CHAPTER 1

Solution of Equations

1. Introduction. 2. General properties. 3. Transformation of equations. 4. Reciprocal equations. 5. Solution of cubic equations–Cardan's method. 6. Solution of biquadratic equations—Ferrari's method ; Descarte's method. 7. Graphical solution of equations. 8. Objective Type Questions.

1.1 INTRODUCTION

The expression $f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}\, x + a_n$
where a's are constants ($a_0 \neq 0$) and n is a positive integer, is called a *polynomial in x* of degree n. The polynomial $f(x) = 0$ is called an *algebraic equation of degree n*. If $f(x)$ contains some other functions such as trigonometric, logarithmic, exponential etc. ; then $f(x) = 0$ is called a *transcendental equation*.

The value of x which satisfies $f(x) = 0$, ...(1)
is called its root. Geometrically, a root of (1) is that value of x where the graph of $y = f(x)$ crosses the x-axis. The process of finding the roots of an equation is known as *solution* of that equation. This is a problem of basic importance in applied mathematics. We often come across problems in deflection of beams, electrical circuits and mechanical vibrations which depend upon the solution of equations. As such, a brief account of solution of equations is given in this chapter.

1.2 GENERAL PROPERTIES

I. *If α is a root of the equation f(x) = 0, then the polynomial f(x) is exactly divisible by x – α and conversely.*
For instance, 3 is a root of the equation $x^4 - 6x^2 - 8x - 3 = 0$, because $x = 3$ satisfies this equation.
$\therefore$ $x - 3$ divides $x^4 - 6x^2 - 8x - 3$ completely, *i.e.*, $x - 3$ is its factor.

II. *Every equation of the nth degree has n roots (real or imaginary).*
Conversely if $\alpha_1, \alpha_2, ..., \alpha_n$ be the roots of the nth degree equation $f(x) = 0$, then

$$f(x) = A\,(x - \alpha_1)\,(x - \alpha_2) ... (x - \alpha_n) \text{ where } A \text{ is a constant.}$$

Obs. If a polynomial of degree n vanishes for more than n value of x, it must be identically zero.

Example 1.1. *Solve the equation* $2x^3 + x^2 - 13x + 6 = 0$.

Solution. By inspection, we find $x = 2$ satisfies the given equation.
$\therefore$ 2 is its root, *i.e.* $x - 2$ is a factor of $2x^3 + x^2 - 13x + 6$. Dividing this polynomial by $x - 2$, we get the quotient $2x^2 + 5x - 3$ and remainder 0.

Equating the quotient to zero, we get $2x^2 + 5x - 3 = 0$.

Solving this quadratic, we get $x = \dfrac{-5 \pm \sqrt{[5^2 - 4\,.\,(2)\,.\,(-3)]}}{2 \times 2} = \dfrac{-5 \pm 7}{4} = -3, \dfrac{1}{2}$.

Hence, the roots of the given equation are 2, – 3, 1/2.

Note. *The labour of dividing the polynomial by $x - 2$ can be saved considerably by the following simple device called* **synthetic division.**

2	1	–13	6	2
	4	10	– 6	
2	5	– 3	0	

[*Explanation* : (*i*) Write down the coefficient of the powers of x in order (supplying the missing powers of x by zero coefficients and write 2 on extreme right.

(*ii*) Put 2 as the first term of 3rd row and multiply it by 2, write 4 under 1 and add, giving 5.

(*iii*) Multiply 5 by 2, write 10 under – 13 and add, giving – 3.

(*iv*) Multiply – 3 by 2, write – 6 under 6 and add given zero].

Thus the quotient is $2x^2 + 5x - 3$ and remainder is zero.

Obs. To divide a polynomial by $x + h$, we write $-h$ on the extreme right.

III. Intermediate value property. If $f(a)$ and $f(b)$ *have different signs, then the equation $f(x) = 0$ has atleast one root between $x = a$ and $x = b$.*

The polynomial $f(x)$ is a continuous function of x (Fig. 1.1). So while x changes from a to b, $f(x)$ must pass through all the values from $f(a)$ to $f(b)$. But since one of these quantities $f(a)$ or $f(b)$ is positive and the other negative, it follows that at least for one value of x(say α) lying between a and b, $f(x)$ must be zero. Then α is the required root.

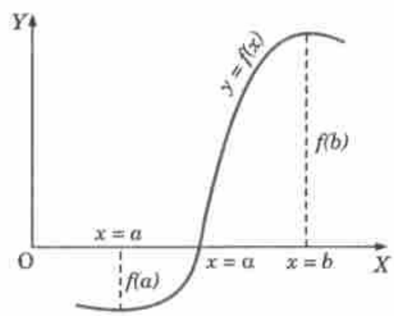

Fig. 1.1

IV. *In an equation with real coefficients, imaginary roots occur in conjugate pairs, i.e.*, if $\alpha + i\beta$ is a root of the equation $f(x) = 0$, then $\alpha - i\beta$ must also be its root. (See p. 534)

Similarly if $a + \sqrt{b}$ is an irrational root of an equation, then $a - \sqrt{b}$ must also be its root.

Obs. *Every equation of the odd degree has at least one real root.*

This follows from the fact that imaginary roots occur in conjugate pairs.

Example 1.2. *Solve the equation $3x^3 - 4x^2 + x + 88 = 0$, one root being $2 + \sqrt{7}i$.*

Solution. Since one root is $2 + \sqrt{7}i$, the other root must be $2 - \sqrt{7}i$.

∴ The factors corresponding to these roots are

$$(x - 2 - \sqrt{7}i) \text{ and } (x - 2 + \sqrt{7}i)$$

or $$(x - 2 - \sqrt{7}i)(x - 2 + \sqrt{7}i) = (x - 2)^2 + 7 = x^2 - 4x + 11,$$

which is a divisor of $3x^3 - 4x^2 + x + 88$ ...(*i*)

∴ Division of (*i*) by $x^2 - 4x + 11$ gives $3x + 8$ as the quotient.

Thus the depressed equation is $3x + 8 = 0$. Its root is $-8/3$. Hence the roots of the given equation are $2 \pm \sqrt{7}i, -8/3$.

V. Descarte's rule of signs. **The equation $f(x) = 0$ cannot have more positive roots than the changes of signs in $f(x)$; and more negative roots than the changes of signs in $f(-x)$.*

For instance, consider the equation $f(x) = 2x^7 - x^5 + 4x^3 - 5 = 0$ ...(1)

Sign of $f(x)$ are + – + –

Clearly, $f(x)$ has 3 changes of signs (from + to – or – to +).

Thus (*i*) cannot have more than 3 positive roots.

*After the French mathematician and philosopher *Rene Descartes* (1596–1650), who invented Analytic geometry in 1637.

Also $$f(-x) = 2(-x)^7 - (-x)^5 + 4(-x)^3 - 5$$
$$= -2x^7 + x^5 - 4x^3 - 5$$

This shows that $f(x)$ has 2 changes of signs. Thus (i) cannot have more than 2 negative roots.

Obs. Existence of imaginary roots. *If an equation of the nth degree has at the most p positive roots and at the most q negative roots, then it follows that the equation has at least n – (p + q) imaginary roots.*

Evidently (1) above is an equation of the 7th degree and has at the most 3 positive roots and 2 negative roots. Thus (1) has at least 2 imaginary roots.

VI. Relations between roots and coefficients, *If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ be the roots of the equation*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0 \qquad \ldots(1)$$

then $$\Sigma\alpha_1 = -\frac{a_1}{a_0}, \quad \Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0}, \quad \Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$$

....

$$\alpha_1\alpha_2\alpha_3 \ldots\ldots \alpha_n = (-1)^n \frac{a_n}{a_0}.$$

Example 1.3. *Solve the equation $x^3 - 7x^2 + 36 = 0$, given that one root is double of another.*

Solution. Let the roots be α, β, γ such that $\beta = 2\alpha$.

Also $$\alpha + \beta + \gamma = 7, \ \alpha\beta + \beta\gamma + \gamma\alpha = 0, \ \alpha\beta\gamma = -36$$

$$\therefore \quad 3\alpha + \gamma = 7 \qquad \ldots(i)$$
$$2\alpha^2 + 3\alpha\gamma = 0 \qquad \ldots(ii)$$
$$2\alpha^2\gamma = -36 \qquad \ldots(iii)$$

Solving (i) and (ii), we get $\alpha = 3, \gamma = -2$.

[The values $\alpha = 0, \gamma = 7$ are inadmissible, as they do not satisfy (iii)].

Hence the required roots are 3, 6 and – 2.

Example 1.4. *Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$, given that the sum of two of its roots is zero.* *(Cochin, 2005 ; Madras, 2003)*

Solution. Let the roots be $\alpha, \beta, \gamma, \delta$ such that $\alpha + \beta = 0$.

Also $$\alpha + \beta + \gamma + \delta = 2 \quad \therefore \quad \gamma + \delta = 2$$

Thus the quadratic factor corresponding to α, β is of the form $x^2 - 0x + p$, and that corresponding to γ, δ is of the form of $x^2 - 2x + q$.

$$\therefore \quad x^4 - 2x^3 + 4x^2 + 6x - 21 = (x^2 + p)(x^2 - 2x + q) \qquad \ldots(i)$$

Equating the coefficients of x^2 and x from both sides of (i), we get

$$4 = p + q, \quad 6 = -2p.$$

$$\therefore \quad p = -3, \quad q = 7.$$

Hence the given equation is equivalent to $(x^2 - 3)(x^2 - 2x + 7) = 0$

$\therefore$ The roots are $x = \pm\sqrt{3}, 1 \pm i\sqrt{6}$.

Example 1.5. *Find the condition that the cubic $x^3 - lx^2 + mx - n = 0$ should have its roots in*

(a) arithmetical progression. *(Madras, 2000 S)*

(b) geometrical progression.

Solution. (a) Let the roots be $a - d, a, a + d$ so that the sum of the roots $= 3a = l$ *i.e.*, $a = l/3$.

Since a is the root of the given equation

$$\therefore \quad a^3 - la^2 + ma - n = 0 \qquad \ldots(i)$$

Substituting $a = l/3$, we get $(l/3)^3 - l(l/3)^2 + m(l/3) - n = 0$.

or $$2l^3 - 9lm + 27n = 0,$$ which is the required condition.

(*b*) Let the roots be a/r, a, ar, so that the product of the roots $= a^3 = n$.

Putting $a = (n)^{1/3}$, in (*i*), we get $n - ln^{2/3} + mn^{1/3} - n = 0$ or $m = ln^{1/3}$

Cubing both sides, we get $m^3 = l^3 n$, which is the required condition.

Example 1.6. *Solve the equation $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$ whose roots are in A.P.*

Solution. Let the roots be $a - 3d$, $a - d$, $a + d$, $a + 3d$, so that the sum of the roots $= 4a = 2$.

$\therefore \quad a = 1/2$

Also product of the roots $= (a^2 - 9d^2)(a^2 - d^2) = 40$

or $$\left(\frac{1}{4} - 9d^2\right)\left(\frac{1}{4} - d^2\right) = 40 \quad \text{or} \quad 144d^4 - 40d^2 - 639 = 0$$

$\therefore \quad d^2 = 9/4 \quad \text{or} \quad -7/36$

Thus, $d = \pm 3/2$, the other value is not admissible.

Hence the required roots are $-4, -1, 2, 5$.

Example 1.7. *Solve the equation $2x^4 - 15x^3 + 35x^2 - 30x + 8 = 0$, whose roots are in G.P.*

Solution. Let the roots be a/r^3, a/r, ar, ar^3, so that product of the roots $= a^4 = 4$.

Also the product of a/r^3, ar^3 and a/r, ar are each $= a^2 = 2$.

$\therefore$ The factors corresponding to a/r^3, ar^3 and a/r, ar are $x^2 + px + 2$, $x^2 + qx + 2$.

Thus, $\quad 2x^4 - 15x^3 + 35x^2 - 30x + 8 = 2(x^2 + px + 2)(x^2 + qx + 2)$

Equating the coefficients of x^3 and x^2

$$-15 = 2p + 2q \quad \text{and} \quad 35 = 8 + 2pq$$

$\therefore \quad p = -9/2, q = -3.$

Thus the given equation is $2\left(x^2 - \dfrac{9}{2}x + 2\right)(x^2 - 3x + 2) = 0$

Hence the required roots are 1/2, 4 and 1, 2 *i.e.*, $\dfrac{1}{2}$, 1, 2, 4.

Example 1.8. *If α, β, γ be the roots of the equation $x^3 + px + q = 0$, find the value of*
(*a*) $\Sigma\alpha^2\beta$, (*b*) $\Sigma\alpha^4$ (*c*) $\Sigma\alpha^3\beta$.

Solution. We have $\quad \alpha + \beta + \gamma = 0 \quad$ ...(*i*)

$$\alpha\beta + \beta\gamma + \gamma\alpha = p \quad \text{...}(ii)$$

$$\alpha\beta\gamma = -q \quad \text{...}(iii)$$

(*a*) Multiplying (*i*) and (*ii*), we get

$\alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta + 3\alpha\beta\gamma = 0$

or $$\Sigma\alpha^2\beta = -3\alpha\beta\gamma = 3q \quad \text{[By (iii)]}$$

(*b*) Multiplying the given equation by x, we get $x^4 + px^2 + qx = 0$

Putting $x = \alpha, \beta, \gamma$ successively and adding, we get $\Sigma\alpha^4 + p\Sigma\alpha^2 + q\Sigma\alpha = 0$

or $$\Sigma\alpha^4 = -p\Sigma\alpha^2 - q(0) \quad \text{...}(iv)$$

Now squaring (*i*), we get $\alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 0$

or $$\Sigma\alpha^2 = -2p \quad \text{[By (ii)]}$$

Hence, substituting the value of $\Sigma\alpha^2$ in (*iv*), we obtain

$$\Sigma\alpha^4 = -p(-2p) = 2p^2.$$

(*c*) $\Sigma\alpha^3\beta = \Sigma\alpha^2\,\Sigma\alpha\beta - \alpha\beta\gamma\Sigma\alpha = -2p(p) - (-q)(0) = -2p^2$.

PROBLEMS 1.1

1. Form the equation of the fourth degree whose roots are $3 + i$ and $\sqrt{7}$. *(Madras, 2000 S)*
2. Solve the equation (*i*) $x^3 + 6x + 20 = 0$, one root being $1 + 3i$.
   (*ii*) $x^4 - 2x^3 - 22x^2 + 62x - 15 = 0$, given that $2 + \sqrt{3}$ is a root.
3. Show that $x^7 - 3x^4 + 2x^3 - 1 = 0$ has at least four imaginary roots. *(Cochin, 2005)*
4. Show that the equation $x^4 + 15x^2 + 7x - 11 = 0$ has one positive, one negative and two imaginary roots.
5. Find the number and position of real roots of $x^4 + 4x^3 - 4x - 13 = 0$.
6. Solve the equation $3x^3 - 11x^2 + 8x + 4 = 0$, given that two of its roots are equal.
7. If r_1, r_2, r_3 are the roots of the equation $2x^3 - 3x^2 + kx - 1 = 0$, find constant k if sum of two roots is 1. *(S.V.T.U., 2009)*
8. The equation $x^4 - 4x^3 + ax^2 + 4x + b = 0$ has two pairs of equal roots. Find the values of a and b.

Solve the following equations 9–14 :

9. $x^3 - 9x^2 + 14x + 24 = 0$, given that two of its roots are in the ratio 3 : 2.
10. $x^3 - 4x^2 - 20x + 48 = 0$ given that the roots α and β are connected by the relation $\alpha + 2\beta = 0$. *(S.V.T.U., 2007)*
11. $x^4 - 6x^3 + 13x^2 - 12x + 4 = 0$, given that it has two parts of equal roots. *(Madras, 2003)*
12. $x^4 - 8x^3 + 21x^2 - 20x + 5 = 0$ given that the sum of two of the roots is equal to the sum of the other two.
13. $x^3 - 12x^2 + 39x - 28 = 0$, roots being in arithmetical progression. *(Madras, 2001 S)*
14. $8x^3 - 14x^2 + 7x - 1 = 0$, roots being in geometrical progression. *(Osmania, 1999)*
15. O, A, B, C are the four points on a straight line such that the distances of A, B, C from O are the roots of equation $ax^3 + 3bx^2 + 3cx + d = 0$. If B is the middle point of AC, show that $a^2d - 3abc + 2b^3 = 0$. *(S.V.T.U., 2006)*
16. Solve the equations (*i*) $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$ whose roots are in A.P.
    (*ii*) $x^4 + 5x^3 - 30x^2 + 40x + 64 = 0$ whose roots are in G.P.
17. If α, β, γ be the roots of the equation $x^3 - lx^2 + mx - n = 0$, find the value of
    (*i*) $\Sigma\alpha^2\beta^2$, (*ii*) $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$
18. Find the sum of the cubes of the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$.
19. If α, β, γ are the roots of $x^3 + 4x - 3 = 0$, find the value of (*i*) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ (*ii*) $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$.
20. If α, β, γ be the roots of $x^3 + px + q = 0$, show that
    (*i*) $\alpha^5 + \beta^5 + \gamma^5 = 5\alpha\beta\gamma(\beta\gamma + \gamma\alpha + \alpha\beta)$, (*ii*) $3\Sigma\alpha^2\,\Sigma\alpha^5 = 5\Sigma\alpha^3\,\Sigma\alpha^4$.

1.3 TRANSFORMATION OF EQUATIONS

(1) To find an equation whose roots are *m* times the roots of the given equation, *multiply the second term by m, third term by m^2 and so on* (all missing terms supplied with zero coefficients).

For instance, let the given equation be

$$3x^4 + 6x^3 + 4x^2 - 8x + 11 = 0 \qquad ...(i)$$

To multiply its roots by m, put $y = mx$ (or $x = y/m$) in (*i*).

Then $$3(y/m)^4 + 6(y/m)^3 + 4(y/m)^2 + 8(y/m) + 11 = 0$$

Multiplying by m^4, we get $$3y^4 + m(6y^3) + m^2(4y^2) - m^3(8y) + m^4(11) = 0$$

This is same as multiplying the second term by m, third term by m^2 and so on in (*i*).

Cor. To find an equation whose roots are with opposite signs to those of the given equation, *change the signs of the every alternative term of the given equation beginning with the second.*

Changing the signs of the roots of (*i*) is same as multiplying its roots by – 1.

$\therefore$ The required equation will be

$$3x^4 + (-1)6x^3 + (-1)^2\,4x^2 - (-1)^3\,8x + (-1)^4\,11 = 0$$

or $$3x^4 - 6x^3 + 4x^3 + 8x + 11 = 0$$

which is (*i*) with signs of every alternate term changed beginning with the second.

(2) To find an equation whose roots are reciprocal of the root of the given equation, *change x to 1/x.*

Example 1.9. *Solve $6x^3 - 11x^2 - 3x + 2 = 0$, given that its roots are in harmonic progression.*

Solution. Since the roots of the given equation are in H.P., the roots of the equation having reciprocal roots will be in A.P.

The equation with reciprocal roots is $6(1/x)^3 - 11(1/x)^2 - 3(1/x) + 2 = 0$

or $$2x^3 - 3x^2 - 11x + 6 = 0 \qquad ...(i)$$

Since the roots of the given equation are in H.P., therefore, the roots of (i) are in A.P. Let the root be $a - d$, a, $a + d$. Then

$$3a = 3/2 \text{ and } a(a^2 - d^2) = -3.$$

Solving these equations, we get $a = 1/2$, $d = 5/2$.

Thus the roots of (i) are -2, $1/2$, 3.

Hence the required roots of the given equation are $-1/2$, 2, $1/3$.

Example 1.10. *If α, β, γ be the roots of the cubic equation $x^3 - px^2 + qx - r = 0$, form the equation whose roots are $\beta\gamma + 1/\alpha$, $\gamma\alpha + 1/\beta$, $\alpha\beta + 1/\gamma$.*

Hence evaluate $\Sigma(\alpha\beta + 1/\gamma)(\beta\gamma + 1/\alpha)$. (S.V.T.U., 2008)

Solution. If x is a root of the given equation and y a root of the required equation, then

$$y = \beta\gamma + 1/\alpha = \frac{\alpha\beta\gamma + 1}{\alpha} = \frac{r+1}{\alpha} \qquad [\because \ \alpha\beta\gamma = r]$$

or $$y = \frac{r+1}{x} \Rightarrow x = \frac{r+1}{y}$$

Thus substituting $x = (r + 1)/y$ in the given equation, we get

$$\left(\frac{r+1}{y}\right)^3 - p\left(\frac{r+1}{y}\right)^2 + q\left(\frac{r+1}{y}\right) - r = 0$$

or $ry^3 - q(r + 1)y^2 + p(r + 1)^2 y - (r + 1)^3 = 0$, which is the required equation.

Hence $\Sigma(\alpha\beta + 1/\gamma)(\beta\gamma + 1/\alpha) = p(r + 1)^2/r$.

Example 1.11. *Form an equation whose roots are cubes of the roots of $x^3 - 3x^2 + 1 = 0$.* ...(i)

Solution. If y be a root of the required equation, then $y = x^3$ ...(ii)

Now we have to eliminate x from (i) and (ii)

$\therefore$ Rewriting (i) as $x^3 + 1 = 3x^2$

Cubing both sides, $x^9 + 3x^6 + 3x^3 + 1 = 27x^6$

Substituting $x^3 = y$, we get $y^3 - 24y^2 + 3y + 1 = 0$, which is the required equation.

(3) To diminish the roots of an equation $f(x) = 0$ by h, *divide $f(x)$ by $x - h$ successively. Then the successive remainders determine the coefficients of the required equation.*

Let the given equation be

$$a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0 \qquad ...(i)$$

To diminish its roots by h, put $y = x - h$ (or $x = y + h$) in (i) so that

$$a_0(y + h)^n + a_1(y + h)^{n-1} + ... + a_n = 0 \qquad ...(ii)$$

On simplification, it takes the form

$$A_0 y^n + A_1 y^{n-1} + ... + A_n = 0 \qquad ...(iii)$$

Its coefficient $A_0, A_1, ... A_n$ can easily be found with the help of *synthetic division* (p. 2). For this, we put $y = x - h$ in (iii) so that

$$A_0(x - h)^n + A_1(x - h)^{n-1} + ... + A_n = 0 \qquad ...(iv)$$

Clearly, (i) and (iv) are identical. If we divide L.H.S. of (iv) by $x - h$, the remainder is A_n and the quotient $Q = A_0(x - h)^{n-1} + A_1(x - h)^{n-2} + ... + A_{n-1}$. Similarly, if we divide Q by $x - h$, the remainder is A_{n-1} and the quotient is Q_1(say). Again dividing Q_1 by $x - h$, A_{n-2} will be obtained as remainder and so on.

Obs. To increase the roots by h, we take h negative.

Example 1.12. *Transform the equation $x^3 - 6x^2 + 5x + 8 = 0$ into another in which the second term is missing. Hence find the equation of its squared differences.* (Cochin, 2005)

Solution. Sum of the roots of the given equation = 6.

In order that the second term in the transformed equation is missing, the sum of the roots is to be zero.

Since the equation has 3 roots, if we decrease each root by 2, the sum of the roots of the new equation will become zero.

$\therefore$ Dividing $x^3 - 6x^2 + 5x + 8$ by $x - 2$ successively, we have

1	– 6	5	8	(2
	2	– 8	– 6	
	– 4	– 3	**2**	
	2	– 4		
	– 2	**– 7**		
	2			
1	**0**			

Thus the transformed equation is $x^3 - 7x + 2 = 0$. ...(*i*)

If α, β, γ be the roots of the given equation, then the roots of (*i*) are $\alpha - 2, \beta - 2, \gamma - 2$.

Let these roots be denoted by a, b, c.

Then $b - c = \beta - \gamma$. Also $a + b + c = 0$, $abc = -2$.

Now $(b-c)^2 = (b+c)^2 - 2bc = (a+b+c-a)^2 - \dfrac{2abc}{a} = a^2 + 4/a$

$\therefore$ The equation of squared differences of (*i*) is given by the transformation $y = x^2 + 4/x$

or $$x^3 - xy + 4 = 0 \qquad ...(ii)$$

Subtracting (*ii*) from (*i*), we get $-7x + xy - 2 = 0$ or $x = 2/(y-7)$

Substituting for x in (*i*), the equation becomes

$$[2/(y-7)]^3 - 7[2/(y-7)] + 2 = 0 \quad \text{or} \quad y^3 - 28y^2 + 245y - 682 = 0 \qquad ...(iii)$$

Roots of this equation are $(b-c)^2, (c-a)^2, (a-b)^2$ *i.e.*, $(\beta-\gamma)^2, (\gamma-\alpha)^2, (\alpha-\beta)^2$.

Hence (*iii*) is the required equation.

1.4 RECIPROCAL EQUATIONS

If an equation remains unaltered on changing x to $1/x$, it is called a **reciprocal equation.**

Such equations are of the following standard types :

I. *A reciprocal equation of an odd degree having coefficients of terms equidistant from the beginning and end equal.* It has a root = – 1.

II. *A reciprocal equation of an odd degree having coefficients of terms equidistant from the beginning and end equal but opposite in sign.* It has root = 1.

III. *A reciprocal equation of an even degree having coefficients of terms equidistant from the beginning and end equal but opposite in sign.* Such an equation has two roots = 1 and – 1.

The substitution $x + 1/x = y$ reduces the degree of the equation of half its former degree.

Example 1.13. *Solve $6x^5 - 41x^4 + 97x^3 - 97x^2 + 41x - 6 = 0$.* (Coimbatore, 2001 S)

Solution. This is a reciprocal equation of odd degree with opposite signs. $\therefore$ $x = 1$ is a root.

Dividing L.H.S. by $x - 1$, the given equation reduces to

$$6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$$

Dividing by x^2, we have

$$6(x^2 + 1/x^2) - 35(x + 1/x) + 62 = 0$$

Putting $x + 1/x = y$ and $x^2 + 1/x^2 = y^2 - 2$, we get

$$6(y^2 - 2) - 35y + 62 = 0 \quad \text{or} \quad 6y^2 - 35y + 50 = 0 \quad \text{or} \quad (3y-1)(2y-5) = 0$$

$\therefore$ $$x + 1/x = y = 1/3 \quad \text{or} \quad 5/2$$

i.e., $3x^2 - 10x + 3 = 0$ or $2x^2 - 5x + 2 = 0$

i.e., $(3x - 1)(x - 3) = 0$ or $(2x - 1)(x - 2) = 0$

$\therefore$ $x = 1/3, 3$ or $1/2, 2$

Hence the required roots are 1, 1/3, 3, 1/2, 2.

Example 1.14. *Solve* $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0.$ *(Madras, 2003)*

Solution. This is a reciprocal equation of even degree with opposite signs. $\therefore$ $x = 1, -1$ are its roots.

Dividing L.H.S. by $x - 1$ and $x + 1$, the given equation reduces to

$$6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0$$

Dividing by x^2, we get

$$6(x^2 + 1/x^2) - 25(x + 1/x) + 37 = 0.$$

Putting $x + 1/x = y$ and $x^2 + 1/x^2 = y^2 - 2$, it becomes

$$6(y^2 - 2) - 25y + 37 = 0 \quad \text{or} \quad 6y^2 - 25y + 25 = 0$$

or $(2y - 5)(3y - 5) = 0$

$\therefore$ $x + 1/x = y = 5/2$ or $5/3$.

i.e., $2x^2 - 5x + 2 = 0$ or $3x^2 - 5x + 3 = 0$

$\therefore$ $x = 2, 1/2$ or $x = \dfrac{5 \pm i\sqrt{11}}{6}$

Hence the required roots of the given equation are $1, -1, 2, 1/2, \dfrac{5 \pm i\sqrt{11}}{6}$.

PROBLEMS 1.2

1. Find the equation whose roots are 3 times the roots of $x^3 + 2x^2 - 4x + 1 = 0$.
2. Form the equation whose roots are the reciprocals of the roots of $2x^5 + 4x^3 - 13x^2 + 7x - 6 = 0$. *(S.V.T.U., 2009)*
3. Find the equation whose roots are the negative reciprocals of the roots of
$$x^4 + 7x^3 + 8x^2 - 9x + 10 = 0.$$
4. Solve the equation $6x^3 - 11x^2 - 3x + 2 = 0$, given that its roots are in H.P.
5. Find the equation whose roots are the roots of
   (*i*) $x^3 - 6x^2 + 11x - 6 = 0$ each increased by 1. *(S.V.T.U., 2009)*
   (*ii*) $x^4 + x^3 - 3x^2 - x + 2 = 0$ each diminished by 3.
   (*iii*) $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x + 6 = 0$ each diminished by 1.
6. Find the equation whose roots are the squares of the roots of $x^3 - x^2 + 8x - 6 = 0$.
7. Find the equation whose roots are the cubes of the roots of $x^3 + px^2 + q = 0$.
8. If α, β, γ are the roots of the equation $2x^3 + 3x^2 - x - 1 = 0$, form the equation whose roots are $(1-\alpha)^{-1}$, $(1-\beta)^{-1}$ and $(1-\gamma)^{-1}$.
9. If a, b, c are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are ab, bc and ca. *(Madras, 2003)*
10. If α, β, γ be the roots of $x^3 + mx + n = 0$, form the equation whose roots are
    (*a*) $\alpha + \beta - \gamma, \beta + \gamma - \alpha, \gamma + \alpha - \beta$, (*b*) $\beta\gamma/\alpha, \gamma\alpha/\beta, \alpha\beta/\gamma$ (*c*) $\dfrac{1}{\beta} + \dfrac{1}{\gamma}, \dfrac{1}{\gamma} + \dfrac{1}{\alpha}, \dfrac{1}{\alpha} + \dfrac{1}{\beta}$.
11. Find the equation of squared differences of the roots of the cubic $x^3 + 6x^2 + 7x + 2 = 0$.
12. Solve the equations :
    (*i*) $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$ (*ii*) $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$. *(Madras, 2003)*
    (*iii*) $8x^5 - 22x^4 - 55x^3 + 55x^2 + 22x - 8 = 0$. (*iv*) $6x^5 + x^4 - 43x^3 - 43x^2 + x + 6 = 0$ *(S.V.T.U., 2006)*
    (*v*) $3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0$.
13. Show that the equation $x^4 - 10x^3 + 23x^2 - 6x - 15 = 0$ can be transformed into reciprocal equation by diminishing the roots by 2. Hence solve the equation.
14. By suitable transformation, reduce the equation $x^4 + 16x^3 + 83x^2 + 152x + 84 = 0$ to an equation in which term in x^3 is absent and hence solve it. *(Madras, 2002)*

1.5 SOLUTION OF CUBIC EQUATIONS–CARDAN'S METHOD*

Consider the equation $ax^3 + bx^2 + cx + d = 0$ ...(1)

Dividing by a, we get an equation of the form $x^3 + lx^2 + mx + n = 0$.

To remove the x^2 term, put $y = x - (-l/3)$ or $x = y - l/3$ so that the resulting equation is of the form

$$y^3 + py + q = 0 \quad ...(2)$$

To solve (2), put $y = u + v$

so that $y^3 = u^3 + v^3 + 3uv(u + v) = u^3 + v^3 + 3uvy$

or $y^3 - 3uvy - (u^3 + v^3) = 0$ ...(3)

Comparing (2) and (3), we get

$$uv = -p/3,\ u^3 + v^3 = -q \text{ or } u^3 + v^3 = -q \text{ and } u^3v^3 = -p^3/27$$

$\therefore$ u^3, v^3 are the roots of the equation $t^2 + qt - p^3/27 = 0$

which gives $u^3 = \frac{1}{2}(-q + \sqrt{q^2 + 4p^3/27}) = \lambda^3$ (say)

and $v^3 = \frac{1}{2}(-q - \sqrt{q^2 + 4p^3/27})$

$\therefore$ The three values of u are $\lambda, \lambda\omega, \lambda\omega^2$, where ω is one of the imaginary cube roots of unity.

From $uv = -p/3$, we have $v = -p/3u$

$\therefore$ When $u = \lambda, \lambda\omega$ and $\lambda\omega^2$,

$$v = -\frac{p}{3\lambda}, -\frac{p\omega^2}{3\lambda} \text{ and } -\frac{p\omega}{3\lambda}. \qquad [\because\ \omega^3 = 1]$$

Hence the three roots of (2) are $\lambda - \frac{p}{3\lambda}, \lambda\omega - \frac{p\omega^2}{3\lambda}, \lambda\omega^2 - \frac{p\omega}{3\lambda}$ (Being $= u + v$)

Having known y, the corresponding values of x can be found from the relation $x = y - l/3$.

Obs. 1. If one value of u is found to be a rational number, find the corresponding value of v giving one root $y = u + v$. Then find the corresponding root $x = \alpha$ (say). Finally, divide the left hand side of (1) by $x - \alpha$, giving the remaining quadratic equation from which the other two roots can be found readily.

Obs. 2. If u^3 and v^3 turn out to be conjugate complex numbers, the roots of the given cubic can be obtained in neat forms by employing De Moivre's theorem. (§ 19.5)

Example 1.15. *Solve by Cardan's method $x^3 - 3x^2 + 12x + 16 = 0$.* *(U.P.T.U., 2008)*

Solution. Given equation is $x^3 - 3x^2 + 12x + 16 = 0$ ...(*i*)

To remove the second term from (i), diminish each root of (*i*) by $3/3 = 1$, *i.e.*, put $y = x - 1$ or $x = y + 1$ [$\therefore$ Sum of roots = 3]. Then (*i*) becomes

$$(y + 1)^3 - 3(y + 1) + 12(y + 1) + 16 = 0 \text{ or } y^3 + 9y^2 + 26 = 0 \quad ...(ii)$$

To solve (*ii*), put $y = u + v$ so that $y^3 - 3uvy - (u^3 + v^3) = 0$ ...(*iii*)

Comparing (*ii*) and (*iii*), we get $uv = -3$ and $u^3 + v^3 = -26$

$\therefore$ u^3, v^3 are the roots of the equation $t^2 + 26t - 27 = 0$

or $(t + 27)(t - 1) = 0$ whence $t = -27, t = 1$.

or $u^3 = -27$ *i.e.*, $u = -3$ and $v^3 = 1$ *i.e.*, $v = 1$

$\therefore$ $y = u + v = -3 + 1 = -2$ and $x = y + 1 = -1$

Dividing L.H.S. of (*i*) by $x + 1$, we obtain $x^2 - 4x + 16 = 0$

or $$x = \frac{4 \pm \sqrt{(16 - 64)}}{2} = 2 \pm i\ 2\sqrt{3}$$

Hence the required roots of the given equation are $-1, 2 \pm i\ 2\sqrt{3}$.

*Named after an Italian mathematician *Girolamo Cardan* (1501–1576) who was the first to use complex number as roots of an equation.

Example 1.16. *Solve the cubic equation $28x^3 - 9x^2 + 1 = 0$ by Cardan's method.*

Solution. Since the term in x is missing, let us put $x = 1/y$ in the given equation so that the transformed equation is $y^3 - 9y + 28 = 0$ ...(*i*)

To solve (*i*), put $y = u + v$ so that $y^3 - 3uvy - (u^3 + v^3) = 0$ ...(*ii*)

Comparing (*ii*) and (*iii*), we get $uv = 3$ and $u^3 + v^3 = -28$.

$\therefore$ u^3, v^3 are the roots of $t^2 + 28t + 27 = 0$

or $(t + 1)(t + 27) = 0$ or $t = -1, -27$ or $u = -1, v = -3$

$\therefore$ $y = u + v = -4$. Dividing L.H.S. of (*i*) by $y + 4$, we obtain $y^2 - 4y + 7 = 0$ whence $y = 2 \pm i\sqrt{3}$.

$\therefore$ Roots of (*i*) are $-4, 2 \pm i\sqrt{3}$.

Hence the roots of the given cubic equation are $-\frac{1}{4}, \frac{1}{2 \pm i\sqrt{3}}$ or $-\frac{1}{4}, (2 - i\sqrt{3})/7, (2 + i\sqrt{3})/7$.

Example 1.17. *Solve the equation $x^3 + x^2 - 16x + 20 = 0$.*

Solution. Instead of diminishing the roots of the given equation by $-1/3$, we first multiply its roots by 3, so that the equation becomes

$$x^3 + 3x^2 - 144x + 540 = 0 \quad ...(i)$$

To remove the x^2 term, put $y = x - (-3/3)$ or $x = y - 1$ in (*i*)

so that $(y - 1)^3 + 3(y - 1)^2 - 144(y - 1) + 540 = 0$

or $y^3 - 147y + 686 = 0$ ...(*ii*)

To solve (iii), let $y = u + v$, so that

$$y^3 - 3uvy - (u^3 + v^3) = 0 \quad ...(iii)$$

Comparing (*ii*) and (*iii*), we get

$$uv = 49, u^3 + v^3 = -686, \text{ so that } u^3 v^3 = (343)^2.$$

$\therefore$ u^3, v^3 are the roots of the quadratic

$$t^2 + 686t + (343)^2 = 0 \quad \text{or} \quad (t + 343)^2 = 0$$

$\therefore$ $t = -343$ *i.e.*, $u^3 = v^3 = -343$ or $u = v = -7$.

Thus $y = u + v = -14$ and $x = y - 1 = -15$.

Dividing L.H.S. of (*i*) by $x + 15$, we get

$$(x - 6)^2 = 0 \quad \text{or} \quad x = 6, 6.$$

$\therefore$ The root of (*i*) are $-15, 6, 6$.

Hence the roots of the given equation are $-5, 2, 2$.

Example 1.18. *Solve $x^3 - 3x^2 + 3 = 0$.* (*S.V.T.U., 2006*)

Solution. Given equation is $x^3 - 3x^2 + 3 = 0$ ...(*i*)

To remove the x^2 term, put $y = x - 3/3$ or $x = y + 1$,

so that (*i*) becomes $(y + 1)^3 - 3(y + 1)^2 + 3 = 0$

or $y^3 - 3y + 1 = 0$ ...(*ii*)

To solve it, put $y = u + v$

so that $y^3 - 3uvy - (u^3 + v^3) = 0$ ...(*iii*)

Comparing (*ii*) and (*iii*), we get $uv = 1, u^3 + v^3 = -1$

$\therefore$ u^3, v^3 are the roots of the equation $t^2 + t + 1 = 0$

Hence $u^3 = \frac{-1 + i\sqrt{3}}{2}$ and $v^3 = \frac{-1 - i\sqrt{3}}{2}$

$\therefore$ $u = \left(\frac{-1 + i\sqrt{3}}{2}\right)^{1/3}$ put $-\frac{1}{2} = r\cos\theta$ and $\sqrt{3}/2 = r\sin\theta$

$= [r(\cos\theta + i\sin\theta)]^{1/3}$ so that $r = 1, \theta = 2\pi/3$

$= [\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)]^{1/3}$,

where n is any integer or zero. Using De Moivre's theorem (p. 647).

$$u = \cos\left(\frac{\theta + 2n\pi}{3}\right) + i \sin\left(\frac{\theta + 2n\pi}{3}\right)$$

Giving n the value 0, 1, 2 successively we get the three values of u to be

$$\cos\frac{\theta}{3} + i\sin\frac{\theta}{3},\ \cos\frac{\theta + 2\pi}{3} + i\sin\frac{\theta + 2\pi}{3},\ \cos\frac{\theta + 4\pi}{3} + i\sin\frac{\theta + 4\pi}{3}$$

i.e.,
$$\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9},\ \cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9},\ \cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}.$$

The corresponding values of v are

$$\cos\frac{2\pi}{9} - i\sin\frac{2\pi}{9},\ \cos\frac{8\pi}{9} - i\sin\frac{8\pi}{9},\ \cos\frac{14\pi}{9} - i\sin\frac{14\pi}{9}.$$

$\therefore$ The three values of $y = u + v$ are $2\cos 2\pi/9$, $2\cos 8\pi/9$, $2\cos 14\pi/9$.

Hence the roots of (i) are found from $x = 1 + y$ to be

$$1 + 2\cos 2\pi/9,\ 1 + 2\cos 8\pi/9,\ 1 + 2\cos 14\pi/9.$$

PROBLEMS 1.3

Solve the following equations by Cardan's method :

1. $x^3 - 27x + 54 = 0$. (*U.P.T.U., 2003*)
2. $x^3 - 18x + 35 = 0$ (*Osmania, 2003*)
3. $x^3 - 15x = 126$ (*S.V.T.U., 2009*)
4. $2x^3 + 5x^2 + x - 2 = 0$ (*U.P.T.U., 2003*)
5. $9x^3 + 6x^2 - 1 = 0$ (*S.V.T.U., 2008*)
6. $x^3 - 6x^2 + 6x - 5 = 0$
7. $x^3 - 3x + 1 = 0$
8. $27x^3 + 54x^2 + 198x - 73 = 0$

1.6 SOLUTION OF BIQUADRATIC EQUATIONS

(1) Ferrari's method

This method of solving a biquadratic equation is illustrated by the following examples :

Example 1.19. *Solve the equation $x^4 - 12x^3 + 41x^2 - 18x - 72 = 0$ by Ferrari's method.* (*S.V.T.U., 2007*)

Solution. Combining x^4 and x^3 terms into a perfect square, the given equation can be written as

$$(x^2 - 6x + \lambda)^2 + (5 - 2\lambda)x^2 + (12\lambda - 18)x - (\lambda^2 + 72) = 0$$

or
$$(x^2 - 6x + \lambda)^2 = \{(2\lambda - 5)x^2 + (18 - 12\lambda)x + (\lambda^2 + 72)\} \quad ...(i)$$

This equation can be factorised if R.H.S. is a perfect square

i.e., if
$$(18 - 12\lambda)^2 = 4(2\lambda - 5)(\lambda^2 + 72) \quad [b^2 = 4ac]$$

i.e., if
$$2\lambda^3 - 41\lambda^2 + 252\lambda - 441 = 0 \quad \text{which gives} \quad \lambda = 3.$$

$\therefore$ (i) reduces to $(x^2 - 6x + 3)^2 = (x - 9)^2$

i.e.,
$$(x^2 - 5x - 6)(x^2 - 7x + 12) = 0.$$

Hence the roots of the given equation are $-1, 3, 4$ and 6.

Example 1.20. *Solve the equation $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$ by Ferrari's method.*

Solution. Combining x^4 and x^3 terms into a perfect square, the given equation can be written as $(x^2 - x + \lambda)^2 = (2\lambda + 6)x^2 - (2\lambda + 10)x + (\lambda^2 + 3)$. This equation can be factorised, if R.H.S. is a perfect square *i.e.*, if $(2\lambda + 10)^2 = 4(2\lambda + 6)(\lambda^2 + 3)$ $[b^2 = 4ac]$

or
$$2\lambda^3 + 5\lambda^2 - 4\lambda - 7 = 0, \text{ which gives } \lambda = -1.$$

$\therefore$ (i) reduces to $(x^2 - x - 1)^2 = 4x^2 - 8x + 4$

or
$$(x^2 - x - 1)^2 - (2x - 2)^2 = 0 \quad \text{or} \quad (x^2 + x - 3)(x^2 - 3x + 1) = 0$$

$$\therefore \quad x = \frac{-1 \pm \sqrt{1 + 12}}{2} \quad \text{or} \quad \frac{3 \pm \sqrt{9 - 4}}{2}$$

Hence the roots are $\dfrac{-1 \pm \sqrt{13}}{2}, \dfrac{3 \pm \sqrt{5}}{2}$.

(2) Descarte's method

This method of solving a biquadratic equations consists in removing the term in x^3 and then expressing the new equation as product of two quadratics. It has been best illustrated by the following examples :

Example 1.21. *Solve the equation $x^4 - 8x^2 - 24x + 7 = 0$ by Descarte's method.* (U.P.T.U., 2001)

Solution. In the given equation, the term in x^3 is already absent so we assume that

$$x^4 - 8x^2 - 24x + 7 = (x^2 + px + q)(x^2 - px + q') \quad ...(i)$$

Equating coefficients of the like powers of x in (*i*), we get

$$-8 = q + q' - p^2, -24 = p(q' - q); 7 = qq'$$

$\therefore \quad q + q' = p^2 - 8, q - q' = 24/p$

$\therefore \quad (p^2 - 8)^2 - (24/p)^2 = 4 \times 7$

$$p^2 - 16p^4 + 36p^2 - 576 = 0 \quad \text{or} \quad t^3 - 16t^2 + 36t - 576 = 0 \text{ where } t = p^2$$

Now $t = 16$ satisfies this cubic so that $p = 4$.

$\therefore \quad q + q' = 8, q - q' = 6 \quad \therefore \quad q = 7, q' = 1$

Thus (*i*) takes the form $(x^2 + 4x + 7)(x^2 - 4x + 1) = 0$

whence $\quad x = \dfrac{-4 \pm \sqrt{(16-28)}}{2}$ or $x = \dfrac{4 \pm \sqrt{(16-4)}}{2}$

Hence $x = -2 \pm \sqrt{3}i, 2 \pm \sqrt{3}$.

Example 1.22. *Solve the equation $x^4 - 6x^3 - 3x^2 + 22x - 6 = 0$ by Descarte's method.*

Solution. Here sum of roots = 6 and number of roots = 4

$\therefore$ To remove the second term, we have to diminish the roots by 6/4 (= 3/2) which will be a problem. Therefore, we first multiply the roots by 2. $\therefore$ $y^4 - 12y^3 + 12y^2 + 176y - 96 = 0$ where $y = 2x$. Now diminishing the roots by 3, we obtain $z^4 - 42z^2 + 32z + 297 = 0$ where $z = y - 3$.

Assuming that $\quad z^4 - 42z^2 + 32z + 297 = (z^2 + pz + q)(z^2 - pz + q') \quad ...(i)$

and comparing coefficients, we get

$$-42 = q + q' - p^2 ; 32 = p(q' - q) ; 297 = q q'$$

$\therefore \quad q + q' = p^2 - 42 ; q - q' = -32/p, q q' = 297$

$\therefore \quad (p^2 - 42)^2 - (-32/p)^2 = 4 \times 297$

or $\quad t^3 - 84t^2 + 576t - 1024 = 0 \quad$ where $t = p^2$

Now $t = 4$ satisfies this cubic so that $p = 2$.

$\therefore \quad q + q' = -38, q - q' = -16, \quad \therefore \quad q = -27, q' = -11.$

Thus (*i*) takes the form $(z^2 + 2z - 27)(z^2 - 2z - 11) = 0$

Whence $\quad z = \dfrac{-2 \pm \sqrt{(4+108)}}{2}$ or $z = \dfrac{2 \pm \sqrt{(4+44)}}{2}$

or $\quad x = \dfrac{1}{2} y = \dfrac{1}{2}(z+3) = \dfrac{1}{2}(2 \pm \sqrt{28}) = \dfrac{1}{2}(4 \pm \sqrt{12})$

Hence $\quad x = 1 \pm \sqrt{7}, 2 \pm \sqrt{3}$.

PROBLEMS 1.4

Solve by Ferrari's method, the equations :

1. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$ (U.P.T.U., 2003)
2. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$ (U.P.T.U., 2002)
3. $x^4 - 10x^2 - 20x - 16 = 0$
4. $x^4 - 8x^3 - 12x^2 + 60x + 63 = 0$ (U.P.T.U., 2005)

Solve the following equations by Descartes method :

5. $x^4 - 6x^3 + 3x^2 + 22x - 6 = 0$
6. $x^4 + 12x - 5 = 0$
7. $x^4 - 8x^3 - 24x + 7 = 0$ (U.P.T.U., 2001)
8. $x^4 - 10x^3 + 44x^2 - 104x + 96 = 0$

Obs. We have obtained algebraic solutions of cubic and biquadratic equations. But the need often arises to solve higher degree or transcendental equations for which no algebraic methods are available in general. Such equations can be best solved by *graphical method* (explained below) or by *numerical methods* (§28.2).

1.7 GRAPHICAL SOLUTION OF EQUATIONS

Let the equation be $f(x) = 0$.

(*i*) *Find the interval (a, b) in which a root of $f(x) = 0$ lies.*

[At least one root of $f(x) = 0$ lies in (a, b) if $f(a)$ and $f(b)$ are of opposite signs—§1.2(III) p. 2].

(*ii*) *Write the equation $f(x) = 0$ as $\phi(x) = \psi(x)$ where $\psi(x)$ contains only terms in x* and the constants.

(*iii*) *Draw the graphs of $y = \phi(x)$ and $y = \psi(x)$ on the same scale and with respect to the same axes.*

(*iv*) *Read the abscissae of the points of intersection of the curves $y = \phi(x)$ and $y = \psi(x)$. These are required real roots of $f(x) = 0$.*

Sometimes it may not be convenient to write the given equation $f(x) = 0$ in the form $\phi(x) = \psi(x)$. In such cases, we proceed as follows :

(*i*) Form a table for the value of x and $y = f(x)$ directly.

(*ii*) Plot these points and pass a smooth curve through them.

(*iii*) Read the abscissae of the points where this curve cuts the x-axis. These are the required roots of $f(x) = 0$.

Obs. The roots, thus located graphically are approximate and to improve their accuracy, the curves are replotted on the larger scale in the immediate vicinity of each point of intersection. This gives a better approximation to the value of desired root. The above graphical operation may be repeated until the root is obtained correct upto required number of decimal places. But this method of repeatedly drawing graphs is very tedious. It is, therefore, advisable to improve upon the accuracy of an approximate root by numerical method of §28.2.

Example 1.23. *Find graphically an approximate value of the root of the equation*

$$3 - x = e^{x-1}.$$

Solution. Let $f(x) = e^{x-1} + x - 3 = 0$ ...(*i*)

$f(1) = 1 + 1 - 3 = -\text{ve}$

and $f(2) = e + 2 - 3 = 2.718 - 1 = +\text{ve}$

$\therefore$ A root of (*i*), lies between $x = 1$ and $x = 2$.

Let us rewrite (*i*) as $e^{x-1} = 3 - x$.

The abscissa of the point of intersection of the curves

$$y = e^{x-1} \quad ...(ii)$$

and

$$y = 3 - x \quad ...(iii)$$

will give the required root.

To plot (*ii*), we form the following table of values :

$x =$	$y = e^{x-1}$
1.1	1.11
1.2	1.22
1.3	1.35
1.4	1.49
1.5	1.65
1.6	1.82
1.7	2.01
1.8	2.23
1.9	2.46
2.0	2.72

Taking the origin at (1, 1) and 1 small unit along either axis = 0.02, we plot these points and pass a smooth curve through them as shown in Fig. 1.2.

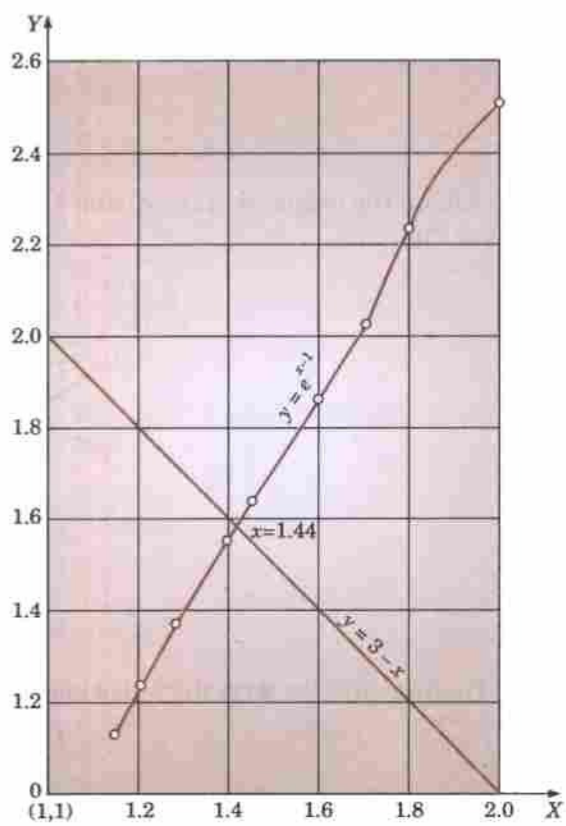

Fig. 1.2

To draw the line (iii), we join the points (1, 2) and (2, 1) on the same scale and with the same axes.

From the figure, we get the required root to be $x = 1.44$ nearly.

Example 1.24. *Obtain graphically an approximate value of the root of $x = \sin x + \pi/2$.*

Solution. Let us write the given equation as $\sin x = x - \pi/2$

The abscissa of the point of intersection of the curve $y = \sin x$ and the line $y = x - \pi/2$ will give a rough estimate of the root.

To draw a curve $y = \sin x$, we form the following table :

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
y	0	0.71	1	0.71	0

Taking 1 unit along either axis $= \pi/4 = 0.8$ nearly, we plot the curve as shown in Fig. 1.3.

Also we draw the line $y = x - \pi/2$ to the same scale and with the same axis.

From the graph, we get $x = 2.3$ radians approximately.

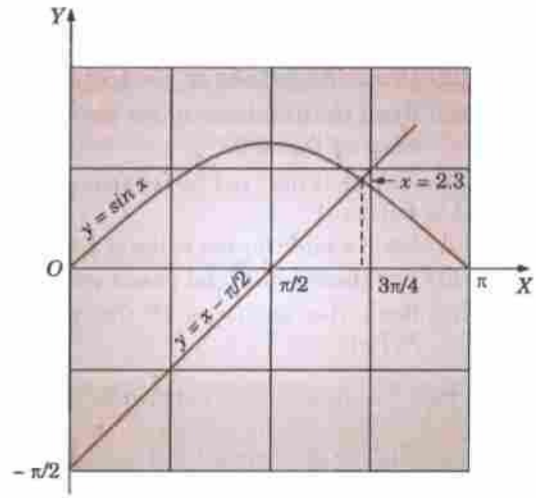

Fig. 1.3

Example 1.25. *Obtain graphically the lowest root of $\cos x \cosh x = -1$.*

Solution. Let $f(x) = \cos x \cosh x + 1 = 0$ ...(*i*)

$\because$ $f(0) = +$ ve, $f(\pi/2) = +$ ve and $f(\pi) = -$ ve.

$\therefore$ The lowest root of (*i*) lies between $x = \pi/2$ and $x = \pi$.

Let us write (*i*) as $\cos x = -\operatorname{sech} x$.

The abscissa of the point of intersection of the curves

$y = \cos x$ ...(*ii*) and $y = -\operatorname{sech} x$ ...(*iii*)

will give the required root. To draw (*ii*), we form the following table of values :

$x =$	$\pi/2 = 1.57$	$3\pi/4 = 2.36$	$\pi = 3.14$
$y = \cos x$	0	-0.71	-1

Taking the origin at (1.57, 0) and 1 unit along either axes $= \pi/8 = 0.4$ nearly, we plot the cosine curve as shown in Fig. 1.4.

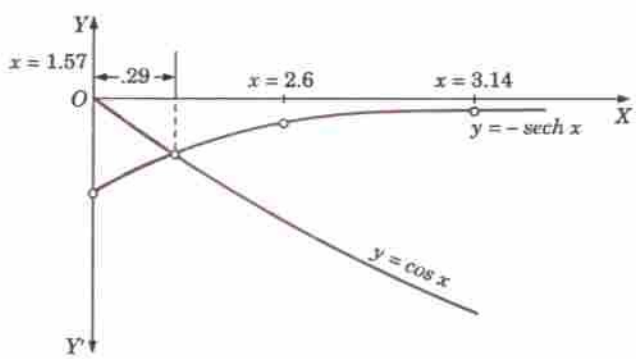

Fig. 1.4

To draw (iii), we form the following table :

$x =$	1.57	2.36	3.14
$\cosh x =$	2.58	5.56	11.12
$y = -\operatorname{sech} x$	-0.39	-0.18	-0.09

Then we plot the curve (*iii*) to the same scale with the same axes.

From the figure we get the lowest root to be approximately $x = 1.57 + 0.29 = 1.86$.

PROBLEMS 1.5

Solve the following equations graphically :

1. $x^3 - x - 1 = 0$ (*Madras, 2000 S*)
2. $x^3 - 3x - 5 = 0$
3. $x^3 - 6x^2 + 9x - 3 = 0$.
4. $\tan x = 1.2\,x$
5. $x = 3\cos(x - \pi/4)$
6. $e^x = 5x$ which is near $x = 0.2$.

1.8 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 1.6

Choose the correct answer or fill up the blanks in the following problems :

1. If for the equation $x^3 - 3x^2 + kx + 3 = 0$, one root is the negative of another, then the value of k is
   (*a*) 3 (*b*) -3 (*c*) 1 (*d*) -1.
2. If the roots of the equation $x^n - 1 = 0$ are $1, \alpha_1, \alpha_2, \ldots\ldots, \alpha_{n-1}$, then $(1 - \alpha_1)(1 - \alpha_2) \ldots\ldots (1 - \alpha_{n-1})$ is equal to
   (*a*) 0 (*b*) 1 (*c*) n (*d*) $n + 1$.
3. If α, β, γ are the roots of $2x^3 - 3x^2 + 6x + 1 = 0$, then $\alpha^2 + \beta^2 + \gamma^2$ is
   (*a*) 15/4 (*b*) -3 (*c*) $-15/4$ (*d*) 33/4.
4. $x + 2$ is a factor of
   (*a*) $x^4 + 2$ (*b*) $x^4 - x^2 + 12$
   (*c*) $x^4 - 2x^3 - x + 2$ (*d*) $x^4 + 2x^3 - x - 2$
5. If $\alpha + \beta + \gamma = 5$; $\alpha\beta + \beta\gamma + \gamma\alpha = 7$; $\alpha\beta\gamma = 3$, then the equation whose roots are α, β and γ is
   (*a*) $x^3 - 7 = 0$ (*b*) $x^3 - 7x^2 + 3 = 0$
   (*c*) $x^3 - 5x^2 + 7x - 3 = 0$ (*d*) $x^3 + 7x^2 - 3 = 0$.
6. If one of the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$ is 2, then the other two roots are
   (*a*) 1 and 3 (*b*) 0 and 4
   (*c*) -1 and 5 (*d*) -2 and 6.
7. The equation whose roots are the reciprocals of the roots of $x^3 + px^2 + r = 0$ is
   (*a*) $x^3 + 1/p.x^2 + 1/r = 0$ (*b*) $1/r \,.\, x^3 + 1/p.x + 1 = 0$
   (*c*) $rx^3 + px^2 + 1 = 0$ (*d*) $rx^3 + px + 1 = 0$.
8. If 1 and 2 are two roots of the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, then the remaining two roots are
   (*a*) -3 and 5 (*b*) 3 and -5
   (*c*) -6 and 5 (*d*) 6 and -5.
9. If the roots of $x^3 - 3x^2 + px + 1 = 0$, are in arithmetic progression, then the sum of squares of the largest and the smallest roots is
   (*a*) 3 (*b*) 5 (*c*) 6 (*d*) 10.
10. A root of $x^3 - 8x^2 + px + q = 0$ where p and q are real numbers is $3 + i\sqrt{3}$. The real root is
    (*a*) 2 (*b*) 6 (*c*) 9 (*d*) 12.
11. One of the roots of the equation $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$ where $a_0, a_1, \ldots a_{n-1}$ are real, is given to be $2 - 3i$. Of the remaining, the next $n - 2$ roots are given to be $1, 2, 3, \ldots, n - 2$. The nth root is
    (*a*) n (*b*) $n - 1$ (*c*) $2 + 3i$ (*d*) $-2 + 3i$.
12. If a real root of $f(x) = 0$ lies in $[a, b]$, then the sign of $f(a) . f(b)$ is ......
13. Descartes rule of signs states that ......
14. If α, β, γ are the roots of the equation $x^3 - px + q = 0$, then $\Sigma\, 1/\alpha =$ ......
15. If α, β, γ are the roots of $x^3 = 7$, then $\Sigma\alpha^3$ is ......
16. One real root of the equation $x^3 + 2x^2 + 5 = 0$ lies between ....

17. In an equation with real coefficients, imaginary roots must occur in ......
18. If $f(\alpha)$ and $f(\beta)$ are of opposite signs, then $f(x) = 0$ has at least one root between α and β provided ......
19. If α, β, γ are the roots of the equation $x^3 + 2x + 3 = 0$, then $\alpha + 3, \beta + 3, \gamma + 3$ are the roots of the equation ......
20. If one root is double of another in $x^3 - 7x^2 + 36 = 0$, then its roots are ......
21. The equation whose roots are 10 times those $x^3 - 2x - 7 = 0$, is ......
22. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, then $\Sigma\,(1/\alpha\beta) =$ ......
23. $\sqrt{3}$ and $-1 + i$ are the roots of the biquadratic equation ......
24. If α, β, γ are the roots of $x^3 - 3x + 2 = 0$, then the value of $\alpha^2 + \beta^2 + \gamma^2$ is ......
25. If there is a root of $f(x) = 0$ in the interval $[a, b]$, then sign of $f(a)/f(b)$ is ......
26. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, then the condition for $\alpha + \beta = 0$ is ......
27. The three roots of $x^3 = 1$ are ......
28. One real root of the equation $x^3 + x - 5 = 0$ lies in the interval
    (*i*) (2, 3), (*ii*) (3, 4), (*iii*) (1, 2), (*iv*) (– 3, – 2)
29. If two roots of $x^3 - 3x^2 + 2 = 0$ are equal, then its roots are ......
30. The cubic equation whose two roots are 5 and $1 - i$ is ......
31. The sum and product of the roots of the equation $x^5 = 2$ are ...... and .....
32. If the roots of the equation $x^4 + 2x^3 - ax^2 - 22x + 40 = 0$ are $-5, -2, 1$ and 4, then $a =$ ......
33. A root of $x^3 - 3x^2 + 2.5 = 0$ lies between 1.1 and 1.2. (True or False)
34. The equation $x^6 - x^5 - 10x + 7 = 0$ has four imaginary roots. (True or False)