.

$$\left[\because \quad u_n - u_{n-1} = \frac{n}{2n-1} - \frac{n-1}{2n-3} = \frac{-1}{(2n-1)(n-3)} < 0\right.$$

But $\quad \underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}} \frac{n}{2n-1} = \underset{n\to\infty}{\text{Lt}} \frac{1}{2-1/n} = \frac{1}{2}$ which is not zero.

Hence the given series is oscillatory.

(*ii*) The terms of the given series are alternately positive and negative

$$u_n - u_{n-1} = \frac{x^n}{n(n-1)} - \frac{x^{n-1}}{(n-1)(n-2)} = \frac{x^{n-1}[(n-2)x-n]}{n(n-1)(n-2)} < 0 \quad \text{for} \quad n \geq 2, \qquad (\because \ 0 < x < 1)$$

i.e., $\quad u_n < u_{n-1}$ for $n \geq 2$. Also $\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}} \frac{x^n}{n(n-1)} = 0 \qquad (\because \ 0 < x < 1)$

Hence the given series is convergent.

PROBLEMS 9.7

Discuss the convergence of the following series :

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \infty.$ (*P.T.U., 2009*)
2. $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \ldots \infty.$ (*V.T.U., 2010*)
3. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ (*Delhi, 2002*)
4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$.
5. $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + \ldots \infty$ (*Osmania, 2003*)
6. $\frac{1}{6} - \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \frac{5}{26} - \ldots \infty.$
7. $1 - 2x + 3x^2 - 4x^3 + \ldots + \infty, \left(x < \frac{1}{2}\right).$ (*Cochin, 2005*)
8. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$.
9. $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \ldots \infty \ (0 < x < 1).$ (*V.T.U., 2004 ; Delhi, 2002*)
10. $\left(\frac{1}{2} - \frac{1}{\log 2}\right) - \left(\frac{1}{2} - \frac{1}{\log 3}\right) + \left(\frac{1}{2} - \frac{1}{\log 4}\right) - \left(\frac{1}{2} - \frac{1}{\log 5}\right) + \ldots \infty.$

9.13 SERIES OF POSITIVE AND NEGATIVE TERMS

The series of positive terms and the alternating series are special types of these series with arbitrary signs.

Def. (1) *If the series of arbitrary terms* $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$
be such that the series $\quad |u_1| + |u_2| + |u_3| + \ldots + |u_n| + \ldots$
is convergent, then the series Σu_n *is said to be* **absolutely convergent.**

(2) *If* $\Sigma\, |u_n|$ *is divergent but* Σu_n *is convergent, then* Σu_n *is said to be* **conditionally convergent.**

For instance, the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots$ *is absolutely convergent,* since the series

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots$ is known to be convergent.

Again, since the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$ is convergent, and the series of absolute values $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ is divergent, so the original series is conditionally convergent.

Obs. 1. *An absolutely convergent series is necessarily convergent but not conversely.*

Let Σu be an absolutely convergent series.

Clearly $\quad u_1 + u_2 + u_3 + \ldots + u_n + \ldots$

$\leq | u_1 | + | u_2 | + | u_3 | + \ldots + | u_n | + \ldots$ which is known to be convergent.

Hence the series Σu_n is also convergent.

Obs. 2. As the series $\Sigma | u_n |$ is of positive terms, the tests already established for positive term series can be applied to examine Σu_n for its absolute convergence. For instance, Ratio test can be restated as follows :

The series Σu_n is absolutely convergent if $\underset{n \to \infty}{\text{Lt}} \frac{|u_{n+1}|}{|u_n|} < 1$,

and is divergent if $\underset{n \to \infty}{\text{Lt}} \frac{|u_{n+1}|}{|u_n|} > 1$. This test fails when the limit is unity.

Example 9.18. *Examine the following series for convergence :*

(*i*) $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \ldots \infty$ (*V.T.U., 2006*)

(*ii*) $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \ldots \infty.$

Solution. (*i*) The series of absolute terms is $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ldots$ which is, evidently convergent.

$\therefore$ the given series is absolutely convergent and hence it is convergent.

(*ii*) Here
$$u_n = (-1)^{n-1} \frac{(1+2+3+\ldots+n)}{(n+1)^3}$$

$$= (-1)^{n-1} \frac{n(n+1)}{2(n+1)^3} = (-1)^{n-1} \frac{n}{2(n+1)^2} = (-1)^{n-1} a_n \text{ (Say).}$$

Then
$$a_n - a_{n+1} = \frac{1}{2}\left[\frac{n}{(n+1)^2} - \frac{n+1}{(n+2)^2}\right] = \frac{1}{2} \frac{n^2+n-1}{(n+1)^2 (n+2)^2} > 0.$$

i.e.,
$$a_{n+1} < a_n. \text{ Also } \underset{n \to \infty}{\text{Lt}} a_n = \frac{1}{2} \underset{n \to \infty}{\text{Lt}} \frac{n}{(n+1)^2} = 0.$$

Thus by Leibnitz's rule, Σa_n and therefore Σu_n is convergent.

Also
$$| u_n | = \frac{1}{2} \frac{n}{n^2+1}. \text{ Taking } v_n = \frac{1}{n}, \text{ we note that}$$

$$\underset{n \to \infty}{\text{Lt}} \frac{|u_n|}{v_n} = \frac{1}{2} \underset{n \to \infty}{\text{Lt}} \frac{n^2}{n^2+1} = \frac{1}{2} \neq 0$$

Since Σv_n is divergent, therefore $\Sigma | u_n |$ is also divergent.

i.e., Σu_n is convergent but $\Sigma | u_n |$ is divergent.

Thus the given series Σu_n is conditionally convergent.

Example 9.19. *Test whether the following series are absolutely convergent or not ?*

(*i*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ (*ii*) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2}$.

Solution. (*i*) Given series is $\Sigma u_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \infty$

This is an alternating series of which terms go on decreasing and $\underset{n \to \infty}{\text{Lt}}\ u_n = \underset{n \to \infty}{\text{Lt}}\ \frac{1}{2n-1} = 0$

$\therefore$ by Leibnitz's rule, Σu_n converges.

The series of absolute terms is $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots \infty$

Here $u_n = \frac{1}{2n-1}$. Taking $v_n = \frac{1}{n}$, we have

$$\underset{n \to \infty}{\text{Lt}}\ \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \left(\frac{n}{2n-1}\right) = \underset{n \to \infty}{\text{Lt}} \left(\frac{1}{2 - \frac{1}{n}}\right) = \frac{1}{2} \neq 0 \text{ and finite.}$$

$\therefore$ by Comparison test, Σu_n diverges [$\because$ Σv_n diverges.

Hence the given series converges and the series of absolute terms diverges, therefore the given series converges conditionally.

(*ii*) The terms of given series are alternately positive and negative. Also each term is numerically less than the preceding term and $\underset{n \to \infty}{\text{Lt}}\ |u_n| = \underset{n \to \infty}{\text{Lt}}\ [1/n(\log n)^2] = 0$.

$\therefore$ by Leibnitz's rule, the given series converges.

Also $$\int_2^\infty \frac{dx}{x(\log x)^2} = \left| -\frac{1}{\log x} \right|_2^\infty = \frac{1}{\log 2} = 0 \text{ and finite.}$$

i.e., the series of absolute terms converges.

Hence, the given series converges absolutely.

9.14 POWER SERIES

(1) Def. *A series of the form* $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$ ...(*i*)

where the a's are independent of x, is called a **power series** *in x.* Such a series may converge for some or all values of x.

(2) Interval of convergence

In the power series (*i*), $u_n = a_n x^n$.

$$\therefore \quad \underset{n \to \infty}{\text{Lt}}\ \frac{u_{n+1}}{u_n} = \underset{n \to \infty}{\text{Lt}}\ \frac{a_{n+1}x^{n+1}}{a_n x^n} = \underset{n \to \infty}{\text{Lt}} \left(\frac{a_{n+1}}{a_n}\right) . x$$

If $\underset{n \to \infty}{\text{Lt}} \left(\frac{a_{n+1}}{a_n}\right) = l$, then by Ratio test, the series (*i*) converges, when lx is numerically less than 1, *i.e.*, when $|x| < 1/l$ and diverges for other values.

Thus the power series (*i*) has an interval $-1/l < x < 1/l$ within which it converges and diverges for values of x outside this interval. Such an interval is called the *interval of convergence of the power series.*

Example 9.20. *State the values of x for which the following series converge :*

(*i*) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots \infty$, (*ii*) $\frac{1}{1-x} + \frac{1}{2(1-x)^2} + \frac{1}{3(1-x)^3} + \ldots \infty$.

Solution. (*i*) Here $u_n = (-1)^{n-1} \frac{x^n}{n}$ and $u_{n+1} = (-1)^n \frac{x^{n+1}}{n+1}$

$$\therefore \quad \frac{u_{n+1}}{u_n} = -\frac{n}{n+1}x \quad \text{and} \quad \underset{n \to \infty}{\text{Lt}} \left| \frac{u_{n+1}}{u_n} \right| = \left(\underset{n \to \infty}{\text{Lt}}\ \frac{1}{1+1/n} \right) |x| = |x|$$

$\therefore$ by Ratio test the given series converges for $|x| < 1$ and diverges for $|x| > 1$.

Let us examine the series for $x = \pm 1$.

For $x = 1$, the series reduces to $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \ldots$

which is an alternating series and is convergent.

For $x = -1$, the series becomes $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\right)$

which is a divergent series as can be seen by comparison with p-series when $p = 1$.

Hence the given series converges for $-1 < x \le 1$.

(*ii*) Here $\quad u_n = \dfrac{1}{n(1-x)^n}$

$$\underset{n\to\infty}{\text{Lt}} \left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{\text{Lt}} \left|\frac{1}{(n+1)(1-x)^{n+1}} \cdot n(1-x)^n\right| = \left|\frac{1}{1-x}\right| \underset{n\to\infty}{\text{Lt}} \frac{n}{n+1} = \left|\frac{1}{1-x}\right|$$

By Ratio test, Σu_n converges for $\left|\dfrac{1}{1-x}\right| < 1$, *i.e.*, $|1 - x| > 1$

i.e., for $-1 > 1 - x > 1$ or $x < 0$ and $x > 2$.

Let us examine the series for $x = 0$ and $x = 2$.

For $x = 0$, the given series becomes $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$ which is a divergent harmonic series.

For $x = 2$, the given series becomes $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots + \frac{(-1)^n}{n} + \ldots$

It is an alternating series which is convergent by Leibnitz's rule

$\left[\because \quad u_n < u_{n-1} \text{ for all } n \text{ and } \underset{n\to\infty}{\text{Lt}} u_n = 0.\right]$

Hence the given series converges for $x < 0$ and $x \ge 2$.

Example 9.21. *Test the series* $\frac{x}{\sqrt{3}} - \frac{x^2}{\sqrt{5}} + \frac{x^3}{\sqrt{7}} - \ldots$ *for absolute convergence and conditional convergence.*

(V.T.U., 2010)

Solution. We have $u_n = (-1)^{n-1} \dfrac{x^n}{\sqrt{(2n+1)}}$ and $u_{n+1} = \dfrac{(-1)^n x^{n+1}}{\sqrt{(2n+3)}}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{\text{Lt}} \left|\frac{\dfrac{(-1)^n x^{n+1}}{\sqrt{(2n+3)}} \cdot \sqrt{(2n+1)}}{(-1)^{n-1} x^n}\right| = \underset{n\to\infty}{\text{Lt}} \left|(-1)\sqrt{\left(\frac{2n+1}{2n+3}\right)}x\right|$$

$$= \underset{n\to\infty}{\text{Lt}} \left|\sqrt{\left(\frac{2+1/n}{2+3/n}\right)}x\right| = |x|$$

Hence *the given series is absolutely convergent for $|x| < 1$ and is divergent for $|x| > 1$* and the test fails for $|x| = 1$.

For $x = 1$, $\quad u_n = \dfrac{(-1)^{n-1}}{\sqrt{(2n+1)}}$. Since $2n + 1 < 2n + 3$ or $(2n+1)^{-1/2} > (2n+3)^{-1/2}$

i.e., $\quad u_n > u_{n+1}$. Also $\underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}} \dfrac{1}{\sqrt{(2n+1)}} = 0.$

$\therefore$ the series is convergent by Leibnitz's test.

But $\quad \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{7}} + \ldots$ has $u_n = \dfrac{1}{\sqrt{(2n+1)}} = \dfrac{1}{\sqrt{n}} \cdot \dfrac{1}{\sqrt{(2+1/n)}}$

On comparing it with $v_n = \frac{1}{\sqrt{n}}$, Σu_n is divergent.

Hence the *given series is conditionally convergent for x = 1.*

For x = – 1, the series becomes $-\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \ldots\right)$

But we have seen that the series $\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \ldots$ is divergent.

Hence, *the given series is divergent when x = – 1.*

9.15 (1) CONVERGENCE OF EXPONENTIAL SERIES

The series $1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \infty$ *is convergent for all values of x.* (*J.N.T.U., 2006*)

Here $$\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \left[\frac{x^n}{n!} \div \frac{x^{n-1}}{(n-1)!}\right] = \underset{n\to\infty}{\text{Lt}} \frac{x}{n} = 0$$

Hence the series converges, whatever be the value of x.

(2) Convergence of logarithmic series

The series $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^n}{n} + \ldots \infty$ *is convergent for* $-1 < x \le 1$.

Here $$\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \frac{(-1)^{n+1}x^{n+1}}{n+1} \cdot \frac{n}{(-1)^n x^n} = -x \underset{n\to\infty}{\text{Lt}} \frac{n}{n+1} = -x \underset{n\to\infty}{\text{Lt}} \left\{\frac{1}{1+1/n}\right\} = -x.$$

Hence the series converges for $|x| < 1$ and diverges for $|x| > 1$.

When $x = 1$, the series being $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$, is convergent.

When $x = -1$, the series being $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right)$, is divergent.

Hence the series converges for $-1 < x \le 1$.

(3) Convergence of binomial series

The series $1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!} x^r + \ldots \infty$

converges for $|x| < 1$.

Here $$u_r = \frac{n(n-1)\ldots(n-r)}{(r-1)!} x^{r-1} \text{ and } u_{r+1} = \frac{n(n-1)\ldots(n-r+1)}{r!} x^r$$

$$\therefore \quad \underset{r\to\infty}{\text{Lt}} \frac{u_{r+1}}{u_r} = \underset{r\to\infty}{\text{Lt}} \frac{n-r+1}{r} x = \underset{r\to\infty}{\text{Lt}} \left(\frac{n+1}{r} - 1\right) x = -x \text{ for } r > n+1.$$

Hence, the series converges for $|x| < 1$.

PROBLEMS 9.8

1. Test the following series for conditional convergence : (*i*) $\sum \frac{(-1)^{n-1}}{\sqrt{n}}$ (*ii*) $\sum \frac{(-1)^{n-1} n}{n^2+1}$.
2. Prove that the series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots$ converges absolutely. (*Rohtak, 2006 S*)
3. Test the following series for conditional convergence :

   (*i*) $1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots \infty$ (*ii*) $1 - \frac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + \ldots \infty$

4. Discuss the absolute convergence of (*i*) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ (*Hissar, 2005 S*)

(*ii*) $x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \dots \infty$ (*iii*) $\frac{1}{\sqrt{(1^3+1)}} - \frac{1}{\sqrt{(2^3+1)}}x + \frac{1}{\sqrt{(3^3+1)}}x^2 - \dots \infty$

5. Find the nature of the series $\frac{x}{1\cdot 2} - \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} - \frac{x^4}{4\cdot 5} + \dots \infty$ (*V.T.U., 2009*)

6. For what values of x are the following series convergent :

(*i*) $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \dots \infty$ (*P.T.U., 2009 S ; V.T.U., 2008*)

(*ii*) $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \dots \infty$

7. Find the radius of convergence of the series $\sum \frac{n!}{n^n} x^n$. (*Calicut, 2005*)

8. Prove that $\frac{1}{a} + \frac{1}{a+1} - \frac{1}{a+2} + \frac{1}{a+3} + \frac{1}{a+4} - \frac{1}{a+5} + \dots$ is a divergent series.

9. Test the series $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}}$ for

(*i*) absolute convergence and (*ii*) conditional convergence. (*V.T.U., 2007 ; Rohtak, 2005*)

9.16 PROCEDURE FOR TESTING A SERIES FOR CONVERGENCE

First see whether the given series is

(*i*) a series with terms alternately positive and negative ;

(*ii*) a series of positive terms excluding power series ;

or (*iii*) a power series.

For alternating series (*i*), apply the Leibnitz's rule (§ 9.12).

For series (*ii*), first find u_n and if possible evaluate Lt u_n. If Lt $u_n \neq 0$, the series is divergent. If Lt $u_n = 0$, compare Σu_n with $\Sigma 1/n^p$ and apply the comparison tests (§ 9.6).

If the comparison tests are not applicable, apply the Ratio test (§ 9.9). If Lt $u_n/u_{n+1} = 1$, *i.e.*, the ratio test fails, apply Raabe's teat (§ 9.10). If Raabe's test fails for a similar reason, apply Logarithmic test (§ 9.10). If this also fails, apply Cauchy's root test (§ 9.11).

For the power series (*iii*), apply the Ratio test as in § 9.14. If the Ratio test fails, examine the series as in case (*ii*) above.

PROBLEMS 9.9

Test the convergence of the following series :

1. $\sum_{n=1}^{\infty} \frac{2^n - 2}{2^n + 1} \cdot x^{n-1} (x > 0)$. (*Osmania, 1999*)

2. $\sum \left(\frac{1}{\sqrt{n}} - \sqrt{\frac{n}{n+1}}\right)$.

3. $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \dots$

4. $\sum_{n=1}^{\infty} \sqrt{\left\{\frac{2^n+1}{3^n+1}\right\}}$

5. $\frac{1}{1+\sqrt{2}} + \frac{2}{1+2\sqrt{3}} + \frac{3}{1+3\sqrt{4}} + \dots \infty$.

6. $\frac{x}{1+\sqrt{1}} + \frac{x^2}{2+\sqrt{2}} + \frac{x^3}{3+\sqrt{3}} + \dots \infty$.

7. $1 + \frac{2^2}{3^2}x + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2}x^2 + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2}x^3 + \dots \infty$.

8. $\sum_{1}^{\infty} \frac{nx^n}{(n+1)(n+2)} (x > 0)$.

9. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$.

10. $\sum_{n=1}^{\infty} \frac{x^n}{(2n-1)^2 2^n}$.

11. $\sum_{0}^{\infty} \frac{(3x+5)^n}{(n+1)!}$.

12. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n n}$.

13. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$.

14. $\sum \frac{(-1)^{n-1} \sin nx}{n^3}$.

15. $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \dots \infty$. (V.T.U., 2003)

16. $\sum_{n=2}^{\infty} \frac{1}{(n \log n)(\log \log n)^p}$.

9.17 UNIFORM CONVERGENCE

Let $$u_1(x) + u_2(x) + \dots \infty = \sum_{n=1}^{\infty} u_n(x) \qquad \dots(1)$$

be an infinite series of functions each of which is defined in the interval (a, b). Let $s_n(x)$ be the sum of its first n terms, *i.e.*, $s_n(x) = u_1(x) + u_2(x) + \dots + u_n(x)$

At some point $x = x_1$, if $\underset{n \to \infty}{\text{Lt}}\, s_n(x_1) = s(x_1)$,

then the series (1) is said to converge to sum $s(x_1)$ at that point. This means at $x = x_1$ given a positive number ε, we can find a number N such that $| s(x_1) - s_n(x_1) | < \varepsilon$ for $n > N$ ...(2)

Evidently N will depend on ε but generally it will also depend on x_1. Now if we keep the same ε but take some other value x_2 of x for which (1) is convergent, then we may have to change N for the inequality (2) to hold. If we wish to approximate the sum $s(x)$ of the series by its partial sums $s_n(x)$, we shall require different partial sums at different points of the interval and the problem will become quite complicated. If, however, we choose an N which is independent of the values of x, the problem becomes simpler. Then the partial sum $s_n(x)$, $(n > N)$ approximates to $s(x)$ for all values of x in the interval (a, b) and ε is uniform throughout this interval. Thus we have

Definition. *The series $\sum u_n(x)$ is said to be uniformly convergent in the interval (a, b), if for a given $\varepsilon > 0$, a number N can be found* **independent of x,** *such that for every x in the interval (a, b),*

$$| s(x) - s_n(x) | < \varepsilon \text{ for all } n > N.$$

Example 9.21. *Examine the geometric series $1 + x + x^2 + \dots + x^{n-1} + \dots \infty$ for uniform convergence in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.*

Solution. We have $$s_n(x) = 1 + x + x^2 + \dots + x^{n-1} = \frac{1 - x^n}{1 - x}.$$

and $$s(x) = \underset{n \to \infty}{\text{Lt}} \frac{1 - x^n}{1 - x} = \frac{1}{1 - x} \text{ for } | x | < 1$$

$\therefore$ $$| s(x) - s_n(x) | = \left| \frac{x^n}{1 - x} \right| = \frac{| x^n |}{1 - x} = \frac{| x |^n}{1 - x} \text{ which will be } < \varepsilon, \text{ if } | x |^n < \varepsilon (1 - x).$$

Choose N such that $| x |^N = \varepsilon (1 - x)$

or $$N = \log [\varepsilon (1 - x)] / \log | x | \qquad \dots(i)$$

Evidently N increases with the increase of $| x |$ and in the interval $-\frac{1}{2} \le x \le \frac{1}{2}$, it assumes a maximum value $N' = \log (\varepsilon/2)/\log \frac{1}{2}$ at $x = \frac{1}{2}$ for a given ε.

Thus $| s(x) - s_n(x) | < \varepsilon$ for all $n \ge N'$ for every value of x in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Hence the geometric series converges uniformly in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Obs. *The geometric series though convergent in the interval $(-1, 1)$, is not uniformly convergent in this interval,* since we cannot find a fixed number N for every x in this interval

($\because$ N given by (*i*) $\to \infty$ as $| x | \to 1$).

9.18 WEIERSTRASS'S M-TEST*

A series $\Sigma u_n(x)$ is uniformly convergent in an interval (a, b), if there exists a convergent series ΣM_n of positive constants such that $|u_n(x)| \le M_n$ for all values of x in (a, b).

Since ΣM_n is convergent, therefore, for a given $\varepsilon > 0$, we can find a number N, such that $|s - s_n| < \varepsilon$ for every $n > N$,

where $s = M_1 + M_2 + \dots + M_n + M_{n+1} + \dots$ and $s_n = M_1 + M_2 + \dots + M_n$

This implies that $|M_{n+1} + M_{n+2} + \dots| < \varepsilon$ for every $n > N$.

Since $|u_n(x)| \le M_n$

$$\therefore \quad |u_{n+1}(x)| + u_{n+2}(x) + \dots | \le |u_{n+1}(x)| + |u_{n+2}(x)| + \dots$$
$$\le M_{n+1} + M_{n+2} + \dots < \varepsilon \text{ for every } n > N.$$

i.e., $|s(x) - s_n(x)| < \varepsilon$ for every $n > N$, where $s(x)$ is the sum of the series $\Sigma u_n(x)$.

Since N does not depend on x, the series $\Sigma u_n(x)$ converges uniformly in (a, b).

Obs. *$\Sigma u_n(x)$ is also absolutely convergent for every x, since $|u_n(x)| \le M_n$.*

Example 9.22. *Show that the following series converges uniformly in any interval :*

(i) $\sum \dfrac{\cos nx}{n^p}$ (Andhra, 1999) (ii) $\sum \dfrac{1}{n^3 + n^4x^2}$.

Solution. (i) $\left|\dfrac{\cos nx}{n^p}\right| = \dfrac{|\cos nx|}{n^p} \le \dfrac{1}{n^p}$ $(= M_n)$ for all values of x.

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges for p > 1,

$\therefore$ By M-test, the given series converges uniformly for all real values of x and $p > 1$.

(ii) For all values of x, $n^3 + n^4x^2 > n^3$

$\therefore$ $\left|\dfrac{1}{n^3 + n^4x^2}\right| < \dfrac{1}{n^3}$ $(= M_n)$. But ΣM_n being p-series with $p > 1$, is convergent.

$\therefore$ By M-test, the given series converges uniformly in any interval.

Example 9.23. *Examine the following series for uniform convergence :*

(i) $\sum_{n=1}^{\infty} \dfrac{\sin(nx + x^2)}{n(n+2)}$ (P.T.U., 2009) (ii) $\sum_{n=1}^{\infty} \dfrac{1}{n^p + n^q x^2}$ (P.T.U., 2005 S)

Solution. (i) $\left|\dfrac{\sin(nx + x^2)}{n(n+2)}\right| = \dfrac{|\sin(nx + x^2)|}{n^2 + 2n} \le \dfrac{1}{n^2}$ $(= M_n)$ for all real x.

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ is convergent, therefore, by M-test, the given series is uniformly convergent for all real values of x.

(ii) For all real values of x, $x^2 \ge 0$, *i.e.*, $n^q x^2 \ge 0$

i.e., $n^p + n^q x^2 \ge n^p$ or $\dfrac{1}{n^p + n^q x^2} \le \dfrac{1}{n^p}$ $(= M_n)$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \dfrac{1}{n^p}$ is convergent for $p > 1$,

$\therefore$ by M-test, the given series is uniformly convergent for all real values of x and $p > 1$.

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* Named after the great German mathematician *Karl Weierstrass* (1815–1897) who made basic contributions to Calculus, Approximation theory, Differential geometry and Calculus of variations. He was also one of the founders of Complex analysis.

9.19 PROPERTIES OF UNIFORMLY CONVERGENT SERIES

I. If the series $\Sigma u_n(x)$ converges uniformly to sum $s(x)$ in the interval (a, b) and each of the functions $u_n(x)$ is continuous in this interval, then the sum $s(x)$ is also continuous in (a, b).

II. If the series $\Sigma u_n(x)$ converges uniformly in the interval (a, b) and each of the functions $u_n(x)$ is continuous in this interval, then the series can be integrated term by term

i.e.,
$$\int_a^b [u_1(x) + u_2(x) + \dots]\, dx = \int_a^b u_1(x)\, dx + \int_a^b u_2(x)\, dx + \dots$$

III. If $\Sigma u_n(x)$ is a convergent series having continuous derivatives of its terms, and the series $\Sigma u_n'(x)$ converges uniformly, then the series can be differentiated term by term

$$\frac{d}{dx}[u_1(x) + u_2(x) + \dots] = u_1'(x) + u_2'(x) + \dots.$$

Example 9.24. *Prove that* $\int_0^1 \left(\sum \frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}.$

Solution. $|x^n| \le 1$ for $0 \le x \le 1$

$\therefore$ $\left|\frac{x^n}{n^2}\right| \le \frac{1}{n^2}$ $(= M_n)$ for $0 \le x \le 1$. But ΣM_n is a convergent series.

$\therefore$ by M-test, the series $\Sigma(x^n/n^2)$ is uniformly convergent in $0 \le x \le 1$. Also x^n/n^2 is continuous in this interval.

$\therefore$ the series $\Sigma(x^n/n^2)$ can be integrated term by term in the interval $0 \le x \le 1$.

i.e.,
$$\int_0^1 \left(\sum \frac{x^n}{n^2}\right) dx = \sum \left(\int_0^1 \frac{x^n}{n^2}\, dx\right) = \sum \left(\frac{1}{n^2}\int_0^1 x^n\, dx\right) = \sum \frac{1}{n^2(n+1)}.$$

Imp. Obs. *There is no relation between absolute and uniform convergence.* In fact, a series may converge absolutely but not uniformly while another series may converge uniformly but not absolutely.

For instance, the series

$\frac{1}{x^2+1} - \frac{1}{x^2+2} + \frac{1}{x^2+3} - \dots$ can be seen to converge uniformly but not absolutely, while the series

$x^2 + \frac{x^2}{x^2+1} + \frac{x^2}{(x^2+1)^2} + \frac{x^2}{(x^2+1)^3} + \dots$ can be shown to converge absolutely but not uniformly.

PROBLEMS 9.10

Test for uniform convergence the series :

1. $\sum_{n=1}^{\infty} \frac{x^n}{n^{3/2}}$.
2. $\sum \frac{\cos nx}{2^n}$.
3. $\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \frac{\sin 4x}{4^2} + \dots \infty$. *(P.T.U., 2003 ; Andhra, 2000)*
4. $\sin x - \frac{\sin 2x}{2\sqrt{2}} + \frac{\sin 3x}{3\sqrt{3}} - \frac{\sin 4x}{4\sqrt{4}} + \dots \infty$.
5. $\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \infty$.
6. $\frac{ax}{2} + \frac{a^2x^2}{5} + \frac{a^3x^3}{10} + \dots + \frac{a^nx^n}{n^2+1} + \dots \infty$.
7. Show that the series $\Sigma r^n \sin n\theta$ and $\Sigma r^n \cos n\theta$ converge uniformly for all real values of θ if $0 < r < 1$.
8. Show that $\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \frac{1}{4+x^2} + \dots$ converges uniformly in the interval $x \ge 0$ but not absolutely.
9. Prove that $\sum \frac{x}{n(1+nx^2)}$ is uniformly convergent for all real values of x.

10. Examine the following series for uniform convergence :

(i) $\sum_{n=1}^{\infty} \frac{1}{n^4 + n^3x^2}$ (ii) $\sum_{n=1}^{\infty} \frac{\cos(x^2 + n^2x)}{n(n^2+2)}$.

11. Show that

(i) $\int_0^1 \left(\sum \frac{\sin x}{x}\right) dx = 1 - \frac{1}{3.3!} + \frac{1}{5.5!} - \frac{1}{7.7!} + \dots \infty$; (ii) $\int_0^{\pi} \left(\sum \frac{\sin n\theta}{n^3}\right) d\theta = 2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.

9.20 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 9.11

Choose the correct answer or fill up the blanks in each of the following problems :

1. The series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$ converges if

(a) $p > 0$ (b) $p < 1$ (c) $p > 1$ (d) $p \le 1$.

2. The series $\sum_{n=0}^{\infty} (2x)^n$ converges if

(a) $-1 \le x \le 1$ (b) $-\frac{1}{2} < x < \frac{1}{2}$ (c) $-2 < x < 2$ (d) $-\frac{1}{2} \le x \le \frac{1}{2}$.

3. The series $\frac{2}{1^2} - \frac{3}{2^2} + \frac{4}{3^2} - \frac{5}{4^2} + \dots$ is

(a) conditionally convergent (b) absolutely convergent
(c) divergent (d) none of the above.

4. Which one of the following series is *not* convergent ?

(a) $\frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \dots \infty$ (b) $1\frac{1}{2} - 1\frac{1}{3} + 1\frac{1}{4} - 1\frac{1}{5} + \dots \infty$

(c) $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots \infty$ (d) $x + x^2 + x^3 + x^4 + \dots \infty$ where $|x| < 1$.

5. The sum of the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is

(a) zero (b) infinite (c) log 2
(d) not defined as the series is not convergent.

6. Let Σu_n be a series of positive terms. Given that Σu_n is convergent and also

$\underset{n \to \infty}{\text{Lt}} \frac{u_{n+1}}{u_n}$ exists, then the said limit is

(a) necessarily equal to 1 (b) necessarily greater than 1
(c) may be equal to 1 or less than 1 (d) necessarily less than 1.

7. $\sum \left(1 + \frac{1}{n}\right)^{-n^2}$ is

(a) convergent (b) oscillatory (c) divergent.

8. $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$ is

(a) oscillatory (b) conditionally convergent
(c) divergent (d) absolutely convergent.

9. $1+\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}-\frac{1}{7^2}-\frac{1}{8^2}+\ldots\infty$ is

(a) conditionally convergent (b) convergent
(c) oscillatory (d) divergent.

10. $\int_0^1 \left(\sum \frac{x^n}{n^2}\right) dx =$

(a) $\sum_{n=0}^{\infty} \frac{1}{n(n+1)}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^2(n-1)}$ (c) $\sum_{n=0}^{\infty} \frac{1}{n(n-1)}$ (d) $\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$.

11. If Σu_n is a convergent series of positive terms, then $\underset{n\to\infty}{\text{Lt}}\ u_n$ is

(a) 1 (b) ± 1 (c) 0 (d) 0. (V.T.U., 2010)

12. Geometric series $1+x+x^2+\ldots+x^{n-1}+\ldots\infty$

(a) converges in the interval .... (b) converges uniformly in the interval ....

13. The series $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\infty$ converges in the interval ....

14. If $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}}=k$, then Σu_n converges for k ....

15. A sequence (a_n) is said to be bounded, if there exists a number k such that for every n, a_n is ....

16. The series $2-5+3+2-5+3-5+\ldots\infty$ is .... (Convergent etc.)

17. The series $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\infty$ converges for ....

18. If $\underset{n\to\infty}{\text{Lt}}\ n\left\{\frac{u_n}{u_{n+1}}-1\right\}=k$, then Σu_n diverges for k ....

19. A sequence which is monotonic and bounded is ....

20. The series $\frac{1}{1.2}+\frac{2}{3.4}+\frac{3}{5.6}+\ldots\infty$ is .... (Convergent etc.)

21. The series $\frac{2^p}{1^q}+\frac{3^p}{2^q}+\frac{4^p}{3^q}+\ldots\infty$ converges for ....

22. The series $\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+\ldots\infty$ is .... (Convergent etc.)

23. The series $\sqrt{\left(\frac{2^n-1}{3^n-1}\right)}$ is ... (Convergent etc.)

24. The series $1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2\ldots+\left(-\frac{1}{2}\right)^n(x-2)^n+\ldots\infty$ converges in the interval ......

25. Is the series $\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}$ convergent ?

26. The exponential series $1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots\infty$ is absolutely convergent. (True/False)

27. The series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}+\ldots\infty$, is .... (Convergent/divergent/oscillatory)

28. Is the series $\sum n \tan 1/n$ convergent ?

29. The series $\sum \frac{1}{nx^n}$ converges for x ......

30. The series $\sum_{n=1}^{\infty} \frac{x^n}{n^3}$ converges uniformly when x lies in the interval ....

31. Every absolutely convergent series is necessarily

(*a*) divergent (*b*) convergent

(*c*) conditionally convergent (*d*) none of these. *(V.T.U., 2009)*

32. The convergence of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ is tested by

(*a*) Ratio test (*b*) Raabe's test (*c*) Leibnitz's (*d*) Cauchy root test. *(V.T.U., 2009)*

33. The series $\sum \frac{x^n}{(n+1)^n}, x > 0$ is

(*a*) divergent (*b*) convergent (*c*) oscillatory (*d*) none of these. *(V.T.U., 2010)*

34. $\sum \sin\left(\frac{1}{n}\right)$ is

(*a*) convergent (*b*) divergent (*c*) oscillatory (*d*) none of these.

35. $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ is convergent. (True or False)