CHAPTER 36

Integral Equations

1. Introduction. 2. Definition. 3. Conversion of a linear differential equation to an integral equation and vice versa. 4. Conversion of boundary value problems to integral equations using Green's function. 5. Solution of an integral equation. 6. Integral equations of the convolution type. 7. Abel's integral equation. 8. Integro-differential equations. 9. Integral equations with separable kernels. 10. Solution of Fredholm equations with separable kernels. 11. Solution of Fredholm and Volterra equations by the method of successive approximations.

36.1 INTRODUCTION

Integral equations play an effective role in the study of boundary value problems. Such equations also occur in many fields of mechanics and mathematical physics. Integral equations may be obtained directly from physical problems *e.g.*, radiation transfer problem and neutron diffusion problem etc. They also arise as representation formulae for the solutions of differential equations. A differential equation can be replaced by an integral equation with the help of initial and boundary conditions.

Integral equations were first encountered in the theory of Fourier integrals. In 1826, another integral equation was obtained by *Abel*. Actual development of the theory of integral equations began with the works of the Italian mathematician V. Volterra (1896) and the Swedish mathematician I. Fredholm (1900).

36.2 DEFINITION

An integral equation is an equation in which an unknown function appears under the integral sign. We shall take up integral equations in which only linear functions of the unknown function are involved. The general type of *linear integral equation* is of the form

$$y(x) = F(x) + \lambda \int_a^b K(x, t)\, y(t)\, dt$$

where $F(x)$ and $K(x, t)$ are known functions while $y(x)$ is to be determined. The function $K(x, t)$ is called the *Kernel of the integral equation.*

If a and b are constants, the equation is known as *Fredholm integral equation.*

If a is a constant while b is a variable, it is called a *Volterra integral equation.*

36.3 CONVERSION OF A LINEAR DIFFERENTIAL EQUATION TO AN INTEGRAL EQUATION AND VICE VERSA

To make this transformation, the use of the following formula is necessary :

$$\int_a^x \int_a^x f(x)\, dx\, dx = \int_a^x (x-t)\, f(t)\, dt \qquad ...(I)$$

In general,
$$\int_a^x \int_a^x ... \int_a^x f(x)\, dx^n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt \qquad ...(II)$$

Proof. Let $$I_n = \int_a^x (x-t)^{n-1} f(t)\, dt \quad ...(1)$$

where n is a positive integer and a is a constant.

Differentiating both sides w.r.t. x, using Leibnitz's rule (p. 139)

$$\frac{dI_n}{dx} = \int_a^x \frac{\partial}{\partial x}(x-t)^{n-1} f(t)\, dt + [(x-t)^{n-1} f(t)]_{t=x} \cdot 1 - [(x-t)^{n-1} f(t)]_{t=a} \cdot 0$$

$$= \int_a^x (n-1)(x-t)^{n-2} f(t)\, dt = (n-1) I_{n-1}(x) \quad ...(2)$$

Again differentiating (2) w.r.t. x

$$\frac{d^2 I_n}{dx^2} = (n-1)\frac{d}{dx}[I_{n-1}, (x)] = (n-1)(n-2) I_{n-2}, \text{ using (1)}$$

Proceeding in this way, we get

$$\frac{d^{n-1} I_n}{dx^{n-1}} = (n-1)(n-2) \ldots 1 \,.\, I_1(x) = (n-1)!\, I_1(x)$$

Now taking $n = 1$ in (1), we get

$$I_1 = \int_a^x f(t)\, dt = \int_a^x f(x_1)\, dx_1 \quad ...(3)$$

Putting $x = a$ in (1), we obtain

$$I_n(a) = 0 \text{ for all } n$$

Taking $n = 2$ in (2), we get $\dfrac{dI_2}{dx} = I_1(x)$

$$\therefore \quad I_2 = \int_a^x I_1(x_2)\, dx_2 \quad [\because \ I_2(a) = 0]$$

$$= \int_a^x \int_a^{x_2} f(x_1)\, dx_1\, dx_2 \quad \text{[Using (3)]} \ ...(4)$$

Putting $n = 3$ in (2), we have $\dfrac{dI_3}{dx} = 2I_2(x)$

$$\therefore \quad I_3 = 2\int_a^x I_2(x)\, dx \quad [\because \ I_3(a) = 0]$$

$$= 2\int_a^x \int_a^{x_3} \int_a^{x_2} f(x_1) dx_1\, dx_2\, dx_3 \quad \text{[Using (4)]}$$

Proceeding in this way, we get

$$I_n = (n-1)! \int_a^x \int_a^{x_n} \cdots \int_a^{x_2} F(x_1)\, dx_1\, dx_2 \ldots dx_n$$

i.e., $$\int_a^x \int_a^{x_n} \cdots \int_a^{x_2} f(x_1)\, dx_1\, dx_2 \ldots dx_n = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\, dt$$

If $x_2, x_3, \ldots x_n$ be the same as x, we get the formula (II) above.

Example 36.1. *Convert the differential equation $y''(x) - 3y'(x) + 2y(x) = 5 \sin x$, $y(0) = 1$, $y'(0) = -2$ into an integral equation.*

Solution. Integrating both sides of the given differential equation, we get

$$[y'(x) - y'(0)] - 3[y(x) - y(0)] + 2\int_0^x y(t)\, dt = 5(1 - \cos x)$$

Since $y'(0) = -2$ and $y(0) = 1$, it becomes

$$y'(x) + 2 - 3y(x) + 3 + 2\int_0^x y(t)\, dt = 5 - 5\cos x$$

i.e., $$y'(x) - 3y(x) + 2\int_0^x y(t)\,dt = -5\cos x$$

Integrating again as before, we have

$$[y(x) - y(0)] - 3\int_0^x y(t)\,dt + 2\int_0^x\int_0^x y(t)\,dt = -5\sin x$$

or $$y(x) - 1 - 3\int_0^x y(t)\,dt + 2\int_0^x (x-t)\,y(t)\,dt = -5\sin x \qquad \text{[Using (I) above]}$$

or $$y(x) + \int_0^x [2(x-t) - 3]\,y(t)\,dt = 1 - 5\sin x$$

which is the desired integral equation.

Example 36.2. *Show that the integral equation*

$$y(x) = \int_0^x (x+t)\,y(t)\,dt + 1 \qquad ...(i)$$

is equivalent to the differential equation

$$y''(x) - 2x\,y'(x) - 3y(x) = 0,\ y(0) = 1,\ y'(0) = 0. \qquad \textit{(Kerala, M. Tech., 2005)}$$

Solution. Differentiating (i) by Leibnitz's rule (p. 139), we have

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x}(x+t)\,y(t)\,dt + (x+x)\,y(x)\frac{d}{dx}(x) - (x+0)\,y(0)\frac{d}{dx}(0)$$

$$= \int_0^x y(t)\,dt + 2xy(x) = \int_0^x y(x)\,dx + 2xy(x) \qquad ...(ii)$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = y(x) + 2[xy'(x) + 1\,.\,y(x)] = 2xy'(x) + 3y(x)$$

or $$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 3y(x) = 0 \qquad ...(iii)$$

Putting $x = 0$ in (i), we obtain

$$y(0) = \int_0^0 (x+t)\,y(t)\,dt + 1 \quad i.e.,\quad y(0) = 1$$

and putting $x = 0$ in (ii), we get $y'(0) = 0$.

Hence (i) is equivalent to (iii) with initial conditions $y(0) = 1$, $y'(0) = 0$.

Example 36.3. *Show that the integral equation*

$$y(x) = \int_0^x t(t-x)\,y(t)\,dt + \frac{1}{2}x^2, \qquad ...(i)$$

is equivalent to the differential equation

$$\frac{d^2y}{dx^2} + xy = 1 \text{ and the conditions } y(0) = y'(0) = 0.$$

Solution. Differentiating (i) w.r.t. x by Leibnitz's rule (p. 139), we have

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x}[t(t-x)]\,y(t)\,dt + [t(t-x)\,y(t)]_{t=x}\,.\,1 + x - [t(t-x)\,y(t)]_{t=0}\,.\,0$$

$$= \int_0^x t(-1)\,y(t)\,dt + x = x - \int_0^x ty(t)\,dt \qquad ...(ii)$$

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 1 - \left\{\int_0^x \frac{\partial}{\partial x}[t\,y(t)]\,dt - [t\,y(t)]_{t=0}\cdot 0 + [ty(t)]_{t=x}\cdot 1\right\} = 1 - xy(x)$$

or $$\frac{d^2y}{dx^2} + xy = 1 \text{ which is the differential equation corresponding to } (i).$$

Also $y(0) = 0$ and $y'(0) = 0$.

Example 36.4. *Find the integral equation corresponding to the boundary value problem*

$$y''(x) + \lambda y(x) = 0, \qquad y(0) = y(1) = 0.$$

Solution. Integrating both sides of the given differential equation w.r.t. x over $(0, x)$, we get

$$y'(x) - y'(0) + \lambda \int_0^x y(x)\, dx = 0$$

or $$y'(x) = c - \lambda \int_0^x y(x)\, dx, \text{ taking } y'(0) = c \qquad ...(i)$$

Again integrating (i) w.r.t. x in $(0, x)$, we obtain

$$y(x) - y(0) = cx - \lambda \int_0^x \int_0^x y(x)\, dx$$

$$y(x) = cx - \lambda \int_0^x (x - t)\, y(t)\, dt \qquad ...(ii)$$

[Using (I) of § 36.3 and noting that $y(0) = 0$]

Putting $x = 1$ in (ii), we get

$$y(1) = c - \lambda \int_0^1 (1 - t)\, y(t)\, dt \qquad [\because \ y(1) = 0]$$

$$\therefore \qquad c = \lambda \int_0^1 (1 - t)\, y(t)\, dt \qquad ...(iii)$$

Substituting the value of c from (iii) in (ii), we get

$$y(x) = \lambda x \int_0^1 (1 - t)\, y(t)\, dt - \lambda \int_0^x (x - t)\, y(t)\, dt$$

$$= \lambda x \left\{ \int_0^x (1 - t)\, y(t)\, dt + \int_x^1 (1 - t)\, y(t)\, dt \right\} - \lambda \int_0^x (x - t)\, y(t)\, dt$$

$$= \lambda \int_0^x t(1 - x)\, y(t)\, dt + \lambda \int_x^1 (1 - t)\, y(t)\, dt$$

$$= \lambda \left\{ \int_0^x K(x, t)\, y(t)\, dt + \int_x^1 K(x, t)\, y(t)\, dt \right\}$$

where $$K(x, t) = \begin{cases} t(1 - x) & \text{when } t < x \\ x(1 - t) & \text{when } t > x \end{cases}$$

Hence $$y(x) = \lambda \int_0^1 K(x, t)\, y(t)\, dt$$

which is a *Fredholm integral equation* with a symmetric kernel $K(x, t)$.

PROBLEMS 36.1

Transform each of the following boundary value problems into corresponding integral equations :

1. $y'' + xy' + y = 0$, given that $y(0) = 1$, $y'(0) = 0$. *(Madras, M.E., 2000)*
2. $y'' - \sin xy' + e^y = x$, given that $y = 1$, $\frac{dy}{dx} = -1$ when $x = 0$.
3. $y'' + xy' = 1$, given that $y(0) = y'(0) = 0$. *(Madras, 2000 S)*
4. $y'' + (1 - x)\, y' + e^{-x} y = x^3 - 5x$, given that $y = -3$, $\frac{dy}{dx} = 4$ when $x = 0$.
5. $\frac{d^3y}{dx^3} + y = \cos x$ given that $y = 0$, $y' = 1$, $y'' = 2$ at $x = 0$. *(Kerala, M. Tech, 2005)*

6. $\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}-xy=\sin x$ given that $y=1, y'=-1, y''=\dfrac{1}{2}$ at $x=0$.

7. $\dfrac{d^4y}{dx^4}-4\dfrac{d^3y}{dx^3}+6\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+y=4\cos 2x$ given that
$y=-1, y'=4, y''=0, y'''=2$ when $x=0$.

Convert each of the following integral equations into differential equations alongwith initial conditions :

8. $y(x)=\int_0^x (x+t)\,y(t)\,dt-1.$

9. $y(x)=\int_0^x (x-t)\,y(t)\,dt+3\sin x.$

10. $y(x)+3\int_0^x (x-t)^2\,y(t)\,dt=x^2-3x+4.$

11. $y(x)+\int_0^x [(x-t)^2+4(x-t)-3]\,y(t)\,dt=e^{-x}.$

12. If $y''(x)=f(x)$; $y(0)=y(l)=0$, show that $y(x)=\int_0^l K(x,t)\,f(t)\,dt,$

where $K(x,t)=\begin{cases}\dfrac{t}{l}(x-l) & \text{when } t<x\\[2mm] \dfrac{x}{l}(t-l) & \text{when } t>x\end{cases}$

36.4 CONVERSION OF BOUNDARY VALUE PROBLEMS TO INTEGRAL EQUATIONS USING GREEN'S FUNCTION

Consider the linear homogeneous differential equation

$$L(y)+\phi(x)=0 \qquad \text{...(1)}$$

where $$L(y)=\left[\frac{d}{dx}\left(p\frac{d}{dx}\right)+q\right]y=py''+p'y'+qy \qquad \text{...(2)}$$

together with the homogeneous boundary conditions of the form

$$\alpha y+\beta\frac{dy}{dx}=0 \qquad \text{...(3)}$$

Now let us find a function $G(x,t)$ which for a given number t, is given by $G_1(x)$ when $x<t$ and by $G_2(x)$ when $x>t$ and which has the following properties :

I. G_1 and G_2 satisfy the equation $L(G)=0$ in their defined intervals *i.e.*, $L(G_1)=0$ when $x<t$; $L(G_2)=0$ when $x>t$.

II. G_1 and G_2 satisfy the boundary conditions at the end points $x=a$ and $x=b$ respectively.

III. $G(x,t)$ is continuous at $x=t$ *i.e.*, $G_1(t)=G_2(t)$.

IV. The derivative of G is continuous at every point within the range of x except at $x=t$ *i.e.*, $G_2'(t)-G_1'(t)=-1/p(t)$

Def. *$G(x,t)$ as defined above is called the* **Green's function.**

If $G(x,t)$ exists, then the solution of the given problem can be transformed to the integral equation

$$y(x)=\int_a^b G(x,t)\,\phi(t)\,dt \qquad \text{...(4)}$$

Let $y=y_1(x)$ and $y=y_2(x)$ be the non-trivial solutions of the equations $L(y)=0$ which satisfy the homogeneous conditions at $x=a$ and $x=b$ respectively.

The above conditions I and II are satisfied if we write

$$G=\begin{cases}C_1y_1(x), & \text{when } x<t\\ C_2y_2(x), & \text{when } x>t\end{cases} \qquad \text{...(5)}$$

Imposing the condition III on (5), we get

$$C_2y_2(t)-C_1y_1(t)=0 \qquad \text{...(6)}$$

Imposing the condition IV on (5), we have

$$C_2y_2'(t)-C_1y_1'(t)=-1/p(t) \qquad \text{...(7)}$$

Equations (6) and (7) give a unique solution, if

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t) \neq 0$$

By Abel's formula*, we find that

$$y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t) = C/p(t) \qquad ...(8)$$

Now $[(7) \times y_2(t) - (6) \times y_1(t)]$ gives on using (8),

$$C_1 = -\frac{1}{C}\, y_2(t) \text{ and } C_2 = -\frac{1}{C}\, y_1(t)$$

Thus (5) reduces to $$G(x, t) = \begin{cases} -\dfrac{1}{C}\, y_1(x)\, y_2(t), & x < t \\ -\dfrac{1}{C}\, y_1(t)\, y_2(x), & x > t \end{cases} \qquad ...(9)$$

Conversely it can be shown that the integral equation.

$$y(x) = \int_a^b G(x, t)\, \phi(t)\, dt$$

where $G(x, t)$ is as defined by (9), satisfies the differential equation $L(y) + \phi(x) = 0$ together with the prescribed boundary conditions.

Example 36.5. *Transform the differential equation $y'' + y = x$, $y(0) = 1$, $y'(1) = 0$ to a Fredholm integral equation, finding the corresponding Green's function.* (Madras M.E., 2000 S)

Solution. Given equation is $\quad L(y) + \phi(x) = 0 \qquad ...(i)$

with the conditions $y(0) = 0$, $y'(1) = 0$, where $L(y) = y''$ and $\phi(x) = y - x$

The associated equation $\quad L(y) = 0$ is $y''(y) = 0 \qquad ...(ii)$

Its solution is $\quad y = C_1 x + C_2 \qquad ...(iii)$

Now $y_1(x)$ is a particular solution of (*ii*) satisfying the condition $y(0) = 0$.

$\therefore$ Taking $C_2 = 0$ and $C_1 = 1$, we get $y_1(x) = x \qquad ...(iv)$

$y_2(x)$ is another solution of (*ii*) satisfying the condition $y'(1) = 0$.

$\therefore$ From (*iii*), $y'(1) = C_1 = 0$. Then $y(x) = C_2$

Taking $C_2 = 1$, a particular solution is $y_2(x) = 1$.

The constant C is found from $y_1(x)\, y_2'(x) - y_2(x) y_1'(x) = \dfrac{C}{p(x)}$

Since $L(y) = py'' + p'y' + qy = y''$ (given), $\quad \therefore \quad p = 1$.

Thus $C = y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x) = x \,.\, 0 - 1.1 = -1$.

$\therefore$ Green's function is given by

$$G(x, t) = \begin{cases} \dfrac{-y_1(x)\, y_2(t)}{C}, & x < t \\ \dfrac{-y_1(t)\, y_2(x)}{C}, & x > t \end{cases} = \begin{cases} x, & x < t \\ t, & x > t \end{cases} \qquad ...(v)$$

Hence the equation (*i*) is equivalent to the integral equation

$$y(x) = \int_0^1 G(x, t)\, \phi(t)\, dt = \int_0^1 G(x, t) \,.\, (y - t)\, dt$$

$$= \int_0^1 G(x, t)\, y(t)\, dt - \left\{ \int_0^x x \,.\, t\, dt + \int_x^1 t \,.\, t\, dt \right\}$$

---

* The conditions that $y_1(x)$ and $y_2(x)$ satisfy the equation

$L(y) = 0$ are $(p\, y_1')' + q y_1 = 0 \qquad ...(i), \qquad (p y_2')' + q y_2 = 0 \qquad ...(ii)$

$[(i) \times y_2 - (ii) \times y_1]$ gives $y_2 (p y_1')' - y_1 (p y'_2)' = 0$

or $\qquad [p(y_1 y_2' - y_2 y_1')]' = 0 \quad$ or $\quad y_1 y_2' - y_2 y_1' = C/p$

$$= \int_0^1 G(x,t)\, y(t)\, dt - \left\{ x \left| \frac{t^2}{2} \right|_0^x + \left| \frac{t^3}{3} \right|_x^1 \right\}$$

$$= \int_0^1 G(x,t)\, y(t)\, dt - \frac{1}{6}(x^3 + 2)$$

where $G(x, t)$ is given by (v).

Example 36.6. *Find the Green's function for the boundary value problem*

$$d^2y/dx^2 + \mu^2 x = 0, \qquad y(0) = 0 = y(1).$$

Solution. We observe that the solution $y_1 = \sin \mu x$ satisfies the boundary condition $y(0) = 0$ and the solution $y_2 = \sin \mu(x - 1)$ satisfies the second condition $y(1) = 0$. Also both these solutions are linearly independent.

The constant C is found from $y_1 y_2' - y_2 y_1' = C/p\,(x)$.

Since $L(y) = py'' + p'y' + qy = y''$ (given), $\quad \therefore \; p = 1$

$\therefore \quad C = y_1 y_2' - y_2 y_1' = \mu \sin \mu x \cos \mu(x-1) - \mu \sin \mu(x-1) \cos \mu x = \mu \sin \mu$

Hence the Green's function is given by

$$G(x,t) = \begin{cases} -\dfrac{y_1(x)\, y_2(t)}{C}, & x < t \\ -\dfrac{y_1(t)\, y_2(x)}{C}, & x > t \end{cases} = \begin{cases} -\dfrac{\sin \mu x \sin \mu(t-1)}{\mu \sin \mu}, & x < t \\ -\dfrac{\sin \mu t \sin \mu(x-1)}{\mu \sin \mu}, & x > t \end{cases}$$

PROBLEMS 36.2

1. Transform the problem $d^2y/dx^2 + xy = 1$; $y(0) = 0 = y(1)$ to an integral equation, finding the corresponding Green's function.
2. Transform the problem $y'' + y = x$; $y(0) = 1$, $y'(1) = 0$ to an integral equation using Green's function.
3. Construct an integral equation corresponding to the boundary value problem

$$\frac{d^2u}{dx^2} + e^u u = x \; ; \; u(0) = 0, \; u(1) = 1.$$

4. Find the Green's function for the boundary value problem $d^2y/dx^2 - y = 0$ with $y(0) = y(1) = 0$.
5. Transform the boundary value problem $x^2 \dfrac{d^2u}{dx^2} + x \dfrac{du}{dx} + (\lambda x^2 - 1)\, u = 0$; $u(0) = u(1) = 0$, to an integral equation.

[**Hint.** $u_1(x) = x$, $u_2(x) = \dfrac{1}{x} - x$ and $C = -2$]

36.5 SOLUTION OF AN INTEGRAL EQUATION

The solution of the integral equation $y(x) = F(x) + \lambda \int_a^b K(x,t)\; y(t)\, dt$ *is a function* $y(x)$ *which when substituted in the equation reduces it to an identity w.r.t.* x.

Example 36.7. *Show that* $y(x) = 2 - x$ *is a solution of the integral equation*

$$\int_0^x e^{x-t}\, y(t)\, dt = e^x + x - 1. \qquad \ldots(i)$$

Solution. Since $\qquad y(t) = 2 - t$

$$\therefore \quad \int_0^x e^{x-t}\; y(t)\, dt = \int_0^x e^{x-t}\; (2 - t)\, dt$$

$$= 2e^x \int_0^x e^{-t}\, dt - e^x \int_0^x t\, e^{-t}\, dt$$

$$= 2e^x \Big| -e^{-t} \Big|_0^x - e^x \left\{ \Big| t.(-e^{-t}) \Big|_0^x - \int_0^x 1.(-e^{-t})\, dt \right\}$$

$$= 2e^x(-e^{-x}+1) + e^x(xe^{-x}) - e^x \Big| -e^{-t} \Big|_0^x$$

$$= -2 + 2e^x + x + e^x(e^{-x}-1) = e^x + x - 1.$$

Thus (i) is identically satisfied by $y(x) = 2 - x$. Hence $y(x) = 2 - x$ is a solution of (i).

Example 36.8. *Show that the function $y(x) = (1+x^2)^{-3/2}$ is a solution of the Volterra integral equation :*

$$y(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2}\, y(t)\, dt. \qquad ...(i)$$

Solution. Substituting $y(x) = (1+x^2)^{-3/2}$ in the R.H.S. of (i), we have

$$\frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} \cdot \frac{1}{(1+t^2)^{3/2}}\, dt$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2} \left\{ \frac{1}{(1+t^2)^{1/2}} \right\}_0^x$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2} \cdot \frac{1}{(1+x^2)^{1/2}} - \frac{1}{1+x^2} = \frac{1}{(1+x^2)^{3/2}} = y(x)$$

Thus $y(x) = (1+x^2)^{-3/2}$ is a solution of the integral equation (i).

Example 36.9. *Show that the function $y(x) = xe^x$ is a solution of the integral equation*

$$y(x) = \sin x + 2\int_0^x \cos(x-t)\, y(t)\, dt. \qquad ...(i)$$

Solution. Substituting $y(x) = xe^x$ in the R.H.S. of (i), we have

$$\sin x + 2\int_0^x \cos(x-t) . te^t\, dt$$

$$= \sin x + 2\left\{ \cos x \int_0^x t . e^t \cos t\, dt + \sin x \int_0^x te^t \sin t\, dt \right\} \qquad \text{(Integrating by parts)}$$

$$= \sin x + 2\cos x \left\{ \frac{1}{2} \Big| t . e^t(\cos t + \sin t) \Big|_0^x - \frac{1}{2}\int_0^x e^t(\cos t + \sin t)\, dt \right\}$$

$$+ 2\sin x \left\{ \frac{1}{2} \Big| t . e^t(\sin t - \cos t) \Big|_0^x - \frac{1}{2}\int_0^x e^t(\sin t - \cos t)\, dt \right\}$$

$$= \sin x + xe^x(\cos^2 x + \cos x \sin x + \sin^2 x - \sin x \cos x)$$

$$- \cos x \int_0^x e^t(\cos t + \sin t)\, dt - \sin x \int_0^x t(\sin t - \cos t)\, dt$$

$$= \sin x + xe^x - \cos x \Big| e^t \sin t \Big|_0^x + \sin x \Big| e^t \cos t \Big|_0^x$$

$$= \sin x + xe^x - e^x \cos x \sin x + e^t \sin x \cos x - \sin x = xe^x$$

Thus $y(x) = xe^x$ is a solution of the integral equation (i).

36.6 INTEGRAL EQUATIONS OF THE CONVOLUTION TYPE

$$y(x) = F(x) + \int_0^x K(x-t)\, y(t)\, dt$$

is an *integral equation of convolution type* and can be written as

$$y(x) = F(x) + K(x) * y(x)$$ [See p. 748]

It is a special integral equation of importance in applications.

Taking Laplace transform of both sides, assuming that $L\,F(x) = f(s)$ and $L[K(x)] = k(s)$ both exist, and using convolution theorem

$$\bar{y}(s) = f(s) + k(s)\,.\,y(s) \quad \text{or} \quad \bar{y}(s) = \frac{f(s)}{1 - k(s)}$$

Now taking the inverse transform of both sides, we get the required solution.

Example 36.10. *Solve the integral equation*

$$y(x) = 3x^2 + \int_0^x y(t) \sin(x - t)\, dt.$$

Solution. Given integral equation can be written as

$$y(x) = 3x^2 + y(x) * \sin(x - t) \qquad ...(i)$$

Taking Laplace transform of both sides and using the convolution theorem (p. 748), we get

$$\bar{y} = \frac{6}{s^3} + y \cdot \frac{s}{s^2 + 1} \quad \text{or} \quad \bar{y} = \frac{6(s^2 + 1)}{s^5} = 6\left(\frac{1}{s^3} + \frac{1}{s^5}\right)$$

On inversion, we get $\quad y = 6\left(\dfrac{x^2}{2!} + \dfrac{x^4}{4!}\right) = 3x^2 + x^4/4$

which is the required solution of (*i*).

Obs. The above solution can also be verified by direct substitution in the given integral equation.

Example 36.11. *Solve* $y(x) = x + 2\int_0^x \cos(x - t)\, y(t)\, dt.$

Solution. The given equation can be written as

$$y(x) = x + 2\cos(x) * y(x)$$

Taking Laplace transform of both sides and using convolution theorem, we get

$$\bar{y}(s) = \frac{1}{s^2} + 2\frac{s}{s^2 + 1} \cdot \bar{y}(s) \quad \text{or} \quad \bar{y}\left(1 - \frac{2s}{s^2 + 1}\right) = \frac{1}{s^2}$$

or

$$\bar{y} = \frac{s^2 + 1}{s^2(s - 1)^2} = \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s - 1} + \frac{2}{(s - 1)^2}$$

On inversion, we obtain $\quad y = 2 + x - 2e^x + 2xe^x$

Hence $y = x + 2 + 2(x - 1)e^x$ is the desired solution.

Example 36.12. *Solve the integral equation* $\int_0^x y(t)\, y(x - t)\, dt = 4 \sin 9x.$

Solution. The given integral equation can be written as

$$y(x) * y(x) = 4 \sin 9x \qquad ...(i)$$

Taking Laplace transform of both sides and using the convolution theorem, we get

$$\{\bar{y}(s)\}^2 = \frac{36}{s^2 + 81} \quad \text{or} \quad \bar{y} = \pm\frac{6}{\sqrt{s^2 + 81}}$$

On inversion and noting that $L^{-1}\dfrac{1}{\sqrt{(s^2 + a^2)}} = J_0(ax)$, we get

$$y = \pm 6\, L^{-1}\left\{\frac{1}{\sqrt{(s^2 + 9^2)}}\right\} = \pm 6\, J_0(9x)$$

Thus $y = 6J_0(9x)$ and $y = -6\, J_0(9x)$ are both solutions of (*i*).

36.7 ABEL'S INTEGRAL EQUATION

The integral equation $\int_0^x \frac{y(t)}{(x-t)^\alpha}\, dx = G(x)$

such that $G(x)$ is given and α is a constant $(0 < \alpha < 1)$, is known as *Abel's integral equation*. This is an important integral equation of the convolution type. An application of this equation is in finding the shape of a smooth wire lying in a vertical plane such that a bead placed anywhere on the wire slides to the lowest point in the same time. This is the well known *tautochrone problem* and the shape of the wire is a *cycloid.*

Example 36.13. *Solve the Abel's integral equation* $\int_0^x \frac{y(t)}{\sqrt{(x-t)}}\, dt = 1 + 2x - x^2$.

Solution. The given equation can be written as

$$y(x) * x^{-1/2} = 1 + 2x - x^2$$

Taking the Laplace transform of both sides and using convolution theorem, we get

$$\bar{y} \, . \, L(x^{-1/2}) = L(1 + 2x - x^2)$$

or $$\bar{y}\frac{\Gamma(1/2)}{s^{1/2}} = \frac{1}{s} + \frac{2}{s^2} - \frac{2}{s^3} \quad \text{or} \quad \bar{y} = \frac{1}{\Gamma(1/2)}\left(\frac{1}{s^{1/2}} + 2\cdot\frac{1}{s^{3/2}} - 2\frac{1}{s^{5/2}}\right)$$

On inversion, and noting that $L^{-1}\frac{1}{s^{n+1}} = \frac{x^n}{\Gamma(n+1)}$, we have

$$y = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{x^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} + 2.\frac{x^{1/2}}{\Gamma\left(\frac{3}{2}\right)} - 2\frac{x^{3/2}}{\Gamma\left(\frac{5}{2}\right)}$$

$$= \frac{1}{\pi}\left(x^{-1/2} + 4x^{1/2} - \frac{8}{3}x^{3/2}\right)$$

Hence $$y = \frac{1}{3\pi\sqrt{x}}(3 + 12x - 8x^3) \text{ is the desired solution.}$$

36.8 INTEGRO-DIFFERENTIAL EQUATIONS

An integral equation in which various derivatives of the unknown function $y(x)$ are also present, is called an *integro-differential equation*. An example of such an equation is

$$y'(x) = y(x) - \cos x + \int_0^x \sin(x-t)\, y(t)\, dt$$

The solution of integro-differential equation subject to given initial conditions can also be found by Laplace transforms as illustrated below :

Example 36.14. *Solve* $\frac{dy}{dx} + 3y + 2\int_0^x y\, dx = x$, *given* $y(0) = 1$.

Solution. Given equation can be written as $y'(x) + 3y(x) + 2\int_0^x y\, dx = x$

Taking Laplace transform of both sides, we get

$$L[y'(x)] + 3\,L[y(x)] + 2L\left\{\int_0^x y(x)\, dx\right\} = L(x)$$

or $$\{s\,\bar{y}(s) - y(0)\} + 3\,\bar{y}(s) + 2\frac{1}{s}\bar{y}(s) = \frac{1}{s^2}$$ [Using § 21.6]

$$s\bar{y} + 3\bar{y} + 2\frac{1}{s}\bar{y} = 1 + \frac{1}{s^2}$$ [$\because\ y(0) = 1$]

or $$\bar{y} = \frac{1+s^2}{s(s^2+3s+2)} = \frac{1+s^2}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{2}{s+1} + \frac{5}{2(s+2)}$$

On inversion, we obtain $y = \frac{1}{2} L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s+1}\right) + \frac{5}{2} L^{-1}\left(\frac{1}{s+2}\right)$

Hence $y = \frac{1}{2} - 2e^{-x} + \frac{5}{2} e^{-2x}$ is the required solution.

Example 36.15. *Solve* $\frac{dy}{dx} = 3 \int_0^x \cos 2(x-t)\, y(t)\, dt + 2$ *given* $y(0) = 1$.

Solution. Given equation can be written as
$$y'(x) = 3 \cos 2x * y(x) + 2$$
Taking Laplace transform of both sides, we get
$$L[(y'(x)] = 3L(\cos 2x)\,.\, \bar{y}(s) + \frac{2}{s}$$
$$s\,\bar{y}(s) - y(0) = \frac{3s\bar{y}(s)}{s^2+4} + \frac{2}{s} \quad \text{or} \quad \bar{y} = \frac{(s+2)(s^4+4)}{s^2(s^2+1)} \qquad [\because \; y(0) = 1]$$
$$= \frac{4}{s} + \frac{8}{s^2} - 3\frac{s}{s^2+1} - 6\frac{1}{s^2+1}$$

On inversion, we obtain $y = 4 + 8x - 3\cos x - 6 \sin x$
which is the required solution.

Obs. The given integro-differential equation can be converted into the following integral equation by integrating it from 0 to x and using $y(0) = 1$.
$$y(x) = 2x + 1 + 3\int_0^x (x-t)\cos 2(x-t)\, y(t)\, dt.$$

PROBLEMS 36.3

1. Show that $y(x) = 1 - x$ is a solution of the integral equation $\int_0^x e^{x-t}\, y(t)\, dt = x$.
2. Show that $y(x) = 1$ is a solution of the Fredholm integral equation
$$y(x) + \int_0^1 x(e^{tx} - 1)\, y(t)\, dt = e^x - x.$$
3. Show that $y(x) = \frac{1}{\pi\sqrt{x}}$ is a solution of the integral equation $\int_0^x \frac{y(t)}{\sqrt{(x-t)}}\, dt = 1$.
4. Show that $y(x) = e^x(2x - 2/3)$ is a solution of the Fredholm integral equation
$$y(x) + \int_0^1 e^{x-t}\, y(t)\, dt = 2xe^x.$$

Solve each of the following integral equations :

5. $y(x) = x + \frac{1}{6}\int_0^x (x-t)^3\, y(t)\, dt$.
6. $y(x) = x^2 + \int_0^x y(t) \sin(x-t)\, dt$.
7. $\int_0^x y(t)\, y(x-t)\, dt = 2y(x) + x - 2$.
8. $\int_0^x y(t)\, y(x-t)\, dt = 9 \sin 4x$.
9. Find a solution of the integral equation $y(x) = \frac{1}{2}\sin 2x + \int_0^x y(t)\, y(x-t)\, dt$.

Solve the following integral equations :

10. $\frac{dy}{dx} + 4y + 5\int_0^x y(t)\, dt = e^{-x}$, $y(0) = 0$.
11. $\frac{dy}{dx} + 2y + \int_0^x y(t)\, dt = \sin x$, $y(0) = 1$. *(Mumbai, 2006)*

12. $y'(x) = x + \int_0^x y(x-t)\cos t\, dt,\ y(0) = 1.$

13. $y'(x) = \int_0^x y(t)\cos(x-t)\, dt,\ y(0) = 1.$

14. $\int_0^x \frac{y(t)}{\sqrt{(x-t)}}\, dt = \sqrt{x}.$

15. $\int_0^x \frac{y(t)}{(x-t)^{1/3}}\, dt = x(x+1).$

36.9 INTEGRAL EQUATIONS WITH SEPARABLE KERNELS

A kernel $K(x, t)$ of Fredholm integral equation is said to be *separable* (or *degenerate*) if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of x alone and a function of t alone *i.e.*, if it is of the form $K(x, t) = \sum_{n=1}^{m} f_n(x)\, g_n(t)$.

Also since $\cos(x + t) = \cos x \cos t - \sin x \sin t$, $\cos(x + t)$ is a separable kernel.

36.10 SOLUTION OF FREDHOLM EQUATIONS WITH SEPARABLE KERNELS

Consider the integral equation

$$y(x) = \lambda \int_a^b K(x, t) \cdot y(t)\, dt + F(x) \qquad \text{...(1)}$$

where

$$K(x, t) = f_1(x) \cdot g_1(t) + f_2(x) \cdot g_2(t) + \dots + f_m(x) \cdot g_m(t) \qquad \text{...(2)}$$

Substituting (2) in (1), we get

$$y(x) = \lambda \int_a^b \left[\sum_{n=1}^{m} f_n(x) \cdot g_n(t) \right] y(t)\, dt + F(x)$$

$$= \lambda \sum_{n=1}^{m} f_n(x) \left\{ \int_a^b g_n(t)\, y(t)\, dt \right\} + F(x) \qquad \text{...(3)}$$

Evidently $\int_a^b g_n(t)\, y(t)\, dt = C_n$ (say), is a constant and will be different for different values of n. Then (3) takes the form

$$y(x) = \lambda \sum_{n=1}^{m} C_n\, f_n(x) + F(x) \qquad \text{...(4)}$$

This is a solution of (1) in which m constants $C_1, C_2, \dots, C_m$ are to be determined.

Now multiplying both sides of (4) by $g_k(x)$ and integrating from a to b, we obtain

$$\int_a^b y(x)\, g_k(x)\, dx = \lambda \int_a^b \sum_{n=1}^{m} C_n\, f_n(x)\, g_k(x)\, dx + \int_a^b F(x)\, g_k(x)\, dx$$

Writing $\int_a^b g_k(x)\, f_n(x)\, dx = \alpha_{kn}$ and $\int_a^b g_k(x)\, F(x)\, dx = \beta_k$, the above equation becomes

$$C_k = \lambda \sum_{n=1}^{m} C_n \alpha_{kn} + \beta_k$$

$$= \lambda(C_1\alpha_{k1} + C_2\alpha_{k2} + \dots + C_m\alpha_{km}) + \beta_k$$

Taking $k = 1, 2, \dots, m$, we get the following m equations which determine $C_1, C_2, \dots C_m$:

$$\left.\begin{aligned} (1 - \lambda\alpha_{11})\, C_1 - \lambda\alpha_{12}\, C_2 - \dots - \lambda\, \alpha_{1m} C_m &= \beta_1 \\ -\lambda a_{21}\, C_1 + (1 - \lambda a_{22})\, C_2 - \dots - \lambda\, a_{2m}\, C_m &= \beta_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ -\lambda\alpha_{m1} C_1 - \lambda\alpha_{m2}\, C_2 - \dots + (1 - \lambda\alpha_{mm})\, C_m &= \beta_m \end{aligned}\right\} \qquad \text{...(5)}$$

Equations (5) will give a unique solution if the determinant Δ of the coefficients of $C_1, C_2, \dots C_m$ is not zero.

Now the following cases arise :

I. When $F(x) = 0$, (1) is said to be a *homogeneous integral equation* and all β's are zero.

(*i*) *If* $\Delta \neq 0$, the only solution of (4) is the trivial solution $C_1 = C_2 = \ldots = C_m = 0$. Then $y(x) = 0$ is the trivial solution of (1).

(*ii*) *If* $\Delta = 0$, at least one of the C's can be assigned any value and the remaining C's can be found accordingly. Then (4) gives infinitely many solutions. The values of λ for which $\Delta = 0$ are known as the *eigen values*. Any non-trivial solution of the homogeneous integral equation for a certain value of λ is called the corresponding *eigen function*. The solutions corresponding to eigen values of λ can be expressed as arbitrary multiples of eigen functions.

II. When $F(x) \neq 0$. Let us assume that $\int_a^b g_m(x) F(x)\, dx = 0$ so that $\beta_m = 0$

(*i*) *If* $\Delta \neq 0$, the only solution of (*i*) is the trivial solution $C_1 = C_2 = \ldots = C_m = 0$.

Then $y = F(x)$ is the desired solution of (1).

(*ii*) *If* $\Delta = 0$, atleast one of the C's can be given any value and the remaining C's can be found.

Then (4) gives infinitely many solutions.

III. When atleast one of the β's $\neq 0$

(*i*) *If* $\Delta \neq 0$, then equations (5) give a unique solution of the constants C.

Hence there is a unique solution of (1).

(*ii*) *If* $\Delta = 0$, then equations (5) will be inconsistent.

$\therefore$ Either there is no solution or infinitely many solutions of (*i*) exist.

Example 36.16. *Find the eigen values and eigen functions of the following homogeneous integral equations with degenerate kernels :*

(*i*) $y(x) = \lambda \int_0^1 (2xt - 4x^2)\, y(t)\, dt$ (*ii*) $y(x) = \dfrac{1}{e^2 - 1} \int_0^1 e^{x+t}\, y(t)\, dt.$

Solution. (*i*) Given equation may be written as

$$y(x) = \lambda \left\{ 2x \int_0^1 t y(t)\, dt - 4x^2 \int_0^1 y(t)\, dt \right\}$$

or
$$y(x) = (2\lambda x)\, C_1 - (4\lambda x^2)\, C_2 \qquad \ldots(i)$$

where $C_1 = \int_0^1 t y(t)\, dt,\ C_2 = \int_0^1 y(t)\, dt$

Substituting $y(x)$ in C_1, C_2 we get

$$C_1 = \int_0^1 t\{(2\lambda t)\, C_1 - (4\lambda t^2)\, C_2\}\, dt$$

$$C_2 = \int_0^1 \{(2\lambda t)\, C_1 - (4\lambda t^2)\, C_2\}\, dt$$

or
$$C_1 \left\{1 - 2\lambda \int_0^1 t^2 dt\right\} + C_2 \left\{4\lambda \int_0^1 t^3\, dt\right\} = 0$$

$$C_1 \left\{-2\lambda \int_0^1 t\, dt\right\} + C_2 \left\{1 + 4\lambda \int_0^1 t^2\, dt\right\} = 0$$

or
$$C_1 \left(1 - \frac{2\lambda}{3}\right) + \lambda C_2 = 0\ ;\ -\lambda C_1 + C_2\,(1 + 4\lambda/3) = 0 \qquad \ldots(ii)$$

$\therefore$ The determinant of eigen values will be

$$\begin{vmatrix} 1 - 2\lambda/3 & \lambda \\ -\lambda & 1 + 4\lambda/3 \end{vmatrix} = 0 \text{ or } (\lambda + 3)^2 = 0$$

$\therefore$ The eigen values are $\lambda_1 = -3,\ \lambda_2 = -3$

For $\lambda_1 = \lambda_2 = -3$, the equations (*ii*) reduce to

$$3C_1 - 3C_2 = 0\ ;\ 3C_1 - 3C_2 = 0\ \textit{i.e.},\ C_1 = C_2$$

$\therefore$ From (*i*) $y(x) = -6C_1(x - 2x^2) = x - 2x^2$ if $C_1 = -1/6$

Hence the eigen function corresponding to $\lambda_1 = \lambda_2 = -3$, is

$$y(x) = x - 2x^2$$

(*ii*) Given integral equation may be written as

$$y(x) = \frac{e^x}{e^2 - 1} \int_0^1 e^5 e^t \cdot y(t)\, dt = \frac{e^x}{e^2 - 1} C, \quad ...(i)$$

where $C = \int_0^1 e^t\, y(t)\, dt$

Substituting the value of $y(t)$ from (*i*) in C, we get

$$C = \int_0^1 e^t \left(\frac{e^t C}{e^2 - 1}\right) dt = \frac{C}{e^2 - 1} \int_0^1 e^{t^2}\, dt$$

or

$$C\left[1 - \frac{1}{e^2 - 1} \int_0^1 e^{t^2}\, dt\right] = 0 \quad i.e. \quad C = 0$$

Hence from (*i*), $y(x) = 0$

which shows that the given integral equation has only trivial solution.

Thus it does not contain any eigen values or eigen functions.

Example 36.17. *Solve the integral equation*

$$y(x) = \cos x + \lambda \int_0^\pi \sin (x - t)\, y(t)\, dt.$$

Solution. Writing the given equation in the following form :

$$y(x) = \cos x + \lambda \left\{\sin x \int_0^\pi \cos t\; y(t)\, dt - \cos x \int_0^\pi \sin t\; y(t)\, dt\right\}$$

or

$$y(x) = \cos x + (\lambda \sin x)\, C_1 - (\lambda \cos x)\, C_2 \quad ...(i)$$

where $C_1 = \int_0^\pi \cos t \cdot y(t)\, dt$, $\quad C_2 = \int_0^\pi \sin t \cdot y(t)\, dt$

Substituting $y(x)$ in C_1 and C_2, we get

$$C_1 = \int_0^\pi \cos t\, \{\cos t + (\lambda \sin t)\, C_1 - (\lambda \cos t)\, C_2\}\, dt$$

$$C_2 = \int_0^\pi \sin t\, \{\cos t + (\lambda \sin t)\, C_1 - (\lambda \cos t)\, C_2\}\, dt$$

or

$$\left.\begin{aligned} C_1 \left\{1 - \lambda \int_0^\pi \cos t \sin t\, dt\right\} + C_2 \left(\lambda \int_0^\pi \cos^2 t\, dt\right) &= \int_0^\pi \cos^2 t\, dt \\ C_1 \left\{-\lambda \int_0^\pi \sin^2 t\, dt\right\} + C_2 \left\{1 + \lambda \int_0^\pi \sin t \cos t\, dt\right\} &= \int_0^\pi \sin t \cos t\, dt \end{aligned}\right\} \quad ...(ii)$$

By evaluating each of the integrals in (*ii*), we get

$$C_1 + \frac{1}{2} C_2\, \lambda\pi = \frac{1}{2}\pi\, ; -\frac{1}{2} C_1\, \lambda\pi + C_2 = 0 \quad ...(iii)$$

The determinant of the equations (*iii*) is given by

$$\begin{vmatrix} 1 & \frac{1}{2}\lambda\pi \\ -\frac{1}{2}\lambda\pi & 1 \end{vmatrix} = 1 + \frac{1}{4}\lambda^2\pi^2 \neq 0$$

Thus the equations (*iii*) have a unique solution

$$C_1 = \frac{2\pi}{4 + \lambda^2\pi^2}\, ; C_2 = \frac{\lambda\pi^2}{4 + \lambda^2\pi^2}$$

Substituting these values of C_1 and C_2 in (*i*), we obtain the required solution

$$y(x) = \cos x + \frac{\lambda}{4 + \lambda^2\pi^2}(2\pi \sin x - \lambda\pi^2 \cos x)$$

or

$$y(x) = \frac{2}{4 + \lambda^2\pi^2}(2 \cos x + \pi\lambda \sin x).$$

PROBLEMS 36.4

Determine the eigen values and eigen functions for the following homogeneous integral equations with degenerate kernels :

1. $y(x) = \lambda \int_0^1 (3x - 2)\ t \,.\, y(t)\, dt.$
2. $y(x) = \lambda \int_{-1}^1 (5x\, t^3 + 4x^2\, t + 3xt)\, y(t)\, dt.$
3. $y(x) = \lambda \int_0^{\pi/4} \sin^2 x \cdot y(t)\, dt.$
4. $y(x) - \lambda \int_0^{2\pi} \sin x \, \sin t\, y(t)\, dt = 0.$ (*Madras M.E., 2000 S*)
5. $y(x) = \lambda \int_0^{\pi} \sin x \cos t \,.\, y(t)\, dt.$
6. $y(x) = \lambda \int_0^{2\pi} \sin(x + t) \,.\, y(t)\, dt.$ (*Madras M.E., 2000*)

Solve the following integral equations :

7. $y(x) = x + \lambda \int_0^1 (x - t)\, y(t)\, dt.$
8. $y(x) = x + \lambda \int_0^1 (1 + x + t)\, y(t)\, dt.$
9. $y(x) = x + \lambda \int_0^{\pi} (1 + \sin x \sin t)\, y(t)\, dt.$
10. $y(x) = (2x - \pi) + 4 \int_0^{\pi/2} \sin^2 x \cdot y(t)\, dt.$
11. $y(x) = \sin x + \lambda \int_0^{\pi/2} \sin x \cos t \cdot y(t)\, dt.$
12. $y(x) = x + \lambda \int_{-\pi}^{\pi} (x \cos t + t^2 \sin x + \cos x \sin t)\, y(t)\, dt.$
13. Obtain the solution of $y(x) = 1 + \lambda \int_0^1 xt \cdot y(t)\ dt$ in the form $y(x) = 1 + \frac{3\lambda x}{2(3 - \lambda)}\ (\lambda \neq 3).$

    What happens when $\lambda = 3$?
14. For the integral equation

$$y(x) = F(x) + \lambda \int_0^1 (1 - 3xt)\ y(t)\, dt,$$

    find the eigen values of λ and the corresponding eigen functions.
15. Obtain the most general solution of the equation

$$y(x) = F(x) + \lambda \int_0^{2\pi} \sin(x + t)\ y(t)\, dt$$

    where (*i*) $F(x) = x$ (*ii*) $F(x) = 1$, under the assumption that $\lambda \neq \pm 1/\pi$.

36.11 SOLUTION OF FREDHOLM INTEGRAL EQUATION BY THE METHOD OF SUCCESSIVE APPROXIMATIONS

Consider the Fredholm equation $y(x) = F(x) + \lambda \int_a^b K(x, t)\, y(t)\, dt$ ...(1)

where $F(x)$ is continuous in $a \leq x \leq b$ and $K(x, t)$ is finite and continuous in the rectangle $a \leq x \leq b$ and $a \leq t \leq b$.

Replacing y under the integral sign by an initial approximation $y(0)$, we get the first approximation

$$y^{(1)}(x) = F(x) + \lambda \int_a^b K(x, t)\, y^{(0)}(t)\, dt \qquad ...(2)$$

Replacing y under the integral sign in (1) by $y^{(1)}$, we get the next approximation

$$y^{(2)}(x) = F(x) + \lambda \int_a^b K(x, t)\, y^{(1)}(t)\, dt \qquad ...(3)$$

Proceeding in this manner, we get the general formula for successive approximations as

$$y^{(n)}(x) = F(x) + \lambda \int_a^b K(x, t)\, y^{(n-1)}(t)\, dt \quad \text{...(4)}$$

We now, obtain the condition for the convergency of the sequence $y^{(n)}(x)$.

Replacing x by t and t by a dummy variable t_1, (2) becomes

$$y^{(1)}(t) = F(t) + \lambda \int_a^b K(t, t_1)\, y^{(0)}(t_1)\, dt_1$$

Then (3) takes the form

$$y^{(2)}(x) = F(x) + \lambda \int_a^b K(x, t) \left\{ F(t) + \lambda \int_a^b K(t, t_1)\, y^{(0)}(t_1)\, dt_1 \right\} dt$$

$$= F(x) + \lambda \int_a^b K(x, t)\, F(t)\, dt + \lambda^2 \int_a^b K(x, t) \,.\, \int_a^b K(t, t_1) y^{(0)}(t_1)\, dt_1\, dt \quad \text{...(5)}$$

Writing $K^* \phi(x) = \int_a^b K(x, t)\ \phi(t)\, dt$, the equations (1), (2) and (5) become

$$y(x) = F(x) + \lambda K^* y(x)$$

$$y^{(1)}(x) = F(x) + \lambda K^* y^{(0)}(x)$$

$$y^{(2)}(x) = F(x) + \lambda K^* F(x) + \lambda^2 K^{*2} y^{(0)}(x)$$

Similarly $\quad y^{(3)}(x) = F(x) + \lambda K^* F(x) + \lambda^2 K^{*2} F(x) + \lambda^3 K^{*3} y^{(0)} x$

In general $\quad y^{(n)}(x) = F(x) + \lambda K^* F(x) + \lambda^2 K^{*2} F(x) + \lambda^3 K^{*3} F(x) + \dots + \lambda^n K^{*n} y^{(0)}(x)$

As $n \to \infty$, we get

$$y(x) = \underset{n \to \infty}{\text{Lt}}\ y^n(x) = F(x) + \underset{n \to \infty}{\text{Lt}}\ \{\lambda K^* F(x) + \lambda^2 K^{*2} F(x) + \dots \infty\} \quad \text{...(6)}$$

Now $F(x)$ and $K(x, t)$ being continuous for all values of x and t in (a, b), $F(x) \le M$ and $|K(x, t)| \le m$ where M, m are their respective maximum values in (a, b).

$$\therefore \quad |K^* F(x)| = \left| \int_a^b K(x, t)\ F(x)\, dt \right|$$

$$\le mM \left| \int_a^b dt \right| \le mM(b - a)$$

Similarly $\quad K^{*r} F(x) \le m^r M \,.\, (b - a)^r$

Then $\quad |\lambda^r K^{*r} F(x)| \le |\lambda^r| \,.\, m^r M(b - a)^r$

$$\le M\{|\lambda|\ m\,(b - a)\}^r$$

$\therefore$ In (6), $\quad \sum_1^\infty \lambda^r K^{*r}\ F(x) \le M \sum_1^\infty [|\lambda|\, m(b - a)]^r$

Now the series on R.H.S. being a geometric series, converges for $|\lambda|\ m\,(b - a) < 1$.

Thus by comparison test, $\sum_1^\infty \lambda^r K^{*r}\ F(x)$ also converges for $|\lambda|\ m(b - a) < 1$

i.e., for
$$|\lambda| < \frac{1}{m\,(b - a)} \quad \text{...(7)}$$

Hence the given integral equation (1) will have a continuous solution when the condition (7) is satisfied.

Obs. 1. To evaluate the successive terms in the series (6) conveniently, we choose $y^{(0)}(x) = F(x)$.

Obs. 2. Volterra integral equations can also be solved by following exactly similar procedure as above (See Example 36.19).

Example 36.18. *Solve, by using the method of successive approximations, the integral equation*

$$y(x) = 1 + \lambda \int_0^1 xt\ y(t)\, dt. \quad \text{...(i)}$$

Solution. Taking the initial approximation $y^{(0)}(x) = 1$ and substituting it in the R.H.S. of (i), we get

$$y^{(1)}(x) = 1 + \lambda \int_0^1 xt \cdot 1\, dt = 1 + \lambda x \left| \frac{t^2}{2} \right|_0^1 = 1 + \frac{\lambda x}{2}$$

Substituting $y^{(1)}(x)$ in the R.H.S. of (i), we have

$$y^{(2)}(x) = 1 + \lambda \int_0^1 xt\left(1 + \lambda \frac{t}{2}\right) dt = 1 + \lambda x \int_0^1 \left(t + \frac{\lambda t^2}{2}\right) dt$$

$$= 1 + \lambda x \left| \frac{t^2}{2} + \frac{\lambda}{2} \cdot \frac{t^3}{3} \right|_0^1 = 1 + \frac{\lambda x}{2} + \frac{\lambda^2 x}{6}$$

Substituting $y^{(2)}(x)$ in the R.H.S. of (i), we get

$$y^{(3)}(x) = 1 + \lambda \int_0^1 xt\left(1 + \frac{\lambda t}{2} + \frac{\lambda^2 t}{6}\right) dt = 1 + \lambda x \left| \frac{t^2}{2} + \left(\frac{\lambda}{2} + \frac{\lambda^2}{6}\right)\frac{t^3}{3} \right|_0^1$$

$$= 1 + \frac{\lambda x}{2} + \frac{\lambda^2 x}{6} + \frac{\lambda^3 x}{18} = 1 + \frac{\lambda x}{2}\left(1 + \frac{\lambda}{3} + \frac{\lambda^2}{3^2} + \cdots\right)$$

Hence the solution of (i) is

$$y(x) = 1 + \frac{\lambda x}{2}\left(1 + \frac{\lambda}{3} + \left(\frac{\lambda}{3}\right)^2 + \left(\frac{\lambda}{3}\right)^3 + \cdots\right)$$

As the number of terms tends to infinity, the exact solution is

$$y(x) = 1 + \frac{\lambda x}{2}\left(1 + \frac{\lambda}{3} + \left(\frac{\lambda}{3}\right)^2 + \left(\frac{\lambda}{3}\right)^3 + \cdots \infty\right)$$

$$= 1 + \frac{\lambda x}{2} \frac{1}{1 - \lambda/3} \qquad \text{[Summing up the G.P. which converges for } \lambda/3 < 1]$$

or $$y(x) = 1 + \frac{3\lambda x}{2(3 - \lambda)} \quad \text{only if } \lambda < 3.$$

Example 36.19. *Using the method of successive approximations, solve the Volterra integral equation*

$$y(x) = 1 + x + \int_0^x (x - t)\, y(t)\, dt.$$

Solution. Taking the initial approximation $y^{(0)}(x) = 1 + x$ and substituting it in the R.H.S. of (i), we get

$$y^{(1)}(x) = 1 + x + \int_0^x (x - t)(1 + t)\, dt$$

$$= 1 + x + x\left(x + \frac{x^2}{2}\right) - \left(\frac{x^2}{2} + \frac{x^3}{3}\right) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Substituting $y^{(1)}(x)$ in the R.H.S. of (i), we obtain

$$y^{(2)}(x) = 1 + x + \int_0^x (x - t)\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6}\right) dt$$

$$= 1 + x + x\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right) - \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{30}\right)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

Proceeding in this manner, we get

$$y^{(n)}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!}$$

As $n \to \infty$, the exact solution of (i) is

$$y(x) = \underset{n\to\infty}{\text{Lt}}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty = e^x.$$

PROBLEMS 36.5

Apply the method of successive approximations, to solve the following Fredholm integral equations : (1 to 3)

1. $y(x) = 1 - \lambda \int_0^1 xt\, y(t)\, dt.$
2. $y(x) = \sin x + \lambda \int_0^{2\pi} \cos(x+t)\, y(t)\, dt.$
3. $y(x) = 1 + \lambda \int_0^1 (1 - 3xt)\, y(t)\, dt.$

Using the iterative method, solve the following Volterra integral equations : (4 to 6)

4. $y(x) = 1 + \int_0^x y(t)\, dt.$ *(Madras M.E., 2000 S)*
5. $y(x) = 1 + x - \int_0^x y(t)\, dt.$
6. $y(x) = x + \int_0^x (t - x)\, y(t)\, dt.$
7. $y(x) = 2(1 + x^2) - \int_0^x xy(t)\, dt.$
8. Choosing the initial approximation $y^{(0)}(x) = 0$, for the solution of the integral equation

$$y(x) = \int_0^x t(t-x)\, y(t)\, dt + \frac{x^2}{2}, \text{ show that } y^{(2)}(x) = \frac{x^2}{2} - \frac{x^5}{40}.$$

9. Starting with the initial approximation $y^{(0)}(x) = 1$, for the solution of the integral equation

$$y(x) = 1 + \int_0^x (x+t)\, y(t)\, dt,$$

show that $\quad y^{(3)}(x) = 1 + \dfrac{3x^2}{2} + \dfrac{7x^4}{8} + \dfrac{77x^6}{240}.$ *(Madras M.E., 2000)*