CHAPTER 37

Discrete Mathematics

I. SET THEORY

37.1 SETS

(1) The concept and the language of sets play a very important role in expressing mathematical ideas concisely and precisely. It was Cantor* who first introduced and developed the notion of sets in mathematical investigations. It is, therefore, essential for a student of engineering to grasp the basic ideas of *Set Theory*.

Def. *A collection of objects defined by some property, is called a* **set**. *The objects belonging to a set are called its* **elements** *or* **members**.

Examples of a set are (*i*) the set of positive integers less than 25, (*ii*) set of pages in a book, and (*iii*) set of women students in a college.

A set is denoted by a single capital letter *e.g.* $A, B, ..., S, X, Y$ and the elements of a set are generally denoted by small letters $a, b, c, ..., x, y, z$.

When e is an element of a set S, we write $e \in S$ and read as 'e belongs to S'. When e is not an element of S, we write $e \notin S$.

If S be a set of odd integers, $3, 7, 11 \in S$ but $4, 6 \notin S$.

(2) Representation of a set

(*i*) *Tabular form of a set.* In this, the elements are enclosed in curly brackets after separating them by commas, *e.g.*, the set of positive even integers less than 9 is written as $S = \{2, 4, 6, 8\}$ and the set of prime numbers between 4 and 14 is $T = \{5, 7, 11, 13\}$.

(*ii*) *Symbolic form of a set.* In this, the set is written as $\{x/P(x)\}$ where x is a typical element of the set and $P(x)$ is the property satisfied by this element. In symbolic form, the above two sets are

$$S = \{x/x = a \text{ positive even number} < 9\}$$
$$T = \{x/x = a \text{ prime number between 4 and 14}\}.$$

*The great German mathematician George Cantor (1845–1918), the creator of Set theory.

(3) Empty set or null set. *A set which has no elements is called an* **empty set** *or the* **null set** *and is denoted by the symbol* ϕ.

(4) Finite *and* **infinite sets.** *A set is said to be* **finite** *if it has a finite number of elements. Otherwise a set is said to be* **infinite.**

The number of distinct elements in a finite set A is called its **cardinality** *and is denoted by* $|A|$.

For instance, the set of days in a year is *finite*, the set of points in a line is an *infinite set.*

(5) Subset. *If every element of a set A is also an element of set B, then A is called a* **subset** *of B* and this relationship is denoted by $A \subset B$ or $B \supset A$; which is read as 'A is contained in B'.

Another definition : If A and B are two sets such that

$$x \in A \quad \Rightarrow \quad x \in B,$$

then A is called a subset of B.

The notation $\Rightarrow$ stands for the word '*implies*'.

For instance, the set V of vowels is a subset of the set A of the English alphabet and we write $V \subset A$.

(6) Power set. *For a set A, collection of all subsets of A is called the power set of A and is denoted by P(A).*

If $A = [1, 2, 3]$ then $P(A)$ consists of 2^3 *i.e.* 8 elements ϕ, [1], [2], [3], [1, 2], [2, 3], [3, 1] and [1, 2, 3].

In general, if A has n elements, then $P(A)$ has 2^n elements.

(7) Equality of sets. *Two sets A and B are said to be equal if the elements of both are the same i.e.,* if each element of A is also an element of B and *vice versa*, and we write $A = B$.

In other words, *if A and B are two sets such that*

$$A \subset B \quad \text{and} \quad B \subset A \Leftrightarrow A = B.$$

Here $\Leftrightarrow$ stands for '*implies and is implied by*' or '*if and only if*'.

For instance, $\{2, 3, 5\} = \{3, 2, 5, 3\} = \{2, 5, 3, 2\}$, since the change in the order of elements or the repetition of an element is immaterial and all these contain the same elements 2, 3, 5.

(8) Proper and improper subsets. *When the set B contains all the elements of A and some others, A is said to be a* **proper subset** *of B* and is denoted by $A \subseteq B$.

i.e., if $A \subset B$ and $A \neq B$ then $A \subseteq B$.

If $A \subset B$ and every element of B is also an element of A i.e., $B \subset A$, then A is said to be an **improper subset** *of B i.e., $A = B$.*

For instance, the set of positive odd integers and the set of positive even integers are both proper subsets of the set of natural numbers.

(9) Universal set *is that which has all the sets under investigation as its subsets. It is generally denoted by 'U'.*

For instance the set of all letters of English alphabet is a universal set of the sets of the form $\{a, i, e, u\}$, $\{b, x, u, m\}$ etc.

Example 37.1. *If A, B, C are sets such that $A \subseteq B$ and $B \subseteq C$, then show that $A \subseteq C$.*

Solution. Let x be any element of A.

Since $A \subseteq B$ *i.e.*, all the elements of A belong to B,

so $$x \in A \quad \Rightarrow \quad x \in B \qquad \text{...}(i)$$

Again as $B \subseteq C$ *i.e.*, all elements of B belong to C,

so $$x \in B \quad \Rightarrow \quad x \in C \qquad \text{...}(ii)$$

$\therefore$ It follows from (*i*) and (*ii*) that $x \in A \;\Rightarrow\; x \in C$

i.e., $$A \subseteq C.$$

Example 37.2. *Which of the following sets are equal ?*

$$S_1 = \{1, 2, 2, 3\},\ S_2 = \{x : x^2 - 2x + 1 = 0\}$$
$$S_3 = \{3, 2, 1\} \text{ and } S_4 = \{x : x^3 - 6x^2 + 11x - 6 = 0\}.$$

Solution. Here $S_1 = \{1, 2, 2, 3\} = \{1, 2, 3\}$

$$S_2 = \{x : (x-1)^2 = 0\} = \{1\},\ S_3 = \{1, 2, 3\}$$
$$S_4 = \{x : (x-1)(x-2)(x-3) = 0\} = \{1, 2, 3\}$$

From these we find that S_1, S_3, S_4 are equal.

37.2 SET OPERATIONS

(1) Union *of two sets A and B is the set of all elements which belong to A or to B or to both. It is denoted by* $A \cup B$ read as 'A union B' and is represented by the shaded portion in Fig. 37.1.

Symbolically $A \cup B = \{x/x \in A \text{ or } x \in B\}$.

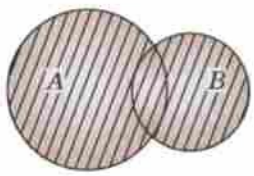


Fig. 37.1

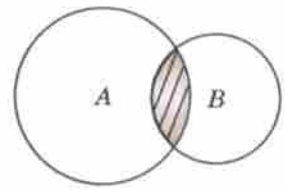


Fig. 37.2

(2) Intersection *of two sets A and B is the set of elements which are common to both A and B.* It is denoted by $A \cap B$ read as 'A intersection B' and is represented by the shaded portion in Fig. 37.2.

Symbolically $\qquad A \cap B = \{x/x \in A \text{ and } x \in B\}$

Such diagrams as Figs. 37.1 and 37.2 which exhibit the various relations between the sets are known as **Venn diagrams**.

(3) Disjoint sets. *If the sets A and B have no common elements, they are called* **disjoint sets**. Their intersection is an empty set.

For instance, if A be a set of boys in a college and B the set of girls in the same college, then A and B are disjoint sets *i.e.* $A \cap B = \phi$.

(4) Complement of a set. *If* $B \subset A$, *the set of elements of A which are not in B is called the* **complement of B in A** *and is denoted by* B^c *in A.* It is also known as the difference $A - B$ of sets A and B. Thus

$$B^c \text{ in } A = \{x/x \in A \text{ and } x \notin B\}$$

which is shown shaded in Fig. 37.3 (*i*).

If U be a universal set, then the set '$U - A$' is called the complement of A and is denoted by A^c, which is shown shaded in Fig. 37.3 (*ii*).

For instance, if $U = \{1, 2, 3, 4, 5, \ldots\}$ and $A = (1, 3, 5, 7, \ldots)$, then $A^c = \{2, 4, 6, 8 \ldots\}$.

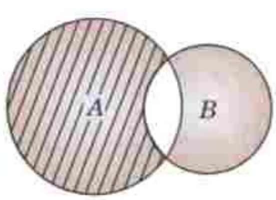


Fig. 37.3 (i)

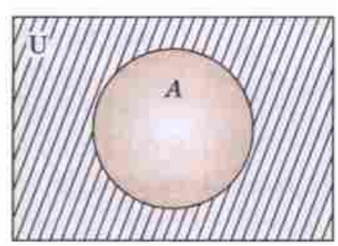


Fig. 37.3 (ii)

(5) Cartesian product *of two sets A and B denoted by* $A \times B$ *is defined to be set of all ordered pairs (a, b), where* $a \in A$ *and* $b \in B$ *i.e.,*

$$A \times B = \{(a, b) : a \in A \quad \text{and} \quad b \in B\}$$

For instance if $A = \{1, 2\}$, $B = \{1, 2, 3\}$, then $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$, $B \times A = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\}$ $\therefore$ $A \times B \neq B \times A$.

Example 37.3. *If* $A = \{2, 5, 6, 7\}$, $B = \{0, 2, 5, 7, 8\}$, $C = \{1, 2, 3, 5, 6\}$, *show that*
$$A \cup (B \cup C) = (A \cup B) \cap (A \cup C).$$

Solution. Here $\qquad B \cap C = \{2, 5\}$

$\therefore \qquad A \cup (B \cap C) = \{2, 5, 6, 7\} \qquad \ldots(i)$

Again $\qquad A \cup B = \{0, 2, 5, 6, 7, 8\}$,

$\qquad A \cup C = \{1, 2, 3, 5, 6, 7\}$

$\therefore \qquad (A \cup B) \cap (A \cup C) = [2, 5, 6, 7] \qquad \ldots(ii)$

Hence from (*i*) and (*ii*), we get

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Example 37.4. *With the help of Venn diagram, show that*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

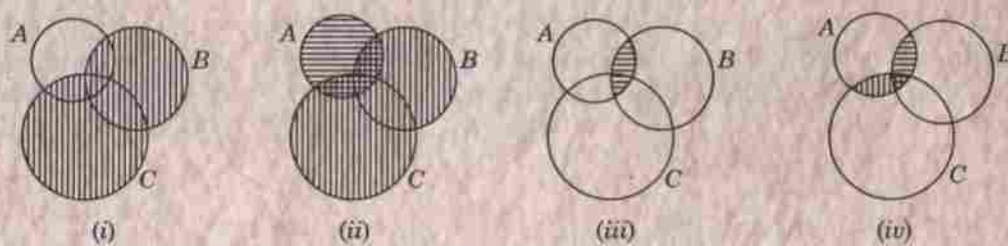


Fig. 37.4

Solution. First we draw vertical lines in the whole areas of B and C so as to represent $B \cup C$. [Fig. 37.4 (*i*)].

Now draw horizontal lines in the whole area A. Thus the double hatched area in [Fig. 37.4 (*ii*)] represents area common to A and $B \cup C$ *i.e.*, $A \cap (B \cup C)$.

Again we draw horizontal lines in the area common to A and B so as to represent $A \cap B$ [Fig. 37.4 (*iii*)].

Now draw vertical lines in the area common to A and C, so as to represent $A \cap C$. Then the whole hatched area in [Fig. 37.4 (*iv*)] represents $(A \cap B) \cup (A \cap C)$.

Hence we observe that the double hatched area in Fig. 37.4 (*ii*) is equal to the total hatched area in Fig. 37.4 (*iv*).

Example 37.5. *Prove that (i)* $A - (B \cap C) = (A - B) \cup (A - C)$.

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$. *(Tiruputi, 2001)*

Solution. (*i*) Let x be an arbitrary element of the set $A - (B \cap C)$, then

$$\begin{aligned} x \in A - (B \cap C) \;&\Rightarrow\; x \in A \text{ and } x \notin (B \cap C) \qquad [\because\; x \notin (A - B) \Rightarrow x \in A \text{ and } x \notin B] \\ &\Rightarrow\; x \in A \text{ and } (x \notin B \;\text{ or }\; x \notin C) \\ &\Rightarrow\; (x \in A \text{ and } x \notin B) \;\text{ or }\; (x \in A \text{ and } x \notin C) \\ &\Rightarrow\; x \in (A - B) \;\text{ or }\; x \in (A - C) \\ &\Rightarrow\; x \in (A - B) \cup (A - C). \end{aligned}$$

$$\therefore \qquad A - (B \cap C) \subset (A - B) \cup (A - C) \qquad \text{...}(i)$$

Again if x be an arbitrary element of the set $(A - B) \cup (A - C)$, then

$$\begin{aligned} x \in (A - B) \cup (A - C) \;&\Rightarrow\; x \in (A - B) \;\text{ or }\; x \in (A - C) \\ &\Rightarrow\; (x \in A \text{ and } x \notin B) \;\text{ or }\; (x \in A \text{ and } x \notin C) \\ &\Rightarrow\; x \in A \text{ and } (x \notin B \;\text{ or }\; x \notin C) \\ &\Rightarrow\; x \in A \text{ and } x \notin B \cap C \\ &\Rightarrow\; x \in A - (B \cap C) \end{aligned}$$

$$\therefore \qquad (A - B) \cup (A - C) \subset A - (B \cap C) \qquad \text{...}(ii)$$

From (*i*) and (*ii*), we get $A - (B \cap C) = (A - B) \cup (A - C)$.

(*ii*)

$$\begin{aligned} (x, y) \in A \times (B \cap C) \;&\Rightarrow\; x \in A \text{ and } y \in (B \cap C) \\ &\Rightarrow\; (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C) \\ &\Rightarrow\; (x, y) \in (A \times B) \text{ and } (x, y) \in (A \times C) \\ &\Rightarrow\; (x, y) \in (A \times B) \cap (A \times C) \end{aligned}$$

Hence $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

37.3 LAWS OF SET THEORY

1. *Commutative Law*

$$A \cup B = B \cup A \,;\, A \cap B = B \cap A.$$

2. *Associative Law*

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

3. *Distributive Law*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

4. *Complement Law*

$$A \cup A^c = U \,;\, A \cap A^c = \phi.$$

5. *Identity Law*

$$A \cup \phi = A = \phi \cup A$$
$$A \cap U = A = U \cap A.$$

6. *Absorption Law*

$$A \cup (A \cap B) = A \,;\, A \cap (A \cap B) = A$$

7. *De Morgan's Law*

$$(A \cup B)^c = A^c \cap B^c \,;\, (A \cap B)^c = A^c \cup B^c$$

8. *Involution Law*

$$(A^c)^c = A.$$

37.4 PRINCIPLE OF INCLUSION

(1) *If A and B be sets with cardinalities* $|A|$ *and* $|B|$*, then*

$$|A \cup B| = |A| + |B| - |A \cup B|$$

Proof. The number of common elements in A and B is $|A \cap B|$. Each of these elements is counted twice in $|A| + |B|$, once in $|A|$ and once in $|B|$. This should be adjusted by subtracting the term $|A \cap B|$ from $|A| + |B|$.

Hence $$|A \cup B| = |A| + |B| - |A \cap B|.$$

Obs. Using the distributive law, we can extend the above result for three sets A, B, C so that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

(V.T.U., 2002)

For
$$\begin{aligned} |A \cup B \cup C| &= |(A \cup B) \cup C| \\ &= |A \cup B| + |C| - |(A \cup B) \cap C| \\ &= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)| \\ &= |A| + |B| + |C| - |A \cap B| - [|A \cap C| + |B \cap C| - |A \cap B \cap C|] \end{aligned}$$

whence follows the result.

(2) Duality. *If S be any identity involving sets and operations* (*e.g.* complement, intersection $\cap$ and union $\cup$ etc.) *and a new set S^* is obtained by replacing $\cap$ by $\cup$, $\cup$ by $\cap$, ϕ by U and U by ϕ in S, then the statement S^* is true and is called the* **dual** *of the statement S.*

For instance, the dual of $A \cap (B \cup A) = A$ is $A \cup (B \cap A) = A$.

Example 37.6. *In a survey conducted on 250 persons, it was found that 180 drink tea and 70 drink coffee and 50 take both tea and coffee. How many drink atleast one bevarage and how many drink neither ?*

Solution. Let A be the set of tea drinkers and B the set of coffee drinkers. Then

$$|A \cup B| = |A| + |B| - |A \cap B| = 180 + 70 - 50 = 200$$

Hence 200 persons drink at least one beverage and 250 – 200 = 50 persons drink neither tea nor coffee.

Example 37.7. *How many integers between 1 and 468 are divisible by 3 but not by 5.*

Solution. Number of integers between 1 and 468 which are divisible by 3 $= \left[\frac{468}{3}\right] = 156$

Number of integers between 1 and 468 which are divisible by 3 and 5 $= \left[\frac{468}{3 \times 5}\right] = 31$

Hence the number of integers between 1 and 468 divisible by 3 but not by 5 = 156 – 31 = 125.

Example 37.8. *How many integers are between 1 and 200 which are divisible by any one of the integers 2, 3 and 5 ?*

Solution. Let A_1, A_2, A_3 denote the set of integers between 1 and 200 which are divisible by 2, 3, 5 respectively.

$$[A_1] = \left[\frac{200}{2}\right] = 100,\ [A_2] = \left[\frac{200}{3}\right] = 66,\ [A_3] = \left[\frac{200}{5}\right] = 40$$

$$[A_1 \cap A_2] = \left[\frac{200}{2 \times 3}\right] = 33,\ [A_1 \cap A_3] = \left[\frac{200}{2 \times 5}\right] = 20$$

$$[A_2 \cap A_3] = \left[\frac{200}{3 \times 5}\right] = 13,\ [A_1 \cap A_2 \cap A_3] = \left[\frac{200}{2 \times 3 \times 5}\right] = 6$$

Hence
$$[A_1 \cup A_2 \cup A_3] = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + [A_1 \cap A_2 \cap A_3]$$
$$= 100 + 66 + 40 - 33 - 13 - 20 + 6 = 146.$$

PROBLEMS 37.1

1. Show that the following sets are equal :
$A = \{2, 1\}$, $B = \{1, 2, 1, 2, 1, 2\}$, $C = \{x : x^2 - 3x + 2 = 0\}$.
2. Which of the following statements are true ? Give reason to support your answer.
(*i*) $\{a\} \subset \{a, b, c\}$ (*ii*) $a \subset \{a, b, c\}$ (*iii*) $a \subseteq \{a, b, c\}$
(*iv*) $\{a, b\} \subset \{a, b, c\}$ (*v*) $\{a, b\} \in \{a, b, c\}$ (*vi*) $\phi \subset \{a, b, c\}$.
3. Prove that
(*i*) $B - A$ is a subset of A^c. (*ii*) $B - A^c = B \cap A$ *(Andhra, 2004)*
(*iii*) $\{A \subset B, B \subset C, C \subset A\} \Rightarrow A = C$.
4. If $A = \{a, b, c, d, e\}$, $B = \{a, c, e, g\}$ and $C = \{b, e, f, g\}$, prove that
(*i*) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ (*ii*) $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$.
5. If $A = A \cup B$ then prove that $B = A \cap B$.
6. Prove that $A \cup B' = B \Leftrightarrow A \subset B$.
7. With the help of the Venn-diagram, prove that
(*i*) $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. *(Andhra, 2004)*
(*ii*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. *(V.T.U., 2001 S)*
8. If $B \subset A$, prove that
(*i*) $B \cup C \subset A \cup C$ (*ii*) $B \cap C \subset A \cap C$.
9. (*i*) If $A \cup B = A \cap B$, show that $A = B$.
(*ii*) If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, show that $B = C$.
10. (*i*) Prove that (*i*) $A - B = A - A \cap B$. *(V.T.U., 2001 ; Madras, 2000)*
(*ii*) $A - (B \cup C) = (A - B) \cap (A - C)$.
11. Show that for any two sets A and B
(*i*) $A - B = A \cap B^c$ (*ii*) $A \subseteq B \Leftrightarrow B^c \leq A^c$
(*iii*) $A \cup B = (A \cap \overline{B}) \cup (B \cap \overline{A}) \cup (A \cap B)$.
12. If A, B, C be sets such that $A \subset B$, $B \cap C = \phi$, show that $A \cap C = \phi$.
13. Show that $A \cup (B \cup C)^c = (A \cup B^c) \cap (A \cup C^c)$.

14. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$. *(Andhra, 2001)*
15. If S is any set and $P(s)$ is its power set and A and B belong to $P(s)$, prove that $B \cap (A - B) = \phi$.
16. If A and B are finite sets then prove that $A \cup B$ and $A \cap B$ are finite sets and
$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$ *(Andhra, 2004)*
17. In survey conducted on 200 people, it was found that 140 are smokers while 80 are alcoholic and 40 are both smokers and alcoholics. Find how many are neither smokers not alcoholics.
18. How many integers between 1 and 789 are divisible by 5 but not by 7.
19. How many integers are between 1 and 250 while are divisible by any of the integers 3, 5, and 7.
20. Out of a class of 153 students, 54 have taken History, 63 have taken Geography, 62 have taken Economics, and 43 have taken Geography and History, 45 have taken History and Economics, 46 have taken Geography and Economics and 37 have taken all the three subjects. How many of the students have not taken any of these three subjects ? Use a Venn diagram.

II. ALGEBRA OF LOGIC

37.5 INTRODUCTION

(1) Logic is concerned with all types of reasoning such as valid statements, mathematical proofs, valid conclusions etc. Logical reasoning is used to prove theorems, to verify the correctness of computer programs and to draw conclusions from experiments. Later on, we shall observe that the algebra of sets and logic is analogous to the algebra of switching circuits which is similar to 'Boolean Algebra'.

(2) Propositions and Statements. A *proposition* is a declarative sentence which is either true (1) or false (0). Some authors use T and F respectively for 1 and 0. The truth or falsity of a proposition is defined as its *truth value*.

All the declarative sentences to which it is possible to assign one and only one of the two possible truth values are called *statements*.

Example 37.9. *Which of the following are statements : (a) Agra is in India (b) $3 + 4 = 5$ (c) Where do you live ? (d) Do you speak Hindi ?*

Solution. (*a*) and (*b*) are statements that happen (*a*) is true and (*b*) is false.

(*c*) and (*d*) are questions so they are not statements.

(3) Compound statements. The statement which is composed of sub-statements and logical connectives is called a *compound statement.*

e.g., 'It is raining and it is cold' is a compound statement as it is comprised of two sub-statements 'It is raining' and 'it is cold'.

(4) Truth table. The truth value of a compound statement is completely determined by the truth value of its substatements. *A convenient way to represent a compound statement is by means of the* **truth table** *wherein the values of a compound statement are specified for all possible choices of the values of the sub statements.*

We shall use the numbers 0 and 1 to denote the false and true statements. Also we use letters $p, q, r, \ldots$ to represent a proposition or logical variable.

37.6 LOGICAL OPERATORS

(1) Conjunction. *If p and q are two statements then their conjunction p and q written as $p \wedge q$, is defined by the truth table 1.*

Table 1

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Table 2

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

For example, the conjunction of p : it is raining and q : I am cold is $p \wedge q$: It is raining and I am cold.

(2) Disconjunction. *If p and q are two statements, then their disconjunction p or q written as $p \vee q$ is defined by the truth table 2.*

For example, the disjunction of p and q for p : it is raining today ; q : 3 is an odd integer is

$p \vee q$: it is raining today or 3 is an odd integer.

(3) Negation. *If p is a given statement and its negative 'not p', written as ~ p (or Np or ⌉p) is defined by the following truth table :*

p	$\sim p$
0	1
1	0

For example, the negation of the following statement

(a) $p : 2 + 3 > 1$ is $\sim p : 2 + 3 \leq 1$

(b) q : it is hot is $\sim q$: it is cold.

Example 37.10. *If p be 'it is hot' and q be 'it is raining', describe each of the following statements by a sentence :*

(a) $q \vee \sim p$ (b) $\sim p \wedge \sim q$ (c) $\sim (\sim p \vee q)$.

Solution. (a) It is raining or it is not hot. (b) It is not hot and it is not raining.

(c) It is hot but not raining.

(4) Conditional operator. *The conditional statement 'if p then q' written as $p \to q$ is defined by the truth table 4 :*

Table 4

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Table 5

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Obs. The contrapositive of conditional statement $p \to q$ is the statement $\sim p \to \sim q$.

(5) Biconditional operator. If p and q be two statements, then the statement 'p if and only if q' denoted by $p \leftrightarrow q$ and abbreviated as 'p if q' is called a *biconditional statement*. The truth table for biconditional statement is table 5.

Example 37.11. *Construct the truth tables for*

(a) $p \wedge \sim q$ (b) $(p \vee q) \vee \sim p$

(c) $(p \to q) \wedge (q \to p)$ (d) $(p \to q) \vee \sim (p \leftrightarrow \sim q)$.

Solution. (a) The truth table is

p	q	$\sim q$	$p \wedge \sim q$
1	1	0	0
1	0	1	1
0	1	0	0
0	0	1	0

(b) The truth table is

p	q	$p \vee q$	$\sim p$	$(p \vee q) \vee \sim p$
1	1	1	0	1
1	0	1	0	1
0	1	1	1	1
0	0	0	1	1

(*c*) The truth table is

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
1	1	1	1	1
1	0	0	1	0
0	1	1	0	0
0	0	1	1	1

(*d*) The truth table in this case is

p	q	$p \to q$	$\sim q$	$p \to \sim q$	$(p \to \sim q)$	$p \to q \vee \sim (p \to \sim q)$
1	1	1	0	0	1	1
1	0	0	1	1	0	0
0	1	1	0	1	0	1
0	0	1	1	0	1	1

37.7 STATEMENTS GENERATED BY A SET

(1) *If S be a set of statements, then any valid combination of statements in S with conjunction, disconjunction or negation is a statement generated by S.*

A statement generated by a set S need not include each element of S in its expression.

For example, if p, q, r are statements in S then

(*a*) $(p \wedge q) \wedge r$ (*b*) $\sim q \wedge r$ (*c*) $(p \wedge q) \vee (\sim q \wedge r)$

are statements generated by S. Their truth tables are :

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$ (*a*)	$\sim q$	$\sim q \wedge r$ (*b*)	(*c*)
1	1	1	1	1	0	0	1
1	1	0	1	0	0	0	1
1	0	1	0	0	1	1	1
0	1	1	0	0	0	0	0
1	0	0	0	0	1	0	0
0	1	0	0	0	0	0	0
0	0	1	0	0	1	1	1
0	0	0	0	0	1	0	0

(2) Tautology *is an expression involving logical variables which is true for all cases in its truth table.* It is also called a *logical truth.*

(3) Contradiction is an expression involving logical variables which is false for all cases in its truth table. Obviously, the negation of a contradiction is a tautology.

In other words, a statement formula which is a tautology is identically true, while a formula which is a contradiction is identically false.

Obs. The conjunction of two tautologies is also a tautology.

Example 37.12. *Show that (a)* $p \vee \sim p$ *is a tautology (b)* $p \to q \leftrightarrow (\sim p \vee q)$.

Solution. (*a*) The truth table is

p	$\sim p$	$p \vee \sim p$
1	0	1
0	1	1

Hence $p \vee \sim p$ is a tautology.

(*b*) The truth table is

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
1	1	1	0	1	1
1	0	0	0	0	1
0	1	1	1	1	1
0	0	1	1	1	1

Hence $(p \to q) \leftrightarrow (\sim p \vee q)$ is true.

37.8 EQUIVALENCE

(1) *If p and q be statements generated by the set of statements S, then p and q are equivalent if $p \leftrightarrow q$ is a tautology which is denoted by $p \Leftrightarrow q$.*

If $p \to q$ is a tautology, then we say that p *implies* q and write it as $p \Rightarrow q$.

Obs. All tautologies are equivalent to each other and all contradictions are equivalent to each other.

(2) Equivalent formulae. Some basic equivalent formulae are given below which can be proved by using truth tables :

1. $p \vee p \Leftrightarrow p$	$p \wedge p \Leftrightarrow p$	*Idempotent laws*
2. $p \vee q \Leftrightarrow q \wedge p$	$p \wedge q \Leftrightarrow q \wedge p$	*Commutative laws*
3. $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	*Associative laws*
4. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	*Distributive laws*
5. $p \vee \sim p \Leftrightarrow 1$	$p \wedge \sim p \Leftrightarrow 0$	*Negation law*
6. $p \vee 0 \Leftrightarrow p$	$p \wedge 1 \Leftrightarrow p$	*Identity laws*
7. $p \vee 1 \Leftrightarrow 1$	$p \wedge 0 \Leftrightarrow 0$	*Null laws*
8. $p \vee (p \wedge q) \Leftrightarrow p$	$p \wedge (p \vee q) \Leftrightarrow p$	*Absorption laws*
9. $\sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$	$\sim (p \vee q) \Leftrightarrow \sim p \vee \sim q$	*De Morgan's laws*
10. $p \Rightarrow p \vee q$	$q \Rightarrow p \vee q$	*Disjunctive addition*
11. $p \wedge q \Rightarrow q$	$p \wedge q \Rightarrow q$	
12. $(p \vee q) \wedge \sim p \Rightarrow q$	$(p \vee q) \wedge \sim q \Rightarrow p$	
13. $(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$		*Chain rule*
14. $p \to q \Leftrightarrow \sim p \vee q$		*Conditional equivalence*
15. $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$		*Biconditional equivalence*

Example 37.13. *Show that*

(a) $p \to (q \to r) \Leftrightarrow p \to (\sim q \vee r) \Leftrightarrow (p \wedge q) \to r$

(b) $[\sim p \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$

Solution. (*a*) By conditional equivalence $q \to r \Leftrightarrow \sim q \vee r$

Replacing $q \to r$ by $\sim q \vee r$, we get $p \to (\sim q \vee r)$ which is equivalent to $\sim p \vee (\sim q \vee r)$ by the same rule.

Thus $\sim p \vee (\sim q \vee r) \Leftrightarrow (\sim p \vee \sim q) \vee r$ [By (3)]

$\Leftrightarrow \sim (p \wedge q) \vee r$ [By (9)]

$\Leftrightarrow (p \wedge q) \to r$ [By (14)]

(*b*) $[\sim p \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (p \wedge r)$

$\Leftrightarrow [\sim p \wedge (\sim q \wedge r)] \vee (q \vee p) \wedge r$ [By (4)]

$\Leftrightarrow [\sim p \wedge \sim q \wedge r] \vee (q \vee p) \wedge r$ [By (3)]

$\Leftrightarrow [\sim (p \vee q) \wedge r] \vee (q \vee p) \wedge r$ [By (9)]

$\Leftrightarrow [\sim (p \vee q) \vee (p \vee q)] \wedge r$ [By (4)]

$\Leftrightarrow 1 \wedge r$ [By (5)]

$\Leftrightarrow r$ [By (6)]

37.9 DUALITY LAW

(1) *Two formulae A and A* are said to be duals of each other if either one can be obtained from the other by replacing* $\wedge$ *by* $\vee$ *and* $\vee$ *by* $\wedge$.

If the formula A contains special variables 1 or 0, then its dual A* is obtained by replacing 1 by 0 and 0 by 1.

e.g., (*i*) Dual of $(p \vee q) \wedge r$ is $(p \wedge q) \vee r$

(*ii*) Dual of $(p \wedge q) \vee 0$ is $(p \vee q) \wedge 1$.

(2) Tautology implications. *A statement A is said to tautologically imply a statement B if and only if* $A \to B$ *is a tautology which is read as "A implies B".*

The implications listed below have important applications which can be proved by truth tables :

1. $p \wedge q \Rightarrow p$ $\qquad$ $p \Rightarrow p \vee q$
2. $\sim p \Rightarrow p \to q$ $\qquad$ $q \Rightarrow p \to q$
3. $\sim (p \to q) \Rightarrow p$ $\qquad$ $\sim (p \to q) \Rightarrow \sim q$
4. $p \wedge (p \to q) \Rightarrow q$ $\qquad$ $\sim p \wedge (p \vee q) \Rightarrow q$
5. $(p \to q) \wedge (q \to r) \Rightarrow p \to r$

37.10 ARGUMENTS

(1) *An argument is an assertion that a given set of propositions* $p_1, p_2, ..., p_n$ (called *premises*) *yields another proposition* q (called *conclusion*). The argument is symbolically written as "$p_1, p_2, ..., p_n \vdash q$".

An argument $p_1, p_2, ..., p_n \vdash q$ is true provided q is true whenever all the premises $p_1, p_2, ..., p_n$ are true. An argument which is true is said to be '*valid argument*'. Otherwise it is called a *fallacy*.

Example 37.14. *Show that*

(*a*) *the argument* $p \leftrightarrow q, q \vdash p$ *is valid.*

(*b*) *the argument* $p \to q, q \vdash p$ *is a fallacy.*

Solution. (*a*) Let us first prepare the truth table as follows :

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Since $p \leftrightarrow q$ is true in cases (rows) 1 and 4, and q is true in cases 1 and 3, therefore $p \leftrightarrow q$ and q both are true in case 1 only when p is also true. This shows that the given argument is valid.

(*b*) Let us first prepare the truth table below :

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

This table shows that $p \to q$ and q both are true in case 3 only while the conclusion p is false. Hence the given argument is a fallacy.

(2) Theorem. *The argument* $p_1, p_2, ..., p_n \vdash q$ *is valid if and only if the proposition* $(p_1 \wedge p_2 \wedge p_3 \wedge ..., \wedge p_n) \to q$ *is a tautology.*

The proportions $p_1, p_2, ..., p_n$ are simultaneously true if and only if the proposition $p_1 \wedge p_2 \wedge ..., \wedge p_n$ is true *i.e.*, if the proposition $(p_1 \wedge p_2 \wedge ..., \wedge p_n) \to q$ is a tautology.

Obs. The validity of an argument depends upon the particular form of the argument, not on the truth values of the statement appearing in the argument.

Example 37.15. *Test the validity of the following argument :*

S_1 *: If 5 is less than 3, then 6 is not a prime number*

S_2 *: 5 is not less than 6*

S_3 *: 5 is a prime number.*

Solution. Let '5 is less than 3' be p and '5 is a prime number' be q. Then the given argument is of the form $p \to \sim q, \sim p \vdash q$.

Since in the last line of the truth table, the premises $p \to \sim q$ and $\sim q$ are true but the conclusion q is false, therefore the argument is fallacy.

p	q	$\sim q$	$p \to \sim q$	$\sim p$
1	1	0	0	0
1	0	1	1	0
0	1	0	1	1
0	0	1	1	1

PROBLEMS 37.2

1. If $p \equiv$ Sam is a teacher, $q \equiv$ John is an honest boy, then translate the following into logical sentences :
   (*a*) $\sim (p \wedge q)$, (*b*) $p \vee \sim q$, (*c*) $\sim p \Leftrightarrow q$, (*d*) $p \Rightarrow \sim q$.
2. Change the following sentence into symbols :
   (*a*) 'If I do not have car or I do not wear good dress then I am not a millionaire'.
   (*b*) Everyone who is healthy can do all kinds of work. (*Anna, 2004 S*)
3. Prepare truth tables for the following statements (*a*) $(p \Rightarrow q) \wedge \sim q$, (*b*) $(p \Leftrightarrow q) \wedge (r \vee q)$.
4. Write down the truth table of
   (*a*) $p \vee q$ (*Madras, 1997*) (*b*) $p \wedge (p \wedge q)$ (*Madras, 2005 S*)
5. Verify that the following are tautologies :
   (*a*) $p \to [q \to (p \wedge q)]$ (*Anna, 2005*)
   (*b*) $(p \wedge q \Rightarrow r) \Leftrightarrow (p \Rightarrow r) \vee (q \Rightarrow r)$ (*c*) $(p \Rightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$.
6. Show that $Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$ is a tautology. (*Anna, 2004 S ; Madras, 2003 S*)
7. Over the universe of positive integers
   $p(n)$: n is prime and $n < 32$.
   $q(n)$: n is a power of 3.
   $r(n)$: n is a divisor of 27.
   (*i*) What are the truth sets of these propositions ?
   (*ii*) Which of the three propositions implies one of the others ?
8. Given the propositions over the natural numbers
   $p : n < 4$, $q : 2n > 17$ and $r : n$ is a divisor of 18, what are the truth sets of
   (*i*) q, (*ii*) $p \wedge q$,
   (*iii*) r, (*iv*) $q \to r$. (*Madras, 1999*)
9. Show that (*a*) $\sim Q, P \to Q \Rightarrow \sim P$. (*Madras, 2003*)
   (*b*) $(P \to R) \wedge (Q \to R) \Leftrightarrow (P \vee Q) \to R$. (*Madras, 2001*)
10. Construct the truth table for (*i*) $(\sim p \to q) \wedge (q \rightleftarrows p)$. (*Bharthian, MSc. 2001*)
    (*ii*) $\rceil [P \vee (Q \wedge R)] \rightleftarrows (P \vee Q) \wedge (P \vee R)$. (*Andhra, 2004*)
11. Prove that the following statement is a contradiction :
    $S = [(p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q) \wedge (\sim p \vee \sim q)]$.
12. If p, q, r are three statements then prove that
    (*a*) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (*b*) $(p \Rightarrow q) \vee r \equiv (p \vee r) \Rightarrow (q \vee r)$
    (*c*) $\sim (p \vee q) \equiv \sim p \wedge \sim q$.
13. Define conjunction, conditional, biconditional and negation, with examples.

14. Show that (*i*) $p \wedge q$ logically implies $p \leftrightarrow q$.
    (*ii*) $p \leftrightarrow \sim q$ does not logically imply $p \to q$. *(U.P.T.U., 2001)*
15. Write the duals of $(p \vee q) \wedge r$ and $(p \wedge q) \vee t$.
16. Show that $s \vee r$ is tautologically implied by $(p \vee q) \wedge (q \to r) \wedge (q \to s)$. *(Andhra, 2004 ; Bharathian, 2001)*
17. Let $P(n)$ be '$8^n - 3^n$ is a multiple of 5'. Prove that $P(n)$ is a tautology on n.
18. Prove that $P \to \rceil Q, R \to Q, R \vdash \rceil P$ is a valid argument. *(Madras, 2003)*
19. Prove that $p \to (q \vee r) \Leftrightarrow (p \wedge \rceil q) \to r$. *(Anna, 2005)*
20. Show that $R \vee S$ follows logically from the premises $C \vee D, C \vee D \to \sim H, H \to A \wedge \sim B, A \wedge \sim B \to R \vee S$.

37.11 PREDICATES

Statements involving variables such as '$x > 7$' and '$x = y + 7$' are neither true nor false so long as the values of the variable x, y are not specified. We now, discuss the ways that propositions can be evolved from such statements. The statement '$x > 7$' has two parts : First part—'the variable x' is the *subject* of the statement ; Second part–'is greater than 7' is the *predicate* which refers to the property that the subject of statement can have. If $P(x)$ denotes the statement '$x > 7$' then P is the *predicate* and x is the *variable*. The statement $P(x)$ is also known as the value of the propositional function P at x. The *predicates* are denoted by capital letters and the *objects* by the variables (denoted by small letters in brackets).

Thus *predicates are simple statements which turn out to be propositions involving variables whose values are not well specified.*

In other words, *predicate is a variable statement which becomes specific when particular values are assigned to the variables.*

There are statements which involves more than one variable consider the statement '$x = y + 7$' which is denoted by $Q(x, y)$ where Q is the predicate and x, y are the variables. When values are assigned to the variables x, y, the statement $Q(x, y)$ has the truth value.

Similarly $R(x, y, z)$ denotes the statement of the type '$x + y = z$'. When values are assigned to x, y, z, this statement has a truth value.

For example, consider the statements (i) Ram is fair ; (ii) Sham is fair.

Here in (*i*) and (*ii*) 'is fair' is the predicate while Ram and Sham are the objects. If we denote the predicate by F and the objects by r and s, then the above statements can be symbolically expressed as (*i*) $F(r)$; (*ii*) $F(s)$.

Now consider the statement Ram is fair and the house is pink.

Writing 'the house is pink' as $P(h)$, the given statement can be expressed as $F(r) \wedge P(h)$.

37.12 QUANTIFIERS

(1) In a propositional function, when all the variables are assigned values, the resulting statement has a truth value. However, there is another method to create a proposition from a propositional function which is called *quantification*. It is of two types : *Universal quantification* and *Existential quantification*.

(2) Universal quantification. Many statements assert that a property is true for all values of a variable in a certain domain. This domain is termed as the universe of discourse and such a statement is expressed using universal quantification.

Thus *the universal quantification of P(x) is the proposition 'P(x) is true for all values of x in the universe of discourse'.*

The universal quantification of $P(x)$ is denoted by $\forall\, xP(x)$. The symbol $\forall$ is called the *universal quantifier*.

Obs. When it is possible to list all the elements in the universe of discourse say : $x_1, x_2, \ldots, x_n$, then the universal quantification $\forall\, x\, P(x)$ is same as the conjunction $P(x_1) \wedge P(x_2) \wedge \ldots, P(x_n)$ for this conjunction is true if and only if $P(x_1)$, $P(x_2), \ldots, P(x_n)$ are all true.

Example 37.16. *What is the truth value of the quantification $\forall\, x\, P(x)$ where*
(a) $P(x)$ is the statement '$x < 5$' and universe of discourse is the set of real numbers.
(b) $P(x)$ is the statement '$x^2 < 18$' and the universe of discourse consists of positive integers not exceeding 5 ?

Solution. (*a*) For instance, $P(6)$ is false ; therefore $P(x)$ is not true for all real numbers x.
Thus $\forall\, xP(x)$ is false.

(*b*) The statement $\forall\, xP(x)$ is same as the conjunction $P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5)$

Here the universe of discourse is 1, 2, 3, 4, 5 and $P(5)$ is the statement '$5^2 < 18$' which is false. Hence $\forall\, xP(x)$ is also false.

(3) Existential quantification. Many statements assert that there is an element with a certain property. To express such statements we use *existential quantification*. In such cases, we form a proposition which is true if and only if $P(x)$ is true for at least one value of x in the universe of discourse.

Thus *the existential quantification of P(x) is the proposition 'these exists an element x in the universe of discourse such that P(x) is true'.*

The notation $\exists xP(x)$ is used for the *existential quantification* wherein $\exists$ is called the *existential quantifier*.

Obs. When it is possible to list all the elements in the universe say : $x_1, x_2, \ldots, x_n$, then the existential quantification $\exists x\, P(x)$ is same as the disjunction $P(x_1) \vee P(x_2) \vee \ldots, P(x_n)$.

Example 37.17. *What is the truth value of the quantification $\exists xP(x)$ where*
(a) P(x) is the statement '$x > 5$' and universe of discourse is the set of real numbers.
(b) P(x) is the statement '$x^2 > 18$' and the universe of discourse consists of positive integers not exceeding 5 ?

Solution. (*a*) Since '$x > 5$' is true, say : for $x = 6$, 8 etc.

$\therefore\quad \exists xP(x)$ is true.

(*b*) The statement $\exists xP(x)$ is same as the disjunction $P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5)$

The universe of discourse is 1, 2, 3, 4, 5 and $P(5)$ is the statement $5^2 > 18$, which is true.

Hence $\exists xP(x)$ is also true.

Example 37.18. *Find the truth value of each of the following statements :*
(*a*) $\exists x,\ x^2 = 1$ (*b*) $\forall x,\ |\, x\, | = x$ (*c*) $\exists x,\ x + 4 = x$.

Solution. (*a*) If $x_0 = 1$, then $x_0^{\ 2} = 1$, therefore the given statement is true.

(*b*) If $x_0 = -3$, then $|\, x_0\, | \neq x_0$, therefore the given statement is false.

(*c*) As there is no solution to $x + 4 = x$, the given statement is false.

Example 37.19. *Given P = {2, 3, 4, 5, 6}, state the truth values of each of the following statements :*
(*a*) $(\forall\, x \in P)\ (x + 3 < 10)$ (*b*) $(\forall\, x \in P)\ (x + 2 \leq 7)$
(*c*) $(\exists x \in P)\ (x + 3 = 10)$ (*d*) $(\exists x \in P)\ (x + 2 < 7)$.

Solution. (*a*) True, for each number in P satisfies $x + 3 < 10$.

(*b*) False, for if $x_0 = 6$, then $x_0 + 2$ is not ≤ 7.

(*c*) False, for no number in P is a solution to $x + 3 = 10$.

(*d*) True, for if $x_0 = 2$ then $x_0 + 2 < 7$ is a solution.

Example 37.20. *Negate each of the following statements :*
(*a*) $\forall x,\ x^2 = x,$ (*b*) $\forall\, x,\ x + 4 > x$ (*c*) $\forall x,\ |\, x\, | = x.$

Solution. (*a*) $\sim \forall\, x, x^2 = x \equiv \forall\, x \sim (x^2 = x) \equiv \forall\, x, x^2 \neq x.$

(*b*) $\sim \forall\, x, x + 4 > x \equiv \exists\, x \sim (x + 4 > x) \equiv \rightarrow x\ , x + 4 \leq x.$

(*c*) $\sim \forall\, x,\ |\, x\, | = x \equiv \exists\, x \sim (|\, x\, | = x) \equiv \exists\, x,\ |\, x\, | \neq x.$

Example 37.21. *Symbolise using quantifiers :*
(i) Every even number is divisible by 2.
(ii) There is no prime number between 23 and 29. (*V.T.U., MCA, 2001 S*)

Solution. (*i*) $E(x)$: x is even number ; $D(x)$: x is divisible by 2.

$(\forall\, x)\ [E(x) \rightarrow D(x)]$

(*ii*) If p denotes the set of prime numbers, then $(\nexists_n)_p\ (23 < n < 29)$.

Example 37.22. *Symbolise the expression 'All the world loves a lover.'* (Madras, 2001)

Solution. Let $p(x)$: x is a person ;

$L(x)$: x is a lover.

and $Q(x, y)$: x loves y

Then the required expression is

$$(\forall x)\ [p(x) \to (y)\ (p(y) \wedge L(y)] \to Q(x, y).$$

Summary. (*i*) $\forall\ Q(x)$ means that the predicate $Q(x)$ is true *for all values* in the universe of x.

(*ii*) $\exists Q(x)$ means that the predicate $Q(x)$ is satisfied if there is *at least one value* in the universe of x.

37.13 NORMAL FORMS

(1) For the given variables $p_1, p_2, ..., p_n$, we may form a statement $S(p_1, p_2, ..., p_n)$. The truth table for S will contain 2^n rows for all possible truth values of the n variables. The expression S may have the truth value 1 in all cases or may have the truth value 0 in all cases or have the truth value 1 for at least one combination of truth values assigned to the n variables. (Here S is said to be satisfiable). The problem of finding in a finite number of steps whether a given expression is a tautology or a contradiction or at least satisfiable is known as a *decision problem*. As the formation of a truth table is quite cumbersome, we go for an alternate approach called *normal form.*

In this approach, we use the word 'sum' in place of disjunction and 'product' in place of conjunction.

A sum of the variables and their negations is called an *elementary sum*. Similarly a product of the variables and their negation is called an *elementary product*.

(2) Disjunctive normal form *of a given formula is the formula which is equivalent to the given formula and which contains the sum of the elementary products.* The disjunctive normal form of a given form is not unique. In fact, several disjunctive normal forms can be obtained for a given formula by applying the distributive laws in different ways.

A given formula is however, identically false if every elementary product appearing in its disjunctive normal form is identically false.

Example 37.23. *Obtain the disjunctive normal forms of*

(i) $p \wedge (p \to q)$ *(ii)* $\sim (p \vee q) \leftrightarrow (p \wedge q)$.

Solution. (*i*) $p \wedge (p \to q) \Leftrightarrow p \wedge (\sim p \vee q) \Leftrightarrow (p \wedge \sim p) \vee (p \wedge q)$.

which is the desired disjunctive normal form.

(*ii*) $\sim (p \vee q) \leftrightarrow (p \wedge q) \Leftrightarrow \sim (p \vee q) \wedge (p \wedge q) \vee (p \vee q) \wedge \sim (p \wedge q)$ $[\because\ E \leftrightarrow F \Leftrightarrow (E \wedge F) \vee (\sim E \wedge \sim F)$

$$\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee [(p \vee q) \wedge (\sim p \vee \sim q)]$$
$$\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee [(p \vee q) \wedge \sim p] \vee [(p \vee q) \wedge \sim q]$$
$$\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee (p \wedge \sim p) \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim q)$$

which is the desired disjunctive normal form.

(3) Conjunctive normal form *of a given formula is that formula which is equivalent to the given formula and contains the product of elementary sums.*

Example 37.24. *Find a conjunctive normal form of* $\sim (p \vee q) \leftrightarrow (p \wedge q)$.

Solution. $\sim (p \vee q) \leftrightarrow (p \wedge q) \Leftrightarrow [\sim (p \vee q) \to (p \wedge q)] \wedge [p \wedge q \to \sim (p \vee q)]$

$$\Leftrightarrow [(p \vee q) \vee (p \wedge q)] \vee [\sim (p \wedge q) \vee \sim (p \vee q)]$$ [By conditional equivalence]
$$\Leftrightarrow (p \vee q \vee p) \wedge (p \vee q \vee q) \wedge [(\sim p \vee \sim q) \vee (\sim p \wedge \sim q)]$$
$$\Leftrightarrow p \vee q \vee p \wedge (p \vee q \vee q) \wedge (\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)$$

which is the required conjunctive normal form.

(4) Principal disjunctive normal form. Consider a formula for the propositions p and q using conjunction as $p \wedge q, p \wedge \sim q, \sim p \wedge q, \sim p \wedge \sim q$. These terms are called *minterms* or *Boolean conjunction* of p and q.

An equivalent formula for a given formula, consisting of disjunctions of minterms only is called the **principal disjunctive normal form** (*pdnf*) or **sum of products canonical form**.

Procedure *to obtain the principle disjunctive normal form* : (*i*) Replace the conditions and biconditions by their equivalent formulae containing $\wedge, \vee, \sim$ only.

(*ii*) Using DeMorgans laws, apply negations to the variables.

(*iii*) Apply the distribution laws.

(*iv*) Introduce the missing factors to obtain minterms in the disjunctions.

(*v*) Delete identical minterms appearing in the disjunctions.

Example 37.25. *Obtain the pdnf for*

(*i*) $p \vee q$ (*ii*) $\sim(p \wedge q)$ (*iii*) $\sim p \vee q$ *i.e.*, $p \to q$.

Solution. (*i*)
$$p \vee q \Leftrightarrow [p \wedge (q \vee \sim q)] \vee [q \wedge (p \vee \sim p)]$$
$$\Leftrightarrow (p \wedge q) \vee (p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p)$$
$$\Leftrightarrow (p \wedge q) \vee (p \wedge \sim q) \vee (q \wedge \sim p)$$

(*ii*)
$$\sim (p \wedge q) \Leftrightarrow (\sim p \vee \sim q) \Leftrightarrow [\sim p \wedge (\sim q \vee q)] \vee [\sim q \wedge (p \vee \sim p)]$$
$$\Leftrightarrow (\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim q \wedge p) \vee (\sim q \wedge \sim p)$$
$$\Leftrightarrow (\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim q \wedge p)$$

(*iii*)
$$\sim p \vee q \Leftrightarrow \sim p \wedge (q \vee \sim q) \vee [q \wedge (p \vee \sim p)]$$
$$\Leftrightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p)$$
$$\Leftrightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p).$$

(5) Principal conjunctive normal form. Consider a formula for the propositions p and q using disconjunction as $p \vee q$, $\sim p \vee q$, $p \vee \sim q$, $\sim p \vee \sim q$. These terms are called *max terms.*

An equivalent formula for a given formula, consisting of conjunctions of the max terms only is called the **principal conjunctive normal form** (*pcnf*) or **product of sum canonical form**.

Procedure *for obtaining pcnf* for a given formula is similar to the one for *pdnf* as all assertions made for *pdnf* can be made for *pcnf* using duality principle.

Example 37.26. *Obtain the principal disjunctive and conjunctive normal forms of*
$$p \to [(p \to q) \wedge \sim (\sim q \vee \sim p)].$$
(*Bharathiar, 2001*)

Solution. (*i*)
$$p \to [(p \to q) \wedge \sim (\sim q \vee \sim p)]$$
$$\Leftrightarrow \sim p \wedge [(\sim p \vee q) \wedge (q \wedge p)]$$
[Using DeMorgan's law and equivalence $p \to q \Leftrightarrow \sim p \vee q$.]
$$\Leftrightarrow \sim p \vee [\sim p \wedge (q \wedge p)] \vee [q \wedge (q \wedge p)]$$
$$\Leftrightarrow \sim p \vee (q \wedge p)$$
$$\Leftrightarrow [\sim p \wedge (q \vee \sim q)] \vee (q \wedge p)$$
$$\Leftrightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p)$$

This is the desired *pdnf.*

(*ii*)
$$p \to [(p \to q) \wedge \sim (\sim q \vee \sim p)]$$
$$\Leftrightarrow \sim p \vee [(\sim p \vee q) \wedge (q \wedge p)]$$
$$\Leftrightarrow [\sim p \vee (\sim p \vee q)] \wedge [\sim p \vee (q \wedge p)]$$
$$\Leftrightarrow (\sim p \vee q) \wedge (\sim p \vee q) \wedge (\sim p \vee p)$$
$$\Leftrightarrow \sim p \vee q$$

This is the desired *pcnf.*

Example 37.27. *Obtain the principal conjunctive normal form for* $(Q \to P) \wedge (\sim P \wedge Q)$. (*Andhra, 2004*)

Solution.
$$(Q \to P) \wedge (\sim P \wedge Q) \Leftrightarrow (\sim Q \vee P) \wedge (\sim P \wedge Q)$$
$$\Leftrightarrow (\sim Q \vee P) \wedge [\sim P \vee (Q \wedge \sim Q) \wedge Q \vee (P \wedge \sim P)]$$
$$\Leftrightarrow (\sim Q \vee P) \wedge (\sim P \vee Q) \wedge (\sim P \vee \sim Q) \wedge (Q \vee P) \wedge (Q \vee \sim P)$$
$$\Leftrightarrow (\sim Q \vee P) \wedge (\sim P \vee Q) \wedge (\sim P \vee \sim Q) \wedge (Q \vee P).$$

37.14 INFERENCE THEORY

Inferring the conclusions from certain premises is known as the *inference theory*. When conclusion is reached from a set of premises by using the accepted rules of reasoning, then this process is called a *deduction* or a *formal proof*. A proof of a theorem is a valid argument.

The criteria for finding 'whether an argument is valid' are called *rules* which are expressed in terms of premises and conclusions or in terms of statement formulae.

The proofs are of two types : *Direct* or *Indirect*.

(*i*) If in a proof the truth of the premises directly shows the truth of the conclusions, then it is called a *direct proof*.

(*ii*) *An indirect proof* proceeds by assuming that p is true and also C is false, and then deduce a contradiction using p and $\sim C$, along with other premises.

The only difference between assumptions in a direct proof or an indirect proof is the negated conclusion.

Rules for deriving direct and indirect proofs :

(*i*) the proof should have finite steps only ;

(*ii*) each step must be either a premise or a proposition which is implied from previous steps using valid equivalence or implication ;

(*iii*) the last step for a direct proof must be conclusion while for an indirect proof it must be a contradiction.

Example 37.28. *State whether the conclusion C follows logically from the premises R and S*

(*a*) $R : p \to q,$ $S : p,$ $C : \sim q$

(*b*) $R : p,$ $S : p \leftrightarrow q,$ $C : \sim (p \wedge q)$

(*c*) $R : p \to q,$ $S : p,$ $C : q$

Solution. Let us first form the following truth table :

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim (p \wedge q)$	$p \leftrightarrow q$
1	1	0	0	1	0	1
1	0	0	1	0	1	0
0	1	1	0	1	0	0
0	0	1	1	1	1	1

(*a*) Here only the first row of premises R and S contains 1 but not the conclusion C.

Hence C is not valid.

(*b*) Only the first row of both R and S contains 1 but not the conclusion C.

Hence C is not valid.

(*c*) Here both R and S contain '1' only in the first row and the conclusion C also has '1' in that row. Hence our conclusion is valid.

Example 37.29. *Find the direct and indirect proofs of* $p \to (q \to r), \sim r \vee p, q \Rightarrow r \to s.$

Solution. *Direct proof*

(*i*) $\sim r \vee p$ (premise) (*ii*) r (another premise)

(*iii*) p [By (*i*) and (*ii*)] (*iv*) $p \to (q \to r)$ (premise)

(*v*) $q \to r$ [By (*iii*) and (*iv*)] (*vi*) q (premise)

(*vii*) r [By (*v*) and (*vi*)], which is a conclusion.

Indirect proof

(*i*) $\sim (s \to r)$ (negative of conclusion)

(*ii*) $s \wedge \sim r$ (By conditional equivalence)

(*iii*) $\sim r \vee p$ (premise)

(*iv*) $\sim r$ and (*v*) s [By conjunctive simplification]

(*vi*) p [By disconjunction of (*iii*) and (*iv*)]

(*vii*) $p \to (q \to r)$ (premise)

(viii) $q \to r$ [By (vi) and (vii)]

(ix) q (premise)

(x) r [By (viii) and (ix)]

(xi) $r \wedge \sim r$ [By (x) and (iv)]

This is a contradiction.

PROBLEMS 37.3

1. If $A = \{1, 2, 3, 4, 5\}$ be the universal set, determine the truth values of each of the following statements :
   (a) $(\forall x \in A)\ (x + 2 < 10)$
   (b) $\exists x \in A\ (x + 2 = 10)$
2. Negate each of the following statements :
   (a) $\forall x, x^3 = x$;
   (b) $\forall x,\ x + 5 > x$
   (c) Some students are 26 or older.
   (d) All students live in the hostels.
3. What is the truth value of $\forall x\ P(x)$ where $P(x)$ is a statement '$x^2 < 10$' and the universe of discourse consists of positive integers not exceeding 4.
4. Use universal quantifier to state 'the sum of any two rational numbers is rational'.
5. Over the universe of real numbers, use quantifier to say that the equation $a + x = b$ has a solution for all values of a and b.
6. Translate the following statements involving quantifiers, into formulae :
   (a) All rationals are reals.
   (b) No rationals are reals.
   (c) Some rationals are reals.
   (d) Some rationals are not reals.
7. Show that $Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$ is a tautology.
8. Convert $\sim A \wedge (\sim B \to C) \Rightarrow 0$ into CNF. (V.T.U., MCA, 2001)
9. Without constructing truth tables, obtain the product of sums canonical form of the formula $(\neg P \to R) \wedge (Q \leftrightarrow P)$. Hence find the sum of products canonicals form. (Anna, 2004 S)
10. Find the direct proof of $p \to r, q \to s, p \vee q \Rightarrow s \vee r$.
11. Prove that $P \to Q, Q \to R, P \vee R \Rightarrow R$ by using indirect method. (Anna, 2004 S)
12. Using quantifier say $\sqrt{2}$ is not a real number ?
13. State whether the conclusion C follows logically from the premises R and S
    (a) $R : p \to q,\quad S : \sim q,\quad C : q$
    (b) $R : p \to q,\quad S : q,\quad C : \sim p$.

III. BOOLEAN ALGEBRA

37.15 INTRODUCTION

(1) The concept of Boolean algebra was first introduced by George Boole* in 1854 through his paper 'An investigation of the laws of thought'. It is basically two values *i.e.*, (0, 1) set. Earlier it had applications to statements and sets which are either true or false. In 1938 Claude Shannon showed that basic rules given by Boole could be used to design circuits. These days however, Boolean algebra has wide applications to switching circuits, electrical networks and electronic computers.

Basically there are three operations in the Boolean algebra (i) *AND*, (ii) *OR*, and (iii) *NOT*, which are symbolically represented by $\wedge$, $\vee$ and / respectively. Some authors, use the symbols (+), (.) and (/) for the same operations.

Here ′ denotes the complement of an element and is defined by $0' = 1$ and $1' = 0$.

The operator $\wedge$ (*i.e.*, '*AND*') has the following values $1 \wedge 1 = 1, 1 \wedge 0 = 0, 0 \wedge 1 = 0, 0 \wedge 0 = 0$; while the operator $\vee$ (*i.e.*, *OR*) has the values $1 \vee 1 = 1$, $1 \vee 0 = 1$, $0 \vee 1 = 1$, $0 \vee 0 = 0$.

Def. *Any non-empty set B with the binary operations '$\wedge$' and '$\vee$' and the uninary operation '/' is called the Boolean algebra [B, $\wedge$, $\vee$, /] if the following axioms hold where a, b, c are elements in B :*

1. *Commutative law* : $a \wedge b = b \wedge a$; $a \vee b = b \vee a$
2. *Associative law* : $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
   $a \vee (b \vee c) = (a \vee b) \vee c$

*A British mathematician George Boole (1813–1864) who created Boolean algebra.

3. *Distributive law* : $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$a \wedge b \vee c = (a \wedge b) \vee (a \wedge c)$

4. *Complement law* : $a \wedge a' = 0, a \vee a' = 1$

The operations $\wedge$, $\vee$ and / are called sum, product, and complement respectively. We shall follow the usual practice that / has precedence over $\vee$, and $\vee$ has precedence over $\wedge$, unless guided by brackets. *e.g.*, $a \wedge b \vee c$ means $a \wedge (b \vee c)$ not $(a \wedge b) \vee c$ while $a \vee b'$ implies $a \vee b'$ not $(a \vee b)'$.

(2) Boolean function. The variable x is called a *Boolean variable* if it assumes values only from B {0, 1}.

Def. *A function from the set* $\{(x_1, x_2, ..., x_n) : x_i \in B, 1 \le i \le n\}$ *is called a Boolean function of degree n.* Boolean functions can be represented by expressions comprised of variables and Boolean operations.

e.g., $0, 1, x_1, x_2, ..., x_n$ are Boolean expressions in the variables x_i $(1 \le i \le n)$. If p and q are Boolean expressions then $p \wedge q$, $p \vee q$ and p' are also Boolean expressions and each represents a Boolean function.

By substituting 0 and 1 for the variables in the expression, the values of this function can be found.

(3) *If f and g be Boolean functions of degree n, then*

(*i*) *Complement of f* is the function f' where

$$f'(x_1, x_2, ..., x_n) = [f(x_1, x_2, ..., x_n)]'$$

(*ii*) *f and g are equal* if $f(x_1, x_2, ..., x_n) = g(x_1, x_2, ..., x_n)$

(*iii*) *Boolean sum* $f \vee g$ *is*

$$(f \vee g)(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) \vee g(x_1, x_2, ..., x_n)$$

(*iv*) *Boolean product* $f \wedge g$ *is*

$$(f \wedge g)(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) \wedge g(x_1, x_2, ..., x_n).$$

(4) Power in a Boolean function : $x^2 = x \vee x = x$

$$x^3 = x^2 \vee x = x \vee x = x, ..., x^n = x$$

Similarly, $2x = x$, $3x = x$ etc.

Example 37.30. *Find the values of the Boolean function* $f = (x \wedge y') \vee z'$.

Solution. f being third degree Boolean function has 2^3 *i.e.* 8 values which are shown in the following table :

x	y	z	y'	z'	$x \wedge y'$	$(x \wedge y') \vee z'$
1	1	1	0	0	0	0
1	1	0	0	1	0	1
0	1	1	0	0	0	0
1	0	1	1	0	1	1
0	0	1	1	0	0	0
1	0	0	1	1	1	1
0	1	0	0	1	0	1
0	0	0	1	1	0	1

37.16 DUALITY

(1) The dual of any Boolean function is obtained by interchanging Boolean sums and Boolean products along with the interchange of zeros and ones.

For example the dual of $x \vee (y \wedge 0)$ is $x \wedge (y \vee 1)$.

The dual of any theorem of a Boolean algebra is also its theorem. This implies that the dual of any theorem in Boolean algebra is always true.

(2) Principle of duality. *The dual of any theorem (or property) in Boolean algebra is also a theorem (or property).*

37.17 BOOLEAN IDENTITIES

There are many identities in Boolean algebra which are quite useful in simplifying electrical circuits. Some of the important ones are given below :

1. *Identity law* :	$x \vee 0 = x$; $x \wedge 1 = x$
2. *Dominance laws* :	$x \vee 1 = 1$; $x \wedge 0 = 0$
3. *Complement law* :	$x \vee x' = 1$; $x \wedge x' = 0$
4. *Idempotent law* :	$x \vee x = x$; $x \wedge x = x$
5. *Double complement law* :	$(x')' = x$
6. *Commutative law* :	$x \vee y = y \vee x$; $x \wedge y = y \wedge x$
7. *Associative law* :	$x \vee (y \vee z) = (x \vee y) \vee z$ $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
8. *Distributive law* :	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
9. *De-Morgan's law* :	$x \wedge y = x' \vee y'$; $(x \vee y)' = x' \wedge y'$ (*Bhopal, 2008*)
10. *Absorption law* :	$x \wedge (x \vee y) = x$

Example 37.31. *Let B be a Boolean algebra. Show that for all $a \in B$, there exists a unique complement a'.* (*Andhra, 2004*)

Solution. Let b and c be two complements of a.

Then
$$\begin{aligned} b &= b \wedge 1 && [\because\ 0 \text{ is an additive identity}]\\ &= b \wedge (a \vee c) && [\because\ c \text{ is complement of } a]\\ &= (b \wedge a) \vee (b \wedge c) = (a \wedge b) \vee (b \wedge c)\\ &= 0 \vee (b \wedge c) && [\because\ a \wedge b = a \wedge a' = 0]\\ &= b \wedge c && \text{...}(i) \end{aligned}$$

Similarly
$$\begin{aligned} c &= c \wedge 1 = c \wedge (a \vee b) && [\because\ a \vee b = a \vee a' = 1]\\ &= (c \wedge a) \vee (c \wedge b)\\ &= (a \wedge c) \vee (b \wedge c) && [\because\ a \wedge c = a \wedge a' = 0]\\ &= 0 \vee (b \wedge c)\\ &= b \wedge c && \text{...}(ii) \end{aligned}$$

From (*i*) and (*ii*), we find that $b = c$.

Thus the complement of a is unique.

Example 37.32. *In a Boolean algebra, show that*

(*i*) $x + (x \,.\, y) = x$ (*ii*) $x \,.\, (x + y) = x$. (*Bhopal, 2008*)

Solution. (*i*)
$$\begin{aligned} x + (x \,.\, y) &= x \wedge (x \vee y) = (x \vee 0) \wedge (x \vee y) && [\because\ x \vee 0 = x]\\ &= x \vee (0 \wedge y) && [\text{By distributive law}]\\ &= x \vee (y \wedge 0) && [\text{By commutative law}]\\ &= x \vee 0 && [\because\ y \wedge 0 = 0\\ &= x. \end{aligned}$$

(*ii*)
$$\begin{aligned} x \,.\, (x + y) &= x \vee (x \wedge y)\\ &= (x \wedge 1) \vee (x \wedge y) && [\because\ x \wedge 1 = x\\ &= x \wedge (1 \vee y) && [\text{By distributive law}]\\ &= x \wedge (y \vee 1) && [\text{By commutative law}]\\ &= x \wedge 1 && [\because\ y \vee 1 = 1\\ &= x. \end{aligned}$$

Example 37.33. *Simplify the following :*

(*i*) $(x + y) \,.\, x' \,.\, y'$ (*ii*) $x \vee y \wedge y \vee z \wedge y \vee z'$ (*iii*) $x \vee y \wedge [(x \wedge y') \vee y]'$.

Solution. (*i*) $(x \wedge y) \vee x' \vee y' = (x \wedge y) \vee (x' \vee y')$
$= (x \wedge y) \vee (x \wedge y)'$ [By De Morgan's law]
$= 1.$ [$\because \quad p \vee p' = 1$]

(*ii*) $x \vee y \wedge y \vee z \wedge y \vee z' = (y \vee x) \wedge (y \vee z) \wedge (y \vee z')$ [By commutative law]
$= [y \vee (x \wedge z)] \wedge (y \vee z')$ [By distributive law]
$= y \vee [x \wedge z \wedge z']$
$= y \vee [x \wedge (z \wedge z')]$ [By associative law]
$= y \vee (x \wedge 0)$ [$\because \quad z \wedge z' = 0$]
$= y \vee 0$ [$\because \quad x \wedge 0 = 0$]
$= y.$

(*iii*) $x \vee y \wedge [(x \wedge y') \vee y]' = x \vee y \wedge [y \vee (x \wedge y')]'$ [By commutative law]
$= x \vee y \wedge [(y \vee x) \wedge (y \vee y')]'$ [By distributive law]
$= (x \vee y) \wedge [(x \vee y) \wedge 1]'$ [$\because \quad y \vee y' = 1$]
$= (x \vee y) \wedge (x \vee y)' = 0.$

Example 37.34. *Show that*
(*i*) $x \vee y \wedge y \vee z \wedge z \vee x = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$
(*ii*) $(x \wedge y) \vee (x' \wedge z) = (x' \vee y) \wedge (x \vee z).$ *(Bhopal, 2008)*

Solution. (*i*) R.H.S. $= (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$
$= (x \wedge y) \vee (z \wedge y) \vee (z \wedge x)$ [By commutative law]
$= (x \wedge y) \vee (z \wedge y \vee x)$ [By distributive law]
$= (x \vee z) \wedge (y \vee z) \wedge [x \vee (y \vee x)] \wedge [y \vee (y \wedge x)]$
$= (x \vee z) \wedge (y \vee z) \wedge [(x \vee y) \wedge (y \vee x)]$ [$\because \quad x \vee x = x$ etc.]
$= (x \vee z) \wedge (y \vee z) \wedge [(x \vee y) \wedge (x \vee y)]$
$= (x \vee z) \wedge (y \vee z) \wedge (x \vee y)$ [$\because \quad p \wedge p = p$
$= (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ (By commutative law)
= L.H.S.

(*ii*) L.H.S. $= (x \wedge y) \vee (x' \wedge z)$
$= [x \vee (x' \wedge z)] \wedge [y \vee (x' \wedge z)]$ [By distributive law]
$= [(x \vee x') \vee (x \vee z)] \wedge [(y \vee x') \wedge (y \vee z)]$
$= [1 \vee (x \vee z)] \wedge [(y \vee x') \wedge (y \vee z)]$
$= (x \vee z) \wedge (y \vee x') \wedge (y \vee z) \vee 1$
$= (x \vee z) \wedge (y \vee x') \wedge [(y \vee z) \vee (x \wedge x')]$
$= (x \vee z) \wedge (y \vee x') \wedge [(y \vee z) \vee x] \vee (y \vee z) \vee x']$
$= (x \vee z) \wedge [y \vee z \vee x] \wedge (x' \vee y) \vee [x' \wedge (y \vee z)]$
$= (x \vee z) \vee (1 \wedge y) \wedge (x' \vee y) \vee (1 \wedge z)$
$= [(x \vee z) \wedge 1] \wedge [(x' \vee y) \vee 1]$
$= (x \vee z) \wedge (x' \vee y) =$ R.H.S.

Example 37.35. *Show that*
(*i*) $x \vee y \wedge x' \vee y' = (x' \wedge y) \vee (x \wedge y')$
(*ii*) $[x \wedge (x' \vee y)] \vee [x' \wedge (x \vee y)] = y.$

Solution. (*i*) $(x \vee y) \wedge (x' \vee y') = [(x \vee y) \wedge x'] \vee [(x \vee y) \wedge y']$
$= [(x \wedge x') \vee (y \wedge x')] \vee \{(x \wedge y') \vee (y \wedge y')\}$ [By distributive law]
$= \{0 \vee (x' \wedge y)\} \vee \{(x \wedge y') \vee 0\}$ [$\because \quad x \wedge x' = 0$
$= (x' \wedge y) \vee (x \wedge y').$

(*ii*) $[x \wedge (x' \vee y)] \vee [x' \wedge (x \vee y)]$
$= [(x \wedge x') \vee (x \wedge y)] \vee [(x' \wedge x) \vee (x' \wedge y)]$
$= [0 \vee (x \wedge y)] \vee [0 \vee (x' \wedge y)]$ [$\because \quad x \wedge x' = 0$
$= (x \vee x') \wedge y = y \wedge 1 = y.$

Example 37.36. *If $a \vee x = b \vee x$ and $a \vee x' = b \vee x'$, then prove that $a = b$.*

Solution. Since $a \vee x = b \vee x$ and $a \vee x' = b \vee x'$

$\therefore$ $$(a \vee x) \wedge (a \vee x') = (b \vee x) \wedge (b \vee x')$$

i.e., $$a \vee (x \wedge x') = b \vee (x \wedge x')$$ [By distributive law]

or $$a \vee 0 = b \vee 0$$ [$\because$ $x \wedge x' = 0$]

or $$a = b.$$

Example 37.37. *In Boolean algebra [B, +, ., /], show that*

$$(x \,.\, y' + y \,.\, z) \,.\, (x \,.\, z + y \,.\, z') = x \,.\, z$$ *(Bhopal, 2008 ; M.P.T.U., 2001)*

Solution. $\{(x \vee y') \wedge (y \vee z)\} \vee \{(x \vee z) \wedge (y \vee z')\}$

$$= \{(x \vee y') \wedge y\} \vee \{(x \vee y') \wedge z\} \vee \{(x \wedge z) \wedge y\} \vee \{(x \vee z) \wedge z'\}$$
$$= \{(x \wedge y) \vee (y' \wedge y)\} \vee \{(x \wedge z) \vee (y' \wedge z)\} \vee \{(x \wedge y) \vee (z \wedge y)\} \vee \{(x \wedge z') \vee (z \wedge z')\}$$
$$= \{(x \wedge y) \vee 0\} \vee \{(x \wedge z) \vee (y' \wedge z)\} \vee \{(x \wedge y) \vee (z \wedge y)\} \vee \{(x \wedge z') \vee 0\}$$
$$= \{(x \wedge y) \vee (x \wedge y)\} \vee \{(x \wedge z) \vee (x \wedge z')\} \vee \{(z \wedge y) \vee (z \wedge y')\}$$
$$= (x \wedge y) \vee (x \wedge 1) \vee (z \wedge 1) = (x \wedge y) \vee x \vee z$$
$$= (x \vee x \vee z) \wedge (y \vee x \vee z) = (x \vee z) \wedge (y \vee x \vee z)$$
$$= (x \vee z) \wedge (1 \vee y) = (x \vee z) \wedge 1$$
$$= x \vee z.$$

PROBLEMS 37.4

1. Find the truth table for the Boolean function $f(x, y, z) = (x \wedge y) \vee (y \wedge z')$.
2. Write the dual of the Boolean expression $x + x' \,.\, y = x + y$. *(Andhra, 2004)*
3. Simplify the following :
   (*i*) $(x \wedge y \wedge z)'$ (*ii*) $(x \vee y \vee z) \wedge (x' \wedge y' \wedge z')$.
4. In a Boolean algebra $[B, \wedge, \vee, /]$, prove that
   (*i*) $(x \wedge y) \vee (x \wedge y') = x$. *(Anna, 2005)* (*ii*) $x' \wedge (x \vee y) = x' \wedge y$.
5. If $a \wedge x = b \wedge x$ and $a \wedge x' = b \wedge x'$, then show that $a = b$.
6. In a Boolean algebra $[B, \wedge, \vee, /]$, show that
   (*i*) $x \wedge (x \wedge y) = x \wedge y$ (*ii*) $x \vee (x \vee y) = x \vee y$.
7. In Boolean algebra, prove that
   (*i*) $x \wedge (x' \vee y) = x \wedge y$ (*ii*) $x' \wedge y = x' \wedge (x \vee y)$.
8. Show that $(x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y') = x' \vee y'$.
9. If B be a Boolean algebra and $x, y, z \in B$, prove that
   $$(x \vee y) \wedge (x \vee y') \wedge (x' \vee y) = x \wedge y$$
10. Prove that $(x \vee y) \wedge [z \vee (x' \wedge y')] = (x \vee y) \wedge z$.
11. In a Boolean algebra B, prove that $(a + b)' = a' \,.\, b' \;\forall\; a, b \in B$. *(Bhopal, 2009)*
12. In any Boolean algebra, show that $a = b$ if and only if $a \,.\, b' + a' \,.\, b = 0$. *(Madras, 2001)*
13. In Boolean algebra, show that
    (*i*) $(a + b) \,.\, (a' + c) = a \,.\, c + a' \,.\, b + b \,.\, c$.
    (*ii*) $(a + b') \,.\, (b + c') \,.\, (c + a') = (a' + b) \,.\, (b' + c) \,.\, (c' + a)$. *(Andhra, 2004)*
14. Give the truth table for the Boolean function
    $f : B_3 \to B$ determined by the polynomial
    $$P(x_1, x_2, x_3) = (x_1 \vee x_3)' \wedge (x_1 \wedge (x_2 \vee x'_3)).$$ *(V.T.U., 2001)*

37.18 MINIMAL BOOLEAN FUNCTION

Def. *A minimal Boolean function in n variables is the product of $x_1, x_2, ..., x_n$.* It is also called *minterm.*

If x, y are two variables and x', y' are their complementary variables respectively, then $x \vee y$, $x' \vee y$, $x' \wedge y$, $x \wedge y'$, $x' \vee y'$ are each a minimal Boolean function.

Similarly there are 2^3 *i.e.* 8 minimal Boolean functions in the three variables x, y, z *i.e.*, $x \vee y \vee z, x' \vee y \vee z, x \vee y' \vee z, x \vee y \vee z', x \vee y' \vee z, x' \vee y \vee z', x' \vee y' \vee z, x' \vee y' \vee z'$.

In general, *there are 2^n minimal Boolean functions (or minterms) in n variables.*

Similarly the join of the variables $x_1, x_2, ... x_n$ is called a *maxterm* and there will be 2^n *maxterms.*

37.19 DISJUNCTIVE NORMAL FORM

(1) Def. *A Boolean function which can be expressed as sum of minimal Boolean functions is called a* **Disjunctive normal form** *or* **minterm normal form** or **Canonical form.**

(2) *If the number of distinct terms in a disjunctive normal form of Boolean function in n variables are 2^n, then it is called a* **complete disjunctive normal form.**

(3) Complement function of a disjunctive normal form *function f is the sum of all those terms of a complete disjunctive normal form which are not present in the disjunctive normal form of f.* The complement of f is denoted by f'.

For example, if $f = (x \vee y) \wedge (x \vee y')$

then its complete disjunctive normal form in variables x and y is

$$(x \vee y) \wedge (x' \vee y) \wedge (x \wedge y') \wedge (x' \vee y')$$

$\therefore$ The complement function of this disjunctive normal form is $f'(x, y) = (x' \wedge y) \wedge (x' \vee y')$.

Example 37.38. *Find the value of the complete disjunctive normal form in three variables x, y, z.*

Solution. The complete disjunctive normal form in three variables x, y, z is

$$\begin{aligned} f(x, y, z) &= (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x' \vee y \vee z) \\ &\quad \wedge (x \vee y' \vee z') \wedge (x' \vee y \vee z') \wedge (x' \vee y' \vee z) \wedge (x' \vee y' \vee z') \\ &= [(x \vee y) \vee (z \wedge z')] \wedge [(x \wedge y') \vee (z \wedge z')] \wedge [(x' \wedge y) \vee (z \wedge z')] \wedge [(x' \vee y') \vee (z \wedge z')] \\ &= [(x \vee y) \vee 0] \wedge [(x \wedge y') \vee 0] \wedge [(x' \wedge y) \vee 0] \wedge [(x' \vee y') \vee 0] \\ &= [x \vee (y \wedge y')] \wedge [x' \vee (y \wedge y')] \\ &= (x \vee 0) \wedge (x' \vee 0) = x \wedge x' = 0. \end{aligned}$$

37.20 CONJUNCTIVE NORMAL FORM

Def. *If a Boolean function $f(x_1, x_2, ..., x_n)$ is expressed in the form of factors and each factor is the sum of all the n-variables, then such a function is called a* **conjective normal form** or **maxterm normal form** or **dual canonical form.**

(2) *If a conjective normal of a function of n variables contains all the 2^n distinct factors, then such a function is called a* **complete conjective normal form.**

(3) Complement function of a conjective normal form *function f is a Boolean function which is the product of all those terms of complete conjective normal form which are not present in conjective normal form of f.*

The complement of conjective normal form f is denoted by f'.

For example, if $f(x, y) = (x \wedge y) \vee (x \wedge y')$

then its complete conjunctive normal form in x and y is

$$(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$$

$\therefore$ The complement function of this conjunctive normal form is

$$f'(x, y) = (x' \wedge y) \vee (x' \wedge y')$$

Example 37.39. *Given Boolean expression f, where $f(x_1, x_2, x_3) = (x_3' \wedge x_2) \vee (x_1' \wedge x_3) \vee (x_2 \wedge x_3)$, simplify this expression stating the laws used and obtain the minterm normal form.* (Bharathiar, 1997)

Solution. Given

$$\begin{aligned} f(x_1, x_2, x_3) &= (x_3' \wedge x_2) \vee (x_1' \wedge x_3) \vee (x_2 \wedge x_3) \\ &= (x_3' \wedge x_2) \vee (x_2 \wedge x_3) \vee (x_1' \wedge x_3) && \text{[By commutative law]} \\ &= (x_2 \wedge x_3') \vee (x_2 \wedge x_3) \vee (x_1' \wedge x_3) \\ &= [x_2 \wedge (x_3' \vee x_3)] \vee (x_1' \wedge x_3) && \text{[By distributive law]} \end{aligned}$$

$= (x_2 \wedge 1) \vee (x_1' \wedge x_3)$ $\quad [\because \ p \vee p' = 1]$

$= x_2 \vee x_1' \wedge x_3$ $\quad$ [By identity law]

Minterm normal form of $f(x_1, x_2, x_3)$

$$= [(x_3' \wedge x_2) \wedge (x_1 \vee x_1')] \vee [(x_1' \wedge x_3) \wedge (x_2 \vee x_2')] \vee [(x_2 \wedge x_3) \wedge (x_1 \vee x_1')]$$
$$= (x_3' \wedge x_2 \wedge x_1) \vee (x_3' \wedge x_2 \wedge x_1') \vee (x_1' \wedge x_3 \wedge x_2) \vee (x_1' \wedge x_3 \wedge x_2') \vee (x_2 \wedge x_3 \wedge x_1) \vee (x_2 \wedge x_3 \wedge x_1')$$
$$= (x_3' \wedge x_2 \wedge x_1) \vee (x_3' \wedge x_2 \wedge x_1') \vee (x_1' \wedge x_3 \wedge x_2) \vee (x_1' \wedge x_3 \wedge x_2') \vee (x_2 \wedge x_3 \wedge x_1).$$

Example 37.40. *Express the following functions into conjunctive normal forms :*

(i) $x' \wedge y$ *(ii)* $(x \wedge y) \vee (x' \wedge y')$.

Solution. *(i)* $x' \wedge y = x' \wedge y \wedge (z \vee z') = (x' \wedge y \wedge z) \vee (x' \wedge y \wedge z')$.

(ii) $(x \wedge y) \vee (x' \wedge y') = [x \wedge y \wedge (z \vee z')] \vee [x' \wedge y' \wedge (z \vee z')]$

$= (x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z')$.

Example 37.41. *The function* $f = (x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z') \vee (x' \wedge y' \wedge z)$ *is in conjunctive normal form. Write its complement ?*

Solution. The complete conjunctive normal form in three variables x, y, z is $(x \wedge y \wedge z) \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \vee y \vee z') \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z')$.

$\therefore$ The complement of the given function F is

$$F' = (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z').$$

PROBLEMS 37.5

1. Find the value of a complete disjunctive normal form in
   (i) two variables x, y. *(ii)* three variables x, y, z.
2. Express the following functions into disjunctive normal form :
   (i) $x \vee y$ *(ii)* $x \wedge (x' \vee y)$.
3. Express the Boolean function $F = A \vee (B' \wedge C)$ in a sum of minterms.
4. Convert the function $x \wedge y'$ to disjunctive normal form in three variables x, y, z.
5. Express the function $f = (x \vee y') \wedge (x \vee z) \wedge (x \vee y)$ into conjunctive normal form in which maximum number of variables are used.
6. Write the complement of the conjunctive normal form function $(x \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x \wedge y \wedge z')$.

37.21 SWITCHING CIRCUITS

(1) *A switching network is an arrangement of wires and switches* (or gates) *which connect two terminals.* A switch can be either closed or open. A closed switch permits and an open switch stops flow of current

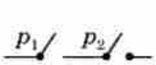
(2) If p denotes a switch, then p' denotes that switch which is open when p is closed and p' is closed when p is open.

If x denotes the state of the switch p, then x' represents the state of the switch p'. x is called the Boolean variable which is a binary variable.

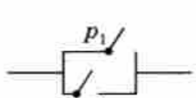
If $x = 1$ denotes the switch is closed or current flows, then $x = 0$ denotes that the switch is open or current stops.

(3) *Two switches p_1 and p_2 are either connected in series* (represented by $\wedge$) *or connected in parallel* (represented by $\vee$).

These are shown as follows :

(i) 

(p_1 & p_2 in series : $p_1 \wedge p_2$)

(ii) 

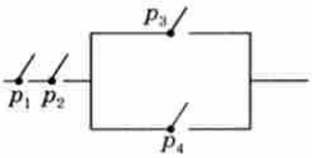
(p_1 & p_2 in parallel : $p_1 \vee p_2$)

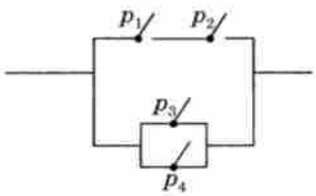
If B [0, 1] is non-empty set and $\wedge$, $\vee$, $/$ are the operations on B, then the system [{0, 1}, $\wedge$, $\vee$, 1] is usually called *Boolean switching algebra.*

(4) Simplification of circuits. The simplification of a circuit means the least complicated circuit with minimum cost and best results. This depends on the cost of the equipment, number of switches and the type of the material used. Thus the simplification of circuits implies the use of lesser number of switches which can be achieved by using different properties of Boolean algebra. In other words, *the simplification of switching circuits is equivalent to simplification of the corresponding Boolean function.*

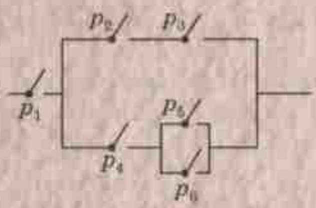
Example 37.42. *Draw the circuit which represents the Boolean function :*
(*i*) $(p_1 \wedge p_2) \wedge (p_3 \vee p_4)$ (*ii*) $(p_1 \wedge p_2) \vee (p_3 \vee p_4)$.

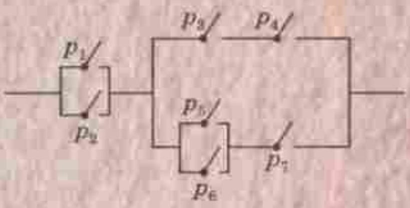
Solution. Here $p_1 \wedge p_2$ is a series circuit while $p_3 \vee p_4$ is a parallel circuit.
The required circuits are as follows :

(*i*) 

(*ii*) 

Example 37.43. *Write the Boolean functions representing the following circuits :*

(*i*) 

(*ii*) 

Also draw the circuit diagram which would be the complement of the circuit in (ii).

Solution. (*i*) The given circuit is represented by the Boolean function :

$$f = p_1 \wedge (p_2 \wedge p_3) \vee [p_4 \wedge (p_5 \vee p_6)]$$

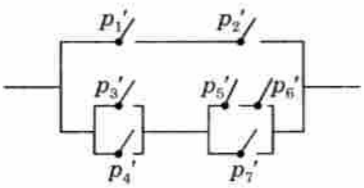
(*ii*) The Boolean function for the given circuit is

$$f = (p_1 \vee p_2) \wedge [(p_3 \wedge p_4) \vee [(p_5 \vee p_6) \wedge p_7]$$

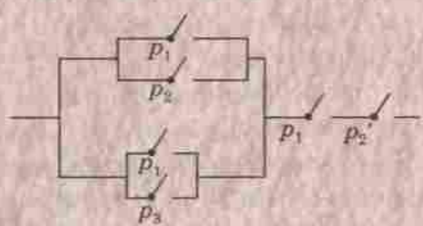
The complement of f *i.e.,*

$$\begin{aligned} f' &= (p_1 \vee p_2)' \vee [(p_3 \wedge p_4) \vee ((p_5 \vee p_6) \wedge p_7)]' \\ &= (p_1' \wedge p_2') \vee [(p_3 \wedge p_4)' \wedge ((p_5 \vee p_6) \wedge p_7)'] \\ &= (p_1' \wedge p_2') \vee [(p_3' \vee p_4') \wedge ((p_5' \wedge p_6') \vee p_7')] \end{aligned}$$

Its circuit diagram is as follows :



Example 37.44. *Simplify the following circuit and draw the diagram of the resulting circuit :*



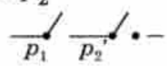
Solution. The given circuit is represented by the Boolean function f

$$= [(p_1 \vee p_2) \vee (p_1 \vee p_3)] \wedge (p_1 \wedge p_2') = (p_1 \vee p_2 \vee p_3) \wedge (p_1 \wedge p_2')$$

$$= (p_1 \wedge p_1 \wedge p_2') \vee (p_2 \wedge p_1 \wedge p_2') \vee (p_3 \wedge p_1 \wedge p_2') \quad \text{(By distributive law)}$$

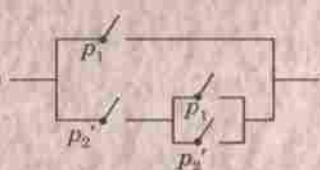
$$= (p_1 \wedge p_2') \vee (p_3 \wedge p_1 \wedge p_2') = p_1 \wedge [p_2' \vee (p_3 \wedge p_2')]$$

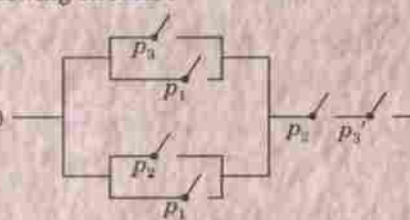
$$= p_1 \wedge p_2' \quad \text{[By absorption law]}$$

Its circuit diagram is 

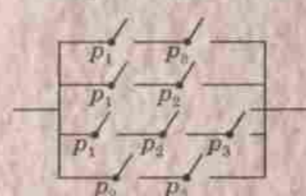
PROBLEMS 37.6

1. Draw the circuit diagram represented by the Boolean functions :
   (i) $[p_1 \wedge (p_1 \vee p_2)] \vee [p_2 \wedge (p_1' \vee p_2)]$ (ii) $p_1 \wedge [(p_2 \vee p_4') \vee (p_3' \wedge (p_1 \vee p_4 \vee p_3'))] \wedge p_2$.
2. Write the Boolean functions representing the following circuits :

(i) 

(ii) 

3. Simplify the Boolean functions, $p \vee (p' \wedge q) \vee (p \wedge q)$
4. Simplify the following circuit and draw the diagram of the resulting circuit :



5. Draw the simplified network of $f(x, y, z) = (x \vee y \vee z) \wedge (x \vee y' \vee z) \wedge (x' \vee y' \vee z)$. *(M.P.T.U., 2001)*
6. Consider the function $f(x_1, x_2, x_3) = [(x_1 \wedge x_2) \wedge (x_1 \wedge x_3)] \vee (x_1 \vee x_2')$
   (a) Simplify f algebraically. (b) Draw the switching circuit of f.
   (c) Also find the minterm normal form of f. *(Madras, 1998)*

IV. FUZZY SETS

37.22 FUZZY LOGIC

We have so far dealt with the fundamentals of classical logic. Besides this, we have *crisp logic* which deals with propositions that are required to be either true or false. There is however another type of logic which includes not only the crisp values but all the values between true (1) and false (0). But there is some degree of vagueness about the exact value between [0, 1]. *The logic to infer a definite outcome from such vague inputs is called* **fuzzy logic**.

(2) Fuzzy set. To provide a mathematical modelling to fuzzy logic, L.A. Zudeh introduced the concept of 'Fuzzy sets' in 1965 on the basis of a *membership function*. The theory of 'fuzzy sets' is now fully developed.

Def. *A fuzzy set F of a non-zero set X(x) is defined as* $F = \{x, \mu_F(x)\} : x \in X$.

Here $\mu_F : X \to [0, 1]$ is a function called the *membership function* of F and $\mu_F(x)$ is the degree of membership of $x \in X$ in F.

In particular $\mu(x) = 1$ implies full membership

$\mu(x) = 0$ implies non-membership

and $0 < \mu(x) < 1$ means intermediate membership.

A fuzzy set F is, therefore, a set of pairs consisting of a particular element of the universe X and its degree of membership *i.e.*, each x is assigned a value in the range (0, 1) indicating the extent to which x has the attribute F. It can also be represented as $F = \{[x_1, \mu_F(x_1)], [x_2, \mu_F(x_2)], ..., [x_n, \mu_F(x_n)]\}$.

For example, if x is the number of cars in a lane, 'small' may be taken as a particular value of the fuzzy variable x and to each x is assigned a number in the range (0, 1) then $\mu_{\text{small}}(x) \in (0, 1)$ is the *membership function.*

Example 37.45. *In a car-race, all the cars complete the race in four time-groups : shortest time, moderate time, long time and longest time.,If we note the time taken by each car in a group, it will give rise to a distribution of times. Now let us find the outcome of the race based on engine power, car speed and road conditions. Each of these variables may further be divided into :*

(i) low, medium and high for the variable engine power,

(ii) slow, moderate and fast for the variable car speed,

(iii) rough, bumpy and smooth for the variable road conditions.

Now we try to predict on some basis, in which of the four groups the car will finish, if it has low engine power, moderate speed and rough road.

Then the distribution for the engine power would correspond to the membership function for low, medium and high. Similarly the distribution for the speed would depend on the membership function for slow, moderate and fast, while the distribution for road conditions would depend on membership function for rough, bumpy and smooth.

37.23 FUZZY SET OPERATIONS

(1) A fuzzy set is said to be *normalised* when the largest element of the set (called *supremum*) is unity.

For instance, the set of members {5, 10 , 15, 20, 25} is normalised to {0.2, 0.4, 0.6, 0.8, 1} by dividing each member by 25, the supremum in the set.

The normalization of a fuzzy set F is expressed as $\sup_{x \in X} F(x) = 1$.

(2) Complement. *The complement of a fuzzy set F is the set F^c with degree of membership of an element in F^c equal to one minus degree of membership of this element in F. (Fig. 37.5)*

For example, if F = {0.4 Ram, 0.6 Sham, 0.8 Jyoti, 0.9 Ritu}
be a set of intelligent students, then F^c = {0.6 Ram, 0.4 Sham, 0.2 Jyoti, 0.1 Ritu}
is a set of non-intelligent students.

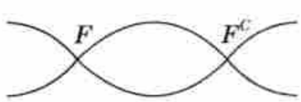


Fig. 37.5

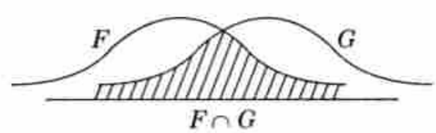


Fig. 37.6

(3) Intersection. *The intersection of two fuzzy sets F and G is the set $F \cap G$, where the degree of membership of an element in $F \cap G$ is the minimum of the degrees of membership of this element in F and G. (Fig. 37.6)*

(4) Union. *The union of two fuzzy sets F and G is the set $F \cup G$, where the degree of membership of an element in $F \cup G$, is the maximum of this element in F and G. (Fig. 37.7)*

For example, if

F = {0.5 Rani, 0.2 Suman, 0.4 Anita, 0.8 Sunita}

be a set of fat girls, and

G = {0.1 Rani, 0.6 Suman, 0.9 Anita, 0.5 Sunita}

be a set of tall girls, then

$F \cap G$ = {0.1 Rani, 0.2 Suman, 0.4 Anita, 0.5 Sunita}

and $F \cup G$ = {0.5 Rani, 0.6 Suman, 0.9 Anita, 0.8 Sunita}

(5) Equality. Two fuzzy sets F and G are said to be equal if and only if $F(x) = G(x)$ for all x in X.

(6) Subset. The fuzzy set F is said to be a subset of the fuzzy set G (*i.e.*, $F \subseteq G$) if and only if $F(x) \le G(x)$ for all $x \in X$.

(7) Double negation. If F is a fuzzy set, then $(F^c)^c = 1$.

(8) De Morgan's laws. If F and G are two fuzzy sets then

$$(F \cup G)^c = F^c \cap G^c \; ; (F \cap G)^c = F^c \cup G^c.$$

Example 37.46. *Let the membership functions for the fuzzy sets F and G be as in the following table :*

X	1	2	3	4	5	6	7	8	9	10
F	0	0	0.1	0.5	0.8	1	0.3	0.5	0	0
G	0	0	0	0	0.1	0.3	0.5	0.8	1	1
F^c	1	1	0.9	0.5	0.2	0	0.7	0.5	1	1

Then the corresponding

$$F \cap G = [0, 0.1, 0.3, 0.5]$$
$$F \cup G = [0.1, 0.5, 0.8, 1, 0.5, 0.8, 1, 1]$$

Clearly F is not a subset of G and G is not a subset of F.

37.24 TRUTH VALUE

(1) Truth value *of the negation of a proposition in fuzzy logic is one minus the truth value of the proposition.*

For example, if the truth value of the statement 'Ram is happy' is 0.8, then the truth value of 'Ram is not happy' is 0.2.

(2) Truth value *of the conjunction of two propositions in the fuzzy logic is the minimum of the truth values of the two propositions.*

(3) Truth value of the disconjunction of two propositions *in the fuzzy logic is the maximum of the truth values of two propositions.*

For example, if the truth value of 'Ram is fat' is 0.6, and the truth value of 'John is fat' is 0.9, then the truth value of the statement

(*a*) '*Ram and John are fat*' is 0.6.

(*b*) '*Ram or John is fat*' is 0.9.

(*c*) '*neither Ram nor John is fat*' is negation of minimum of negation of 'Ram is fat' (*i.e.*, 0.4) and negation of 'John is, fat' (*i.e.*, 0.1) = 0.1.

(*d*) '*Ram is not fat or John is not fat*' is maximum of 0.4 and 0.1 *i.e.*, 0.4.

37.25 ALGEBRAIC OPERATIONS ON FUZZY SETS

(1) Algebraic sum *of two fuzzy sets F and G is defined by the membership function*

$$\mu_{F+G}(x) = \mu_F(x) + \mu_G(x) - \mu_F(x)\,\mu_G(x)$$

and is written as $F + G$.

Algebraic product *of two fuzzy sets F and G is defined by the membership function.*

$$\mu_{F.G}(x) = \mu_F(x).\mu_G(x)$$

and is written as $F.G$.

(2) Properties of fuzzy set operations which are common to crisp set operations, are as under

1. *Idempotent* : $F \cup F = F$, $F \cap F = F$
2. *Identity* : $F \cup \phi = F$, $F \cap U = F$
3. *Commutative* : $F \cup G = G \cup F$, $F \cap G = G \cap F$
4. *Distributive* : $F \cap (G \cap H) = (F \cap G) \cup (F \cap H)$, $F \cup (G \cap H) = (F \cup G) \cap (F \cup H)$
5. *Associative* : $(F \cup G) \cup H = F \cup (G \cup H)$, $(F \cap G) \cap H = F \cap (G \cap H)$
6. *Absorption* : $F \cup (F \cap G) = F$, $F \cap (F \cup G) = F$.

37.26 GENERATION OF RULES FOR FUZZY PROBLEMS

We should know before hand all possible input-output relations while dealing with problems concerning fuzzy engines or fuzzy controls. These input-output rules are then expressed with 'if ... then' statements.

For instance, if F's and G's are inputs of fuzzy problems and R's are the actions taken for each rule, then the set of 'if ... then' rules with two input variables F_1 and G_1 and the actions taken are shown in table 1.

i.e., if F_1 and or G_1, then R_{11}, else
if F_2 and or G_1, then R_{21}, else
if F_1 and or G_2, then R_{12}, else
if F_2 and or G_2, then R_{22}.

Table 1

F_1	R_{11}	R_{12}
F_2	R_{21}	R_{22}
	G_1	G_2

In case, the fuzzy statements have more variables, then 'if ... then' rules becomes more complicated to tabulate.

However such a tabulation can be simplified by following a *decomposition process* as follows :

The decomposition process of three fuzzy variables F, G and H with actions R's taken is shown below :

Table 2

F_1	R_{11}	R_{12}
F_2	R_{21}	R_{22}
	G_1	G_2

Table 3

H_1	R_{111}	R_{121}	R_{211}	R_{221}
H_2	R_{112}	R_{122}	R_{212}	R_{222}
	R_{11}	R_{12}	R_{21}	R_{22}

Here the statements F_2 and G_1 and H_1 then R_{211} is decomposed into 'if F_2 and G_1 then R_{21}' and 'if R_{21} and H_1, then R_{211}.

Similarly R_{21} and H_2 then R_{212}.

This decomposition process can easily be extended to any number of input variables.

37.27 FUZZY PROPOSITIONS

(1) *A fuzzy number is a fuzzy set $R \to [0, 1]$.* We can easily extend classical two-valued logic to three-valued logic. Fuzzy logic, however is an extension of multi-valued logic. It provides foundations for approximate reasoning with imprecise fuzzy propositions using fuzzy set theory.

The classical propositions are statements which are either true or false. In fuzzy logic, the truth or falsity of fuzzy propositions is assigned different degrees *i.e.*, the truth and falsity are expressed by numbers in [0, 1].

A variable whose values are 'words' or 'sentences' is called a *linguistic variable*. For example 'height' is a linguistic variable and its values are tall, very tall, quite tall, not tall, short, not very short, not quite tall etc.

(2) Classification of fuzzy propositions. The classification propositions are statements which are either true or false. In fuzzy logic, the truth or falsity of fuzzy propositions is assigned different degrees *i.e.*, the truth and falsity are expressed by numbers in [0, 1].

The fuzzy propositions of simple nature can be classified into the following four types. In each case, we introduce the relevant canonical form and then discuss its interpretation.

Type I. *Unconditional and unqualified propositions.*

The standard canonical form of this type of proposition is expressed as $p : u$ if F ...(1)

Here u is the variable that takes value u from some universal set U and F is a fuzzy set on U which represents a fuzzy predicate such as young, tall, low, high etc. Given a particular value u (say v), this value belongs to F with membership grade $F(v)$. This membership grade is then interpreted as the degree of truth $T(p)$ of proposition p

i.e., $$T(p) = F(v) \qquad \text{...(2)}$$

Here T is a fuzzy set on [0, 1] which assigns the membership grade $F(v)$ to each value v of u.

In some fuzzy propositions, values of variable u in (1) are assigned to individuals in a given set I *i.e.*, variable u becomes a function $u : I \to u$ where $u(i)$ is the value of v for individual i in U. Accordingly the canonical form (2) is modified to the form

$$p : u\ (i) \text{ is } F \text{ where } i \in I \qquad \text{...(3)}$$

Example 37.47. *Consider a set I of persons, each person is characterized by his 'age' and a fuzzy set expressing the predicate 'young' is given. Denoting our variable by 'age' and fuzzy set by 'young', the canonical form is*

$$p : \text{age } (i) \text{ is young.}$$

Solution. The degree of truth of this proposition $T(p)$ is then determined for each person i in I by means of the equation $T(p)$ = young [age (i)].

Example 37.48. *At a particular place on the earth, consider the air temperature u (in °C). Let Fig. 37.8 represent the membership function as predicate 'high'. Assuming that all relevant temperature readings are given, the corresponding fuzzy proposition is expressed as,*

$$p : \text{temp } (u) \text{ is high } (°C)$$

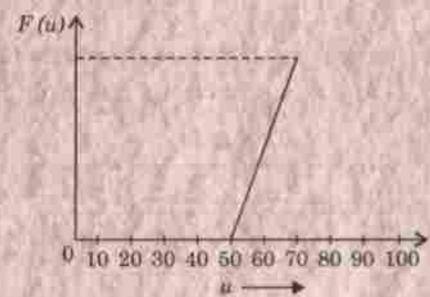


Fig. 37.8

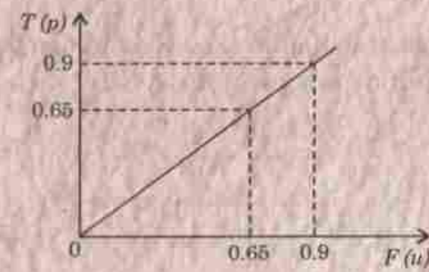


Fig. 37.9

Solution. The degree of truth $T(p)$ depends upon the actual value of the temperature and on the nature of predicate 'high' which is defined by the membership function T in Fig. 37.9.

e.g., if $u = 75$ then $F(75) = 0.65$ and $T(p) = 0.65$.

Type II. *Conditional and unqualified propositions.*

A proposition p of this type is expressed by the canonical form

$$p : \text{If } x \text{ is } F \text{ then } y \text{ is } G \qquad ...(4)$$

where x, y are variables whose values are in the sets X, Y and F, G are fuzzy sets on X and Y respectively. These propositions may also be viewed as propositions of the form

$$\{x, y\} \text{ is } R \qquad ...(5)$$

where R is a fuzzy set on $X \times Y$ which is determined for each $x \in X$ and each $y \in Y$ by the formula

$$R(x, y) = B[F(x), G(y)], \qquad ...(6)$$

where B is a binary operation as [0, 1] representing a suitable *fuzzy implication.*

Type III. *Unconditional and qualified propositions*

A proposition of this type is expressed by either of the following canonical forms :

$$p : u \text{ is } F \text{ is } S \qquad ...(7)$$

or

$$p : \text{Prob } (U \text{ is } F) \text{ is } P \qquad ...(8)$$

where u is a variable that takes value v from some universal set U and F is a fuzzy set on U which represents a fuzzy predicate such as small, young, daughter etc.

Prob. (U is F) is the probability of fuzzy set event 'u is F' ; S is the fuzzy truth qualifier and P is the fuzzy probability qualifier. S and P are both represented by fuzzy set on [0, 1].

Example 37.49. *Mary is 'young' is 'very true' where the predicate 'young' and the truth qualifier 'very true' are represented by the respective fuzzy sets shown in Fig. 37.10.*

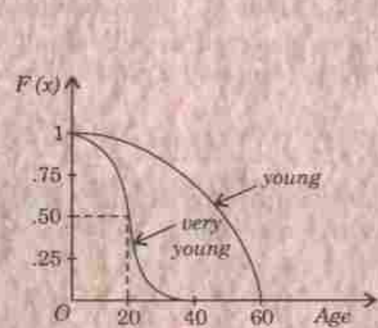


Fig. 37.10

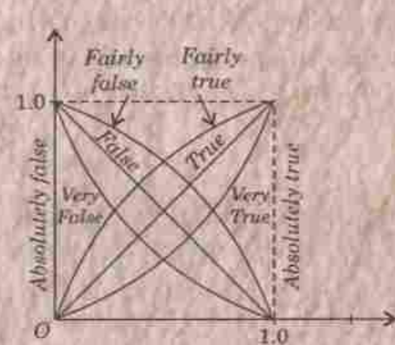


Fig. 37.11

Solution. The degree of truth $T(p)$ of any truth-qualified proposition p is given for each $u \in U$ by $T(p) = S\{F(u)\}$.

Assuming that Mary's age is 20, she belongs to the set representing the predicate 'very young' with membership grade 0.50, our proposition belongs to the set of propositions which are 'very true' with membership grade 0.50 as shown in Fig. 37.11. This implies that the degree of truth of our truth qualified proposition is 0.50.

If the proposition be modified by changing the predicate to 'young' or the truth qualifier to 'fairly true', we would obtain the corresponding degree of truth of such propositions by the same method.

Type IV. *Conditional and qualified propositions*

This type of propositions can either be expressed in the canonical form

$$p : \text{If } x \text{ is } F, \text{ then } y \text{ is } G \text{ in } S$$

or

$$p : \text{Prob } \{x \text{ is } F/y \text{ is } G\} \text{ in } P,$$

where Prob $\{x$ is F/y is $G\}$ is conditional probability.

37.28 APPLICATIONS OF FUZZY SETS

The concept of fuzzy sets has already influenced all engineering disciplines to various degrees.

Electrical engineering is the first such discipline where the utility of fuzzy logic and fuzzy sets has been recognized by developing controllers. Electronic circuits for fuzzy image processing have also been developed.

Some ideas regarding the application of fuzzy sets *in civil engineering* emerged around 1970. In the construction of bridges, dams, buildings etc. a designer has to take into account the safety factor for which the fuzzy theory has an effective role to play. Fuzzy set theory has also proved quite useful for assessing the life of existing constructions.

In *mechanical engineering* design problems, the utility of Fuzzy set theory was realised during mid 1980's. The membership function is expressed in terms of thermal expansion or corrosion or cost of different materials etc.

When the utility of fuzzy controllers was increasingly felt around mid 1980's, the need for computer hardware to implement the various operations involving fuzzy logic, had been recognized. In digital mode, fuzzy sets have been expressed as vectors of (0, 1) members.

Fuzzy control and fuzzy decision making are two well-developed areas of fuzzy set theory. These are directly relevant to *industrial engineering problems.* The utility of fuzzy sets has also been recognized for estimating the service life of given equipment under various conditions.

Modern Reliability theory has also been developed on the assumption of fuzzy sets. At any given time, an engineering product may be in functioning state to some degree or in failed state to another degree. The behaviour of an engineering product with respect to its functioning state and failed state has been characterized as based on fuzzy set theory.

The use of fuzzy set theory in *Robotics* includes approximate reasoning, fuzzy controllers, fuzzy pattern recognition and fuzzy data bases.

PROBLEMS 37.7

1. Given fuzzy sets $F_1 = \{0.6 \text{ Sonu}, 0.9 \text{ Renu}, 0.7 \text{ Paul}, 0.3 \text{ Sham}\}$
$F_2 = \{0.3 \text{ Sham}, 0.8 \text{ Paul}, 0.9 \text{ Renu}, 0.5 \text{ Sonu}\}$
and $F_3 = \{0.8 \text{ Paul}, 0.3 \text{ Sham}, 0.5 \text{ Sonu}, 0.9 \text{ Renu}\}$
Which of the above two sets are equal ?
2. Write the complement set of the fuzzy set F, if $F = \{0.8 \text{ Ram}, 0.3 \text{ Sham}, 0.6 \text{ John}, 0.7 \text{ Charu}\}$.
3. If $F = \{0.3x_1, 0.7x_2, 0.5x_3, 0.8x_4\}$ and $G = \{0.4x_1, 0.6x_2, 0.1x_3, 0.9x_4\}$ been two fuzzy sets, then write down $F \cup G$ and $F \cap G$.
4. State the truth values of the negation of the following propositions :
(*i*) Truth value of 'F is rich' is 0.8
(*ii*) Truth value of 'G is fat' is 0.6
(*iii*) Truth value of 'Mary is beautiful' is 0.7.
5. Let the membership functions of fuzzy sets F and G be as follows :
$X : \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$F : [0, 0, 0, 0, 0.1, 0.3, 0.5, 0.9, 1, 1]$

$G : [0, 0, 0.1, 0.5, 0.9, 1, 0.9, 0.5, 0.0]$

State whether (*i*) $F = G$ (*ii*) F is a subset of G.

Also write down F', $F \cap G$ and $F \cup G$.

6. The truth values of the statements

'Latif is a good player' is 0.7

and 'John is a good player' is 0.6.

What is the truth value of

(*i*) the conjunction of the above two prepositions.

(*ii*) the disjunction of the above prepositions.

7. Define a Fuzzy set and the standard operations on Fuzzy sets. (*Bhopal, 2009*)
8. State the constituents of the pair in a fuzzy set.
9. Write a note on 'Fuzzy logic affects many disciplines' ? (*Bhopal, 2001*)