CHAPTER 19

Complex Numbers and Functions

1. Complex Numbers. 2. Argand's diagram. 3. Geometric representation of $z_1 \pm z_2$; z_1z_2 and z_1/z_2. 4. De Moivre's theorem. 5. Roots of a complex number. 6. To expand sin $n\theta$, cos $n\theta$ and tan $n\theta$ in powers of sin θ, cos θ and tan θ respectively; Addition formulae for any number of angles; To expand $\sin^m \theta$, $\cos^n \theta$ and $\sin^m \theta \cos^n \theta$ in a series of sines or cosines of multiples of θ. 7. Complex function: Definition. 8. Exponential function of a complex variable. 9. Circular functions of a complex variable. 10. Hyperbolic functions. 11. Inverse hyperbolic functions. 12. Real and imaginary parts of circular and hyperbolic functions. 13. Logarithmic functions of a complex variable. 14. Summation of series – '$C + iS$' method. 15. Approximations and Limits. 16. Objective Type of Questions.

19.1 COMPLEX NUMBERS

Definition. *A number of the form $x + iy$, where x and y are real numbers and $i = \sqrt{(-1)}$, is called a* **complex number.**

x is called the *real part* of $x + iy$ and is written as $R(x + iy)$ and y is called the *imaginary part* and is written as $I(x + iy)$.

A pair of complex numbers $x + iy$ and $x - iy$ are said to be conjugate of each other.

Properties : (1) *If $x_1 + iy_1 = x_2 + iy_2$, then $x_1 - iy_1 = x_2 - iy_2$.*

(2) *Two complex numbers $x_1 + iy_1$ = and $x_2 + iy_2$ are said to be equal when*

$$R(x_1 + iy_1) = R(x_2 + iy_2), \text{ i.e., } x_1 = x_2$$

and

$$I(x_1 + iy_1) = I(x_2 + iy_2), \text{ i.e., } y_1 = y_2.$$

(3) *Sum, difference, product and quotient of any two complex numbers is itself a complex number.*

If $x_1 + iy_1$ and $x_2 + iy_2$ be two given complex numbers, then

(*i*) their sum $= (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

(*ii*) their difference $= (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

(*i*) their product $= (x_1 + iy_1) + (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$

and (*iv*) their quotient $= \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\dfrac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$

(4) *Every complex number $x + iy$ can always be expressed in the form r (cos θ + i sin θ).*

Put $\quad R(x + iy)$, *i.e.*, $x = r \cos \theta$ ...(*i*)

and $\quad I(x + iy)$, *i.e.*, $y = r \sin \theta$ ...(*ii*)

Squaring and adding, we get $x^2 + y^2 = r^2$ *i.e.* $r = \sqrt{(x^2 + y^2)}$ (taking positive square root only)

Dividing (*ii*) by (*i*), we get $y/x = \tan\theta$ *i.e.* $\theta = \tan^{-1}(y/x)$.

Thus $x + iy = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{(x^2 + y^2)}$ and $\theta = \tan^{-1}(y/x)$.

Definitions. *The number* $r = +\sqrt{(x^2 + y^2)}$ *is called the* **modulus** *of* $x + iy$ *and is written as mod* $(x + iy)$ *or* $|x + iy|$.

The angle θ *is called the amplitude or argument of* $x + iy$ *and is written as amp* $(x + iy)$ *or arg* $(x + iy)$.

Evidently the amplitude θ has an infinite number of values. The value of θ which lies between $-\pi$ and π is called the *principal value of the amplitude*. Unless otherwise specified, we shall take amp (z) to mean the principal value.

Note. *$\cos\theta + i\sin\theta$ is briefly written as cis θ (pronounced as 'sis θ')*

(5) *If the conjugate of* $z = x + iy$ be $\bar{z}$, *then*

(*i*) $R(z) = \frac{1}{2}(z + \bar{z}), I(z) = \frac{1}{2i}(z - \bar{z})$ (*ii*) $|z| = \sqrt{R^2(z) + I^2(z)} = |\bar{z}|$

(*iii*) $z\bar{z} = |z|^2$ (*iv*) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(*v*) $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$ (*vi*) $\overline{(z_1 / z_2)} = \bar{z}_1 / \bar{z}_2$, where $\bar{z}_2 \neq 0$.

Example 19.1. *Reduce* $1 - \cos\alpha + i\sin\alpha$ *to the modulus amplitude form.*

Solution. Put $1 - \cos\alpha = r\cos\theta$ and $\sin\alpha = r\sin\theta$

$\therefore$ $$r = (1 - \cos\alpha)^2 + \sin^2\alpha = 2 - 2\cos\alpha = 4\sin^2\alpha/2$$

i.e., $$r = 2\sin\alpha/2$$

and $$\tan\theta = \frac{\sin\alpha}{1 - \cos\alpha} = \frac{2\sin\alpha/2\cos\alpha/2}{2\sin^2\alpha/2} = \cot\alpha/2$$

$$= \tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) \quad \therefore \theta = \frac{\pi - \alpha}{2}.$$

Thus $$1 - \cos\alpha + i\sin\alpha = 2\sin\frac{\alpha}{2}\left[\cos\frac{\pi - \alpha}{2} + i\sin\frac{\pi - \alpha}{2}\right].$$

Example 19.2. *Find the complex number* z *if arg* $(z + 1) = \pi/6$ *and arg* $(z - 1) = 2\pi/3$. (*Mumbai, 2009*)

Solution. Let $z = x + iy$ so that $z + 1 = (x + 1) + iy$ and $(z - 1) = (x - 1) + iy$

Since $arg\,(z + 1) = \pi/6$, $\therefore \tan^{-1}\left(\frac{y}{x+1}\right) = 30°$

i.e., $$\frac{y}{x+1} = \tan 30° = 1/\sqrt{3}, \text{ or } \sqrt{3y} = x + 1 \qquad ...(i)$$

Also since $\arg(z - 1) = 2\pi/3$, $\therefore \tan^{-1}\left(\frac{y}{x+1}\right) = 120°$

i.e., $$\frac{y}{x-1} = \tan 120° = -\sqrt{3}, \text{ or } y = -\sqrt{3}x + \sqrt{3} \text{ or } \sqrt{3}y = -3x + 3 \qquad ...(ii)$$

Subtracting (*ii*) from (*i*), we get $4x - 2 = 0$ *i.e.,* $x = 1/2$

From (*i*), $\sqrt{3}\,y = 1/2 + 1$, *i.e.,* $y = \sqrt{3}/2$

Hence $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Example 19.3. *Find the real values of* x, y *so that* $-3 + ix^2y$ *and* $x^2 + y + 4i$ *may represent complex conjugate numbers.*

Solution. If $z = -3 + ix^2y$, then $\bar{z} = x^2 + y + 4i$

so that $$z = (x^2 + y) - 4i$$

$\therefore$ $$-3 + ix^2y = x^2 + y - 4i$$

Equating real and imaginary parts from both sides, we get

$$-3 = x^2 + y, x^2y = -4$$

Eliminating x, $(y+3)y = -4$

or $y^2 + 3y - 4 = 0$ *i.e.*, $y = 1$ or -4

When $y = 1$, $x^2 = -3 - 1$ or $x = +2i$ which is not feasible

When $y = -4$, $x^2 = 1$ or $x = \pm 1$

Hence $x = 1$, $y - 4$ or $x = -1, y = -4$.

19.2 (1) GEOMETRIC REPRESENTATION OF IMAGINARY NUMBERS

Let all the real numbers be represented along $X'OX$, the positive real numbers being along OX and negative ones along OX'. Let OA be equal to one unit of measurement (Fig. 19.1).

Take a point L on OX such that $OL = x\,(OA)$.

Then L on OX represents the positive real number x and $i.ix = i^2x = -x$ is represented by a point L' on OX' distant OL from O.

From this we infer that the multiplication of the real number x by i twice amounts to the rotation of OL through two right angles to the position OL'.

Thus it naturally follows that the multiplication of a real number by i is equivalent to the rotation of OL through one right angle to the position OL''.

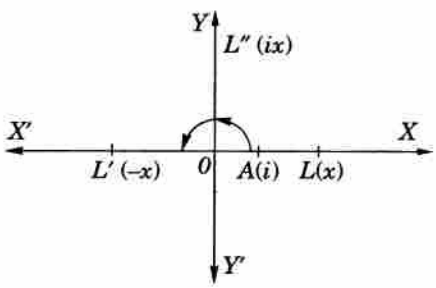

Fig. 19.1

Hence, if $Y'OY$ be a line perpendicular to the real axis $X'OX$, then all imaginary numbers are represented by points on $Y'OY$, called the **imaginary axis**, *the positive ones along OY and negative ones along OY'.**

Obs. Geometric interpretation of i*. From the above, it is clear that i is an operation which when multiplied to any real number makes it imaginary and rotates its direction through a right angle on the complex plane.

(2) Geometric representation of complex numbers†

Consider two lines $X'OX$, $Y'OY$ at right angles to each other.

Let all the real numbers be represented by points on the line $X'OX$ (called the *real axis*), positive real numbers being along OX and negative ones along OX'. Let the point L on OX represent the real number x (Fig. 19.2).

Since the multiplication of a real number by i is equivalent to the rotation of its direction through a right angle. Therefore, let all the imaginary numbers be represented by points on the line $Y'OY$ (called the *imaginary axis*), the positive ones along OY and negative ones along OY'. Let the point M on OY represent the imaginary number iy.

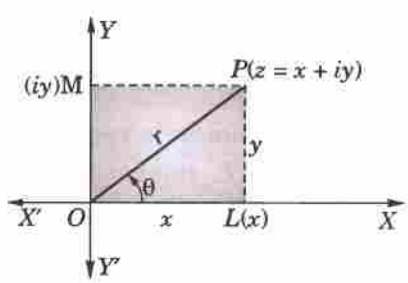

Fig. 19.2

Complete the rectangle $OLPM$. *Then the point whose cartesian coordinates are (x, y) uniquely represents the complex number $z = x + iy$ on the complex plane z.* The diagram in which this representation is carried out is called the **Argand's diagram**.

If (r, θ) be the polar coordinates of P, then r is the modulus of z and θ is its amplitude.

Obs. Since a complex number has magnitude and direction, therefore, it can be represented like a vector. *Hereafter we shall often refer to the complex number $z = x + iy$ as*

(i) the point z whose co-ordinates are (x, y) or (ii) the vector z from O to $P(x, y)$.

Example 19.4. *The centre of a regular hexagon is at the origin and one vertex is given by $\sqrt{3} + i$ on the Argand diagram. Determine the other vertices.*

* The first mathematician to propose a geometric representation of imaginary number i was *Kuhn of Denzig* (1750–51).

† The geometric representation of complex numbers came into mathematics through the memoire of *Jean Robert Argand*, Paris 1806.

Solution. Let $\vec{OA} = \sqrt{3} + i$ so that

$OA = 2$ and $\angle XOA = \tan^{-1} 1/\sqrt{3} = 30°$. (Fig. 19.3)

Being a regular hexagon, $OB = OC = 2$

$\angle XOB = 30° + 60° = 90°$

and $\angle XOC = 30° + 120° = 150°$

$\therefore \quad \vec{OB} = 2(\cos 90° + i \sin 90°) = 2i$

$\vec{OC} = 2(\cos 150° + i \sin 150°) = -\sqrt{3} + i$

Since $\vec{AD}, \vec{BE}, \vec{CF}$ are bisected at O,

$\therefore \quad \vec{OD} = -\vec{OA} = -\sqrt{3} - i$

$\vec{OE} = -\vec{OB} = -2i$ and $\vec{OF} = -\vec{OC} = \sqrt{3} - i$.

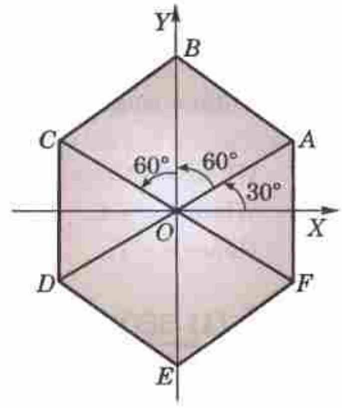

Fig. 19.3

19.3 (1) GEOMETRIC REPRESENTATION OF $z_1 + z_2$

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. (Fig. 19.4)

Complete the parallelogram OP_1PP_2. Draw P_1L, P_2M and $PN \perp$ s to OX. Also draw $P_1K \perp PN$.

Since $ON = OL + LN = OL + OM = x_1 + x_2 \quad [\because LN = P_1K = OM]$

and $NP = NK + KP = LP_1 + MP_2 = y_1 + y_2$.

The coordinates of P are $(x_1 + x_2, y_1 + y_2)$ and it represents the complex number

$$z = x_1 + x_2 + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) = z_1 + z_2.$$

Thus the point P which is the extremity of the diagonal of the parallelogram having OP_1 and OP_2 as adjacent sides, represents the sum of the complex numbers $P_1(z_1)$ and $P_2(z_2)$ such that

$$|z_1 + z_2| = OP \text{ and amp } (z_1 + z_2) = \angle XOP.$$

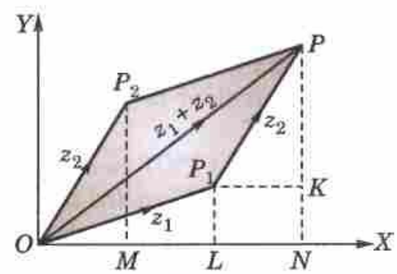

Fig. 19.4

Obs. Vectorially, we have $\vec{OP_1} + \vec{P_1P} = \vec{OP}$.

(2) Geometric representation of $z_1 - z_2$

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ (Fig. 19.5). Then the subtraction of z_2 from z_1 may be taken as addition of z_1 to $-z_2$.

Produce P_2O backwards to R such that $OR = OP_2$. Then the coordinates of R are evidently $(-x_2, -y_2)$ and so it corresponds to the complex number $-x_2 - iy_2 = -z_2$.

Complete the parallelogram $ORQP_1$, then the sum of z_1 and $(-z_2)$ is represented by OQ *i.e.*, $z_1 - z_2 = \vec{OQ} = \vec{P_2P_1}$.

Hence the complex number $z_1 - z_2$ is represented by the vector P_2P_1.

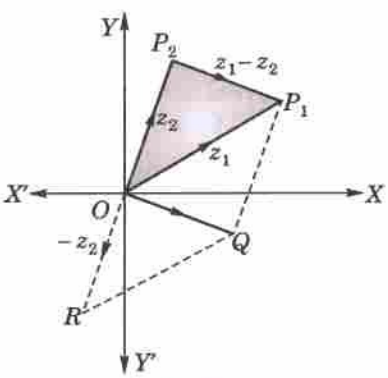

Fig. 19.5

Obs. By means of the relation $\vec{P_2P_1} = \vec{OP_1} - \vec{OP_2}$, any vector $\vec{P_2P_1}$ may be referred to the origin.

Example 19.5. *Find the locus of P(z) when*
(*i*) $|z - a| = k$;
(*ii*) *amp* $(z - a) = \alpha$, where k and α are constants. *(Gorakhpur, 1999)*

Solution. Let a, z be represented by A and P in the complex plane, O being the origin (Fig. 19.6).

Then $z - a = \vec{OP} - \vec{OA} = \vec{AP}$

(*i*) $|z - a| = k$ means that $AP = k$.

Thus the locus of $P(z)$ is a circle whose centre is $A(a)$ and radius k.

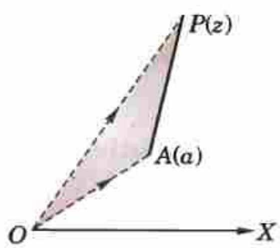

Fig. 19.6

(*ii*) amp $(z - a)$, *i.e.*, amp $(\overrightarrow{AP}) = \alpha$, means that AP always makes a constant angle with the X-axis.

Thus the locus of $P(z)$ is a straight line through $A(a)$ making an $\angle\alpha$ with OX.

Example 19.6. *Determine the region in the z-plane represented by*
(i) $1 < |z + 2i| \le 3$ (ii) $R(z) > 3$ (iii) $\pi/6 \le \text{amp}(z) \le \pi/3$.

Solution. (*i*) $|z + 2i| = 1$ is a circle with centre $(-2i)$ and radius 1 and $|z + 2i| = 3$ is a circle with the same centre and radius 3.

Hence $1 < |z + 2i| \le 3$ represents the region outside the circle $|z + 2i| = 1$ and inside (including circumference of) the circle $|z + 2i| = 3$ [Fig. 19.7].

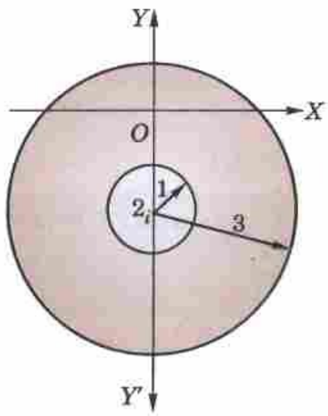

Fig. 19.7

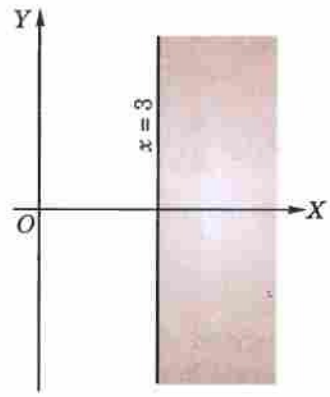

Fig. 19.8

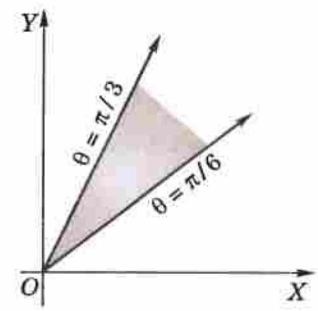

Fig. 19.9

(*ii*) $R(z) > 3$, defines all points (z) whose real part is greater than 3. Hence it represents the region of the complex plane to the right of the line $x = 3$ [Fig. 19.8].

(*iii*) If $z = r(\cos\theta + i\sin\theta)$, then amp $(z) = \theta$.

$\therefore$ $\pi/6 \le \text{amp}(z) \le \pi/3$ defines the region bounded by and including the lines $\theta = \pi/6$ and $\theta = \pi/3$. [Fig. 19.9].

Example 19.7. *If z_1, z_2 be any two complex numbers, prove that*
(*i*) $|\mathbf{z_1 + z_2}| \le |\mathbf{z_1}| + |\mathbf{z_2}|$ [*i.e., the modulus of the sum of two complex numbers is less than or at the most equal to the sum of their moduli*].
(*ii*) $|\mathbf{z_1 - z_2}| \ge |\mathbf{z_1}| - |\mathbf{z_2}|$ [*i.e., the modulus of the difference of two complex numbers is greater than or at the most equal to the difference of their moduli*].

Solution. Let P_1, P_2 represent the complex numbers z_1, z_2 (Fig. 19.10). Complete the parallelogram OP_1PP_2, so that

$$|z_1| = OP_1,\ |z_2| = OP_2 = P_1P,$$

and
$$|z_1 + z_2| = OP.$$

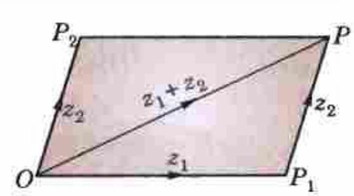

Fig. 19.10

Now from ΔOP_1P, $OP \le OP_1 + P_1P$, the sign of equality corresponding to the case when O, P_1, P are collinear.

Hence $|z_1 + z_2| \le |z_1| + |z_2|$ ...(*i*)

Again $|z_1| = |(z_1 - z_2) + z_2| \le |z_1 - z_2| + |z_2|$ [By (*i*)]

Thus $|z_1 - z_2| \ge |z_1| - |z_2|$ ...(*ii*)

Obs. $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$.
In general, $|z_1 + z_2 + ... + z_n| \le |z_1| + |z_2| + ... + |z_n|$.

Example 19.8. *If $|z_1+z_2| = |z_1-z_2|$, prove that the difference of amplitudes of z_1 and z_2 is $\pi/2$.* (*Mumbai, 2007*)

Solution. Let $z_1 + z_2 = r(\cos\theta + i\sin\theta)$ and $z_1 - z_2 = r(\cos\phi + i\sin\phi)$

Then
$$2z_1 = r[(\cos\theta + \cos\phi) + i(\sin\theta + \sin\phi)]$$
$$= r\left\{2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} + 2i\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right\}$$

or
$$z_1 = r\cos\frac{\theta-\phi}{2}\left(\cos\frac{\theta+\phi}{2} + i\sin\frac{\theta+\phi}{2}\right) \text{ i.e., } \operatorname{amp}(z_1) = \frac{\theta+\phi}{2} \quad ...(i)$$

Also
$$2z_2 = r(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)$$
$$= 2r\sin\frac{\theta-\phi}{2}\left(-\sin\frac{\theta+\phi}{2} + i\cos\frac{\theta+\phi}{2}\right)$$

or
$$z_2 = r\sin\frac{\theta-\phi}{2}\left\{\cos\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right)\right\}$$

i.e.,
$$\operatorname{amp}(z_2) = \frac{\pi}{2} + \frac{\theta+\phi}{2} \quad ...(ii)$$

Hence [(*ii*) – (*i*)], gives $\operatorname{amp}(z_2) - \operatorname{amp}(z_1) = \dfrac{\pi}{2}$.

Example 19.9. *Show that the equation of the ellipse having foci at z_1, z_2 and major axis $2a$, is $|z-z_1| + |z-z_2| = 2a$.*

Also find its eccentricity.

Solution. Let $P(z)$ be any point on the given ellipse (Fig. 19.11) having foci at $S(z_1)$ and $S'(z_2)$ so that $SP = |z - z_1|$ and $S'P = |z - z_2|$.

We know that $SP + S'P = AA' (= 2a)$

i.e., $\quad |z-z_1| + |z-z_2| = 2a$

which is the desired equation of the ellipse.

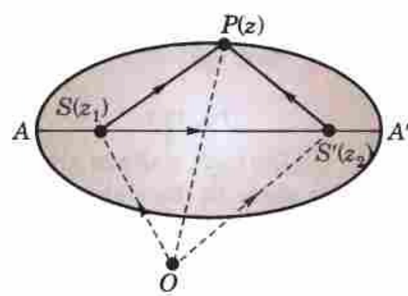

Fig. 19.11

Also we know that $SS' = 2ae$, e being the eccentricity.

or $\quad \left|\vec{OS'} - \vec{OS}\right| = 2ae \quad$ or $\quad |z_2 - z_1| = 2ae$

or $\quad |z_1 - z_2| = 2ae$ whence $e = |z_1 - z_2|/2a$.

(3) Geometric Representation of $\mathbf{z_1 z_2}$. Let P_1, P_2 represent the complex numbers
$$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
and
$$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Measure off $OA = 1$ along OX (Fig. 19.12). Construct ΔOP_2P on OP_2 directly similar to ΔOAP_1,

so that $\quad OP/OP_1 = OP_2/OA$ *i.e.*, $OP = OP_1 \cdot OP_2 = r_1 r_2$

and $\quad \angle AOP = \angle AOP_2 + \angle P_2OP = \angle AOP_2 + \angle AOP_1 = \theta_2 + \theta_1$

$\therefore$ P represents the number
$$r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].$$

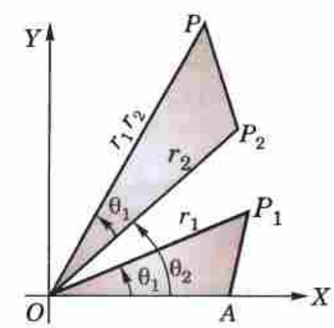

Fig. 19.12

Hence *the product of two complex numbers z_1, z_2 is represented by the point P, such that* (*i*) $|\mathbf{z_1 z_2}| = |\mathbf{z_1}| \cdot |\mathbf{z_2}|$.

(*ii*) **amp $(\mathbf{z_1 z_2})$ = amp $(\mathbf{z_1})$ + amp $(\mathbf{z_2})$.**

Cor. The effect of multiplication of any complex number z by $\cos\theta + i\sin\theta$ is to rotate its direction through an angle θ, for the modulus of $\cos\theta + i\sin\theta$ is unity.

(4) Geometric representation of $\mathbf{z_1/z_2}$.

Let P_1, P_2 represent the complex numbers

$$z_1 = x_1 + iy_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$$

and
$$z_2 = x_2 + iy_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$$

Measure off $OA = 1$, construct triangle OAP on OA directly similar to the triangle OP_2P_1 (Fig. 19.13), so that

$$\frac{OP}{OA} = \frac{OP_1}{OP_2} \quad i.e., \quad OP = \frac{OP_1}{OP_2} = \frac{r_1}{r_2}$$

and
$$\angle XOP = \angle P_2OP_1 = \angle AOP_1 - \angle AOP_2 = \theta_1 - \theta_2.$$

$\therefore$ P represents the number

$$(r_1/r_2) [\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)].$$

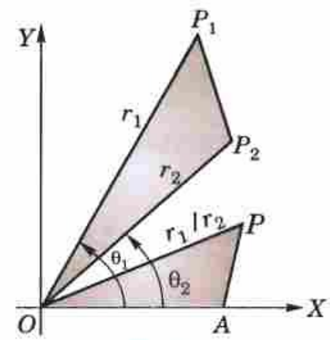

Fig. 19.13

Hence the complex number z_1/z_2 is represented by the point P, such that

(*i*) $|\mathbf{z_1/z_2}| = |\mathbf{z_1}|/|\mathbf{z_2}|$

(*ii*) **amp** $(\mathbf{z_1/z_2})$ **= amp** $(\mathbf{z_1})$ **– amp** $(\mathbf{z_2})$**.**

Note. *If $P_1(z_1)$, $P_2(z_2)$ and $P_3(z_3)$ be any three points, then*

$$\text{amp}\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \angle P_1P_2P_3.$$

Join O, the origin, to P_1, P_2, and P_3. Then from the figure 19.14, we have

$$\vec{P_2P_1} = z_1 - z_2 \quad \text{and} \quad \vec{P_2P_3} = z_3 - z_2$$

$$\therefore \quad \text{amp}\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \text{amp}\left[\frac{\vec{P_2P_3}}{\vec{P_2P_1}}\right]$$

$$= \text{amp}\ (\vec{P_2P_3}) - \text{amp}\ (\vec{P_2P_1}) = \beta - \alpha = \angle P_1P_2P_3.$$

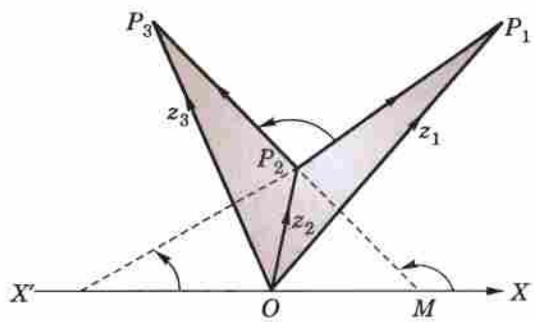

Fig. 19.14

Example 19.10. *Find the locus of the point z, when*

(*i*) $\left|\frac{z-a}{z-b}\right| = k$ (*ii*) *amp* $\left(\frac{z-a}{z-b}\right) = \alpha$ *where k and* α *are constants.*

Solution. Let $A(a)$ and $B(b)$ be any two fixed points on the complex plane and let $P(z)$ be any variable point (Fig. 19.15).

(*i*) Since $|z - a| = AP$ and $|z - b| = BP$.

$\therefore$ The point P moves so that $\left|\frac{z-a}{z-b}\right| = \left|\frac{z-a}{z-b}\right| = \frac{AP}{BP} = k$

i.e., P moves so that its distances from two fixed points are in a constant ratio, which is obviously the *Appollonius* circle.

When $k = 1$, $BP = AP$ *i.e.*, P moves so that its distance from two fixed points are always equal and thus the locus of P is the right bisector of AB.

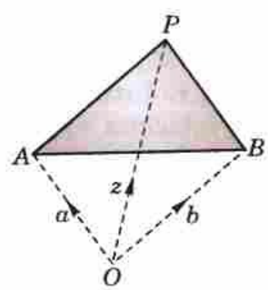

Fig. 19.15

Hence the locus of P(z) is a circle (unless k = 1, when the locus is the right bisector of AB).

Obs. For different values of k, the equation represents family of non-intersecting coaxial circles having A and B as its limiting points.

(*ii*) From the figure 19.16, we have amp $\left(\frac{z-a}{z-b}\right) = \angle APB = \alpha$.

Hence the locus of P(z) is the arc APB of the circle which passes through the fixed points A and B.

If, however, P' (z') be a point on the lower arc AB of this circle, then

amp $\left(\frac{z'-a}{z'-b}\right) = \angle BP'A = \alpha - \pi$, which shows that the locus of P' is the arc $AP'B$ of the same circle.

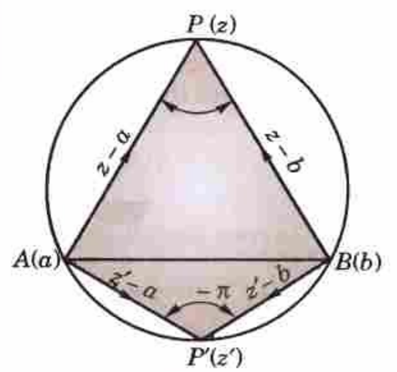

Fig. 19.16

Obs. For different values of α from $-\pi$ to π, the equation represents a family of intersecting coaxial circles having AB as their common radical axis.

Example 19.11. *If z_1, z_2 be two complex numbers, show that*

$$(z_1 + z_2)^2 + (z_1 - z_2)^2 = 2[|z_1|^2 + |z_2|^2].$$

Solution. Let $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$ so that

$$|z_1 + z_2|^2 = (r_1 \cos\theta_1 + r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 + r_2 \sin\theta_2)^2$$
$$= r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

and
$$|z_1 - z_2|^2 = (r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2$$
$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

$$\therefore \quad |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(r_1^2 + r_2^2) = 2\left\{|z_1|^2 + |z_2|^2\right\}.$$

Example 19.12. *If z_1, z_2, z_3 be the vertices of an isosceles triangle, right angled at z_2, prove that*

$$z_1^2 + z_3^2 + 2z_2^2 = 2z_2(z_1 + z_3).$$

Solution. Let $A(z_1)$, $B(z_2)$, $C(z_3)$ be the vertices of ΔABC such that

$$AB = BC \text{ and } \angle ABC = \pi/2. \text{ (Fig. 19.17)}$$

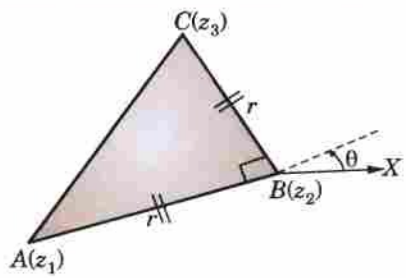

Fig. 19.17

Then $\quad |z_1 - z_2| = |z_3 - z_2| = r$ (say).

If amp $(z_1 - z_2) = \theta$ then amp $(z_3 - z_2) = \pi/2 + \theta$

$$\therefore \quad z_1 - z_2 = r(\cos\theta + i\sin\theta),$$

and
$$z_3 - z_2 = r\left[\cos\left(\frac{\pi}{2} + \theta\right) + i \sin\left(\frac{\pi}{2} + \theta\right)\right] = r(-\sin\theta + i\cos\theta)$$

i.e.,
$$z_3 - z_2 = ir(\cos\theta + i\sin\theta) = i(z_1 - z_2)$$

or
$$(z_3 - z_2)^2 = -(z_1 - z_2)^2 \text{ or } z_1^2 + z_3^2 + 2z_2^2 = 2z_3(z_1 + z_2).$$

Example 19.13. *If z_1, z_2, z_3 be the vertices of an equilateral triangle, prove that*

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Solution. Since ΔABC is equilateral, therefore, BC when rotated through $60°$ coincides with BA (Fig. 19.18). But to turn the direction of a complex number through an $\angle\theta$, we multiply it by $\cos\theta + i\sin\theta$.

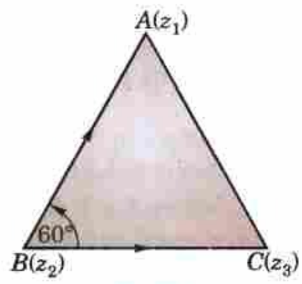

Fig. 19.18

$$\therefore \quad \vec{BC}(\cos\pi/3 + i\sin\pi/3) = \vec{BA}$$

i.e.,
$$(z_3 - z_2)\left(\frac{1 + i\sqrt{3}}{2}\right) = z_1 - z_2$$

or
$$i\sqrt{3}(z_3 - z_2) = 2z_1 - z_2 - z_3$$

Squaring,
$$-3(z_3 - z_2)^2 = (2z_1 - z_2 - z_3)^2$$

or
$$4(z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1) = 0$$

whence follows the required condition.

PROBLEMS 19.1

1. Express the following in the modulus-amplitude form:

   (*i*) $1 + \sin\alpha + i\cos\alpha$ $\qquad$ (*ii*) $\dfrac{1}{(2+1)^2} - \dfrac{1}{(2-1)^2}$. $\qquad$ (*V.T.U., 2011 S*)

2. If $\dfrac{1}{x + iy} + \dfrac{1}{u + iv} = 1$; x, y, u, v being real quantities, express v in terms of x and y.

3. If x and y are real, solve the equation $\frac{iy}{ix+1} - \frac{3y+4i}{3x+y} = 0$.
4. If $\alpha - i\beta = \frac{1}{a-ib}$, prove that $(\alpha^2 + \beta^2)(a^2 + b^2) = 1$. *(Mumbai, 2008 S)*
5. Find what curve $z\bar{z} + (1+i)z + (1-i)\bar{z} = 0$ represents?
6. In an Argand diagram, show that $9 + i$, $4 + 13i$, $-8 + 8i$ and $-3 - 4i$ form a square.
7. If $|z_1| = |z_2|$ and amp (z_1) + amp $(z_2) = 0$, then show that z_1 and z_2 are conjugate complex numbers.
8. A rectangle is constructed in the complex plane and its sides parallel to the axes and its centre is situated at the origin. If one of the vertices of the rectangle is $1 + i\sqrt{3}$, find the complex numbers representing the other three vertices of the rectangle. Find also the area of the rectangle.
9. An equilateral triangle constructed in the complex plane has its one vertex at the point $1 + i\sqrt{3}$. Find the complex numbers representing the other two vertices, O the origin being its circumcentre.
10. The centre of a regular hexagon is at the origin and one vertex is given by $1 + i$ on the Argand diagram. Find the remaining vertices.
11. What domain of the z-plane is represented by
    (*i*) $2 \le |z+3| < 4$ (*ii*) $I(z) > 2$
    (*iii*) $\pi/3 <$ amp $(z) < \pi/2$ (*iv*) $|z+2| + |z-2| < 4$.
12. If $|z^2 - 1| = |z|^2 + 1$, prove that z lies on the imaginary axis. *(Mumbai, 2007)*
13. What are the loci given by (*i*) $|z-1| + |z+1| = 3$ (*ii*) $|z-3| = k\,|z+1|$ for $k = 1$ and 2.
14. Find the locus of z given by :
    (*i*) $|z| = |z-2|$. (*ii*) $|3z-1| = |z-3|$.
15. Find the locus of z :
    (*i*) when $\frac{z+i}{z+2}$ is real, (*ii*) when $\frac{z-i}{z-2}$ is purely imaginary. *(Osmania, 2003 S)*

19.4 DE MOIVRE'S THEOREM*

Statement : *If n be (i) an integer, positive or negative* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$;
(ii) a fraction, positive or negative, one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

Proof. Case I. *When n is a positive integer.*

By actual multiplication

$$\text{cis}\,\theta_1\,\text{cis}\,\theta_2 = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2), \text{ i.e., cis}\,(\theta_1+\theta_2)$$

Similarly $\text{cis}\,\theta_1\,\text{cis}\,\theta_2\,\text{cis}\,\theta_3 = \text{cis}\,(\theta_1+\theta_2)\,\text{cis}\,\theta_3 = \text{cis}\,(\theta_1+\theta_2+\theta_3)$

Proceeding in this way,

$$\text{cis}\,\theta_1\,\text{cis}\,\theta_2\,\text{cis}\,\theta_3 \ldots \text{cis}\,\theta_n = \text{cis}\,(\theta_1+\theta_2+\theta_3+\ldots+\theta_n)$$

Now putting $\theta_1 = \theta_2 = \theta_3 = \ldots = \theta_n = \theta$, we obtain $(\text{cis}\,\theta)^n = \text{cis}\,n\theta$.

Case II. *When n is a negative integer.*

Let $n = -m$, where m is a + ve integer.

$$\therefore \quad (\text{cis}\,\theta)^n = (\text{cis}\,\theta)^{-m} = \frac{1}{(\text{cis}\,\theta)^m} = \frac{1}{\text{cis}\,m\theta} \qquad \text{(By case I)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

[Multiplying the num. and denom. by $(\cos m\theta - i\sin m\theta)$]

*One of the remarkable theorems in mathematics; called after the name of its discoverer *Abraham De Moivre* (1667–1754), a French Mathematician.

$$= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos m\theta - i \sin m\theta$$

$$= \cos(-m\theta) + i \sin(-m\theta) = \text{cis}(-m\theta) = \text{cis } n\theta \qquad [\because \ -m = n]$$

Case III. *When n is a fraction, positive or negative.*

Let $n = p/q$, where q is a + ve integer and p is any integer + ve or – ve

Now $(\text{cis } \theta/q)^q = \text{cis } (q \, . \, \theta/q) = \text{cis } \theta$

$\therefore$ Taking qth root of both sides cis (θ/q) is one of the q values of $(\text{cis } \theta)^{1/q}$,

i.e., one of the values of $(\text{cis } \theta)^{1/q} = \text{cis } \theta/p$

Raise both sides to power p, then one of the values of $(\text{cis } \theta)^{p/q} = (\text{cis } \theta/q)^p = \text{cis } (p/q)\, \theta$ *i.e.*, one of the values of $(\text{cis } \theta)^n = \text{cis } n\theta$. (By case I and II)

Thus *the theorem is completely established for all rational values of n.*

Cor. **1.** $\text{cis } \theta_1 \, . \, \text{cis } \theta_2 \ldots \text{cis } \theta_n = \text{cis } (\theta_1 + \theta_2 + \ldots + \theta_n)$

2. $(\cos\theta - i \sin\theta)^n = \cos n\theta - i \sin n\theta = (\cos\theta + i \sin\theta)^{-n}$

3. $(\text{cis } m\theta)^n = \text{cis } mn\theta = (\text{cis } n\theta)^m$.

Example 19.14. *Simplify* $\dfrac{(\cos 3\theta + i \sin 3\theta)^4 (\cos 4\theta - i \sin 4\theta)^5}{(\cos 4\theta + i \sin 4\theta)^3 (\cos 5\theta + i \sin 5\theta)^{-4}}$.

Solution. We have, $(\cos 3\theta + i \sin 3\theta)^4 = \cos 12\theta + i \sin 12\theta = (\cos\theta + i \sin\theta)^{12}$

$(\cos 4\theta - i \sin 4\theta)^5 = \cos 20\theta - i \sin 20\theta = (\cos\theta + i \sin\theta)^{-20}$

$(\cos 4\theta + i \sin 4\theta)^3 = \cos 12\theta + i \sin 12\theta = (\cos\theta + i \sin\theta)^{12}$

$(\cos 5\theta + i \sin 5\theta)^{-4} = \cos 20\theta - i \sin 20\theta = (\cos\theta + i \sin\theta)^{-20}$

$$\therefore \quad \text{The given expression} = \frac{(\cos\theta + i \sin\theta)^{12} (\cos\theta + i \sin\theta)^{-20}}{(\cos\theta + i \sin\theta)^{12} (\cos\theta + i \sin\theta)^{-20}} = 1.$$

Example 19.15. *Prove that*

$$(1 + \cos\theta + i \sin\theta)^n + (1 + \cos\theta - i \sin\theta)^n = 2^{n+1} \cos^n (\theta/2) \, . (\cos n\theta/2).$$

Solution. Put $1 + \cos\theta = r \cos\alpha$, $\sin\theta = r \sin\alpha$.

$$\therefore \quad r^2 = (1 + \cos\theta)^2 + \sin^2\theta = 2 + 2\cos\theta = 4\cos^2 \theta/2 \quad i.e., \quad r = 2\cos\theta/2$$

and
$$\tan\alpha = \frac{\sin\theta}{1 + \cos\theta} = \frac{2 \sin\theta/2 \, . \cos\theta/2}{2\cos^2 \theta/2} = \tan\theta/2 \quad i.e., \quad \alpha = \theta/2.$$

$$\therefore \quad \text{L.H.S.} = [r(\cos\alpha + i \sin\alpha)]^n + [r(\cos\alpha - i \sin\alpha)]^n$$

$$= r^n[(\cos\alpha + i \sin\alpha)^n + (\cos\alpha - i \sin\alpha)^n] = r^n (\cos n\alpha + i \sin n\alpha + \cos n\alpha - i \sin n\alpha)$$

$$= r^n \, . \, 2\cos n\alpha \qquad [\text{Substituting the values of } r \text{ and } \alpha]$$

$$= 2^{n+1} \cos^n (\theta/2) \cos (n\theta/2).$$

Example 19.16. *If* $2\cos\theta = x + \dfrac{1}{x}$, *prove that*

(*i*) $2\cos r\theta = x^r + 1/x^r$, (*ii*) $\dfrac{x^{2n} + 1}{x^{2n-1} + x} = \dfrac{\cos n\theta}{\cos (n-1)\theta}$ (*Madras, 2000 S*)

Solution. Since $x + 1/x = 2\cos\theta$ $\therefore$ $x^2 - 2x\cos\theta + 1 = 0$

whence
$$x = \frac{2\cos\theta \pm \sqrt{(4\cos^2\theta - 4)}}{2} = \cos\theta \pm i \sin\theta.$$

(*i*) Taking the + ve sign, $x^r = (\cos\theta + i \sin\theta)^r = \cos r\theta + i \sin r\theta$ (*S.V.T.U., 2009*)

and $x^{-r} = (\cos\theta + i \sin\theta)^{-r} = \cos r\theta - i \sin r\theta$

Adding $x^r + 1/x^r = 2\cos r\theta$. Similarly with the – ve sign, the same result follows.

(*ii*) $$\frac{x^{2n}+1}{x^{2n-1}+x}=\frac{(\cos\theta+i\sin\theta)^{2n}+1}{(\cos\theta+i\sin\theta)^{2n-1}+\cos\theta+i\sin\theta}$$

$$=\frac{\cos 2n\theta+i\sin 2n\theta+1}{\cos(2n-1)\theta+i\sin(2n-1)\theta+\cos\theta+i\sin\theta}$$

$$=\frac{(1+\cos 2n\theta)+i\sin 2n\theta}{(\cos\overline{2n-1}\,\theta+\cos\theta)+i(\sin\overline{2n-1}\,\theta+\sin\theta)}$$

$$=\frac{2\cos^2 n\theta+2i\sin n\theta\cos\theta}{2\cos n\theta\cos\overline{n-1}\,\theta+2i\sin n\theta\cos\overline{n-1}\,\theta}$$

$$=\frac{\cos n\theta\,(2\cos n\theta+2i\sin n\theta)}{\cos\overline{n-1}\,\theta\,(2\cos n\theta+2i\sin n\theta)}=\frac{\cos n\theta}{\cos\overline{n-1}\,\theta}.$$

Example 19.17. *If* $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0$,
prove that (*i*) $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$
(*ii*) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$
(*iii*) $\sin 4\alpha+\sin 4\beta+\sin 4\gamma=2\Sigma\sin 2(\alpha+\beta)$
(*iv*) $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$. *(Mumbai, 2009)*

Solution. Let $a=\text{cis}\,\alpha,\ b=\text{cis}\,\beta$ and $c=\text{cis}\,\gamma$.

Then $a+b+c=(\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)=0$ ...(1)

(*i*) $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=(\cos\alpha+i\sin\alpha)^{-1}+(\cos\beta+i\sin\beta)^{-1}+(\cos\gamma+i\sin\gamma)^{-1}$$

$$=\sum\frac{\cos\alpha-i\sin\alpha}{\cos\alpha-i\sin\alpha}\cdot\frac{1}{\cos\alpha+i\sin\alpha}=\sum(\cos\alpha-i\sin\alpha)$$

$$=(\cos\alpha+\cos\beta+\cos\gamma)-i(\sin\alpha+\sin\beta+\sin\gamma)=0 \quad \text{(Given)}$$

or $bc+ca+ab=0$ ...(2)

$\therefore\ a^2+b^2+c^2=(a+b+c)^2-2(bc+ca+ab)=0$ [By (1) & (2) ...(3)]

or $(cis\,\alpha)^2+(cis\,\beta)^2+(cis\,\gamma)^2=cis\,2\alpha+cis\,2\beta+cis\,2\gamma=0$

Equating imaginary parts from both sides, we get

$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$$

(*ii*) Since $a+b+c=0$, $\therefore\ a^3+b^3+c^3=3abc$

or $(cis\,\alpha)^3+(cis\,\beta)^3+(cis\,\gamma)^3=3\,cis\,\alpha\,cis\,\beta\,cis\,\gamma$

or $cis\,3\alpha+cis\,3\beta+cis\,3\gamma=3\,cis\,(\alpha+\beta+\gamma)$

Equating imaginary parts from both sides, we get

$$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$$

(*iii*) From (1), $a+b=-c$ or $(a+b)^2=c^2$ or $a^2+b^2-c^2=-2ab$

Again squaring, $a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2=4a^2b^2$

i.e., $a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$

or $(cis\,\alpha)^4+(cis\,\beta)^4+(cis\,\gamma)^4=2\sum(cos\,\alpha)^2\,(cis\,\beta)^2$

or $cis\,4\alpha+cis\,4\beta+cis\,4\gamma=2\sum cis\,2\alpha\,cis\,2\beta=2\sum cis\,2(\alpha+\beta)$

Equating imaginary parts from both sides, we get

$$\sin 4\alpha+\sin 4\beta+\sin 4\gamma=2\sum\sin 2(\alpha+\beta)$$

(*iv*) From (2), $ab+bc+ca=0$

or $cis\,\alpha\,cis\,\beta+cis\,\beta\,cis\,\gamma+cis\,\gamma\,cis\,\alpha=0$

or $cis\,(\alpha+\beta)+cis\,(\beta+\gamma)+cis\,(\gamma+\alpha)=0$

Equating imaginary parts from both sides, we get

$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$$

PROBLEMS 19.2

1. Prove that (*i*) $\dfrac{(\cos 5\theta - i\sin 5\theta)^2 (\cos 7\theta + i\sin 7\theta)^{-3}}{(\cos 4\theta - i\sin 4\theta)^9 (\cos\theta + i\sin\theta)^5} = 1$

(*ii*) $\dfrac{(\cos\alpha + i\sin\alpha)^4}{(\sin\beta + i\cos\beta)^5} = \sin(4\alpha + 5\beta) - i\cos(4\alpha + 5\beta).$ (*iii*) $\left(\dfrac{\cos\theta + i\sin\theta)^4}{\sin\theta + i\cos\theta}\right) = \cos 8\theta + i\sin 8\theta.$

2. If $p = \text{cis}\,\theta$ and $q = \text{cis}\,\phi$, show that

(*i*) $\dfrac{p-q}{p+q} = i\tan\dfrac{\theta-\phi}{2}$ (*Mumbai, 2008*) (*ii*) $\dfrac{(p+q)(pq-1)}{(p-q)(pq+1)} = \dfrac{\sin\theta + \sin\phi}{\sin\theta - \sin\phi}.$ (*Kurukshetra, 2005*)

3. If $a = \text{cis}\,2\alpha$, $b = \text{cis}\,2\beta$, $c = \text{cis}\,2\gamma$ and $d = \text{cis}\,2\delta$, prove that

(*i*) $\sqrt{\dfrac{ab}{c}} + \sqrt{\dfrac{c}{ab}} = 2\cos(\alpha + \beta - \gamma)$ (*ii*) $\sqrt{\dfrac{ab}{cd}} + \sqrt{\dfrac{cd}{ab}} = 2\cos(\alpha + \beta - \gamma - \delta).$

4. If $x_r = \text{cis}\,(\pi/2^r)$, show that $\underset{n\to\infty}{\text{Lt}}\; x_1x_2x_3 \ldots x_n = -1.$ (*S.V.T.U., 2009 ; Mumbai, 2007*)

5. Find the general value of θ which satisfies the equation
$$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \ldots (\cos n\theta + i\sin n\theta) = 1.$$

6. Prove that (*i*) $(a + ib)^{m/n} + (a - ib)^{m/n} = 2(a^2 + b^2)^{m/2n}\cos\left(\dfrac{m}{n}\tan^{-1}\dfrac{b}{a}\right).$

(*ii*) $(1 + i)^n + (1 - i)^n = 2^{n/2+1}\cos n\pi/4.$

7. Simplify $[\cos\alpha - \cos\beta + i(\sin\alpha - \sin\beta)]^n + [\cos\alpha - \cos\beta - i(\sin\alpha - \sin\beta)]^n$

8. Prove that (*i*) $(1 + \sin\theta + i\cos\theta)^n + (1 + \sin\theta - i\cos\theta)^n = 2^{n+1}\cos^n\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)\cos\left(\dfrac{n\pi}{4} - \dfrac{n\theta}{2}\right).$

(*ii*) $\left[\dfrac{1 + \sin\alpha + i\cos\alpha}{1 + \sin\alpha - i\cos\alpha}\right]^n = \cos\left(\dfrac{n\pi}{2} - n\alpha\right) + i\sin\left(\dfrac{n\pi}{2} - n\alpha\right).$ (*S.V.T.U., 2006*)

9. If $2\cos\theta = x + 1/x$ and $2\cos\phi = y + 1/y$, show that one of the values of

(*i*) $x^m y^n + \dfrac{1}{x^m y^n}$ is $2\cos(m\theta + n\phi).$ (*S.V.T.U., 2007*)

(*ii*) $\dfrac{x^m}{y^n} + \dfrac{y^n}{x^m}$ is $2\cos(m\theta - n\phi).$ (*Nagpur, 2009*)

10. If α, β be the roots of $x^2 - 2x + 4 = 0$, prove that $\alpha^n + \beta^n = 2^{n+1}\cos n\pi/3.$ (*Delhi, 2002*)

11. If α, β are the roots of the equation $z^2\sin^2\theta - z\sin\theta + 1 = 0$, then prove that

(*i*) $\alpha^n + \beta^n = 2\cos n\theta\,\text{cosec}^n\,\theta$ (*ii*) $\alpha^n\beta^n = \text{cosec}^{2n}\,\theta.$ (*Mumbai, 2009*)

12. If $x^2 - 2x\cos\theta + 1 = 0$, show that $x^{2n} - 2x^n\cos n\theta + 1 = 0.$

13. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and $x + y + z = 0$, then prove that
$$x^{-1} + y^{-1} + z^{-1} = 0.$$

14. If $\sin\theta + \sin\phi + \sin\psi = 0 = \cos\theta + \cos\phi + \cos\psi$, prove that

(*i*) $\cos 2\theta + \cos 2\phi + \cos 2\psi = 0$ (*Mumbai, 2009*)

(*ii*) $\cos 3\theta + \cos 3\phi + \cos 3\psi = 3\cos(\theta + \phi + \psi)$

(*iii*) $\cos 4\theta + \cos 4\phi + \cos 4\psi = 2\sum \cos 2(\phi + \psi).$

15. If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$, prove that

(*i*) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 3/2$

(*ii*) $\cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha) = 0$ (*Mumbai, 2009 ; S.V.T.U., 2008*)

16. If $\sin\alpha + 2\sin\beta + 3\sin\gamma = 0$, $\cos\alpha + 2\cos\beta + 3\cos\gamma = 0$, prove that $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha + \beta + \gamma)$ and $\cos 3\alpha + 8\cos 3\beta + 27\sin 3\gamma = 18\cos(\alpha + \beta + \gamma).$

19.5 ROOTS OF A COMPLEX NUMBER

There are q and only q distinct values of $(\cos\theta + i\sin\theta)^{1/q}$, *q being an integer.*

Since $\cos\theta = \cos(2n\pi + \theta)$ and $\sin\theta = \sin(2n\pi + \theta)$, where n is any integer.

$\therefore$ $$\text{cis}\,\theta = \text{cis}\,(2n\pi + \theta).$$

By De Moivre's theorem one of the values of

$$(\text{cis } \theta)^{1/q} = [\text{cis } (2n\pi + \theta)]^{1/q} = \text{cis } (2n\pi + \theta)/q \qquad ...(1)$$

Giving n the values 0, 1, 2, 3, ..., $(q - 1)$ successively, we get the following q values of $(\text{cis } \theta)^{1/q}$;

$$\left.\begin{array}{ll} \text{cis } \theta/q & (\text{for } n = 0) \\ \text{cis } (2\pi + \theta)/q & (\text{for } n = 1) \\ \text{cis } (4\pi + \theta)/q & (\text{for } n = 2) \\ \dots\dots\dots & \dots\dots\dots \\ \text{cis } [2(q-1)\pi + \theta]/q & (\text{for } n = q - 1) \end{array}\right\} \qquad ...(2)$$

Putting $n = q$ in (1), we get a value of $(\text{cis } \theta)^{1/q} = \text{cis } (2\pi + \theta/q) = \text{cis } \theta/q$, which is the same as the value of $n = 0$.

Similarly for $n = q + 1$, we get a value of $(\text{cis } \theta)^{1/q}$ to be cis $(2\pi + \theta)/q$, which is the same as the value for $n = 1$ and so on.

Thus, the values of $(\text{cis } \theta)^{1/q}$ for $n = q, q + 1, q + 2$ etc. are the mere repetition of the q values obtained in (2).

Moreover, the q values given by (2) are clearly distinct from each other, for no two of the angles involved therein are equal or differ by a multiple of 2π.

Hence $(\text{cis } \theta)^{1/q}$ has q and only q distinct values given by (2).

Obs. $(\text{cis } \theta)^{p/q}$ *where p/q is a rational fraction in its lowest terms, has also q and only q distinct values;* which are obtained by putting $n = 0, 1, 2, ..., q - 1$ successively in cis $p(2n\pi + \theta)/q$.

Note that $(\text{cis } \theta)^{6/15}$ has only 5 distinct values and not 15; because 6/15 in its lowest terms = 2/5

$\therefore$ *In order to find the distinct values of* $(\text{cis } \theta)^{p/q}$ *always see that* p/q *is in its lowest terms.*

Note. The above discussion can usefully be employed for extracting any assigned root of a given quantity. We have only to express it in the form $r(\cos \theta + i \sin \theta)$ and proceed as above.

Example 19.18. *Find the cube roots of unity and show that they form an equilateral triangle in the Argand diagram.*

Solution. If x be a cube root of unity, then

$$x = (1)^{1/3} = (\cos 0 + i \sin 0)^{1/3} = (\text{cis } 0)^{1/3} = (\text{cis } 2n\pi)^{1/3} = \text{cis } 2n\pi/3$$

where $n = 0, 1, 2$.

$\therefore$ the three vlaues of x are cis 0 = 1,

$$\text{cis } 2\pi/3 = \cos 120° + i \sin 120° = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

and
$$\text{cis } 4\pi/3 = \cos 240° + i \sin 240° = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

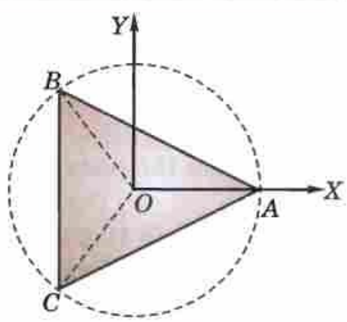

Fig. 19.19

These three cube roots are represented by the points A, B, C on the Argand diagram such that $OA = OB = OC$ and $\angle AOB = 120°$, $\angle AOC = 240°$ (Fig. 19.19).

$\therefore$ these points lie on a circle with centre O and unit radius such that $\angle AOB = \angle BOC = \angle COA = 120°$ *i.e.*, $AB = BC = CA$.

Hence A, B, C form an equilateral triangle.

Example 19.19. *Find all the values of* $\left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{3/4}$.

Also show that the continued product of these values is 1. *(Nagpur, 2009)*

Solution. Put $1/2 = r \cos \theta$ and $\sqrt{3}/2 = r \sin \theta$ so that $r = 1$ and $\theta = \pi/3$

$$\therefore \quad (1/2 + \sqrt{3}i/2)^{3/4} = [(\cos \pi/3 + i \sin \pi/3)^3]^{1/4} = (\text{cis } \pi)^{1/4}$$

$$= [\text{cis } (2n + 1)\pi]^{1/4} = \text{cis } (2n + 1)\,\pi/4 \text{ where } n = 0, 1, 2, 3.$$

Hence the required values are cis $\pi/4$, cis $3\pi/4$, cis $5\pi/4$ and cis $7\pi/4$.

$$\therefore \quad \text{their continued product} = \text{cis}\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) = \text{cis } 4\pi = 1.$$

Example 19.20. *Use De Moivre's theorem to solve the equation.* *(P.T.U., 2005)*

$$x^4 - x^3 + x^2 - x + 1 = 0.$$

Solution. '$x^4 - x^3 + x^2 - x + 1$' is a G.P. with common ratio $(-x)$, therefore

$$\frac{1-(-x)^5}{1-(-x)} = 0, \quad x \neq -1 \quad \text{or} \quad x^5 + 1 = 0$$

i.e., $$x^5 = -1 = \text{cis } \pi = \text{cis } (2n+1)\pi$$

$\therefore$ $$x = [\text{cis } (2n+1)\pi]^{1/5} = \text{cis } (2n+1)\pi/5, \text{ where } n = 0, 1, 2, 3, 4$$

Hence the values are cis $\pi/5$, cis $3\pi/5$, cis π, cis $7\pi/5$, cis $9\pi/5$

or $$\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}, \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}, -1, \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}, \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$$

Rejecting the value -1 which corresponds to the factor $x + 1$, the required roots are :

$$\cos \pi/5 \pm i \sin \pi/5, \cos 3\pi/5 \pm i \sin 3\pi/5.$$

Example 19.21. *Show that the roots of the equation $(x-1)^n = x^n$, n being a positive integer are $\frac{1}{2}(1 + i \cot r\pi/n)$, where r has the values 1, 2, 3, ..., n − 1.*

Solution. Given equation is $\left(\frac{x-1}{x}\right)^n = 1$ or $1 - \frac{1}{x} = (1)^{1/n}$

or $$\frac{1}{x} = 1 - (1)^{1/n} = 1 - \text{cis }\frac{2r\pi}{n}, r = 0, 1, 2, \ldots (n-1). \qquad [\because \ 1 = \text{cis } 2\pi r]$$

or $$= \left(1 - \cos\frac{2r\pi}{n}\right) - i\sin\frac{2r\pi}{n} = 2\sin^2\frac{r\pi}{n} - 2i\sin\frac{r\pi}{n}\cos\frac{r\pi}{n}$$

$\therefore$ $$x = \frac{1}{2\sin\frac{r\pi}{n}} \cdot \frac{1}{\left(\sin\frac{r\pi}{n} - i\cos\frac{r\pi}{n}\right)} = \frac{\sin\frac{r\pi}{n} + i\cos\frac{r\pi}{n}}{2\sin\frac{r\pi}{n}}$$

$$= \frac{1}{2}\left(1 + i\cot\frac{r\pi}{n}\right), r = 1, 2, \ldots (n-1). \qquad [\because \ \cot 0 \to \infty]$$

Hence the roots of the given equation are $\frac{1}{2}(1 + i \cot r\pi/n)$ where $r = 1, 2, 3, \ldots (n-1)$.

Example 19.22. *Find the 7th roots of unity and prove that the sum of their nth powers always vanishes unless n be a multiple number of 7, n being an integer, and then the sum is 7.*

(Mumbai, 2008 ; Kurukshetra, 2005)

Solution. We have $\quad (1)^{1/7} = (\cos 2r\pi + i \sin 2r\pi)^{1/7} = \text{cis }\frac{2r\pi}{7} = \left(\text{cis }\frac{2\pi}{7}\right)^r$

Putting $r = 0, 1, 2, 3, 4, 5, 6$, we find that 7th roots of unity are $1, \rho, \rho^2, \rho^3, \rho^4, \rho^5, \rho^6$ where $\rho = \cos 2\pi/7$.

$\therefore$ sum S of the nth powers of these roots $= 1 + \rho^n + \rho^{2n} + \ldots + \rho^{6n}$ ...(*i*)

$$= \frac{1-\rho^{7n}}{1-\rho^n}, \text{ being a G.P. with common ratio } \rho$$

When n is not a multiple of 7, $\rho^{7n} = (\rho^7)^n = (\text{cis } 2\pi)^n = 1$.

i.e., $$1 - \rho^{7n} = 0 \text{ and } 1 - \rho^n \neq 0, \text{ as } n \text{ is not a multiple of 7.}$$

Thus $S = 0$.

When n is a multiple of 7 = 7p (say)

From (*i*), $\quad S = 1 + (\rho^7)^p + (\rho^7)^{2p} + \ldots + (\rho^7)^{6p} = 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7.$

Example 19.23. *Find the equation whose roots are $2\cos\pi/7$, $2\cos 3\pi/7$, $2\cos 5\pi/7$.*

Solution. Let $y = \cos\theta + i\sin\theta$, where $\theta = \pi/7, 3\pi/7, \ldots, 13\pi/7$.

Then $y^7 = (\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta = -1$ or $y^7 + 1 = 0$

or $(y+1)(y^6 - y^5 + y^4 - y^3 + y^2 - y + 1) = 0$

Leaving the factor $y + 1$ which corresponds to $\theta = \pi$,

We get $y^6 - y^5 + y^4 - y^3 + y^2 - y + 1 = 0$ ...(*i*)

Its roots are $y = \text{cis}\,\theta$ where $\theta = \pi/7, 3\pi/7, 5\pi/7, 9\pi/7, 11\pi/7, 13\pi/7$.

Dividing (*i*) by y^3, $(y^3 + 1/y^3) - (y^2 + 1/y^2) + (y + 1/y) - 1 = 0$

or $\{(y + 1/y)^3 - 3(y + 1/y)\} - \{(y + 1/y)^2 - 2\} - (y + 1/y) - 1 = 0$

or $x^3 - x^2 - 2x + 1 = 0$ ...(*ii*)

where $x = y + 1/y = 2\cos\theta$.

Now since $\cos 13\pi/7 = \cos\pi/7$, $\cos 11\pi/7 = \cos 3\pi/7$, $\cos 9\pi/7 = \cos 5\pi/7$

Hence the roots of (*ii*) are $2\cos\dfrac{\pi}{7}, 2\cos\dfrac{3\pi}{7}, 2\cos\dfrac{5\pi}{7}$.

PROBLEMS 19.3

1. Find all the values of
   (*i*) $(1+i)^{1/4}$ (*ii*) $(-1+i)^{2/5}$
   (*iii*) $(-1+i\sqrt{3})^{3/2}$ (*iv*) $(1+i\sqrt{3})^{1/3} + (1-i\sqrt{3})^{1/3}$.
2. If w is a complex cube root of unity, prove that $1 + w + w^2 = 0$.
3. Find all the values of $(-1)^{1/6}$.
4. Mark by points on the Argand diagram, all the values of $(1+i\sqrt{3})^{1/5}$ and verify that they form a pentagon.
5. Use De Moivre's theorem to solve the following equations :
   (*i*) $x^5 + 1 = 0$ (*ii*) $x^7 + x^4 + x^3 + 1 = 0$
   (*iii*) $x^9 + x^5 - x^4 - 1 = 0$ (*Madras, 2000*) (*iv*) $(x-1)^5 + x^5 = 0$.
6. Find the roots common to the equations $x^4 + 1 = 0$ and $x^6 - i = 0$.
7. Solve the equation $x^{12} - 1 = 0$ and find which of its roots satisfy the equation $x^4 + x^2 + 1 = 0$.
8. Show that the roots of $(x+1)^7 = (x-1)^7$ are given by $\pm i\cot r\pi/7$, $r = 1, 2, 3$. (*Mumbai, 2008*)
9. Prove that the nth roots of unity form a geometric progression. (*Mumbai, 2007*)
   Also show that the sum of these n roots is zero and their product is $(-1)^{n-1}$.
10. Find the equation whose roots are $2\cos 2\pi/7$, $2\cos 4\pi/7$, $2\cos 6\pi/7$.

19.6 (1) TO EXPAND sin nθ, cos nθ AND tan nθ IN POWERS OF sin θ, cos θ AND tan θ RESPECTIVELY (n BEING A POSITIVE INTEGER)

We have $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$ (By De Moivre's theorem)

$$= \cos^n\theta + {}^nC_1\cos^{n-1}\theta\,(i\sin\theta) + {}^nC_2\cos^{n-2}\theta\,(i\sin\theta)^2 + {}^nC_3\cos^{n-3}\theta\,(i\sin\theta)^3 + \ldots$$

(By Binomial theorem)

$$= (\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots) + i({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots)$$

Equating real and imaginary parts from both sides, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots \quad \ldots(1)$$

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \ldots \quad \ldots(2)$$

Replacing every $\sin^2\theta$ by $1-\cos^2\theta$ in (1) and every $\cos^2\theta$ by $1-\sin^2\theta$ in (2), we get the desired expansions of $\cos n\theta$ and $\sin n\theta$.

Dividing (2) by (1),

$$\tan n\theta = \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \ldots}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots}$$

and dividing numerator and denominator by $\cos^n\theta$, we get

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - \ldots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \ldots}.$$

Example 19.24. *Express cos 6θ in terms of cos θ.* *(Madras, 2002)*

Solution. We know that $\cos n\theta = \cos^n \theta - {}^nC_2 \cos^{n-2} \theta \sin^2 \theta + {}^nC_4 \cos^{n-4} \theta \sin^4 \theta - \ldots$

Put $n = 6$, then $\cos 6\theta = \cos^6 \theta - {}^6C_2 \cos^4 \theta \sin^2 \theta + {}^6C_4 \cos^2 \theta \sin^4 \theta - {}^6C_6 \sin^6 \theta$

$$= \cos^6 \theta - 15 \cos^4 \theta (1 - \cos^2 \theta) + 15 \cos^2 \theta (1 - \cos^2 \theta)^2 - (1 - \cos^2 \theta)^3$$

$$= 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1.$$

(2) Addition formulae for any number of angles

We have, $\cos (\theta_1 + \theta_2 + \ldots + \theta_n) + i \sin (\theta_1 + \theta_2 + \ldots + \theta_n)$

$$= (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \ldots (\cos \theta_n + i \sin \theta_n)$$

Now $\cos \theta_1 + i \sin \theta_1 = \cos \theta_1 (1 + i \tan \theta_1)$, $\cos \theta_2 + i \sin \theta_2 = \cos \theta_2 (1 + i \tan \theta_2)$ and so on.

$\therefore$ $\cos (\theta_1 + \theta_2 + \ldots + \theta_n) + i \sin (\theta_1 + \theta_2 + \ldots + \theta_n)$

$$= \cos \theta_1 \cos \theta_2 \ldots \cos \theta_n (1 + i \tan \theta_1)(1 + i \tan \theta_2) \ldots (1 + i \tan \theta_n)$$

$$= \cos \theta_1 \cos \theta_2 \ldots \cos \theta_n [1 + i (\tan \theta_1 + \tan \theta_2 + \ldots + \tan \theta_n)$$
$$+ i^2(\tan \theta_1 \tan \theta_2 + \tan \theta_2 \tan \theta_3 + \ldots) + i^3(\tan \theta_1 \tan \theta_2 \tan \theta_3 + \ldots) + \ldots + \ldots]$$

$$= \cos \theta_1 \cos \theta_2 \ldots \cos \theta_n (1 + is_1 - s_2 - is_3 + s_4 + \ldots)$$

where $s_1 = \tan \theta_1 + \tan \theta_2 + \ldots + \tan \theta_n$, $s_2 = \Sigma \tan \theta_1 \tan \theta_2$, $s_3 = \Sigma \tan \theta_1 \tan \theta_2 \tan \theta_3$ etc.

Equating real and imaginary parts, we have

$$\cos (\theta_1 + \theta_2 + \ldots + \theta_n) = \cos \theta_1 \cos \theta_2 \ldots \cos \theta_n (1 - s_2 + s_4 - \ldots)$$

$$\sin (\theta_1 + \theta_2 + \ldots + \theta_n) = \cos \theta_1 \cos \theta_2 \ldots \cos \theta_n (s_1 - s_3 + s_5 - \ldots)$$

and by division, we get $\mathbf{tan} \; (\theta_1 + \theta_2 + \ldots + \theta_n) = \dfrac{\mathbf{s_1 - s_3 + s_5 - \ldots}}{\mathbf{1 - s_2 + s_4 - s_6 + \ldots}}$.

Example 19.25. *If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi/2$, show that $xy + yz + zx = 1$.* *(P.T.U., 2003)*

Solution. Let $\tan^{-1} x = \alpha$, $\tan^{-1} y = \beta$, $\tan^{-1} z = \gamma$ so that $x = \tan \alpha$, $y \tan \beta$, $z = \tan \gamma$

We know that $$\tan (\alpha + \beta + \gamma) = \frac{\tan \alpha + \tan \beta + \tan \gamma - \tan \alpha \tan \beta \tan \gamma}{1 - \tan \alpha \tan \beta - \tan \beta \tan \gamma - \tan \gamma \tan \alpha}$$

$\therefore$ $$\tan \pi/2 = \frac{x + y + z - xyz}{1 - xy - yz - zx} \quad \text{or} \quad 1 - xy - yz - zx = 0$$

Hence $xy + yz + zx = 1$.

Example 19.26. *If $\theta_1, \theta_2, \theta_3$ be three values of θ which satisfy the equation $\tan 2\theta = \lambda \tan (\theta + \alpha)$ and such that no two of them differ by a multiple of π, show that $\theta_1 + \theta_2 + \theta_3 + \alpha$ is a multiple of π.*

Solution. Given equation can be written as $\dfrac{2t}{1 - t^2} = \lambda \dfrac{t + \tan \alpha}{1 - t \,.\, \tan \alpha}$ where $t = \tan \theta$

or $$\lambda t^3 + (\lambda - 2) \tan \alpha \,.\, t^2 + (2 - \lambda) t - \lambda \tan \alpha = 0$$

$\therefore$ $\tan \theta_1$, $\tan \theta_2$, $\tan \theta_3$, being its roots, we have

$$s_1 = \Sigma \tan \theta_1 = -\frac{\lambda - 2}{\lambda} \tan \alpha \qquad \text{[By § 1.3]}$$

$$s_2 = \Sigma \tan \theta_1 \tan \theta_2 = \frac{2 - \lambda}{\lambda} \quad \text{and} \quad s_3 = \tan \alpha$$

$\therefore$ $$\tan (\theta_1 + \theta_2 + \theta_3) = \frac{s_1 - s_3}{1 - s_2} = \frac{(-1 + 2/\lambda) \tan \alpha - \tan \alpha}{1 - (2/\lambda - 1)}$$

$$= -\tan \alpha = \tan (n\pi - \alpha)$$

Thus $\theta_1 + \theta_2 + \theta_3 = n\pi - \alpha$, whence follows the result.

(3) To expand $\sin^m \theta$, $\cos^n \theta$ or $\sin^m \theta \cos^n \theta$ in a series of sines or cosines of multiples of θ

If $z = \cos \theta + i \sin \theta$ then $1/z = \cos \theta - i \sin \theta$.

By De Moivre's theorem, $z^p = \cos p\theta + i \sin p\theta$ and $1/z^p = \cos p\theta - i \sin p\theta$

$\therefore$ $z + 1/z = 2 \cos \theta$, $z - 1/z = 2i \sin \theta$; $z^p + 1/z^p = 2 \cos p\theta$, $z^p - 1/z^p = 2i \sin p\theta$

These results are used to expand the powers of $\sin \theta$ or $\cos \theta$ or their products in a series of sines or cosines of multiples of θ.

Example 19.27. *Expand $\cos^8 \theta$ in a series of cosines of multiples of θ.*

Solution. Let $z = \cos \theta + i \sin \theta$, so that $z + 1/z = 2 \cos \theta$ and $z^p + 1/z^p = 2 \cos p\theta$.

$\therefore$ $(2 \cos \theta)^8 = (z + 1/z)^8$

$$= z^8 + {}^8C_1 z^7 \cdot \frac{1}{z} + {}^8C_2 z^6 \cdot \frac{1}{z^2} + {}^8C_3 z^5 \cdot \frac{1}{z^3} + {}^8C_4 z^4 \cdot \frac{1}{z^4} + {}^8C_5 z^3 \cdot \frac{1}{z^5} + {}^8C_6 z^2 \cdot \frac{1}{z^6} + {}^8C_7 z \cdot \frac{1}{z^7} + \frac{1}{z^8}$$

$$= (z^8 + 1/z^8) + {}^8C_1(z^6 + 1/z^6) + {}^8C_2(z^4 + 1/z^4) + {}^8C_3(z^2 + 1/z^2) + {}^8C_4$$

$$= (2 \cos 8\theta) + 8(2 \cos 6\theta) + 28(2 \cos 4\theta) + 56(2 \cos 2\theta) + 70.$$

Hence $$\cos^8 \theta = \frac{1}{128} [\cos 8\theta + 8 \cos 6\theta + 28 \cos 4\theta + 56 \cos 2\theta + 35].$$

Example 19.28. *Expand $\sin^7 \theta \cos^3 \theta$ in a series of sines of multiples of θ.*

Solution. Let $z = \cos \theta + i \sin \theta$

so that $z + 1/z = 2 \cos \theta$, $z - 1/z = 2i \sin \theta$ and $z^p - 1/z^p = 2i \sin p\theta$.

$\therefore$ $(2i \sin \theta)^7 (2 \cos \theta)^3 = (z - 1/z)^7 (z + 1/z)^3$

$$= (z - 1/z)^4 [(z - 1/z)(z + 1/z)]^3 = (z - 1/z)^4 (z^2 - 1/z^2)^3$$

$$= \left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right)\left(z^6 - 3z^2 + \frac{3}{z^2} - \frac{1}{z^6}\right)$$

$$= \left(z^{10} - \frac{1}{z^{10}}\right) - 4\left(z^8 - \frac{1}{z^8}\right) + 3\left(z^6 - \frac{1}{z^6}\right) + 8\left(z^4 - \frac{1}{z^4}\right) - 14\left(z^2 - \frac{1}{z^2}\right)$$

$$= 2i \sin 10\theta - 4(2i \sin 8\theta) + 3(2i \sin 6\theta) + 8(2i \sin 4\theta) - 14(2i \sin 2\theta)$$

Since $i^7 = -i$,

$\therefore$ $$\sin^7 \theta \cos^3 \theta = -\frac{1}{2^9} [\sin 10\theta - 4 \sin 8\theta + 3 \sin 6\theta + 8 \sin 4\theta - 14 \sin 2\theta].$$

Obs. The expansion of $\sin^m \theta \cos^n \theta$ is a series of sines or cosines of multiples of θ according as m is odd or even.

PROBLEMS 19.4

1. Express $\sin 6\theta/\sin \theta$ as a polynomial in $\cos \theta$?

Prove that (2–5) :

2. $\sin 7\theta/\sin \theta = 7 - 56 \sin^2 \theta + 112 \sin^4 \theta - 64 \sin^6 \theta$.
3. $\dfrac{1 + \cos 7\theta}{1 + \cos \theta} = (x^3 - x^2 - 2x + 1)^2$, where $x = 2 \cos \theta$. *(Madras, 2002)*
4. $2(1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2$ where $x = 2 \cos \theta$.
5. $\tan 5\theta = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ where $t = \tan \theta$.
6. If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, show that $x + y + z = xyz$.
7. If α, β, γ be the roots of the equation $x^3 + px^2 + qx + p = 0$, prove that $\tan^{-1} \alpha + \tan^{-1} \beta + \tan^{-1} \gamma = n\pi$ radians except in one particular case.

Prove that (8-12) :

8. $\cos^7 \theta = \dfrac{1}{16} (\cos 7\theta + 7 \cos 5\theta + 21 \cos 3\theta + 35 \cos \theta)$. *(Madras, 2003 S)*
9. $\cos^6 \theta - \sin^6 \theta = \dfrac{1}{16} (\cos 6\theta + 15 \cos 2\theta)$. *(Mumbai, 2007)*
10. $\sin^8 \theta = 2^{-7} (\cos 8\theta - 8 \cos 6\theta + 28 \cos 4\theta - 56 \cos 2\theta + 35)$.
11. $32 \sin^4 \theta \cos^2 \theta = \cos 6\theta - 2 \cos 4\theta - \cos 2\theta + 2$.
12. $\sin^5 \theta \cos^2 \theta = \dfrac{1}{64} (\sin 7\theta - 3 \sin 5\theta + \sin 3\theta + 5 \sin \theta)$. *(Madras, 2003)*

13. Expand $\cos^5 \theta \sin^7 \theta$ in a series of sines of multiples of θ ? *(Madras, 2002)*
14. If $\cos^5 \theta = A \cos \theta + B \cos 3\theta + C \cos 5\theta$, find $\sin^5 \theta$ in terms of A, B, C.
15. If $\sin^4 \theta \cos^3 \theta = A_1 \cos \theta + A_3 \cos 3\theta + A_5 \cos 5\theta + A_7 \cos 7\theta$, prove that
$$A_1 + 9A_3 + 25A_5 + 49A_7 = 0.$$

19.7 COMPLEX FUNCTION

Definition. *If for each value of the complex variable z $(= x + iy)$ in a given region R, we have one or more values of w $(= u + iv)$, then w is said to be a* **complex function** *of z* and we write $w = u(x, y) + iv(x, y) = f(z)$ where u, v are real functions of x and y.

If to each value of z, there corresponds one and only one value of w, then w is said to be a *single-valued function of z* otherwise a *multi-valued function*. For example, $w = 1/z$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function of z. The former is defined at all points of the z-plane except at $z = 0$ and the latter assumes two values for each value of z except at $z = 0$.

19.8 EXPONENTIAL FUNCTION OF A COMPLEX VARIABLE

(1) Definition. When x is real, we are already familiar with the exponential function
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \infty .$$
Similarly, *we define the exponential function of the complex variable $z = x + iy$, as*
$$\mathbf{e^z \text{ or } \exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots \infty} \qquad \dots(i)$$

(2) Properties :

I. Exponential form of $z = re^{i\theta}$

Putting $x = 0$ in (i), we get
$$e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \dots \infty$$
$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right) = \cos y + i \sin y$$

Thus $\quad e^z = e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$

Also $\quad x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$. Thus, $z = re^{i\theta}$

II. e^z is periodic function having imaginary period $2\pi i$, [$\because \; e^{z + 2n\pi i} = e^z \cdot e^{2n\pi i} = e^z$].

III. e^z is not zero for any value of z.

Since $\quad e^z = e^{x + iy} = re^{i\theta}$ or $e^x \cdot e^{iy} = re^{i\theta}$

$\therefore \quad r = e^x > 0, y = \theta, |e^{iy}| = 1,$

Thus $\quad |e^z| = |e^x| \cdot |e^{iy}| = e^x \neq 0.$

IV. $\quad e^{\bar{z}} = \overline{e^z}$

Since $\quad e^{\bar{z}} = e^{x - iy} = e^x \cdot e^{-iy} = e^x (\cos y - i \sin y)$
$$= \overline{e^x (\cos y + i \sin y)} = \overline{e^z}$$

19.9 CIRCULAR FUNCTIONS OF A COMPLEX VARIABLE

(1) Definitions:

Since $\quad e^{iy} = \cos y + i \sin y$ and $e^{-iy} = \cos y - i \sin y$.

$\therefore$ the circular functions of real angles can be written as

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and so on.}$$

It is, therefore, natural to *define the circular functions of the complex variable z by the equations :*

$$\mathbf{\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}, \tan z = \frac{\sin z}{\cos z}}$$

with cosec z, sec z and cot z as their respective reciprocals.

(2) Properties :

I. Circular functions are periodic : $\sin z$, $\cos z$ are periodic functions having real period 2π while $\tan z$, $\cot z$ have period π. [a $\sin(z + 2n\pi) = \sin z$, $\tan(z + n\pi) = \tan z$ etc.]

II. Even and odd functions : cos z, sec z are even functions while sin z, cosec z are odd functions. [$\because$ $\cos z$ $= \dfrac{e^{-iz} + e^{iz}}{2} = \cos z$, and $\sin(-z) = \dfrac{e^{-iz} - e^{iz}}{2i} = \dfrac{e^{iz} - e^{-iz}}{2i} = -\sin z$]

III. Zeros of sin z are given by $z = \pm 2n\pi$ *and zeros of cos z are given by* $z = \pm \frac{1}{2}(2n + 1)\pi$, $n = 0, 1, 2, ...$

IV. All the formulae for real circular functions are valid for complex circular functions

e.g., $\sin^2 z + \cos^2 z = 1$, $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$.

(3) Euler's theorem $e^{iz} = \cos z + i \sin z$.

By definition $\cos z + i \sin z = \dfrac{e^{iz} + e^{-iz}}{2} + i\,\dfrac{e^{iz} - e^{-iz}}{2i} = e^{iz}$ where $z = x + iy$.

Also we have shown that $e^{iy} = \cos y + i \sin y$, where y is real.

Thus $e^{i\theta} = \mathbf{\cos\theta + \mathit{i} \sin\theta}$, *where θ is real or complex. This is called the Euler's theorem.**

Cor. *De Moivre's theorem for complex numbers*

Whether θ is real or complex, we have

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

Thus De Moivre's theorem is true for all θ (real or complex).

Example 19.29. *Prove that (i)* $[\sin(\alpha + \theta) - e^{i\alpha}\sin\theta]^n = \sin^n\alpha\, e^{-in\theta}$

(ii) $\sin(\alpha - n\theta) + e^{-i\alpha}\sin n\theta = e^{-in\theta}\sin\alpha$.

Solution. (*i*) L.H.S. $= [\sin\alpha\cos\theta + \cos\alpha\sin\theta - (\cos\alpha + i\sin\alpha)\sin\theta]^n$

$= (\sin\alpha\cos\theta - i\sin\alpha\sin\theta)^n$

$= \sin^n\alpha(\cos\theta - i\sin\theta)^n = \sin^n\alpha(e^{-i\theta})^n = \sin^n\alpha e^{-in\theta}$

(*ii*) L.H.S. $= \sin\alpha\cos n\theta - \cos\alpha\sin n\theta + (\cos\alpha - i\sin\alpha)\sin n\theta$

$= \sin\alpha\cos n\theta - i\sin\alpha\sin n\theta$

$= \sin\alpha(\cos n\theta - i\sin n\theta) = \sin\alpha \,.\, e^{-in\theta}$.

Example 19.30. *Given* $\dfrac{1}{\rho} = \dfrac{1}{L\rho i} + C\rho i + \dfrac{1}{R}$, *where L, ρ, R are real, express ρ in the form* $Ae^{i\theta}$ *giving the values of A and θ.*

Solution.
$$\frac{1}{\rho} = \frac{R + L\rho^2 CR(-1) + L\rho i}{L\rho Ri} = \frac{(R - L\rho^2 CR) + iLR}{L\rho Ri}$$

or
$$\rho = L\rho\frac{Ri}{(R - L\rho^2 CR) + iL\rho} \times \frac{(R - L\rho^2 CR) - iL\rho}{(R - L\rho^2 CR) - iL\rho}$$

$$= \frac{L^2\rho^2 R + iL\rho R(R - L\rho^2 CR)}{(R - L\rho^2 CR)^2 + (L\rho)^2} = A(\cos\theta + i\sin\theta), \text{ say}$$

*See footnote p. 205.

Equating real and imaginary parts, we have

$$A\cos\theta = \frac{L^2\rho^2 R}{(R - L\rho^2 CR)^2 + (L\rho)^2} \quad ...(i)$$

$$A\sin\theta = \frac{L\rho R(R - L\rho^2 CR)}{(R - L\rho^2 CR)^2 + (L\rho)^2} \quad ...(ii)$$

Squaring and adding (*i*) and (*ii*),

$$A^2 = \frac{(L^2\rho^2 R)^2 + (L\rho R)^2 (R - L\rho^2 CR)^2}{[(R - L\rho^2 CR)^2 + (L\rho)^2]^2} \quad \text{or} \quad A = \frac{L\rho R}{\sqrt{[(R - L\rho^2 CR)^2 + (L\rho)]^2}} \quad ...(iii)$$

Dividing (*ii*) by (*i*),

$$\tan\theta = \frac{R - L\rho^2 CR}{L\rho} \quad \text{or} \quad \theta = \tan^{-1}\left\{\frac{R(1 - LC\rho^2)}{L\rho}\right\} \quad ...(iv)$$

Hence $\quad P = A(\cos\theta + i\sin\theta) = Ae^{i\theta}$

where A and θ are given by (*iii*) and (*iv*).

19.10 HYPERBOLIC FUNCTIONS

(1) Definitions: *If x be real or complex,*

(*i*) $\dfrac{e^x - e^{-x}}{2}$ *is defined as* **hyperbolic sine of x** and is written as **sinh x.**

(*ii*) $\dfrac{e^x + e^{-x}}{2}$ *is defined as* **hyperbolic cosine of x** *and is written as* **cosh x.**

Thus $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Also we define,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \ \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}; \ \text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

(2) Properties

I. Periodic functions : sinh z and cosh z are periodic functions having imaginary period 2πi.

[$\because \quad \sinh(z + 2\pi i) = \sinh z$; $\cosh(z + 2\pi i) = \cosh z$]

II. Even and odd functions : cosh z is an even function while sinh z is an odd function

III. $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$.

IV. **Relations between hyperbolic and circular functions.**

Since for all values of θ, $\quad \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

∴ Putting $\theta = ix$, we have $\quad \sin ix = \dfrac{e^{-x} - e^{x}}{2i} = -\dfrac{e^x - e^{-x}}{2i}$ $\quad [\because \ e^{i\theta} = e^{i \cdot ix} = e^{-x}]$

$$= i^2 \frac{e^x - e^{-x}}{2i} = i \cdot \frac{e^x - e^{-x}}{2} = i\sinh x$$

and, therefore, $\quad \cos ix = \dfrac{e^{-x} + e^{x}}{2} = \cosh x$

Thus $\quad$ **sin ix = i sinh x** $\quad ...(i)$

cos ix = cosh x $\quad ...(ii)$

and ∴ $\quad$ **tan ix = i tanh x** $\quad ...(iii)$

Cor. $\quad$ **sinh ix = i sin x** $\quad ...(iv)$

cosh ix = cos x $\quad ...(v)$

tanh ix = i tan x $\quad ...(vi)$

V. Formulae of hyperbolic functions

(a) Fundamental formulae

(1) $\cosh^2 x - \sinh^2 x = 1$ (2) $\text{sech}^2 x + \tanh^2 x = 1$ (3) $\coth^2 x - \text{cosech}^2 x = 1$.

(b) Addition formulae

(4) $\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$ (5) $\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(6) $\tanh (x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

(c) Functions of 2x.

(7) $\sinh 2x = 2 \sinh x \cosh x$

(8) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x$

(9) $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

(d) Functions of 3x

(10) $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$

(11) $\cosh 3x = 4 \cosh^3 x - 3 \cosh x$ (12) $\tanh 3x = \dfrac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$

(e) (13) $\sinh x + \sinh y = 2 \sinh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$ (14) $\sinh x - \sinh y = 2 \cosh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

(15) $\cosh x + \cosh y = 2 \cosh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$ (16) $\cosh x - \cosh y = 2 \sinh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$.

Proofs. (1) Since, for all values of θ, we have $\cos^2 \theta + \sin^2 \theta = 1$.

∴ putting $\theta = ix$, we get $\cos^2 ix + \sin^2 ix = 1$ or $\cosh^2 x - \sinh^2 x = 1$

Otherwise : $\cosh^2 x - \sinh^2 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{e^x - e^{-x}}{2}\right)^2 = \dfrac{1}{4} [e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2] = 1$.

Similarly we can establish the formulae (2) and (3).

(4) $\sinh (x + y) = (1/i) \sin i(x + y) = - i[\sin ix \cos iy + \cos ix \sin iy]$

$= - i[i \sinh x \,.\, \cosh y + \cosh x \,.\, i \sinh y] = \sinh x \cosh y + \cosh x \sinh y$.

Otherwise : $\sinh x \cosh y + \cosh x \sinh y$

$$= \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh (x + y)$$

Similarly we can establish the formulae (5) and (6).

(12) $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

Putting $A = ix$, $\tan 3ix = \dfrac{3 \tan ix - \tan^3 ix}{1 - 3 \tan^2 ix}$ or $i \tanh 3x = \dfrac{3(i \tanh x) - (i \tanh x)^3}{1 - 3(i \tanh x)^2}$

$$\tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$$

Similarly, we can establish the formulae (7) to (11).

(16) $\cos C - \cos D = - 2 \sin \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$

Putting $C = ix$, and $D = iy$, $\cos ix - \cos iy = - 2 \sin i \dfrac{x+y}{2} \sin i \dfrac{x-y}{2}$

or $\cosh x - \cosh y = - 2\left(i \sinh \dfrac{x+y}{2}\right)\left(i \sinh \dfrac{x-y}{2}\right) = 2 \sinh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

Similarly, we can establish the formulae (13) to (15).

19.11 INVERSE HYPERBOLIC FUNCTIONS

(1) Definitions: If *sinh* $u = z$, then u is called the hyperbolic sine inverse of z and is written as $u = \sinh^{-1} z$. Similarly we define $\cosh^{-1} z$, $\tanh^{-1} z$, etc.

The inverse hyperbolic functions like other inverse functions are many-valued, but we shall consider only their principal values.

(2) *To show that (i)* $\sinh^{-1} z = \log [z + \sqrt{(z^2 + 1)}]$ *(Mumbai, 2009)*

(ii) $\cosh^{-1} z = \log [z + \sqrt{(z^2 - 1)}]$, *(iii)* $\tanh^{-1} z = \dfrac{1}{2} \log \dfrac{1+z}{1-z}$.

(i) Let $\sinh^{-1} z = u$, then $z = \sinh u = \dfrac{1}{2}(e^u - e^{-u})$

or $$2z = e^u - 1/e^u \quad \text{or} \quad e^{2u} - 2ze^u - 1 = 0$$

This being a quadratic in e^u, we have

$$e^u = \frac{2z \pm \sqrt{(4z^2 + 4)}}{2} = z \pm \sqrt{(z^2 + 1)}$$

$\therefore$ Taking the positive sign only, we have

$$e^u = z + \sqrt{(z^2 + 1)} \quad \text{or} \quad u = \log [z + \sqrt{(z^2 + 1)}]$$

Similarly we can establish *(ii)*

(iii) Let $\tanh^{-1} z = u$, then $z = \tanh u$

i.e., $$z = \frac{e^u - e^{-u}}{e^u + e^{-u}}.$$

Applying componendo and dividendo, we get $\dfrac{1+z}{1-z} = e^u/e^{-u} = e^{2u}$

or $$2u = \log \left(\frac{1+z}{1-z}\right) \text{ whence follows the result.}$$ *(P.T.U., 2005)*

Example 19.31. *If* $u = \log \tan (\pi/4 + \theta/2)$, *prove that*

(i) $\tanh u/2 = \tan \theta/2$ *(Mumbai, 2008 ; P.T.U., 2006 ; Madras, 2003)*

(ii) $\theta = -i \log \tan \left(\dfrac{\pi}{4} + \dfrac{iu}{2}\right)$. *(Kurukshetra, 2006)*

Solution. We have $\quad e^u = \tan \left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right) \quad$ or $\quad \dfrac{e^{u/2}}{e^{-u/2}} = \dfrac{1 + \tan \theta/2}{1 - \tan \theta/2}$

By componendo and dividendo, we get

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \tan \theta/2 \quad i.e., \quad \tanh \frac{u}{2} = \tan \frac{\theta}{2} \qquad ...(i)$$

or $$\frac{1}{i} \tan \frac{iu}{2} = \frac{1}{i} \tanh \frac{i\theta}{2} \quad \text{or} \quad \frac{i\theta}{2} = \tanh^{-1} \left(\tan \frac{iu}{2}\right) = \frac{1}{2} \log \frac{1 + \tan iu/2}{1 - \tan iu/2}$$

or $$\theta = \frac{1}{i} \log \tan \left(\frac{\pi}{4} + \frac{iu}{2}\right) = -i \log \tan \left(\frac{\pi}{4} + \frac{iu}{2}\right). \qquad ...(ii)$$

Example 19.32. *Show that* $\tanh^{-1} (\cos \theta) = \cosh^{-1} (\text{cosec } \theta)$. *(Kurukshetra, 2005)*

Solution. Let $\tanh^{-1} (\cos \theta) = \phi$ so that $\cos \theta = \tanh \phi$

or $\quad \tanh^2 \phi = \cos^2 \theta \quad$ or $\quad 1 - \text{sech}^2 \phi = \cos^2 \theta$

or $\quad \text{sech}^2 \phi = 1 - \cos^2 \theta = \sin^2 \theta \quad$ or $\quad \text{sech } \phi = \sin \theta$

or $\quad \cosh \phi = \text{cosec } \theta \quad$ or $\quad \phi = \cosh^{-1} (\text{cosec } \theta)$.

Example 19.33. *Find tanh x, if 5 sinh x – cosh x = 5.* (*Mumbai, 2004*)

Solution. We have $5(\sinh x - 1) = \cosh x$

or $$25(\sinh x - 1)^2 = \cosh^2 x = 1 + \sinh^2 x$$

or $$24 \sinh^2 x - 50 \sinh x + 24 = 0 \quad \text{or} \quad 12 \sinh^2 x - 25 \sinh x + 12 = 0$$

or $$(3 \sinh x - 4)(4 \sinh x - 3) = 0 \quad \text{whence } \sinh x = 4/3 \text{ or } 3/4.$$

$\therefore$ $\cosh x = \sqrt{(1 + \sinh^2 x)} = 5/3$ or $-5/4$ [$\because$ $\cosh x = 5/4$ doesn't satisfy (*i*)]

Hence $\tanh x = \dfrac{4}{5}$ or $-\dfrac{3}{5}$.

PROBLEMS 19.5

1. Separate into real and imaginary parts
   (*i*) $\exp(z^2)$ where $z = x + iy$ (*ii*) $\exp(5 + i\pi/2)$ (*iii*) $\exp(5 + 3i)^2$.
2. From the definitions of sin z and cos z, prove that
   (*i*) $\cos 2z = 2\cos^2 z - 1$ (*ii*) $\dfrac{\sin 2z}{1 - \cos 2z} = \cot z$ (*iii*) $\sin 3z = 3 \sin z - 4 \sin^3 z$.
3. Prove that $[\sin(\alpha - \theta) + e^{-i\alpha} \sin \theta]^n = \sin^{n-1} \alpha \{\sin(\alpha - n\theta) + e^{-i\alpha} \sin n\theta\}$
4. If $z = e^{i\theta}$, show that $\dfrac{z^2 - 1}{z^2 + 1} = i \tan \theta$.
5. Eliminate z from p cosech $z + q$ sech $z + r = 0$, p' cosech $z + q'$ sech $z + r' = 0$.
6. If $y = \log \tan x$, show that $\sinh ny = \dfrac{1}{2}(\tan^n x - \cot^n x)$.
7. If $\tan y = \tan \alpha \tanh \beta$ and $\tan z = \cot \alpha \tanh \beta$, prove that $\tan(y + z) = \sinh 2\beta \operatorname{cosec} 2\alpha$.
8. Prove that
   (*i*) $\cosh(\alpha + \beta) - \cosh(\alpha - \beta) = 2 \sinh \alpha \sinh \beta$
   (*ii*) $\sinh(\alpha + \beta) \cosh(\alpha - \beta) = \dfrac{1}{2}(\sinh 2\alpha + \sinh 2\beta)$.
9. Prove that (*i*) $(\cosh \theta \pm \sinh \theta)^n = \cosh n\theta + \sinh n\theta$; (*ii*) $\left(\dfrac{1 + \tanh \theta}{1 - \tanh \theta}\right)^3 = \cosh 6\theta + \sinh 6\theta$.
10. Express $\cosh^7 \theta$ in terms of hyperbolic cosines of multiples of θ.
11. If $\sin \theta = \tanh x$, prove that $\tan \theta = \sinh x$.
12. If $\tan x/2 = \tanh u/2$, prove that
    (*i*) $\tan x = \sinh u$ and $\cos x \cosh u = 1$; (*ii*) $u = \log_e \tan(\pi/4 + x/2)$.
13. If $\cosh x = \sec \theta$, prove that
    (*i*) $\tanh^2 x/2 = \tan^2 \theta/2$ (*ii*) $x = \log_e \tan(\pi/4 + \theta/2)$.
14. Show that $\tan^{-1} z = \dfrac{i}{2} \log \dfrac{i + z}{i - z}$.
15. Prove that
    (*i*) $\sinh^{-1} x = \cosh^{-1} \sqrt{(1 + x^2)} = \tanh^{-1} \dfrac{x}{\sqrt{(1 - x^2)}} = \dfrac{1}{2} \operatorname{cosech}^{-1} \dfrac{1}{2x\sqrt{(1 + x^2)}}$
    (*ii*) $\tanh^{-1} x = \sinh^{-1} \dfrac{x}{\sqrt{(1 - x^2)}}$.
16. Show that
    (*i*) $\sinh^{-1}(\tan \theta) = \log \tan(\pi/4 + \theta/2)$ (*ii*) $\operatorname{sech}^{-1}(\sin \theta) = \log \cot \theta/2$.
17. Solve the equation $7 \cosh x + 8 \sinh x = 1$ for real values of x. (*Mumbai, 2008*)
18. Find $\tanh x$ if $\sinh x - \cosh x = 5$.

19.12 REAL AND IMAGINARY PARTS OF CIRCULAR AND HYPERBOLIC FUNCTIONS

(1) *To separate the real and imaginary parts of*

(i) sin $(x + iy)$; (ii) cos $(x + iy)$; (iii) tan $(x + iy)$; (iv) cot $(x + iy)$; (v) sec $(x + iy)$; (vi) cosec $(x + iy)$.

Proofs. (*i*) $\sin(x + iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$.

Similarly, $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$

(*iii*) Let $\alpha + i\beta = \tan(x + iy)$ then $\alpha - i\beta = \tan(x - iy)$

Adding, $2\alpha = \tan(x + iy) + \tan(x - iy)$

i.e.,
$$\alpha = \frac{\sin(x + iy + x - iy)}{2\cos(x + iy)\cos(x - iy)} = \frac{\sin 2x}{\cos 2x + \cos 2iy} = \frac{\sin 2x}{\cos 2x + \cosh 2y}$$

Subtracting, $2i\beta = \tan(x + iy) - \tan(x - iy)$

i.e.,
$$i\beta = \frac{\sin 2iy}{2\cos(x + iy)\cos(x - iy)} = \frac{i \sinh 2y}{\cos 2x + \cosh 2y}$$

$\therefore$
$$\beta = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

Similarly, $\cot(x + iy) = \dfrac{\sin 2x - i \sinh 2y}{\cosh 2y - \cos 2x}$.

(*v*) Let $\alpha + i\beta = \sec(x + iy)$ then $\alpha - i\beta = \sec(x - iy)$

Adding, $2\alpha = \sec(x + iy) + \sec(x - iy)$

i.e.,
$$\alpha = \frac{\cos(x - iy) + \cos(x + iy)}{2\cos(x + iy)\cos(x - iy)} = \frac{2\cos x \cos iy}{\cos 2x + \cos 2iy} = \frac{2\cos x \cosh y}{\cos 2x + \cosh 2y}$$

Subtracting, $2i\beta = \sec(x + iy) - \sec(x - iy)$

i.e.,
$$i\beta = \frac{\cos(x - iy) - \cos(x + iy)}{2\cos(x + iy)\cos(x - iy)} = \frac{2\sin x \sin iy}{\cos 2x + \cos 2iy} = \frac{2i \sin x \sinh y}{\cos 2x + \cosh 2y}$$

$\therefore$
$$\beta = \frac{2\sin x \sinh y}{\cos 2x + \cosh 2y}$$

Similarly, $\operatorname{cosec}(x + iy) = 2\,\dfrac{\sin x \cosh y - i \cos x \sinh y}{\cosh 2y - \cos 2x}$.

(2) *To separate the real and imaginary parts of*

(i) sinh $(x + iy)$; (ii) cosh $(x + iy)$; (iii) tanh $(x + iy)$.

Proofs. (*i*) $\sinh(x + iy) = (1/i)\sin i(x + iy) = (1/i)\sin(ix - y)$

$= (1/i)[\sin ix \cos y - \cos ix \sin y] = (1/i)[i \sinh x \cos y - \cosh x \sin y]$

$= \sinh x \cos y + i \cosh x \sin y$

Similarly, $\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$.

(*iii*) If $\alpha + i\beta = \tanh(x + iy) = (1/i)\tan(ix - y)$

then $\alpha - i\beta = \tanh(x - iy) = (1/i)\tan(ix + y)$

Adding, $2\alpha = (1/i)[\tan(ix - y) + \tan(ix + y)]$

$$\alpha = \frac{\sin(ix - y + ix + y)}{i \,.\, 2\cos(ix - y)\cos(ix + y)} = \frac{(1/i)\sin 2ix}{\cos 2ix + \cos 2y} = \frac{\sinh 2x}{\cosh 2x + \cos 2y}.$$

Subtracting, $2i\beta = (1/i)[\tan(ix - y) - \tan(ix + y)]$

i.e.,
$$i\beta = -\frac{\sin[(ix + y) - (ix - y)]}{i \,.\, 2\cos(ix + y)\cos(ix - y)}$$

$\therefore$
$$\beta = \frac{\sin 2y}{\cos 2ix + \cos 2y} = \frac{\sin 2y}{\cosh 2x + \cos 2y}.$$

Example 19.34. *If* $\cosh(u + iv) = x + iy$, *prove that*

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1 \quad \textit{(P.T.U., 2009 S)} \qquad \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1.$$ *(Madras, 2000)*

Solution. Since $x + iy = \cosh(u + iv) = \cos(iu - v)$

$= \cos iu \cos v + \sin iu \sin v = \cosh u \cos v + i \sinh u \sin v.$

$\therefore$ equating real and imaginary parts, we get $x = \cosh u \cos v$; $y = \sinh u \sin v$

i.e.,
$$\frac{x}{\cosh u} = \cos v \quad \text{and} \quad \frac{y}{\sinh u} = \sin v$$

Squaring and adding, we get the first result.

Again
$$\frac{x}{\cos v} = \cosh u \quad \text{and} \quad \frac{v}{\sin v} = \sinh u.$$

$\therefore$ squaring and subtracting, we get the second result.

Example 19.35. *If* $\tan(\theta + i\phi) = e^{i\alpha}$, *show that*

$$\theta = (n + 1/2)\pi/2 \quad \text{and} \quad \phi = \frac{1}{2} \log \tan(\pi/4 + \alpha/2).$$ *(S.V.T.U., 2007 ; Rohtak, 2005)*

Solution. Since $\tan(\theta + i\phi) = \cos\alpha + i \sin\alpha \quad \therefore \quad \tan(\theta - i\phi) = \cos\alpha - i\sin\alpha$

$\therefore$
$$\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$$

$$= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)} = \frac{2\cos\alpha}{1 - (\cos^2\alpha + \sin^2\alpha)} = \frac{2\cos\alpha}{0} \to \infty$$

i.e.,
$$2\theta = n\pi + \pi/2 \quad \text{or} \quad \theta = (n + 1/2)\pi/2$$

Also
$$\tan 2i\phi = \tan[(\theta + i\phi) - (\theta - i\phi)] = \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

or
$$i \tanh 2\phi = \frac{2i \sin\alpha}{1 + (\cos^2\alpha + \sin^2\alpha)} = i\sin\alpha \quad \text{or} \quad \frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \frac{\sin\alpha}{1}$$

By componendo and dividendo, we get

$$\frac{e^{2\phi}}{e^{-2\phi}} = \frac{1 + \sin\alpha}{1 - \sin\alpha} = \frac{\cos^2\alpha/2 + \sin^2\alpha/2 + 2\sin\alpha/2\cos\alpha/2}{\cos^2\alpha/2 + \sin^2\alpha/2 - 2\sin\alpha/2\cos\alpha/2}$$

or
$$e^{4\phi} = \frac{(\cos\alpha/2 + \sin\alpha/2)^2}{(\cos\alpha/2 - \sin\alpha/2)^2} = \left(\frac{1 + \tan\alpha/2}{1 - \tan\alpha/2}\right)^2$$

or
$$e^{2\phi} = \frac{1 + \tan\alpha/2}{1 - \tan\alpha/2} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right). \quad \text{Hence } \phi = \frac{1}{2}\log\tan(\pi/4 + \alpha/2).$$

Example 19.36. *Separate* $\tan^{-1}(x + iy)$ *into real and imaginary parts.* *(S.V.T.U., 2009)*

Solution. Let $\alpha + i\beta = \tan^{-1}(x + iy)$. Then $\alpha - i\beta = \tan^{-1}(x - iy)$

Adding,
$$2\alpha = \tan^{-1}(x + iy) + \tan^{-1}(x - iy)^* = \tan^{-1}\frac{(x + iy) + (x - iy)}{1 - (x + iy)(x - iy)}$$

$\therefore$
$$\alpha = \frac{1}{2}\tan^{-1}\frac{2x}{1 - x^2 - y^2}$$

Subtracting,
$$2i\beta = \tan^{-1}(x + iy) - \tan^{-1}(x - iy) = \tan^{-1}\frac{(x + iy) - (x - iy)}{1 + (x + iy)(x - iy)}$$

$$= \tan^{-1} i\frac{2y}{1 + x^2 + y^2} = i\tanh^{-1}\frac{2y}{1 + x^2 + y^2} \qquad [\because \tan^{-1} iz = i\tanh^{-1} z]$$

$\therefore$
$$\beta = \frac{1}{2}\tanh^{-1}\frac{2y}{1 + x^2 + y^2}.$$

Example 19.37. *Separate* $\sin^{-1}(\cos\theta + i\sin\theta)$ *into real and imaginary parts, where* θ *is a positive acute angle.*

---

* $\tan^{-1} A \pm \tan^{-1} B = \tan^{-1}\dfrac{A \pm B}{1 \mp AB}$

Solution. Let $\sin^{-1}(\cos\theta + i\sin\theta) = x + iy$

Then $\cos\theta + i\sin\theta = \sin(x + iy) = \sin x\cosh y + i\cos x\sinh y$

$\therefore$ $\cos\theta = \sin x\cosh y$ ...(*i*) and $\sin\theta = \cos x\sinh y$ ...(*ii*)

Squaring and adding, we have

$$1 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = \sin^2 x(1 + \sinh^2 y) + \cos^2 x\sinh^2 y$$
$$= \sin^2 x + \sinh^2 y(\sin^2 x + \cos^2 x)$$

or $$1 - \sin^2 x = \sinh^2 y, \quad i.e. \quad \cos^2 x = \sinh^2 y.$$

Hence from (*ii*), we have $\sin^2\theta = \cos^4 x$, *i.e.*, $\cos^2 x = \sin\theta$ because θ being a positive acute angle, $\sin\theta$ is positive.

As x is to be between $-\pi/2$ and $\pi/2$, therefore, we have

$$\cos x = +\sqrt{(\sin\theta)} \quad \text{or} \quad x = \cos^{-1}\sqrt{(\sin\theta)}$$

The relation (*ii*), then, gives $\sinh y = \sqrt{(\sin\theta)}$ so that $y = \log[\sqrt{(\sin\theta)} + \sqrt{(1 + \sin\theta)}]$.

PROBLEMS 19.6

1. If $\sin(A + iB) = x + iy$, prove that

   (*i*) $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$ (*ii*) $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = 1.$ (*P.T.U., 2010*)

2. If $\cos(\alpha + i\beta) = r(\cos\theta + i\sin\theta)$, prove that (*i*) $e^{2\beta} = \dfrac{\sin(\alpha - \theta)}{\sin(\alpha + \theta)}$. (*Kurukshetra, 2005 ; Madras, 2003*)

   (*ii*) $\beta = \dfrac{1}{2}\log\dfrac{\sin(\alpha - \theta)}{\sin(\alpha + \theta)}$. (*V.T.U., 2006*)

3. If $\cos(\theta + i\phi) = \cos\alpha + i\sin\alpha$, prove that

   (*i*) $\sin^2\theta = \pm\sin\alpha$ (*Madras, 2003*) (*ii*) $\cos 2\theta + \cosh 2\phi = 2$.

4. If $\tan(A + iB) = x + iy$, prove that

   (*i*) $x^2 + y^2 + 2x\cot 2A = 1$. (*ii*) $x^2 + y^2 - 2y\coth 2B + 1 = 0$. (*iii*) $x\sinh 2B = y\sin 2A$.

5. If $\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha$, prove that $e^{2\phi} = \pm\cot\alpha/2$ and $2\theta = \left(n + \dfrac{1}{2}\right)\pi + \alpha$. (*Nagpur, 2009 ; S.V.T.U., 2008*)

6. If $\tan(x + iy) = \sin(u + iv)$, prove that $\dfrac{\sin 2x}{\sinh 2y} = \dfrac{\tan u}{\tan v}$. (*S.V.T.U., 2006*)

7. If $\operatorname{cosec}(\pi/4 + ix) = u + iv$, prove that $(u^2 + v^2) = 2(u^2 - v^2)$. (*Mumbai, 2009*)

8. If $x = 2\cos\alpha\cosh\beta$, $y = 2\sin\alpha\sinh\beta$, prove that $\sec(\alpha + i\beta) + \sec(\alpha - i\beta) = \dfrac{4x}{x^2 + y^2}$.

9. If $a + ib = \tanh(v + i\pi/4)$, prove that $a^2 + b^2 = 1$.

10. Reduce $\tan^{-1}(\cos\theta + i\sin\theta)$ to the form $a + ib$. (*Mumbai, 2009*)

    Hence show that $\tan^{-1}(e^{i\theta}) = \dfrac{n\pi}{2} + \dfrac{\pi}{4} - \dfrac{i}{2}\log\tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$.

11. Separate $\cos^{-1}(\cos\theta + i\sin\theta)$ into real and imaginary parts, where θ is a positive acute angle.

12. If $\sin^{-1}(u + iv) = \alpha + i\beta$, prove that $\sin^2\alpha$ and $\cosh^2\beta$ are the roots of the equation

    $$x^2 - x(1 + u^2 + v^2) + u^2 = 0.$$

13. If $\cos^{-1}(x + iy) = \alpha + i\beta$, show that

    (*i*) $x^2\sec^2\alpha - y^2\operatorname{cosec}^2\alpha = 1$, (*ii*) $x^2\operatorname{sech}^2\beta + y^2\operatorname{cosech}^2\beta = 1$.

14. Prove that (*i*) $\sin^{-1}(ix) = 2n\pi + i\log(\sqrt{1 + x^2} + x)$ (*ii*) $\sin^{-1}(\operatorname{cosec}\theta) = \pi/2 + i\log\cot\theta/2$.

19.13 LOGARITHMIC FUNCTION OF A COMPLEX VARIABLE

(1) *Definition. If $z(= x + iy)$ and $w(= u + iv)$ be so related that $e^w = z$, then w is said to be a logarithm of z to the base e and is written as $w = \log_e z$.* ...(*i*)

Also $$e^{w + 2in\pi} = e^w \cdot e^{2in\pi} = z \qquad [\because e^{2in\pi} = 1]$$

$\therefore$ $$\log z = w + 2in\pi \qquad ...(ii)$$

i.e., the logarithm of a complex number has an infinite number of values and is, therefore, a multi-valued function.

The general value of the logarithm of z is written as $Log\ z$ (beginning with capital L) so as to distinguish it from its principal value which is written as $\log z$. This principal value is obtained by taking $n = 0$ in $Log\ z$.

Thus from (*i*) and (*ii*), $Log\ (x + iy) = 2in\pi + \log (x + iy)$.

Obs. 1. If $y = 0$, then $Log\ x = 2in\pi + \log x$.

This shows that the logarithm of a real quantity is also multi-valued. Its principal value is real while all other values are imaginary.

2. *We know that the logarithm of a negative quantity has no real value. But we can now evaluate this.*

e.g. $\log_e (-2) = \log_e 2(-1) = \log_e 2 + \log_e (-1) = \log_e 2 + i\pi$ $\quad [\because\ -1 = \cos \pi + i \sin \pi = e^{i\pi}]$

$= 0.6931 + i\ (3.1416)$.

(2) *Real and imaginary parts of Log* $(x + iy)$.

$$Log\ (x + iy) = 2in\pi + \log (x + iy)$$

$$= 2in\pi + \log [r (\cos \theta + i \sin \theta)] \qquad \begin{cases} \text{Put } x = r\cos\theta,\ y = r\sin\theta \text{ so that} \\ r = \sqrt{(x^2 + y^2)} \text{ and } \theta = \tan^{-1}(y/x) \end{cases}$$

$$= 2in\pi + \log (re^{i\theta})$$

$$= 2in\pi + \log r + i\theta = \log \sqrt{(x^2 + y^2)} + i\ \{2n\pi + \tan^{-1}(y/x)\}$$

(3) *Real and imaginary parts of* $(\alpha + i\beta)^{x + iy}$

$$(\alpha + i\beta)^{x+iy} = e^{(x+iy)\,Log\,(\alpha + i\beta)} = e^{(x+iy)\,[2in\pi + \log(\alpha + i\beta)]} \qquad \begin{cases} \text{Put } \alpha = r\cos\theta,\ \beta = r\sin\theta \text{ so that} \\ r = \sqrt{(\alpha^2 + \beta^2)} \text{ and } \theta = \tan^{-1}\beta/\alpha \end{cases}$$

$$= e^{(x+iy)[2in\pi + \log re^{i\theta}]} = e^{(x+iy)\,[\log r + i\,(2n\pi + \theta)]}$$

$$= e^{A + iB} = e^A (\cos B + i \sin B).$$

where $A = x \log r - y\ (2n\pi + \theta)$ and $B = y \log r + x\ (2n\pi + \theta)$.

$\therefore$ the required real part $= e^A \cos B$ and the imaginary part $= e^A \sin B$.

Example 19.38. *Find the general value of log* $(-i)$.

Solution. $\quad Log\ (-i) = 2in\pi + \log [0 + i(-1)] \qquad \begin{cases} \text{Put } 0 = r\cos\theta,\ -1 = r\sin\theta 2 \\ \text{so that } r = 1 \text{ and } \theta = -\pi/2 \end{cases}$

$$= 2in\pi + \log [r (\cos \theta + i \sin \theta)] = 2in\pi + \log (re^{i\theta})$$

$$= 2in\pi + \log r + i\theta = 2in\pi + \log 1 + i\ (-\pi/2) = i\left(2n - \frac{1}{2}\right)\pi.$$

Example 19.39. *Prove that* (*i*) $i^i = e^{-(4n+1)\pi/2}$ *and* $Log\ i^i = -\left(2n + \frac{1}{2}\right)\pi$.

(*ii*) $(\sqrt{i})^{\sqrt{i}} = e^{-\alpha}$ *cis* α *where* $\alpha = \pi/4\sqrt{2}$. *(Mumbai, 2008)*

Solution. (*i*) By definition, we have

$$i^i = e^{i\,Log\,i} = e^{i\,(2in\pi + \log i)} = e^{-2n\pi + i\log[\exp(i\pi/2)]} \qquad [\because\ i = \text{cis}\ \pi/2 = \exp(i\pi/2)]$$

$$= e^{-2n\pi + i(i\pi/2)} = e^{-(2n + 1/2)\pi}$$

Taking logarithms, we get (*ii*)

(*ii*) $\qquad (\sqrt{i})^{\sqrt{i}} = e^{\sqrt{i}\log\sqrt{i}}$

Now $\qquad \sqrt{i}\ \log\sqrt{i} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\pi/2\right)^{1/2} \log\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$$= \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\log(e^{i\pi/2}) = \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\frac{i\pi}{2}$$

$$= \frac{i\pi}{4}\left(\frac{i}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -\frac{\pi}{4\sqrt{2}} + i\frac{\pi}{4\sqrt{2}}$$

Hence $\quad (\sqrt{i})^{\sqrt{i}} = e^{-\alpha + i\alpha}$ where $\alpha = \pi/4\sqrt{2}$

$$= e^{-\alpha} \cdot e^{i\alpha} = e^{-\alpha}(\cos\alpha + i\sin\alpha).$$

Example 19.40. *If $(a + ib)^p = m^{x+iy}$, prove that one of the values of y/x is $2\tan^{-1}(b/a) \div \log(a^2 + b^2)$.*

Solution. Taking logarithms, $(a + ib)^p = m^{x+iy}$ gives $p\log(a + ib) = (x + iy)\log m$

or
$$p\left(\frac{1}{2}\log(a^2 + b^2) + i\tan^{-1}\frac{b}{a}\right) = x\log m + iy\log m$$

Equating real and imaginary parts from both sides, we get

$$\frac{p}{2}\log(a^2 + b^2) = x\log m \quad ...(i), \qquad p\tan^{-1}\frac{b}{a} = y\log m \quad ...(ii)$$

Division of (ii) by (i) gives

$$y/x = 2\tan^{-1}(b/a)/\log(a^2 + b^2).$$

Example 19.41. *If $i^{i^{\cdots\infty}} = A + iB$, prove that $\tan \pi A/2 = B/A$ and $A^2 + B^2 = e^{-\pi/B}$.* (S.V.T.U., 2006 S)

Solution. $\quad i^{i^{\cdots\infty}} = A + iB \quad i.e. \quad i^{A+iB} = A + iB$

or
$$A + iB = e^{(A+iB)\log i} = e^{(A+iB)\log(\cos\pi/2 + i\sin\pi/2)}$$
$$= \exp[(A + iB)\log(e^{i\pi/2})] = e^{(A+iB)(i\pi/2)}$$
$$= e^{-B\pi/2} \cdot e^{i\pi A/2} = e^{-B\pi/2}\left(\cos\frac{\pi A}{2} + i\sin\frac{\pi A}{2}\right)$$

Equating real and imaginary parts, we get

$$A = e^{-B\pi/2}\cos\frac{\pi A}{2} \quad ...(i) \qquad B = e^{-B\pi/2}\sin\frac{\pi A}{2} \quad ...(ii)$$

Division of (ii) by (i) gives $B/A = \tan\pi A/2$

Squaring and adding (i) and (ii), $A^2 + B^2 = e^{-B\pi}$.

Example 19.42. *Prove that $\log\left(\frac{a+ib}{a-ib}\right) = 2i\tan^{-1}\left(\frac{b}{a}\right)$. Hence evaluate $\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right]$.* (P.T.U., 2006)

Solution. Putting $a = r\cos\theta$, $b = r\sin\theta$ so that $\theta = \tan^{-1} b/a$, we have

$$\log\left(\frac{a+ib}{a-ib}\right) = \log\frac{r(\cos\theta + i\sin\theta)}{r(\cos\theta - i\sin\theta)} = \log(e^{i\theta} \div e^{-i\theta})$$
$$= \log e^{2i\theta} = 2i\theta = 2i\tan^{-1} b/a.$$

Thus
$$\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right] = \cos[i(2i\theta)] = \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-(b/a)^2}{1+(b/a)^2} = \frac{a^2-b^2}{a^2+b^2}.$$

Example 19.43. *Separate into real and imaginary parts $\log\sin(x + iy)$.*

Solution.
$$\log\sin(x + iy) = \log(\sin x\cos iy + \cos x\sin iy)$$
$$= \log(\sin x\cosh y + i\cos x\sinh y) = \log r(\cos\theta + i\sin\theta),$$

where
$$r\cos\theta = \sin x\cosh y \text{ and } r\sin\theta = \cos x\sinh y,$$

so that
$$r = \sqrt{(\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y)}$$
$$= \sqrt{\frac{1-\cos 2x}{2}\cdot\frac{1+\cosh 2y}{2} + \frac{1+\cos 2x}{2}\cdot\frac{\cosh 2y - 1}{2}} = \sqrt{\left[\frac{1}{2}(\cosh 2y - \cos 2x)\right]}$$

and
$$\theta = \tan^{-1}(\cot x\tanh y).$$

Thus $\log\sin(x + iy) = \log(re^{i\theta}) = \log r + i\theta$

$$= \frac{1}{2}\log\left[\frac{1}{2}(\cosh 2y - \cos 2x)\right] + i\tan^{-1}(\cot x\tanh y).$$

Example 19.44. *Find all the roots of the equation*
(*i*) *sin z = cosh 4* (*ii*) *sinh z = i.*

Solution. (*i*) $\sin z = \cosh 4 = \cos 4i = \sin(\pi/2 - 4i)$

$\therefore$ $z = n\pi + (-1)^n (\pi/2 - 4i)$ $\left\{\because \text{ If } \sin\theta = \sin\alpha \text{ then } \theta = n\pi + (-1)^n \alpha\right.$

(*ii*) $i = \sinh z = \dfrac{e^z - e^{-z}}{2}$

or $e^{2z} - 2ie^z - 1 = 0$, *i.e.* $(e^z - i)^2 = 0$ *i.e.*, $e^z = i$

or $z = Log\ i = 2in\pi + \log i = 2in\pi + \log e^{i\pi/2} = 2in\pi + i\pi/2 = i\left(2n + \dfrac{1}{2}\right)\pi.$

PROBLEMS 19.7

1. Find the general value of
(*i*) $\log(6 + 8i)$ (*Rohtak, 2006*) (*ii*) $\log(-1)$. (*J.N.T.U., 2003*)
2. Show that (*i*) $\log(1 + i\tan\alpha) = \log(\sec\alpha) + i\alpha$, where α is an acute angle.
(*ii*) $\text{Log}_e \dfrac{3-i}{3+i} = 2i\left(n\pi - \tan^{-1}\dfrac{1}{3}\right)$.
3. If $(a_1 + ib_1)(a_2 + ib_2)......(a_n + ib_n) = A + iB$, prove that
(*i*) $(a_1^2 + b_1^2)(a_2^2 + b_2^2)......(a_n^2 + b_n^2) = A^2 + B^2$
(*ii*) $\tan^{-1}\dfrac{b_1}{a_1} + \tan^{-1}\dfrac{b_2}{a_2} + ... + \tan^{-1}\dfrac{b_n}{a_n} = \tan^{-1}\dfrac{B}{A}$.
4. Find the modulus and argument of (*i*) $(1 - i)^{1+i}$. (*P.T.U., 2010*) (*ii*) $i^{\log(1+i)}$
5. If $i^{\alpha + i\beta} = \alpha + i\beta$, prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$. (*Kurukshetra, 2005*)
6. Prove that $\log\left\{\dfrac{\sin(x + iy)}{\sin(x - iy)}\right\} = 2i\tan^{-1}(\cot x \tanh y)$. (*Mumbai, 2007*)
7. Prove that $\tan\left[i\log\left(\dfrac{a - ib}{a + ib}\right)\right] = \dfrac{2ab}{a^2 - b^2}$.
8. If $\tan\log(x + iy) = a + ib$ where $a^2 + b^2 \neq 1$, show that $\tan\log(x^2 + y^2) = \dfrac{2a}{1 - a^2 - b^2}$.
9. If $\sin^{-1}(x + iy) = \log(A + iB)$, show that $\dfrac{x^2}{\sin^2 u} - \dfrac{y^2}{\cos^2 u} = 1$, where $A^2 + B^2 = e^{2u}$.
10. Separate into real and imaginary parts $\log\cos(x + iy)$.
11. Find all the roots of the equation, (*i*) $\cos z = 2$, (*ii*) $\tanh z + 2 = 0$.

19.14 SUMMATION OF SERIES – 'C + iS' METHOD

This is the most general method and is applied to find the sum of a series of the form

$$a_0 \sin\alpha + a_1 \sin(\alpha + \beta) + a_2 \sin(\alpha + 2\beta) + ...$$

or

$$a_0 \cos\alpha + a_1 \cos(\alpha + \beta) + a_2 \cos(\alpha + 2\beta) + ...$$

Procedure. (*i*) Put the given series = S (or C) according as it is a series of sines (or cosines).
Then write C (or S) = a similar series of cosines (or sines).

e.g., If $S = a_0 \sin\alpha + a_1 \sin(\alpha + \beta) + a_2 \sin(\alpha + 2\beta) + ...$

then $C = a_0 \cos\alpha + a_1 \cos(\alpha + \beta) + a_2 \cos(\alpha + 2\beta)...$

(*ii*) Multiply the series of sines by i and add to the series of cosines, so that

$$C + iS = a_0[\cos\alpha + i\sin\alpha] + a_1[\cos(\alpha + \beta) + i\sin(\alpha + \beta)] + ...$$
$$= a_0 e^{i\alpha} + a_1 e^{i(\alpha + \beta)} + a_2 e^{i(\alpha + 2\beta)} + ...$$

(*iii*) Sum up this last series using any of the following standard series :

(1) Exponential series *i.e.*, $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty = e^x$

(2) Sine, cosine, sinh or cosh series

i.e., $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \infty = \sin x$, $\quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \infty = \cos x$

$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \infty = \sinh x$, $\quad 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \infty = \cosh x$

(3) Logarithmic series

i.e., $x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \infty = \log(1+x)$, $\quad -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \infty\right) = \log(1-x)$

(4) Gregory's series

i.e., $x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \infty = \tan^{-1} x$, $\quad x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \infty = \tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}$

(5) Binomial series

i.e., $1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \dots \infty = (1+x)^n$

$1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \dots \infty = (1+x)^{-n}$

$1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \dots \infty = (1-x)^{-n}$

(6) Geometric series

i.e., $a + ar + ar^2 + \dots$ to n terms $= a\,\frac{1-r^n}{1-r}$, $a + ar + ar^2 + \dots \infty = \frac{a}{1-r}, |r| < 1.$

(*iv*) Finally express the sum thus obtained in the form $A + iB$ so that by equating the real and imaginary parts, we get $C = A$ and $S = B$.

Series depending on exponential series

Example 19.45. *Sum the series* $\sin\alpha + x\sin(\alpha+\beta) + \frac{x^2}{2!}\sin(\alpha+2\beta) + \dots \infty$.

Solution. Let $S = \sin\alpha + x\sin(\alpha+\beta) + \frac{x^2}{2!}\sin(\alpha+2\beta) + \dots \infty$

and $C = \cos\alpha + x\cos(\alpha+\beta) + \frac{x^2}{2!}\cos(\alpha+2\beta) + \dots \infty$

$$\therefore \quad C + iS = [\cos\alpha + i\sin\alpha] + x[\cos(\alpha+\beta) + i\sin(\alpha+\beta)] + \frac{x^2}{2!}[\cos(\alpha+2\beta) + i\sin(\alpha+2\beta)] + \dots\dots \infty$$

$$= e^{i\alpha} + xe^{i(\alpha+\beta)} + \frac{x^2}{2!}\cdot e^{i(\alpha+2\beta)} + \dots \infty = e^{i\alpha}\left[1 + \frac{xe^{i\beta}}{1!} + \frac{x^2e^{2i\beta}}{2!} + \dots \infty\right]$$

$$= e^{i\alpha}\cdot e^{xe^{i\beta}} = e^{i\alpha} e^{x(\cos\beta + i\sin\beta)} = e^{x\cos\beta + i(\alpha + x\sin\beta)} = e^{x\cos\beta} e^{i(\alpha + x\sin\beta)}$$

$$= e^{x\cos\beta}[\cos(\alpha + x\sin\beta) + i\sin(\alpha + x\sin\beta)]$$

Equating imaginary parts from both sides, we have $S = e^{x\cos\beta}\sin(\alpha + x\sin\beta)$.

Series depending on logarithmic series

Example 19.46. *Sum the series*

$$\sin^2\theta - \frac{1}{2}\sin 2\theta \sin^2\theta + \frac{1}{3}\sin 3\theta \sin^3\theta - \frac{1}{4}\sin 4\theta \sin^4\theta + \dots \infty.$$

(*P.T.U., 2010 ; V.T.U., 2006 S*)

Solution. Let $S = \sin\theta \,.\, \sin\theta - \frac{1}{2}\sin 2\theta \,.\, \sin^2\theta + \frac{1}{3}\sin 3\theta \,.\, \sin^3\theta - \ldots \infty$

and $C = \cos\theta \,.\, \sin\theta - \frac{1}{2}\cos 2\theta \,.\, \sin^2\theta + \frac{1}{3}\cos 3\theta \,.\, \sin^3\theta - \ldots \infty$

$\therefore$ $C + iS = e^{i\theta}\sin\theta - \frac{e^{2i\theta}\sin^2\theta}{2} + \frac{e^{3i\theta}\sin^3\theta}{3} - \ldots \infty$

$= \log(1 + e^{i\theta}\sin\theta) = \log[1 + (\cos\theta + i\sin\theta)\sin\theta]$

$= \log[1 + \cos\theta\sin\theta + i\sin^2\theta]$ [Put $1 + \cos\theta\sin\theta = r\cos\alpha$; $\sin^2\theta = r\sin\alpha$] ...(*i*)

$= \log r(\cos\alpha + i\sin\alpha) = \log re^{i\alpha} = \log r + i\alpha$

Equating imaginary parts, we have $S = \alpha = \tan^{-1}\left(\frac{\sin^2\theta}{1+\cos\theta\sin\theta}\right)$. [from (*i*)]

Series depending on binomial series

Example 19.47. *Find the sum to infinity of the series*

$$1 - \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta - \frac{1.3.5}{2.4.6}\cos 3\theta + \ldots (-\pi < \theta < \pi).$$ *(S.V.T.U., 2009)*

Solution. Let $C = 1 - \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta - \frac{1.3.5}{2.4.6}\cos 3\theta + \ldots \infty$

and $S = 0 - \frac{1}{2}\sin\theta + \frac{1.3}{2.4}\sin 2\theta - \frac{1.3.5}{2.4.6}\sin 3\theta + \ldots \infty$

$\therefore$ $C + iS = 1 - \frac{1}{2}e^{i\theta} + \frac{1.3}{2.4}e^{2i\theta} - \frac{1.3.5}{2.4.6}e^{3i\theta} - \ldots$

$$= 1 + \left(-\frac{1}{2}\right)e^{i\theta} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{1.2}e^{2i\theta} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{1.2.3}e^{3i\theta} + \ldots$$

$$= (1 + e^{i\theta})^{-1/2} = (1 + \cos\theta + i\sin\theta)^{-1/2} = \left(2\cos^2\frac{\theta}{2} + i\,.\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-1/2}$$

$$= \left(2\cos\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1/2} = \left(2\cos\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{4} - i\sin\frac{\theta}{4}\right).$$

Equating real parts, we have $C = (2\cos\theta/2)^{-1/2}\cos\theta/4$.

PROBLEMS 19.8

Sum the following series :

1. $\cos\theta + \sin\theta\cos 2\theta + \frac{\sin^2\theta}{1.2}\cos 3\theta + \ldots \infty$. *(P.T.U., 2005)*
2. $\sin\alpha - \frac{\sin(\alpha+2\beta)}{2!} + \frac{\sin(\alpha+4\beta)}{4!} - \ldots \infty$.
3. $x\sin\theta - \frac{1}{2}x^2\sin 2\theta + \frac{1}{3}x^3\sin 3\theta - \ldots \infty$. *(Kurukshetra, 2005)*
4. $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta \ldots \infty$. *(S.V.T.U., 2006)*
5. $e^{\alpha}\cos\beta - \frac{e^{3\alpha}}{3}\cos 3\beta + \frac{e^{5\alpha}}{5}\cos 5\beta - \ldots \infty$.
6. $c\sin\alpha + \frac{c^3}{3}\sin 3\alpha + \frac{c^5}{5}\sin 5\alpha + \ldots \infty$.
7. $1 - \frac{1}{2}\cos 2\theta + \frac{1.3}{2.4}\cos 4\theta - \frac{1.3.5}{2.4.6}\cos 6\theta + \ldots \infty$. *(Kurukshetra, 2006)*
8. $n\sin\alpha + \frac{n(n+1)}{1.2}\sin 2\alpha + \frac{n(n+1)(n+2)}{1.2.3}\sin 3\alpha + \ldots \infty$.
9. $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots \sin(\alpha + \overline{n-1}\,\beta)$ *(P.T.U., 2009 S)*
10. $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots$ to n terms. *(Kurukshetra, 2006)*
11. $\sin\alpha\cos\alpha + \sin^2\alpha\cos 2\alpha + \sin^3\alpha\cos 3\alpha + \ldots\infty$.
12. $1 + x\cos\theta + x^2\cos 2\theta + \ldots + x^{n-1}\cos(n-1)\theta$.

19.15 APPROXIMATIONS AND LIMITS

Example 19.48. *If* $\frac{\sin\theta}{\theta} = \frac{599}{600}$, *find an approximate value of* θ *in radians.*

Solution. Since $\quad \frac{\sin\theta}{\theta} = 1 - \frac{1}{600}$ which is nearly equal to 1. $\quad \therefore \quad \theta$ must be very small.

We know that $\quad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots$

$$\therefore \quad \frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} + \frac{\theta^4}{5!}$$

Omitting θ^4 and higher powers, we have

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} = 1 - \frac{1}{600} \quad \text{or} \quad \theta^2 = \frac{1}{100}. \text{ Hence } \theta = 0.1 \text{ radians.}$$

Example 19.49. *Solve approximately* $\sin\left(\frac{\pi}{6} + \theta\right) = 0.51.$

Solution. Since 0.51 is nearly equal to 1/2, which is the value of $\sin \pi/6$, so θ must be very small.

$$\therefore \quad \sin\left(\frac{\pi}{6} + \theta\right) = \sin\frac{\pi}{6}\cos\theta + \cos\frac{\pi}{6}\sin\theta = \frac{1}{2}\left(1 - \frac{\theta^2}{2!} + \ldots\right) + \frac{\sqrt{3}}{2}\left(\theta - \frac{\theta^3}{3!} + \ldots\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\theta, \text{ omitting } \theta^2 \text{ and higher powers of } \theta.$$

Hence the given equation becomes,

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\theta = 0.51 \quad \text{or} \quad \theta = \frac{1}{50\sqrt{3}}$$

or

$$\theta = \frac{1}{50\sqrt{3}} \text{ radian} = \frac{\sqrt{3}}{150} \times 57.29 \text{ degrees nearly} = 39.7'.$$

PROBLEMS 19.9

1. Given $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$, show that θ is 1°58′ nearly.
2. If $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$, find an approximate value of θ in radians. *(Madras, 2003)*
3. If $\cos\theta = \frac{1681}{1682}$, find θ approximately.
4. Solve approximately the equation $\cos\left(\frac{\pi}{3} + \theta\right) = 0.49$.

19.16 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 19.10

Choose the correct answer or fill up the blanks in each of the following problems :

1. If $x + iy = \sqrt{2} + 3i$, then $x^2 + y$ is
   (*a*) 7 (*b*) 5 (*c*) 13 (*d*) $\sqrt{2} + 3$
2. The real part of $(\sin x + i\cos x)^5$ is
   (*a*) $-\cos 5x$ (*b*) $-\sin 5x$ (*c*) $\sin 5x$ (*d*) $\cos 5x$.

3. The number $(i)^i$ is
   (a) a purely imaginary number (b) an irrational number
   (c) a rational number (d) an integer.
4. The relation $|3-z|+|3+z|=5$ represents
   (a) a circle (b) a parabola (c) an ellipse (d) a hyperbola.
5. z is a complex number with $|z|=1$ and $\arg(z)=3\pi/4$. The value of z is
   (a) $(1+i)/\sqrt{2}$ (b) $(-1+i)/\sqrt{2}$ (c) $(1-i)/\sqrt{2}$ (d) $(-1-i)/\sqrt{2}$.
6. If $f(z)=e^{2z}$, then the imaginary part of $f(z)$ is
   (a) $e^y \sin x$ (b) $e^x \cos y$ (c) $e^{2x} \cos 2y$ (d) $e^{2x} \sin 2y$.
7. Expansion of $\sin^m \theta \cos^n \theta$ is a series of sines of multiples of θ when m is ......
8. Expansion of $\cos 6\theta$ in terms of $\cos\theta$ is ......
9. If $f(z)=3\bar{z}$, then the value of $f(z)$ at $z=2+4i$ is ......
10. If $x=\cos\theta+i\sin\theta$, then $x^n-1/x^n=$ ......
11. Imaginary part of $(2+i3)/(3-i4)$ is ......
12. Real part of $\cosh(x+iy)$ is ......
13. If $\dfrac{\sin\theta}{\theta}=\dfrac{2165}{2166}$, then $\theta=$ ...... approximately.
14. If $\tan x/2=\tanh y/2$, then $\cos x \cosh y=$ ......
15. Imaginary part of $\sin\bar{z}$ is ......
16. Modulus of $(\sqrt{i})^{\sqrt{i}}=$ ....
17. If $\sin\alpha+\sin\beta+\sin\gamma=0=\cos\alpha+\cos\beta+\cos\gamma$, then $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos$ (.............)
18. $\log(-1)=$ ....
19. $(i)^i$ is purely real or imaginary
20. If $\sin\theta=\tanh\phi$, then $\tan\theta=$ ....
21. Imaginary part of $\tan(\theta+i\phi)=$ ....
22. $\cos 5\alpha=$ (.........) $\cos^5\alpha+$ (.....) $\cos^3\alpha+$ (.....) $\cos\alpha$.
23. Cube roots of unity form .... triangle.
24. If $|z_1+z_2|=|z_1-z_2|$ then $\text{amp}(z_1)-\text{amp}(z_2)$ is ....
25. If $-3+ix^2y$ and x^2+y+4i represent conjugate complex number then $x=$ .... and $y=$ ....
26. If $\left|\dfrac{z-a}{z-b}\right|=k\ (\neq 1)$, then the locus of z is ....
27. $(-i)^{-i}$ is purely real. (True or False)
28. The statements $\text{Re}\, z>0$ and $|z-1|<|z+1|$ are equivalent. (*Mumbai, 2007*) (True or False)
29. Hyperbolic functions are periodic. (True or False)
30. nth roots of unity form a G.P. (True or False)
31. $\sin ix=-i\sinh x$. (*Mumbai, 2008*) (True or False)
32. If the sum and product of two complex numbers are real, then the two numbers must be either real or conjugate. (*Mumbai, 2008*) (True or False)
33. The modulus of the sum of two complex numbers $\geq$ to the sum of their moduli. (True or False)