CHAPTER 33

Numerical Solution of Partial Differential Equations

33.1 INTRODUCTION

There are many boundary value problems which involve partial differential equations. Only a few of these equations can be solved by analytical methods. In most cases, we depend on the numerical solution of such partial differential equations. Of the various numerical methods available for solving these equations, the method of finite differences is most commonly used.

In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite-difference approximations. Then the given equation is changed to a difference equation which is solved by iterative procedures. This process is slow but gives good results of boundary value problems. An added advantage of this method is that the computation can be done by electronic computer. Here we shall apply this method to the solution of important applied partial differential equations. For a detailed study, the reader should refer to author's book '*Numerical Methods in Engineering and Science*'.

33.2 CLASSIFICATION OF SECOND ORDER EQUATIONS

The general linear partial differential equation of the second order in two independent variables is of the form.

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Such a partial differential equation is said to be

(*i*) **elliptic,** *if* $B^2 - 4AC < 0$,

(*ii*) **parabolic,** *if* $B^2 - 4AC = 0$,

and (*iii*) **hyperbolic,** *if* $B^2 - 4AC > 0$.

Example 33.1. *Classify the following equations :*

(*i*) $\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x \partial y} + 4\dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$ (*ii*) $x^2\dfrac{\partial^2 u}{\partial x^2} + (1 - y^2)\dfrac{\partial^2 u}{\partial y^2} = 0, -\infty < x < \infty, -1 < y < 1$

(*Madras, 2003*)

(*iii*) $(1 + x^2)\dfrac{\partial^2 u}{\partial x^2} + (5 + 2x^2)\dfrac{\partial^2 u}{\partial x \partial t} + (4 + x^2)\dfrac{\partial^2 u}{\partial t^2} = 0.$

Solution. (*i*) Comparing this equation with (1) above, we find that

$$A = 1, B = 4, C = 4$$

$$\therefore \quad B^2 - 4AC = (4)^2 - 4 \times 1 \times 4 = 0$$

So the equation is parabolic.

(*ii*) Here $\quad A = x^2, B = 0, C = 1 - y^2$

$$\therefore \quad B^2 - 4AC = 0 - 4x^2(1 - y^2) = 4x^2(y^2 - 1)$$

For all x between $-\infty$ and ∞, x^2 is positive

For all y between -1 and 1, $y^2 < 1 \quad \therefore \quad B^2 - 4AC < 0$

Hence the equation is elliptic.

(*iii*) Here $A = 1 + x^2, B = 5 + 2x^2, C = 4 + x^2$

$$\therefore \quad B^2 - 4AC = (5 + 2x^2)^2 - 4(1 + x^2)(4 + x^2) = 9 \text{ i.e. } > 0$$

So the equation is hyperbolic.

PROBLEMS 33.1

1. What is the classification of the equation $f_{xx} + 2f_{xy} + f_{yy} = 0$.
2. Determine whether the following equation is elliptic or hyperbolic ?
$$(x + 1)u_{xx} - 2(x + 2)u_{xy} + (x + 3)u_{yy} = 0.$$
3. Classify the equations (*i*) $y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} + 2u_x - 3u = 0$ (*Madras, 2000 S*)

(*ii*) $x^2 \dfrac{\partial^2 u}{\partial x^2} + y^2 \dfrac{\partial^2 u}{\partial y^2} = x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y}$ (*P.T.U., 2009 S*)

(*iii*) $\dfrac{3\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x \partial y} + 6\dfrac{\partial^2 u}{\partial y^2} - 2\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} - u = 0.$ (*Anna, 2008*)

4. In which parts of the (x, y) plane is the following equation elliptic ?
$$\partial^2 u/\partial x^2 + \partial^2 u/\partial x\partial y + (x^2 + 4y^2)\,\partial^2 u/\partial y^2 = 2\sin(xy).$$

33.3 FINITE-DIFFERENCE APPROXIMATIONS TO DERIVATIVES

Consider a rectangular region R in the x-y plane. Divide this region into a rectangular network of sides $\Delta x = h$ and $\Delta y = k$ as shown in Fig. 33.1. The points of intersection of the dividing lines are called *mesh points, nodal points* or *grid points.*

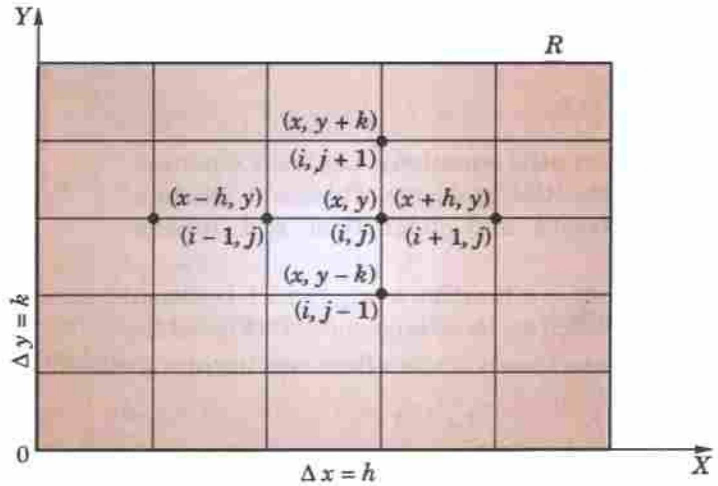

Fig. 33.1

Then we have the finite difference approximations for the partial derivatives in x-direction (§ 32.12) :

$$\frac{\partial u}{\partial x} = \frac{u(x + h, y) - u(x, y)}{h} + O(h)$$

$$= \frac{u(x, y) - u(x-h, y)}{h} + O(h) = \frac{u(x+h, y) - u(x-h, y)}{2h} + O(h^2)$$

and
$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x-h, y) - 2u(x, y) + u(x+h, y)}{h^2} + O(h^2)$$

Writing $u(x, y) = u(ih, jk)$ as simply $u_{i,j}$, the above approximations become

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h) \quad ...(1)$$

$$= \frac{u_{i,j} - u_{i-1,j}}{h} + O(h) \quad ...(2)$$

$$= \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2) \quad ...(3)$$

and
$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2) \quad ...(4)$$

Similarly we have the approximations for the derivatives w.r.t. y :

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k) \quad ...(5)$$

$$= \frac{u_{i,j} - u_{i,j-1}}{k} + O(k) \quad ...(6)$$

$$= \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2) \quad ...(7)$$

and
$$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + O(k^2) \quad ...(8)$$

Replacing the derivatives in any partial differential equation by their corresponding difference approximations (1) to (8), we obtain the finite-difference analogues of the given equations.

33.4 ELLIPTIC EQUATIONS

The Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and the Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

are examples of elliptic partial differential equations. Laplace's equation arises in steady-state flow and potential problems. Poisson's equation arises in fluid mechanics, electricity and magnetism and torsion problems.

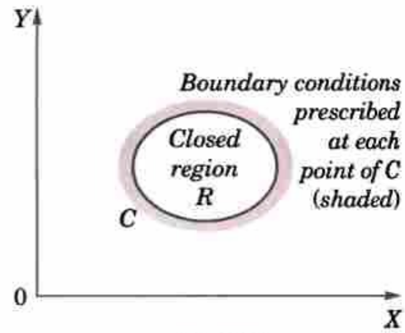

Fig. 33.2

The solution of these equations is a function $u(x, y)$ which is satisfied at every point of a region R subject to certain boundary conditions specified on the closed curve C (Fig. 33.2).

In general, problems concerning steady viscous flow, equilibrium stresses in elastic structures etc., lead to elliptic type of equations.

33.5 SOLUTION OF LAPLACE'S EQUATION*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(1)$$

* See p. 619

Consider a rectangular region R for which $u(x, y)$ is known at the boundary. Divide this region into a network of square mesh of side h, as shown in Fig. 33.3 (assuming that an exact sub-division of R is possible). Replacing the derivatives in (1) by their difference approximations, we have

$$\frac{1}{h^2}\,[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] + \frac{1}{h^2}\,[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}] = 0$$

or
$$u_{i,j} = \frac{1}{4}\,[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}] \quad ...(2)$$

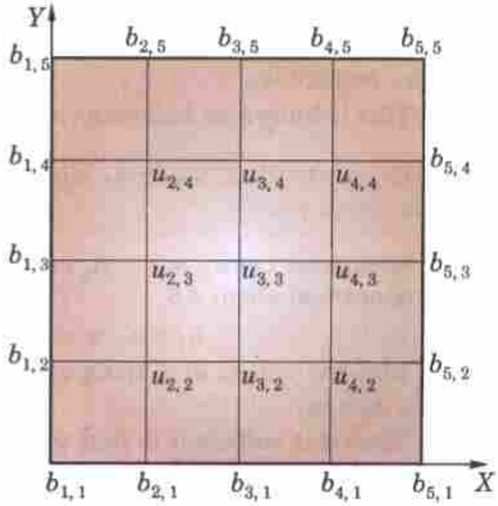

Fig. 33.3

This shows that the value of $u_{i,j}$ at any interior mesh point is the average of its values at four neighbouring points to the left, right, above and below. (2) is called the **standard 5-point formula** which is exhibited in Fig. 33.4.

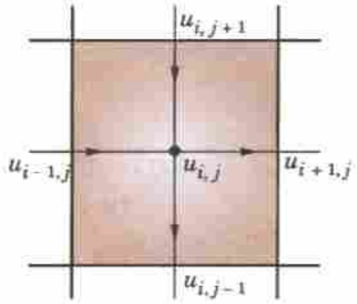

Fig. 33.4

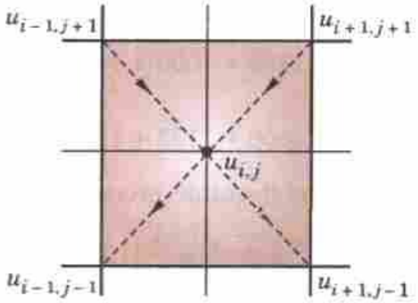

Fig. 33.5

Sometimes a formula similar to (2) is used which is given by

$$u_{i,j} = \frac{1}{4}(u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}) \quad ...(3)$$

This shows that the value of $u_{i,j}$ is the average of its values at the four neighbouring diagonal mesh points. (3) is called the **diagonal 5-point formula** which is represented in Fig. 33.5. Although (3) is less accurate than (2), yet it serves as a reasonably good approximation for obtaining the starting values at the mesh points.

Now to find the initial values of u at the interior mesh points, we first use diagonal five point formula (3) and compute $u_{3,3}, u_{2,4}, u_{4,4}, u_{4,2}$ and $u_{2,2}$, in this order. Thus we get,

$$u_{3,3} = \frac{1}{4}(b_{1,5} + b_{5,1} + b_{5,5} + b_{1,1})\,;\; u_{2,4} = \frac{1}{4}(b_{1,5} + u_{3,3} + b_{3,5} + b_{1,3})$$

$$u_{4,4} = \frac{1}{4}(b_{3,5} + b_{5,3} + b_{5,5} + u_{3,3})\,;\; u_{4,2} = \frac{1}{4}(u_{3,3} + b_{5,1} + b_{3,1} + b_{5,3})$$

$$u_{2,2} = \frac{1}{4}(b_{1,3} + b_{3,1} + u_{3,3} + b_{1,1})$$

The values at the remaining interior points *i.e.* $u_{2,3}, u_{3,4}, u_{4,3}$ and $u_{3,2}$ are computed by the standard five-point formula (2). Thus, we obtain

$$u_{2,3} = \frac{1}{4}(b_{1,3} + u_{3,3} + u_{2,4} + u_{2,2}),\; u_{3,4} = \frac{1}{4}(u_{2,4} + u_{4,4} + b_{3,5} + u_{3,3})$$

$$u_{4,3} = \frac{1}{4}(u_{3,3} + b_{5,3} + u_{4,4} + u_{4,2}),\; u_{3,2} = \frac{1}{4}(u_{2,2} + u_{4,2} + u_{3,3} + u_{3,1})$$

Having found all the nine values of $u_{i,j}$ once, their accuracy is improved by repeated application of (2) in the form

$$u^{(n+1)}_{i,j} = \frac{1}{4}[u^{(n+1)}_{i-1,j} + u^{(n)}_{i+1,j} + u^{(n+1)}_{i,j+1} + u^{(n)}_{i,j-1}]$$

This formula utilises the latest iterative value available and scans the mesh points symmetrically from left to right along successive rows. This process is repeated till the difference of values in one round and the next becomes negligible.

This is known as *Liebmann's iteration process.*

Example 33.2. *Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the square mash of Fig. 33.6 with boundary values as shown.* *(Rohtak, 2005 ; V.T.U., 2005)*

Solution. Let $u_1, u_2, \ldots, u_9$ be the values of u at the interior mesh-points. Since the boundary values of u are symmetrical about AB.

$\therefore \quad u_7 = u_1, u_8 = u_2, u_9 = u_3.$

Also the values of u being symmetrical about CD, $u_3 = u_1$, $u_6 = u_4$, $u_9 = u_7$.

Thus it is sufficient to find the values u_1, u_2, u_4 and u_5.

Now we find their initial value in the following order :

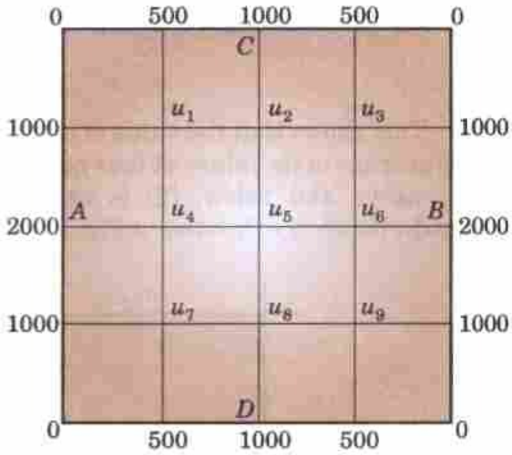

Fig. 33.6

$$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500 \quad \text{(Std. formula)}$$

$$u_1 = \frac{1}{4}(0 + 1500 + 1000 + 2000) = 1125 \quad \text{(Diag. formula)}$$

$$u_2 = \frac{1}{4}(1125 + 1125 + 1000 + 1500) \approx 1188 \quad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(2000 + 1500 + 1125 + 1125) \approx 1438 \quad \text{(Std. formula)}$$

We carry out the iteration process using the formulae :

$$u_1^{(n+1)} = \frac{1}{4}[1000 + u_2^{(n)} + 500 + u_4^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + u_1^{(n)} + 1000 + u_5^{(n)}]$$

$$u_4^{(n+1)} = \frac{1}{4}[2000 + u_5^{(n)} + u_1^{(n+1)} + u_1^{(n)}]$$

$$u_5^{(n+1)} = \frac{1}{4}[u_4^{(n+1)} + u_4^{(n)} + u_2^{(n+1)} + u_2^{(n)}]$$

First iteration : (put $n = 0$)

$$u_1^{(1)} = \frac{1}{4}(1000 + 1188 + 500 + 1438) \approx 1032$$

$$u_2^{(1)} = \frac{1}{4}(1032 + 1032 + 1000 + 1500) = 1141$$

$$u_4^{(1)} = \frac{1}{4}(2000 + 1500 + 1032 + 1032) = 1391$$

$$u_5^{(1)} = \frac{1}{4}(1391 + 1391 + 1141 + 1141) = 1266$$

Second iteration : (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(1000 + 1141 + 500 + 1391) = 1008$$

$$u_2^{(2)} = \frac{1}{4}(1008 + 1008 + 1000 + 1266) = 1069$$

$$u_4^{(2)} = \frac{1}{4}(2000 + 1266 + 1008 + 1008) = 1321$$

$$u_5^{(2)} = \frac{1}{4}(1321 + 1321 + 1069 + 1069) = 1195$$

Third iteration :

$$u_1^{(3)} = \frac{1}{4}(1000 + 1069 + 500 + 1321) = 973$$

$$u_2^{(3)} = \frac{1}{4}(973 + 973 + 1000 + 1195) = 1035$$

$$u_4^{(3)} = \frac{1}{4}(2000 + 1195 + 973 + 973) = 1288$$

$$u_5^{(3)} = \frac{1}{4}(1288 + 1288 + 1035 + 1035) = 1162$$

Fourth iteration :

$$u_1^{(4)} = \frac{1}{4}(1000 + 1135 + 500 + 1288) = 956$$

$$u_2^{(4)} = \frac{1}{4}(956 + 956 + 1000 + 1162) = 1019$$

$$u_4^{(4)} = \frac{1}{4}(2000 + 1162 + 956 + 956) = 1269$$

$$u_5^{(4)} = \frac{1}{4}(1269 + 1269 + 1019 + 1019) = 1144$$

Fifth iteration :

$$u_1^{(5)} = \frac{1}{4}(1000 + 1019 + 500 + 1269) = 947$$

$$u_2^{(5)} = \frac{1}{4}(947 + 947 + 1000 + 1144) \approx 1010$$

$$u_4^{(5)} = \frac{1}{4}(2000 + 1144 + 947 + 947) \approx 1260$$

$$u_5^{(5)} = \frac{1}{4}(1260 + 1260 + 1010 + 1010) = 1135$$

Similarly, $u_1^{(6)} = 942,\ u_2^{(6)} = 1005,\ u_4^{(6)} = 1255,\ u_5^{(6)} = 1130$

$u_1^{(7)} = 940,\ u_2^{(7)} = 1003,\ u_4^{(7)} = 1253,\ u_5^{(7)} = 1128$

$u_1^{(8)} = 939,\ u_2^{(8)} = 1002,\ u_4^{(8)} = 1252,\ u_5^{(8)} = 1127$

$u_1^{(9)} = 939,\ u_2^{(6)} = 1001,\ u_4^{(9)} = 1251,\ u_5^{(9)} = 1126$

Thus there is negligible difference between the values obtained in the eighth and ninth iterations.

Hence $u_1 = 939$, $u_2 = 1001$, $u_4 = 1251$ and $u_5 = 1126$.

Example 33.3. *Given the values of u(x, y) on the boundary of the square in the Fig. 33.7, evaluate the function u(x, y) satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal points of this figure.*

(Bhopal, 2009 ; Madras, 2003)

Solution. To get the initial values of u_1, u_2, u_3, u_4, we assume $u_4 = 0$. Then

$$u_1 = \frac{1}{4}(1000 + 0 + 1000 + 2000) = 1000 \qquad \text{(Diag. formula)}$$

$$u_2 = \frac{1}{4}(1000 + 500 + 1000 + 0) = 625 \qquad \text{(Std. formula)}$$

$$u_3 = \frac{1}{4}(2000 + 0 + 1000 + 500) = 875 \qquad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375 \qquad \text{(Std. formula)}$$

We carry out the successive iterations, using the formulae

$$u_1^{(n+1)} = \frac{1}{4}[2000 + u_2^{(n)} + 1000 + u_3^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + 500 + 1000 + u_4^{(n)}]$$

$$u_3^{(n+1)} = \frac{1}{4}[2000 + u_4^{(n)} + u_1^{(n+1)} + 500]$$

$$u_4^{(n+1)} = \frac{1}{4}[u_3^{(n+1)} + 0 + u_2^{(n+1)} + 0]$$

First iteration : (put $n = 0$)

$$u_1^{(1)} = \frac{1}{4}(2000 + 625 + 1000 + 875) = 1125$$

$$u_2^{(1)} = \frac{1}{4}(1125 + 500 + 1000 + 375) = 750$$

$$u_3^{(1)} = \frac{1}{4}(2000 + 375 + 1125 + 500) = 1000$$

$$u_4^{(1)} = \frac{1}{4}(1000 + 0 + 750 + 0) \approx 438$$

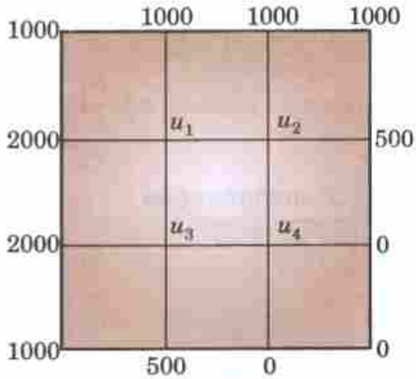

Fig. 33.7

Second iteration : (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(2000 + 750 + 1000 + 1000) \approx 1188$$

$$u_2^{(2)} = \frac{1}{4}(1188 + 500 + 1000 + 438) \approx 782$$

$$u_3^{(2)} = \frac{1}{4}(2000 + 438 + 1188 + 500) \approx 1032$$

$$u_4^{(2)} = \frac{1}{4}(1032 + 0 + 782 + 0) \approx 454$$

Third iteration : (put $n = 2$)

$$u_1^{(3)} = \frac{1}{4}(2000 + 782 + 1000 + 1032) \approx 1204$$

$$u_2^{(3)} = \frac{1}{4}(1204 + 500 + 1000 + 454) \approx 789$$

$$u_3^{(3)} = \frac{1}{4}(2000 + 454 + 1204 + 500) \approx 1040$$

$$u_4^{(3)} = \frac{1}{4}(1040 + 0 + 789 + 0) \approx 458$$

Similarly, $u_1^{(4)} \approx 1207$, $u_2^{(4)} \approx 791$, $u_3^{(4)} \approx 1041$, $u_4^{(4)} = 458$

and $u_1^{(5)} = 1208$, $u_2^{(5)} = 791.5$, $u_3^{(5)} = 1041.5$, $u_4^{(5)} = 458.25$

Thus there is no significant difference between the fourth and fifth iteration values.

Hence $u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$ and $u_4 = 458$.

Example 33.4. *Solve the Laplace equation $u_{xx} + u_{yy} = 0$ given that (Fig. 33.8).*

Solution. We first find the initial values in the following order :

$$u_5 = \frac{1}{4}(0 + 17 + 21 + 12.1) = 12.5 \quad \text{(Std. formula)}$$

$$u_1 = \frac{1}{4}(0 + 12.5 + 0 + 17) = 7.4 \quad \text{(Diag. formula)}$$

$$u_3 = \frac{1}{4}(12.5 + 18.6 + 17 + 21) = 17.28 \quad \text{(Diag. formula)}$$

$$u_7 = \frac{1}{4}(12.5 + 0 + 0 + 12.1) = 6.15 \quad \text{(Diag. formula)}$$

$$u_9 = \frac{1}{4}(12.5 + 9 + 21 + 12.1) = 13.65 \quad \text{(Diag. formula)}$$

$$u_2 = \frac{1}{4}(17 + 12.5 + 7.4 + 17.3) = 13.55 \quad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(7.4 + 6.2 + 0 + 12.5) = 6.52 \quad \text{(Std. formula)}$$

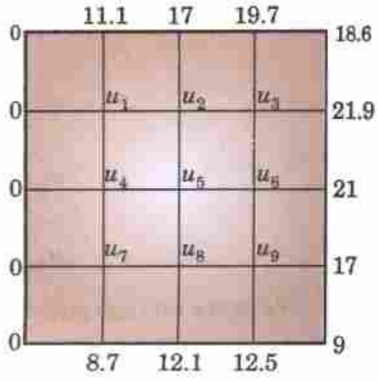

Fig. 33.8

$$u_6 = \frac{1}{4}(17.3 + 13.7 + 12.5 + 21) = 16.12 \quad \text{(Std. formula)}$$

$$u_8 = \frac{1}{4}(12.5 + 12.1 + 6.2 + 13.7) = 11.12 \quad \text{(Std. formula)}$$

Now we carry out the iteration process using the Standard formula :

$$u_1^{(n+1)} = \frac{1}{4}\left[(0 + 11.1 + u_4^{(n)} + u_2^{(n)})\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[(u_1^{(n+1)} + 17 + u_5^{(n)} + u_3^{(n)})\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[(u_2^{(n+1)} + 19.7 + u_6^{(n)} + 21.9)\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[(u_1^{(n+1)} + 19.7 + u_7^{(n)} + u_5^{(n)})\right]$$

$$u_5^{(n+1)} = \frac{1}{4}\left[(u_4^{(n+1)} + u_2^{(n+1)} + u_8^{(n)} + u_6^{(n)})\right]$$

$$u_6^{(n+1)} = \frac{1}{4}\left[(u_5^{(n+1)} + u_3^{(n+1)} + u_9^{(n)} + 21)\right]$$

$$u_7^{(n+1)} = \frac{1}{4}\left[0 + (u_4^{(n+1)} + 8.7 + u_8^{(n)})\right]$$

$$u_8^{(n+1)} = \frac{1}{4}\left[(u_7^{(n+1)} + u_5^{(n+1)} + 12.1 + u_9^{(n)})\right]$$

$$u_9^{(n+1)} = \frac{1}{4}\left[(u_8^{(n+1)} + u_6^{(n)} + 12.8 + 17)\right]$$

First iteration (put $n = 0$, in the above results)

$$u_1^{(1)} = \frac{1}{4}\left(0 + 11.1 + u_4^{(o)} + u_2^{(o)}\right) = \frac{1}{4}\ (0 + 11.1 + 6.52 + 13.55) = 7.79$$

$$u_2^{(1)} = \frac{1}{4}\ (7.79 + 17 + 12.5 + 17.28) = 13.64$$

$$u_3^{(1)} = \frac{1}{4}\ (13.64 + 19.7 + 16.12 + 21.9) = 12.84$$

$$u_4^{(1)} = \frac{1}{4}\ (0 + 7.79 + 6.15 + 12.5) = 6.61$$

$$u_5^{(1)} = \frac{1}{4}\ (6.61 + 13.64 + 11.12 + 16.12) = 11.88$$

$$u_6^{(1)} = \frac{1}{4}\ (11.88 + 17.84 + 13.65 + 21) = 16.09$$

$$u_7^{(1)} = \frac{1}{4}\ (0 + 6.61 + 8.7 + 11.12) = 6.61$$

$$u_8^{(1)} = \frac{1}{4}\ (6.61 + 11.88 + 12.1 + 13.65) = 11.06$$

$$u_9^{(1)} = \frac{1}{4}\ (11.06 + 16.09 + 12.8 + 17) = 12.238$$

Second iteration (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}\ (0 + 11.1 + 6.61 + 13.64) = 7.84$$

$$u_2^{(2)} = \frac{1}{4}\ (7.84 + 17 + 11.88 + 17.84) = 16.64$$

$$u_3^{(2)} = \frac{1}{4}\ (13.64 + 19.7 + 16.09 + 21.9) = 17.83$$

$$u_4^{(2)} = \frac{1}{4}\ (0 + 7.84 + 6.61 + 11.88) = 6.58$$

$$u_5^{(2)} = \frac{1}{4}(6.58 + 13.64 + 11.06 + 16.09) = 11.84$$

$$u_6^{(2)} = \frac{1}{4}(11.84 + 17.83 + 14.24 + 21) = 16.23$$

$$u_7^{(2)} = \frac{1}{4}(0 + 6.58 + 8.7 + 11.06) = 6.58$$

$$u_8^{(2)} = \frac{1}{4}(6.58 + 11.84 + 12.1 + 14.24) = 11.19$$

$$u_9^{(2)} = \frac{1}{4}(11.19 + 16.23 + 12.8 + 17) = 14.30$$

Third iteration (put $n = 2$)

$$u_1^{(3)} = \frac{1}{4}(0 + 11.1 + 6.58 + 13.64) = 7.83$$

$$u_2^{(3)} = \frac{1}{4}(7.83 + 17 + 11.84 + 17.83) = 13.637$$

$$u_3^{(4)} = \frac{1}{4}(13.63 + 19.7 + 16.23 + 21.9) = 17.86$$

$$u_4^{(3)} = \frac{1}{4}(0 + 7.83 + 6.58 + 11.84) = 6.56$$

$$u_5^{(3)} = \frac{1}{4}(6.56 + 13.63 + 11.19 + 16.23) = 11.90$$

$$u_6^{(3)} = \frac{1}{4}(11.90 + 17.86 + 14.30 + 21) = 16.27$$

$$u_7^{(3)} = \frac{1}{4}(0 + 6.56 + 8.7 + 11.19) = 6.61$$

$$u_8^{(3)} = \frac{1}{4}(6.61 + 11.90 + 12.1 + 14.30) = 11.23$$

$$u_9^{(3)} = \frac{1}{4}(11.23 + 16.27 + 12.8 + 17) = 14.32$$

Similarly, $u_1^{(4)} = 7.82, u_2^{(4)} = 13.65, u_3^{(4)} = 17.88, u_4^{(4)} = 6.58, u_5^{(4)} = 11.94, u_6^{(4)} = 16.28,$

$u_7^{(4)} = 6.63, u_8^{(4)} = 11.25, u_9^{(4)} = 14.33$

$u_1^{(5)} = 7.83, u_2^{(5)} = 13.66, u_3^{(5)} = 17.89, u_4^{(5)} = 6.50, u_5^{(5)} = 11.95, u_6^{(5)} = 16.29,$

$u_7^{(5)} = 6.64, u_8^{(5)} = 11.25, u_9^{(5)} = 14.34$

33.6 SOLUTION OF POISSON'S EQUATION*

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{f(x, y)} \qquad ...(1)$$

This is an *elliptic equation* which can be solved numerically at the interior mesh points of a square network when the boundary values are known. The standard 5-point formula for (1) takes the form

$$u_{i-1,j} + u_{i+1,j} + u_{i,\,j+1} + u_{i,\,j-1} - 4u_{i,j} = h^2 f(ih, jh) \qquad ...(2)$$

By applying (2) at each mesh-point, we arrive at linear equations in the pivotal values i, j. These equations can be solved by Gauss-Seidal iteration method (p. 938).

Example 33.5. *Solve the Poisson equation $u_{xx} + u_{yy} = -81xy$, $0 < x < 1$, $0 < y < 1$ given that $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = 100$, $u(x, 1) = 100$ and $h = 1/3$.* (Anna, 2005)

Solution. Here $h = 1/3$ $u_{i,j-1}$ $u_{i,j}$ (Fig. 33.9)

* See p. 882.

The standard 5-point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4\,u_{i,j} = h^2 f(ih, jh) = h^2\,[-81\,(ih\,.\,jh)] = h^4\,(-81)\,ij = -ij \quad ...(i)$$

For u_1 $(i = 1, j = 2)$, (i) gives $0 + u_2 + u_3 + 100 - 4u_1 = -2$

i.e., $$-4u_1 + u_2 + u_3 = -102 \quad ...(ii)$$

For u_2 $(i = 2, j = 2)$, (i) gives $u_1 + 100 + u_4 + 100 - 4u_2 = -4$

i.e., $$u_1 - 4u_2 + u_4 = -204 \quad ...(iii)$$

For u_3 $(i = 1, j = 1)$, (i) gives $0 + u_4 + 0 + u_1 - 4u_3 = -1$

i.e., $$u_1 - 4u_3 + u_4 = -1 \quad ...(iv)$$

For u_4 $(i = 2, j = 1)$, gives $u_3 + 100 + u_2 - 4u_4 = -2$

$$u_2 + u_3 - 4u_4 = -102 \quad ...(v)$$

Subtracting (v) from (ii), $-4u_1 + 4u_4 = 0$ *i.e.* $u_1 = u_4$

Then (iii) becomes $2u_1 - 4u_2 = -204$ ...(vi)

and (iv) becomes $2u_1 - 4u_3 = -1$ ...(vii)

Now $(4) \times (ii) + (vi)$ gives $-14u_1 + 4u_3 = -612$ ...$(viii)$

$(vii) + (viii)$ gives $-12u_1 =- 613$

Thus $$u_1 = 613/12 = 51.0833 = u_4.$$

From (vi), $$u_2 = \frac{1}{2}\,(u_1 + 102) = 76.5477$$

From (vii), $$u_3 = \frac{1}{2}\left(u_1 + \frac{1}{2}\right) = 25.7916.$$

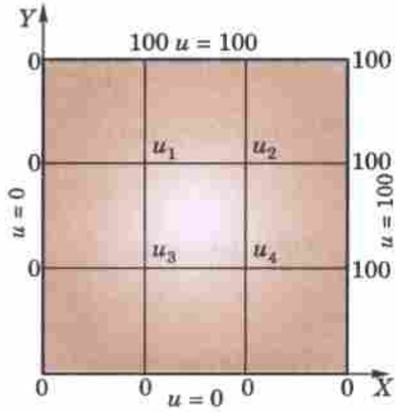

Fig. 33.9

Example 33.6. *Solve the partial differential equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square with sides $x = 0 = y$, $x = 3 = y$ with $u = 0$ on the boundary and mesh length = 1.*

(Anna, 2007 ; P.T.U., 2007 ; Delhi, 2002)

Solution. Here $h = 1$ (Fig. 33.10).

$\therefore$ The standard 5-point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -10(i^2 + j^2 + 10) \quad ...(i)$$

For $u_1 (i = 1, j = 2)$, (i) gives

$$0 + u_2 + 0 + u_3 - 4u_1 = -10(1 + 4 + 10)$$

i.e., $$u_1 = \frac{1}{4}(u_2 + u_3 + 150) \quad ...(ii)$$

For $u_2 (i = 2, j = 2)$, (i) gives $u_2 = \dfrac{1}{4}(u_1 + u_4 + 180)$ ...(iii)

For $u_3 (i = 1, j = 1)$, we have $u_3 = \dfrac{1}{4}(u_1 + u_4 + 120)$ ...(iv)

For $u_4 (i = 2, j = 1)$, we have $u_4 = \dfrac{1}{4}(u_2 + u_3 + 150)$ ...(v)

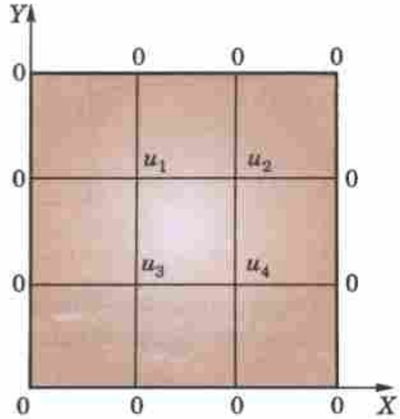

Fig. 33.10

Equations (ii) and (v) show that $u_4 = u_1$. Thus the above equations reduce to

$$u_1 = \frac{1}{4}(u_2 + u_3 + 150) \quad ...(vi)$$

$$u_2 = \frac{1}{2}(u_1 + 90) \quad ...(vii)$$

$$u_3 = \frac{1}{2}(u_1 + 60) \quad ...(viii)$$

Now let us solve these equations by Gauss-Seidal iteration method.

First iteration : Starting from the approximations $u_2 = 0$, $u_3 = 0$, we obtain

$$u_1^{(1)} = 37.5. \text{ Then } u_2^{(1)} = \frac{1}{2}(37.5 + 90) \approx 64\ ;\ u_3^{(1)} = \frac{1}{2}(37.5 + 60) \approx 49$$

Second iteration :

$$u_1^{(2)} = \frac{1}{4}(64 + 49 + 150) = 66 \; ; \; u_2^{(2)} = \frac{1}{2}(66 + 90) = 78 \; ; \; u_3^{(2)} = \frac{1}{2}(66 + 60) = 63$$

Third iteration :

$$u_1^{(3)} = \frac{1}{4}(78 + 63 + 150) \approx 73 \; ; \; u_2^{(3)} = \frac{1}{2}(73 + 90) \approx 82 \; ; \; u_3^{(2)} = \frac{1}{2}(73 + 60) \approx 67$$

Fourth iteration :

$$u_1^{(4)} = \frac{1}{4}(82 + 67 + 150) \approx 75 \; ; \; u_2^{(4)} = \frac{1}{2}(75 + 90) = 82.5 \; ; \; u_3^{(4)} = \frac{1}{2}(75 + 60) = 67.5$$

Fifth iteration :

$$u_1^{(5)} = \frac{1}{4}(82.5 + 67.5 + 150) \approx 75 \; ; \; u_2^{(5)} = \frac{1}{2}(75 + 90) = 82.5 \; ; \; u_3^{(5)} = \frac{1}{2}(75 + 60) = 67.5$$

Since these values are the same as those of fourth iteration, we have

$$u_1 = 75, u_2 = 82.5, u_3 = 67.5 \text{ and } u_4 = 75.$$

PROBLEMS 33.2

1. Solve the equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in Fig. 33.11. Iterate until the maximum difference between the successive values at any point is less than 0.001. *(Delhi, 2002)*
2. Solve $\nabla^2 u = 0$ under the conditions $(h = 1, k = 1)$, $u(0, y) = 0$, $u(4, y) = 12 + y$ for $0 \le y \le 4$; $u(x, 0) = 3x$, $u(x, 4) = x^2$ for $0 \le x \le 4$. *(Cusat, 2008)*
3. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the square mesh with boundary values as shown in Fig. 33.12. Iterate until the maximum difference between successive values at any point is less than 0.005.

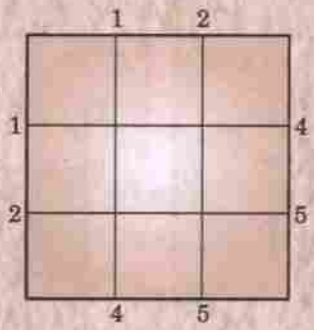

Fig. 33.11

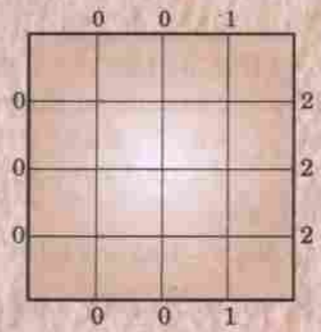

Fig. 33.12

4. Using central-difference approximation solve $\nabla^2 u = 0$ at the nodal points of the square grid of Fig. 33.13 using the boundary values indicated. *(V.T.U., 2000)*

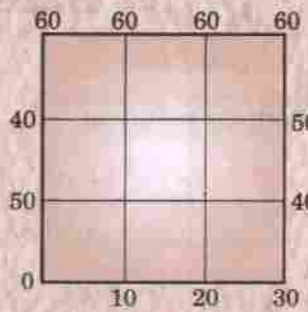

Fig. 33.13

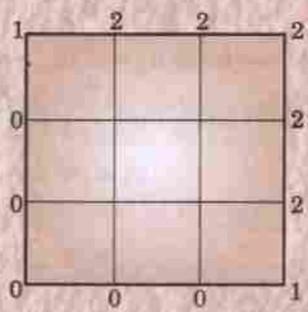

Fig. 33.14

5. Solve $u_{xx} + u_{yy} = 0$ for the square mesh with boundary values as shown in Fig. 33.14. Iterate till the mesh values are correct to two decimal places.

6. Solve the Laplace's equation $u_{xx} + u_{yy} = 0$ in the domain of Fig. 33.15.

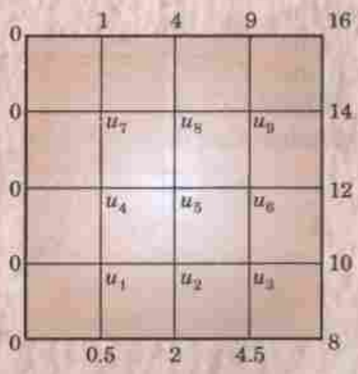

Fig. 33.15

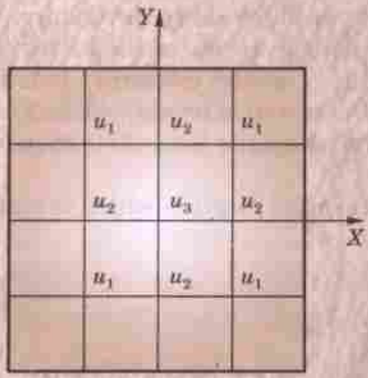

Fig. 33.16

7. Solve the Poisson's equation $\nabla^2 u = 8x^2y^2$ for the square mash of Fig. 33.16 with $u(x, y) = 0$ on the boundary and mesh length = 1. *(J.N.T.U., 2004 S)*

Note. Solution of elliptic equations by *Relaxation method* is given in author's book *'Numerical Methods in Engineering and Science'*.

33.7 PARABOLIC EQUATIONS

The one-dimensional heat conduction equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is a well-known example of parabolic partial differential equations. The solution of this equation is a temperature function $u(x, t)$ which is defined for values of x from 0 to l and for values of time t from 0 to ∞. The solution is not defined in a closed domain but advances in an open-ended region from initial values, satisfying the prescribed boundary conditions. (Fig. 33.17).

In general, the study of pressure waves in a fluid, propagation of heat and unsteady state problems lead to parabolic type of equations.

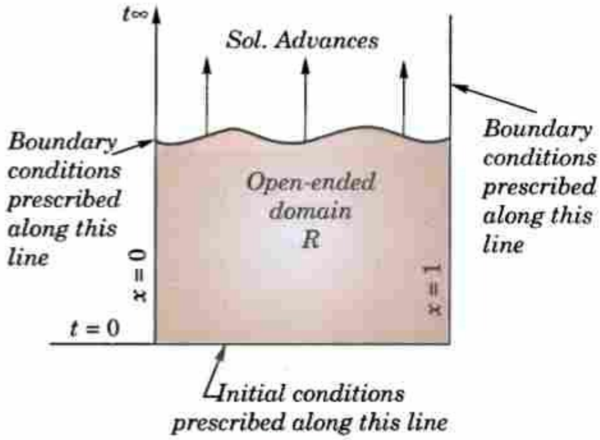

Fig. 33.17

33.8 SOLUTION OF HEAT EQUATION

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

where $c^2 = k/s\rho$ is the diffusivity of the substance (cm²/sec.)

Consider a rectangular mesh in the x-t plane with spacing h along x direction and k along time t direction. Denoting a mesh point $(x, t) = (ih, jk)$ as simply i, j, we have

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k} \qquad \text{[By (5) § 33.3]}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \qquad \text{[By (4) § 33.3]}$$

Substituting these in (1), we obtain

$$u_{i,j+1} - u_{i,j} = \frac{kc^2}{h^2}\,[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$$

or $$u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha)\, u_{i,j} + \alpha u_{i+1,j} \qquad ...(2)$$

where $\alpha = kc^2/h^2$ is the mesh ratio parameter.

This formula enables us to determine the value of u at the $(i, j+1)$th mesh point in terms of the known function values at the points x_{i-1}, x_i and x_{i+1} at the instant t_j. It is a relation between the function values at the two time levels $j+1$ and j and is therefore, called a *2-level formula*. In schematic form (2) is shown in Fig. 33.18 which is called the *Schmidt explicit formula* which is valid only for $0 < \alpha \le \dfrac{1}{2}$.

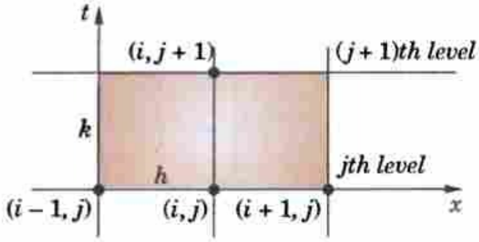

Fig. 33.18

Obs. In particular when $\alpha = \dfrac{1}{2}$, (2) reduces to

$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j}) \qquad ...(3)$$

which shows that the *value of u at x_i at time t_{j+1} is the mean of the u-values at x_{i-1} and x_{i+1} at time t_j*. This relation, known as *Bendre-Schmidt recurrence relation*, gives the values of u at the internal mesh points with the help of boundary conditions.

Note. The other formulae (*i.e. Crank-Nicolson formula* and *Du Fort-Frankel formula*) are given in author's book *'Numerical Methods in Engineering and Science'*.

Example 33.7. *Solve* $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial^2 x^2}$ *in* $0 < x < 5$, $t \ge 0$ *given that* $u(x, 0) = 20$, $u(0, t) = 0$, $u(5, t) = 100$. *Compute u for the time-step with h = 1 by Crank-Nicholson method.* (Anna, 2006)

Solution. Here $c^2 = 1$ and $h = 1$.

Taking α (*i.e.*, c^2k/h) = 1, we get $k = 1$

Also we have

j \ i	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

Then Crank-Nicholson formula becomes

$$4u_{i,j+1} = u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}$$

$$4u_1 = 0 + 20 + 0 + u_2 \quad i.e. \quad 4u_1 - u_2 = 20 \qquad ...(1)$$

$$4u_2 = 20 + 20 + u_1 + u_3 \quad i.e. \quad 4u_1 - 4u_2 + u_3 = -40 \qquad ...(2)$$

$$4u_3 = 20 + 20 + u_2 + u_4 \quad i.e. \quad u_2 - 4u_3 + u_4 = -40 \qquad ...(3)$$

$$4u_4 = 20 + 100 + u_3 + 100 \quad i.e. \quad 4u_3 - 4u_4 + = -220 \qquad ...(4)$$

Now (1) – 4(2) gives $15u_2 - 4u_3 = 180$ ...(5)

4(3) + (4) gives $4u_2 - 15u_3 = -380$ ...(6)

Then 15(5) – 4(6) gives $209u_2 = 4220$ *i.e.* $u_2 = 20.2$

From (5), we get $4u_3 = 15 \times 20.2 - 180$ *i.e.* $u_3 = 30.75$

From (1), $4u_1 = 20 + 20.2$ *i.e.* $u_1 = 10.05$

From (4), $4u_4 = 220 + 30.75$ *i.e.* $u_4 = 62.69$

Thus the required values are 10.05, 20.2, 30.75 and 62.68.

Example 33.8. *Solve the boundary value problem $u_t = u_{xx}$ under the conditions $u(0, t) = u(1, t) = 0$ and $u(x, 0) = \sin \pi x$, $0 \le x \le 1$ using Schmidt method (Take h = 0.2 and α = 1/2).* (Rohtak, 2003)

Solution. Since $h = 0.2$ and $\alpha = 1/2$

$\therefore$ $\alpha = \dfrac{k}{h^2}$ gives $k = 0.02$

Since $\alpha = 1/2$, we use Bendre-Schmidt relation

$$u_{i,j+1} = \frac{1}{2}(i_{i-1,j} + u_{i+1,j}) \quad ...(i)$$

We have $u(0, 0) = 0, u(0.2, 0) = \sin \pi/5 = 0.5875$

$$u(0.4, 0) = \sin 2\pi/5 = 0.95111, u(0.6, 0) = \sin 3\pi/5 = 0.9511$$

$$u(0.8, 0) = \sin 4\pi/5 = 0.5875, u(1, 0) = \sin \pi = 0$$

The value of u at the mesh points can be obtained by using the recurrence relation (*i*) as shown in table below :

	$x \to 0$		0.2	0.4	0.6	0.8	1.0
$t \downarrow$	$j \backslash i$	0	1	2	3	4	5
0	0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	1	0	0.4756	0.7695	0.7695	0.4756	0
0.04	2	0	0.3848	0.6225	0.6225	0.3848	0
0.06	3	0	0.3113	0.5036	0.5036	0.3113	0
0.08	4	0	0.2518	0.4074	0.4074	0.2518	0
0.1	5	0	0.2037	0.3296	0.3296	0.2037	0

Example 33.9. *Find the values of $u(x, t)$ satisfying the parabolic equation $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0, t) = 0 = u(8, t)$ and $u(x, 0) = 4x - \frac{1}{2}x^2$ at the points $x = i : i = 0, 1, 2, ..., 8$ and $t = \frac{1}{8}j : j = 0, 1, 2, ..., 5$.*

Solution. Here $c^2 = 4$, $h = 1$ and $k = 1/8$. Then $\alpha = c^2k/h^2 = 1/2$.

$\therefore$ we have Bendre-Schmidt's recurrence relation

$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j}) \quad ...(i)$$

Now since $u(0, t) = 0 = u(8, t)$

$\therefore$ $u_{0,i} = 0$ and $u_{8,j} = 0$ for all values of j, *i.e.*, the entries in the first and last column are zero.

Since $u(x, 0) = 4x - \frac{1}{2}x^2$

$$\therefore \quad u_{i,0} = 4i - \frac{1}{2}i^2 = 0, 3.5, 6, 7.5, 8, 7.5, 6, 3.5$$

for $i = 0, 1, 2, 3, 4, 5, 6, 7$ at $t = 0$

These are the entries of the first row.

$j \backslash i$	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0

Putting $j = 0$ in (i), we have

$$u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$$

Taking $i = 1, 2, \ldots, 7$ successively, we get

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0 + 6) = 3$$

$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{1}{2}(3.5 + 7.5) = 5.5$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = \frac{1}{2}(6 + 8) = 7$$

$$u_{4,1} = 7.5,\ u_{5,1} = 7,\ u_{6,1} = 5.5,\ u_{7,1} = 3.$$

These are the entries in the second row.

Putting $j = 1$ in (i), the entries of the third row are given by

$$u_{i,2} = \frac{1}{2}(u_{i-1,1} + u_{i+1,1})$$

Similarly putting $j = 2, 3, 4$ successively in (i), the entries of the fourth, fifth and sixth rows are obtained. Hence the values of $u_{i,j}$ are as given in the above table.

Example 33.10. *Solve the equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions $u(x, 0) = \sin \pi x$, $0 \le x \le 1$; $u(0, t) = u(1, t) = 0$. Carry out computations for two levels, taking $h = 1/3$, $k = 1/36$. (V.T.U., 2005)

Solution. Here $c^2 = 1$, $h = 1/3$, $k = 1/36$ so that

$$\alpha = kc^2/h^2 = 1/4.$$

Also $u_{1,0} = \sin \pi/3 = \sqrt{3}/2$, $u_{2,0} = \sin 2\pi/3 = \sqrt{3}/2$ and all other boundary values are zero as shown in Fig. 33.19.

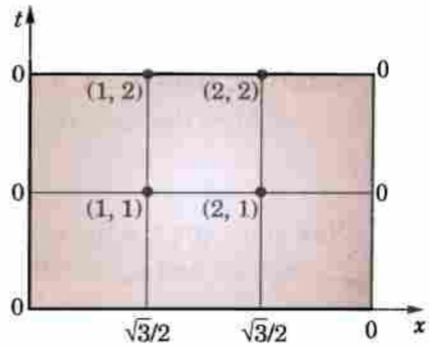

Fig. 33.19

Schmidt's formula [(2) of § 33.8]

$$u_{i,j+1} = \alpha u_{i-1,j} + (1 - 2\alpha)\, u_{i,j} + \alpha u_{i+1,j}$$

becomes $$u_{i,j+1} = \frac{1}{4}[u_{i-1,j} + 2u_{i,j} + u_{i+1,j}]$$

For $i = 1, 2$; $j = 0$:

$$u_{1,1} = \frac{1}{4}[u_{0,0} + 2u_{1,0} + u_{2,0}] = \frac{1}{4}(0 + 2 \times \sqrt{3/2} + \sqrt{3/2}) = 0.65$$

$$u_{2,1} = \frac{1}{4}[u_{1,0} + 2u_{2,0} + u_{3,0}] = \frac{1}{4}(\sqrt{3/2} + 2 \times \sqrt{3/2} + 0) = 0.65$$

For $i = 1, 2, j = 1$:

$$u_{1,2} = \frac{1}{4}(u_{0,1} + 2u_{1,1} + u_{2,1}) = 0.49$$

$$u_{2,2} = \frac{1}{4}(u_{1,1} + 2u_{2,1} + u_{3,1}) = 0.49.$$

PROBLEMS 33.3

1. Find the solution of the parabolic equation $u_{xx} = 2u_t$ when $u(0, t) = u(4, t) = 0$ and $u(x, 0) = x(4 - x)$, taking $h = 1$. Find the values upto $t = 5$. (*Madras, 2001*)

2. Solve the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ with the conditions $u(0, t) = 0$, $u(x, 0) = x(1 - x)$ and $u(1, t) = 0$. Assume $h = 0.1$. Tabulate u for $t = k$, $2k$ and $3k$ choosing an appropriate value of k. (*Anna, 2004*)

3. Given $\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 0$; $f(0, t) = f(5, t) = 0$, $f(x, 0) = x^2(25 - x^2)$; find the values of f for $x = ih(i = 0, 1, ..., 5)$ and $t = jk(j = 0, 1, ..., 6)$ with $h = 1$ and $k = \frac{1}{2}$, using the explicit method.

4. Solve the heat equation $\partial u/\partial t = \partial^2 u/\partial x^2$ subject to the conditions $u(0, t) = u(1, t) = 0$ and

$$u(x, 0) = 2x \text{ for } 0 \le x \le \frac{1}{2}$$
$$= 2(1 - x) \text{ for } \frac{1}{2} \le x \le 1.$$

Take $h = 1/4$ and k according to Bendre-Schmidt equation.

33.9 HYPERBOLIC EQUATIONS

The wave equation : $\frac{\partial^2 u}{dt^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is the simplest example of hyperbolic partial differential equations. Its solution is the displacement function $u(x, t)$ defined for values of x from 0 to l and for t from 0 to ∞, satisfying the initial and boundary conditions. The solution as for parabolic equations, advances in an open-ended region (Fig. 33.17). In the case of hyperbolic equations however, we have two initial conditions and two boundary conditions.

Such equations arise from convective type of problems in vibrations, wave mechanics and gas dynamics.

33.10 SOLUTION OF WAVE EQUATION

$$\frac{\partial^2 u}{dt^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

subject to the initial conditions : $u = f(x)$, $\frac{\partial u}{\partial t} = g(x)$, $0 \le x \le 1$ at $t = 0$ ...(2)

and the boundary conditions : $u(0, t) = \phi(t)$, $u(1, t) = \psi(t)$. ...(3)

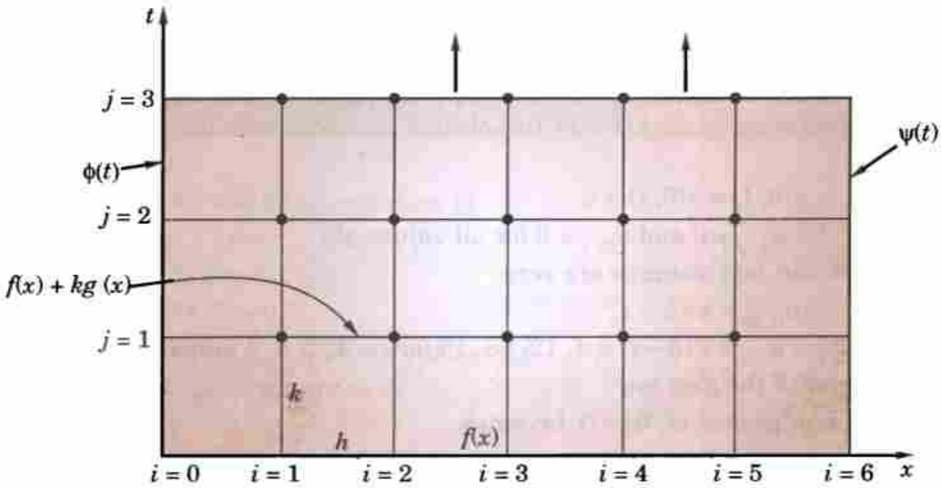

Fig. 33.20

Consider a rectangular mesh in the x-t plane spacing h along x direction and k along time t direction. Denoting a mesh point $(x, t) = (ih, jk)$ as simply i, j, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

Replacing the derivatives in (1) by their above approximations, we obtain

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = \frac{c^2k^2}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i,j+1})$$

or
$$u_{i,j+1} = 2(1-\alpha^2c^2)\,u_{i,j} + \alpha^2c^2\,(u_{i-1,j} + u_{i+1,j} - u_{i,j-1}) \qquad ...(4)$$

where $\alpha = k/h$.

Now replacing the derivative in (2) by its central difference approximation, we get

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{\partial u}{\partial t} = g(x)$$ [See (7) p. 1042]

or
$$u_{i,j+1} = u_{i,j-1} + 2k\,g(x) \quad \text{at } t=0 \quad i.e.\ u_{i,1} = u_{i,-1} + 2kg(x) \text{ for } j=0 \qquad ...(5)$$

Also initial condition $u = f(x)$ at $t = 0$ becomes $u_{i,0} = f(x)$ ...(6)

Combining (5) and (6), we have $u_{i,1} = f(x) + 2kg(x)$ ...(7)

Also (3) gives $\quad u_{0,j} = \phi(t)$ and $u_{1,j} = \psi(t)$

Hence (4) gives the values of $u_{i,j+1}$ at the $(j+1)$th level when the nodal values at $(j-1)$th and jth levels are known from (6) and (7) as shown in Fig. 32.20. Thus (4) gives an ***implicit scheme*** *for the solution of the wave equation.*

A special case : The coefficient of $u_{i,j}$ in (4) will vanish if $\alpha = 1/c$ or $k = h/c$. Then (4) reduces to the *simple form*

$$\mathbf{u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}} \qquad ...(8)$$

This result provides an ***explicit scheme*** *for the solution of the wave equation.*

Obs. 1. *For $\alpha = 1/c$,* the solution of (4) is stable and coincides with the solution of (1).
For $\alpha < 1/c$, the solution is stable but inaccurate.
For $\alpha > 1/c$, the solution is unstable.

Obs. 2. *The formula (4) converges for $\alpha \le 1$ i.e. for $k \le h$.*

Example 33.11. *Evaluate the pivotal values of the equation $u_{tt} = 16u_{xx}$, taking $h = 1$ upto $t = 1.25$. The boundary conditions are $u(0, t) = u(5, t) = 0$, $u_t(x, 0) = 0$ and $u(x, 0) = x^2(5-x)$.* (Madras, 2006)

Solution. Here $c^2 = 16$.

$\therefore$ The difference equation for the given equation is

$$u_{i,j+1} = 2(1-16\alpha^2)\,u_{i,j} + 16\alpha^2\,(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad \text{where } \alpha = k/h \qquad ...(i)$$

Taking $h = 1$ and choosing k so that the coefficient of $u_{i,j}$ vanishes, we have $16\alpha^2 = 1$, *i.e.*, $k = h/4, = 1/4$.

$\therefore$ (*i*) reduces to $\quad u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ ...(*ii*)

which gives a convergent solution (since $k/h < 1$). Its solution coincides with the solution of the given differential equation.

Now since $\quad u(0, t) = u(5, t) = 0.$

$\therefore \quad u_{0,j} = 0$ and $u_{5,j} = 0$ for all values of j

i.e. the entries in the first and last columns are zero.

Since $\quad u_{(x,0)} = x^2(5-x)$

$\therefore \quad u_{i,0} = i^2(5-i) = 4, 12, 18, 16$ for $i = 1, 2, 3, 4$ at $t = 0$.

These are the entries of the *first row.*

Finally the initial condition $u_t(x, 0) = 0$, becomes

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0, \text{ when } j = 0, \text{ giving } u_{i,1} = u_{i,-1} \qquad ...(iii)$$

Putting $j = 0$ in (*ii*), $\quad u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$

$\qquad = u_{i-1,0} + u_{i+1,0} - u_{i,1}$ using (*iii*)

or
$$u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0}) \qquad ...(iv)$$

Taking $i = 1, 2, 3, 4$ successively, we obtain

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0 + 12) = 6$$

$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{1}{2}(4 + 18) = 11$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = \frac{1}{2}(12 + 16) = 14$$

$$u_{4,1} = \frac{1}{2}(u_{3,0} + u_{5,0}) = \frac{1}{2}(18 + 0) = 9$$

These are the entries of the *second row.*

Putting $j = 1$ in (*ii*), we get

$$u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$$

Taking $i = 1, 2, 3, 4$ scccessively, we obtain

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 11 - 4 = 7$$
$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 6 + 14 - 12 = 8$$
$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 11 + 9 - 18 = 2$$
$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 14 + 0 - 16 = -2$$

These are the entries of the *third row.*

Similarly putting $j = 2, 3, 4$ successively in (*ii*), the entries of the fourth, fifth and sixth rows are obtained.

Hence the values of $u_{i,j}$ are as shown in the table below :

j \ i	0	1	2	3	4	5
0	0	4	12	18	16	0
1	0	6	11	14	9	0
2	0	7	8	2	−2	0
3	0	2	−2	−8	−7	0
4	0	−9	−14	−11	−6	0
5	0	−16	−18	−12	−4	0

Example 33.12. *The transverse displacement u of a point at a distance x from one end and at any time t of a vibrating string satisfies the equation $\partial^2u/\partial t^2 = 4\partial^2u/\partial x^2$, with boundary conditions $u = 0$ at $x = 0, t > 0$ and $u = 0$ at $x = 4, t > 0$ and initial conditions $u = x(4 - x)$ and $\partial u/\partial t = 0$ at $t = 0, 0 \le x \le 4$. Solve this equation numerically for one half period of vibration, taking $h = 1$ and $k = 1/2$.*

Solution. Here, $h/k = 2 = c$.

$\therefore$ the difference equation for the given equation is

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \quad \text{...(i)}$$

which gives a convergent solution (since $k < h$).

Now since $u(0, t) = u(4, t) = 0,$

$\therefore$ $u_{0,j} = 0$ and $u_{4,j} = 0$ for all values of j.

i.e., the entries in the first and last columns are zero.

Since $u_{(x,0)} = x(4 - x),$

$\therefore$ $u_{i,0} = i(4 - i) = 3, 4, 3$ for $i = 1, 2, 3$ at $t = 0$.

These are the entries of the *first row.*

Also $u_t(x, 0) = 0$ becomes

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0 \text{ when } j = 0, \text{ giving } u_{i,1} = u_{i,-1} \quad \text{...(ii)}$$

Putting $j = 0$ in (*i*), $u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$

$$= u_{i-1,0} + u_{i+1,0} - u_{i,1}, \text{ using (ii)}$$

or

$$u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0}) \quad \text{...(iii)}$$

Taking $i = 1, 2, 3$ successively, we obtain

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{1,0}) = 2 \; ; \; u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = 3$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = 2$$

These are the entries of the *2nd row*.

Putting $j = 1$ in (i), $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Taking $i = 1, 2, 3$, successively, we get

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 3 - 3 = 0$$
$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 2 + 2 - 4 = 0$$
$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 3 + 0 - 3 = 0$$

These are the entries of the *3rd row* and so on.

Now the equation of the vibrating string of length l is $u_{tt} = c^2 u_{xx}$.

$\therefore$ Its period of vibration $= \dfrac{2l}{c} = \dfrac{2 \times 4}{2} = 4$ sec. $[\because \; l = 4 \text{ and } c = 2]$

This shows that we have to compute $u_{(x,t)}$ upto $t = 2$.

i.e., similarly we obtain the values of $u_{i,2}$ (4th row) and $u_{i,3}$ (5th row).

Hence the values of $u_{i,j}$ are as shown in the table below :

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	0	0	0	0
3	0	−2	−3	−2	0
4	0	−3	−4	−3	0

Example 33.13. *Find the solution of the initial boundary value problem ; $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$, $0 \le x \le 1$; subject to the initial conditions $u(x, 0) = \sin \pi x$, $0 \le x \le 1$ and the boundary conditions $u(0, t) = 0$, $u(1, t) = 0$, $t > 0$; by using in the (a) the explicit scheme.*

(b) the implicit scheme. *(Anna, 2007)*

Solution. (*a*) *Explicit scheme*

Take $h = 0.2$, $k = \dfrac{h}{c} = 0.2$ $[\because \; c = 1]$

$\therefore$ We use the formula, $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ ...(*i*)

Since $u(0, t) = 0$, $u(1, t) = 0$, $u_{0,j} = 0$, $u_{1,j} = 0$ for all values of j

i.e., the entries in the first and last columns are zero.

Since $u(x, 0) = \sin \pi x$, $u_{i,0} = \sin \pi x$

$\therefore$ $u_{1,0} = 0$, $u_{2,0} = \sin(0.2\pi) = 0.5878$, $u_{3,0} = \sin(0.4\pi) = 0.9511$, $u_{4,0} = \sin(0.6\pi) = 0.5878$.

These are the entries of the first row.

Since $u_t(x, 0) = 0$ we have $\dfrac{1}{2}(u_{i,j+1} - u_{i,j-1} = 0$, when $j = 0$

i.e., $u_{i,1} = u_{i,-1}$ ...(*ii*)

Putting $j = 0$ in (*i*), $u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$

Using (*ii*) $u_{i,1} = \dfrac{1}{2}(u_{i-1,0} + u_{i+1,} u_{i+1,0})$

Taking $i = 1, 2, 3, 4$ successively, we obtain the entries of the second row.

Putting $j = 1$ in (*i*), $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Now taking $i = 1, 2, 3, 4$ successively, we get the entries of the third row.

Similarly taking $j = 2, j = 3, j = 4$ successively, we obtain the entries of the fourth, fifth and sixth rows respectively.

j \ i	0	1	2	3	4	5
0	0	0.5878	0.9511	0.9511	0.5878	0
1	0	0.4756	0.7695	0.9511	0.7695	0
2	0	0.1817	0.4756	0.5878	0.3633	0
3	0	0	0.0001	– 0.1122	– 0.1816	0
4	0	– 0.1816	– 0.5878	– 0.7694	0.4755	0
5	0	– 0.5878	– 0.9511	– 0.9511	– 0.5878	0

(*b*) *Implicit scheme*

We have the formula :

$$u_{i,j+1} = 2(1-\alpha^2 c^2)\, u_{i,j} + \alpha^2 c^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1}, \text{ where } \alpha = k/h \quad ...(i)$$

Here $c^2 = 1$. Take $h = 0.25$ and $k = 0.5$ so that $\alpha = k/h = 2$.

$\therefore$ (*i*) reduces to

$$u_{i,j+1} = -6u_{i,j} + 4(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad ...(ii)$$

Since $\quad u_{i,0} = \sin \pi x$

$\therefore \quad u_{(1,0)} = 0.7071, u_{(2,0)} = 0.5, u_{3,0} = 0.7071$

There are the entries of the first row.

Since $u_t(x, 0) = 0$, we have $\frac{1}{2}(y_{i,i+1} - y_{i,i-1}) = 0$, where $j = 0$

$$\therefore \quad y_{i,1} = y_{i,-1} \quad ...(iii)$$

Put $j = 0$ and using (*iii*), (*ii*) reduces to

$$u_{i,+1} = -3u_{i,0} + 2(u_{i-1,0} + 2u_{i+1,0})$$

Now taking
$$i = 1, u_{1,1} = -3u_{1,0} + 2(u_{0,0} + u_{2,0}) = 0.1213$$
$$i = 2, u_{2,1} = -3u_{2,0} + 2(u_{1,0} + u_{3,0}) = 0.1716$$
$$i = 3, u_{3,1} = -3u_{3,0} + 2(u_{2,0} + u_{4,0}) = 0.1213$$

These are the entries of the second row.

Putting $j = 1$, (*ii*) reduces to

$$u_{i,2} = -6u_{i,1} + 4(u_{i-1,2} + u_{i+1,1})$$

Now taking
$$i = 1, u_{1,2} = -6u_{1,1} + 4(u_{0,1} + u_{2,1}) = 0.414$$
$$i = 2, u_{2,2} = -6u_{2,1} + 4(u_{1,1} + u_{3,1}) = 0.0592$$
$$i = 3, u_{3,2} = -6u_{3,1} + 4(u_{2,1} + u_{4,1}) = 0.414$$

These are the entries of the third row.

Putting $j = 2$, (*ii*) reduces to

$$u_{i,3} = -6u_{i,2} + 4(u_{i-1,2} + u_{i+1,2}) - u_{i,1}$$

Now taking
$$i = 1, u_{1,3} = -6u_{1,2} + 4(u_{0,2} + u_{2,2}) - u_{1,1} = 0.1097$$
$$i = 2, u_{2,3} = -6u_{2,2} + 4(u_{1,2} + u_{3,2}) - u_{2,1} = 0.1476$$
$$i = 3, u_{3,3} = -6u_{3,2} + 4(u_{2,2} + u_{4,2}) - u_{4,2} = 0.1097$$

These are the entries of the third row.

Hence the value of $u_{i,j}$ are as tabulated below :

j \ i	0	1	2	3	4
0	0	0.7071	0.5	0.7071	0
1	0	– 0.1213	– 0.1716	– 0.1213	0
2	0	0.0414	0.0592	0.0414	0

PROBLEMS 33.4

1. Solve the boundary value problem $u_{tt} = u_{xx}$ with the conditions $u(0, t) = u(1, t) = 0$, $u(x, 0) = \frac{1}{2}x(1-x)$ and $u_t(x, 0) = 0$, taking $h = k = 0.1$ for $0 \le t \le 0.4$. Compare your solution with the exact solution at $x = 0.5$ and $t = 0.3$. *(V.T.U., 2000)*
2. The transverse diaplacement u of a point at a distance x from one end and at any time t of a vibrating string satisfies the equation $\partial^2u/\partial t^2 = 25\, \partial^2u/\partial x^2$, with the boundary conditions $u(0, t) = u(5, t) = 0$ and the initial conditions $u(x, 0) = \begin{cases} 20x & \text{for } 0 \le x \le 1 \\ 5(5-x) & \text{for } 1 \le x \le 5 \end{cases}$ and $u_t(x, 0) = 0$. Solve this equation numerically for one half period of vibration, taking $h = 1$, $k = 0.2$.
3. Solve $y_{tt} = y_{xx}$ upto $t = 0.5$ with a spacing of 0.1 subject to $y(0, t) = 0$, $y(1, t) = 0$, $y_t(x, 0) = 0$, and $y(x, 0) = 10 + x(1-x)$. *(Anna, 2004)*
4. The function u satisfies the equation
$$\partial^2u/\partial t^2 = \partial^2u/\partial x^2$$
and the conditions : $u(x, 0) = \frac{1}{8}\sin \pi x$, $u_t(x, 0) = 0$ for $0 \le x \le 1$,
$$u(0, t) = u(1, t) = 0 \text{ for } t \ge 0.$$
Use the explicit scheme to calculate u for $x = 0(0.1)\,1$ and $t = 0(0.1)\,0.5$.

33.11 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 33.5

Fill up the blanks or select the correct answer from each of the following questions :

1. Which of the following equations is parabolic :
(a) $f_{xy} - f_x = 0$ (b) $f_{xx} + 2f_{xy} + f_{yy} = 0$ (c) $f_{xx} + 2f_{xy} + 4f_{yy} = 0$ (d) none
2. $u_{ij} = \frac{1}{4}(u_{i+1,j} - u_{i-1,j} + u_{i,j+1} - u_{i,j-1})$ is Leibmann's five point formula. (True or False)
3. $u_{xx} + 3u_{xy} + u_{yy} = 0$ is classified as ......
4. $\nabla^2 u = f(x, y)$ is known as ......
5. The simplest formula to solve $u_{tt} = \alpha^2 u_{xx}$ is ......
6. The finite difference form of $\partial^2 u/\partial x^2$ is ......
7. Schmidt's finite difference scheme to solve $u_t = c^2 u_{xx}$ is ......
8. The 5-point diagonal formula gives u_{ij} = ...
9. The partial differential equation $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$ is classified as ......
10. $u_{i,j+1} = \frac{1}{2}(u_{i+1,j} + u_{i-1,j})$ is called ...... recurrence relation.
11. Interms of difference quotients $4u_{xx} = u_{tt}$ is ......
12. Bendre-Schmidt recurrence relation for one dimensional heat equation is ......
13. The diagonal 5-point formula to solve the Laplace equation $u_{xx} + u_{yy} = 0$ is ......
14. In the parabolic equation $u_t = \alpha^2 u_{xx}$ if $\lambda = k\alpha^2/h^2$, where $k = \Delta t$, and $h = \Delta x$, then explicit method is stable if λ = ...
15. $2\frac{\partial^2 u}{\partial t^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial x^2} = 0$ is classified as ...... *(P.T.U., 2007)*
16. The boundary conditions of one-dimensional wave equation are ......
17. The explicit formula for one-dimensional wave equation with $1 - \lambda^2\alpha^2 = 0$ and $\lambda = k/h$ is ......
18. The general form of Poisson's equation in partial derivations is ......
19. If u satisfies Laplace equation and $u = 100$ on the boundary of a square, the value of u at an interior grid point is ......
20. The Laplace equation $u_{xx} + u_{yy} = 0$ in difference quotients is ......
21. The equation $yu_{xx} + u_{yy} = 0$ is hyperbolic in the region ......
22. To solve $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$ by Bendre-Schmidt method with $h = 1$, the value of k is ......