CHAPTER 12

Applications of Differential Equations of First Order

1. Introduction. 2. Geometric applications. 3. Orthogonal trajectories. 4. Physical applications. 5. Simple electric circuits. 6. Newton's law of cooling. 7. Heat flow. 8. Rate of decay of radio-active materials. 9. Chemical reactions and solutions. 10. Objective Type of Questions.

12.1 INTRODUCTION

In this chapter, we shall consider only such practical problems which give rise to differential equations of the first order. The fundamental principles required for the formation of such differential equations are given in each case and are followed by illustrative examples.

12.2 GEOMETRIC APPLICATIONS

(*a*) *Cartesian coordinates.* Let $P(x, y)$ be any point on the curve $f(x, y) = 0$ (Fig. 12.1), then [as per 4.6 §(1) & 4.11(1) & (4)], we have

(*i*) slope of the tangent at $P(= \tan \psi) = dy/dx$

(*ii*) equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x)$$

so that its x-intercept ($= OT$)

$$= x - y \cdot dx/dy$$

and y-intercept ($= OT'$) $= y - x \cdot dy/dx$

(*iii*) equation of the normal at P is $Y - y = -\dfrac{dx}{dy}(X - x)$

(*iv*) length of the tangent ($= PT$) $= y\sqrt{[1 + (dx/dy)^2]}$

(*v*) length of the normal ($= PN$) $= y\sqrt{[1 + (dy/dx)^2]}$

(*vi*) length of the sub-tangent ($= TM$) $= y \cdot dx/dy$

(*vii*) length of the sub-normal ($= MN$) $= y \cdot dy/dx$

(*viii*) $\dfrac{ds}{dx} = [1 + (dy/dx)^2]$; $\dfrac{ds}{dy} = \sqrt{[1 + (dx/dy)^2]}$

(*ix*) differential of the area $= ydx$ or xdy

(*x*) ρ, radius of curvature at $P = \dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}$

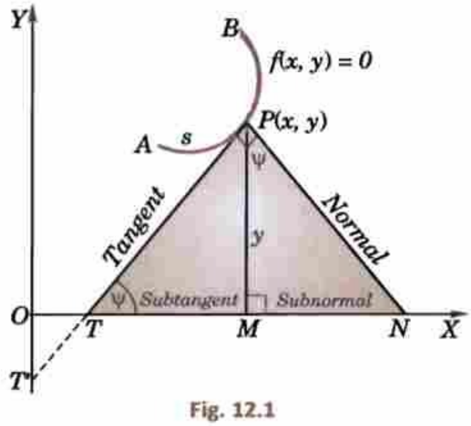

Fig. 12.1

(*b*) *Polar coordinates*. Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$ (Fig. 12.2), then [as per § 4.7, 4.9 (2) & 4.11 (4)], we have

(*i*) $\psi = \theta + \phi$

(*ii*) $\tan\phi = r d\theta/dr$, $p = r \sin\phi$

(*iii*) $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

(*iv*) polar sub-tangent $(= OT) = r^2 d\theta/dr$

(*v*) polar sub-normal $(ON) = dr/d\theta$

(*vi*) $\dfrac{ds}{dr} = \sqrt{\left[1 + \left(r\dfrac{d\theta}{dr}\right)^2\right]}$; $\dfrac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]}$

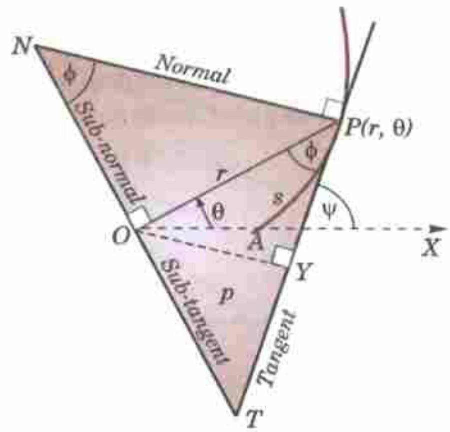

Fig. 12.2

Example 12.1. *Show that the curve in which the portion of the tangent included between the co-ordinates axes is bisected at the point of contact is a rectangular hyperbola.*

Solution. Let the tangent at any point $P(x, y)$ of a curve cut the axes at T and T' (Fig. 12.3).

We know that its x-intercept $(= OT) = x - y \,.\, dx/dy$

and $\qquad$ y-intercept $(= OT') = y - x \,.\, dy/dx$

$\therefore$ the co-ordinates of T and T' are

$$(x - y \,.\, dx/dy, 0), (0, y - x\, dy/dx)$$

Since P is the mid-point of TT'

$$\therefore \qquad \frac{[x - y.dx/dy] + 0}{2} = x$$

or $\qquad x - y \,.\, dx/dy = 2x$ or $x\,dy + y\,dx = 0$

or $\qquad d(xy) = 0$ Integrating, $xy = c$

which is the equation of a rectangular hyperbola, having x and y axes as its asymptotes.

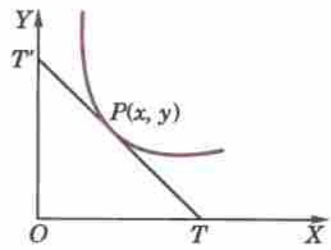

Fig. 12.3

Example 12.2. *Find the curve for which the normal makes equal angles with the radius vector and the initial line.*

Solution. Let PT and PN be the tangent and normal at $P(r, \theta)$ of the curve so that

$$\tan\phi = r\, d\theta/dr$$

By the condition of the problem,

$$\angle OPN = 90° - \phi = \angle ONP \text{ (Fig. 12.4).}$$

$$\therefore \qquad \theta = \angle PON = 180° - (180° - 2\phi) = 2\phi$$

or $\qquad \theta/2 = \phi \quad \therefore \quad \tan\dfrac{\theta}{2} = \tan\phi = r\dfrac{d\theta}{dr}.$

Here the variables are separable.

$$\therefore \qquad \frac{dr}{r} = \frac{\cos\theta/2}{\sin\theta/2}\, d\theta$$

Integrating both sides $\log r = 2 \log \sin\theta/2 + \log c$

or $\qquad r = c \sin^2\theta/2 = \dfrac{1}{2}\, c(1 - \cos\theta)$

Thus the curve is the cardioid $r = a(1 - \cos\theta)$.

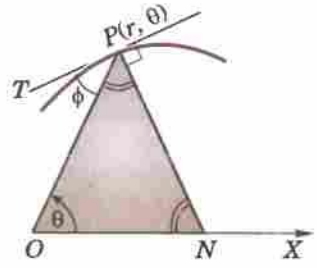

Fig. 12.4

Example 12.3. *Find the shape of a reflector such that light coming from a fixed source is reflected in parallel rays.*

Solution. Taking the fixed source of light as the origin and the X-axis parallel to the reflected rays; the reflector will be a surface generated by the revolution of a curve $f(x, y) = 0$ about X-axis (Fig. 12.5).

In the XY-plane, let PP' be the reflected ray, where P is the point (x, y) on the curve $f(x, y) = 0$.

If TPT' be the tangent at P, then

$\because$ angle of incidence = angle of reflection,

$$\therefore \quad \phi = \angle OPT = \angle P'PT' = \angle OTP = \psi$$

i.e.,
$$p = \frac{dy}{dx} = \tan \angle XOP = \tan 2\phi$$
$$= \frac{2 \tan \phi}{1 - \tan^2 \phi} = \frac{2p}{1 - p^2}$$

or
$$2x = \frac{y}{p} - yp \text{ which is solvable for } x \quad \ldots(i)$$

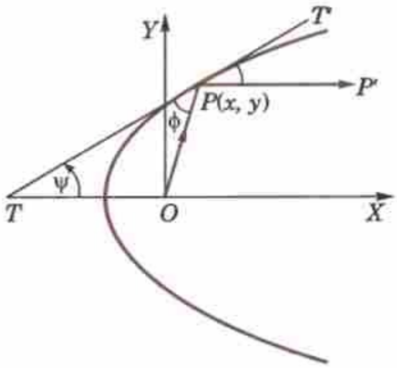

Fig. 12.5

$\therefore$ differentiating (i) w.r.t. y, $\dfrac{2}{p} = \dfrac{1}{p} - \dfrac{y}{p^2}\dfrac{dp}{dy} - p - y\dfrac{dp}{dy}$

i.e.,
$$\left(\frac{1}{p} + p\right) + \left(\frac{1}{p^2} + 1\right) y \frac{dp}{dy} = 0 \quad \text{or} \quad \left(\frac{1}{p} + p\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

This gives $\quad dp/p = - dy/y$

Integrating, $\quad \log p = \log c - \log y, \quad$ i.e., $p = c/y \quad \ldots(ii)$

Thus eliminating p from (i) and (ii), we have family of curves $y^2 = 2cx + c^2$.

Hence the reflector is a member of the family of paraboloids of revolution $y^2 + z^2 = 2cx + c^2$.

PROBLEMS 12.1

1. Find the equation of the curve which passes through
   (i) the point $(3, -4)$ and has the slope $2y/x$ at the point (x, y) on it.
   (ii) the origin and has the slope $x + 3y - 1$.
2. At every point on a curve the slope is the sum of the abscissa and the product of the ordinate and the abscissa, and the curve passes through $(0, 1)$. Find the equation of the curve.
3. A curve is such that the length of the perpendicular from origin on the tangent at any point P of the curve is equal to the abscissa of P. Prove that the differential equation of the curve is
$$y^2 - 2xy\frac{dy}{dx} - x^2 = 0, \text{ and hence find the curve.}$$
4. A plane curve has the property that the tangents from any point on the y-axis to the curve are of constant length a. Find the differential equation of the family to which the curve belongs and hence obtain the curve.
5. Determine the curve whose sub-tangent is twice the abscissa of the point of contact and passes through the point $(1, 2)$. (*Sambalpur, 1998*)
6. Determine the curve in which the length of the sub-normal is proportional to the square of the ordinate.
7. The tangent at any point of a certain curve forms with the coordinate axes a triangle of constant area A. Find the equation to the curve.
8. Find the curve which passes through the origin and is such that the area included between the curve, the ordinate and the x-axis is twice the cube of that ordinate.
9. Find the curve whose (i) polar sub-tangent is constant.
   (ii) polar sub-normal is proportional to the sine of the vectorial angle.
10. Determine the curve for which the angle between the tangent and the radius vector is twice the vectorial angle. (*Kanpur, 1996*)
11. Find the curve for which the tangent at any point P on it bisects the angle between the ordinate at P and the line joining P to the origin.
12. Find the curve for which the tangent, the radius vector r and the perpendicular from the origin on the tangent form a triangle of area kr^2.

12.3 (1) ORTHOGONAL TRAJECTORIES

Two families of curves such that every member of either family cuts each member of the other family at right angles are called **orthogonal trajectories** *of each other (Fig. 12.6).*

The concept of the orthogonal trajectories is of wide use in applied mathematics especially in field problems. For instance, in an electric field, the paths along which the current flows are the orthogonal trajectories of the equipotential curves and *vice versa*. In fluid flow, the stream lines and the equipotential lines (lines of constant velocity potential) are orthogonal trajectories. Likewise, the lines of heat flow for a body are perpendicular to the isothermal curves. The problem of finding the orthogonal trajectories of a given family of curves depends on the solution of the first order differential equations.

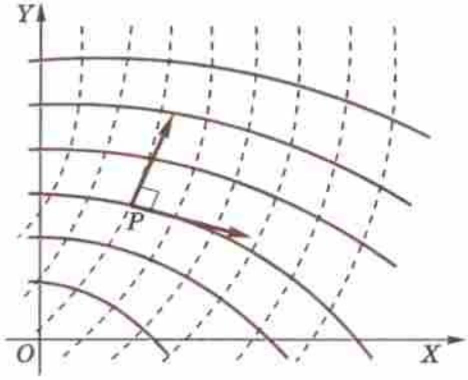

Fig. 12.6

(2) To find the orthogonal trajectories of the family of curves F(x, y, c) = 0.

(*i*) *Form its differential equation in the form* $f(x, y, dy/dx) = 0$ *by eliminating c.*

(*ii*) *Replace, in this differential equation,* dy/dx *by* $-dx/dy$, (so that the product of their slopes at each point of intersection is -1).

(*iii*) *Solve the differential equation of the orthogonal trajectories i.e.,* $f(x, y, -dx/dy) = 0$.

Example 12.4. *If the stream lines (paths of fluid particles) of a flow around a corner are xy = constant find their orthogonal trajectories (called equipotential lines-§ 20.6)* (*Marathwada, 2008*)

Solution. Taking the axes as the walls, the stream lines of the flow around the corner of the walls is

$$xy = c \qquad ...(i)$$

Defferentiating, we get, $$x\frac{dy}{dx} + y = 0 \qquad ...(ii)$$

as the differential equation of the given family (*i*).

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (*ii*), we obtain $x\left(-\frac{dx}{dy}\right) + y = 0$

or $$xdx - ydy = 0 \qquad ...(iii)$$

as the differential equation of the orthogonal trajectories.

Integrating (*iii*), we get $x^2 - y^2 = c'$ as the required orthogonal trajectories of (*i*) *i.e.*, the *equipotential lines*, shown dotted in Fig. 12.7.

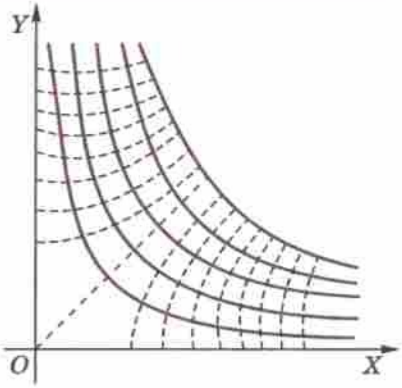

Fig. 12.7

Example 12.5. *Find the orthogonal trajectories of the family of confocal conics* $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$, *where λ is the parameter.* (V.T.U., 2009 S)

Solution. Differentiating the given equation, we get $\frac{2x}{a^2} + \frac{2y}{a^2+\lambda}\frac{dy}{dx} = 0$

or $$\frac{y}{a^2+\lambda} = -\frac{x}{a^2\,(dy/dx)} \quad \text{or} \quad \frac{y^2}{a^2+\lambda} = \frac{-xy}{a^2\,(dy/dx)}$$

Substituting this in the given equation, we get

$$\frac{x^2}{a^2} - \frac{xy}{a^2\,(dy/dx)} = 1 \quad \text{or} \quad (x^2 - a^2)\frac{dy}{dx} = xy \qquad ...(i)$$

which is the differential equation of the given family.

Changing dy/dx to $-dx/dy$ in (*i*), we get $(a^2 - x^2)\,dx/dy = xy$ as the differential equation of the orthogonal trajectories.

Separating the variables and integrating, we obtain

$$\int y\,dy = \int \frac{a^2 - x^2}{x}\,dx + c \quad \text{or} \quad \frac{1}{2}y^2 = a^2\log x - \frac{1}{2}x^2 + c$$

or $$x^2 + y^2 = 2a^2\log x + c' \qquad [c' = 2c]$$

which is the equation of the required orthogonal trajectories.

Example 12.6. *Find the orthogonal trajectories of a system of confocal and coaxial parabolas.*

Solution. The equation of the family of confocal parabolas having x-axis as their axis, is of the form

$$y^2 = 4a(x + a) \quad \text{...(i)}$$

Differentiating, $$y\frac{dy}{dx} = 2a \quad \text{...(ii)}$$

Substituting the value of a from (*ii*) in (*i*), we get $y^2 = 2y\dfrac{dy}{dx}\left(x + \dfrac{1}{2}y\dfrac{dy}{dx}\right)$

i.e., $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \text{ as the differential equation of the family.} \quad \text{...(iii)}$$

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in (*iii*), we obtain $y\left(\dfrac{dx}{dy}\right)^2 - 2x\dfrac{dx}{dy} - y = 0$

or $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \text{ which is the same as (iii).}$$

Thus we see that a system of confocal and coaxial parabolas is *self-orthogonal, i.e.*, each member of the family (*i*) cuts every other member of the same family orthogonally.

(3) To find the orthogonal trajectories of the curves F(r, θ, c) = 0.

(*i*) *Form its differential equation in the form $f(r, \theta, dr/d\theta) = 0$ by eliminating c.*

(*ii*) *Replace in this differential equation,*

$$\frac{dr}{d\theta} \text{ by } -r^2\frac{d\theta}{dr}$$

[∵ for the given curve through $P(r, \theta)$ $\tan\phi = r\,d\theta/dr$ and for the orthogonal trajectory through P

$$\tan\phi' = \tan(90° + \phi) = -\cot\phi = -\frac{1}{r}\frac{dr}{d\theta}$$

Thus for getting the differential equation of the orthogonal trajectory

$$r\frac{d\theta}{dr} \text{ is to be replaced by } -\frac{1}{r}\frac{dr}{d\theta}$$

or $$\frac{dr}{d\theta} \text{ is to be replaced by } -r^2\frac{d\theta}{dr}.]$$

(*iii*) *Solve the differential equation of the orthogonal trajectories*

i.e., $$f(r, \theta, -r^2 d\theta/dr) = 0.$$

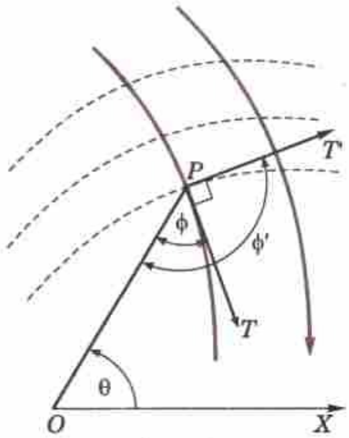

Fig. 12.8

Example 12.7. *Find the orthogonal trajectory of the cardioids $r = a(1 - \cos\theta)$.* (*Kurukshetra, 2005*)

Solution. Differentiating $$r = a(1 - \cos\theta). \quad \text{...(i)}$$

with respect to θ, we get $$\frac{dr}{d\theta} = a\sin\theta \quad \text{...(ii)}$$

Eliminating a from (*i*) and (*ii*), we obtain

$$\frac{dr}{d\theta}\cdot\frac{1}{r} = \frac{\sin\theta}{1 - \cos\theta} = \cot\frac{\theta}{2} \text{ which is the differential equation of the given family.}$$

Replacing $dr/d\theta$ by $-r^2\,d\theta/dr$, we obtain

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot\frac{\theta}{2} \quad \text{or} \quad \frac{dr}{r} + \tan\frac{\theta}{2}\,d\theta = 0$$

as the differential equation of orthogonal trajectories. It can be rewritten as

$$\frac{dr}{r} = -\frac{(\sin\theta/2)d\theta}{\cos\theta/2}$$

Integrating, $\log r = 2 \log \cos \theta/2 + \log c$

or $$r = c \cos^2 \theta/2 = \frac{1}{2} c(1 + \cos \theta) \quad \text{or} \quad r = a'(1 + \cos \theta)$$

which is the required orthogonal trajectory.

Example 12.8. *Find the orthogonal trajectory of the family of curves $r^n = a \sin n\theta$.* (V.T.U., 2006)

Solution. We have $n \log r = \log a + \log \sin n\theta$.

Differentiating w.r.t. θ, we have

$$\frac{n}{r} \frac{dr}{d\theta} = \frac{n \cos n\theta}{\sin n\theta} \quad \text{or} \quad \frac{1}{r} \frac{dr}{d\theta} = \cot n\theta$$

Replacing $dr/d\theta$ by $- r^2\, d\theta/dr$, we obtain

$$\frac{1}{r}\left(-r^2 \frac{d\theta}{dr}\right) = \cot n\theta \quad \text{or} \quad \tan n\theta.\ d\theta - \frac{dr}{r} = 0$$

Integrating, $$\int \frac{dr}{r} + \int \frac{\sin n\theta}{\cos n\theta}\, d\theta = c,$$

i.e., $$\log r - \frac{1}{n} \log \cos n\theta = c \quad \text{or} \quad \log (r^n/\cos n\, \theta) = nc = \log b. \text{ (say)}$$

or $r^n = b \cos n\theta$, which is the required orthogonal trajectory.

PROBLEMS 12.2

Find the orthogonal trajectories of the family of :

1. Parabolas $y^2 = 4ax$. (*Marathwada, 2009*)
2. Parabolas $y = ax^2$. (*J.N.T.U., 2006*)
3. Semi-cubical parabolas $ay^2 = x^3$. (*J.N.T.U., 2005*)
4. Coaxial circles $x^2 + y^2 + 2\lambda x + c = 2$, λ being the parameter. (*J.N.T.U., 2006*)
5. Confocal conics $\dfrac{x^2}{a^2 + \lambda} + \dfrac{y^2}{b^2 + \lambda} = 1$, λ being the parameter. (*Kurukshetra, 2006*)
6. Cardioids $r = a\,(1 + \cos \theta)$. (*J.N.T.U., 2003*)
7. $r = 2a\,(\cos \theta + \sin \theta)$ (*V.T.U., 2010 S*)
8. Confocal and coaxial parabolas $r = 2a/(1 + \cos \theta)$. (*Nagpur, 2008*)
9. Curves $r^2 = a^2 \cos 2\theta$. (*V.T.U., 2009 S*)
10. $r^n \cos n\theta = a^n$. (*V.T.U. 2011*)
11. Show that the family of parabolas $x^2 = 4a\,(y + a)$ is self orthogonal. (*Kerala, 2005*)
12. Show that the family of curves $r^n = a \sec n\theta$ and $r^n = b\, \mathrm{cosec}\, n\theta$ are orthogonal. (*Mumbai, 2005*)
13. The *electric lines of force* of two opposite charges of the same strength at ($\pm$ 1, 0) are circles (through these points) of the form $x^2 + y^2 - ay = 1$. Find their *equipotential lines* (orthogonal trajectories).

    [**Isogonal trajectories.** *Two families of curves such that every member of either family cuts each member of the other family at a constant angle α (Say), are called* ***isogonal trajectories*** *of each other.* The slopes m, m' of the tangents to the corresponding curves at each point, are connected by the relation $\dfrac{m \square m'}{1 + mm'} = \tan \alpha = \text{const.}$]
14. Find the isogonal trajectories of the family of circles $x^2 + y^2 = a^2$ which intersect at 45°.

12.4 PHYSICAL APPLICATIONS

(1) Let a body of mass m start moving from O along the straight line OX under the action of a force F. After any time t, let it be moving at P where $OP = x$, then

(*i*) its velocity $(v) = \dfrac{dx}{dt}$

(*ii*) its acceleration $(a) = \dfrac{dv}{dt}$ or $\dfrac{vdv}{dx}$ or $\dfrac{d^2x}{dt^2}$

If, however, the body be moving along a curve, then

(*i*) its velocity $(v) = ds/dt$ and

(*ii*) its acceleration $(a) = \dfrac{dv}{dt}, v\dfrac{dv}{ds}$ or $\dfrac{d^2s}{dt^2}$.

The quantity mv is called the *momentum*.

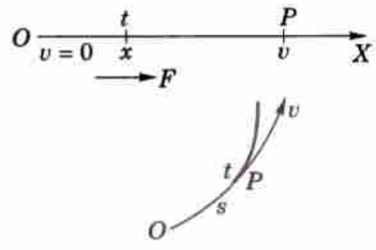

Fig. 12.9

(2) Newton's second law *states that* $F = \dfrac{d}{dt}(mv)$.

If m is constant, then $F = m\dfrac{dv}{dt} = ma$, *i.e., net force = mass × acceleration.*

(3) Hooke's law* *states that tension of an elastic string (or a spring) is proportional to extension of the string (or the spring) beyond its natural length.*

Thus $\qquad$ $\mathbf{T = \lambda e/1,}$

where e is the extension beyond the natural length l and λ is the *modulus of elasticity.*

Sometimes for a spring, we write $\mathbf{T = ke,}$

where e is the extension beyond the natural length and k is the *stiffness of the spring.*

(4) Systems of units

I. **F.P.S.** [foot (*ft.*) pound (*lb.*), second (*sec.*)] **system**. If mass m is in *pounds* and acceleration (a) is in *ft/sec*^{2}, then the force $F(= ma)$ is in *poundals*.

II. **C.G.S.** [centimetre (*cm.*), gram (*g*), second (*sec*)] **system**. If mass m is in *grams* and acceleration a is in *cm/sec*^{2} then the force $F(= ma)$ is *dynes*.

III. **M.K.S.** [metre (*m*), kilogram (*kg.*), second (*sec*)] **system**. If mass m is in kilograms and acceleration a in *m/sec*^{2}, then the force $F(= ma)$ is in *newtons* (*nt*).

These are called *absolute units*. If g is the acceleration due to gravity and w is the weight of the body, then w/g is the mass of the body in *gravitational units.*

$$g = 32 \text{ ft/sec}^2 = 980 \text{ cm/sec}^2 = 9.8 \text{ m/sec}^2 \text{ approx.}$$

Example 12.9. Motion of a boat across a stream. *A boat is rowed with a velocity u directly across a stream of width a. If the velocity of the current is directly proportional to the product of the distances from the two banks, find the path of the boat and the distance down stream to the point where it lands.*

Solution. Taking the origin at the point from where the boat starts, let the axes be chosen as in Fig. 12.10.

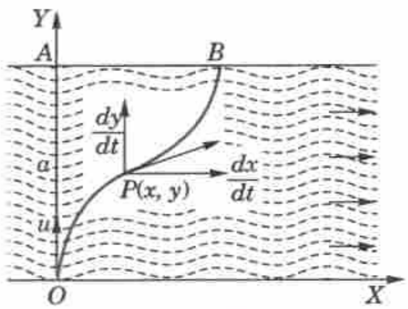

Fig. 12.10

At any time t after its start from O, let the boat be at $P(x, y)$, so that

dx/dt = velocity of the current $= ky(a - y)$

dy/dt = velocity with which the boat is being rowed $= u$.

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{u}{ky(a-y)} \qquad \text{...}(i)$$

This gives the direction of the resultant velocity of the boat which is also the direction of the tangent to the path of the boat.

Now (*i*) is of variables separable form and we can write it as

$$y(a - y)dy = \frac{u}{k}\, dx$$

Integrating, we get $\qquad \dfrac{ay^2}{2} - \dfrac{y^3}{3} = \dfrac{u}{k}x + c$

Since $y = 0$ $\quad$ when $\quad x = 0, \quad \therefore \quad c = 0.$

Hence the equation to the path of the boat is $x = \dfrac{k}{6u}y^2(3a - 2y)$

Putting $y = a$, we get the distance AB, down stream where the boat lands $= ka^3/6u$.

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*Named after an English physicist *Robert Hooke* (1635–1703) who had discovered the law of gravitation earlier than Newton.

Example 12.10. Resisted motion. *A moving body is opposed by a force per unit mass of value cx and resistance per unit of mass of value bv^2 where x and v are the displacement and velocity of the particle at that instant. Find the velocity of the particle in terms of x, if it starts from rest.* (*Marathwada, 2008*)

Solution. By Newton's second law, the equation of motion of the body is $v\dfrac{dv}{dx} = -cx - bv^2$

or $$v\frac{dv}{dx} + bv^2 = -cx \qquad ...(i)$$

This is Bernoulli's equation. $\therefore$ Put $v^2 = z$ and $2v\, dv/dx = dz/dx$, so that (i) becomes

$$\frac{dz}{dx} + 2bz = -2cx \qquad ...(ii)$$

This is Leibnitz's linear equation and I.F. $= e^{2bx}$.

$\therefore$ the solution of (ii) is $\quad ze^{2bx} = -\int 2cxe^{2bx}\,dx + c'$ [Integrate by parts]

$$= -2c\left[x\cdot\frac{e^{2bx}}{2b} - \int 1\cdot\frac{e^{2bx}}{2b}\,dx\right] + c' = -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2bx} + c'$$

or $$v^2 = \frac{c}{2b^2} + c'e^{-2bx} - \frac{cx}{b} \qquad ...(iii)$$

Initially $v = 0$ when $x = 0$ $\therefore$ $0 = c/2b^2 + c'$.

Thus, substituting $c' = -c/2b^2$ in (iii), we get $v^2 = \dfrac{c}{2b^2}(1 - e^{-2bx}) - \dfrac{cx}{b}$.

Example 12.11. Resisted vertical motion. *A particle falls under gravity in a resisting medium whose resistance varies with velocity. Find the relation between distance and velocity if initially the particle starts from rest.* (*U.P.T.U., 2003*)

Solution. After falling a distance s in time t from rest, let v be velocity of the particle. The forces acting on the particle are its weight mg downwards and resistance $m\lambda v$ upwards.

$\therefore$ equating of motion is $\quad m\dfrac{dv}{dt} = mg - m\lambda v$

or $$\frac{dv}{dt} = g - \lambda v \quad \text{or} \quad \frac{dv}{g - \lambda v} = dt$$

Integrating, $\displaystyle\int\frac{dv}{g - \lambda v} = \int dt + c$ or $\displaystyle -\frac{1}{\lambda}\log(g - \lambda v) = t + c$

Since $v = 0$ when $t = 0$, $\quad\therefore\quad c = -\dfrac{1}{\lambda}\log g$

Thus $$\frac{1}{\lambda}\log\left[\frac{g}{g - \lambda v}\right] = t \quad \text{or} \quad \frac{g - \lambda v}{g} = e^{-\lambda t}$$

or $$\frac{ds}{dt} = v = \frac{g}{\lambda}(1 - e^{-\lambda t}) \qquad ...(i)$$

Integrating, $$s = \frac{g}{\lambda}\int(1 - e^{-\lambda t})dt + c' \quad \text{or} \quad s = \frac{g}{\lambda}\left(t + \frac{1}{\lambda}e^{-\lambda t}\right) + c'$$

Since $s = 0$ when $t = 0$, $\quad\therefore\quad c' = -g/\lambda^2$

Thus $$s = \frac{g}{\lambda}t + \frac{g}{\lambda^2}(e^{-\lambda t} - 1) \qquad ...(ii)$$

Eliminating t from (i) and (ii), we get

$$s = \frac{g}{\lambda^2}\log\left(\frac{g}{g - \lambda v}\right) - \frac{v}{\lambda}$$

which is the desired relation between s and v.

Example 12.12. *A body of mass m, falling from rest is subject to the force of gravity and an air resistance proportional to the square of the velocity (i.e., kv^2). If it falls through a distance x and possesses a velocity v at that instant, prove that*

$$\frac{2kx}{m} = \log \frac{a^2}{a^2 - v^2}, \text{ where } mg = ka^2.$$

Solution. If the body be moving with the velocity v after having fallen through a distance x, then its equation of motion is

$$mv\frac{dv}{dx} = mg - kv^2 \quad \text{or} \quad mv\frac{dv}{dx} = k(a^2 - v^2). \qquad [\because mg = ka^2] \quad ...(i)$$

$\therefore$ separating the variables and integrating, we get $\int \frac{vdv}{a^2 - v^2} = \int \frac{k}{m} dx + c$

or $$-\frac{1}{2}\log(a^2 - v^2) = \frac{kx}{m} + c \qquad ...(ii)$$

Initially, when $x = 0, v = 0$. $\therefore -\frac{1}{2}\log a^2 = c$ ...(iii)

Subtracting (iii) from (ii), we have $\frac{1}{2}[\log a^2 - \log(a^2 - v^2)] = kx/m$

or $$\frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)$$

Obs. *When the resistance becomes equal to the weight, the acceleration becomes zero and particle continues to fall with a constant velocity, called the* **limiting** *or* **terminal** *velocity. From (i), it follows that the acceleration will become zero when $v = a$. Thus, the limiting velocity, i.e., the maximum velocity which the particle can attain is a.*

Example 12.13. Velocity of escape from the earth. *Find the initial velocity of a particle which is fired in radial direction from the earth's centre and is supposed to escape from the earth. Assume that it is acted upon by the gravitational attraction of the earth only.*

Solution. According to Newton's law of gravitation, the acceleration α of the particle is proportional to $1/r^2$ where r is the variable distance of the particle from the earth's centre. Thus

$$\alpha = v\frac{dv}{dr} = -\frac{\mu}{r^2}$$

where v is the velocity when at a distance r from the earth's centre. The acceleration is negative because v is decreasing. When $r = R$, the earth's radius then $\alpha = -g$, the acceleration of gravity at the surface.

i.e., $$-g = -\mu/R^2, \text{ i.e., } \mu = gR^2 \quad \therefore \quad v\frac{dv}{dr} = -\frac{gR^2}{r^2}$$

Separating the variables and integrating, we obtain $\int vdv = -gR^2 \int \frac{dr}{r^2} + c$

i.e., $$v^2 = \frac{2gR^2}{r} + 2c \qquad ...(i)$$

On the earth's surface $r = R$ and $v = v_0$ (say), the initial velocity. Then

$$v_0^2 = 2gR + 2c, \quad \text{i.e.,} \quad 2c = v_0^2 - 2gR$$

Inserting this value of c in (i), we get $v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR$

When v vanishes, the particle stops and the velocity will change from positive to negative and the particle will return to the earth. Thus the velocity will remain positive, if and only if $v_0^2 \geq 2gR$ and then the particle projected fron the earth with this velocity will escape from the earth. Hence the minimum such velocity of projection $v_0 = \sqrt{(2gR)}$ *is called the velocity of escape* from the earth [See Problem 9, page 454].

Example 12.14. Rotating cylinder containing liquid. *A cylindrical tank of radius r is filled with water to a depth h. When the tank is rotated with angular velocity ω about its axis, centrifugal force tends to drive the water outwards from the centre of the tank. Under steady conditions of uniform rotation, show that the section of the free surface of the water by a plane through the axis, is the curve*

$$y = \frac{\omega^2}{2g}\left(x^2 - \frac{r^2}{2}\right) + h.$$

Solution. Let the figure represent an axial section of the cylindrical tank. Forces acting on a particle of mass m at $P(x, y)$ on the curve, cut out from the free surface of water, are :

(*i*) the weight mg acting vertically downwards,

(*ii*) the centrifugal force $m\omega^2 x$ acting horizontally outwards.

As the motion is steady, P moves just on the surface of the water and, therefore, there is no force along the tangent to the curve. Thus the resultant R of mg and $m\omega^2 x$ is along the outward normal to the curve.

$\therefore \quad R \cos \psi = mg$ and $R \sin \psi = m\omega^2 x$

whence $$\frac{dy}{dx} = \tan \psi = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g} \qquad ...(i)$$

This is the differential equation of the surface of the rotating liquid.

Integrating (*i*), we get

$$\int dy = \frac{\omega^2}{g} \int x\, dx + c$$

i.e., $$y = \frac{\omega^2 x^2}{2g} + c \qquad ...(ii)$$

To find c, we note that the volume of the liquid remains the same in both cases (Fig. 12.11).

When $x = 0$ in (*ii*), $OA\ (= y) = c$. When $x = r$ in (*ii*), $$h'\ (= y) = \frac{\omega^2 r^2}{2g} + c \qquad ...(iii)$$

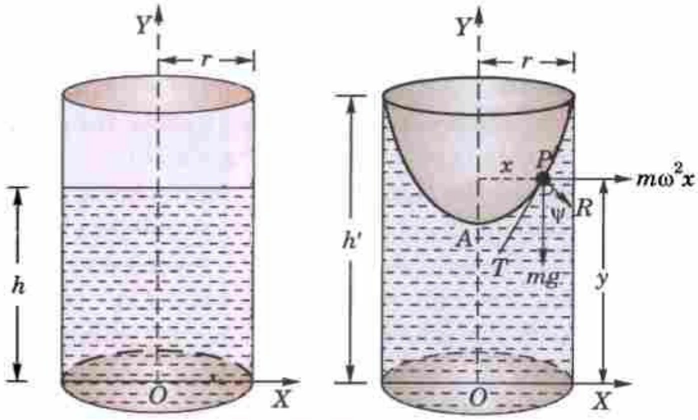


Fig 12.11

Now the volume of the liquid in the non-rotational case $= \pi r^2 h$, and the volume of the liquid in the rotational case

$$= \pi r^2 h' - \int_{OA}^{h'} \pi x^2 dy = \pi r^2 h' - \frac{2\pi g}{\omega^2} \int_c^{h'} (y - c)\, dy \qquad \text{[From (ii)]}$$

$$= \pi r^2 h' - \frac{\pi g}{\omega^2}(h' - c)^2 = \pi r^2 \left(\frac{\omega^2 r^2}{4g} + c\right) \qquad \text{[By (iii)]}$$

Thus $$\pi r^2 h = \pi r^2 \left(\frac{\omega^2 r^2}{4g} + c\right) \text{ whence } c = h - \frac{\omega^2 r^2}{4g}$$

$\therefore$ (*ii*) becomes, $$y = \frac{\omega^2 x^2}{2g} + h - \frac{\omega^2 x^2}{4g} \quad \text{or} \quad y = \frac{\omega^2}{2g}\left(x^2 - \frac{r^2}{2}\right) + h$$

which is the desired equation of the curve.

Example 12.15. Discharge of water through a small hole. *If the velocity of flow of water through a small hole is $0.6\sqrt{2gy}$ where g is the gravitational acceleration and y is the height of water level above the hole, find the time required to empty a tank having the shape of a right circular cone of base radius a and height h filled completely with water and having a hole of area A_0 in the base.*

Solution. At any time t, let the height of the water level be y and radius of its surface be r (Fig. 12.12) so that

$$\frac{h - y}{r} = \frac{h}{a} \quad \text{or} \quad r = a(h - y)/h$$

$\therefore$ surface area of the liquid $= \pi r^2 = \pi a^2 (1 - y/h)^2$

Volume of water drained through the hole per unit time

$$= 0.6\sqrt{(2gy)}\,A_0 = 4.8\sqrt{y}\,A_0 \qquad [\because \; g = 32$$

$\therefore$ rate of fall of liquid level $= 4.8\,A_0\sqrt{y} + \pi a^2(1 - y/h)^2$

i.e., $\dfrac{dy}{dt} = -\dfrac{4.8\,A_0\sqrt{y}}{\pi a^2(1 - y/h)^2}$ (– ve is taken since the water level decreases)

Hence time to empty the tank (= t)

$$= -\int_h^0 \frac{\pi a^2 (1 - y/h)^2}{4.8\,A_0\sqrt{y}}\,dy = \frac{\pi a^2}{4.8\,A_0}\int_0^h (y^{-1/2} - 2y^{1/2}/h + y^{3/2}/h^2)\,dy$$

$$= \frac{\pi a^2}{4.8\,A_0}\left|2y^{1/2} - \frac{4}{3h}y^{3/2} + \frac{2}{5h^2}y^{5/2}\right|_0^h = 0.2\,\pi a^2\sqrt{h}/A_0.$$

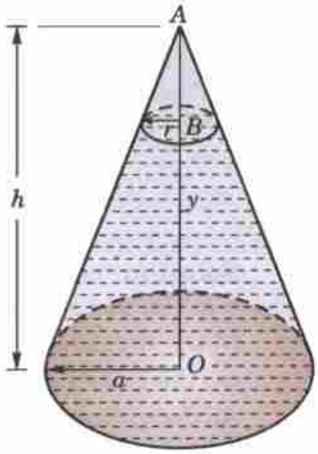


Fig. 12.12

Example 12.16. Atmospheric pressure. *Find the atmospheric pressure p lb. per ft. at a height z ft. above the sea-level, both when the temperature is constant or variable.*

Solution. Take a vertical column of air of unit cross-section.

Let p be the pressure at a height z above the sea-level and $p + \delta p$ at height $z + \delta z$. Let ρ be the density at a height z. (Fig. 12.13)

Now since the thin column δz of air is being pressured upwards with pressure p and downwards with $p + \delta p$, we get by considering its equilibrium;

$$p = p + \delta p + g\rho\delta z.$$

Taking the limit, we get $\quad dp/dz = -g\rho \quad$ ...(*i*)

which is the differential equation giving the atmospheric pressure at height z.

(*i*) *When the temperature is constant*, we have by Boyle's law*, $p = k\rho$ ...(*ii*)

$\therefore$ Substituting the value of ρ from (*ii*) in (*i*), we get

$$\frac{dp}{dz} = -gp/k \quad \text{or} \quad \int\frac{dp}{p} = -\frac{g}{k}\int dz + c \quad \text{or} \quad \log p = -\frac{g}{k}z + c$$

At the sea-level, where $z = 0$, $p = p_0$ (say) then $c = \log p_0$

$$\therefore \qquad \log p - \log p_0 = -\frac{g}{k}z \;\; i.e., \;\; \log p/p_0 = -gz/k$$

Hence p is given by $p = p_0 e^{-gz/k}$.

(*ii*) *When the temperature varies*, we have $p = k\rho^n$.

Proceeding as above, we shall find that p is given by $\dfrac{n}{n-1}(p_0^{1-1/n} - p^{1-1/n}) = gk^{-1/n}z$.

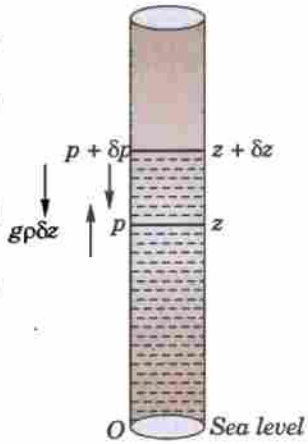


Fig. 12.13

PROBLEMS 12.3

1. A particle of mass m moves under gravity in a medium whose resistance is k times its velocity, where k is a constant. If the particle is projected vertically upwards with a velocity v, show that the time to reach the highest point is $\dfrac{m}{k}\log_e\left(1 + \dfrac{kv}{mg}\right)$.
2. A body of mass m falls from rest under gravity and air resistance proportional to square of velocity. Find velocity as function of time. (*Marathwada, 2008*)
3. A body of mass m falls from rest under gravity in a field whose resistance is mk times the velocity of the body. Find the terminal velocity of the body and also the time taken to acquire one half of its limiting speed.
4. A particle is projected with velocity v along a smooth horizontal plane in the medium whose resistance per unit mass is μ times the cube of the velocity. Show that the distance it has described in time t is $\dfrac{1}{\mu v}(\sqrt{1 + 2\mu v^2 t} - 1)$.

*Named after the English physicist *Robert Boyle* (1627–1691) who was one of the founders of the Royal Society.

5. When a bullet is fired into a sand tank, its retardation is proportional to the square root of its velocity. How long will it take to come to rest if it enters the sand bank with velocity v_0?
6. A particle of mass m is attached to the lower end of a light spring (whose upper end is fixed) and is released. Express the velocity v as a function of the stretch x feet.
7. A chain coiled up near the edge of a smooth table just starts to fall over the edge. The velocity v when a length x has fallen is given by $xv \dfrac{dv}{dx} + v^2 = gx$.

   Show that $v = 8\sqrt{(x/3)}$ ft/sec.
8. A toboggan weighing 200 lb., descends from rest on a uniform slope of 5 in 13 which is 15 yards long. If the coefficient of friction is 1/10 and the air resistance varies as the square of the velocity and is 3 lb. weight when the velocity is 10 ft/sec.; prove that its velocity at the bottom is 38.6 ft/sec and show that however long, the slope is the velocity cannot exceed 44 ft per sec.
   [**Hint.** Fig. 12.14. Equation of motion is
   $$\frac{W}{g}\cdot v\frac{dv}{dx} = -\mu R - kv^2 + W\sin\alpha]$$

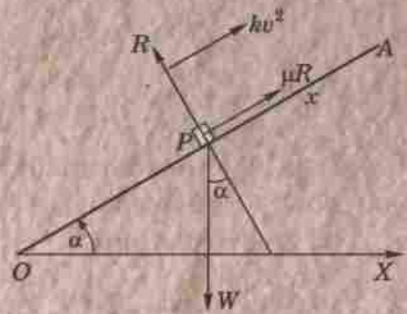


Fig. 12.14

9. Show that a particle projected from the earth's surface with a velocity of 7 miles/sec. will not return to the earth. [Take earth's radius = 3960 miles and $g = 32.17$ ft/sec^2].
10. A cylindrical tank 1.5 m. high stands on its circular base of diameter 1 m. and is initially filled with water. At the bottom of the tank there is a hole of diameter 1 cm., which is opened at some instant, so that the water starts draining under gravity. Find the height of water in the tank at any time t sec. Find the times at which the tank is one-half full, one quarter full, and empty.

    [**Hint.** Take $g = 980$ cm/sec^2 in $v = 0.6\sqrt{(2gy)}$]
11. The rate at which water flows from a small hole at the bottom of a tank is proportional to the square root of the depth of the water. If half the water flows from a cylindrical tank (with vertical axis) in 5 minutes, find the time required to empty the tank.
12. A conical cistern of height h and semi-vertical angle α is filled with water and is held in vertical position with vertex downwards. Water leaks out from the bottom at the rate of kx^2 cubic cms per second, k is a constant and x is the height of water level from the vertex. Prove that the cistern will be empty in $(\pi h \tan^2\alpha)/k$ seconds.
13. Upto a certain height in the atmosphere, it is found that the pressure p and the density ρ are connected by the relation $p = k\rho^n$ $(n > 1)$. If this relation continued to hold upto any height, show that the density would vanish at a finite height.

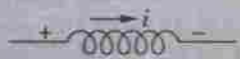
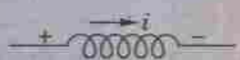
12.5 SIMPLE ELECTRIC CIRCUITS

We shall consider circuits made up of

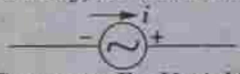
(*i*) three passive elements—resistance, inductance, capacitance and

(*ii*) an active element—voltage source which may be a battery or a generator.

(1) Symbols

	Element	*Symbol*	*Unit**
1.	*Quantity of electricity*	q	coulomb
2.	*Current* (= time rate flow of electricity)	i	ampere (A)
3.	*Resistance, R*	 R	ohm (Ω)
4.	*Inductance, L*	 L	henry (H)
5.	*Capacitance, C*	 C	farad (F)

*These units are respectively named after the French engineer and physicist *Charles Augustin de Coulomb* (1736–1806); French physicist *Andre Marie Ampere* (1775–1836); German physicist *George Simon Ohm* (1789–1854); Italian physicist *Joseph Henry* (1797–1878); American physicist *Michael Faraday* (1791–1867) and the Italian physicist *Alessandro Volta* (1745–1827).

Element	*Symbol*	*Unit*
6. *Electromotive force (e.m.f.) or voltage, E*	 *Battery, E = Constant*  *Generator, E = Variable*	volt (V)

7. *Loop* is any closed path formed by passing through two or more elements in series.
8. *Nodes* are the terminals of any of these elements.

(2) Basic relations

(*i*) $i = \dfrac{dq}{dt}$ or $q = \int i dt$ [$\because$ current is the rate of flow of electricity]

(*ii*) Voltage drop across resistance $R = Ri$ [Ohm's Law]

(*iii*) Voltage drop across inductance $L = L\dfrac{di}{dt}$

(*iv*) Voltage drop across capacitance $C = \dfrac{q}{C}$.

(3) Kirchhoff's laws*. The formulation of differential equations for an electrical circuit depends on the following two Kirchhoff's laws which are of cardinal importance :

I. *The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit.*

II. *The algebraic sum of the currents flowing into (or from) any node is zero.*

(4) Differential equations

(*i*) *R, L series circuit.* Consider a circuit containing resistance R and inductance L in series with a voltage source (battery) E. (Fig. 12.15).

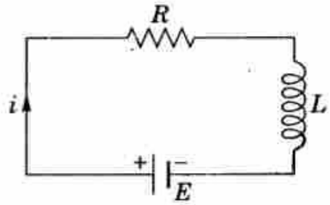


Fig. 12.15

Let i be the current flowing in the circuit at any time t. Then by Kirchhoff's first law, we have sum of voltage drops across R and $L = E$

i.e.,
$$Ri + L\frac{di}{dt} = E \quad \text{or} \quad \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \qquad ...(1)$$

This is a Leibnitz's linear equation.

$$\text{I.F.} = e^{\int \frac{R}{L}dt} = e^{Rt/L} \text{ and therefore, its solution is } i(\text{I.F.}) = \int \frac{E}{L}(\text{I.F.})dt + c$$

or
$$i \cdot e^{Rt/L} = \int \frac{E}{L} e^{Rt/L}\, dt + c = \frac{E}{L}\cdot\frac{L}{R}\cdot e^{Rt/L} + c \text{ whence } i = \frac{E}{R} + ce^{-Rt/L} \qquad ...(2)$$

If initially there is no current in the circuit, *i.e.*, $i = 0$, when $t = 0$, we have $c = -E/R$.

Thus (2) becomes $i = \dfrac{E}{R}\left(1 - e^{-Rt/L}\right)$ which shows that i increases with t and attains the maximum value E/R.

(*ii*) *R, L, C series circuit.* Now consider a circuit containing resistance R, inductance L and capacitance C all in series with a constant e.m.f. E (Fig. 12.16)

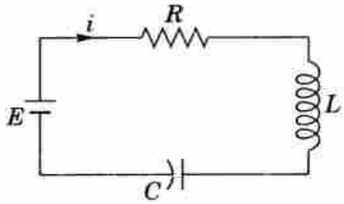


Fig. 12.16

If i be the current in the circuit at time t, then the charge q on the condenser $= \int i\, dt$, *i.e.*, $i = \dfrac{dq}{dt}$.

Applying Kirchhoff's law, we have, sum of the voltage drops across R, L and $C = E$.

i.e.,
$$Ri + L\frac{di}{dt} + \frac{q}{C} = E$$

*Named after the German physicist *Gustav Robert Kirchhoff* (1824–1887).

or $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E.$$

This is the desired differential equation of the circuit and will be solved in § 14.5.

Example 12.17. *Show that the differential equation for the current i in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force E sin ωt satisfies the equation L di/dt + Ri = E sin ωt.*

Find the value of the current at any time t, if initially there is no current in the circuit.

(*Kurukshetra, 2005*)

Solution. By Kirchhoff's first law, we have sum of voltage drops across R and $L = E \sin \omega t$

i.e., $$Ri + L\frac{di}{dt} = E \sin \omega t.$$

This is the required differential equation which can be written as $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\sin \omega t$

This is a *Leibnitz's equation*. Its I.F. $= e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$

$\therefore$ the solution is $i(\text{I.F.}) = \int \frac{E}{L}\sin \omega t \cdot (\text{I.F.})dt + c$

or $$ie^{Rt/L} = \frac{E}{L}\int e^{Rt/L}\sin \omega t\, dt + c = \frac{E}{L}\frac{e^{Rt/L}}{\sqrt{[(R/L)^2 + \omega^2]}}\sin\left(\omega t - \tan^{-1}\frac{L\omega}{R}\right) + c$$

or $$i = \frac{E}{\sqrt{(R^2 + \omega^2 L^2)}}\sin(\omega t - \phi) + ce^{-Rt/L} \text{ where } \tan\phi = L\omega/R \quad ...(i)$$

Initially when $t = 0$; $i = 0$. $\therefore$ $0 = \frac{E\sin(-\phi)}{\sqrt{(R^2 + \omega^2L^2)}} + c$, *i.e.*, $c = \frac{E\sin\phi}{\sqrt{(R^2 + \omega^2L^2)}}$

Thus (*i*) takes the form $i = \frac{E\sin(\omega t - \phi)}{\sqrt{(R^2 + \omega^2L^2)}} + \frac{E\sin\phi}{\sqrt{(R^2 + \omega^2L^2)}}\cdot e^{-Rt/L}$

or $$i = \frac{E}{\sqrt{(R^2 + \omega^2L^2)}}[\sin(\omega t - \phi) + \sin\phi \,.\, e^{-Rt/L}]$$ which gives the current at any time t.

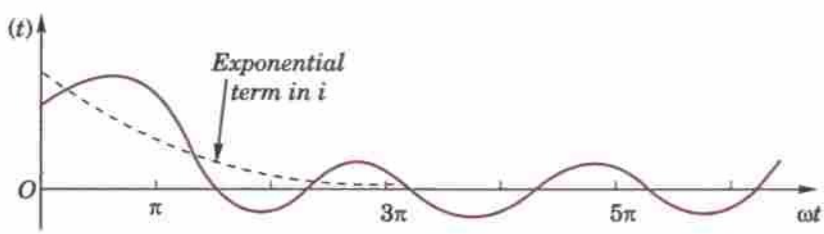


Fig. 12.17

Obs. As t increases indefinitely, the exponential term will approach zero. This implies that after sometime the current $i(t)$ will execute nearly harmonic oscillations only (Fig. 12.17).

PROBLEMS 12.4

1. When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current i builds up at a rate given by $L\, di/dt + Ri = E$.
   Find i as a function of t. How long will it be, before the current has reached one-half its final value if $E = 6$ volts, $R = 100$ ohms and $L = 0.1$ henry ?
2. When a resistance R ohms is connected in series with an inductance L henries with an e.m.f. of E volts, the current i amperes at time t is given by $L\, di/dt + Ri = E$.
   If $E = 10 \sin t$ volts and $i = 0$ when $t = 0$, find i as a function of t.

3. A resistance of 100 Ω, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in the circuit at $t = 0.5$ sec, if $i = 0$ at $t = 0$. *(Marathwada, 2008)*
4. The equation of electromotive force in terms of current i for an electrical circuit having resistance R and condenser of capacity C in series, is

$$E = Ri + \int \frac{idt}{C}$$

Find the current i at any time t when $E = E_m \sin \omega t$. *(S.V.T.U., 2008, P.T.U., 2006)*

5. A resistance R in series with inductance L is shunted by an equal resistance R with capacity C. An alternating e.m.f. $E \sin pt$ produces currents i_1 and i_2 in two branches. If initially there is no current, determine i_1 and i_2 from the equations

$$L\frac{di_1}{dt} + Ri_1 = E \sin pt \quad \text{and} \quad \frac{i_2}{C} + R\frac{di_2}{dt} = pE \cos pt.$$

Verify that if $R^2C = L$, the total current $i_1 + i_2$ will be $(E \sin pt)/R$.

12.6 NEWTON'S LAW OF COOLING*

According to this law, *the temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.*

If θ_0 is the temperature of the surroundings and θ that of the body at any time t, then

$$\frac{d\theta}{dt} = -k(\theta - \theta_0), \text{ where } k \text{ is a constant.}$$

Example 12.18. *A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of the air being 40°C. What will be the temperature of the body after 40 minutes from the original ?*

Solution. If θ be the temperature of the body at any time t, then

$$\frac{d\theta}{dt} = -k(\theta - 40), \qquad \text{where } k \text{ is a constant.}$$

Integrating, $$\int \frac{d\theta}{\theta - 40} = -k \int dt + \log c, \qquad \text{where } c \text{ is a constant.}$$

or $$\log(\theta - 40) = -kt + \log c \quad i.e., \quad \theta - 40 = ce^{-kt} \qquad ...(i)$$

When $t = 0$, $\theta = 80°$ and when $t = 20$, $\theta = 60°$, $\therefore$ $40 = c$, and $20 = ce^{-20k}$; $k = \frac{1}{20}\log 2$.

Thus (i) becomes $$\theta - 40 = 40e^{-\left(\frac{1}{20}\log 2\right)t}$$

When $t = 40$ min., $$\theta = 40 + 40e^{-2\log 2} = 40 + 40e^{\log(1/4)} = 40 + 40 \times \tfrac{1}{4} = 50°\text{C}.$$

12.7 HEAT FLOW

The fundamental principles involved in the problems of heat conduction are :

(i) *Heat flows from a higher temperature to the lower temperature.*

(ii) *The quantity of heat in a body is proportional to its mass and temperature.*

(iii) *The rate of heat-flow across an area is proportional to the area and to the rate of change of temperature with respect to its distance normal to the area.*

If q (cal./sec.) be the quantity of heat that flows across a slab of area α (cm^2) and thickness δx in one second, where the difference of temperature at the faces is δT, then by (iii) above

$$q = -k\alpha dT/dx \qquad ...(A)$$

where k is a constant depending upon the material of the body and is called the *thermal conductivity*.

*Named after the great English mathematician and physicist *Sir Issac Newton* (1642–1727) whose contributions are of utmost importance. He discovered many physical laws, invented Calculus alongwith *Leibnitz* (see footnote p. 139) and created analytical methods of investigating physical problems. He became professor at Cambridge in1699, but his *'Mathematical Principles of Natural Philosophy'* containing development of classical mechanics had been completed in 1687.

Example 12.19. *A pipe 20 cm in diameter contains steam at 150°C and is protected with a covering 5 cm thick for which $k = 0.0025$. If the temperature of the outer surface of the covering is 40°C, find the temperature half-way through the covering under steady state conditions.*

Solution. Let q cal./sec. be the constant quantity of heat flowing out radially through a surface of the pipe having radius x cm. and length 1 cm (Fig. 12.18). Then the area of the lateral surface (belt) $= 2\pi x$.

$\therefore$ the equation (A) above gives

$$q = -k \cdot 2\pi x \cdot \frac{dT}{dx} \quad \text{or} \quad dT = -\frac{q}{2\pi k} \cdot \frac{dx}{x}$$

Integrating, we have

$$T = -\frac{q}{2\pi k} \log_e x + c$$

Since $T = 150$, when $x = 10$. $\therefore$ $150 = -\frac{q}{2\pi k} \log_e 10 + c$ ...(*i*)

Again since $T = 40$, when $x = 15$, $40 = -\frac{q}{2\pi k} \log_e 15 + c$ ...(*ii*)

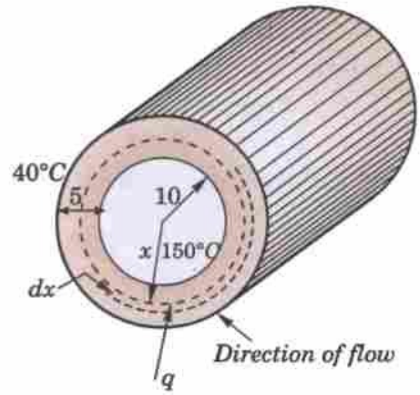


Fig. 12.18

Subtracting (*ii*) from (*i*), $110 = \frac{q}{2\pi k} \log_e 1.5$ ...(*iii*)

Let $T = t$, when $x = 12.5$ $\therefore$ $t = -\frac{q}{2\pi k} \log_e 12.5 + c$ ...(*iv*)

Subtracting (*i*) from (*iv*), $t - 150 = -\frac{q}{2\pi k} \log_e 1.25$ ...(*v*)

Dividing (*v*) by (*iii*), $\frac{t-150}{110} = -\frac{\log_e 1.25}{\log_e 1.5}$, whence $t = 89.5$°C.

PROBLEMS 12.5

1. If the temperature of the air is 30°C and the substance cools from 100°C to 70°C in 15 minutes, find when the temperature will 40°C.
2. If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after 24 minutes.
3. Two friends A and B order coffee and receive cups of equal temperature at the same time. A adds a small amount of cool cream immediately but does not drink his coffee until 10 minutes later, B waits for 10 minutes and adds the same amount of cool cream and begins to drink. Assuming the Newton's law of cooling, decide who drinks the hotter coffee ?
4. A pipe 20 cm. in diameter contains steam at 200°C. It is covered by a layer of insulations 6 cm thick and thermal conductivity 0.0003. If the temperature of the outer surface is 30°C, find the heat loss per hour from two metre length of the pipe.
5. A steam pipe 20 cm. in diameter contains steam at 150°C and is covered with asbestos 5 cm thick. The outside temperature is kept at 60°C. By how much should the thickness of the covering be increased in order that the rate of heat loss should be decreased by 25% ?

12.8 RATE OF DECAY OF RADIO-ACTIVE MATERIALS

This law states that disintegration at any instant is proportional to the amount of material present.

of material at any time t, then $\frac{du}{dt} = -ku$, where k is a constant.

Example 12.20. *Uranium disintegrates at a rate proportional to the amount then present at any instant. If M_1 and M_2 grams of uranium are present at times T_1 and T_2 respectively, find the half-life of uranium.*

Solution. Let the mass of uranium at any time t be m grams.

Then the equation of disintegration of uranium is $\dfrac{dm}{dt} = -\mu m$, where μ is a constant.

Integrating, we get $\int \dfrac{dm}{dt} = -\mu \int dt + c$ or $\log m = c - \mu t$ ...(i)

Initially, when $t = 0$, $m = M$ (say) so that $c = \log M$ $\therefore$ (i) becomes, $\mu t = \log M - \log m$ ...(ii)

Also when $t = T_1$, $m = M_1$ and when $t = T_2$, $m = M_2$

$\therefore$ From (ii), we get $\mu T_1 = \log M - \log M_1$ ...(iii)

$\mu T_2 = \log M - \log M_2$ ...(iv)

Subtracting (iii) from (iv), we get

$$\mu(T_2 - T_1) = \log M_1 - \log M_2 = \log (M_1/M_2) \text{ whence } \mu = \frac{\log (M_1/M_2)}{T_2 - T_1}$$

Let the mass reduce to half its initial value in time T. *i.e.*, when $t = T$, $m = \dfrac{1}{2}M$.

$\therefore$ from (ii), we get $\mu T = \log M - \log (M/2) = \log 2$.

Thus $$T = \frac{1}{\mu} \log 2 = \frac{(T_2 - T_1)\log 2}{\log (M_1/M_2)}.$$

12.9 CHEMICAL REACTIONS AND SOLUTIONS

A type of problems which are especially important to chemical engineers are those concerning either chemical reactions or chemical solutions. These can be best explained through the following example :

Example 12.21. *A tank initially contains 50 gallons of fresh water. Brine, containing 2 pounds per gallon of salt, flows into the tank at the rate of 2 gallons per minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it take for the quantity of salt in the tank to increase from 40 to 80 pounds ?* (Andhra, 1997)

Solution. Let the salt content at time t be u lb. so that its rate of change is du/dt

$= 2$ gal. $\times$ 2 lb. $= 4$ lb./min.

If C be the concentration of the brine at time t, the rate at which the salt content decreases due to the out-flow

$= 2$ gal. $\times$ C lb. $= 2C$ lb./min.

$$\therefore \quad \frac{du}{dt} = 4 - 2C \quad \text{...(i)}$$

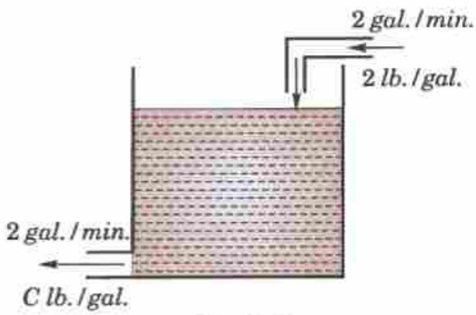


Fig. 12.19

Also since there is no increase in the volume of the liquid, the concentration $C = u/50$.

$\therefore$ (i) becomes $\dfrac{du}{dt} = 4 - 2\dfrac{u}{50}$

Separating the variables and integrating, we have

$$\int dt = 25 \int \frac{du}{100 - u} + k \quad \text{or} \quad t = -25 \log_e (100 - u) + k \quad \text{...(ii)}$$

Initially when $t = 0$, $u = 0$ $\therefore$ $0 = -25 \log_e 100 + k$ ...(iii)

Eliminating k from (ii) and (iii), we get $t = 25 \log_e \dfrac{100}{100 - u}$.

Taking $t = t_1$ when $u = 40$ and $t = t_2$ when $u = 80$, we have

$$t_1 = 25 \log_e \frac{100}{60} \text{ and } t_2 = 25 \log_e \frac{100}{20}$$

$\therefore$ The required time $(t_2 - t_1) = 25 \log_e 5 - 25 \log_e 5/3$

$= 25 \log_e 3 = 25 \times 1.0986 = 27$ min. 28 sec.

PROBLEMS 12.6

1. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What would be the value of N after $1\frac{1}{2}$ hours ? *(Nagarjuna, 2008 ; J.N.T.U., 2003)*
2. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in 2 hours, in how many hours will it triple ? *(Andhra, 2000)*
3. Radium decomposes at a rate proportional to the amount present. If a fraction p of the original amount disappears in 1 year, how much will remain at the end of 21 years ?
4. If 30% of radio active substance disappeared in 10 days, low long will it take for 90% of it to disappear ? *(Madras, 2000 S)*
5. Under certain conditions cane-sugar in water is converted into dextrose at a rate which is proportional to the amount unconverted at any time. If of 75 gm. at time $t = 0$, 8 gm. are converted during the first 30 minutes, find the amount converted in $1\frac{1}{2}$ hours.
6. In a chemical reaction in which two substances A and B initially of amounts a and b respectively are concerned, the velocity of transformation dx/dt at any time t is known to be equal to the product $(a-x)(b-x)$ of the amounts of the two substances then remaining untransformed. Find t in terms of x if $a = 0.7$, $b = 0.6$ and $x = 0.3$ when $t = 300$ seconds.
7. A tank contains 1000 gallons of brine in which 500 lt. of salt are dissolved. Fresh water runs into the tank at the rate of 10 gallons /minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it be before only 50 lt. of salt is left in the tank ?

   [**Hint.** If u be the amount of salt after t minutes, then $du/dt = -10u/1000$.]
8. A tank is initially filled with 100 gallons of salt solution containing 1 lb. of salt per gallon. Fresh brine containing 2 lb. of salt per gallon runs into the tank at the rate of 5 gallons per minute and the mixture assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time, and determine how long it will take for this amount to reach 150 lb.

12.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 12.7

Fill up the blanks or choose the correct answer in the following problems :

1. If a coil having a resistance of 15 ohms and an inductance of 10 henries is connected to 90 volts supply then the current after 2 secs is .....
2. A tennis ball dropped from a height of 6 m, rebounds infinitely often. If it rebounds 80% of the distance that it falls, then the total distance for these bounces is .....
3. Radium decomposes at a rate proportional to the amount present. If 5% of the original amount disappears in 50 years then ......% will remain after 100 years.
4. The curve whose polar subtangent is constant is ......
5. The curve in which the length of the subnormal is proportional to the square of the ordinate, is ......
6. The curve in which the portion of the tangent between the axes is bisected at the point of contact, is ......
7. If the stream lines of a flow around a corner are $xy = c$, then the equipotential lines are ......
8. The orthogonal trajectories of a system of confocal and coaxial parabolas is ......
9. When a bullet is fired into a sand tank, its retardation is proportional to $\sqrt{(\text{velocity})}$. If it enters the sand tank with velocity v_o, it will come to rest after ...... seconds.
10. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in two hours, then it will triple after ..... hours.
11. Ram and Sunil order coffee and receive cups simultaneously at equal temperature. Ram adds a spoon of cold cream but doesn't drink for 10 minutes, Sunil waits for 10 minutes and adds a spoon of cold cream and begins to drink. Who drinks the hotter coffee ?
12. The equation $y - 2x = c$ represents the orthogonal trajectories of the family

    (*i*) $y = ae^{-2x}$ (*ii*) $x^2 + 2y^2 = a$ (*iii*) $xy = a$ (*iv*) $x + 2y = a$.

13. In order to keep a body in air above the earth for 12 seconds, the body should be thrown vertically up with a velocity of

(a) $\sqrt{6}$ g m/sec (b) $\sqrt{12}$ g m/sec (c) 6 g m/sec (d) 12g m/sec.

14. The orthogonal trajectory of the family $x^2 + y^2 = c^2$ is

(a) $x + y = c$ (b) $xy = c$ (c) $x^2 + y^2 = x + y$ (d) $y = cx$. *(V.T.U., 2010)*

15. If a thermometer is taken outdoors where the temperature is 0°C, from a room having temperature 21°C and the reading drops to 10°C in 1 minute then its reading will be 5°C after ......minutes.

16. The equation of the curve for which the angle between the tangent and the radius vector is twice the vectorial angle is $r^2 = 2a \sin 2\theta$. This satisfies the differential equation

(a) $r\dfrac{dr}{d\theta} = \tan 2\theta$ (b) $r\dfrac{dr}{d\theta} = \cos 2\theta$ (c) $r\dfrac{d\theta}{dr} = \tan 2\theta$ (d) $r\dfrac{d\theta}{dr} = \cos 2\theta$.

17. Two balls of m_1 and m_2 grams are projected vertically upwards such that the velocity of projection of m_1 is double that of m_2. If the maximum height to which m_1 and m_2 rise be h_1 and h_2 respectively then

(a) $h_1 = 2h_2$ (b) $2h_1 = h_2$ (c) $h_1 = 4h_2$ (d) $4h_1 = h_2$.

18. Two balls are projected simultaneously with same velocity from the top of a tower, one vertically upwards and the other vertically downwards. If they reach the ground in times t_1 and t_2, then the height of the tower is

(a) $\frac{1}{2} g t_1 t_2$ (b) $\frac{1}{2} g(t_1^2 + t_2^2)$ (c) $\frac{1}{2} g(t_1^2 - t_2^2)$ (d) $\frac{1}{2} g(t_1 + t_2)^2$.

19. A particle projected from the earth's surface with a velocity of 7 miles/sec will return to the earth.
(Taking $g = 32.17$ and earth's radius = 3960 miles) (True/False)

20. If a particle falls under gravity with air resistance k times its velocity, then its velocity cannot exceed g/k. (True/False)