CHAPTER 6

Integral Calculus and Its Applications

1. Reduction formulae. 2. Reduction formulae for $\int \sin^n x\, dx$, $\int \cos^n x\, dx$ and evaluation of $\int_0^{\pi/2} \sin^n x\, dx$, $\int_0^{\pi/2} \cos^n x\, dx$. 3. Reduction formula for $\int \sin^m x \cos^n x\, dx$ and evaluation of $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$. 4. Reduction formulae for $\int \tan^n x\, dx$ $\int \cot^n x\, dx$. 5. Reduction formulae for $\int \sec^n x\, dx$, $\int \operatorname{cosec}^n x\, dx$. 6. Reduction formulae for $\int x^n e^{ax}\, dx$, $\int x^m (\log x)^n\, dx$. 7. Reduction formulae for $\int x^n \sin mx\, dx$, $\int x^n \cos nx\, dx$ and $\int \cos^m x \sin nx\, dx$. 8. Definite integrals. 9. Integral as the limit of a sum. 10. Areas of curves. 11. Lengths of curves. 12. Volumes of revolution. 13. Surface areas of revolution. 14. Objective Type of Questions.

6.1 REDUCTION FORMULAE

The reader is already familiar with some standard methods of integrating functions of a single variable. However, there are some integrals which cannot be evaluated by the afore-said methods. In such cases, the method of reduction formulae proves useful. A reduction formula connects an integral with another of the same type but of lower order. The successive application of the reduction formula enables us to evaluate the given integral. Now we shall derive some standard reduction formulae.

6.2 (1) REDUCTION FORMULAE for

(a) $\int \sin^n x\, dx$ **(b)** $\int \cos^n x\, dx.$

(a) $\int \sin^n x\, dx = \int \sin^{n-1} x \cdot \sin x\, dx$ [Integrated by parts]

$$= \sin^{n-1} x \cdot (-\cos x) - \int (n-1) \sin^{n-2} x \cos x\, (-\cos x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, (1 - \sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^n x\, dx$$

Transposing

$$n \int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx$$

or

$$\int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx \qquad ...(i)$$

(b) Similarly, $$\int \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

Thus we have the required reduction formulae.

Obs. To integrate $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$,

(a) when the index of sin x is odd put cos x = t
when the index of cos x is odd, put sin x = t
(b) when the index is an even positive integer, express the integrand as a series of cosines of multiple angles and integrate term by term if n is small, *otherwise use the method of reduction formulae.*

(2) To show that $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$= \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots} \times \left(\frac{\pi}{2}, \text{only if } n \text{ is even}\right)$$

From (i), we have

$$I_n = \int_0^{\pi/2} \sin^n x\,dx = -\left|\frac{\sin^{n-1} x \cos x}{n}\right|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

i.e. $$I_n = \frac{n-1}{n} I_{n-2}$$

Case I. *When n is odd*

Similarly $$I_{n-2} = \frac{n-3}{n-2} I_{n-4},\quad I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

....

$$I_5 = \frac{4}{5} I_3,\ I_3 = \frac{2}{3} I_1 = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \frac{2}{3}\left| -\cos x \right|_0^{\pi/2} = \frac{2}{3}.$$

Form these, we get $I_n = \dfrac{(n-1)(n-3)(n-5)\ldots 2}{n(n-2)(n-4)\ldots 3}$ ...(ii)

Case II. *When n is even*

We have $$I_{n-2} = \frac{n-3}{n-2} I_{n-4},\ I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

....

$$I_4 = \frac{3}{4} I_2,\ I_2 = \frac{1}{2} I_0 = \int_0^{\pi/2} \sin^0 x\,dx = \frac{1}{2}\int_0^{\pi/2} dx = \frac{1}{2}\cdot\frac{\pi}{2}.$$

Form these, we obtain $I_n = \dfrac{(n-1)(n-3)(n-5)\ldots 3\cdot 1}{n(n-2)(n-4)\ldots 4\cdot 2}\cdot\dfrac{\pi}{2}$ ...(iii)

Combining (ii) and (iii), we get the required result for $\int_0^{\pi/2} \sin^n x\,dx$.

Proceeding exactly as above, we get the result for $\int_0^{\pi/2} \cos^n x\,dx$.

Example 6.1. *Integrate (i) $\int \sin^4 x\,dx$ (ii) $\int_0^{\pi/2} \cos^6 x\,dx$.*

Solution. (i) We have the reduction formula

$$\int \sin^n x\,dx = \frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

Putting $n = 4, 2$ successively,

$$\int \sin^4 x\,dx = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\int \sin^2 x\,dx \quad ...(\alpha)$$

$$\int \sin^2 x\,dx = -\frac{\sin x \cos x}{2} + \frac{1}{2}\int (\sin x)^0\,dx$$

But $\int (\sin x)^0\,dx = \int dx = x. \quad \therefore \quad \int \sin^2 x\,dx = -\frac{\sin x\cos x}{2} + \frac{x}{2}$

Substituting this in (α), we get

$$\int \sin^4 x\,dx = -\frac{\sin^3 x\cos x}{4} + \frac{3}{4}\left(-\frac{\sin x\cos x}{2} + \frac{x}{2}\right)$$

(*ii*) We know that $\int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots}\left(\frac{\pi}{2} \text{ if } n \text{ is even}\right)$

Putting $n = 6$, we get

$$\int_0^{\pi/2} \cos^6 x\,dx = \frac{5.3.1}{6.4.2}\frac{\pi}{2} = \frac{5\pi}{16}.$$

Example 6.2. *Evaluate*

(*i*) $\int_0^a \frac{x^7\,dx}{\sqrt{(a^2-x^2)}}$ (*V.T.U., 2006*) (*ii*) $\int_0^\pi \frac{\sqrt{(1-\cos x)}}{1+\cos x}\sin^2 x\,dx$ (*iii*) $\int_0^\infty \frac{dx}{(a^2+x^2)^n}$.

Solution. (*i*) $\int_0^a \frac{x^7}{\sqrt{(a^2-x^2)}}dx$ | Put $x = a\sin\theta$, so that $dx = a\cos\theta\,d\theta$ Also when $x = 0$, $\theta = 0$, when $x = a$, $\theta = \pi/2$

$$= \int_0^{\pi/2} \frac{a^7\sin^7\theta}{a\cos\theta}\cdot a\cos\theta\,d\theta = a^7\int_0^{\pi/2}\sin^7\theta\,d\theta = a^7\cdot\frac{6.4.2}{7.5.3.1} = \frac{16}{35}a^7$$

(*ii*) Putting $x = 2\theta$, we get

$$\int_0^\pi \frac{\sqrt{(1-\cos x)}}{1+\cos x}\sin^2 x\,dx = \int_0^{\pi/2}\frac{\sqrt{(1-\cos 2\theta)}}{1+\cos 2\theta}\sin^2 2\theta\cdot 2d\theta$$

$$= 2\int_0^{\pi/2}\frac{\sqrt{2}\sin\theta}{2\cos^2\theta}\cdot(2\sin\theta\cos\theta)^2\,d\theta = 4\sqrt{2}\int_0^{\pi/2}\sin^3\theta\,d\theta = 4\sqrt{2}\cdot\frac{2}{3} = \frac{8\sqrt{2}}{3}.$$

(*iii*) $\int_0^\infty \frac{dx}{(a^2+x^2)^n}$ | Put $x = a\tan\theta$, so that $dx = a\sec^2\theta\,d\theta$ Also when $x = 0$, $\theta = 0$, when $x = \infty$, $\theta = \pi/2$

$$= \int_0^{\pi/2}\frac{a\sec^2\theta\,d\theta}{a^{2n}\sec^{2n}\theta} = \frac{1}{a^{2n-1}}\int_0^{\pi/2}\cos^{2n-2}\theta\,d\theta = \frac{1}{a^{2n-1}}\cdot\frac{(2n-3)(2n-5)\ldots 3.1}{(2n-2)(2n-4)\ldots 4.2}\cdot\frac{\pi}{2}.$$

Example 6.3. *Evaluate* $\int_0^a \frac{x^n}{\sqrt{(a^2-x^2)}}dx$. *Hence find the value of* $\int_0^1 x^n\sin^{-1}x\,dx$.

Solution. Putting $x = a\sin\theta$, we get

$$\int_0^a \frac{x^n}{\sqrt{(a^2-x^2)}}dx = \int_0^{\pi/2}\frac{(a\sin\theta)^n}{a\cos\theta}(a\cos\theta)\,d\theta = a^n\int_0^{\pi/2}\sin^n\theta\,d\theta$$

$$\left.\begin{aligned} &= \frac{(n-1)(n-3)\ldots.2}{n(n-2)\ldots.3}a^n, \text{ if } n \text{ is odd} \\ &= \frac{(n-1)(n-3)\ldots 1}{n(n-2)\ldots 2}\cdot\frac{\pi}{2}a^n, \text{ if } n \text{ is even}\end{aligned}\right\} \quad \ldots(i)$$

Now integrating by parts, we have

$$\int_0^1 x^n\sin^{-1}x\,dx = \left|(\sin^{-1}x)\cdot\frac{x^{n+1}}{n+1}\right|_0^1 - \int_0^1\frac{x^{n+1}}{n+1}\frac{1}{\sqrt{(1-x^2)}}dx$$

$$= \frac{1}{(n+1)}\left[\frac{\pi}{2} - \int_0^1 \frac{x^{n+1}}{(1-x^2)}\right] \qquad \text{[Using (i) p. 241]}$$

$$= \frac{1}{n+1}\left\{\frac{\pi}{2} - \frac{n(n-2)(n-4)\ldots 1}{(n+1)(n-1)(n-3)\ldots 2}\frac{\pi}{2}\right\} \qquad \text{when } n \text{ is odd}$$

$$= \frac{1}{n+1}\left\{\frac{\pi}{2} - \frac{n(n-2)(n-4)\ldots 2}{(n+1)(n-1)(n-3)\ldots 3}\right\} \qquad \text{when } n \text{ is even}$$

Evaluate 6.4. *Evaluate* $I_n = \int_0^a (a^2 - x^2)^n\, dx$ *where n is a positive integer. Hence show that*

$$I_n = \frac{2n}{2n+1} a^2 I_{n-1}.$$

Solution. Putting $n = a \sin \theta$, we get

$$I_n = \int_0^a (a^2 - x^2)^n\, dx = \int_0^{\pi/2} (a^2 - a^2 \sin^2\theta)^n\, a\cos\theta\, d\theta = a^{2n+1}\int_0^{\pi/2} \cos^{2n+1}\theta\, d\theta$$

$$= a^{2n+1} \cdot \frac{(2n)(2n-2)(2n-4)\ldots 4.2}{(2n+1)(2n-1)(2n-3)\ldots 5.3} \qquad [\because \ (2n+1) \text{ is always odd}]$$

Now replacing n by $n-1$, we get

$$I_{n-1} = a^{2n-1}\frac{(2n-2)(2n-4)\ldots 4.2}{(2n-1)(2n-3)\ldots 5.3} \quad \therefore \quad \frac{I_n}{I_{n-1}} = a^2 \cdot \frac{2n}{2n+1} \quad \text{or} \quad I_n = \frac{2n}{2n+1}a^2 I_{n-1}.$$

which is the second desired result.

6.3 (1) REDUCTION FORMULAE for $\int \sin^m x \cos^n x\, dx$

$$\int \sin^m x \cos^n x\, dx = \int \sin^{m-1} x \cdot \cos^n x \cdot \sin x\, dx \qquad \text{[Integrate by parts]}$$

$$= \sin^{m-1} x \cdot \left(\frac{-\cos^{n+1} x}{n+1}\right) - \int (m-1)\sin^{m-2} x \cos x \cdot \left(-\frac{\cos^{n+1} x}{n+1}\right) dx.$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{m+1}\int \sin^{m-2} x\,(1 - \sin^2 x)\cos^n x\, dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^n x\, dx - \frac{m-1}{n+1}\int \sin^m x \cos^n x\, dx$$

Transposing the last term to the left and dividing by $1 + (m-1)/(n+1)$, *i.e.*, $(m+n)/(n+1)$, we obtain the reduction formula

$$\int \sin^m x \cos^n x\, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\, dx \qquad \ldots(1)$$

Obs. To integrate $\int \sin^m x \cos^n x\, dx$,

(*a*) *when m is odd,* put $\cos x = t$

when n is odd, put $\sin x = t$

(*b*) *when m and n both are even integers,* express the integrand as a series of cosines of multiple angles and integrate term by term if m and n are small, *otherwise* use the method of reduction formulae.

(2) To show that

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{(m-1)(m-3)\ldots \times (n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \times \left(\frac{\pi}{2}, \textit{only if both m and n are even}\right)$$

From (*i*), we have

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\, dx = \left| -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} \right|_0^{\pi/2} + \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2} x \cos^n x\, dx$$

i.e., $$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}.$$

Case I. *When m is odd*

Similarly, $$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n},\ I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,n}$$

....

$$I_{5,n} = \frac{4}{n+5} I_{3,n}$$

Finally $$I_{3,n} = \frac{2}{n+3} I_{1,n} = \frac{2}{n+3}\int_0^{\pi/2} \sin x \cos^n x\, dx$$

$$= \frac{2}{n+3}\left| -\frac{\cos^{n+1} x}{n+1} \right|_0^{\pi/2} = \frac{2}{(n+3)(n+1)} \quad \text{...(ii)}$$

From these, we obtain

$$I_{m,n} = \frac{(m-1)(m-3)(m-5)\ldots 4.2}{(m+n)(m+n-2)(m+n-4)\ldots(n+3)(n+1)}$$

Case II. *When m is even*

We have, $$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n},\quad I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,n}$$

....

$$I_{4,n} = \frac{3}{n+4} I_{2,n},\ I_{2,n} = \frac{1}{n+2} I_{0,n} = \frac{1}{n+2}\int_0^{\pi/2} \cos^n x\, dx$$

From these, we have $$I_{m,n} = \frac{(m-1)(m-3)(m-5)\ldots 1}{(m+n)(m+n-2)(m+n-4)\ldots(n+2)}\int_0^{\pi/2} \cos^n x\, dx$$

$$= \frac{(m-1)(m-3)\ldots 1}{(m+n)(m+n-2)\ldots(n+2)} \cdot \frac{(n-1)(n-3)\ldots}{n(n-2)\ldots} \times (\pi/2 \text{ only if } n \text{ is even}) \quad \text{...(iii)}$$

Combining (*ii*) and (*iii*), we get the desired result.

Example 6.5. *Integrate* (*i*) $\int \sin^4 x \cos^2 x\, dx$ *(Raipur, 2005)*

(*ii*) $\int_0^{\infty} \frac{t^6}{(1+t^2)^7}\, dt$ (*iii*) $\int_0^{\infty} \frac{x^2}{(1+x^2)^{7/2}}\, dx$ *(V.T.U., 2010 S)*

Solution. (*i*) Taking $n = 2$, in (*i*) of page 241, we have the reduction formula :

$$\int \sin^m x \cos^2 x\, dx = \frac{\sin^{m-1} x \cos^3 x}{m+2} + \frac{m-1}{m+2}\int \sin^{m-2} x \cos^2 x\, dx$$

Putting $m = 4, 2$ successively,

$$\int \sin^4 x \cos^2 x\, dx = -\frac{\sin^3 x \cos^3 x}{6} + \frac{3}{6}\int \sin^2 x \cos^2 x\, dx \quad \text{...(1)}$$

$$\int \sin^2 x \cos^2 x\, dx = -\frac{\sin x \cos^3 x}{4} + \frac{1}{4}\int \cos^2 x\, dx$$

But $$\int \cos^2 x\, dx = \frac{1}{2}\int (1+\cos 2x)\, dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)$$

$\therefore \quad \int \sin^2 x \cos^2 x \, dx = -\frac{\sin x \cos^3 x}{4} + \frac{1}{16}(2x + \sin 2x)$

Substituting this in (1), we get

$$\int \sin^4 x \cos^2 x \, dx = -\frac{\sin^3 x \cos^3 x}{6} + \frac{1}{2}\left\{-\frac{\sin x \cos^3 x}{4} + \frac{1}{16}(2x + \sin 2x)\right\}$$

(*ii*) Putting $t = \tan \theta$, so that

$$\int_0^\infty \frac{t^6}{(1+t^2)^7} dt = \int_0^{\pi/2} \frac{\tan^6 \theta}{\sec^{14} \theta} \sec^2 \theta \, d\theta = \int_0^{\pi/2} \sin^6 \theta \cos^6 \theta \, d\theta = \frac{5 \cdot 3 \cdot 1 \times 5 \cdot 3 \cdot 1}{12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \times \frac{\pi}{2} = \frac{5\pi}{2048}.$$

(*iii*) Putting $x = \tan \theta$, so that

$$\int_0^\infty \frac{x^2}{(1+x^2)^{7/2}} dx = \int_0^{\pi/2} \frac{\tan^2 \theta}{\sec^7 \theta} \sec^2 \theta \, d\theta = \int_0^{\pi/2} \sin^2 \theta \cos^3 \theta \, d\theta = \frac{1.2}{53.1} = \frac{2}{15}.$$

Example 6.6. *Evaluate :* (*i*) $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta \, d\theta$ *(V.T.U., 2003 S)*

(*ii*) $\int_0^1 x^4 (1-x^2)^{3/2} dx$ (*iii*) $\int_0^{2a} x^2 \sqrt{(2ax - x^2)} \, dx.$ *(V.T.U., 2010)*

Solution. (*i*) $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta \, d\theta = \int_0^{\pi/6} \cos^4 3\theta (2 \sin 3\theta \cos 3\theta)^3 \, d\theta$

$$= 8 \int_0^{\pi/6} \sin^3 3\theta \cos^7 3\theta \, d\theta$$

Put $3\theta = x$ so that $3d\theta = dx$

Also when $\theta = 0, x = 0$; when $\theta = \pi/6, x = \pi/2$.

$$= \frac{8}{3} \int_0^{\pi/2} \sin^3 x \cos^7 x \, dx$$

$$= \frac{8}{3} \cdot \frac{2 \times 6 \cdot 4 \cdot 2}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} = \frac{1}{15}.$$

(*ii*) $\int_0^1 x^4 (1-x^2)^{3/2} dx$

Put $x = \sin t$ so that $dx = \cos t \, dt$

When $x = 0, t = 0$; when $x = 1, t = \pi/2$

$$= \int_0^{\pi/2} \sin^4 t (\cos^2 t)^{3/2} \cdot \cos t \, dt = \int_0^{\pi/2} \sin^4 t \cos^4 t \, dt$$

$$= \frac{3 \cdot 1 \times 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{3\pi}{256}.$$

(*iii*) $\int_0^{2a} x^2 \sqrt{(2ax - x^2)} \, dx$

$$= \int_0^{\pi/2} x^{5/2} \sqrt{(2a - x)} \, dx$$

Put $x = 2a \sin^2 \theta$

$\therefore \; dx = 4a \sin \theta \cos \theta \, d\theta$

$$= \int_0^{\pi/2} (2a \sin^2 \theta)^{5/2} \sqrt{(2a)} \cos \theta \cdot 4a \sin \theta \cos \theta \, d\theta$$

$$= 2^5 a^4 \int_0^{\pi/2} \sin^6 \theta \cos^2 \theta \, d\theta = 32 a^4 \cdot \frac{5 \cdot 3 \cdot 1 \times 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{5\pi a^4}{8}.$$

PROBLEMS 6.1

Evaluate :

1. (*i*) $\int_0^{\frac{\pi}{2}} \cos^9 x \, dx$ (*ii*) $\int_0^{\pi/6} \sin^5 3\theta \, d\theta$
2. (*i*) $\int_0^1 \frac{x^9}{\sqrt{(1-x^2)}} dx$ (*ii*) $\int_0^1 x^5 \sin^{-1} x \, dx$
3. (*i*) $\int_0^\infty \frac{dx}{(1+x^2)^n} \; (n > 1)$ *(V.T.U., 2008 S)* (*ii*) $\int_0^{\pi/4} \sin^2 x \cos^4 x \, dx.$ *(J.N.T.U., 2003)*

4. If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$ $(m > 0, n > 0)$, show that $I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$.

Hence evaluate $\int_0^{\pi/2} \sin^4 x \cos^8 x \, dx$

Evaluate :

5. (i) $\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$ *(Cochin, 2005)* (ii) $\int_0^{\pi/2} \sin^{15} x \cos^3 x \, dx$

6. (i) $\int_0^1 x^6 \sqrt{(1-x^2)} \, dx$ (ii) $\int_0^{\pi/2} \cos^4 3\theta \sin^3 6\theta \, d\theta$

7. (i) $\int_0^{2a} x^{7/2} (2a-x)^{-1/2} \, dx$ (ii) $\int_0^{2a} \frac{x^3 \, dx}{\sqrt{(2ax-x^2)}}$ *(Madras, 2000 S)*

8. (i) $\int_0^2 x^{5/2} \sqrt{(2-x)} \, dx$ (ii) $\int_0^4 x^3 \sqrt{(4x-x^2)} \, dx$ *(V.T.U., 2004)*

9. If $I_n = \int x^n \sqrt{(a-x)} \, dx$, prove that $(2n+3) I_n = 2an \, I_{n-1} - 2x^n (a-x)^{3/2}$ *(Marathwada, 2008)*

10. If n is a positive integer, show that $\int_0^{2a} x^n \sqrt{(2ax-x^2)} \, dx = \frac{2n+1}{(n+2)\, n!} \cdot \frac{a^{n+2}}{2n} \pi$ *(V.T.U., 2007)*

6.4 REDUCTION FORMULAE for (a) $\int \tan^n x \, dx$ (b) $\int \cot^n x \, dx$

(a) Let $I_n = \int \tan^n x \, dx = \int \tan^{n-2} x \cdot \tan^2 x \, dx = \int \tan^{n-2} x \cdot (\sec^2 x - 1) \, dx$

$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

Thus, $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$ which is the required reduction formula.

(b) Let $I_n = \int \cot^n x \, dx = \int \cot^{n-2} x \cot^2 x \, dx = \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1) \, dx$

$$= \int \cot^{n-2} x \operatorname{cosec}^2 x \, dx - \int \cot^{n-2} x \, dx$$

Thus $I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$

which is the required reduction formula.

Example 6.7. *Evaluate (i) $\tan^5 x \, dx$ (ii) $\int \cot^6 x \, dx$.*

Solution. (i) Putting $n = 5, 3$ successively in the reduction formula for $\int \tan^n x \, dx$, we get

$$I_5 = \frac{1}{4} \tan^4 x - I_3 \, ; \quad I_3 = \frac{1}{2} \tan^2 x - I_1$$

Thus $I_5 = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + I_1$

i.e., $\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \int \tan x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \log \cos x.$

(ii) Putting $n = 6, 4, 2$ successively in the reduction formula for $\int \cot^n x \, dx$, we get

$$I_6 = -\frac{1}{5} \cot^5 x - I_4 \, ; I_4 = -\frac{1}{3} \cot^3 x - I_2 \, ; I_2 = -\cot x - I_0$$

Thus $I_6 = \frac{-1}{5} \cot^5 x + \frac{1}{3} \cot^3 x - \cot x - \int dx$

i.e., $$\int \cot^6 x\,dx = -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - x.$$

Example 6.8. *If* $I_n = \int_0^{\pi/4} \tan^n \theta\,d\theta$, *prove that* $n(I_{n-1} + I_{n+1}) = 1$. (V.T.U., 2003)

Solution. The reduction formula for $\int_0^{\pi/4} \tan^n \theta\,d\theta$ is

$$I_n = \frac{1}{n-1}\left|\tan^n x\right|_0^{\pi/4} - I_{n-2} = \frac{1}{n-1} - I_{n-2} \quad \text{or} \quad I_n + I_{n-2} = \frac{1}{n-1}$$

Changing n to $n + 1$, we obtain

$$I_{n+1} + I_{n-1} = \frac{1}{(n+1)} \quad \text{or} \quad (n+1)(I_{n+1} + I_{n-1}) = 1.$$

6.5 REDUCTION FORMULAE for (a) $\int \sec^n x\,dx$ (b) $\int \text{cosec}^n x\,dx$

(*a*) Let $$I_n = \int \sec^n x\,dx = \int \sec^{n-2} x \,.\, \sec^2 x\,dx$$

Integrating by parts, we have

$$I_n = \sec^{n-2} x.\,\tan x - \int \{(n-2)\sec^{n-3} x \cdot \sec x \tan x\} \tan x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \cdot \tan^2 x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \cdot (\sec^2 x - 1)\,dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

Transposing, we have

$$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

Thus $$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$ which is the desired reduction formula.

(*b*) Let $$I_n = \int \text{cosec}^n x\,dx = \int \text{cosec}^{n-2} x \cdot \text{cosec}^2 x\,dx$$

Integrating by parts, we have

$$I_n = \text{cosec}^{n-2} x.\,(-\cot x) - \int [(n-2)\,\text{cosec}^{n-3} x \cdot (-\text{cosec}\, x \cot x) \cdot (-\cot x)\,dx$$

$$= -\cot x\,\text{cosec}^{n-2} x - (n-2)\int \text{cosec}^{n-2} x\,(\text{cosec}^2 x - 1)\,dx$$

$$= -\cot x\,\text{cosec}^{n-2} x - (n-2)I_n + (n-2)I_{n-2}$$

or $$[1 + (n-2)]I_n = -\cot x\,\text{cosec}^{n-2} x + (n-2)I_{n-2}$$

Thus $$I_n = -\frac{\cot x\,\text{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

which is the required reduction formula.

Example 6.9. *Evaluate* (*i*) $\int_0^{\pi/4} \sec^4 x\,dx$ (*ii*) $\int_{\pi/3}^{\pi/2} \text{cosec}^3 \theta\,d\theta$. (V.T.U., 2008)

Solution. (*i*) Putting $n = 4$ in the reduction formula for $\int \sec^n x\,dx$, we get $I_4 = \frac{\sec^2 x \tan x}{3} + \frac{2}{3} I_2$

$$\therefore \quad \int_0^{\pi/4} \sec^4 x\,dx = \left|\frac{\sec^2 x \tan x}{3}\right|_0^{\pi/4} + \frac{2}{3}\int_0^{\pi/4} \sec^2 x\,dx$$

$$= \frac{2}{3} + \frac{2}{3}\left|\tan x\right|_0^{\pi/4} = \frac{2}{3}(1+1) = 4/3.$$

(*ii*) Putting $n = 3$ in the reduction formula for $\int \operatorname{cosec}^n x\, dx$, we get

$$I_3 = -\frac{1}{2}\cot x \operatorname{cosec} x + \frac{1}{2} I_1$$

$$\therefore \quad \int_{\pi/3}^{\pi/2} \operatorname{cosec}^3 x\, dx = -\frac{1}{2}\left| \cot x \operatorname{cosec} x \right|_{\pi/3}^{\pi/2} + \frac{1}{2}\int_{\pi/3}^{\pi/2} \operatorname{cosec} x\, dx$$

$$= -\frac{1}{2}\left(0 - \frac{2}{3}\right) + \frac{1}{2}\left| \log(\operatorname{cosec} x - \cot x)\right|_{\pi/3}^{\pi/2}$$

$$= \frac{1}{3} + \frac{1}{2}\left[\log 1 - \log\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\right] = \frac{1}{3} + \frac{1}{4}\log 3.$$

PROBLEMS 6.2

1. Evaluate (*i*) $\int \tan^6 x\, dx$ (*V.T.U., 2007*) (*ii*) $\int \cot^5 x\, dx$.
2. Show that $\int_0^{\pi/4} \tan^7 x\, dx = \frac{1}{12}(5 - 6\log 2)$
3. If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, prove that $(n-1)(I_n + I_{n-2}) = 1$. (*V.T.U., 2009*)
   Hence evaluate I_5. (*Madras, 2000*)
4. If $I_n = \int_{\pi/4}^{\pi/2} \cot^n \theta\, d\theta\ (n > 2)$, prove that $I_n = \frac{1}{n-1} - I_{n-1}$. Hence evaluate I_4. (*Marathwada, 2008*)
5. Obtain the reduction formula for $\int_0^{\pi/4} \sec^n \theta\, d\theta$. (*V.T.U., 2010 S*)
6. Evaluate (*i*) $\int \sec^6 \theta\, d\theta$ (*ii*) $\int_{\pi/6}^{\pi/2} \operatorname{cosec}^5 d\theta$.
7. Evaluate $\int_0^a (a^2 + x^2)^{5/2}\, dx$.
8. If $I_n = \int \frac{t^n}{1+t^2}\, dt$, show that $I_{n+2} = \frac{t^{n+1}}{n+1} - I_n$. Hence evaluate I_6.

6.6 REDUCTION FORMULAE for

(a) $\int \mathbf{x^n\, e^{ax}\, dx}$ **(b)** $\int \mathbf{x^m\, (\log x)^n\, dx}$.

(*a*) Let $\quad I_n = \int x^n e^{ax}\, dx$

Integrating by parts, we have

$$I_n = x^n \cdot \frac{e^{ax}}{a} - \int n x^{n-1} \cdot \frac{e^{ax}}{a}\, dx$$

or $\quad I_n = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a} I_{n-1}$ which is the required reduction formula. (*Madras, 2006*)

(*b*) Let $\quad I_{m,n} = \int x^m (\log x)^n\, dx = \int (\log x)^n \cdot x^m\, dx$

Integrating by parts, we have

$$I_{m,n} = (\log x)^n \cdot \frac{x^{m+1}}{m+1} - \int n(\log x)^{n-1} \cdot \frac{1}{x} \cdot \frac{x^{m+1}}{m+1}\, dx$$

$$= \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^m (\log x)^{n-1}\, dx \quad \text{or} \quad I_{m,n} = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1} I_{m,n-1}$$

which is the desired reduction formula.

6.7 REDUCTION FORMULAE for

(a) $\int x^n \sin mx\, dx$ **(b)** $\int x^n \cos mx\, dx$ **(c)** $\int \cos^m x \sin nx\, dx$

(a) Let $I_n = \int x^n \sin mx\, dx$

Integrating by parts, we get

$$I_n = x^n \left(\frac{-\cos mx}{m}\right) - \int n x^{n-1}\left(\frac{-\cos mx}{m}\right) dx$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m}\int x^{n-1} \cos mx\, dx \qquad \text{[Again integrate by parts]}$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m}\left\{x^{n-1}\cdot\frac{\sin mx}{m} - \left\{\int (n-1)x^{n-2}\cdot\frac{\sin mx}{m}\, dx\right\}\right\}$$

or $$I_n = -\frac{x^n \cos mx}{m} + \frac{n}{m^2} x^{n-1}\sin mx - \frac{n(n-1)}{m^2} I_{n-2}$$

which is the desired reduction formula. *(Madras, 2003)*

(b) Let $I_n = \int x^n \cos mx\, dx$

Integrating twice by parts as above, we get

$$I_n = \frac{x^n \sin mx}{m} + \frac{n}{m^2} x^{n-1}\cos mx - \frac{n(n-1)}{m^2} I_{n-2}$$

(c) Let $I_{m,n} = \int \cos^m x \sin nx\, dx$

Integrating by parts,

$$I_{m,n} = -\cos^m x\cdot\frac{\cos nx}{n} - \int m\cos^{m-1} x(-\sin x)\cdot\left(\frac{-\cos nx}{n}\right)dx$$

$$= -\frac{1}{n}\cos^m x\cos nx - \frac{m}{n}\int \cos^{m-1} x\cdot\cos nx \sin x\, dx$$

$$\left[\begin{array}{l}\because\ \sin(n-1)x = \sin nx\cos x - \cos nx \sin x \\ \text{or } \cos nx\sin x = \sin nx\cos x - \sin(n-1)x\end{array}\right.$$

$$= -\frac{1}{n}\cos^m x\cos nx - \frac{m}{n}\int \cos^{m-1} x\ \{\sin nx\cos x - \sin(n-1)x\}\, dx$$

$$= -\frac{1}{n}\cos^m x\cos nx - \frac{m}{n}(I_{m,n} - I_{m-1,n-1})$$

Transposing, we get

$$\left(1+\frac{m}{n}\right) I_{m,n} = -\frac{1}{n}\cos^m x\cos nx + \frac{m}{n} I_{m-1,n-1}$$

or $$I_{m,n} = -\frac{\cos^m x\cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

which is the desired reduction formula.

Example 6.10. *Show that* $\int_0^{\pi/2} \cos^m x\cos nx\, dx = \frac{m}{m+n}\int_0^{\pi/2}\cos^{m-1} x\cos(n-1)x\, dx$

Hence deduce that $\int_0^{\pi/2}\cos^n x\cos nx\, dx = \frac{\pi}{2^{n+1}}$. *(S.V.T.U., 2008)*

Solution. Let $I_{m,n} = \int_0^{\pi/2}\cos^m x\cos nx\, dx$

Integrating by parts

$$I_{m,n} = \left|\cos^m x\cdot\frac{\sin nx}{n}\right|_0^{\pi/2} - \int_0^{\pi/2} m\cos^{m-1} x(-\sin x)\times\frac{\sin nx}{n}\, dx$$

$$= \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x \sin nx \sin x \, dx \qquad \left[\begin{array}{l}\because \cos(n-1)x = \cos nx \cos x + \sin nx \sin x \\ \text{or } \sin nx \sin x = \cos(n-1)x - \cos nx \cos x\end{array}\right.$$

$$= \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x [\cos(n-1)x - \cos nx \cos x] \, dx = \frac{m}{n}(I_{m-1,\,n-1} - I_{m,\,n})$$

Transposing and dividing by $(1 + m/n)$, we get

$$I_{m,\,n} = \frac{m}{m+n} I_{m-1,\,n-1}$$

which is the required result.

Putting $\quad m = n, I_n \left(= \int_0^{\pi/2} \cos^n x \cos nx \, dx\right) = \frac{1}{2} I_{n-1}$

Changing n to $n - 1$,

$$I_{n-1} = \frac{1}{2} I_{n-2}$$

$$\therefore \quad I_n = \frac{1}{2}\left(\frac{1}{2} I_{n-2}\right) = \frac{1}{2^2} I_{n-2} = \frac{1}{2^3} I_{n-3} \ldots\ldots = \frac{1}{2^n} I_{n-n} = \frac{1}{2^n} \cdot \int_0^{\pi/2} (\cos x)^0 \, dx$$

Hence $\quad I_n = \frac{1}{2^n} \cdot \frac{\pi}{2} = \frac{\pi}{2^{n+1}}.$

Example 6.11. *Find a reduction formula for* $\int e^{ax} \sin x \, dx$. *Hence evaluate* $\int e^x \sin^3 x \, dx$.

Solution. Let $I_n = \int e^{ax} \sin^n x \, dx = \int \underset{I}{\sin^n x} \cdot \underset{II}{e^{ax}} \, dx$

Integrating by parts,

$$I_n = \sin^n x \cdot \frac{e^{ax}}{a} - \int (n \sin^{n-1} x \cos x) \cdot \frac{e^{ax}}{a} \, dx$$

$$= \frac{e^{ax} \sin^n x}{a} - \frac{n}{a}\int (\sin^{n-1} x \cos x) \cdot e^{ax} \, dx \qquad \text{[Again integrating by parts]}$$

$$= \frac{e^{ax} \sin^n x}{a} - \frac{n}{a}\left[\sin^{n-1} x \cos x \cdot \frac{e^{ax}}{a} - \int \{(n-1)\sin^{n-2} x \times \cos x \cdot \cos x + \sin^{n-1} x(-\sin x)\} \frac{e^{ax}}{a} \, dx\right]$$

$$= \frac{e^{ax} \sin^{n-1} x}{a^2}(a \sin x - n \cos x) + \frac{n}{a^2}\int [(n-1)\sin^{n-2} x \times (1 - \sin^2 x) - \sin^n x] e^{ax} \, dx$$

$$= \frac{e^{ax} \sin^{n-1} x}{a}(a \sin x - n \cos x) + \frac{n(n-1)}{a^2} I_{n-2} - \frac{n^2}{a^2} I_n$$

Transposing and dividing by $(1 + n^2/a^2)$, we get

$$I_n = \frac{e^{ax} \sin^{n-1} x (a \sin x - n \cos x)}{a^2 + n^2} + \frac{n(n-1)}{a^2 + n^2} I_{n-2}$$

which is the required reduction formula.

Putting $a = 1$ and $n = 3$, we get

$$I_3 = \frac{e^x \sin^2 x (\sin x - 3 \cos x)}{1^2 + 9} + \frac{3 \cdot 2}{1^2 + 9} I_1$$

But $\quad I_1 = \int e^x \sin x \, dx = \frac{e^{ax}}{\sqrt{2}} \sin(x - \tan^{-1} 1).$

$$\therefore \quad I_3 = \frac{e^x \sin^2 x (\sin x - 3 \cos x)}{10} + \frac{3}{5} \cdot \frac{e^{ax}}{\sqrt{2}} \sin(x - \pi/4).$$

PROBLEMS 6.3

1. If $I_n = \int x^n e^x \, dx$, show that $I_n + n I_{n-1} = x^n e^x$. Hence find I_4. *(Madras, 2000)*

2. If $u_n = \int_0^a x^n e^{-x} \, dx$, prove that $u_n - (n + a) u_{n-1} + a(n-1) u_{n-2} = 0$. *(Madras, 2003)*

3. Obtain a reduction formula for $\int x^m (\log x)^n \, dx$. Hence evaluate $\int_0^1 x^5 (\log x)^3 \, dx$. *(S.V.T.U., 2009 ; Bhillai, 2005)*

4. If n is a positive integer, show that $\int_0^1 x^m (\log x)^n \, dx = \frac{(-1)^n \, n!}{(m+1)^{n+1}}$, $m > -1$.

5. If $I_n = \int_0^{\pi/2} x \sin^n x \, dx$ $(n > 1)$, prove that $n^2 I_n = n(n-1) I_{n-2} + 1$. Hence evaluate I_5.

6. If $I_n = \int_0^{\pi/2} x \cos^n x \, dx$ $(n > 1)$, prove that $I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n^2}$. Hence evaluate I_4.

7. If $u_n = \int_0^{\pi/2} x^n \sin x \, dx$, $(n > 1)$, prove that $u_n + n(n-1) u_{n-2} = n(\pi/2)^{n-1}$. Hence evaluate u_2. *(Madras, 2000 S)*

8. If $I_n = \int x^n \sin ax \, dx$, show that $a^2 I_n = -ax^n \cos ax + nx^{n-1} \sin ax - n(n-1) I_{n-2}$. *(Marathwada, 2008)*

9. Prove that $\int_0^{\pi/2} \cos^{n-2} x \sin nx \, dx = \frac{1}{n-1}$, $n > 1$.

10. If $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx \, dx$, prove that $I_{m,n} = \frac{m(m-1)}{m^2 - n^2} I_{m-2,n}$

11. Find a reduction formula for $\int e^{ax} \cos^n x \, dx$. Hence evaluate $\int_0^{\pi/2} e^{2x} \cos^3 x \, dx$.

12. Obtain a reduction formula for $I_m = \int_0^{\infty} e^{-x} \sin^m x \, dx$ where $m \geq 2$ in the form

    $(1 + m^2) I_m = m(m-1) I_{m-2}$. Hence evaluate I_4. *(Gorakhpur, 1999)*

6.8 DEFINITE INTEGRALS

Property I. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(*i.e., the value of a definite integral depends on the limits and not on the variable of integration*).

Let $\int f(x)\,dx = \phi(x)$; $\therefore \int_a^b f(x)\,dx = \phi(b) - \phi(a)$.

Then $\int f(t)\,dt = \phi(t)$; $\therefore \int_a^b f(t)\,dt = \phi(b) - \phi(a)$.

Hence the result.

Property II. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(*i.e., the interchange of limits changes the sign of the integral*).

Let $\int f(x)\,dx = \phi(x)$; $\therefore \int_a^b f(x)\,dx = \phi(b) - \phi(a)$

and $$-\int_b^a f(x)\,dx = -\left|\,\phi(x)\right|_b^a = -[\phi(a) - \phi(b)] = \phi(b) - \phi(a).$$

Hence the result.

Property III. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Let $\int f(x)\,dx = \phi(x)$, so that $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$ ...(1)

Also $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = |\,\phi(x)\,|_a^c + |\,\phi(x)\,|_c^b$$

$$= [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)] = \phi(b) - \phi(a) \quad ...(2)$$

Hence the result follows from (1) and (2).

Property IV. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Put $x = a - t$, so that $dx = -\,dt$. Also when $x = 0, t = a$; when $x = a, t = 0$.

$$\therefore \quad \int_0^a f(x)\,dx = -\int_a^0 f(a-t)\,dt = \int_0^a f(a-t)\,dt = \int_0^a f(a-x)\,dx \quad \text{[Prop. II]}$$

Example 6.12. *Evaluate* $\int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx$.

Solution. Let $$I = \int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx$$

Then $$I = \int_0^{\pi/2} \frac{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]}}{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]} + \sqrt{\left[\cos\left(\frac{1}{2}\pi - x\right)\right]}}\,dx \quad \text{[Prop. IV]}$$

$$= \int_0^{\pi/2} \frac{\sqrt{(\cos x)}}{\sqrt{(\cos x)} + \sqrt{(\sin x)}}\,dx$$

Adding $$2I = \int_0^{\pi/2} \frac{\sqrt{(\sin x)} + \sqrt{(\cos x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx = \int_0^{\pi/2} dx = |\,x\,|_0^{\pi/2} = \frac{\pi}{2}.$$

Hence $I = \pi/4$.

Example 6.13. *Evaluate* $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$. *(Cochin, 2005)*

Solution. Let $$I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$$

$\left[\begin{array}{l}\text{Put } x = \tan\theta \text{ so that } dx = \sec^2\theta\,d\theta \\ \text{When } x = 0, \theta = 0\text{ ; when } x = 1, \theta = \pi/4\end{array}\right.$

$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\,.\sec^2\theta\,d\theta = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$$= \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right] d\theta = \int_0^{\pi/4} \log\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right) d\theta \quad \text{[Prop. IV]}$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta = \log 2\int_0^{\pi/4} d\theta - I$$

Transposing, $2I = \log 2 \cdot |\,\theta\,|_0^{\pi/4} = \frac{\pi}{4}\log 2$. Hence $I = \frac{\pi}{8}\log 2$.

Example 6.14. *Evaluate* $\int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx$. *(Madras, 2006)*

Solution. Let $$I = \int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx$$

Then $$I = \int_0^\pi \frac{(\pi - x)\sin^3(\pi - x)}{1 + \cos^2(\pi - x)}\,dx$$ [Prop. IV]

$$= \int_0^\pi \frac{(\pi - x)\sin^3 x}{1 + \cos^2 x}\,dx = \pi\int_0^\pi \frac{\sin^3 x}{1+\cos^2 x}\,dx - I$$

Transposing, $$2I = \pi\int_0^\pi \frac{\sin^3 x}{1+\cos^2 x}\,dx$$

$$= -\pi\int_1^{-1}(1-t^2)\frac{dt}{1+t^2}$$

Put $\cos x = t$ so that $-\sin x\,dx = dt$
When $x = 0, t = 1$; When $x = \pi, t = -1$;

$$= \pi\int_1^{-1}\frac{-2+(1+t^2)}{1+t^2}\,dt = -2\pi\int_1^{-1}\frac{dt}{1+t^2} + \pi\int_1^{-1} dt$$

$$= -2\pi \left| \tan^{-1} t \right|_1^{-1} + \pi \left| t \right|_1^{-1} = -2\pi\left(-\frac{\pi}{4}-\frac{\pi}{4}\right) - 2\pi.$$ Hence, $I = \pi^2/2 - \pi$.

Property V. $\int_{-a}^{a} \mathbf{f(x)\,dx} = 2\int_0^a \mathbf{f(x)\,dx}$, **if f(x) is an even function,**

$= \mathbf{0}$ **if f(x) is an odd function.** *(Bhopal, 2008)*

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \quad ...(1) \quad \text{[Prop. I]}$$

In $\int_{-a}^{0} f(x)\,dx$, put $x = -t$, so that $dx = -dt$

$$\therefore \quad \int_{-a}^{0} f(x)\,dx = -\int_a^0 f(-t)\,dt = \int_0^a f(-t)\,dt = \int_0^a f(-x)\,dx \quad \text{[Prop. II]}$$

Substituting in (1), we get

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \quad ...(2)$$

(i) If $f(x)$ is an even function, $f(-x) = f(x)$.

$\therefore$ from (2), $\int_{-a}^{a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$

(ii) If $f(x)$ is an odd function, $f(-x) = -f(x)$.

$\therefore$ from (2), $\int_{-a}^{a} f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0.$

Property VI. $\int_0^{2a} \mathbf{f(x)\,dx} = 2\int_0^a \mathbf{f(x)\,dx}$, **if f(2a – x) = f(x)**

$= \mathbf{0}$, **if f(2a – x) = – f(x)**

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \quad ...(1) \quad \text{[Prop. III]}$$

In $\int_0^{2a} f(x)\,dx$, put $x = 2a - t$, so that $dx = -dt$

Also when $x = a, t = a$; when $x = 2a, t = 0$.

$$\therefore \quad \int_0^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt = \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx \quad \text{[Prop. II]}$$

Subsituting in (1), we get

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx \quad ...(2)$$

(i) If $f(2a-x) = f(x)$, then from (2)

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

(*ii*) If $f(2a-x) = -f(x)$, then from (2)

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0.$$

Cor. 1. *If n is even,* $\int_0^{\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi/2} \sin^m x \cos^n x\,dx$ *and if n is odd,* $\int_0^{\pi} \sin^m x \cos^n x\,dx = 0.$

Cor. 2. *If m is odd,* $\int_0^{2\pi} \sin^m x \cos^n x\,dx = 0$

and *if m is even,* $\int_0^{2\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi} \sin^m x \cos^n x\,dx$

$$= 4\int_0^{\pi/2} \sin^m x \cos^n x\,dx\text{, if } n \text{ is even} = 0\text{, if } n \text{ is odd.}$$

Example 6.15. *Evaluate* $\int_0^{\pi} \theta \sin^2 \theta \cos^4 \theta\,d\theta$. *(V.T.U., 2009 S)*

Solution. Let $I = \int_0^{\pi} \theta \sin^2 \theta \cos^4 \theta\,d\theta$

Then $I = \int_0^{\pi} (\pi-\theta)\sin^2(\pi-\theta)\cos^4(\pi-\theta)\,d\theta = \pi\int_0^{\pi} \sin^2\theta\cos^4\theta\,d\theta - I$ [Prop. IV]

or $2I = \pi\int_0^{\pi} \sin^2\theta\cos^4\theta\,d\theta = 2\pi\int_0^{\pi/2} \sin^2\theta\cos^4\theta\,d\theta$ [Prop. VI Cor. 2]

$$= 2\pi \cdot \frac{1.3.1}{6.4.2} = \frac{\pi}{2} = \frac{\pi^2}{16}$$

Hence $I = \dfrac{\pi^2}{32}$

Example 6.16. *Evaluate* $\int_0^{\pi/2} \log \sin x\,dx$. *(Anna, 2005 S)*

Solution. Let $I = \int_0^{\pi/2} \log \sin x\,dx$ ...(*i*)

then $I = \int_0^{\pi/2} \log \sin(\pi/2 - x)\,dx = \int_0^{\pi/2} \log \cos x\,dx$ ...(*ii*)

Adding (*i*) and (*ii*)

$$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x)\,dx$$

$$= \int_0^{\pi/2} \log(\sin x + \cos x)\,dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log \sin 2x\,dx - \int_0^{\pi/2} \log 2\,dx = \int_0^{\pi/2} \log \sin 2x\,dx - \log 2\int_0^{\pi/2} dx$$

$$= \int_0^{\pi/2} \log \sin 2x\,dx - \log 2\,|\,x\,|_0^{\pi/2} = I' - \frac{\pi}{2}\log 2 \quad \text{...(iii)}$$

where $I' = \int_0^{\pi/2} \log \sin 2x\,dx$ [Put, $2x = t$, so that $2dx = dt$. When $x = 0, t = 0$; when $x = \pi/2, t = \pi$]

$$= \frac{1}{2}\int_0^{\pi} \log \sin t\,dt = \frac{1}{2}\int_0^{\pi} \log \sin x\,dx \quad [\because \log \sin(\pi - x) = \log \sin x\text{, Prop. IV}]$$

$$= \frac{1}{2} \cdot 2\int_0^{\pi/2} \log \sin x\,dx = I.$$

Thus from (*iii*), $2I = I - (\pi/2)\log 2$, *i.e.*, $I = -(\pi/2)\log 2$.

Obs. The following are its immediate deductions :

$$\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2} \log 2$$

and $$\int_0^{\pi} \log \sin x \, dx = -\pi \log 2.$$

Example 6.17. *Evaluate* $\int_0^1 \frac{\sin^{-1} x}{x} dx$.

Solution. Put $\sin^{-1} x = \theta$ or $x = \sin \theta$ so that $dx = \cos \theta \, d\theta$

Also when $x = 0, \theta = 0$; when $x = 1, \theta = \pi/2$.

$$\therefore \quad \int_0^1 \frac{\sin^{-1} x}{x} dx = \int_0^{\pi/2} \theta \, . \frac{\cos \theta}{\sin \theta} d\theta \qquad \text{[Integrate by parts]}$$

$$= [\theta \, . \log \sin \theta]_0^{\pi/2} - \int_0^{\pi/2} 1 \, . \log \sin \theta \, d\theta$$

$$= - \int_0^{\pi/2} \log \sin \theta \, d\theta = -\left(-\frac{\pi}{2} \log 2\right) = \frac{\pi}{2} \log 2 \qquad \left[\underset{x \to 0}{\text{Lt}} (x \log x) = 0 \right]$$

PROBLEMS 6.4

Prove that :

1. (i) $\int_0^{\pi/2} \log \tan x \, dx = 0$ (ii) $\int_0^{\pi/2} \sin 2x \log \tan x \, dx = 0$

2. (i) $\int_0^{\infty} \frac{x^7 (1 - x^{12})}{(1 + x)^{28}} dx = 0$ (ii) $\int_0^{\pi/4} \log (1 + \tan \theta) \, d\theta = \frac{\pi}{8} \log_e 2$ (*Madras, 2000*)

3. (i) $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$ (ii) $\int_0^{a} \frac{dx}{x + \sqrt{(a^2 + x^2)}} = \frac{\pi}{4}$

4. (i) $\int_0^{\pi/2} \frac{dx}{1 + \sqrt{\cot x}} = \frac{\pi}{4}$. (ii) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4}$.

5. (i) $\int_0^{\pi/2} \frac{x \tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}$. (ii) $\int_0^{\pi} \frac{x}{1 + \sin x} dx = \pi$ (*Anna, 2002 S*)

6. (i) $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} = \frac{1}{2} \pi (\pi - 2)$. (ii) $\int_0^{\pi/2} \frac{x \, dx}{\sin x + \cos x} = \frac{\pi}{2\sqrt{2}} \log (\sqrt{2} + 1)$.

Evaluate :

7. (i) $\int_0^{\pi} \sin^4 x \, dx$ (ii) $\int_0^{2\pi} \cos^6 x \, dx$

   (iii) $\int_0^{\pi} \sin^6 x \cos^4 x \, dx$ (*V.T.U., 2001*) (iv) $\int_0^{2\pi} \sin^4 x \cos^6 x \, dx$

8. (i) $\int_0^{\pi} x \sin^7 x \, dx$ (*V.T.U., 2009*) (ii) $\int_0^{\pi} x \cos^4 x \sin^5 x \, dx$ (*Marathwada, 2008*)

Prove that :

9. (i) $\int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}$ (ii) $\int_0^{\pi/2} \frac{x \, dx}{2 \sin^2 x + \cos^2 x} = \frac{\pi^2}{2\sqrt{2}}$

10. (i) $\int_0^{\pi} \frac{x \, dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a\sqrt{(a^2 - 1)}}$ $(a > 1)$ (ii) $\int_0^{\pi} \frac{x \, dx}{1 + \sin^2 x} = \frac{\pi^2}{2\sqrt{2}}$.

11. $\int_0^{\pi} \log(1+\cos\theta)\, d\theta = -\pi \log_e 2$ (Madras, 2003)

12. (i) $\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2}\, dx = \pi \log_e 2$ (ii) $\int_0^{\infty} \frac{\log(x+1/x)}{1+x^2}\, dx = \pi \log_e 2$.

6.9 (1) INTEGRAL AS THE LIMIT OF A SUM

We have so far considered integration as inverse of differentiation. We shall now define the definite integral as the limit of a sum :

Def. *If $f(x)$ is continuous and single valued in the interval $[a, b]$, then the definite integral of $f(x)$ between the limits a and b is defined by the equation*

$$\int_a^b \mathbf{f(x)\, dx} = \underset{h \to 0}{\mathrm{Lt}}\ \mathbf{h\,[f(a) + f(a+h) + f(a+2h) + \dots + f(a + \overline{n-1}\,h)\,]},$$

where $nh = b - a$. ...(1)

(2) Evaluation of limits of series

The summation definition of a definite integral enables us to express the limits of sums of certain types of series as definite integrals which can be easily evaluated. We rewrite (1) as follows :

$$\int_a^b f(x)\, dx = \underset{n \to \infty}{\mathrm{Lt}}\ h \sum_{r=0}^{n-1} f(a + rh), \text{ where } nh = b - a.$$

Putting $a = 0$ and $b = 1$, so that $h = 1/n$, we get

$$\underset{n \to \infty}{\mathbf{Lt}}\ \frac{\mathbf{1}}{\mathbf{n}} \sum_{r=0}^{n-1} \mathbf{f}\left(\frac{\mathbf{r}}{\mathbf{n}}\right) = \int_0^1 \mathbf{f(x)\, dx}$$

Thus to express a given series as definite integral:

(i) Write the general term (T_r or T_{r+1} whichever involves r)

i.e., $f(r/n) \, . \, 1/n$

(ii) Replace r/n by x and $1/n$ by dx,

(iii) Integrate the resulting expression, taking

the lower limit = $\underset{n \to \infty}{\mathrm{Lt}}\ (r/n)$ where r is as in the first term,

and the upper limit = $\underset{n \to \infty}{\mathrm{Lt}}\ (r/n)$ where r is as in the last term.

Example 6.18. *Find the limit, when $n \to \infty$, of the series*

$$\frac{n}{n^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \dots + \frac{n}{n^2+(n-1)^2}$$

Solution. Here the general term $(= T_{r+1}) = \dfrac{n}{n^2+r^2} = \dfrac{n}{1+(r/n)^2} \cdot \dfrac{1}{n}$

$$= \frac{1}{1+x^2}\, dx \qquad \text{[Putting } r/n = x \text{ and } 1/n = dx]$$

Now for the first term $r = 0$ and for the last term $r = n - 1$

$\therefore$ the lower limit of integration $= \underset{n \to \infty}{\mathrm{Lt}} \left(\dfrac{0}{n}\right) = 0$

and the upper limit of integration $= \underset{n \to \infty}{\mathrm{Lt}} \left(\dfrac{n-1}{n}\right) = \underset{n \to \infty}{\mathrm{Lt}} \left(1 - \dfrac{1}{n}\right) = 1.$

Hence, the required limit $= \displaystyle\int_0^1 \frac{dx}{1+x^2} = \Big|\tan^{-1} x\Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \pi/4.$

[To find limit of a product by integration :

Let $P = \underset{n \to \infty}{\text{Lt}}$ (given product)

Take logs of both sides, so that

$\log P = \underset{n \to \infty}{\text{Lt}}$ (a series) $= k$ (say). Then $P = e^k$.]

Example 6.19. *Evaluate* $\underset{n \to \infty}{Lt} \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\dots\left(1+\frac{n}{n}\right)\right\}^{1/n}$. *(Bhopal, 2008)*

Solution. Let $$P = \underset{n \to \infty}{\text{Lt}} \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\dots\left(1+\frac{n}{n}\right)\right\}^{1/n}.$$

Taking logs of both sides,

$$\log P = \underset{n \to \infty}{\text{Lt}} \frac{1}{n}\left\{\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \dots + \log\left(1+\frac{n}{n}\right)\right\}$$

Its general term $= \log\left(1+\frac{r}{n}\right) \cdot \frac{1}{n} = \log(1+x) \cdot dx$ [Putting $r/n = x$ and $1/n = dx$]

Also for first term $r = 1$ and for the last term $r = n$.

$\therefore$ The lower limit of integration $= \underset{n \to \infty}{\text{Lt}} (1/n) = 0$ and the upper limit $= \underset{n \to \infty}{\text{Lt}} (n/n) = 1$

Hence $$\log P = \int_0^1 \log(1+x)\,dx = \int_0^1 \log(1+x) \cdot 1\,dx \quad \text{[Integrate by parts]}$$

$$= \left|\log(1+x) \cdot x\right|_0^1 - \int_0^1 \frac{1}{1+x} \cdot x\,dx$$

$$= \log 2 - \int_0^1 \frac{1+x-1}{1+x}\,dx = \log 2 - \int_0^1 dx + \int_0^1 \frac{dx}{1+x}$$

$$= \log 2 - \left|x\right|_0^1 + \left|\log(1+x)\right|_0^1 = \log 2 - 1 + \log 2$$

$$= \log 2^2 - \log_e e = \log(4/e). \text{ Hence, } P = 4/e.$$

PROBLEMS 6.5

Find the limit, as $n \to \infty$, of the series :

1. $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n}$. *(Bhopal, 2009)*

2. $\frac{1}{n^3+1} + \frac{4}{n^3+8} + \frac{9}{n^3+27} + \dots + \frac{r^2}{n^3+r^3} + \dots + \frac{1}{2n}$.

3. $\underset{n \to \infty}{\text{Lt}} \frac{\sqrt{n}}{\sqrt{n^3}} + \frac{\sqrt{n}}{\sqrt{(n+3)^3}} + \frac{\sqrt{n}}{\sqrt{(n+6)^3}} + \dots + \frac{\sqrt{n}}{\sqrt{[n+3(n-1)]^3}}$.

Evaluate :

4. $\underset{n \to \infty}{\text{Lt}} \sum_{r=1}^{n-1} \frac{1}{\sqrt{(n^2-r^2)}}$ *(Bhopal, 2008)*

5. $\underset{n \to \infty}{\text{Lt}} \frac{[(n+1)(n+2)\dots(n+n)]^{1/n}}{n}$.

6. $\underset{n \to \infty}{\text{Lt}} \left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\dots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$ *(Bhopal, 2008)*

6.10 AREAS OF CARTESIAN CURVES

(1) *Area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is* $\int_a^b$ **y dx.**

Let AB be the curve $y = f(x)$ between the ordinates LA $(x = a)$ and MB $(x = b)$. (Fig. 6.1)

Let $P(x, y)$, $P'(x + \delta x, y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates.

Let the area $ALNP$ be A, which depends on the position of P whose abscissa is x. Then the area $PNN'P' = \delta A$.

Complete the rectangles PN' and $P'N'$.

Then the area $PNN'P'$ lies between the areas of the rectangles PN' and $P'N$.

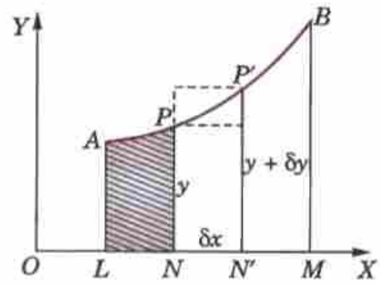

Fig. 6.1

i.e., δA lies between $y\delta x$ and $(y + \delta y)\,\delta x$

$\therefore$ $\dfrac{\delta A}{\delta x}$ lies between y and $y + \delta y$.

Now taking limits as $P' \to P$ *i.e.*, $\delta x \to 0$ (and $\therefore$ $\delta y \to 0$),

$dA/dx = y$

Integrating both sides between the limits $x = a$ and $x = b$, we have

$$\left| A \right|_a^b = \int_a^b y\,dx$$

or (value of A for $x = b$) – (value of A for $x = a$) $= \int_a^b y\,dx$

Thus area $ALMB = \int_a^b y\,dx$.

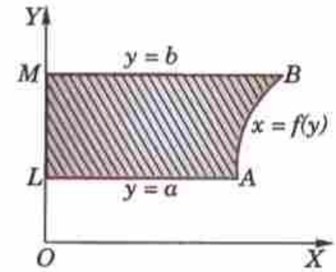

Fig. 6.2

(2) Interchanging x and y in the above formula, we see that *the area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$ is* $\int_a^b$ **x dy.** (Fig. 6.2)

Obs. 1. The area bounded by a curve, the x-axis and two ordinates is called the **area under the curve.** The process of finding the area of plane curves is often called **quadrature.**

Obs. 2. **Sign of an area.** *An area whose boundary is described in the anti-clockwise direction is considered positive and an area whose boundary is described in the clockwise direction is taken as negative.*

In Fig. 6.3, the area $ALMB\left(= \int_a^b y\,dx\right)$ which is described in the anti-clockwise direction and lies above the x-axis, will give a positive result.

In Fig. 6.4, the area $ALMB\left(= \int_a^b y\,dx\right)$ which is described in the clockwise direction and lies below the x-axis, will give a negative result.

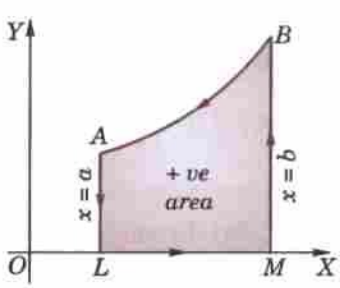

Fig. 6.3

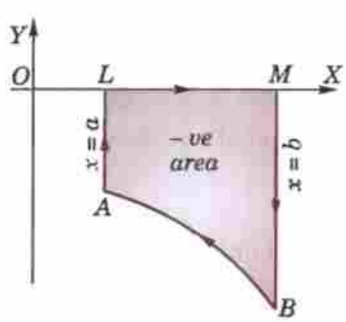

Fig. 6.4

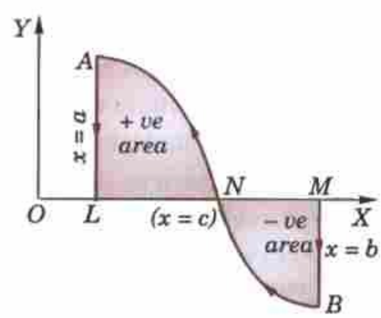

Fig. 6.5

In Fig. 6.5, the area $ALMB\left(= \int_a^b y\,dx\right)$ will not consist of the sum of the area $ALN\left(= \int_a^c y\,dx\right)$ and the area NMB $\left(= \int_c^b y\,dx\right)$, but their difference.

Thus to find the total area in such cases the numerical value of the area of each portion must be evaluated separately and their results added afterwards.

Example 6.20. *Find the area of the loop of the curve $ay^2 = x^2(a - x)$.* (*S.V.T.U., 2009 ; Osmania, 2000*)

Solution. Let us trace the curve roughly to get the limits of integration.

(*i*) The curve is symmetrical about x-axis.

(*ii*) It passes through the origin. The tangents at the origin are $ay^2 = ax^2$ or $y = \pm x$. $\therefore$ Origin is a node.

(*iii*) The curve has no asymptotes.

(*iv*) The curve meets the x-axis at (0, 0) and $(a, 0)$. It meets the y-axis at (0, 0) only.

From the equation of the curve, we have $y = \dfrac{x}{\sqrt{a}}\sqrt{(a-x)}$

For $x > a$, y is imaginary. Thus no portion of the curve lies to the right of the line $x = a$. Also $x \to -\infty$, $y \to \infty$.

Thus the curve is as shown in Fig. 6.6.

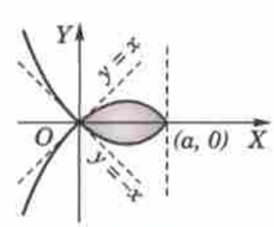

Fig. 6.6

$\therefore$ Area of the loop = 2 (area of upper half of the loop)

$$= 2\int_0^a y\,dx = 2\int_0^a x\sqrt{\left(\frac{a-x}{a}\right)}\,dx = \frac{2}{\sqrt{a}}\int_0^a [a-(a-x)]\sqrt{(a-x)}\,dx$$

$$= \frac{2}{\sqrt{a}}\int_0^a [a(a-x)^{1/2} - (a-x)^{3/2}]\,dx = 2\sqrt{a}\left|\frac{(a-x)^{3/2}}{-3/2}\right|_0^a - \frac{2}{\sqrt{a}}\left|\frac{(a-x)^{5/2}}{-5/2}\right|_0^a$$

$$= -\frac{4}{3}\sqrt{a}\,(0 - a^{3/2}) + \frac{4}{5\sqrt{a}}(0 - a^{5/2}) = \frac{4}{3}a^2 - \frac{4}{5}a^2 = \frac{8}{15}a^2.$$

Example 6.21. *Find the area included between the curve $y^2(2a-x) = x^3$ and its asymptote.* (V.T.U., 2003)

Solution. The curve is as shown in Fig. 4.23.

Area between the curve and the asymptote

$$= 2\int_0^{2a} y\,dx = 2\int_0^{2a}\sqrt{\left(\frac{x^3}{2a-x}\right)}\,dx$$

Put $x = 2a\sin^2\theta$
so that $dx = 4a\sin\theta\cos\theta\,d\theta$

$$= 2\int_0^{\pi/2}\sqrt{\left\{\frac{(2a\sin^2\theta)^3}{2a\cos^2\theta}\right\}}\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= 16a^2\int_0^{\pi/2}\sin^4\theta\,d\theta = 16a^2\cdot\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = 3\pi a^2.$$

Example 6.22. *Find the area enclosed by the curve $a^2x^2 = y^3(2a-y)$.*

Solution. Let us first find the limits of integration.

(*i*) The curve is symmetrical about y-axis.

(*ii*) It passes through the origin and the tangents at the origin are $x^2 = 0$ or $x = 0, x = 0$.

$\therefore$ There is a cusp at the origin.

(*iii*) The curve has no asymptote.

(*iv*) The curve meets the x-axis at the origin only and meets the y-axis at $(0, 2a)$. From the equation of the curve, we have

$$x = \frac{y}{a}\sqrt{\{y(2a-y)\}}$$

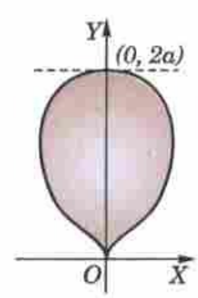

Fig. 6.7

For $y < 0$ or $y > 2a$, x is imaginary. Thus the curve entirely lies between $y = 0$ (x-axis) and $y = 2a$, which is shown in Fig. 6.7.

$$\therefore \text{ Area of the curve } = 2\int_0^{2a} x\,dy = \frac{2}{a}\int_0^{2a} y\sqrt{[y(2a-y)]}\,dy$$

Put $y = 2a\sin^2\theta$
$\therefore$ $dy = 4a\sin\theta\cos\theta\,d\theta$

$$= \frac{2}{a}\int_0^{\pi/2} 2a\sin^2\theta\sqrt{[2a\sin^2\theta\,(2a - 2a\sin^2\theta)]}\times 4a\sin\theta\cos\theta\,d\theta$$

$$= 32a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta = 32a^2\,\frac{3\cdot 1\times 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \pi a^2.$$

Example 6.23. *Find the area enclosed between one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$; and its base.* (V.T.U., 2000)

Solution. To describe its first arch, θ varies from 0 to 2π *i.e.*, x varies from 0 to $2a\pi$ (Fig. 6.8).

$\therefore$ Required area $= \int_{x=0}^{2\pi a} y\,dx$

where $y = a(1 - \cos\theta)$, $dx = a(1 - \cos\theta)\,d\theta$.

$$= \int_{\theta=0}^{\pi/2} a(1 - \cos\theta)\cdot a(1 - \cos\theta)\,d\theta$$

$$= 2a^2 \int_0^{\pi} (1 - \cos\theta)^2\,d\theta = 8a^2 \int_0^{\pi} \sin^4 \frac{\theta}{2}\,d\theta$$

$$= 16a^2 \int_0^{\pi/2} \sin^4 \phi\,d\phi, \text{ putting } \theta/2 = \phi \text{ so that } d\theta = 2d\phi.$$

$$= 16a^2 \cdot \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = 3\pi a^2.$$

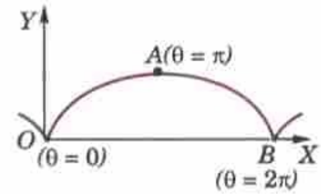

Fig. 6.8

Example 6.24. *Find the area of the tangent cut off from the parabola $x^2 = 8y$ by the line $x - 2y + 8 = 0$.*

Solution. Given parabola is $x^2 = 8y$ ...(*i*)

and the straight line is $x - 2y + 8 = 0$ ...(*ii*)

Substituting the value of y from (*ii*) in (*i*), we get

$$x^2 = 4(x + 8) \text{ or } x^2 - 4x - 32 = 0$$

or $\quad (x - 8)(x + 4) = 0 \quad \therefore \quad x = 8, -4.$

Thus (*i*) and (*ii*) intersect at P and Q where $x = 8$ and $x = -4$. (Fig. 6.9)

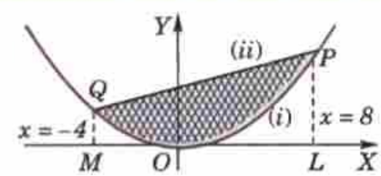

Fig. 6.9

$\therefore$ Required area POQ (*i.e.*, dotted area) = area bounded by straight line (*ii*) and x-axis from $x = -4$ to $x = 8$ – area bounded by parabola (*i*) and x-axis from $x = -4$ to $x = 8$.

$$= \int_{-4}^{8} y\,dx, \text{ from } (ii) - \int_{-4}^{8} y\,dx, \text{ from } (i)$$

$$= \int_{-4}^{8} \frac{x+8}{2}\,dx - \int_{-4}^{8} \frac{x^2}{8}\,dx = \frac{1}{2}\left|\frac{x^2}{2} + 8x\right|_{-4}^{8} - \frac{1}{8}\left|\frac{x^3}{3}\right|_{-4}^{8}$$

$$= \frac{1}{2}\{(32 + 64) - (-24)\} - \frac{1}{24}(512 + 64) = 36.$$

Example 6.25. *Find the area common to the parabola $y^2 = ax$ and the circle $x^2 + y^2 = 4ax$.*

Solution. Given parabola is $\quad y^2 = ax$ ...(*i*)

and the circle is $\quad x^2 + y^2 = 4ax.$ ...(*ii*)

Both these curves are symmetrical about x-axis. Solving (*i*) and (*ii*) for x, we have

$$x^2 + ax = 4ax \text{ or } x(x - 3a) = 0$$

or $\quad x = 0, 3a.$

Thus the two curves intersect at the points where $x = 0$ and $x = 3a$. (Fig. 6.10).

Also (*ii*) meets the x-axis at $A(4a, 0)$.

Area common to (*i*) and (*ii*) *i.e.*, the shaded area

$$= 2[\text{Area } ORP + \text{Area } PRA] \qquad \text{(By symmetry)}$$

$$= 2\left[\int_0^{3a} y\,dx, \text{ from } (i) + \int_{3a}^{4a} y\,dx, \text{ from } (ii)\right]$$

$$= 2\left[\int_0^{3a} \sqrt{(ax)}\,dx + \int_{3a}^{4a} \sqrt{(4ax - x^2)}\,dx\right]$$

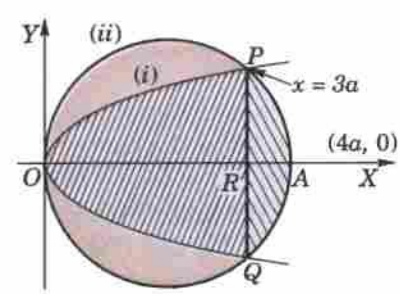

Fig. 6.10

$$= 2\sqrt{a}\left|\frac{x^{3/2}}{3/2}\right|_0^{3a} + 2\int_{3a}^{4a}\sqrt{[4a^2-(x-2a)^2]}\,dx$$

$$= \frac{4\sqrt{a}}{3}(3a)^{3/2} + 2\left[\frac{1}{2}(x-2a)\sqrt{[4a^2-(x-2a)^2]} + \frac{4a^2}{2}\sin^{-1}\frac{x-2a}{2a}\right]_{3a}^{4a}$$

$$= 4\sqrt{3}\,a^2 + 2[\{0 - \frac{1}{2}a\sqrt{3}a\} + 2a^2\{\pi/2 - \pi/6\}]$$

$$= 4\sqrt{3}\,a^2 - \sqrt{3}a^2 + \frac{4}{3}\pi a^2 = \left(3\sqrt{3} + \frac{4}{3}\pi\right)a^2.$$

PROBLEMS 6.6

1. (*i*) Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. (*Kerala, 2005 ; V.T.U., 2003 S*)

   (*ii*) Find the area bounded by the parabola $y^2 = 4ax$ and its latus-rectum.
2. Find the area bounded by the curve $y = x(x-3)(x-5)$ and the x-axis.
3. Find the area included between the curve $ay^2 = x^3$, the x-axis and the ordinates $x = a$.
4. Find the area of the loop of the curve :

   (*i*) $3ay^2 = x(x-a)^2$ (*Rajasthan, 2005*) (*ii*) $x(x^2+y^2) = a(x^2-y^2)$ (*P.T.U., 2010*)
5. Find the whole area of the curve :

   (*i*) $a^2x^2 = y^3(2a-y)$ (*Nagpur, 2009*) (*ii*) $8a^2y^2 = x^2(a^2-x^2)$ (*V.T.U., 2006*)
6. Find the area included between the curve and its asymptotes in each case :

   (*i*) $xy^2 = a^2(a-x)$. (*V.T.U., 2003*) (*ii*) $x^2y^2 = a^2(y^2-x^2)$. (*V.T.U., 2007*)
7. Show that the area of the loop of the curve $y^2(a+x) = x^2(3a-x)$ is equal to the area between the curve and its asymptote.
8. Find the whole area of the *astroid* $x^{2/3} + y^{2/3} = a^{2/3}$ or $x = a\cos^3\theta, y = a\sin^3\theta$. (*V.T.U., 2005*)
9. Find the area bounded by the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ and the coordinate axes.
10. Find the area included between the *cycloid* $x = a(\theta+\sin\theta), y = a(1-\cos\theta)$ *and its base*. Also find the area between the curve and the x-axis. (*Gorakhpur, 1999*)
11. Find the area common to the circles $x^2+y^2=4$ and $x^2+y^2=4x$.
12. Prove that the area common to the parabolas $x^2 = 4ay$ and $y^2 = 4ax$ is $16a^2/3$. (*S.V.T.U., 2008 ; Kurukshetra, 2005*)
13. Find the area included between the circle $x^2+y^2=2ax$ and the parabola $y^2 = ax$.
14. Find the area bounded by the parabola $y^2 = 4ax$ and the line $x+y=3a$.
15. Find the area of the segment cut off from the parabola $y = 4x - x^2$ by the straight line $y = x$. (*V.T.U., 2010 ; S.V.T.U., 2008*)

(2) Areas of polar curves. *Area bounded by the curve $r = f(\theta)$ and the radii vectors*

$$\theta = \alpha, \theta = \beta \textit{ is } \frac{1}{2}\int_\alpha^\beta \mathbf{r^2\,d\theta}.$$

Let AB be the curve $r = f(\theta)$ between the radii vectors $OA\,(\theta=\alpha)$ and $OB\,(\theta=\beta)$. Let $P(r,\theta)$, $P'(r+\delta r, \theta+\delta\theta)$ be any two neighbouring points on the curve. (Fig. 6.11)

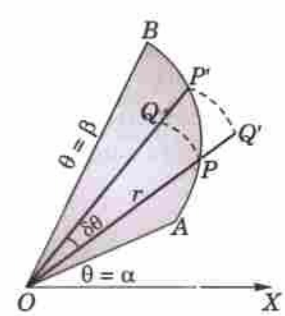

Fig. 6.11

Let the area $OAP = A$ which is a function of θ. Then the area $OPP' = \delta A$. Mark circular arcs PQ and $P'Q'$ with centre O and radii OP and OP'.

Evidently area OPP' lies between the sectors OPQ and $OP'Q'$ *i.e.*, δA lies between $\frac{1}{2}r^2\,\delta\theta$ and $\frac{1}{2}(r+\delta r)^2\,\delta\theta$.

$$\therefore \quad \frac{\delta A}{\delta\theta} \text{ lies between } \tfrac{1}{2}r^2 \text{ and } \tfrac{1}{2}(r+\delta r)^2.$$

Now taking limits as $\delta\theta \to 0$ ($\because \quad \delta r \to 0$), $\dfrac{dA}{d\theta} = \dfrac{1}{2}r^2$

Integrating both sides from $\theta = \alpha$ to $\theta = \beta$, we get $| A |_\alpha^\beta = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$

or (value of A for $\theta = \beta$) – (value of A for $\theta = \alpha$) $= \dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2 \, d\theta$

Hence the required area $OAB = \dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2 \, d\theta$.

Example 6.26. *Find the area of the cardioid $r = a(1 - \cos\theta)$.* (V.T.U., 2004)

Solution. The curve is as shown in Fig. 6.12. Its upper half is traced from $\theta = 0$ to $\theta = \pi$.

$\therefore$ Area of the curve

$$= 2 \cdot \frac{1}{2}\int_0^\pi r^2 \, d\theta = a^2 \int_0^\pi (1 - \cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^\pi (2\sin^2 \theta/2)^2 \, d\theta = 4a^2 \int_0^\pi \sin^4 \theta/2 \cdot d\theta$$

$$= 8a^2 \int_0^{\pi/2} \sin^4 \phi \, d\phi, \text{ putting } \theta/2 = \phi \text{ and } d\theta = 2d\phi.$$

$$= 8a^2 \cdot \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = \frac{3\pi a^2}{2}.$$

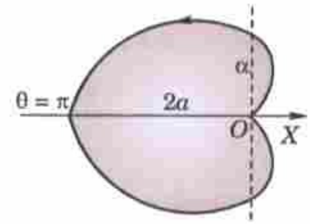

Fig. 6.12

Example 6.27. *Find the area of a loop of the curve $r = a \sin 3\theta$.*

Solution. The curve is as shown in Fig. 4.35. It consists of three loops.

Putting $r = 0$, $\sin 3\theta = 0 \quad \therefore \quad 3\theta = 0$ or π *i.e.*, $\theta = 0$ or $\pi/3$ which are the limits for the first loop.

$\therefore$ Area of a loop

$$= \frac{1}{2}\int_0^{\pi/3} r^2 \, d\theta = \frac{1}{2}a^2 \int_0^{\pi/3} \sin^2 3\theta \, d\theta = \frac{a^2}{4}\int_0^{\pi/3} (1 - \cos 6\theta)\, d\theta$$

$$= \frac{a^2}{4}\left| \theta - \frac{\sin 6\theta}{6} \right|_0^{\pi/3} = \frac{a^2}{4}\left(\frac{\pi}{3} - 0\right) = \frac{\pi a^2}{12}.$$

Obs. The limits of integration for a loop of $r = a \sin n\theta$ or $r = a\cos n\theta$ are the two consecutive values of θ when $r = 0$.

Example 6.28. *Prove that the area of a loop of the curve $x^3 + y^3 = 3axy$ is $3a^2/2$.*

Solution. Changing to polar form (by putting $x = r\cos\theta$, $y = r\sin\theta$), $r = \dfrac{3a\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}$

Putting $r = 0$, $\sin\theta\cos\theta = 0$.

$\therefore \quad \theta = 0, \pi/2$, which are the limits of integration for its loop.

$\therefore$ Area of the loop

$$= \frac{1}{2}\int_0^{\pi/2} r^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{9a^2 \sin^2\theta\cos^2\theta}{(\cos^3\theta + \sin^3\theta)^2}\, d\theta$$

$$= \frac{9a^2}{2}\int_0^{\pi/2} \frac{\tan^2\theta\sec^2\theta}{(1 + \tan^3\theta)^2}\, d\theta$$ [Dividing num. and denom. by $\cos^6\theta$]

$$= \frac{3a^2}{2}\int_1^\infty \frac{dt}{t^2},$$ putting $1 + \tan^3\theta = t$ and $3\tan^2\theta\sec^2\theta \, d\theta = dt$.

$$= \frac{3a^2}{2}\left| \frac{t^{-1}}{-1} \right|_1^\infty = \frac{3a^2}{2}(-0 + 1) = \frac{3a^2}{2}.$$

Example 6.29. *Find the area common to the circles*

$$r = a\sqrt{2} \text{ and } r = 2a\cos\theta$$

Solution. The equations of the circles are $r = a\sqrt{2}$ ...(*i*) and $r = 2a\cos\theta$ ...(*ii*)

(*i*) represents a circle with centre at (0, 0) and radius $a\sqrt{2}$. (*ii*) represents a circle symmetrical about OX, with centre at $(a, 0)$ and radius a.

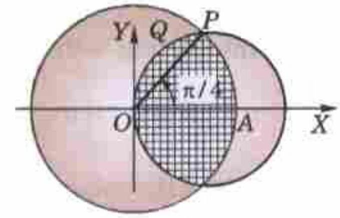

Fig. 6.13

The circles are shown in Fig. 6.13. At their point of intersection P, eliminating r from (*i*) and (*ii*),

$$a\sqrt{2} = 2a\cos\theta \text{ i.e., } \cos\theta = 1\sqrt{2}$$

or $$\theta = \pi/4$$

$\therefore$ Required area $= 2 \times$ area $OAPQ$ (By symmetry)

$$= 2(\text{area } OAP + \text{area } OPQ)$$

$$= 2\left[\frac{1}{2}\int_0^{\pi/4} r^2\, d\theta, \text{ for } (i) + \frac{1}{2}\int_{\pi/4}^{\pi/2} r^2\, d\theta, \text{ for } (ii)\right]$$

$$= \int_0^{\pi/4} (a\sqrt{2})^2\, d\theta + \int_{\pi/4}^{\pi/2} (2a\cos\theta)^2\, d\theta = 2a^2 \,|\, \theta \,|_0^{\pi/4} + 4a^2 \int_{\pi/4}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= 2a^2(\pi/4 - 0) + 2a^2 \left| \theta + \frac{\sin 2\theta}{2} \right|_{\pi/4}^{\pi/2} = \frac{\pi a^2}{2} + 2a^2\left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right) = a^2(\pi - 1).$$

Example 6.30. *Find the area common to the cardioids $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$.* (*Kurukshetra, 2006 ; V.T.U., 2006*)

Solution. The cardioid $r = a(1 + \cos\theta)$ is $ABCOB'A$ and the cardioid $r = a(1 - \cos\theta)$ is $OC'BA'B'O$. Both the cardioids are symmetrical about the initial line OX and intersect at B and B' (Fig. 6.14)

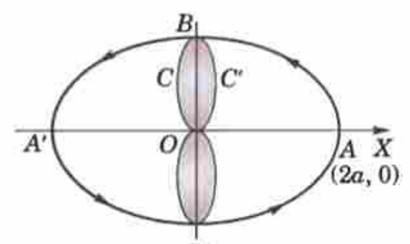

Fig. 6.14

$\therefore$ Required area (shaded) $= 2$ area $OC'BCO$

$$= 2\,[\text{area } OC'BO + \text{area } OBCO]$$

$$= 2\left[\left\{\int_0^{\pi/2} \frac{1}{2} r^2\, d\theta\right\}_{r = a(1-\cos\theta)} + \left\{\int_{\pi/2}^{\pi} \frac{1}{2} r^2\, d\theta\right\}_{r = a(1+\cos\theta)}\right]$$

$$= a^2 \int_0^{\pi/2} (1 - \cos\theta)^2\, d\theta + a^2 \int_{\pi/2}^{\pi} (1 + \cos\theta)^2\, d\theta$$

$$= a^2 \left\{\int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta)\, d\theta + \int_{\pi/2}^{\pi} [1 + 2\cos\theta + \cos^2\theta]\, d\theta\right\}$$

$$= a^2 \left\{\int_0^{\pi} (1 + \cos^2\theta)\, d\theta - 2\int_0^{\pi/2} \cos\theta\, d\theta + 2\int_{\pi/2}^{\pi} \cos\theta\, d\theta\right\}$$

$$= a^2 \left\{\int_0^{\pi} \left(1 + \frac{1 + \cos 2\theta}{2}\right) d\theta - 2\,|\sin\theta\,|_0^{\pi/2} + 2\,|\sin\theta\,|_{\pi/2}^{\pi}\right\}$$

$$= a^2 \left\{\left| \frac{3}{2}\theta + \frac{\sin 2\theta}{4} \right|_0^{\pi} - 2(1 - 0) + 2(0 - 1)\right\} = \left(\frac{3\pi}{2} - 4\right) a^2.$$

PROBLEMS 6.7

1. Find the whole area of
   (*i*) the *cardioid* $r = a(1 + \cos\theta)$ (*V.T.U., 2008*) (*ii*) the *lemniscate* $r^2 = a^2\cos 2\theta$; (*V.T.U., 2006*)
2. Find the area of one loop of the curve
   (*i*) $r = a\sin 2\theta$. (*ii*) $r = a\cos 3\theta$.
3. Show that the area included between the folium $x^3 + y^3 = 3axy$ and its asymptote is equal to the area of loop.
4. Prove that the area of the loop of the curve $x^3 + y^3 = 3axy$ is three times the area of the loop of the curve $r^2 = a^2\cos 2\theta$.
5. Find the area inside the circle $r = a\sin\theta$ and lying outside the *cardioid* $r = a(1 - \cos\theta)$. (*Anna, 2009*)
6. Find the area outside the *circle* $r = 2a\cos\theta$ and inside the *cardioid* $r = a(1 + \cos\theta)$. (*Kurukshetra, 2006*)

6.11 LENGTHS OF CURVES

(1) *The length of the arc of the curve $y = f(x)$ between the points where $x = a$ and $x = b$ is*

$$\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\, dx$$

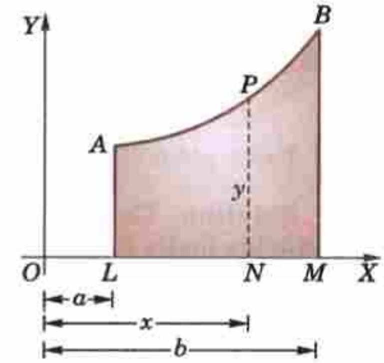

Fig. 6.15

Let AB be the curve $y = f(x)$ between the points A and B where $x = a$ and $x = b$ (Fig. 6.15)

Let $P(x, y)$ be any point on the curve and arc $AP = x$ so that it is a function of x.

Then $$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}$$ [(1) of p. 164]

$$\therefore \quad \int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\, dx = \int_a^b \frac{ds}{dx}\cdot dx = \left| s \right|_{x=a}^{x=b}$$

= (value of s for $x = b$) – (value of s for $x = a$) = arc $AB - 0$

Hence, the arc $AB = \displaystyle\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\, dx.$

(2) *The length of the arc of the curve $x = f(y)$ between the points where $y = a$ and $y = b$, is*

$$\int_a^b \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}\, dy$$ [Use (2) of p. 165]

(3) *The length of the arc of the curve $x = f(t)$, $y = \phi(t)$ between the points where $t = a$ and $t = b$, is*

$$\int_a^b \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}\, dt.$$ [Use (3) p. 165]

(4) *The length of the arc of the curve $r = f(\theta)$ between the points where $\theta = \alpha$ and $\theta = \beta$, is*

$$\int_\alpha^\beta \sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta.$$ [Use (1) of p. 165]

(5) *The length of the arc of the curve $\theta = f(r)$ between the points where $r = a$ and $r = b$, is*

$$\int_a^b \sqrt{\left[1+\left(r\frac{d\theta}{dr}\right)^2\right]}\, dr$$ [Use (2) of p. 166]

Example 6.31. *Find the length of the arc of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus-rectum.* (Delhi, 2002)

Solution. Let A be the vertex and L an extremity of the latus-rectum so that at A, $x = 0$ and at L, $x = 2a$. (Fig. 6.16).

Now $$y = x^2/4a \text{ so that } \frac{dy}{dx} = \frac{1}{4a}\cdot 2x = \frac{x}{2a}$$

$$\therefore \quad \text{arc } AL = \int_0^{2a} \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\, dx$$

$$= \int_0^{2a} \sqrt{\left[1+\left(\frac{x}{2a}\right)^2\right]}\, dx = \frac{1}{2a}\int_0^{2a} \sqrt{[(2a)^2+x^2]}\, dx$$

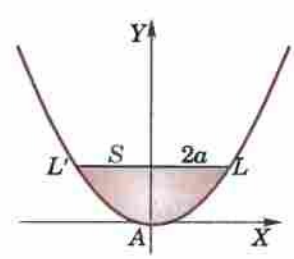

Fig. 6.16

$$= \frac{1}{2a}\left[\frac{x\sqrt{[(2a)^2+x^2]}}{2} + \frac{(2a)^2}{2}\sinh^{-1}\frac{x}{2a}\right]_0^{2a} = \frac{1}{2a}\left[\frac{2a\sqrt{(8a)^2}}{2} + 2a^2\sinh^{-1}1\right]$$

$$= a[\sqrt{2} + \sinh^{-1}1] = a[\sqrt{2} + \log(1+\sqrt{2})] \quad [\because \sinh^{-1}x = \log[x + \sqrt{(1+x^2)}]]$$

Example 6.32. *Find the perimeter of the loop of the curve $3ay^2 = x(x-a)^2$.*

Solution. The curve is symmetrical about the x-axis and the loop lies between the limits $x = 0$ and $x = a$. (Fig. 6.17).

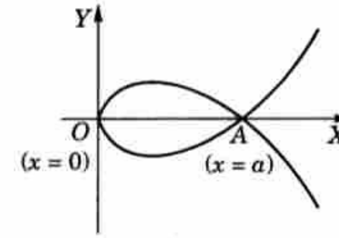

Fig. 6.17

We have $\quad y = \frac{\sqrt{x}(x-a)}{\sqrt{(3a)}}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{(3a)}}\left[\frac{3}{2}x^{1/2} - \frac{a}{2}\cdot x^{-1/2}\right] = \frac{1}{2\sqrt{(3a)}}\frac{3x-a}{\sqrt{x}}$$

$$\therefore \quad \text{Perimeter of the loop} = 2\int_0^a \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx \qquad \text{(By symmetry)}$$

$$= 2\int_0^a \sqrt{\left[1+\frac{(3x-a)^2}{12ax}\right]}dx = 2\int_0^a \frac{\sqrt{(9x^2+6ax+a^2)}}{\sqrt{(12ax)}}dx$$

$$= \frac{1}{\sqrt{(3a)}}\int_0^a \frac{3x+a}{\sqrt{x}}dx = \frac{1}{\sqrt{(3a)}}\int_0^a (3x^{1/2} + ax^{-1/2})\,dx$$

$$= \frac{1}{\sqrt{(3a)}}\left|\frac{3x^{3/2}}{3/2} + a\frac{x^{1/2}}{1/2}\right|_0^a = \frac{1}{\sqrt{(3a)}}(4a^{3/2}) = \frac{4a}{\sqrt{3}}.$$

Example 6.33. *Find the length of one arch of the cycloid*
$x = a(t - \sin t), y = a(1 - \cos t)$. (P.T.U., 2009 ; V.T.U., 2004)

Solution. As a point moves from one end O to the other end of its first arch, the parameter t increases from 0 to 2π. [see Fig. 6.8]

Also $\quad \frac{dx}{dt} = a(1-\cos t),\ \frac{dy}{dt} = a\sin t.$

$$\therefore \quad \text{Length of an arch} = \int_0^{2\pi}\sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}dx$$

$$= \int_0^{2\pi}\sqrt{\{[a(1-\cos t)]^2 + (a\sin t)^2\}}\,dt = a\int_0^{2\pi}\sqrt{[2(1-\cos t)]}\,dt$$

$$= 2a\int_0^{2\pi}\sin t/2\,dt = 2a\left|-\frac{\cos t/2}{1/2}\right|_0^{2\pi} = 4a[(-\cos\pi) - (-\cos 0)] = 8a.$$

Example 6.34. *Find the entire length of the cardioid $r = a(1+\cos\theta)$.*
(P.T.U., 2010 ; Bhopal, 2008 ; Kurukshetra, 2005)
Also show that the upper half is bisected by $\theta = \pi/3$. (Bhillai, 2005)

Solution. The cardioid is symmetrical about the initial line and for its upper half, θ increases from 0 to π (Fig. 6.18)

Also $\quad \frac{dr}{d\theta} = -a\sin\theta.$

$$\therefore \quad \text{Length of the curve} = 2\int_0^{\pi}\sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}d\theta$$

$$= 2\int_0^{\pi} \sqrt{\{[a(1+\cos\theta)]^2 + (-a\sin\theta)^2\}}\, d\theta = 2a\int_0^{\pi} \sqrt{[2(1+\cos\theta)]}\, d\theta$$

$$= 4a\int_0^{\pi} \cos\theta/2\, d\theta = 4a\left|\frac{\sin\theta/2}{1/2}\right|_0^{\pi} = 8a(\sin\pi/2 - \sin 0) = 8a.$$

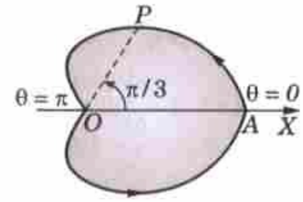

Fig. 6.18

$\therefore$ Length of upper half of the curve is $4a$. Also length of the arc AP from 0 to $\pi/3$.

$$= a\int_0^{\pi/3} \sqrt{[2(1+\cos\theta)]}\, d\theta = 2a\int_0^{\pi/3} \cos\theta/2 \cdot d\theta$$

$$= 4a\left|\sin\theta/2\right|_0^{\pi/3} = 2a = \text{half the length of upper half of the cardioid.}$$

PROBLEMS 6.8

1. Find the length of the arc of the *semi-cubical parabola* $ay^2 = x^3$ from the vertex to the ordinate $x = 5a$.
2. Find the length of the curve (*i*) $y = \log\sec x$ from $x = 0$ to $x = \pi/3$. (*V.T.U., 2010 S ; P.T.U., 2007*)
   (*ii*) $y = \log[(e^x - 1)/(e^x + 1)]$ from $x = 1$ to $x = 2$.
3. Find the length of the arc of the parabola $y^2 = 4ax$ (*i*) from the vertex to one end of the latus-rectum.
   (*ii*) cut off by the line $3y = 8x$. (*V.T.U., 2008 S ; Mumbai, 2006*)
4. Find the perimeter of the loop of the following curves :
   (*i*) $ay^2 = x^2(a - x)$ (*ii*) $9y^2 = (x-2)(x-5)^2$.
5. Find the length of the curve $y^2 = (2x - 1)^2$ cut off by the line $x = 4$. (*V.T.U., 2000 S*)
6. Show that the whole length of the curve $x^2(a^2 - x^2) = 8a^2y^2$ is $\pi a\sqrt{2}$.
7. (*a*) Find the length of an arch of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.
   (*b*) By finding the length of the curve show that the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, is divided in the ratio 1 : 3 at $\theta = 2\pi/3$. (*S.V.T.U., 2009*)
8. Find the whole length of the curve $x = a\cos^3 t$, $y = a\sin^3 t$ *i.e.*, $x^{2/3} + y^{2/3} = a^{2/3}$.
   (*V.T.U., 2010 ; Marathwada, 2008 ; Rajasthan, 2006*)
   Also show that the line $\theta = \pi/3$ divides the length of this *astroid* in the first quadrant in the ratio 1 : 3. (*Mumbai, 2001*)
9. Find the length of the loop of the curve $x = t^2$, $y = t - t^3/3$. (*Mumbai, 2001*)
10. For the curve $r = ae^{\theta}\cot\alpha$, prove that s/r = constant, s being measured from the origin.
11. Find the length of the curve $\theta = \dfrac{1}{2}\left(r + \dfrac{1}{r}\right)$ from $r = 1$ to $r = 3$. (*Marathwada, 2008*)
12. Find the perimeter of the *cardioid* $r = a(1 - \cos\theta)$. Also show that the upper half of the curve is bisected by the line $\theta = 2\pi/3$.
13. Find the whole length of the *lemniscate* $r^2 = a^2\cos 2\theta$.
14. Find the length of the parabola $r(1 + \cos\theta) = 2a$ as cut off by the latus-rectum. (*J.N.T.U., 2003*)

6.12 (1) VOLUMES OF REVOLUTION

(*a*) **Revolution about x-axis.** *The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curve y = f(x), the x-axis and the ordinates x = a, x = b is*

$$\int_a^b \pi y^2\, dx.$$

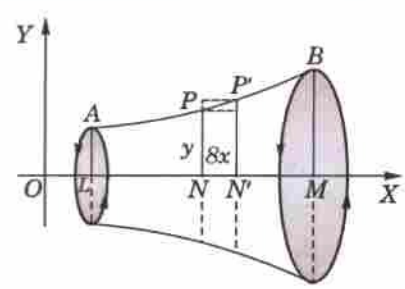

Fig. 6.19

Let AB be the curve $y = f(x)$ between the ordinates $LA(x = a)$ and $MB(x = b)$.

Let $P(x, y)$, $P'(x + \delta x, y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates (Fig. 6.19).

Let the volume of the solid generated by the revolution about x-axis of the area $ALNP$ be V, which is clearly a function of x. Then the volume of the solid generated by the revolution of the area $PNN'P'$ is δV. Complete the rectangles PN' and $P'N$.

The δV lies between the volumes of the right circular cylinders generated by the revolution of rectangles PN' and $P'N$,

i.e., δV lies between $\pi y^2\, \delta x$ and $\pi(y + \delta y)^2\, \delta x$.

$\therefore$ $\dfrac{\delta V}{\delta x}$ lies between πy^2 and $\pi(y + \delta y)^2$.

Now taking limits as $P' \to P$, *i.e.*, $\delta x \to 0$ (and $\therefore$ $\delta y \to 0$), $\dfrac{dV}{dx} = \pi y^2$

$$\therefore \quad \int_a^b \frac{dV}{dx}\, dx = \int_a^b \pi y^2\, dx \quad \text{or} \quad \left| V \right|_{x=a}^{b} = \int_a^b \pi y^2\, dx$$

or (value of V for $x = b$) – (value of V for $x = a$)

i.e., volume of the solid obtained by the revolution of the area $ALMB = \int_a^b \pi y^2\, dx$.

Example 6.35. *Find the volume of a sphere of radius a.* (S.V.T.U., 2007)

Solution. Let the sphere be generated by the revolution of the semi-circle ABC, of radius a about its diameter CA (Fig. 6.20)

Taking CA as the x-axis and its mid-point O as the origin, the equation of the circle ABC is $x^2 + y^2 = a^2$.

$\therefore$ Volume of the sphere = 2 (volume of the solid generated by the revolution about x-axis of the quadrant OAB)

$$= 2 \int_0^a \pi y^2\, dx = 2\pi \int_0^b (a^2 - x^2)\, dx$$

$$= 2\pi \left| a^2 x - \frac{x^3}{3} \right|_0^a = 2\pi \left[a^3 - \frac{a^3}{3} - (0 - 0) \right] = \frac{4}{3}\pi a^3.$$

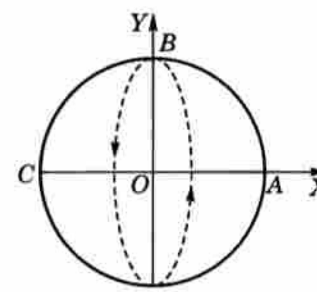

Fig. 6.20

Example 6.36. *Find the volume formed by the revolution of loop of the curve $y^2(a + x) = x^2(3a - x)$, about the x-axis.* (Marathwada, 2008)

Solution. The curve is symmetrical about the x-axis, and for the upper half of its loop x varies from 0 to $3a$ (Fig. 6.21)

$$\therefore \quad \text{Volume of the loop} = \int_0^{3a} \pi y^2\, dx = \pi \int_0^{3a} \frac{x^2(3a - x)}{a + x}\, dx$$

$$= \pi \int_0^{3a} \frac{-x^3 + 3ax^2}{x + a}\, dx$$

[Divide the numerator by the denominator]

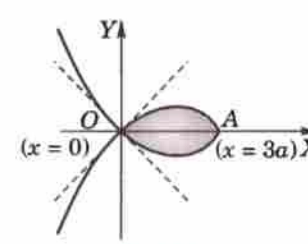

Fig. 6.21

$$= \pi \int_0^{3a} \left[-x^2 + 4ax - 4a^2 + \frac{4a^3}{x + a} \right] dx = \pi \left| -\frac{x^3}{3} + 4a \cdot \frac{x^2}{2} - 4a^2 x + 4a^3 \log (x + a) \right|_0^{3a}$$

$$= \pi \left[\frac{-27a^3}{3} + 2a \cdot 9a^2 - 4a^2 \cdot 3a + 4a^3 \log 4a - (4a^3 \log a) \right]$$

$$= \pi a^3 (-3 + 4 \log 4) = \pi a^3 (8 \log 2 - 3).$$

Example 6.37. *Prove that the volume of the reel formed by the revolution of the cycloid $x = a\,(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ about the tangent at the vertex is $\pi^2 a^3$.* (V.T.U., 2003)

Solution. The arch AOB of the cycloid is symmetrical about the y-axis and the tangent at the vertex is the x-axis. For half the cycloid OA, θ varies from 0 to π. (Fig. 4.31).

Hence the required volume

$$= 2 \int_{\theta=0}^{\theta=\pi} \pi y^2\, dx = 2\pi \int_0^{\pi} a^2 (1 - \cos \theta)^2 \,.\, a\,(1 + \cos \theta)\, d\theta$$

$$= 2\pi a^3 \int_0^\pi (2\sin^2\theta/2)^2 \cdot (2\cos^2\theta/2)\, d\theta$$

$$= 16\pi a^3 \int_0^\pi \sin^4\theta/2 \cdot \cos^2\theta/2 \cdot d\theta \qquad [\text{Put } \theta/2 = \phi,\ d\theta = 2d\phi]$$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^4\phi\cos^2\phi\, d\phi = 32\pi a^3 \cdot \frac{3.1\times 1}{6.4.2}\cdot\frac{\pi}{2} = \pi^2 a^3.$$

Example 6.38. *Find the volume of the solid formed by revolving about x-axis, the area enclosed by the parabola $y^2 = 4ax$, its evolute $27ay^2 = 4(x-2a)^3$ and the x-axis.*

Solution. The curve $27ay^2 = 4(x-2a)^3$ ...(*i*)

is symmetrical about x-axis and is a semi-cubical parabola with vertex at A $(2a, 0)$. The parabola $y^2 = 4ax$ and (*i*) intersect at B and C where $27a\,(4ax) = 4(x-2a)^3$ or $x^3 - 6ax^2 - 15a^2x - 8a^3 = 0$ which gives $x = -a, -a, 8a$. Since x is not negative, therefore we have $x = 8a$ (Fig. 6.22).

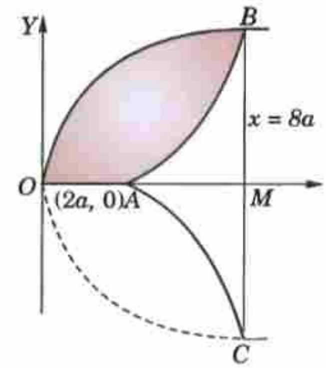

Fig. 6.22

$\therefore$ Required volume = Volume obtained by revolving the shaded area OAB about x-axis = Vol. obtained by revolving area $OMBO$ – Vol. obtained by revolving area $ADBA$

$$= \int_0^{8a} \pi y^2 (= 4ax)\, dx - \int_{2a}^{8a} \pi y^2 \,[\text{for } (i)]\, dx$$

$$= 4a\pi \left| \frac{x^2}{2} \right|_0^{8a} - \frac{4\pi}{27a}\int_{2a}^{8a} (x-2a)^3\, dx$$

$$= 128\pi a^3 - \frac{4\pi}{27a}\left| \frac{(x-2a)^4}{4} \right|_{2a}^{8a}$$

$$= 128\pi a^3 - \frac{\pi}{27a}(6a)^4 = 80\pi a^3.$$

(*b*) **Revolution about the y-axis.** Interchanging x and y in the above formula, we see that the *volume of the solid generated by the revolution about y-axis, of the area, bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$ is*

$$\int_a^b \pi\, \mathbf{x}^2\, \mathbf{dy}.$$

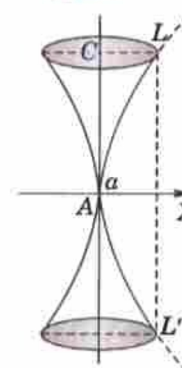

Fig. 6.23

Example 6.39. *Find the volume of the reel-shaped solid formed by the revolution about the y-axis, of the part of the parabola $y^2 = 4ax$ cut off by the latus-rectum.* (Rohtak, 2003)

Solution. Given parabola is $x = y^2/4a$.

Let A be the vertex and L one extremity of the latus-rectum. For the arc AL, y varies from 0 to $2a$ (Fig. 6.23).

$\therefore$ required volume = 2 (volume generated by the revolution about the y-axis of the area ALC)

$$= 2\int_0^{2a} \pi x^2\, dy = 2\pi\int_0^{2a} \frac{y^4}{16a^2}\cdot dy = \frac{\pi}{8a^2}\left| \frac{y^5}{5} \right|_0^{2a} = \frac{\pi}{40a^2}(32a^5 - 0) = \frac{4\pi a^3}{5}.$$

(*c*) **Revolution about any axis.** *The volume of the solid generated by the revolution about any axis LM of the area bounded by the curve AB, the axis LM and the perpendiculars AL, BM on the axis, is*

$$\int_{OL}^{OM} \pi(\mathbf{PN})^2\ \mathbf{d(ON)}$$

where O is a fixed point in LM and PN is perpendicular from any point P of the curve AB on LM.

With O as origin, take OLM as the x-axis and OY, perpendicular to it as the y-axis (Fig. 6.24).

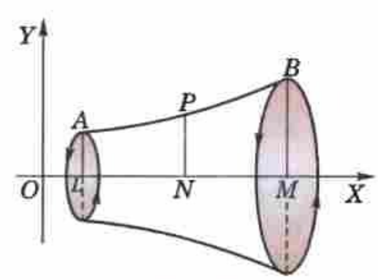

Fig. 6.24

Let the coordinates of P be (x, y) so that $x = ON, y = NP$

If $OL = a$, $OM = b$, then required volume $= \int_a^b \pi y^2\, dx = \int_{OL}^{OM} \pi(PN)^2\, d(ON)$.

Example 6.40. *Find the volume of the solid obtained by revolving the cissoid $y^2(2a - x) = x^3$ about its asymptote.* *(V.T.U., 2000)*

Solution. Given curve is $y = \dfrac{x^3}{2a - x}$ ...(*i*)

It is symmetrical about x-axis and the asymptote is $x = 2a$. (See Fig. 4.23). If $P(x, y)$ be any point on it and PN is perpendicular on the asymptote AN then $PN = 2a - x$ and

$$AN = y = \frac{x^{3/2}}{\sqrt{(2a - x)}} \qquad \text{[From (i)]}$$

$$\therefore \quad d(AN) = dy = \frac{\sqrt{(2a-x)}\,(3/2)\sqrt{x} - x^{3/2}\cdot\frac{1}{2}(2a-x)^{-1/2}(-1)}{2a - x}\, dx$$

$$= \frac{3\sqrt{x}(2a-x) + x^{3/2}}{2(2a-x)^{3/2}}\, dx = \frac{3ax^{1/2} - x^{3/2}}{(2a-x)^{3/2}}\, dx$$

$$\therefore \text{ Required volume } = 2\int_{x=0}^{x=2a} \pi(PN)^2\, d(AN) = 2\pi\int_0^{2a} (2a-x)^2 \cdot \frac{3ax^{1/2} - x^{3/2}}{(2a-x)^{3/2}}\cdot dx$$

$$= 2\pi\int_0^{2a} \sqrt{(2a-x)}\,(3a-x)\sqrt{x}\, dx \qquad \left[\begin{array}{l}\text{Put } x = 2a\sin^2\theta \\ \text{then } dx = 4a\sin\theta\cos\theta\, d\theta\end{array}\right.$$

$$= 2\pi\int_0^{\pi/2} \sqrt{(2a)}\cos\theta\,(3a - 2a\sin^2\theta)\, x\, \sqrt{(2a)}\sin\theta \,.\, 4a\sin\theta\cos\theta\, d\theta$$

$$= 16\pi a^3\left[3\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta - 2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\, d\theta\right]$$

$$= 16\pi a^3\left[3\cdot\frac{1\times 1}{4\cdot 2}\cdot\frac{\pi}{2} - 2\cdot\frac{3\cdot 1\times 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}\right] = 2\pi^2 a^3.$$

(2) Volumes of revolution (polar curves). *The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha, \theta = \beta$ (Fig. 6.25)*

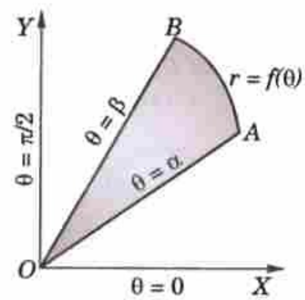

Fig. 6.25

(*a*) *about the initial line OX* $(\theta = 0) = \displaystyle\int_\alpha^\beta \frac{2\pi}{3}\mathbf{r}^3 \sin\theta\, \mathbf{d}\theta$

(*b*) *about the line OY* $(\theta = \pi/2) = \displaystyle\int_\alpha^\beta \frac{2\pi}{3}\mathbf{r}^3\cos\theta\, \mathbf{d}\theta.$

Example 6.41. *Find the volume of the solid generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line.* *(V.T.U., 2010 ; Kurukshetra, 2009 S)*

Solution. The cardioid is symmetrical about the initial line and for its upper half θ varies from 0 to π. [Fig. 6.18].

$$\therefore \text{ required volume } = \int_0^\pi \frac{2}{3}\pi r^3 \sin\theta\, d\theta = \frac{2\pi}{3}\int_0^\pi a^3(1+\cos\theta)^3 \sin\theta\, d\theta$$

$$= -\frac{2\pi a^3}{3}\int_0^\pi (1+\cos\theta)^3\cdot(-\sin\theta)\, d\theta = -\frac{2\pi a^3}{3}\left|\frac{(1+\cos\theta)^4}{4}\right|_0^\pi = -\frac{\pi a^3}{6}[0 - 16] = \frac{8}{3}\pi a^3.$$

Example 6.42. *Find the volume of the solid generated by revolving the lemniscate $r^2 = a^2\cos 2\theta$ about the line $\theta = \pi/2$.* *(V.T.U., 2006)*

Solution. The curve is symmetrical about the pole. For the upper half of the R.H.S. loop, θ varies from 0 to $\pi/4$. (Fig. 4.34).

$\therefore$ required volume = 2(volume generated by the half loop in the first quadrant)

$$= 2\int_0^{\pi/4} \frac{2}{3}\pi r^3 \cos\theta \, d\theta = \frac{4\pi}{3} \cdot \int_0^{\pi/4} a^3 (\cos 2\theta)^{3/2} \cos\theta \, d\theta \qquad [\because \; r = a(\cos 2\theta)^{1/2}]$$

$$= \frac{4\pi a^3}{3} \int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta \, d\theta \qquad \left[\begin{array}{l}\text{Put } \sqrt{2}\sin\theta = \sin\phi \\ \therefore \; \sqrt{2}\cos\theta\, d\theta = \cos\phi\, d\theta\end{array}\right.$$

$$= \frac{4\pi a^3}{3} \int_0^{\pi/2} (1 - \sin^2\phi)^{3/2} \cdot \frac{1}{\sqrt{2}} \cos\phi \, d\phi = \frac{4\pi a^3}{3\sqrt{2}} \int_0^{\pi/2} \cos^4\phi \, d\phi = \frac{4\pi}{3\sqrt{2}} a^3 \frac{3.1}{4.2} \cdot \frac{\pi}{2} = \frac{\pi a^3}{4\sqrt{2}}.$$

PROBLEMS 6.9

1. Find the volume generated by the revolution of the area bounded by x-axis, the *catenary* $y = c \cosh x/c$ and the ordinates $x = \pm c$, about the axis of x.
2. Find the volume of a spherical segment of height h cut off from a sphere of radius a.
3. Find the volume generated by revolving the portion of the parabola $y^2 = 4ax$ cut off by its latus-rectum about the axis of the parabola. *(V.T.U., 2009)*
4. Find the volume generated by revolving the area bounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x = 0$, $y = 0$ about the x-axis.
5. Find the volume of the solid generated by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$.
   (*i*) about the major axis. *(Bhopal, 2002 S)* (*ii*) about the minor axis. *(Bhillai, 2005)*
6. Obtain the volume of the frustrum of a right circular cone whose lower base has radius R, upper base is of radius r and altitude is h.
7. Find the volume generated by the revolution of the curve $27ay^2 = 4(x - 2a)^3$ about the x-axis.
8. Find the volume of the solid formed by the revolution, about the x-axis, of the loop of the curve :
   (*i*) $y^2(a - x) = x^2(a + x)$ (*ii*) $2ay^2 = x(x - a)^2$ (*iii*) $y^2 = x(2x - 1)^2$.
9. Find the volume obtained by revolving one arch of the cycloid
   (*i*) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, about its base. *(Kurukshetra, 2006 ; V.T.U., 2005)*
   (*ii*) $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$, about the x-axis.
10. Find the volume of the spindle-shaped solid generated by the revolution of the *astroid* $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis. *(P.T.U., 2010 ; S.V.T.U., 2008)*
11. Find the volume of the solid formed by the revolution, about the y-axis, of the area enclosed by the curve $xy^2 = 4a^2(2a - x)$ and its asymptote. *(V.T.U., 2006)*
12. Prove that the volume of the solid formed by the revolution of the curve $(a^2 + x^2) = a^3$, about its asymptote is $\frac{1}{2}\pi^2 a^3$.
13. Find the volume generated by the revolution about the initial line of
    (*i*) $r = 2a\cos\theta$ (*ii*) $r = a(1 - \cos\theta)$. *(P.T.U., 2006)*
14. Determine the volume of the solid obtained by revolving the *lemicon* $r = a + b\cos\theta$ $(a > b)$ about the initial line. *(Gorakhpur, 1999)*
15. Find the volume of the solid formed by revolving a loop of the lemniscate $r^2 = a^2\cos 2\theta$ about the initial line. *(J.N.T.U., 2003 ; Delhi, 2002)*

6.13 SURFACE AREAS OF REVOLUTION

(*a*) **Revolution about x-axis.** *The surface area of the solid generated by the revolution about x-axis, of the arc of the curve $y = f(x)$ from $x = a$ to $x = b$, is*

$$\int_{x=a}^{x=b} 2\pi y \, ds.$$

Let AB be the curve $y = f(x)$ between the ordinates LA $(x = a)$ and MB $(x = b)$. Let $P(x, y)$, $P'(x + \delta x y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates (Fig. 6.19).

Let the arc $AP = s$ so that arc $PP' = \delta s$. Let the surface-area generated by the revolution about x-axis of the arc AP be S and that generated by the revolution of the arc PP' be δS.

Since δs is small, the surface area δS may be regarded as lying between the curved surfaces of the right cylinders of radii PN and $P'N'$ and of same thickness δs.

Thus δS lies between $2\pi y\, \delta s$ and $2\pi (y + \delta y)\, \delta s$

$\therefore \quad \dfrac{\delta S}{\delta s}$ lies between $2\pi y$ and $2\pi (y + \delta y)$

Taking limits as $P' \to P$, *i.e.*, $\delta s \to 0$ and $\delta y \to 0$, $dS/dx = 2\pi y$

$$\therefore \quad \int_{x=a}^{x=b} \frac{dS}{ds}\, ds = \int_{x=a}^{x=b} 2\pi y\, ds \quad \text{or} \quad \Big| S \Big|_{x=a}^{x=b} = \int_{x=a}^{x=b} 2\pi y\, ds$$

or (value of S for $x = b$) – (value of S for $x = a$) = $\displaystyle\int_{x=a}^{x=b} 2\pi y\; dx$

or surface area generated by the revolution of the arc $AB - 0 = \displaystyle\int_{x=a}^{x=b} 2\pi y\; ds.$

Hence, the required surface area = $\displaystyle\int_{x=a}^{x=b} 2\pi y\; ds.$

Obs. Practical forms of the formula $S = \int 2\pi y\, ds$.

(*i*) *Cartesian form* [*for the curve* $y = f(x)$]

$$S = \int 2\pi y \frac{ds}{dx}\, dx, \text{ where } \frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$$

(*ii*) *Parametric form* [*for the curve* $x = f(t), y = \phi(t)$]

$$S = \int 2\pi y \frac{ds}{dt}\, dt, \text{ where } \frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}$$

(*iii*) *Polar form* [*for the curve* $r = f(\theta)$]

$$S = \int 2\pi y \frac{ds}{d\theta}\, d\theta, \text{ where } y = r \sin\theta,\ \frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}.$$

Example 6.43. *Find the surface of the solid formed by revolving the cardioid $r = a\,(1 + \cos\theta)$ about the initial line.* (*V.T.U., 2009 ; Rajasthan, 2006 ; J.N.T.U., 2003*)

Solution. The cardioid is symmetrical about the initial line and for its upper half, θ varies from 0 to π (Fig. 6.18).

Also
$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} = \sqrt{[a^2(1 + \cos\theta)^2 + a^2 \sin^2\theta]}$$

$$= a\sqrt{[2(1 + \cos\theta)]} = a\sqrt{[2.2\cos^2\theta/2]} = 2a\cos\theta/2$$

$$\therefore \quad \text{required surface} = \int_0^\pi 2\pi y \frac{ds}{d\theta}\, d\theta = 2\pi \int_0^\pi r \sin\theta \cdot 2a\cos\theta/2\; d\theta$$

$$= 4\pi a \int_0^\pi a(1 + \cos\theta)\, \sin\theta \cdot \cos\theta/2\, d\theta = 4\pi a^2 \int_0^\pi 2\cos^2\frac{\theta}{2} \,.\, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \,.\, \cos\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^\pi \cos^4\frac{\theta}{2} \sin\frac{\theta}{2}\, d\theta = 16\pi a^2 (-2) \int_0^\pi \cos^4\frac{\theta}{2}\left(-\sin\frac{\theta}{2}\cdot\frac{1}{2}\right) d\theta$$

$$= -32\pi a^2 \left| \frac{\cos^5\theta/2}{5} \right|_0^\pi = \frac{-32\pi a^2}{5}(0 - 1) = \frac{32\pi a^2}{5}.$$

(*b*) **Revolution about y-axis.** Interchanging x and y in the above formula, we see that *the surface of the solid generated by the revolution about y-axis, of the arc of the curve $x = f(y)$ from $y = a$ to $y = b$ is*

$$\int_{y=a}^{y=b} 2\pi x\;ds.$$

Example 6.44. *Find the surface area of the solid generated by the revolution of the astroid $x = a\cos^3 t$, $y = a \sin^3 t$, about the y-axis.*

Solution. The astroid is symmetrical about the x-axis, and for its portion in the first quadrant t varies from 0 to $\pi/2$. (Fig. 4.29).

Also $$\frac{dx}{dt} = -3a\cos^2 t \sin t,\ \frac{dy}{dt} = 3a\sin^2 t\cos t.$$

$$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} = \sqrt{[9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t]}$$

$$= 3a\sin t\cos t\sqrt{(\cos^2 t + \sin^2 t)} = 3a\sin t\cos t$$

$\therefore$ required surface $$= 2\int_0^{\pi/2} 2\pi x\frac{ds}{dt}\cdot dt = 4\pi\int_0^{\pi/2} a\cos^3 t\cdot 3a\sin t\cos t\,dt$$

$$= 12\pi a^2\int_0^{\pi/2}\sin t\cos^4 t\,dt = 12\pi a^2\frac{3.1}{5.3.1} = \frac{12\pi a^2}{5}.$$

PROBLEMS 6.10

1. Find the area of the surface generated by revolving the arc of the catenary $y = c\cosh x/c$ from $x = 0$ to $x = c$ about the x-axis.
2. Find the area of the surface formed by the revolution of $y^2 = 4ax$ about its axis, by the arc from the vertex to one end of the latus-rectum.
3. Find the surface of the solid generated by the revolution of the *ellipse* $x^2/a^2 + y^2/b^2 = 1$ about the x-axis. *(Raipur, 2005 ; Bhopal, 2002 S)*
4. Find the volume and surface of the *right circular cone* formed by the revolution of a right-angled triangle about a side which contains the right angle.
5. Obtain the surface area of a *sphere* of radius a.
6. Show that the surface area of the solid generated by the revolution of the curve $x = a\cos^3 t$, $y = a\sin^3 t$ about the x-axis, is $12\pi^2/5$.
7. The arc of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ in the first quadrant revolves about x-axis. Show that the area of the surface generated is $6\pi a^2/5$.
8. Find the surface area of the solid generated by revolving the *cycloid* $x = a(t - \sin t)$, $y = a(1 - \cos t)$ about the base. *(Marathwada, 2008 ; Cochin, 2005 ; Kurukshetra, 2005)*
9. Find the surface area of the solid got by revolving the arch of the *cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about the base. *(V.T.U., 2010 S)*
10. Prove that the surface and volume of the solid generated by the revolution about the x-axis, of the loop of the curve $x = t^2$, $y = t - t^3/3$, [or $9y^2 = x(x-3)^2$], are respectively 3π and $3\pi/4$.
11. Prove that the surface of the solid generated by the revolution of the *tractrix* $x = a\cos t + \frac{a}{2}\log\tan^2 t/2$, $y = a\sin t$, about x-axis is $4\pi a^2$.
12. Find the surface area of the solid of revolution of the curve $r = 2a\cos\theta$ about the initial line. *(V.T.U., 2009)*
13. Find the surface of the solid generated by the revolution of the *cardioid* $r = a(1 - \cos\theta)$ about the initial line.
14. Find the surface of the solid generated by the revolution of the *lemniscate* $r^2 = a^2\cos 2\theta$ about the initial line. *(V.T.U., 2005)*
15. The part of *parabola* $y^2 = 4ax$ cut off by the latus-rectum revolves about the tangent at the vertex. Find the curved surface of the reel thus formed.

6.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 6.11

Choose the correct answer or fill up the blanks in the following problems :

1. If $f(x) = f(2a - x)$, then $\int_0^{2a} f(x)\,dx$ is equal to

(a) $\int_a^0 f(2a-x)\,dx$ (b) $2\int_0^a f(x)\,dx$ (c) $-2\int_0^a f(x)\,dx$ (d) 0.

2. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ is equal to

(a) 0 (b) 1 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$.

3. The value of definite integral $\int_{-a}^{a} |x|\,dx$ is equal to

(a) a (b) a^2 (c) 0 (d) $2a$.

4. $\lim_{n\to\infty}\left[\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+(n-1)^2}\right]$ is equal to

(a) $-\frac{\pi}{4}$ (b) 0 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$.

5. $\int_0^{\pi/2} \frac{\cos 2x}{\cos x+\sin x}\,dx$ equals

(a) -1 (b) 0 (c) 1 (d) 2.

6. $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{3n}\right)$ equals

(a) $\log_e 2$ (b) $2\log_e 2$ (c) $\log_e 3$ (d) $2\log_e 3$.

7. $\int_0^{\pi} \sin^5\left(\frac{x}{2}\right)$ is equal to

(a) $\frac{16}{15}$ (b) $\frac{15}{16}\pi$ (c) $\frac{16}{15}\pi^2$ (d) $\frac{15}{16}$.

8. $\int_0^{\pi/2} \sin^{99} x\,\cos x\,dx$ is equal to

(a) $\frac{1}{99}$ (b) $\frac{\pi}{100}$ (c) $\frac{99}{100}$ (d) None of these. (*V.T.U., 2009*)

9. The value of $\int_{-\pi/2}^{\pi/2} \cos^7 x\,dx$ is

(a) $\frac{32\pi}{35}$ (b) $\frac{32}{35}$ (c) zero.

10. The length of the arc of the equiangular spiral $r = ae^{\theta\cot\alpha}$ between the points for which the radii vectors are r_1 and r_2 is

(a) (r_2-r_1) cosec α (b) $(r_2-r_1)\cos\alpha$ (c) $(r_2-r_1)\sin\alpha$ (d) $(r_2-r_1)\sec\alpha$.

11. The area of the region in the first quadrant bounded by the y-axis and the curves $y = \sin x$ and $y = \cos x$ is

(a) $\sqrt{2}$ (b) $\sqrt{2}+1$ (c) $\sqrt{2}-1$ (d) $2\sqrt{2}-1$.

12. The value of $\int_0^1 x^{3/2}(1-x)^{3/2}\,dx$ is

(a) $\pi/32$ (b) $-\pi/32$ (c) $3\pi/128$ (d) $-3\pi/128$. (*V.T.U., 2010*)

13. The entire length of the cardioid $r = 5(1+\cos\theta)$ is

(a) 40 (b) 30 (c) 20 (d) 5. (*V.T.U., 2009*)

14. The area of the cardioid $r = a(1-\cos\theta)$ is ....

15. If S_1 and S_2 are surface areas of the solids generated by revolving the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ about the y-axis, then

(a) $S_1 > S_2$ (b) $S_1 < S_2$ (c) $S_1 = S_2$ (d) can't predict.

16. The area of the loop of the curve $r = a\sin 3\theta$ is ....

17. If $I_n = \int_0^{\pi/4} \tan^n\theta\,d\theta$, then $n(I_{n-1}+I_{n+1}) =$ ....

18. $\int_0^2 x^3\sqrt{(2x-x^2)}\,dx =$ ....

19. $\int_0^{\pi/2} \sin 2\theta \log \tan \theta \, d\theta$ is equal to
(a) 1 (b) – 1 (c) 0 (d) $\pi/2$.
20. The area of the loop of the folium of Descartes $x^3 + y^3 - 3xy = 0$ is
(a) π (b) $\pi/2$ (c) 1.5 (d) 3.
21. The volume of the frustrum of a right circular cone whose lower base has radius r_1 and upper base has radius r_2 and altitude is h = ...
22. The length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \pi/4$ is
(a) $\log_e 2$ (b) $\log_e 3$ (c) $\log_e (1 + \sqrt{2})$ (d) $\log_e (1 + \sqrt{3})$. (*Bhopal, 2008*)
23. If v_1 = volume of the solid generated by revolving the area included between x-axis and $x^2 + y^2 = a^2$ about x-axis ; v_2 = volume of the solid generated by revolving the entire area of the circle $x^2 + y^2 = a^2$ about x-axis, then
(a) $v_1 = v_2$ (b) $v_2 = 2v_1$ (c) $v_2 = 4v_1$ (d) $v_2 = 16v_1$.
24. If $f(r, \theta) = f(-r, \theta)$, then the curve is symmetrical about the ....
(a) initial line (b) pole (c) origin (d) tangential line. (*V.T.U., 2010*)
25. The volume generated by the revolution of the curve $y = a^3 (a^2 + x^2)^{-1}$ about its asymptote is
(a) $\pi^2 a^3/2$ (b) $\pi a^3/2$ (c) $\pi a^2/2$ (d) $\pi a/2$. (*V.T.U., 2010*)