CHAPTER 31

Difference Equations

31.1 INTRODUCTION

Difference calculus also forms the basis of Difference equations. These equations arise in all situations in which sequential relation exists at various discrete values of the independent variable. The need to work with discrete functions arises because there are physical phenomena which are inherently of a discrete nature. In control engineering, it often happens that the input is in the form of discrete pulses of short duration. The radar tracking devices receive such discrete pulses from the target which is being tracked. As such differences equations arise in the study of electrical networks, in the theory of probability, in statistical problems and many other fields.

Just as the subject of Differential equations grew out Differential calculus to become one of the most powerful instruments in the hands of a practical mathematician when dealing with continuous processes in nature, so the subject of Difference equations is forcing its way to the fore for the treatment of discrete processes. Thus the difference equations may be thought of as the discrete counterparts of the differential equations.

31.2 DEFINITION

(1) **A difference equation** *is a relation between the differences of an unknown function at one or more general values of the argument.*

Thus $\quad \Delta y_{(n+1)} + y_{(n)} = 2 \quad$ ...(1) $\quad$ and $\quad \Delta y_{(n+1)} + \Delta^2 y_{(n-1)} = 1 \quad$ ...(2)

are difference equations.

An alternative way of writing a difference equation is as under :

Since $\Delta y_{(n+1)} = y_{(n+2)} - y_{(n+1)}$, therefore (1) may be written as

$$y_{(n+2)} - y_{(n+1)} + y_{(n)} = 2 \qquad ...(3)$$

Also since, $\Delta^2 y_{(n-1)} = y_{(n+1)} - 2y_{(n)} + y_{(n-1)}$, therefore (2) takes the form :

$$y_{(n+2)} - 2y_{(n)} + y_{(n-1)} = 1 \qquad ...(4)$$

Quite often, difference equations are met under the name of *recurrence relations.*

(2) **Order of a difference equation** *is the difference between the largest and the smallest arguments occuring in the difference equation divided by the unit of increment.*

Thus (3) above is the *second order*, for

$$\frac{\text{largest argument} - \text{smallest argument}}{\text{unit of increment}} = \frac{(n+2)-n}{1} = 2,$$

and (4) is of the *third order*, for $\dfrac{(n+2)-(n-1)}{1} = 3.$

Obs. *While finding the order of a difference equation, it must always be expressed in a form free of Δs, for the highest* power of Δ does not give order of the difference equation.

(3) Solution *of a difference equation is an expression for* $y_{(n)}$ *which satisfies the given difference equation.*

The general solution *of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.*

A particular solution or **particular integral** *is that solution which is obtained from the general solution by giving particular values to the constants.*

31.3 FORMATION OF DIFFERENCE EQUATIONS

The following examples illustrate the way in which difference equations arise and are formed.

Example 31.1. *Form the difference equation corresponding to the family of curves*

$$y = ax + bx^2 \quad \text{...}(i)$$

Solution. We have $\Delta y = a\Delta(x) + b\Delta(x^2) = a(x + 1 - x) + b[(x + 1)^2 - x^2]$

$$= a + b(2x + 1) \quad \text{...}(ii)$$

and $\Delta^2 y = 2b[(x + 1) - x] = 2b$ ...(iii)

To eliminate a and b, we have from (iii), $b = \frac{1}{2}\Delta^2 y$

and form (ii), $a = \Delta y - b(2x + 1) = \Delta y - \frac{1}{2}\Delta^2 y(2x + 1)$

Substituting these values of a and b in (i), we get

$$y = \left[\Delta y - \frac{1}{2}\Delta^2 y(2x+1)\right]x + \frac{1}{2}\Delta^2 y \,.\, x^2$$

or $(x^2 + x)\Delta^2 y - 2x\Delta y + 2y = 0$

This is the desired difference equation which may equally well be written in terms of E as

or $(x^2 + x)y_{x+2} - (2x^2 + 4x)y_{x+1} + (x^2 + 3x + 2)y_x = 0.$

Example 31.2. *From $y_n = A2^n + B(-3)^n$, derive a difference equation by eliminating the constants.*

Solution. We have $y_n = A.2^n + B(-3)^n$, $y_{n+1} = 2A.2^n - 3B(-3)^n$

and $y_{n+2} = 4A.2^n + 9B(-3)^n$.

Eliminating A and B, we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -3 \\ y_{n+2} & 4 & 9 \end{vmatrix} = 0 \quad \text{or} \quad y_{n+2} + y_{n+1} - 6y_n = 0$$

which is the desired difference equation.

PROBLEMS 31.1

1. Write the difference equation $\Delta^3 y_x + \Delta^2 y_x + \Delta y_x + y_x = 0$ in the subscript notation.
2. Assuming $\frac{\log(1-z)}{1+z} = y_0 + y_1 z + y_2 z^2 + \ldots\ldots + y_n z^n, \ldots\ldots$, find the difference equations satisfied by y_n.
3. Form a difference equation by eliminating arbitrary constant from $u_n = a2^{n+1}$. *(Anna, 2008)*
4. Find the difference equation satisfied by
   (i) $y = a/x + b$ *(Tiruchirapalli, 2001)* (ii) $y = ax^2 - bx$.
5. Derive the difference equations in each of the following cases :
   (i) $y_n = A.3^n + B.5^n$ (ii) $y_n = (A + Bx)2^x$. *(Madras, 2001)*
6. Form the difference equations generated by
   (i) $y_n = ax + b2^x$ (ii) $y_n = a2^n + b(-2)^n$ (iii) $y_x = a2^x + b3^x + c$.

31.4 LINEAR DIFFERENCE EQUATIONS

(1) Def. *A **linear difference equation** is that in which y_{n+1}, y_{n+2} etc. occur to the first degree only and are not multiplied together.*

*A **linear difference equation with constant coefficient** is of the form*

$$y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \dots + a_r y_n = f(n) \quad \dots(1)$$

where $a_1, a_2, \dots a_r$ are constants.

Now we shall deal with linear difference equations with constant coefficients only. Their properties are analogous to those of linear differential equations with constant co-efficients.

(2) Elementary properties. If $u_1(n), u_2(n), \dots, u_r(n)$ be r independent solution of the equation

$$y_{n+r} + a_1 y_{n+r-1} + \dots + a_r y_n = 0 \quad \dots(2)$$

then its complete solution is $U_n = c_1 u_1(n) + \dots + c_r u_r(n)$

where $c_1, c_2, \dots, c_r$ are arbitrary constants.

If V_n is a particular solution of (1), then the complete solution of (1) is $y_n = U_n + V_n$. *The part U_n is called the **complementary function (C.F.)** and the part V_n is called the **particular integral (P.I.)** of (1).*

*Thus the **complete solution (C.S.)** of (1) is* $\boldsymbol{y_n}$ **= C.F. + P.I.**

31.5 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

(*i.e., rules to solve a linear difference equation with constant coefficients having right hand side zero*).

(1) To begin with, consider the first order linear equation $y_{n+1} - \lambda y_n = 0$, where λ is a constant.

Rewriting it as $\dfrac{y_{n+1}}{\lambda^{n+1}} - \dfrac{y_n}{\lambda^n} = 0$, we have $\Delta\left(\dfrac{y_n}{\lambda^n}\right) = 0$, which gives $y_n/\lambda^n = c$, a constant.

Thus the solution of $(E - \lambda)\, y_n = 0$ is $\boldsymbol{y_n = c.\lambda^n}$.

(2) Now consider the *second order linear equation* $y_{n+2} + ay_{n+1} + by_n = 0$ which in *symbolic form is*

$$(E^2 + aE + b)y_n = 0 \quad \dots(1)$$

Its symbolic co-efficient equated to zero *i.e.*, $E^2 + aE + b = 0$

is called the *auxiliary equation*. Let its roots be λ_1, λ_2.

Case I. *If these roots are real and distinct,* then (1) is to equivalent to

$$(E - \lambda_1)(E - \lambda_2)\, y_n = 0 \quad \dots(2)$$

$$(E - \lambda_2)(E - \lambda_1)\, y_n = 0 \quad \dots(3)$$

If y_n satisfies the subsidiary equation $(E - \lambda_1)y_n = 0$, then it will also satisfy (3).

Similarly, if y_n satisfies the subsidiary equation $(E - \lambda_2)y_n = 0$, then it will also satisfy (2).

$\therefore$ it follows that we can derive two independent solutions of (1), by solving the two subsidiary equations

$$(E - \lambda_1)\, y_n = 0 \quad \text{and} \quad (E - \lambda_2)\, y_n = 0$$

Their solutions are respectively, $y_n = c_1(\lambda_1)^n$ and $y_n = c_2(\lambda_2)^n$

where c_1 and c_2 are arbitrary constants.

Thus the general solution of (1) is $y_n = c_1(\lambda_1)^n + c_2(\lambda_2)^n$

Case II. *If the roots are real and equal* (*i.e.*, $\lambda_1 = \lambda_2$), then (2) becomes

$$(E - \lambda_1)^2\, y_n = 0 \quad \dots(4)$$

Let $\qquad y_n = (\lambda_1)^n z_n$

where z_n is a new dependent variable. Then (4) takes the form

$$(\lambda_1)^{n+2} z_{n+2} - 2\lambda_1(\lambda_1)^{n+1} z_{n+1} + \lambda_1^2 . (\lambda_1)^n z_n = 0$$

or $$z_{n+2} - 2z_{n+1} + z_n = 0 \quad i.e., \quad \Delta^2 z_n = 0$$

$\therefore$ $\qquad z_n = c_1 + c_2 n$, where c_1, c_2 are arbitrary constants.

Thus the solution of (1) becomes $y_n = (c_1 + c_2 n)(\lambda_1)^n$.

Case III. *If the roots are imaginary,* (*i.e.* $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$) then the solution of (1) is

$$y_n = c_1(\alpha + i\beta)^n + c_2(\alpha - i\beta)^n \quad [\text{Put } \alpha = r\cos\theta \text{ and } \beta = r\sin\theta]$$

$$= r^n\, [c_1(\cos n\theta + i\sin n\theta) + c_2(\cos n\theta - i\sin n\theta)]$$

$$= r^n [A_1 \cos n\theta + A_2 \sin n\theta]$$

where A_1, A_2 are arbitrary constants are $r = \sqrt{(\alpha^2 + \beta^2)}$, $\theta = \tan^{-1}(\beta/\alpha)$.

(3) *In general, to solve the equation* $y_{n+r} + a_1 y_{n+r-2} + ... + a_r y_n = 0$ where a's are constants :

(*i*) Write the equation in the *symbolic form* $(E^r + a_1 E^{r-1} + ... + a_r)y_n = 0$.

(*ii*) Write down the *auxiliary equation i.e.,* $E^r + a_1 E^{r-1} ... + a_r = 0$ and solve it for E.

(*iii*) Write the solution as follows :

Roots of A.E.	*Solution, i.e. C.F.*
1. $\lambda_1, \lambda_2, \lambda_3$, ... (real and distinct roots)	$c_1(\lambda_1)^n + c_2(\lambda_2)^n + c_3(\lambda_3)^n + ...$
2. $\lambda_1, \lambda_1, \lambda_3$, ... (2 real and equal roots)	$(c_1 + c_2 n)(\lambda_1)^n + c_3(\lambda_3)^n + ...$
3. $\lambda_1, \lambda_1, \lambda_1$, ... (3 real and equal roots)	$(c_1 + c_2 n + c_3 n^2)(y_1)^n + ...$
4. $\alpha + i\beta, \alpha - i\beta$, ... (a pair of imaginary roots)	$r^n (c_1 \cos \theta + c_2 \sin n\theta)$ where $r = \sqrt{(\alpha^2 + \beta^2)}$ and $\theta = \tan^{-1}(\beta/\alpha)$

Example 31.3. *Solve the difference equation* $u_{n+3} - 2u_{n+2} - 5u_{n+1} + 6u_n = 0$.

Solution. Given equation in *symbolic form* is $(E^3 - 2E^2 - 5E + 6)u_n = 0$

$\therefore$ its auxiliary equation is $E^3 - 2E^2 - 5E + 6 = 0$

or $$(E - 1)(E + 2)(E - 3) = 0. \qquad \therefore \quad E = 1, -2, 3$$

Thus the complete solution is $u_n = c_1(1)^n + c_2(-2)^n + c_3(3)^n$.

Example 31.4. *Solve* $u_{n+2} - 2u_{n+1} + u_n = 0$.

Solution. Given difference equation in *symbolic form* is $(E^2 - 2E + 1)\, u_n = 0$.

$\therefore$ its auxiliary equation is $E^2 - 2E + 1 = 0$

or $$(E - 1)^2 = 0. \qquad \therefore \quad E = 1, 1$$

Thus the required solution is $u_n = (c_1 + c_2 n)(1)^n$, *i.e.*, $u_n = c_1 + c_2 n$.

Example 31.5. *Solve* $y_{n+1} - 2y_n \cos \alpha + y_{n-1} = 0$.

Solution. This is a second order difference equation in y_{n-1} ; which in *symbolic form* is

$$(E^2 - 2E \cos \alpha + 1)\, y_n = 0$$

The *auxiliary equation* is $E^2 - 2E \cos \alpha + 1 = 0$

$$E = \frac{2 \cos \alpha \pm \sqrt{(4 \cos^2 \alpha - 4)}}{4} = \cos \alpha \pm i \sin \alpha$$

Thus the solution is $y_{n-1} = (1)^{n-1} [c_1 \cos (n-1)\alpha + c_2 \sin (n-1)\alpha]$

or $$y_n = c_1 \cos n\alpha + c_2 \sin n\alpha.$$

Example 31.6. *The integers 0, 1, 1, 2, 3, 5, 8, 13, 21, ... are said to form a Fibonacci sequence. Form the Fibonacci difference equation and solve it.*

Solution. In this sequence, each number beyond the second, is the sum of its two previous number. If y_n be the nth number then $y_n = y_{n-1} + y_{n-2}$ for $n > 2$.

or $$y_{n+2} - y_{n+1} - y_n = 0 \text{ (for } n > 0)$$

or $$(E^2 - E - 1)y_n = 0 \text{ is the difference equation.}$$

Its A.E. is $E^2 - E - 1 = 0$ which gives $E = \frac{1}{2}(1 \pm \sqrt{5})$.

Thus the solution is $$y_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^2 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n, \text{ for } n > 0$$

When $n = 1, y = 0$

$$\therefore \qquad c_1\left(\frac{1+\sqrt{5}}{2}\right)+c_2\left(\frac{1-\sqrt{5}}{2}\right)=0 \qquad ...(i)$$

When $n = 2, y_2 = 0$

$$\therefore \qquad c_1\left(\frac{1+\sqrt{5}}{2}\right)^2+c_2\left(\frac{1-\sqrt{5}}{2}\right)^2=0 \qquad ...(ii)$$

Solving (*i*) and (*ii*), we get

$$c_1 = \frac{5-\sqrt{5}}{10} \text{ and } c_2 = \frac{5+\sqrt{5}}{10}$$

Hence the complete solution is

$$y_n = \frac{5-\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

PROBLEMS 31.2

Solve the following difference equations :

1. $u_{x+2} - 6u_{x+1} + 9u_x = 0.$
2. $y_{n+2} + y_{n+1} + 2y_n = 0.$
3. $\Delta^2 u_n + 2\Delta u_n + u_n = 0.$
4. $(\Delta^2 - 3\Delta + 2)y_n = 0.$
5. $4y_n - y_{n+2} = 0$ given that $y_0 = 0, y_1 = 2.$
6. $u_{k+3} - 3u_{k+2} + 4u_k = 0.$
7. $f(x+3) - 3f(x+1) - 2f(x) = 0.$
8. $u_{n+3} - 3u_{n+1} + 2u_n = 0$, given $u_1 = 0, u_2 = 8$ and $u_3 = -2.$
9. $(E^3 - 5E^2 + 8E - 4)y_n = 0$, given that $y_0 = 3, y_1 = 2, y_3 = 22.$
10. $u_{n+1} - 2u_n + 2u_{n-1} = 0.$
11. $y_{m+3} + 16y_{m-1} = 0.$
    [**Hint.** $E^4 = -16 = 16 [\cos (2n+1)\pi + i \sin (2n+1)\pi]$; use De Moivre's theorem.]
12. Show that the difference equation $I_{m+1} - (2 + r_o/r) I_m + I_{m-1} = 0$ has the solution.
    $I_m = I_o \sinh (n-m)\alpha / \sinh (n-1)\alpha$, if $I = I_0$ and $I_n = 0$, α being $= 2 \sinh^{-1} \frac{1}{2} (r_0/r)^{1/2}$.
13. A series of values of y_n satisfy the relation, $y_{n+2} + ay_{n+1} + by_n$.
    Given that $y_0 = 0, y_1 = 1, y_2 = y_3 = 2$. Show that $y_n = 2^{n/2} \sin n\pi/4$.
14. A plant is such that each of its seeds when one year old produces 8-fold and produces 18-fold when two years old or more. A seed is planted and as soon as a new seed is produced it is planted. Taking y_n to be the number of seeds produced at the end of the nth year, show that $y_{n+1} = 8y_n + 18(y_1 + y_2 + ... + y_{n-1})$.
    Hence show that $y_{n+2} - 9y_{n+1} - 10y_n = 0$ and find y_n.

31.6 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $y_{n+r} + a_1 y_{n+r-1} + ... + a_r y_n = f(n)$
which in *symbolic form* is $\phi(E)y_n = f(n)$ ...(1)
where $\phi(E) = E^r + a_1 E^{r-1} + ... + a_r$

Then the particular integral is given by P.I. $= \dfrac{1}{\phi(E)} f(n)$.

Case I. When $f(n) = a^n$

$$\text{P.I.} = \frac{1}{\phi(E)} a_n, \text{ put } E = a$$

$$= \frac{1}{\phi(a)} a^n, \text{ provided } \phi(a) \neq 0$$

If $\phi(a) = 0$, *then for the equation*

(*i*) $(E - a)y_n = a^n$, P.I. $= \dfrac{1}{E-a} a^n = na^{n-1}$

(ii) $(E-a)^2 y_n = a^n$, $\quad$ P.I. $= \dfrac{1}{(E-a)^2} a^n = \dfrac{n(n-1)}{2!} a^{n-2}$

(iii) $(E-a)^3 y^n = a^n$, $\quad$ P.I. $= \dfrac{1}{(E-a)^3} a^n = \dfrac{n(n-1)(n-2)}{3!} a^{n-3}$

and so on.

Example 31.7. *Solve* $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$.

Solution. Given equation in *symbolic form* is $(E^2 - 4E + 3)y_n = 5^n$

$\therefore$ The auxiliary equation is $E^2 - 4E + 3 = 0$

or $\quad (E-1)(E-3) = 0. \quad \therefore \; E = 1, 3$

$\therefore \quad$ C.F. $= c_1(1)^n + c_2(3)^n = c_1 + c_2 . 3^n$

and $\quad$ P.I. $= \dfrac{1}{E^2 - 4E + 3} 5^n$ $\quad$ [Put $E = 5$]

$$= \frac{1}{25 - 4.5 + 3} 5^n = \frac{1}{8} . 5^n$$

Thus the complete solution is $\quad y_n = c_1 + c_2 . 3^n + 5^n/8$.

Example 31.8. *Solve* $u_{n+2} - 4u_{n+1} + 4u_n = 2^n$.

Solution. Given equation in *symbolic form* is $(E^2 - 4E + 4)u_n = 2^n$.

The auxiliary equation is $E^2 - 4E + 4 = 0. \quad \therefore \; E = 2, 2.$

C.F. $= (c_1 + c_2 n)2^n$

P.I. $= \dfrac{1}{(E-2)^2} . 2^n = \dfrac{n(n-1)}{2!} . 2^{n-2} = n(n-1)\, 2^{n-3}$

Hence the complete solution is $u_n = (c_1 + c_2 n)\, 2^n + n(n-1)\, 2^{n-3}$.

Case II. *When* $f(n) = \sin kn$.

$$\text{P.I.} = \frac{1}{\phi(E)} \sin kn = \frac{1}{\phi(E)} \left(\frac{e^{ikn} - e^{-ikn}}{2i} \right) = \frac{1}{2i} \left[\frac{1}{\phi(E)} a^n - \frac{1}{\phi(E)} b^n \right]$$

where $a = e^{ik}$ and $b = e^{-ik}$.

Now proceed as in case I.

(2) *When* $f(n) = \cos kn \quad$ P.I. $= \dfrac{1}{\phi(E)} \cos kn = \dfrac{1}{\phi(E)} \left(\dfrac{e^{ikn} + e^{-ikn}}{2} \right)$

$$= \frac{1}{2} \left[\frac{1}{\phi(E)} a^n + \frac{1}{\phi(E)} b^n \right] \text{ as before}$$

Now proceed as in case I.

Example 31.9. *Solve* $y_{n+2} - 2 \cos \alpha . y_{n+1} + y_n = \cos \alpha n$. *(Nagpur, 2008)*

Solution. Given equation in *symbolic form* is $(E^2 - 2 \cos \alpha . E + 1)y_n = \cos \alpha n$.

The *auxiliary equation is* $E^2 - 2 \cos \alpha . E + 1 = 0$.

$$\therefore \quad E = \frac{2 \cos \alpha \pm \sqrt{(4 \cos^2 \alpha - 4)}}{2} = \cos \alpha \pm i \sin \alpha$$

$\therefore \quad$ C.F. $= (1)^n [c_1 \cos \alpha n + c_2 \sin \alpha n]$ *i.e.*, $c_1 \cos \alpha n + c_2 \sin \alpha n$

$$\text{P.I.} = \frac{1}{E^2 - 2E \cos \alpha + 1} \cos \alpha n$$

$$= \frac{1}{E^2 - E(e^{i\alpha} + e^{-i\alpha}) + 1} \left(\frac{e^{ian} + e^{-ian}}{2} \right)$$

$$= \frac{1}{2}\left[\frac{1}{(E-e^{i\alpha})(E-e^{-i\alpha})}e^{i\alpha n} + \frac{1}{(E-e^{i\alpha})(E-e^{-i\alpha})}e^{-i\alpha n}\right]$$
$$[\text{Put } E = e^{i\alpha}] \qquad [\text{Put } E = e^{-i\alpha}]$$

$$= \frac{1}{2}\left[\frac{1}{(E-e^{i\alpha})}\cdot\frac{1}{e^{i\alpha}-e^{-i\alpha}}e^{i\alpha n} + \frac{1}{E-e^{-i\alpha}}\cdot\frac{1}{e^{-i\alpha}-e^{i\alpha}}e^{-i\alpha n}\right]$$

$$= \frac{1}{4i\sin\alpha}\left[\frac{1}{E-e^{i\alpha}}e^{i\alpha n} - \frac{1}{E-e^{-i\alpha}}e^{-i\alpha n}\right] = \frac{1}{4i\sin\alpha}\,[n\,.\,e^{i\alpha(n-1)} - n\,e^{-i\alpha(n-1)}]$$

$$= \frac{n}{2\sin\alpha}\left[\frac{e^{i\alpha(n-1)} - e^{-i\alpha(n-1)}}{2i}\right] = \frac{n\sin(n-1)\alpha}{2\sin\alpha}$$

Hence the complete solution is

$$y_n = c_1\cos\alpha n + c_2\sin\alpha n + \frac{n\sin(n-1)\alpha}{2\sin\alpha}.$$

Case III. *When* $f(n) = n^p$. $\qquad$ P.I. $= \frac{1}{\phi(E)}n^p = \frac{1}{\phi(1+\Delta)}n^p$

(1) Expand $[\phi(1+\Delta)]^{-1}$ in ascending powers of Δ by the Binomial theorem as far as the term in Δ^p.

(2) Expand n^p in the factorial form (p. 950) and operate on it with each term of the expansion.

Example 31.10. *Solve* $y_{n+2} - 4y_n = n^2 + n - 1$. *(Madras, 1999)*

Solution. Given equation is $(E^2 - 4)y_n = n^2 + n - 1$.

The *auxiliary equation is* $E^2 - 4 = 0$, $\therefore$ $E = \pm 2$.

$\therefore$ $\qquad$ C.F. $= c_1(2)^n + c_2(-2)^n$.

$\therefore$ $\qquad$ P.I. $= \frac{1}{E^2-4}(n^2+n-1) = \frac{1}{(1+\Delta)^2-4}[n(n-1)+2n-1]$.

$$= \frac{1}{\Delta^2+2\Delta-3}([n]^2+2[n]-1) = -\frac{1}{3}\left[1-\left(\frac{2}{3}\Delta+\frac{\Delta^2}{3}\right)\right]^{-1}\{[n]^2+2[n]-1\}$$

$$= -\frac{1}{3}\left[1+\left(\frac{2}{3}\Delta+\frac{\Delta^2}{3}\right)+\left(\frac{2}{3}\Delta+\frac{\Delta^2}{3}\right)+\ldots\right]\{[n]^2+2[n]-1\}$$

$$= -\frac{1}{3}\left\{1+\frac{2}{3}\Delta+\frac{7}{9}\Delta^2+\ldots\right\}\{[n]^2+2[n]-1\} = -\frac{1}{3}\left\{[n]^2+2[n]-1+\frac{2}{3}(2[n]+2)+\frac{7}{9}\times 2\right\}$$

$$= -\frac{1}{3}\left\{[n]^2+\frac{10}{3}[n]+\frac{17}{9}\right\} = -\frac{n^2}{3}-\frac{7}{9}n-\frac{17}{27}.$$

Hence the *complete solution* is $y_n = c_1 2^n + c_2(-2)^n - \frac{n^2}{3} - \frac{7}{9}n - \frac{17}{27}$.

Case IV. *When* $f(n) = a^n F(n)$, $F(n)$, being a polynomial of finite degree in n.

$$\text{P.I.} = \frac{1}{\phi(E)}a^n F(n) = a^n\frac{1}{\phi(aE)}F(a)$$

Now $F(n)$ being a polynomial in n, proceed as in case III.

Example 31.11. *Solve* $y_{n+2} - 2y_{n+1} + y_n = n^2 . 2^n$. *(Nagpur, 2008)*

Solution. Given equation is $(E^2 - 2E + 1)y_n = n^2 . 2^n$.

Its $\qquad$ C.F. $= c_1 + c_2 n$

and $\qquad$ P.I. $= \frac{1}{(E-1)^2}2^n . n^2 = 2^n\frac{1}{(2E-1)^2}n^2 = 2^n\frac{1}{(1+2\Delta)^2}n^2$

$$= 2^n (1 + 2\Delta)^{-2} n(n-1) + n = 2^n (1 - 4\Delta + 12\Delta^2 - \ldots)([n]^2 + [n])$$
$$= 2^n \{[n]^2 + [n] - 4(2[n] + 1) + 12 \times 2\}$$
$$= 2^n ([n]^2 - 7[n] + 20) = 2^n (n^2 - 8n + 20)$$

Hence the *complete solution* is $y_n = c_1 + c_2 n + 2^n (n^2 - 8n + 20)$.

PROBLEMS 31.3

Solve the following difference equations :

1. $y_{n+2} - 5y_{n+1} - 6y_n = 4^n, y_0 = 0, y_1 = 1.$ *(Madras, 2003)*
2. $y_{n+2} + 6y_{n+1} + 9y_n = 2^n, y_0 = y_1 = 0.$ *(V.T.U., 2009)*
3. $y_{p+3} - 3y_{p+2} 3y_{p+1} - y_p = 1.$ *(Kottayam, 2005)*
4. $y_{n+2} - 2y_{n+1} + 4y_n = 6$, given that $y_0 = 0$ and $y_1 = 2$.
5. $(E^2 - 4E + 3)y = 3^x.$
6. $y_{x+2} - 4y_{x+1} + 4y_x = 3.2^x + 5.4^x.$
7. $u_{n+2} - u_n = \cos n/2.$ *(Madras, 2001 S)*
8. $y_{p+2} - \left(2\cos\frac{1}{2}\right) y_{p+1} + y_p = \sin p/2.$
9. $(E^2 - 4)y_x = x^2 - 1.$
10. $y_{n+3} + y_n = n^2 + 1 y_0 = y_1 = y_2 = 0.$ *(Tirchirapalli, 2001)*
11. $y_{n+3} - 5y_{n+2} + 3y_{n+1} + 9y_n = 2^n + 3n.$ *(Nagpur, 2009)*
12. $(4E^2 - 4E + 1)y = 2^n + 2^{-n}.$ *(Madras, 2001)*
13. $y_{n+2} + 5y_{n+1} + 6y_n = n + 2^n.$ *(Nagpur, 2006)*
14. $u_{x+2} + 6u_{x+1} + 9u_x = x2^x + 3^x + 7.$
15. $y_{n+3} + 8y_n = (2n+3)\, 2^n$ *(Nagpur, 2005)*
16. $u_{n+2} - 4u_{n+1} 4u_n = n^2 2^n.$
17. $(E^2 - 5E + 6)y_k = 4^k (k^2 - k + 5).$
18. $(E^2 - 2E + 4)y_n = -2^n \left\{6\cos\frac{n\pi}{3} + 2\sqrt{3}\sin\frac{n\pi}{3}\right\}.$
19. A beam of length l, supported at n points carries a uniform load w per unit length. The bending moments M_1, M_2 ..... M_n at the supports satisfy the Clapeyron's equation :
$$M_{r+2} + 4M_{r+1} + M_r = -\frac{1}{2} wl^2$$
If a beam weighing 30 kg is supported at its ends and at two other supports dividing the beam into three equal parts of 1 metre length, show that the bending moment at each of the two middle supports is 1 kg metre.

31.7 SIMULTANEOUS DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

The method used for solving simultaneous differential equations with constant coefficients also applies to simultaneous difference equations with constants coefficients. The following example illustrates the technique.

Example 31.12. *Solve the simultaneous difference equations*
$$u_{x+1} + v_x - 3u_x = x,\ 3u_x + v_{x+1} - 5v_x = 4^x$$
subject to the conditions $u_1 = 2$, $v_1 = 0$.

Solution. Given equation in symbolic form, are

$$(E-3)\,u_x + v_x = x \quad \ldots(i)$$
$$3u_x + (E-5)v_x = 4^x \quad \ldots(ii)$$

Operating the first equation with $E - 5$ and subtracting the second from it, we get

$$[(E-5)(E-3) - 3]u_x = (E-5)x - 4^x$$

or
$$(E^2 - 8E + 12)u_x = 1 - 4x - 4^x$$

Its solution is
$$u_x = c_1\, 2^x + c_2\, 6^x - \frac{4}{5}x - \frac{19}{25} + \frac{4^x}{4} \quad \ldots(iii)$$

Substituting the value of u_x from (*iii*) in (*i*), we get

$$v_x = c_1\, 2^x - 3c_2\, 6^x - \frac{3x}{5} - \frac{34}{25} - \frac{4^x}{4} \quad \ldots(iv)$$

Taking $u_1 = 2$, $v_1 = 0$, in (*iii*) and (*iv*), we obtain

$$2c_1 + 6c_2 = \frac{64}{25},\ 2c_1 - 18c_2 = \frac{74}{25}$$

when
$$c_1 = 1.33,\ c_2 = -0.0167$$

Hence $u_x = 1.33.2^x - 0.0167.6^x - 0.8x - 0.76 + 4^{x-1}$
$u_x = 1.33.2^x - 0.05.6^x - 0.6x - 1.36 - 4^{x-1}$.

PROBLEMS 31.4

Solve the following simultaneous difference equations :

1. $y_{x+1} - z_x = 2(x+1), z_{x+1} - y_x = -2(x+1)$.
2. $y_{n+1} - y_n + 2z_{n+1} = 0, z_{n+1} - z_n - 2y_n = 2^n$.
3. $u_{n+1} + n = 3u_n + 2v_n, v_{n+1} - n = u_n + 2v_n$, given $u_0 = 0, v_0 = 3$.
4. $u_{x+1} + v_x + w_x = 1, u_x + v_{x+1} + w_x = x, u_x + v_x + w_{x+1} = 2x$.

31.8 APPLICATION TO DEFLECTION OF A LOADED STRING

Consider a light string of length l stretched tightly between A and B. Let the forces P_i be acting at its equispaced points x_i $(i = 1, 2, ..., n-1)$ and perpendicular to AB resulting in small transverse displacements y_i at these points (Fig. 31.1). Assuming the angle θ_i made by the portion between x_i and x_{i+1} with the horizontal, to be small, we have

$$\sin \theta_i = \tan \theta_i = \theta_i \text{ and } \cos \theta_i = 1$$

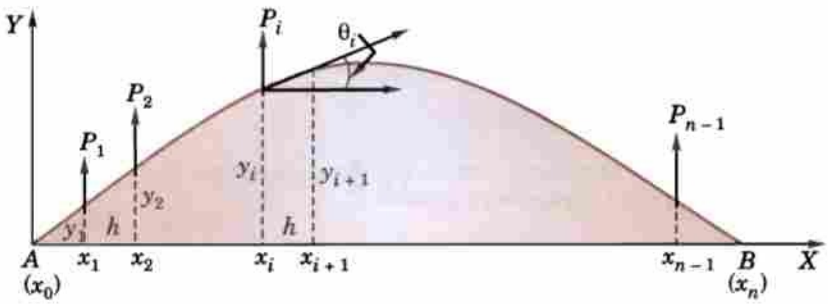

Fig. 31.1

If T be the tension of the string at x_i, then $T \cos \theta_i = T$
i.e., the tension may be taken as uniform.

Taking $x_{i+1} - x_i = h$, we have

$$y_{i+1} - y_i = h \tan \theta_i = h\theta_i \quad ...(1)$$
$$y_i - y_{i-1} = h \tan \theta_{i-1} = h\theta_{i-1} \quad ...(2)$$

Also resolving the forces in equilibrium at $(x_i, y_i) \perp$ to AB, we get

$$T \sin \theta_i - T \sin \theta_{i-1} + P_i = 0 \text{ i.e. } T(\theta_i - \theta_{i-1}) + P_i = 0 \quad ...(3)$$

Eliminating θ_i and θ_{i-1} from (1), (2) and (3), we obtain

$$y_{i+1} - 2y_i + y_{i-1} = -\frac{hP_i}{T} \quad ...(4)$$

which is a difference equation and its solution gives the displacements y_i. To obtain the arbitrary constants in the solution, we take $y_0 = y_n = 0$ as the boundary conditions, since the ends A and B of the string are fixed.

Example 31.13. *A light string stretched between two fixed nails 120 cm apart, carries 11 loads of weight 5 gm each at equal intervals and the resulting tension is 500 gm weight. Show that the sag at the mid-point is 1.8 cm.*

Solution. Taking $h = 10$ cm, $P_i = 5$ gm and $T = 500$ gm wt.,
the above equation (4) becomes $y_{i+1} - 2y_i + y_{i-1} = -1/10$

i.e.,
$$y_{i+2} - 2y_{i+1} + y_i = -\frac{1}{10}$$

Its A.E. is $(E-1)^2 = 0$ *i.e.* $E = 1, 1$. $\therefore$ C.F. $= c_1 + c_2 i$

and $$\text{P.I.} = \frac{1}{(E-1)^2}\left(-\frac{1}{10}\right) = -\frac{1}{10}\frac{1}{(E-1)^2}(1)^i = -\frac{1}{10}\frac{i(i-1)}{2} = \frac{1}{20}(i-i^2)$$

Thus the C.S. is $y_i = c_1 + c_2 i + \frac{1}{20}(i - i^2)$

Since $y_0 = 0, \quad \therefore \quad c_1 = 0$

and $y_{12} = 0, \quad \therefore \quad c_2 = \frac{11}{20}$.

Hence $y_i = \frac{11}{20} i + \frac{1}{20}(i - i^2)$

At the mid-point $i = 6$, we get $y_6 = 1.8$ cm.

PROBLEMS 31.5

1. A light string of length $(n + 1)l$ is stretched between two fixed points with a force P. It is loaded with n equal masses m at distance l. If the system starts rotating with angular velocity ω, find the displacement y_i of the ith mass.

31.9 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 31.6

Select the correct answer or fill up the blanks in the following questions :

1. $y_n = A\,2^n + B\,3^n$, is the solution of the difference equation ....
2. The solution of $(E-1)^3 u_n = 0$ is ....
3. The solution of the difference equation $u_{n+3} - 2u_{n+2} - 5u_{n+1} + 6u_n = 0$ is ....
4. The solution of $y_{n+1} - y_n = 2^n$ is ....
5. The difference equation $y_{n+1} - 2y_n = n$ has y_n = ... as its solution.
6. The difference equation corresponding to the family of curves $y = ax^2 + bx$ is ....
7. The particular integral of the equation $(E-2)\,y_n = 1$.
8. The solution of $4y_n = y_{n+2}$ such that $y_0 = 0, y_1 = 2$, is ....
9. The equation $\Delta^2 u_{n+1} + \frac{1}{2}\Delta^2 u_n = 0$ is of order ....
10. The difference equation satisfied by $y = a + b/x$ is ....
11. The order of the difference equation $y_{n+2} - 2y_{n+1} + y_n = 0$ is ....
12. The solution of $y_{n+2} - 4y_{n+1} + 4y_n = 0$ is ....
13. The particular integral of $u_{x+2} - 6u_{x+1} + 9u_x = 3$ is ....
14. The difference equation generated by $u_n = (a + bn)\,3^n$ is ....
15. Solution of $6y_{n+2} + 5y_{n+1} - 6y_n = 2^n$ is $y_n = A(2/3)^n + B(-3/2)^n + 2^n/28$. (True or False)