CHAPTER 3

Vector Algebra & Solid Geometry

VECTOR ALGEBRA

3.1 (1) VECTORS

A quantity which is completely specified by its magnitude only is called a *scalar*. Length, time, mass, volume, temperature, work, electric charge and numerical data in Statistics are all examples of scalar quantities.

A quantity which is completely specified by its magnitude and direction is called a **vector.** Weight, displacement, velocity, acceleration and electric current density are all vector quantities for each involves magnitude and direction.

A vector is represented by a directed line segment. Thus $\vec{PQ}$ represents a vector whose magnitude is the length PQ and direction is from P (starting point) to Q (end point). We denote a vector by a single letter in capital bold type (or with an arrow on it) and its magnitude (length) by the corresponding small letter in italics type. Thus if **V** is the velocity vector, its magnitude is v, the speed.

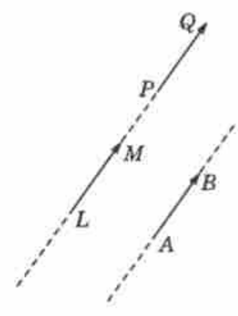


Fig. 3.1

A vector of unit magnitude is called a *unit vector*. The idea of unit vector is often used to represent concisely the direction of any vector. Unit vector corresponding to the vector **A** is written as $\hat{\mathbf{A}}$.

A vector of zero magnitude (which can have no direction associated with it) is called a *zero (or null) vector* and is denoted by **0**–a thick zero.

The vector $\vec{QP}$ represents the negative of $\vec{PQ}$, *i.e.*, **– A.**

Two vectors **A** *and* **B** *having the same magnitude and the same* (or parallel) directions are said to be equal and we write **A** = **B.** Clearly the vectors $\vec{AB}$, $\vec{LM}$ and $\vec{PQ}$ are all equal (Fig. 3.1).

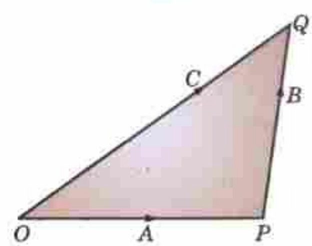


Fig. 3.2

(2) Addition of vectors. Vectors are added according to the *triangle law of addition*, which is a matter of common knowledge. Let **A** and **B** be represented by two vectors $\vec{OP}$ and $\vec{PQ}$ respectively then $\vec{OQ} = \mathbf{C}$ is called the sum or resultant of **A** and **B**. Symbolically, we write,

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

(3) Subtraction of vectors. *The subtraction of a vector* **B** *from* **A** *is taken to be the addition of* –**B** *to* **A** *and we write*

$$\mathbf{A} + (-\mathbf{B}) = \mathbf{A} - \mathbf{B}$$

(4) Multiplication of vectors by scalars.

We have just seen that $\mathbf{A} + \mathbf{A} = 2\mathbf{A}$

and $-\mathbf{A} + (-\mathbf{A}) = -2\,\mathbf{A}$

where both 2**A** and – 2**A** denote vectors of magnitude twice that of **A** ; the former having the same direction as **A** and the latter the opposite direction.

In general, *the product m***A** *of a vector* **A** *and a scalar m is a vector whose magnitude is m times that of* **A** *and direction is the same or opposite to* **A** *according as m is positive or negative.*

Thus $\mathbf{A} = a\,\hat{\mathbf{A}}$.

Example 3.1. *If* **A** *and* **B** *are the vectors determined by two adjacent sides of a regular hexagon. What are the vectors represented by the other sides taken in order ?*

Solution. Let $ABCDEF$ be the given hexagon, such that

$$\vec{AB} = \mathbf{A} \text{ and } \vec{BC} = \mathbf{B}$$

$\therefore$ $\vec{AC} = \vec{AB} + \vec{BC} = \mathbf{A} + \mathbf{B}$

Also $\vec{AD} = 2\vec{BC} = 2\mathbf{B}$

$\therefore$ $\vec{CD} = \vec{AD} - \vec{AC} = 2\mathbf{B} - (\mathbf{A} + \mathbf{B}) = \mathbf{B} - \mathbf{A}$

Now $\vec{DE} = -\vec{AB} = -\mathbf{A}$ [$\because$ AB = and $\parallel$ ED]

$\vec{EF} = -\vec{BC} = -\mathbf{B}$ [$\because$ BC = and $\parallel$ FE]

and $\vec{FA} = -\vec{CD} = -(\mathbf{B} - \mathbf{A}) = \mathbf{A} - \mathbf{B}$ [$\because$ CD = and $\parallel$ AF]

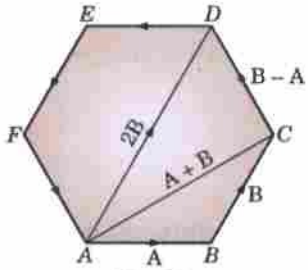


Fig. 3.3

3.2. (1) Space coordinates. Let $X'OX$ and $Y'OY$, $Z'OZ$ be three mutually perpendicular lines which intersect at O. Then O is called the origin.

$X'OX$ is called the **x-axis,** *$Y'OY$ the* **y-axis,** *$Z'OZ$ the* **z-axis** *and taken together these are called the* **coordinate axes.**

The plane YOZ is called the **yz-plane,** *the plane ZOX the* **zx-plane,** *the plane XOY the* **xy-plane** *and taken together these are called the* **coordinate planes.**

Let P be any point in space, Draw PL, PM, PN $\perp$s to the yz, zx and xy-planes. Then LP, MP, NP are respectively called the coordinates of P (Fig. 3.4). In practice, if $OA = x$, $AN\ y$, $NP = z$, then (x, y, z) are the coordinates of P which are positive along OX, OY, OZ respectively and negative along OX', OY', OZ'.

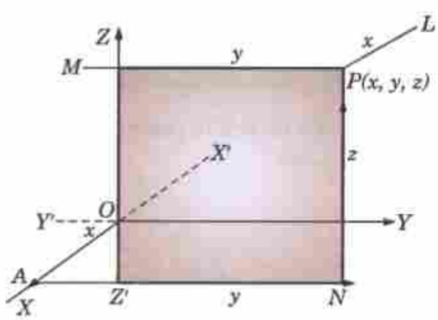


Fig. 3.4

The three coordinate planes divide the space into eight compartments called **octants.** The octant $OXYZ$ in which all the coordinates are positive is called the **positive** or **first octant.**

Note. *Three non-coplanar vectors* **A, B, C** *are said to form a* **right-handed** *(or a* **left-handed***) system according as a right threaded screw rotated through an angle less than 180° from* **A** *to* **B** *will advance along (or opposite to)* **C** as shown in Fig. 3.5.

An area of a closed curve described in a given manner is represented by a vector whose magnitude is the given area and direction normal to the plane of the area. Thus the vector **A** representing the area is taken to be positive or negative according as the direction of description of the boundary of the curve and the sense of **A** correspond to a right-handed or a left-handed system.

We have explained the most commonly used system of coordinates namely the *Rectangular Cartesian Coordinates.* The other two systems of coordinates often used to locate a point in space are the *Polar spherical coordinates and Cylindrical coordinates,* which are explained in § 8.21 and 8.20.

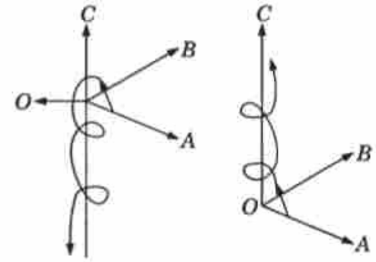


Fig. 3.5

(2) Resolution of vectors. Let **I, J, K** denote unit vectors along OX, OY, OZ respectively. Let $P(x, y, z)$ be a point in space. On OP as diagonal, construct a rectangular parallelopiped with edges OA, OB, OC along the axes so that

$$\vec{OA} = x\mathbf{I},\ \vec{OB} = y\mathbf{J},\ \vec{OC} = z\mathbf{K}$$

Then $$\mathbf{R} = \vec{OP} = \vec{OC'} + \vec{C'P}$$

$$= \vec{OA} = \vec{AC'} + \vec{OC} = \vec{OA} + \vec{OB} + \vec{OC}$$

Hence $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ is called the *position vector* of P relative to origin O and.

$$r = |\mathbf{R}| = \sqrt{(x^2 + y^2 + z^2)}$$

$$[\because\ r^2 = OP^2 = OC'^2 + C'P^2 = OA^2 + AC'^2 + C'P^2]$$

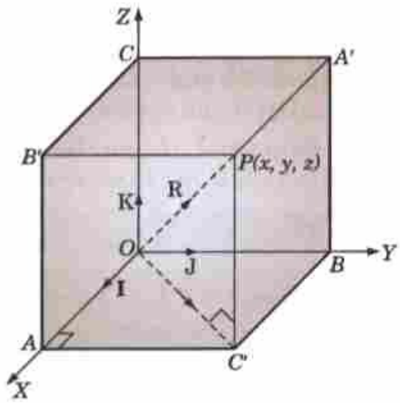


Fig. 3.6

(3) Direction cosines. Let any line L or its parallel OP, make angles α, β, γ with OX, OY, OZ respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ *are called the direction cosines of this line* which are usually denoted by l, m, n.

If l, m, n are direction cosines of a vector **R**, then

$(i)\ \hat{\mathbf{R}} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$, $(ii)\ \mathbf{l^2 + m^2 + n^2 = 1}$

Proof. Let D be the foot of the perpendicular from $P(x, y, z)$ on OY. Then

$$y = OD = r\cos\beta = mr. \text{ Similarly, } z = nr \text{ and } x = lr.$$

$$\therefore \quad \mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K} = r(l\mathbf{I} + m\mathbf{J} + n\mathbf{K})$$

or $$\hat{\mathbf{R}} = \frac{\mathbf{R}}{r} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$$

which expresses a unit vector in terms of its direction cosines.

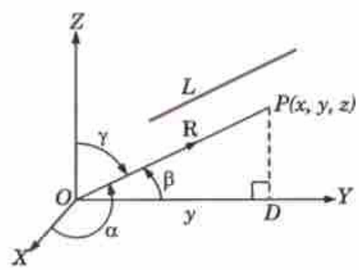


Fig. 3.7

Also $$1 = |\hat{\mathbf{R}}| = \sqrt{(l^2 + m^2 + n^2)} \text{ thus } \mathbf{l^2 + m^2 + n^2 = 1}$$

i.e., $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$ *(V.T.U., 2010)*

Obs. Directions ratios. If the direction cosines of a line be proportional to a, b, c, then these are called proportional direction cosines or direction ratios of the line.

If the direction cosines be l, m, n, then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(a^2 + b^2 + c^2)}} = \frac{1}{\sqrt{(\Sigma a^2)}}$$

$$\therefore \quad l = \frac{a}{\sqrt{(\Sigma a^2)}},\ m = \frac{b}{\sqrt{(\Sigma a^2)}},\ n = \frac{c}{\sqrt{(\Sigma a^2)}}$$

(4) Distance between two points $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ is

$$\sqrt{[(\mathbf{x_2 - x_1})^2 + (\mathbf{y_2 - y_1})^2 + (\mathbf{z_2 - z_1})^2]}$$

and **direction ratios** *of* $\vec{PQ}$ *are* $\mathbf{x_2 - x_1, y_2 - y_1, z_2 - z_1}$

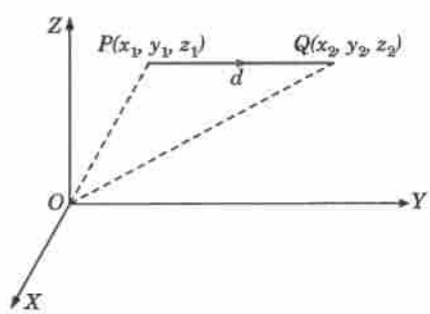


Fig. 3.8

We have

$$\vec{OP} = x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K}$$

and $$\vec{OQ} = x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}$$

$$\therefore \quad \vec{PQ} = \vec{OQ} - \vec{OP}$$

$$= (x_2 - x_1)\mathbf{I} + (y_2 - y_1)\mathbf{J} + (z_2 - z_1)\mathbf{K}$$

Thus, $$d = |\, \vec{PQ} \,| = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

and direction cosines of $\vec{PQ}$ are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

Example 3.2. *Show that the points A(–4, 9, 6), B(–1, 6, 6) and C(0, 7, 10) form a right angled isosceles triangle. Also find the direction cosines of AB.*

Solution. We have $$AB = \sqrt{[(-1+4)^2 + (6-9)^2 + (6-6)^2]} = 3\sqrt{2}$$

$$BC = \sqrt{[(0+1)^2 + (7-6)^2 + (10-6)^2]} = 3\sqrt{2}$$

and $$CA = \sqrt{[(-4-0)^2 + (9-7)^2 + (6-10)^2]} = 6$$

Since $AB^2 + BC^2 = CA^2$ and $AB = BC$, it follows that ΔABC is a right-angled isosceles triangle. The direction ratios of $\vec{AB}$ are $-1+4, 6-9, 6-6$.

$\therefore$ Its direction cosines are $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0$.

3.3 SECTION FORMULAE

The point **R** (x, y, z) dividing the join of the points $\mathbf{A}(x_1, y_1, z_1)$ *and* $\mathbf{B}(x_2, y_2, z_2)$ *in the ratio* $m_1 : m_2$ *is*

$$\mathbf{R} = \frac{m_1\mathbf{B} + m_2\mathbf{A}}{m_1 + m_2}, \quad i.e., \quad \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right) \qquad ...(i)$$

Let $P(\mathbf{A})$ and $Q(\mathbf{B})$ be the given points referred to origin O. Let $R(\mathbf{R})$ be the point dividing the line joining P and Q in the ratio $m_1 : m_2$ so that

$$\frac{PR}{RQ} = \frac{m_1}{m_2}, \quad i.e., \quad m_2 \,.\, PR = m_1 \,.\, RQ$$

$\therefore$ We have $$m_2\, \vec{PR} = m_1\, \vec{RQ}$$

or $$m_2(\vec{OR} - \vec{OP}) = m_1(\vec{OQ} - \vec{OR})$$

or $$m_2(\mathbf{R} - \mathbf{A}) = m_1(\mathbf{B} - \mathbf{R})$$

whence $$\mathbf{R} = \frac{m_1\mathbf{B} + m_2\mathbf{A}}{m_1 + m_2}$$

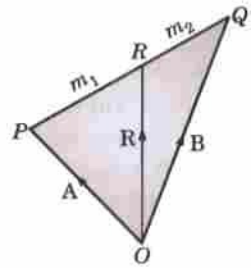


Fig. 3.9

Since $$\mathbf{A} = x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K}, \ \mathbf{B} = x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}$$

and $$\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$$

$$\therefore \quad x\mathbf{I} + y\mathbf{J} + z\mathbf{K} = \frac{m_1(x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}) + m_1(x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K})}{m_1 + m_2}$$

Equating coefficient of **I, J, K,** we get the desired results (i).

Cor. 1. *Mid-point of* $P(\mathbf{A})$ *and* $Q(\mathbf{B})$ *is* $\frac{1}{2}(\mathbf{A} + \mathbf{B})$.

2. Point R dividing the join of $P(\mathbf{A})$ and $Q(\mathbf{B})$ in the ratio $m_1 : m_2$ externally is $\mathbf{R} = \frac{m_1\mathbf{B} - m_2\mathbf{A}}{m_1 - m_2}$.

Obs. Rewriting (i) as $m_2\mathbf{A} + m_1\mathbf{B} - (m_1 + m_2)\mathbf{R} = 0$, we note that the sum of the coefficients of **A, B** and **R** is zero. Hence it follows that any *three points with position vectors* **A, B** *and* **C** *are collinear if*

$$\lambda\mathbf{A} + \mu\mathbf{B} + \gamma\mathbf{C} = 0, \text{ where } \lambda + \mu + \gamma = 0.$$

Example 3.3. *In a trapezium, prove that the straight line joining the mid-points of the diagonals is parallel to the parallel sides and half their difference.*

Solution. Consider a trapezium $OABC$ with parallel sides OA and BC. Take O as the origin and let the other vertices be $A(\mathbf{A}), B(\mathbf{B}), C(\mathbf{C})$.

Since CB is parallel to OA, therefore,

$$\mathbf{B} - \mathbf{C} = \vec{CB} = \lambda \vec{OA} = \lambda \mathbf{A}.$$

The mid-points of the diagonals OB and AC are $D(\mathbf{B}/2)$ and $E(\mathbf{A} + \mathbf{C})/2$.

$$\therefore \quad \vec{DE} = \vec{OE} - \vec{OD} = \frac{1}{2}(\mathbf{A} + \mathbf{C}) - \frac{1}{2}\mathbf{B} = \frac{1}{2}[\mathbf{A} - (\mathbf{B} - \mathbf{C})] \qquad ...(i)$$

$$= \frac{1}{2}(1 - \lambda)\mathbf{A} \qquad ...(ii)$$

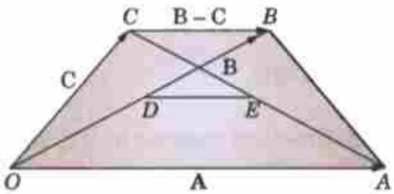


Fig. 3.10

From (*ii*), it is clear that $\vec{DE}$ is parallel to $\vec{OA}$; from (*i*), it follows that $DE = \frac{1}{2}(OA - CB)$.
Hence the result.

Example 3.4. *Show that the line joining one vertex of a parallelogram to the mid-point of an opposite side trisects the diagonal and is itself trisected there at.*

Solution. Consider a parallelogram $OABC$. Take O as the origin and let the other vertices be $A(\mathbf{A})$, $B(\mathbf{B})$ and $C(\mathbf{C})$.

The mid-point D of OA is $\mathbf{A}/2$.

Now since OA is equal to and parallel to CB.

$$\therefore \quad \vec{OA} = \vec{CB}, \text{ i.e., } \mathbf{A} = \mathbf{B} - \mathbf{C}$$

which may be written as $\dfrac{2(\mathbf{A}/2) + 1 . \mathbf{C}}{2+1} = \dfrac{\mathbf{B}}{3} = \mathbf{P}$ so that P trisects DC and OB.

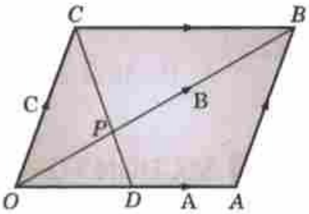


Fig. 3.11

PROBLEMS 3.1

1. Given $\mathbf{R}_1 = 5\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$ and $\mathbf{R}_2 = \mathbf{I} + 3\mathbf{J} + 7\mathbf{K}$, find the magnitude and direction cosines of the vectors $\mathbf{R}_1 + \mathbf{R}_2$ and $2\mathbf{R}_1 - \mathbf{R}_2$.
2. Show that the points (0, 4, 1) ; (2, 3, – . . (4, 5, 0) and (2, 6, 2) are the vertices of a square. (*Osmania, 1999 S*)
3. A straight line is inclined to the axes of x and y at angles of 30° and 60°. Find the inclination of the line to the z-axis. (*Madras, 2003*)
4. If a line makes angles α, β, γ with the axes, prove that
   (*i*) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$. (*V.T.U., 2000 ; Osmania, 1999*)
   (*ii*) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$.
5. If $\mathbf{A}$ and $\mathbf{B}$ are non-collinear vectors and $\mathbf{P} = (2x + 3y - 2)\mathbf{A} + (3x + 2y + 5)\mathbf{B}$ and $\mathbf{Q} = (-x + 4y - 2)\mathbf{A} + (3x - 4y + 7)\mathbf{B}$, find x, y such that $7\mathbf{P} = 3\mathbf{Q}$.
6. Prove that the line joining the mid-points of the two sides of a triangle is parallel to the third side and half of it.
7. Prove that (*i*) the diagonals of a parallelogram bisect each other ;
   (*ii*) a quadrilateral whose diagonals bisect each other is a parallelogram.
8. In a skew quadrilateral, prove that :
   (*i*) the figure formed by joining the mid-points of the adjacent sides is a parallelogram.
   (*ii*) the joins of the mid-points of opposite sides bisect each other.
9. In a trapezium, prove that the straight line joining the mid-points of the non-parallel sides is parallel to the parallel sides and half their sum.
10. Prove that the vectors $\mathbf{A} = 3\mathbf{I} + \mathbf{J} - 2\mathbf{K}$, $\mathbf{B} = -\mathbf{I} + 3\mathbf{J} + 4\mathbf{K}$, $\mathbf{C} = 4\mathbf{I} - 2\mathbf{J} - 6\mathbf{K}$ can form the sides of a triangle. Also find the length of the median bisecting the vector $\mathbf{C}$. (*J.N.T.U., 1995 S*)
11. Find the ratio in which the line joining (2, 4, 16) and (3, 5, – 4) is divided by the plane $2x - 3y + z + 6 = 0$. (*Mysore, 1995*)
12. Show that the three points $\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$, $2\mathbf{I} + 3\mathbf{J} - 4\mathbf{K}$, $-7\mathbf{J} + 10\mathbf{K}$ are collinear.
13. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be the position vectors of the vertices A, B, C of the triangle ABC, show that the three
   (*i*) medians concur at the point $\frac{1}{3}(\mathbf{A} + \mathbf{B} + \mathbf{C})$, called the *centroid*.
   (*ii*) internal bisectors of the angles concur at the point $\dfrac{a\mathbf{A} + b\mathbf{B} + c\mathbf{C}}{a + b + c}$, called the *incentre*.

14. Show that the coordinates of the *centroid of the triangle* whose vertices are $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ are

$$\left[\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right].$$

15. Show that the coordinates of the *centroid of the tetrahedron* whose vertices are $(x_r, y_r, z_r) : r = 1, 2, 3, 4$ are

$$\left[\frac{1}{4}(x_1+x_2+x_3+x_4), \frac{1}{4}(y_1+y_2+y_3+y_4), \frac{1}{4}(z_1+z_2+z_3+z_4)\right].$$

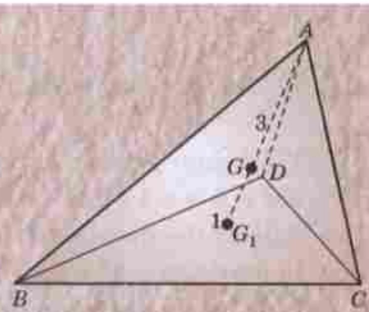


Fig. 3.12

[**Def.** *A* **tetrahedron** *is a solid bounded by four triangular faces.* Thus the tetrahedron $ABCD$ has four faces—the Δs ABC, ACD, ADB, BCD. (Fig. 3.12.)

It has four **vertices** A, B, C, D and three pairs of **opposite edges** AB, CD ; BC, AD ; CA, BD.

The centroid of the tetrahedron divides the join of each vertex to the centroid of the opposite triangular face in the ratio 3 : 1].

16. M and N are the mid-points of the diagonals AC and BD respectively of a quadrilateral $ABCD$. Show that the resultant of the vectors $\vec{AB}, \vec{AD}, \vec{CB}, \vec{CD}$ is $4\vec{MN}$. *(Cochin, 1999)*

3.4 PRODUCTS OF TWO VECTORS

Unlike the product of two scalars or that of a vector by a scalar, the product of two vectors is sometimes seen to result in a scalar quantity and sometimes in a vector. As such, we are led to define two types of such products, called the *scalar product* and the *vector product* respectively.

The scalar and vector products of two vectors **A** and **B** are usually written as **A . B** and **A × B** respectively and are read as **A** dot **B** and **A** cross **B**. In view of this notation, the former is sometimes called the *dot product* and the latter the *cross product.*

In vector algebra, the division of a vector by another vector is not defined.

3.5 SCALAR OR DOT PRODUCT

(1) Definition. *The scalar or dot product of two vectors* **A** *and* **B** *is defined as the scalar ab* cos θ, *where* θ *is the angle between* **A** *and* **B**.

Thus $\quad \mathbf{A.B} = ab \cos\theta.$

(2) Geometrical interpretation. A.B *is the product of the length of one vector and the length of the projection of the other in the direction of the former.*

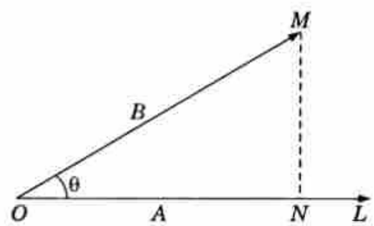


Fig. 3.13

Let $\quad \vec{OL} = \mathbf{A}, \vec{OM} = \mathbf{B} \quad$ then

$\mathbf{A} \cdot \mathbf{B} = ab \cos\theta = a(OM \cos\theta) = a(ON) = |A|$ Proj. of $|\mathbf{B}|$ in the direction of **A**.

Similarly, $\mathbf{A} \cdot \mathbf{B} = |\mathbf{B}|$ Proj. of $|\mathbf{A}|$ in the direction of **B**.

(3) Properties and other results.

I. Scalar product of two vectors is commutative.

i.e., $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ for $\mathbf{A} \cdot \mathbf{B} = ab \cos\theta = ba \cos(-\theta) = \mathbf{B} \cdot \mathbf{A}$

II. The necessary and sufficient condition for two vectors to be perpendicular is that their scalar product should be zero.

When the vectors **A** and **B** are perpendicular, $\mathbf{A} \cdot \mathbf{B} = ab \cos 90° = 0$.

Conversely, when $\mathbf{A} \cdot \mathbf{B} = 0$, $ab \cos\theta = 0$, *i.e.*, $\cos\theta = 0$. $\quad (\because \; a \neq 0, b \neq 0)$, or $\theta = 90°$.)

III. $\mathbf{A} \cdot \mathbf{A} = a^2$ which is written as $\mathbf{A}^2$. Thus *the square of a vector is a scalar which stands for the square of its magnitude.*

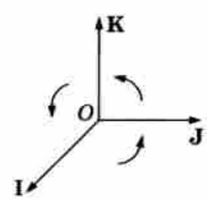


Fig. 3.14

IV. For the mutually perpendicular unit vectors, **I, J, K,** we have the relations.

$$\mathbf{I} \cdot \mathbf{J} = \mathbf{J} \cdot \mathbf{K} = \mathbf{K} \cdot \mathbf{I} = 0$$

and
$$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = 1$$

which are of great utility.

V. Scalar product of two vectors is distributive i.e.,

$$(\mathbf{A}+\mathbf{B}) \,.\, \mathbf{C} = \mathbf{A}\,.\,\mathbf{C} + \mathbf{B}\,.\,\mathbf{C}$$

VI. Schwarz inequality :* $|\mathbf{A}\,.\,\mathbf{B}| \le |\mathbf{A}|\,|\mathbf{B}|$

$$|\mathbf{A}\,.\,\mathbf{B}| = |\mathbf{A}|\,|\mathbf{B}|\,|\cos\theta| \le |\mathbf{A}|\,|\mathbf{B}| \qquad [\because \ |\cos\theta| \le 1]$$

VII. Scalar product of two vectors is equal to the sum of the products of their corresponding components.

For if $\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}$, $\mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$

then by the distributive law, $\mathbf{A}\,.\,\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3$

In particular, $\mathbf{A}^2 = a_1^2 + a_2^2 + a_3^2$.

VIII. **Angle between two lines** *whose direction cosines are l, m, n and l′, m′, n′ is* $\mathbf{cos^{-1}\,(ll' + mm' + nn')}$.

The unit vectors in the direction of the given lines are $\mathbf{U} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$ and $\mathbf{U}' = l'\mathbf{I} + m'\mathbf{J} + n'\mathbf{K}$.

If θ be the angle between the lines, then

$$\mathbf{U}\,.\,\mathbf{U}' = (l\mathbf{I} + m\mathbf{J} + n\mathbf{K}) \cdot (l'\mathbf{I} + m'\mathbf{J} + n'\mathbf{K})$$

or $$1\,.\,1\,.\cos\theta = ll' + mm' + nn' \qquad (V.T.U., 2008)$$

Hence $$\cos\theta = ll' + mm' + nn' \qquad ...(i)$$

Cor. 1.
$$\begin{aligned}\sin^2\theta &= 1 - \cos^2\theta = 1 - (ll' + mm' + nn')^2\\ &= (l^2 + m^2 + n^2)(l'^2 + m'^2 + n'^2) - (ll' + mm' + nn')^2\\ &= (mn' - nm')^2 + (nl' - ln')^2 + (lm' - ml')^2\end{aligned}$$

$\therefore$ $$\sin\theta = \pm\sqrt{\Sigma(mn' - nm')^2}. \qquad ...(ii)$$

Cor. 2. *The condition that the lines whose direction cosines are l, m, n and l′, m′, n′ should be* **perpendicular** *is*

$$\mathbf{ll' + mm' + nn' = 0} \qquad ...(iii)$$

and **parallel** *is* $$\mathbf{l = l',\ m = m',\ n = n'} \qquad ...(iv)$$

These conditions easily follow from (*i*) and (*ii*).

Cor. 3. *The angle θ between two lines whose direction ratios are a, b, c, and a′, b′, c′ is given by*

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{(\Sigma a^2)}\sqrt{(\Sigma a'^2)}}$$

or $$\sin\theta = \frac{\sqrt{[(bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2}}{\sqrt{(\Sigma a^2)}\sqrt{(\Sigma a'^2)}}$$

These lines are (i) perpendicular if $aa' + bb' + cc' = 0$, *(ii) parallel if* $a/a' = b/b' = c/c'$.

IX. **Projection of the line joining two points** (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *on a line whose direction cosines are l, m, n is* $$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1).$$

Let $$\vec{OP} = x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K},\ \vec{OQ} = x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}$$

$\therefore$ $$\vec{PQ} = (x_2 - x_1)\mathbf{I} + (y_2 - y_1)\mathbf{J} + (z_2 - z_1)\mathbf{K}$$

Also unit vector $\mathbf{U}$ along the given lines is $l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$.

$\therefore$ Projection of PQ on the given line $= \vec{PQ}\,.\,\mathbf{U}$.

$$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1).$$

Example 3.5. *Find the sides and angles of the triangle whose vertices are* $\mathbf{I} - 2\mathbf{J} + 2\mathbf{K}$, $2\mathbf{I} + \mathbf{J} - \mathbf{K}$, *and* $3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$.

Solution. Let $\vec{OA} = \mathbf{I} - 2\mathbf{J} + 2\mathbf{K}$, $\vec{OB} = 2\mathbf{I} + \mathbf{J} - \mathbf{K}$, $\vec{OC} = 3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$

Then $$\vec{BC} = \mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$$

$$\vec{CA} = -2\mathbf{I} - \mathbf{J}$$

* Named after the German mathematician *Hermann Amandus Schwarz* (1843—1921) who is known for his work in conformal mapping, calculus of variations and differential geometry. He succeeded *Weierstrass* in Berlin University.

and $\qquad \vec{AB} = \mathbf{I} + 3\mathbf{J} - 3\mathbf{K}$

$\therefore \qquad BC = \sqrt{14}, CA = \sqrt{5}, AB = \sqrt{19}.$

Now d.c.'s of AB and AC being

$1/\sqrt{19}, 3/\sqrt{19}, -3/\sqrt{19}$ and $2/\sqrt{5}, 1/\sqrt{5}, 0,$

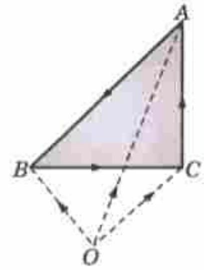


Fig. 3.15

We have $\qquad \cos A = \frac{1}{\sqrt{19}} \cdot \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{19}} \cdot \frac{1}{\sqrt{5}} + \frac{-3}{\sqrt{19}} \cdot 0 = \sqrt{(5/19)}$

i.e., $\angle A = \cos^{-1}\sqrt{(5/19)}$. Again d.c.'s of BC and BA being

$1/\sqrt{14}, -2/\sqrt{14}, 3/\sqrt{14}$ and $-1/\sqrt{19}, -3/\sqrt{19}, 3/\sqrt{19}$;

we have $\qquad \cos B = \frac{1}{\sqrt{14}} \cdot \frac{-1}{\sqrt{19}} + \frac{-2}{\sqrt{14}} \cdot \frac{-3}{\sqrt{19}} + \frac{3}{\sqrt{14}} \cdot \frac{3}{\sqrt{19}} = \sqrt{(14/19)}$, *i.e.,* $\angle B = \cos^{-1}\sqrt{(14/19)}$

Finally, d.c.'s of CA and CB being $-2/\sqrt{5}, -1/\sqrt{5}, 0$ and $-1/\sqrt{14}, 2/\sqrt{14}, -3/\sqrt{14}$;

we have $\qquad \cos C = \frac{-2}{\sqrt{5}} \cdot \frac{-1}{\sqrt{14}} + \frac{-1}{\sqrt{5}} \cdot \frac{2}{\sqrt{14}} + 0 \cdot \frac{-3}{\sqrt{14}} = 0$, *i.e.,* $\angle C = 90°$

Example 3.6. *Prove that the right -bisectors of the sides of a triangle concur at its circumcentre.*

Solution. Let $A(\mathbf{A}), B(\mathbf{B}), C(\mathbf{C})$ be the vertices of any triangle ABC. The mid-points of the sides BC, CA and AB are

$$D\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right), E\left(\frac{\mathbf{C}+\mathbf{A}}{2}\right), F\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right)$$

Let the perpendicular at D and E to BC and CA respectively intersect at the point $P(\mathbf{R})$. Then $\vec{DP} \cdot \vec{BC} = 0$

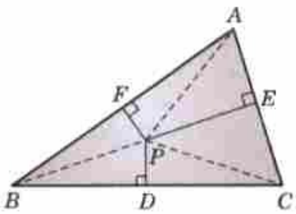


Fig. 3.16

i.e., $\qquad \left(\mathbf{R} - \frac{\mathbf{B}+\mathbf{C}}{2}\right) \cdot (\mathbf{C} - \mathbf{B}) = 0 \qquad$ ...(*i*)

and $\qquad \vec{EP} \cdot \vec{CA} = 0$, *i.e.,* $\left(\mathbf{R} - \frac{\mathbf{C}+\mathbf{A}}{2}\right) \cdot (\mathbf{A} - \mathbf{C}) = 0 \qquad$ ...(*ii*)

Adding (*i*) and (*ii*), we get $\left(\mathbf{R} - \frac{\mathbf{A}+\mathbf{B}}{2}\right) \cdot (\mathbf{A} - \mathbf{B}) = 0$

which shows that FP is perpendicular to AB. Hence the result.

Further $\qquad PA = PB$ if $|\mathbf{A} - \mathbf{R}| = |\mathbf{B} - \mathbf{R}|$

or if, $\qquad (\mathbf{A} - \mathbf{R})^2 = (\mathbf{B} - \mathbf{R})^2$ or if, $\mathbf{A}^2 - 2\mathbf{A} \cdot \mathbf{R} = \mathbf{B}^2 - 2\mathbf{B} \cdot \mathbf{R}$

of if, $\qquad \left(\mathbf{R} - \frac{\mathbf{A}+\mathbf{B}}{2}\right) \cdot (\mathbf{A} - \mathbf{B}) = 0$, which is true.

Example 3.7. *If the distance between two points P and Q is d and the lengths of the projections of PQ on the coordinate planes* d_1, d_2, d_3, *show that* $2d^2 = d_1^2 + d_2^2 + d_3^2$.

Solution. Let P be (x_1, y_1, z_1) and Q be (x_2, y_2, z_2), then

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.$$

The feet of the perpendiculars drawn from P and Q on the XY-plane are the projections of P and Q on this plane. If these are L and M, then L is $(x_1, y_1, 0)$ and M is $(x_2, y_2, 0)$.

$\therefore \qquad d_1 =$ projection of PQ on XY-plane, *i.e.,* LM

or $\qquad d_1^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

Similarly, $\qquad d_2^2 = (y_1 - y_2)^2 + (z_1 - z_2)^2$ and $d_3^2 = (z_1 - z_2)^2 + (x_1 - x_2)^2$

$\therefore \qquad d_1^2 + d_2^2 + d_3^2 = 2[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2] = 2d^2.$

Example 3.8. *A line makes angles α, β, γ, δ with diagonals of a cube, prove that*

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = 4/3.$$ (V.T.U., 2006 ; Osmania, 2000 S)

Solution. Take O, a corner of the cube as origin and OA, OB, OC the three edges through it, as the axes. Let $OA = OB = OC = a$. Then the coordinates of the corners are as shown in Fig. 3.17.

The four diagonals are OP, AA', BB' and CC'.

Clearly, direction cosines of OP are

$$\frac{a-0}{\sqrt{(\Sigma a^2)}}, \frac{a-0}{\sqrt{(\Sigma a^2)}}, \frac{a-0}{\sqrt{(\Sigma a^2)}} \text{ i.e., } \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.$$

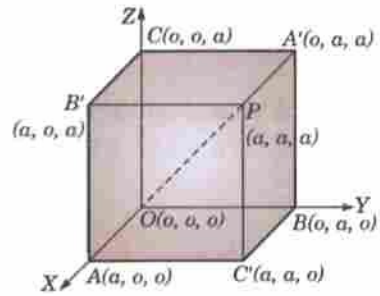


Fig. 3.17

Similarly, direction cosines of AA' are $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$;

Similarly, direction cosines of BB' are $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$;

and Similarly direction cosines of CC' are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$.

Let l, m, n be the direction cosines of the given line which makes angles α, β, γ, δ with OP, AA', BB', CC' respectively. Then

$$\cos \alpha = \frac{1}{\sqrt{3}}(l+m+n)\,;\ \cos \beta = \frac{1}{\sqrt{3}}(-l+m+n)$$

$$\cos \gamma = \frac{1}{\sqrt{3}}(l-m+n);\ \cos \delta = \frac{1}{\sqrt{3}}\ (l+m-n)$$

Squaring and adding, we get

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{1}{3}\,[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2]$$

$$= \frac{1}{3}\ [4(l^2+m^2+n^2)] = \frac{4}{3}. \qquad [\because\ l^2+m^2+n^2 = 1]$$

Example 3.9. *If the edges of a rectangular parallelopiped are a, b, c, show that the angle between the four diagonals are* $\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$.

Solution. Let $OA = a$, $OB = b$, $OC = c$ be the edges of the rectangular parallelopiped. Then the coordinates of the corners are as shown in Fig. 3.18. The four diagonals taken in pairs are (*i*) ($O P$, AA'), (*ii*) (OP, BB'), (*iii*) (OP, CC'), (*iv*) (AA', BB'), (*v*) (AA', CC') and (*vi*) (BB', CC').

Let the angles between these pairs of diagonals be $\theta_1, \theta_2, \ldots \theta_6$ respectively. Clearly d.r.'s OP are a, b, c ; d.r.'s, of AA' are $-a, b, c$, d.r.'s of BB' are $a, -b, c$ and d.r.'s of CC' are $a, b, -c$.

∴ For the pair (*i*) *i.e.*, (OP, AA') ;

$$\cos \theta_1 = \frac{-a^2+b^2+c^2}{\sqrt{(a^2+b^2+c^2}\ \sqrt{(a^2+b^2+c^2)}} = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$$

Similarly, $\cos \theta_2 = \frac{a^2-b^2+c^2}{a^2+b^2+c^2}$, $\cos \theta_3 = \frac{a^2+b^2-c^2}{a^2+b^2+c^2}$;

$\cos \theta_4 = \frac{-a^2-b^2+c^2}{a^2+b^2+c^2}$; $\cos \theta_5 = \frac{-a^2+b^2-c^2}{a^2+b^2+c^2}$;

$\cos \theta_6 = \frac{a^2-b^2-c^2}{a^2+b^2+c^2}$

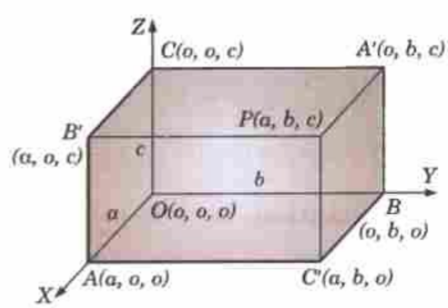


Fig. 3.18

Thus, noting that at least one term in the numerator is negative, we have in general

$$\cos \theta = \frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}.$$

Example 3.10. *Prove that the lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $mn + nl + lm = 0$ are*

(i) Perpendicular if $a^{-1} + b^{-1} + c^{-1} = 0$ (Burdwan, 2003)

(ii) parallel if $\sqrt{a} + \sqrt{b} + \sqrt{c} = 0$.

Solution. Eliminating n from the given relations, we have

$$(m + l)\left(-\frac{al + bm}{c}\right) + lm = 0 \quad \text{or} \quad al^2 + (c - a - b)lm + bm^2 = 0$$

or $$a(l/m)^2 + (c - a - b)(l/m) + b = 0 \qquad \text{...(1)}$$

If l_1, m_1, n_1 ; l_2, m_2, n_2, are the direction cosines of these lines then l_1/m_1, l_2/m_2 are the roots of the quadratic (1).

$$\therefore \quad \frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{b}{a} \quad \text{or} \quad \frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c} \quad \text{(by symmetry)} = k \text{ (say).}$$

The lines will be perpendicular if $l_1 l_2 + m_1 m_2 + n_1 n_2 = k\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 0$

or if, $$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$$

The lines will be parallel if $l_1 = l_2$, $m_1 = m_2$, $n_1 = n_2$.

i.e., if, $l_1/m_1 = l_2/m_2$; *i.e.* if, $(c - a - b)^2 = 4ab$

or if, $c - a - b = \pm 2\sqrt{(ab)}$ or if, $c = a + b \pm 2\sqrt{(ab)} = (\sqrt{a} \pm \sqrt{b})^2$

or if, $\pm\sqrt{c} = \sqrt{a} \pm \sqrt{b}$ or if, $\sqrt{a} + \sqrt{b} + \sqrt{c} = 0$ [Taking necessary signs]

Example 3.11. *Find the angle between the lines whose direction cosines are given by the equation $l + 3m + 5n = 0$ and $5lm - 2mn + 6nl = 0$.*

Solution. Let us eliminate l from the given relations, by substituting $l = -3m - 5n$ in the second relation

$$5m(-3m - 5n) - 2mn + 6n(-3m - 5n) = 0$$

i.e., $$15m^2 + 45mn + 30n^2 = 0 \quad \text{or} \quad m^2 + 3mn + 2n^2 = 0$$

or $$(m + n)(m + 2n) = 0, \quad i.e., \quad m + n = 0 \text{ or } m + 2n = 0$$

Now let us first solve the equations $l + 3m + 5n = 0$ and $m + n = 0$

These give $m = -n$ and $l = -2n$, *i.e.*, $= \dfrac{l}{-2} = \dfrac{m}{-1} = \dfrac{n}{1}$ ...(*i*)

Similarly, solving the equations $l + 3m + 5n = 0$ and $m + 2n = 0$,

We get $$\frac{l}{1} = \frac{m}{-2} = \frac{n}{1} \qquad \text{...(ii)}$$

(*i*) and (*ii*) give the direction ratios of the two lines.

If θ be the angle between these two lines, then

$$\cos\theta = \frac{(-2)\times 1 + (-1)\times(-2) + 1\times 1}{\sqrt{(2^2 + 1^2 + 1^2)}\sqrt{(1^2 + 2^2 + 1^2)}} = \frac{1}{6}, \quad i.e., \quad \theta = \cos^{-1}\left(\frac{1}{6}\right).$$

PROBLEMS 3.2

1. If $\mathbf{A} = \mathbf{I} + 2\mathbf{J} + 3\mathbf{K}$, $\mathbf{B} = -\mathbf{I} + 2\mathbf{J} + \mathbf{K}$ and $\mathbf{C} = 3\mathbf{I} + \mathbf{J}$, find t such that $\mathbf{A} + t\mathbf{B}$ is perpendicular to $\mathbf{C}$.
2. (*i*) Show that $\left(\dfrac{\mathbf{A}}{a^2} - \dfrac{\mathbf{B}}{b^2}\right)^2 = \left(\dfrac{\mathbf{A} - \mathbf{B}}{ab}\right)^2$.

   (*ii*) Interpret geometrically $(\mathbf{C} - \mathbf{A}) \cdot (\mathbf{B} - \mathbf{C}) = 0$.
3. If $|\mathbf{A} + \mathbf{B}| = |\mathbf{A} - \mathbf{B}|$, show that $\mathbf{A}$ and $\mathbf{B}$ are mutually perpendicular.

4. If $\mathbf{A} = \mathbf{I} + 2\mathbf{J} - 3\mathbf{K}$ and $\mathbf{B} = 3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$, show that $\mathbf{A} + \mathbf{B}$ is perpendicular to $\mathbf{A} - \mathbf{B}$. Also calculate the angle between $2\mathbf{A} + \mathbf{B}$ and $\mathbf{A} + 2\mathbf{B}$.
5. Show that the three concurrent lines with direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are coplanar if
$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$
6. Find the projection of the vector $\mathbf{I} - 2\mathbf{J} + \mathbf{K}$ on $4\mathbf{I} - 4\mathbf{J} + 7\mathbf{K}$.
7. The projection of a line on the coordinate axes are 12, 4, 3. Find the length and direction cosines of the line. *(Rajasthan, 2006)*
8. Show (by vector methods) that the mid-point of the hypotenuse of a right-angled triangle is equidistant from its vertices.
9. Prove (by vector methods) that the angle in a semi-circle is a right angle.
10. Show (by vector methods) that the diagonals of a rhombus intersect at right angles.
11. Show that the altitudes of a triangle meet in a point (called the *orthocentre*).
12. $ABCD$ is a tetrahedron having the edges BC and AC at right angles to opposite edges AD and BD respectively. Show that the third pair of opposite edges AB and CD are also at right angles.
13. Find the angle between the lines whose direction cosines are given by the equations $l + m + n = 0$, $l^2 + m^2 + n^2 = 0$. *(Rajasthan, 2005)*
14. Show that the lines whose direction cosines are given by the equations $4lm - 3mn - nl = 0$, and $3l + m + 2n = 0$ are perpendicular. *(Anna, 2005)*
15. Show that the lines whose direction cosines are given by the equations $l + m + n = 0$, $al^2 + bm^2 + cn^2 = 0$ are
(*i*) perpendicular, if $a + b + c = 0$, (*ii*) parallel, if $a^{-1} + b^{-1} + c^{-1} = 0$.
16. Show that the straight lines whose direction cosines are given by the equations
$al + bm + cn = 0$, $fmn + gnl + hlm = 0$ are (*i*) perpendicular if $\dfrac{f}{a} + \dfrac{g}{b} + \dfrac{h}{c} = 0$ *(Osmania, 2003)*
(*ii*) parallel if $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$.
17. Show that the angle between any two diagonals of a cube is $\cos^{-1} 1/3$. *(V.T.U., 2009 ; Assam, 1999)*
18. (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) are the direction cosines of three mutually perpendicular lines. Prove that the line whose d.c.'s are proportional to $l_1 + l_2 + l_3$, $m_1 + m_2 + m_3$, $n_1 + n_2 + n_3$ makes equal angles with the axes. *(V.T.U. 2003)*
19. AB, BC are the diagonals of adjacent faces of a rectangular box with its centre at the origin O, its edges are parallel to the axes. If the angles BOC, COA and AOB are equal to θ, ϕ, ψ respectively, prove that
$$\cos\theta + \cos\phi + \cos\psi = -1.$$

3.6 VECTOR, OR CROSS PRODUCT

(1) Definition. *The vector, or cross product of two vectors* **A** *and* **B** *is defined as a vector such that*

(*i*) *its magnitude is ab* sin θ, θ *being the angle between* **A** *and* **B**,

(*ii*) *its direction is perpendicular to the plane of* **A** *and* **B**,

and (*iii*) *it forms with* **A** *and* **B** *a right-handed system.*

If **N** be a unit vector normal to the plane of **A** and **B** (**A, B, N** forming a right-handed system), then
$$\mathbf{A} \times \mathbf{B} = ab \sin\theta\, \mathbf{N}.$$

(2) Geometrical interpretation. **A × B** *represents twice the vector area of the triangle having the vectors* **A** *and* **B** *as its adjacent sides.*

If **N** be a unit vector normal to the plane of the triangle OAB, then
$$\mathbf{A} \times \mathbf{B} = ab \sin\theta\, \mathbf{N}$$
$$= 2\left(\frac{1}{2} ab \sin\theta\right)\mathbf{N} = 2\Delta\, OAB\, \mathbf{N} = 2\Delta\, \overrightarrow{OA}\, B.$$

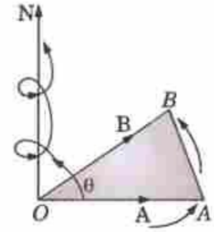


Fig. 3.19

(3) Properties and other results

I. Vector product of two vectors is not commutative,
$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}. \text{ In fact, } \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}.$$

for $\quad \mathbf{A} \times \mathbf{B} = ab \sin\theta\, \mathbf{N}$ or $2\Delta\, \overrightarrow{OAB}$.

and $\quad \mathbf{B} \times \mathbf{A} = ab \sin(-\theta)\, \mathbf{N} = -ab \sin\theta\, \mathbf{N}$ or $2\Delta\, \overrightarrow{OBA}$.

II. The necessary and sufficient condition for two non-zero vectors to be parallel is that their vector product should be zero.

When the vectors **A** and **B** are parallel, the angle θ between them is 0 and 180° so that $\sin\theta = 0$, and as such $\mathbf{A} \times \mathbf{B} = \mathbf{0}$.

Conversely, when $\quad \mathbf{A} \times \mathbf{B} = \mathbf{0}$; $ab \sin\theta = 0$

i.e., $\quad \sin\theta = 0 \qquad (\because \; a \neq 0, b \neq 0)$

or $\quad \theta = 0$ or 180°. In particular, $\mathbf{A} \times \mathbf{A} = 0$.

III. For the orthonormal vector trial **I, J, K,** we have the relations :

$$\mathbf{I} \times \mathbf{I} = \mathbf{J} \times \mathbf{J} = \mathbf{K} \times \mathbf{K} = \mathbf{0}$$
$$\mathbf{I} \times \mathbf{J} = \mathbf{K}, \quad \mathbf{J} \times \mathbf{I} = -\mathbf{K}$$
$$\mathbf{J} \times \mathbf{K} = \mathbf{I}, \quad \mathbf{K} \times \mathbf{J} = -\mathbf{I}$$
$$\mathbf{K} \times \mathbf{I} = \mathbf{J}, \quad \mathbf{I} \times \mathbf{K} = -\mathbf{J}.$$

IV. Relation between scalar and vector products.

We have $\quad (\mathbf{A} \,.\, \mathbf{B})^2 = a^2b^2 \cos\theta = a^2b^2 - a^2b^2 \sin^2\theta = a^2b^2 - (\mathbf{A} \times \mathbf{B}) \,.\, (\mathbf{A} \times \mathbf{B})$

$$(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2\mathbf{B}^2 - (\mathbf{A} \,.\, \mathbf{B})^2.$$

V. Vector product of two vectors is distributive

i.e., $\quad (\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}.$

VI. Analytical expression for the vector product.

If $\quad \mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}, \mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$ *then* $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

For we get

$$\mathbf{A} \times \mathbf{B} = (a_2b_3 - a_3b_2)\mathbf{I} + (a_3b_1 - a_1b_3)\mathbf{J} + (a_1b_2 - a_2b_1)\mathbf{K}$$

whence follows the required result.

Example 3.12. *If* $\mathbf{A} = 4\mathbf{I} + 3\mathbf{J} + \mathbf{K}$, $\mathbf{B} = 2\mathbf{I} - \mathbf{J} + 2\mathbf{K}$, *find a unit vector* **N** *perpendicular to vectors* **A** *and* **B** *such that* **A, B, N** *form a right handed system. Also find the angle between the vectors A and B.*

Solution. Since $\quad \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 7\mathbf{I} - 6\mathbf{J} - 10\mathbf{K}$

and $\quad |\, \mathbf{A} \times \mathbf{B} \,| = \sqrt{[(7)^2 + (-6)^2 + (-10)^2]} = \sqrt{185}$

$\therefore$ Unit vector $\mathbf{N} \perp$ to **A** and **B** $= \dfrac{\mathbf{A} \times \mathbf{B}}{|\, \mathbf{A} \times \mathbf{B} \,|} = (7\mathbf{I} - 6\mathbf{J} - 10\mathbf{K})/\sqrt{185}$

Also $\quad a = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{26}$ and $b = 3$.

If θ be the angle between **A** and **B**, then $|\, \mathbf{A} \times \mathbf{B} \,| = ab \sin\theta$, *i.e.*, $\sin\theta = |\, \mathbf{A} \times \mathbf{B} \,|/ab$

Thus $\quad \sin\theta = \sqrt{185}/3\sqrt{26}$ whence $\theta = 62°\, 40'$.

Example 3.13. (*i*) *Prove that the area of the triangle whose vertices are* **A, B, C** *is*

$$\frac{1}{2}\, |\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B}|$$

(*ii*) *Calculate the area of the triangle whose vertices are A(1, 0 – 1), B(2, 1, 5) and C(0, 1, 2).*

Solution. (*i*) Let $A(\mathbf{A})$, $B(\mathbf{B})$, $C(\mathbf{C})$ be the vertices of the triangle ABC (Fig. 3.20) and O, the origin so that

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \mathbf{C} - \mathbf{B}$$

and $\quad \vec{BA} = \vec{OA} - \vec{OB} = \mathbf{A} - \mathbf{B}$

$\therefore$ Vector area of ΔABC

$$= \frac{1}{2}[\vec{BC} \times \vec{BA}] = \frac{1}{2}[(\mathbf{C}-\mathbf{B}) \times (\mathbf{A}-\mathbf{B})]$$

$$= \frac{1}{2}[\mathbf{C}\times\mathbf{A} - \mathbf{C}\times\mathbf{B} - \mathbf{B}\times\mathbf{A} + \mathbf{B}\times\mathbf{B}]$$

$$= \frac{1}{2}[\mathbf{B}\times\mathbf{C} + \mathbf{C}\times\mathbf{A} + \mathbf{A}\times\mathbf{B}] \quad [\because \mathbf{B}\times\mathbf{B} = \mathbf{0}]$$

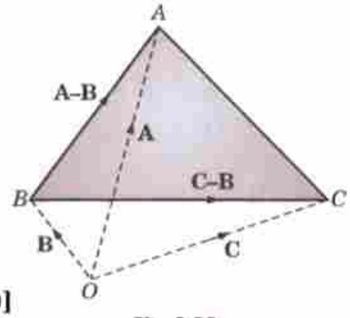


Fig. 3.20

Thus area of $\quad \Delta ABC = \frac{1}{2}\,|\mathbf{B}\times\mathbf{C} + \mathbf{C}\times\mathbf{A} + \mathbf{A}\times\mathbf{B}|$.

(*ii*) Let O be the origin so that

$$\vec{OA} = \mathbf{I} - \mathbf{K},\ \vec{OB} = 2\mathbf{I} + \mathbf{J} + 5\mathbf{K} \text{ and } \vec{OC} = \mathbf{J} + 2\mathbf{K}$$

Then $\quad \vec{BC} = \vec{OC} - \vec{OB} = -2\mathbf{I} - 3\mathbf{K}$

and $\quad \vec{BA} = \vec{OA} - \vec{OB} = -\mathbf{I} - \mathbf{J} - 6\mathbf{K}$

$\therefore$ Vector area of $\quad \Delta ABC = \frac{1}{2}(\vec{BC}\times\vec{BA}) = \frac{1}{2}\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -2 & 0 & -3 \\ -1 & -1 & -6 \end{vmatrix}$

Thus area of $\quad \Delta ABC = \frac{1}{2}\,|-3\mathbf{I} - 9\mathbf{J} + 2\mathbf{K}| = \frac{1}{2}\sqrt{94}$.

Example 3.14. *In a triangle ABC; D, E, F are the mid-points of the sides BC, CA, AB; prove that*

$$\Delta DEF = \Delta FCE = \frac{1}{4}\Delta ABC.$$

Solution. Take B as the origin and let the position vectors of C and A be $\mathbf{C}$ and $\mathbf{A}$ (Fig 3.21); so that the position vectors of D, E, F are

$$\mathbf{C}/2,\ (\mathbf{C}+\mathbf{A})/2,\ \mathbf{A}/2.$$

$$\therefore \quad \Delta \vec{DEF} = \frac{1}{2}(\vec{DE}\times\vec{DF}) = \frac{1}{2}\left(\frac{\mathbf{C}+\mathbf{A}}{2} - \frac{\mathbf{C}}{2}\right)\left(\frac{\mathbf{A}}{2} - \frac{\mathbf{C}}{2}\right)$$

$$= \frac{1}{8}[\mathbf{A}\times(\mathbf{A}-\mathbf{C})] = \frac{1}{8}\,\mathbf{C}\times\mathbf{A} = \frac{1}{4}\Delta\vec{ABC}$$

$$\Delta\vec{FCE} = \frac{1}{2}(\vec{FC}\times\vec{FE}) = \frac{1}{2}[\mathbf{C} - \mathbf{A}/2) \times \mathbf{C}/2]$$

$$= \frac{1}{8}\,\mathbf{C}\times\mathbf{A} = \frac{1}{4}\Delta\vec{ABC}. \text{ Hence the result.}$$

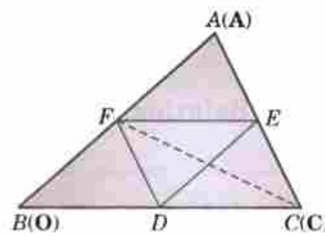


Fig. 3.21

Example 3.15. *Prove that*
(*i*) *sin (A + B) = sin A cos B + cos A sin B.*
(*ii*) *cos (A + B) = cos A cos B – sin A sin B.*

Solution. Let $\mathbf{I}, \mathbf{J}$ denote unit vectors along two perpendicular lines OX, OY so that

$$\mathbf{I}^2 = \mathbf{J}^2 = 1,\ \mathbf{I}.\mathbf{J} = 0$$

and $\quad \mathbf{I}\times\mathbf{I} = \mathbf{J}\times\mathbf{J} = \mathbf{0}$

Let $\quad \angle POX = A$ and $\angle XOQ = B$,

so that $\quad \angle POQ = A + B$.

If $OP = p$ and $OQ = q$, then the coordinates of P are $(p\cos A, -p\sin A)$ and those of Q are $(q\cos B, q\sin B)$ so that

$$\vec{OP} = (p \cos A)\mathbf{I} - (p \sin A)\mathbf{J}$$

$$\vec{OQ} = (q \cos B)\mathbf{I} + (q \sin B)\mathbf{J}$$

Then $|\vec{OP} \times \vec{OQ}| = |[(p \cos A)\mathbf{I} - (p \sin A)\mathbf{J}] \times [(q \cos B)\mathbf{I} + (q \sin B)\mathbf{J}|$

$= pq\ |\cos A \sin B\ (\mathbf{I} \times \mathbf{J}) - \sin A \cos B\ (\mathbf{J} \times \mathbf{I})|$

$= pq\ (\cos A \sin B + \sin A \cos B)$ for $|\ \mathbf{I} \times \mathbf{J}\ | = 1$

Also $|\vec{OP} \times \vec{OQ}| = pq \sin (A + B)$. Equating the two expressions, we get (*i*).

Similarly, (*ii*) follows from $\vec{OP} . \vec{OQ} = pq \cos (A + B)$.

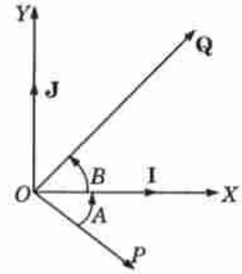


Fig. 3.22

Example 3.16. *In any triangle ABC, prove that*

(*i*) $a/\sin A = b/\sin B = c/\sin C$. (*Sine formula*)

(*ii*) $a = b \cos C + c \cos B$. (*Projection formula*)

(*iii*) $a^2 = b^2 + c^2 - 2bc \cos A$. (*Cosine formula*)

Solution. From ΔABC, we have $\vec{BC} + \vec{CA} + \vec{AB} = 0$

or $$\vec{CA} + \vec{AB} = -\vec{BC} \quad ...(\lambda)$$

(*i*) Multiplying (λ) vectorially by $\vec{AB}$, we get

$$\vec{CA} \times \vec{AB} = -\vec{BC} \times \vec{AB}$$

or $$|\vec{CA} \times \vec{AB}| = |\vec{BC} \times \vec{AB}|$$

$\therefore$ $$bc \sin (\pi - A) = ac \sin (\pi - B)$$

or $$a/\sin A = b/\sin B.$$

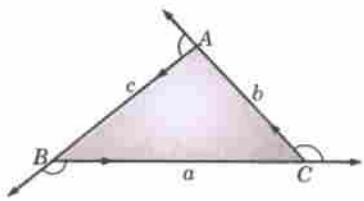


Fig. 3.23

Similarly, multiplying (λ) vectorially by $\vec{CA}$, we get

$a/\sin A = c/\sin C$, whence follows the result.

(*ii*) Multiplying (λ) scalarly by $\vec{BC}$, we get $\vec{CA} . \vec{BC} + \vec{AB} . \vec{BC} = -(\vec{BC})^2$

$\therefore$ $ba \cos (\pi - C) + ca \cos (\pi - B) = -a^2$ or $a = b \cos C + c \cos B$.

(*iii*) Squaring (λ), we get

$$(\vec{CA})^2 + (\vec{AB})^2 + 2\vec{CA} . \vec{AB} = (\vec{BC})^2$$

i.e., $$b^2 + c^2\ 2bc \cos (\pi - A) = a^2 \text{ or } a^2 = b^2 + c^2 - 2bc \cos A.$$

PROBLEMS 3.3

1. Given $\mathbf{A} = 2\mathbf{I} + 2\mathbf{J} - \mathbf{K}$, $\mathbf{B} = 6\mathbf{I} - 3\mathbf{J} + 2\mathbf{K}$, find $\mathbf{A} \times \mathbf{B}$ and the unit vector perpendicular to both **A** and **B**. Also determine the sine of the angle between **A** and **B**.
2. If **A** and **B** are unit vectors and θ is the angle between them, show that $\sin \dfrac{\theta}{2} = \dfrac{1}{2} |\mathbf{A} - \mathbf{B}|$.
3. Find a unit vector normal to the plane of $\mathbf{A} = 3\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$ and $\mathbf{B} = \mathbf{I} + \mathbf{j} - 2\mathbf{K}$.
4. For any vector **A**, show that $|\mathbf{A} \times \mathbf{I}|^2 + |\mathbf{A} \times \mathbf{J}|^2 + |\mathbf{A} \times \mathbf{K}|^2 = 2\ |\mathbf{A}|^2$.
5. By vector method, find the area of the triangle whose vertices are $(3, -1, 2)$, $(1, -1, -3)$ and $(4, -3, 1)$.
6. (*a*) Prove that the vector area of the quadrilateral $ABCD$ is $\dfrac{1}{2}\ \vec{AC} \times \vec{BD}$.

   (*b*) If $3\mathbf{I} + \mathbf{J} - 2\mathbf{K}$ and $\mathbf{I} - 3\mathbf{J} - 4\mathbf{K}$ are the diagonals of a parallelogram. Find its area.

7. Given vectors $\mathbf{A} = \mathbf{I} - 3\mathbf{J} + 2\mathbf{K}$ and $\mathbf{B} = 2\mathbf{I} + \mathbf{J} - \mathbf{K}$. Find the projection of $\mathbf{A} \times \mathbf{B}$ parallel to $5\mathbf{I} - \mathbf{K}$.
8. If $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$, prove that $\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{C} = \mathbf{C} \times \mathbf{A}$, and interpret it geometrically.
9. Show that the perpendicular distance of the point $\mathbf{C}$ from the line joining $\mathbf{A}$ and $\mathbf{B}$ is $|\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B}| \div |\mathbf{B} - \mathbf{A}|$
10. In AC, diagonal of the parallelogram $ABCD$, a point $\mathbf{P}$ is taken. Prove that $\Delta BAP = \Delta DAP$.
11. Prove by vector methods, that
(*i*) $\sin(A - B) = \sin A \cos B - \cos A \sin B$; (*ii*) $\cos(A - B) = \cos A \cos B + \sin A \sin B$. (*Cochin, 1999*)
12. In any triangle ABC, prove by vector methods, that
(*i*) $b = c \cos A + a \cos C$; (*ii*) $c^2 = a^2 + b^2 - 2ab \cos C$.

3.7 PHYSICAL APPLICATIONS

(1) Work done as a scalar product. If constant force $\mathbf{F}$ acting on a particle displaces it from the position A to position B, then

Work done = (resolved part of F in the direction of AB). AB

$$= F \cos \theta \, . \, AB = \mathbf{F} . \overrightarrow{AB}$$

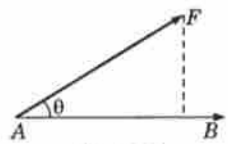


Fig. 3.24

Thus, the work done by a constant force is the scalar (or dot) product of the vectors representing the force and the displacement.

Example 3.17. *Constant forces* $\mathbf{P} = 2\mathbf{I} - 5\mathbf{J} + 6\mathbf{K}$ *and* $\mathbf{Q} = -\mathbf{I} + 2\mathbf{J} - \mathbf{K}$ *act on a particle. Determine the work done when the particle is displaced from A to B the position vectors of A and B being* $4\mathbf{I} - 3\mathbf{J} - 2\mathbf{K}$ *and* $6\mathbf{I} + \mathbf{J} - 3\mathbf{K}$ *respectively.*

Solution. Resultant force $\quad \mathbf{F} = \mathbf{P} + \mathbf{Q} = \mathbf{I} - 3\mathbf{J} + 5\mathbf{K}$

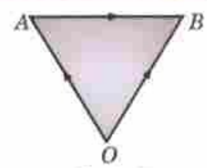


Fig. 3.25

and $\quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (6\mathbf{I} + \mathbf{J} - 3\mathbf{K}) - (4\mathbf{I} - 3\mathbf{J} - 2\mathbf{K}) = 2\mathbf{I} + 4\mathbf{J} - \mathbf{K}$

$\therefore$ *Work done* $\quad = \mathbf{F} . \overrightarrow{AB} = (\mathbf{I} - 3\mathbf{J} + 5\mathbf{K}) . (2\mathbf{I} + 4\mathbf{J} - \mathbf{K})$

$\quad = 1 . 2 - 3 . 4 + 5 . (-1) = -15$ units.

(2) Normal flux. Consider the flow of a liquid through an element of area δs with a velocity $\mathbf{V}$ inclined at an angle θ to the out ward unit normal $\mathbf{N}$ to the surface δs (Fig. 3.26).

$\therefore$ Normal flux of the liquid through δs in unit time

$$\mathbf{V} \cos \theta \, . \, \delta s = \mathbf{V} \, . \, \mathbf{N} \delta s.$$

Thus, *the rate of normal flux per unit area* = $\mathbf{V} . \mathbf{N}$

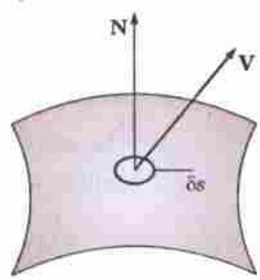


Fig. 3.26

Obs. We can also apply this result to the case of electric or magnetic flux.

(3) Moment of a force about a point. Suppose the moment of the force $\mathbf{F}$ acting at the point P about the point A is required.

Draw $AM \perp$ the line of action of $\mathbf{F}$ (Fig. 3.27). If θ be the angle between $\overrightarrow{AP}$ and $\mathbf{F}$ and $\mathbf{N}$ be a unit vector $\perp$ to their plane, then $\overrightarrow{AP} \times \mathbf{F} = (AP. F \sin \theta) \mathbf{N} = F(AP \sin \theta) \mathbf{N} = (F . AM) \mathbf{N}$

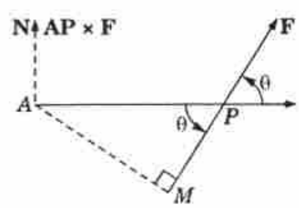


Fig. 3.27

Clearly, (*i*) the magnitude of $\overrightarrow{AP} \times \mathbf{F} = F . AM$ which is the numerical measure of the moment of $\mathbf{F}$ about A.

and (*ii*) the direction of $\overrightarrow{AP} \times \mathbf{F}$ is the direction of the moment of $\mathbf{F}$ about A.

Hence the moment (or torque) of $\mathbf{F}$ about A is $\overrightarrow{AP} \times \mathbf{F}$.

Example 3.18. *Find the torque about the point* $2\mathbf{I} + \mathbf{J} - \mathbf{K}$ *of a force represented by* $4\mathbf{I} + \mathbf{K}$ *acting through the point* $\mathbf{I} - \mathbf{J} + 2\mathbf{K}$.

Solution. Let O be the origin and P be the point, moment about which of the force $\overrightarrow{AB}$ through A, is required (Fig. 3.28).

$\therefore$ $\vec{OP} = 2\mathbf{I} + \mathbf{J} - \mathbf{K}$,

$\vec{OA} = \mathbf{I} - \mathbf{J} + 2\mathbf{K}$, and $\vec{AB} = 4\mathbf{I} + \mathbf{K}$.

Then, $\vec{PA} = \vec{OA} - \vec{OP} = -\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$

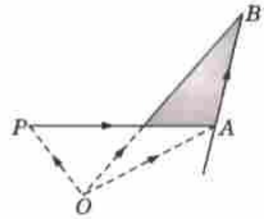


Fig. 3.28

$\therefore$ Moment of the force $\vec{AB}$ about P

$$= \vec{PA} \times \vec{AB} = (-\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}) \times (4\mathbf{I} + \mathbf{K})$$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -1 & -2 & 3 \\ 4 & 0 & 1 \end{vmatrix} = -2\mathbf{I} + 13\mathbf{J} + 8\mathbf{K}$$

$\therefore$ Magnitude of the moment $= \sqrt{(4 + 169 + 64)} = 15.4$

(4) Moment of a force about a line.

Def. *The moment of a force* **F** *about a line* **D** *is the resolved part along* **D** *of the moment of* **F** *about any point on* **D**.

Example 3.19. *Find the moment about a line through the origin having direction of* $2\mathbf{I} + 2\mathbf{J} + \mathbf{K}$, *due to a 30 kg force acting at a point* $(-4, 2, 5)$ *in the direction of* $12\mathbf{I} - 4\mathbf{J} - 3\mathbf{K}$.

Solution. Let **D** be the given line through the origin O and **F** the force through **A**$(-4, 2, 5)$.

Clearly, $\vec{OA} = -4\mathbf{I} + 2\mathbf{J} + 5\mathbf{K}$

and the force $\mathbf{F} = 30\left(\dfrac{12\mathbf{I} - 4\mathbf{J} - 3\mathbf{K}}{13}\right)$

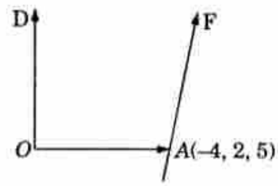


Fig. 3.29

$\therefore$ Moment of **F** about $O = \vec{OA} \times \mathbf{F}$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -4 & 2 & 5 \\ \frac{360}{13} & \frac{-120}{13} & \frac{-90}{13} \end{vmatrix} = \frac{60}{13}(7\mathbf{I} + 24\mathbf{J} - 4\mathbf{K})$$

Thus the moment of **F** *about the line* **D**

= resolved part of the moment of **F** about O along **D**,

i.e., $\dfrac{60}{13}(7\mathbf{I} + 24\mathbf{J} - 4\mathbf{K}) \cdot \hat{\mathbf{D}}$

$$= \frac{60}{13}(7\mathbf{I} + 24\mathbf{J} - 4\mathbf{K}) \cdot \frac{2\mathbf{I} + 2\mathbf{J} + \mathbf{K}}{\sqrt{(4 + 4 + 1)}} = \frac{20}{13}(7 \times 2 + 24 \times 2 - 4 \times 1) = 89.23.$$

(5) Angular velocity of a rigid body

Let a rigid body be rotating about the axis OM with angular velocity ω radians per second (Fig. 3.30). Let P be a point of the body such that $\vec{OP} = \mathbf{R}$ and $\angle MOP = \theta$. Draw $PM \perp OM$.

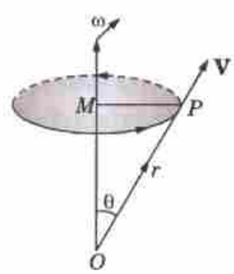


Fig. 3.30

Now if **N** be a unit vector $\perp \vec{\omega}$ **R** then

$$\vec{\omega} \times \mathbf{R} = \omega r \sin\theta \,.\, \mathbf{N} = \omega MP \,.\, \mathbf{N}$$
$$= (\text{speed of } \mathbf{P})\, \mathbf{N}$$
$$= \text{velocity } \mathbf{V} \text{ of } \mathbf{P} \text{ in a direction } \perp \text{ to the plane } MOP.$$

Hence $\mathbf{V} = \vec{\omega} \times \mathbf{R}$.

Example 3.20. *A rigid body is spinning with angular velocity 27 radians per second about an axis parallel to* $2\mathbf{I} + \mathbf{J} - 2\mathbf{K}$ *passing through the point* $\mathbf{I} + 3\mathbf{J} - \mathbf{K}$. *Find the velocity of the point of the body whose position vector is* $4\mathbf{I} + 8\mathbf{J} + \mathbf{K}$.

Solution. Unit vector along the direction of $\vec{\omega} = \dfrac{2\mathbf{I} + \mathbf{J} - 2\mathbf{K}}{\sqrt{(4+1+4)}} = \dfrac{1}{3}(2\mathbf{I} + \mathbf{J} - 2\mathbf{K})$

$\therefore$ Angular velocity $\vec{\omega} = \dfrac{27}{3}(2\mathbf{I} + \mathbf{J} - 2\mathbf{K}) = 9(2\mathbf{I} + \mathbf{J} - 2\mathbf{K})$

Let A be the point $\mathbf{I} + 3\mathbf{J} - \mathbf{K}$ and the point P of the body be $(4\mathbf{I} + 8\mathbf{J} - \mathbf{K})$ so that

$$\vec{AP} = (4\mathbf{I} + 8\mathbf{J} + \mathbf{K}) - (\mathbf{I} + 3\mathbf{J} - \mathbf{K}) = 3\mathbf{I} + 5\mathbf{J} + 2\mathbf{K}$$

$\therefore$ Velocity vector of $P = \mathbf{V} = \vec{\omega} \times \vec{AP} = 9(2\mathbf{I} + \mathbf{J} - 2\mathbf{K}) \times (3\mathbf{I} + 5\mathbf{J} + 2\mathbf{K})$

$$= 9\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 2 & 1 & -2 \\ 3 & 5 & 2 \end{vmatrix} = 9(12\mathbf{I} - 10\mathbf{J} + 7\mathbf{K})$$

and its magnitude $9\sqrt{(144+100+49)} = 9\sqrt{293}$.

PROBLEMS 3.4

1. A particle acted on by constant forces $4\mathbf{I} + \mathbf{J} - 3\mathbf{K}$ and $3\mathbf{I} + \mathbf{J} - \mathbf{K}$ is displace from the point $\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}$ to the point $5\mathbf{I} + 4\mathbf{J} + \mathbf{K}$. Find the total work done by the forces.
2. Forces $2\mathbf{I} - 5\mathbf{J} + 6\mathbf{K}$, $-\mathbf{I} + 2\mathbf{J} - \mathbf{K}$ and $2\mathbf{I} + 7\mathbf{J}$ act on a particle P whose position vector is $4\mathbf{I} - 3\mathbf{J} - 2\mathbf{K}$. Determine the work done by the forces in a displacement of the particle to the point Q (6, 1, –3).
   Also find the vector moment of the resultant of three forces acting at P about the point Q.
3. Forces of magnitudes 5, 3 1 units act in the directions $6\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}$, $3\mathbf{I} - 2\mathbf{J} + 6\mathbf{K}$, $2\mathbf{I} - 3\mathbf{J} - 6\mathbf{K}$ respectively on a particle which is displaced from the point (2, 1, –3) to (5, –1, 1). Find the work done by the forces.
4. The point of application of the force (–2, 4, 7) is displaced from the point (3, –5, 1) to the point (5, 9, 7). But the force is suddenly halved when the point of application moves half the distance. Find the work done.
5. A force $\mathbf{F} = 3\mathbf{I} + 2\mathbf{J} - 4\mathbf{K}$ is applied at the point (1, – 1, 2). Find the moment of the force about the point (2, – 1, 3). *(Assam, 1999)*
6. A force with components (5, – 4, 2) acts at a point P which is at a distance 3 units from the origin. If the moment of the force about origin has components (12, 8, – 14), find the co-ordinates of P.
7. Find the moment of the force $\mathbf{F} = 2\mathbf{I} + 2\mathbf{J} - \mathbf{K}$ acting at the point (1, – 2, 1) about z-axis.
8. A force of 10 kg acts in a direction equally inclined to the co-ordinate axes through the point (3, – 2, 5). Find the magnitude of the moment of the force about a line through the origin and whose direction ratios are (2, –3, 6).
9. A rigid body is rotating at 2.5 radians per second about an axis OR, where R is the point $2\mathbf{I} - 2\mathbf{J} + \mathbf{K}$ relative to O. Find the velocity of the particle of the body at the point $4\mathbf{I} + \mathbf{J} + \mathbf{K}$. (All lengths are in cm).

3.8 PRODUCTS OF THREE OR MORE VECTORS

With any three vectors **A**, **B**, **C**, we can form the products (**A** . **B**) **C**, (**A** × **B**) . **C** and (**A** × **B**) × **C**. The first being the product of a scalar **A** . **B** and a vector **C**, represents a vector in the direction of **C**. The second being the scalar product of vectors **A** × **B** and **C**, represents a scalar and is usually called the *scalar product of three vectors*. The third being the vector product of the vectors **A** × **B** and **C**, represents a vector and is usually known as the *vector product of three vectors*.

The reader must, however, note that the products of the form **A** . (**B** . **C**), **A** × (**B** . **C**) and **A**(**B** × **C**) are meaningless.

In practical applications, we seldom come across products of more than three vectors. Such products if they occur can, in general, be reduced by using successively the expansion formula for vector triple products. As an illustration, we shall consider two products (**A** × **B**). (**C** × **D**) and (**A** × **B**) × (**C** × **D**) of any four vectors, the former being a scalar and a latter a vector.

3.9 SCALAR PRODUCT OF THREE VECTORS

(1) Definition. *If* **A, B, C** *be any three vectors then the scalar or dot product of* **A × B** *with* **C** *is called the scalar product of the three vectors* **A, B, C** *and is written as* (**A × B**) . **C** *or* [**ABC**].

No ambiguity can arise by omitting the brackets in $(A \times B)$. C as $A \times (B \cdot C)$ would be meaningless .

(2) Geometrical interpretation. *The Product* **A × B . C** *represents numerically the volume of a parallelopiped having* **A, B, C** *as coterminous edges.*

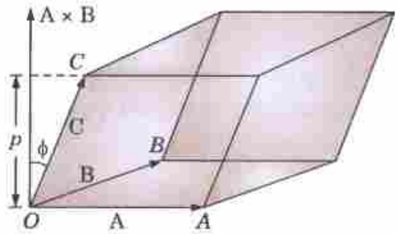


Fig. 3.31

Consider a parallelopiped with $\vec{OA} = \mathbf{A}$, $\vec{OB} = \mathbf{B}$, $\vec{OC} = \mathbf{C}$ as coterminous edges (Fig. 3.31).

Let V be its volume, α the area of each of the two faces parallel to the vectors **A** and **B** and p the perpendicular distance between these faces.

Then $|\mathbf{A} \times \mathbf{B}| = \alpha$ and $|\mathbf{C}| \cos \phi = p$ or $-p$ according as **A, B, C** form a right-handed or left-handed triad.

$$\therefore \quad \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = |\mathbf{A} \times \mathbf{B}| \cdot |\mathbf{C}| \cos \phi = \pm \alpha p = \pm V.$$

Thus [**ABC**] = V *or* $-V$ *according as* **A, B, C** *form a right-handed or left-handed triad.*

(*Kerala, 1990; J.N.T.U., 1988*)

In particular, for an orthonormal right-handed vector triad **I, J, K,**

$$[\mathbf{IJK}] = \mathbf{I} \times \mathbf{J} \cdot \mathbf{K} = \mathbf{K} \cdot \mathbf{K} = I.$$

(3) Properties and other results.

I. The condition for three vectors to be coplanar is that their scalar triple product should vanish.

If three vectors **A, B, C** anti coplanar, then the volume of the parallelopiped so formed is zero, *i.e.,* [**ABC**] = 0.

II. If any two vectors of a scalar triple product are equal, the product vanishes, i.e., [**ABC**] = 0 when either **A** = **B** or **B** = **C**, or **C** = **A**, for in this case the parallelopiped has zero volume.

III. **Two important rules** (*for evaluating a scalar triple product*). *Every scalar triple product*

(*i*) *is independent of the position of the dot or cross.*

and (*ii*) *depends upon the cyclic order of the vectors.*

It is easy to note that if **A, B, C** is a right-handed triad so are **B, C, A** and **C, A, B.**

Moreover a parallelopiped having **A, B, C** as coterminous edges is the same as that having **B, C, A** or **C, A, B** as coterminous edges.

Thus, if V be the volume of this parallelopiped,

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = V, \mathbf{B} \times \mathbf{C} \cdot \mathbf{A} = V, \mathbf{C} \times \mathbf{A} \cdot \mathbf{B} = V$$

Also, since **A . B = B . A**, we have

$$\mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = V$$
$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{B} \times \mathbf{C} \cdot \mathbf{A} = V$$
$$\mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B} = V$$

Thus
$$\left.\begin{array}{l} \mathbf{AA} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C} \\ \mathbf{B} \times \mathbf{C} \cdot \mathbf{A} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} \\ \mathbf{C} \times \mathbf{A} \cdot \mathbf{B} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} \end{array}\right\} = V \qquad \ldots(\alpha)$$

Further a right-handed triad becomes left-handed when the cyclic order of the vectors is changed. Therefore **A, C, B; B, A, C; C, B, A** being left-handed triads, it follows that

$$\mathbf{A} \times \mathbf{C} \cdot \mathbf{B} = -V, \mathbf{B} \times \mathbf{A} \cdot \mathbf{C} = -V, \mathbf{C} \times \mathbf{B} \cdot \mathbf{A} = -V.$$

Thus
$$\left.\begin{array}{l} \mathbf{AA} \times \mathbf{C} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C} \times \mathbf{B} \\ \mathbf{B} \times \mathbf{A} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{A} \times \mathbf{C} \\ \mathbf{C} \times \mathbf{B} \cdot \mathbf{A} = \mathbf{C} \cdot \mathbf{B} \times \mathbf{A} \end{array}\right\} = -V \qquad \ldots(\beta)$$

Obs. In support of the above rules, our notation [**ABC**] indicates the cyclic order of the factors and has nothing to do with position of the dot or the cross.

∴ The relations (α) and (β) can be compactly written as

$$[\mathbf{ABC}] = [\mathbf{BCA}] = [\mathbf{CAB}] = V \quad \text{and} \quad [\mathbf{ACB}] = [\mathbf{BAC}] = [\mathbf{CBA}] = -V.$$

IV. Scalar triple product is distributive

i.e., $[\mathbf{A}, \mathbf{B} + \mathbf{C}, \mathbf{D} - \mathbf{E}] = [\mathbf{ABD}] - [\mathbf{ABE}] + [\mathbf{ACD}] - [\mathbf{ACE}]$

V. If $\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}$, $\mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$, $\mathbf{C} = c_1\mathbf{I} + c_2\mathbf{J} + c_3\mathbf{K}$

then
$$[\mathbf{ABC}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

As $\mathbf{A} \times \mathbf{B} = (a_2b_3 - a_3b_2)\,\mathbf{I} + (a_3b_1 - a_1b_3)\,\mathbf{J} + (a_1b_2 - a_2b_1)\mathbf{K}$

$\therefore$ $[\mathbf{ABC}] = [(a_2b_3 - a_3b_2)\,\mathbf{I} + (a_3b_1 - a_1b_3)\,\mathbf{J} + (a_1b_2 - a_2b_1)\,\mathbf{K}] \cdot (c_1\mathbf{I} + c_2\mathbf{J} + c_3\mathbf{K})$

$= c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1)$ which is the required result.

Obs. Linear dependence of vectors. Any three vectors **A, B, C** are said to be *linearly dependent* if one of these can be expressed as a linear combination of other two *i.e.,*

$$\mathbf{A} = m\mathbf{B} + n\mathbf{C}$$

where m, n are constants. This means that **A** lies in the plane of **B, C** *i.e.,* $[\mathbf{ABC}] = 0$. Thus *three vectors are linearly dependent if their scalar triple product is zero. Otherwise these vectors are linearly independent.*

Example 3.21. *Show that the points* $-6\mathbf{I} + 3\mathbf{J} + 2\mathbf{K}$, $3\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$, $5\mathbf{I} + 7\mathbf{J} + 3\mathbf{K}$ *and* $-13\mathbf{I} + 17\mathbf{J} - \mathbf{K}$ *are coplanar.*

Solution. Let $\vec{OA} = -6\mathbf{I} + 3\mathbf{J} + 2\mathbf{K}$, $\vec{OB} = 3\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$, $\vec{OC} = 5\mathbf{I} + 7\mathbf{J} + 3\mathbf{K}$

and $\vec{OD} = -13\mathbf{I} + 17\mathbf{J} - \mathbf{K}$. Then $\vec{AB} = \vec{OB} - \vec{OA} = 9\mathbf{I} - 5\mathbf{J} + 2\mathbf{K}$

Similarly, $\vec{AC} = 11\mathbf{I} + 4\mathbf{J} + \mathbf{K}$, and $\vec{AD} = -7\mathbf{I} + 14\mathbf{J} - 3\mathbf{K}$.

The given points will be coplanar if $\vec{AB}, \vec{AC}, \vec{AD}$ are coplanar, *i.e.,* if their scalar triple product is zero. Now

$$[\vec{AB}, \vec{AC}, \vec{AD}] = \begin{vmatrix} 9 & -5 & 2 \\ 11 & 4 & 1 \\ -7 & 14 & -3 \end{vmatrix} = 9(-12 - 14) + 5(-33 + 7) + 2(154 + 28) = 0$$

Hence the points A, B, C, D are coplanar.

Example 3.22. *Show that the volume of the tetrahedron ABCD is* $\frac{1}{6}[\vec{AB}, \vec{AC}, \vec{AD}]$.

Hence find the volume of the tetrahedron formed by the points (1, 1, 1), (2, 1, 3), (3, 2, 2) and (3, 3, 4).

Solution. (*i*) Volume of the tetrahedron $ABCD$

$= \frac{1}{3}$ (area of ΔABC) × (height h of D above the plane ABC)

$= \frac{1}{6}$ (2 area of ΔABC)h

$= \frac{1}{6}$ (volume of the parallelopiped with AB, AC, AD as coterminus edges)

$= \frac{1}{6}[\vec{AB}, \vec{AC}, \vec{AD}]$.

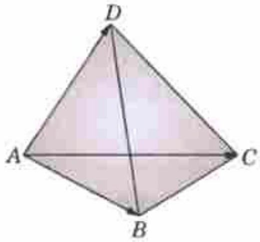


Fig. 3.32

(*ii*) Let $\vec{OA} = \mathbf{I} + \mathbf{J} + \mathbf{K}$, $\vec{OB} = 2\mathbf{I} + \mathbf{J} + 3\mathbf{K}$, $\vec{OC} = 3\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$ and $\vec{OD} = 3\mathbf{I} + 3\mathbf{J} + 4\mathbf{K}$.

Then $\vec{AB} = \vec{OB} - \vec{OA} = \mathbf{I} + 2\mathbf{K}$

Similarly, $AC = 2\mathbf{I} + \mathbf{J} + \mathbf{K}$ and $\vec{AD} = 2\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}$

$\therefore$ Volume of the tetrahedron $ABCD = \frac{1}{6}[\vec{AB}, \vec{AC}, \vec{AD}] = \frac{1}{6}\begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix} = \frac{5}{6}$.

3.10 VECTOR PRODUCT OF THREE VECTORS

(1) Definition. *If* **A, B, C** *be any three vectors, then the vector or cross product of* **A** × **B** *with* **C** *is called the vector product of three vectors* **A, B, C** *and is written as* (**A** × **B**) × **C**.

Here the brackets are essential as $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$, expressing the fact that vector triple product is not associative.

(2) Expansion formula. *If* **A, B, C** *be any three vectors,* $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A}$

In words (extreme × adjacent) × outer = (outer . extreme) adjacent – (outer . adjacent) extreme.

The vector $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ is perpendicular to the vector $\mathbf{A} \times \mathbf{B}$ and the latter is perpendicular to the plane containing **A** and **B**. Hence $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ lies in the plane of **A** and **B**. As such we can write

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = l\mathbf{A} + m\mathbf{B} \qquad ...(1)$$

where l and m are some scalars.

Multiply both sides scalarly by **C**, then $\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = l\mathbf{C} \cdot A + m\mathbf{C} \cdot \mathbf{B}$

The scalar triple product on the left-hand side is zero, since two of its vectors are equal.

$$\therefore \qquad l(\mathbf{C} \cdot \mathbf{A}) + m(\mathbf{C} \cdot \mathbf{B}) = 0$$

or
$$\frac{l}{\mathbf{C} \cdot \mathbf{B}} = \frac{m}{-\mathbf{C} \cdot \mathbf{A}} = n, \text{ say.}$$

Substituting the values of l and m in (1), we get

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = n(\mathbf{C} \cdot \mathbf{B})\mathbf{A} - n(\mathbf{C} \cdot \mathbf{A})\mathbf{B} \qquad ...(2)$$

Evidently n is some numerical constant. To find it, take the special case $\mathbf{A} = \mathbf{I}, \mathbf{B} = \mathbf{C} = \mathbf{J}$. Then (2) gives

$$(\mathbf{I} \times \mathbf{J}) \times \mathbf{J} = n(\mathbf{J} \cdot \mathbf{J})\mathbf{I} - n(\mathbf{J} \cdot \mathbf{I})\mathbf{J}$$

i.e.,
$$\mathbf{K} \times \mathbf{J} = n\mathbf{I} \text{ or } -\mathbf{I} = n\mathbf{I}.$$

This gives $n = -1$. Hence (2) reduces to the required result.

Similarly, it can be shown that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$

Cor. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{0}$.

For L.H.S. $= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} + (\mathbf{B} \cdot \mathbf{A})\mathbf{C} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A} + (\mathbf{C} \cdot \mathbf{B})\mathbf{A} - (\mathbf{C} \cdot \mathbf{A})\mathbf{B}$ which vanishes identically.

Example 3.23. *If A, B, C, D be any four vectors, prove that*

(i) $(A \times B) \cdot (C \times D) = \begin{vmatrix} A \cdot C & B \cdot C \\ A \cdot D & B \cdot D \end{vmatrix}$ *(ii)* $(A \times B) \times (C \times D) = [ACD]\,B - [BCD]A$

Solution. (*i*) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = [\mathbf{A} \times \mathbf{B}) \times \mathbf{C}] \cdot \mathbf{D}$ (interchanging the dot and cross)

$= [(\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}] \cdot \mathbf{D}$

$= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$ whence follows the result.

In particular, we have $(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2\mathbf{B}^2 - (\mathbf{A} \cdot \mathbf{B})^2$ which has already been proved in § 3.6 (3) – IV.

(*ii*) $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{P}$, where $\mathbf{P} = \mathbf{C} \times \mathbf{D}$

$= (\mathbf{A} \cdot \mathbf{P})\mathbf{B} - (\mathbf{B} \cdot \mathbf{P})\mathbf{A} = (\mathbf{A} \cdot \mathbf{C} \times \mathbf{D})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C} \times \mathbf{D})\mathbf{A}$

$= [\mathbf{ACD}]\mathbf{B} - [\mathbf{BCD}]\mathbf{A}.$

Example 3.24. *Show that the components of a vector* **B** *along and perpendicular to a vector* **A**, *in the plane of* **A** *and* **B**, *are*

$$\frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A}^2} \text{ and } \frac{(\mathbf{A} \times \mathbf{B}) \times \mathbf{A}}{\mathbf{A}^2}.$$

Solution. Let $\vec{\mathbf{OA}} = \mathbf{A}$, $\vec{\mathbf{OB}} = \mathbf{B}$ and **OM** be the projection of **B** on **A** (Fig. 3.33)

∴ Component of **B** along **A** = OM (unit vector along **A**)

$$= (\mathbf{B} \cdot \hat{\mathbf{A}})\hat{\mathbf{A}} = \left(\frac{\mathbf{B} \cdot \mathbf{A}}{a}\right)\frac{\mathbf{A}}{a} \qquad [\because \mathbf{A} = a\,\hat{\mathbf{A}}]$$

$$= \frac{\mathbf{B} \cdot \mathbf{A}}{\mathbf{A}^2}\mathbf{A} \qquad [\because a^2 = \mathbf{A}^2]$$

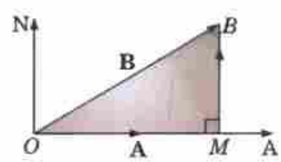


Fig. 3.33

Also component of $\mathbf{B} \perp \mathbf{A} = \overrightarrow{\mathbf{MB}}$

$$= \overrightarrow{OB} - \overrightarrow{OM} = \mathbf{B} - \frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}^2}\,\mathbf{A} = \frac{(\mathbf{A}\cdot\mathbf{A})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{A}}{\mathbf{A}^2} = \frac{(\mathbf{A}\times\mathbf{B})\times\mathbf{A}}{\mathbf{A}^2}.$$

Example 3.25. *Prove the formula*

$$(\mathbf{B}\times\mathbf{C})\cdot(\mathbf{A}\times\mathbf{D}) + (\mathbf{C}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{D}) + (\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = 0.$$

and hence show that $\sin(\theta+\phi)\sin(\theta-\phi) = \sin^2\theta - \sin^2\phi$.

Solution. We know that

$$(\mathbf{B}\times\mathbf{C})\cdot(\mathbf{A}\times\mathbf{D}) = (\mathbf{B}\cdot\mathbf{A})(\mathbf{C}\cdot\mathbf{D}) - (\mathbf{B}\cdot\mathbf{D})(\mathbf{C}\cdot\mathbf{A})$$
$$(\mathbf{C}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{D}) = (\mathbf{C}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{D}) - (\mathbf{C}\cdot\mathbf{D})(\mathbf{A}\cdot\mathbf{B})$$
$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

Adding, we get

$$(\mathbf{B}\times\mathbf{C})\cdot(\mathbf{A}\times\mathbf{D}) + (\mathbf{C}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{D}) + (\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = 0 \qquad ...(i)$$

Let the vectors **A, B, C, D** be acting along coplanar lines *OA, OB, OC, OD* respectively (Fig. 3.34).

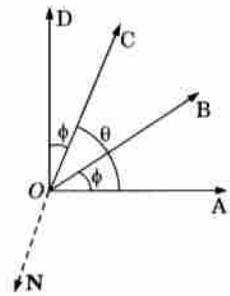


Fig. 3.34

Take $\angle AOC = \theta$ and $\angle AOB = \angle COD = \phi$,

so that $\angle AOD = \theta + \phi$ and $\angle BOC = \theta - \phi$

If **N** be a unit vector normal to the plane of these lines, then

$$(\mathbf{B}\times\mathbf{C})\cdot(\mathbf{A}\times\mathbf{D}) = [bc\sin(\theta-\phi)\mathbf{N}]\cdot[ad\sin(\theta+\phi)\mathbf{N}]$$
$$= abcd\sin(\theta+\phi)\sin(\theta-\phi) \qquad ...(ii)$$
$$(\mathbf{C}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{D}) = [ca\sin(-\theta)\mathbf{N}]\cdot[bd\sin\theta\mathbf{N}]$$
$$= -abcd\sin^2\theta \qquad ...(iii)$$

and
$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = [ab\sin\phi\mathbf{N}]\cdot[cd\sin\phi\mathbf{N}]$$
$$= abcd\sin^2\phi \qquad ...(iv)$$

Substituting the values from (*ii*), (*iii*), (*iv*) in (*i*), we get

$abcd\sin(\theta+\phi)\sin(\theta-\phi) - abcd\sin^2\theta + abcd\sin^2\phi = 0$ whence follows the required result.

Example 3.26. *Prove that*

(*i*) $[\mathbf{B}\times\mathbf{C}, \mathbf{C}\times\mathbf{A}, \mathbf{A}\times\mathbf{B}] = [\mathbf{ABC}]^2$. (*Nagpur, 2009*)

(*ii*) $\mathbf{A}\times[\mathbf{B}\times(\mathbf{C}\times\mathbf{D})] = \mathbf{B}\cdot\mathbf{D}(\mathbf{A}\times\mathbf{C}) - \mathbf{B}\cdot\mathbf{C}(\mathbf{A}\times\mathbf{D})$.

Solution. (*i*) $[\mathbf{B}\times\mathbf{C}, \mathbf{C}\times\mathbf{A}, \mathbf{A}\times\mathbf{B}] = (\mathbf{B}\times\mathbf{C})\cdot(\mathbf{C}\times\mathbf{A})\times(\mathbf{A}\times\mathbf{B})$

$$= (\mathbf{B}\times\mathbf{C})\cdot\{[(\mathbf{C}\times\mathbf{A})\cdot\mathbf{B}]\mathbf{A} - [(\mathbf{C}\times\mathbf{A})\cdot\mathbf{A}]\mathbf{B}\}$$
$$= (\mathbf{B}\times\mathbf{C})\cdot\{[\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})]\mathbf{A}\} \qquad [\because\ [(\mathbf{C}\times\mathbf{A})\cdot\mathbf{A}] = 0$$
$$= [\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}]\,[(\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}] = [\mathbf{BCA}]^2 = [\mathbf{ABC}]^2 \qquad [\because\ [\mathbf{BCA}] = [\mathbf{ABC}]$$

(*ii*)
$$\mathbf{A}\times\{\mathbf{B}\times(\mathbf{C}\times\mathbf{D})\} = \mathbf{A}\times\{(\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{B}\cdot\mathbf{C})\mathbf{D}\}$$
$$= (\mathbf{A}\times\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\times\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) = (\mathbf{B}\cdot\mathbf{D})(\mathbf{A}\times\mathbf{C}) - \mathbf{B}\cdot\mathbf{C}(\mathbf{A}\times\mathbf{D}).$$

PROBLEMS 3.5

1. Find the volume of the parallelopied whose edges are represented by the vectors $\mathbf{A} = 2\mathbf{I} - 3\mathbf{J} + 4\mathbf{K}$, $\mathbf{B} = \mathbf{I} + 2\mathbf{J} - \mathbf{K}$, $\mathbf{C} = 3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$.
2. Find a such that the vectors $2\mathbf{I} - \mathbf{J} + \mathbf{K}$, $\mathbf{I} + 2\mathbf{J} - 3\mathbf{K}$ and $3\mathbf{I} + a\mathbf{J} + 5\mathbf{K}$ are coplanar.
3. (*i*) Prove that the vectors $\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$, $-2\mathbf{I} + 3\mathbf{J} - 4\mathbf{K}$ and $\mathbf{I} - 3\mathbf{J} + 5\mathbf{K}$ are coplanar.
   (*ii*) Do the points (4, – 2, 1), (5, 1, 6), (2, 2 – 5) and (3, 5, 0) lie in a plane.
4. (*a*) Test the linear dependency of the vectors (1, 2, 1), (2, 1, 4), (4, 5, 6) and (1, 8, – 5).
   (*b*) Verify whether the following set of vectors are linearly independent (4, 2, 9), (3, 2, 1), (– 4, 6, 9). (*B.P.T.U., 2005*)
5. Find the volume of the tetrahedron, three of whose coterminus edges are $3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$, $2\mathbf{I} + \mathbf{J} - \mathbf{K}$ and $\mathbf{I} - 2\mathbf{J} + 2\mathbf{K}$.

6. Find the volume of the tetrahedron formed by the points
(i) (1, 3, 6), (3, 7, 12), (8, 8, 9) and (2, 2, 8). *(B.P.T.U., 2005)*
(ii) (2, 1, 1), (1, – 1, 2), (0, 1, – 1) and (1, – 2, 1).
7. If $\mathbf{A}\cdot\mathbf{N}=0$, $\mathbf{B}\cdot\mathbf{N}=0$, $\mathbf{C}\cdot\mathbf{N}=0$, prove that $[\mathbf{ABC}]=0$. Interpret this result geometrically.
8. (a) Prove that $[\mathbf{A}+\mathbf{B}, \mathbf{B}+\mathbf{C}, \mathbf{C}+\mathbf{A}]=2[\mathbf{ABC}]$.
(b) Show that volume of the tetrahedron having $\mathbf{A}+\mathbf{B}$, $\mathbf{B}+\mathbf{C}$ and $\mathbf{C}+\mathbf{A}$ as concurrent edges is twice the volume of the tetrahedron having **A**, **B**, **C** as concurrent edges.
9. If $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$, show that $(\mathbf{A}\times\mathbf{C})\times\mathbf{B}=0$.
10. Show that $\mathbf{I}\times(\mathbf{R}\times\mathbf{I})+\mathbf{J}\times(\mathbf{R}\times\mathbf{J})+\mathbf{K}\times(\mathbf{R}\times\mathbf{K})=2\mathbf{R}$. *(Assam, 1999)*
11. If $\mathbf{A}=\mathbf{I}-2\mathbf{J}-3\mathbf{K}$, $\mathbf{B}=2\mathbf{I}+\mathbf{J}-\mathbf{K}$, $\mathbf{C}=\mathbf{I}+3\mathbf{J}-\mathbf{K}$, find
(i) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ (ii) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{B}\times\mathbf{C})$.
12. (a) Given $\mathbf{A}=2\mathbf{I}-\mathbf{J}+3\mathbf{K}$, $\mathbf{B}=-\mathbf{I}+3\mathbf{J}+3\mathbf{K}$, $\mathbf{C}=\mathbf{I}+\mathbf{J}-2\mathbf{K}$, find the reciprocal triad $(\mathbf{A}', \mathbf{B}', \mathbf{C}')$ and verify that $[\mathbf{ABC}]\,[\mathbf{A}'\mathbf{B}'\mathbf{C}']=1$.
(b) Prove that $\mathbf{A}\times\mathbf{A}'+\mathbf{B}\times\mathbf{B}'+\mathbf{C}\times\mathbf{C}'=0$
13. Prove that (i) $[\mathbf{A}\times\mathbf{B}, \mathbf{C}\times\mathbf{D}, \mathbf{E}\times\mathbf{F}]=[\mathbf{ABD}][\mathbf{CEF}]-[\mathbf{ABC}][\mathbf{DEF}]$
(ii) $\{(\mathbf{A}+\mathbf{B}+\mathbf{C})\times(\mathbf{B}+\mathbf{C})\}\cdot\mathbf{C}=[\mathbf{ABC}]$.
14. Show that
(i) $(\mathbf{B}\times\mathbf{C})\times(\mathbf{A}\times\mathbf{D})+(\mathbf{C}\times\mathbf{A})\times(\mathbf{B}\times\mathbf{D})+(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})=-2[\mathbf{ABC}]\mathbf{D}$. *(Mumbai, 2007)*
(ii) $\mathbf{A}\times[\mathbf{F}\times\mathbf{B})\times(\mathbf{G}\times\mathbf{C})]+\mathbf{B}\times[\mathbf{F}\times\mathbf{C})\times(\mathbf{G}\times\mathbf{A})+\mathbf{C}\times[(\mathbf{F}\times\mathbf{A})\times(\mathbf{G}\times\mathbf{B})]=0$.
15. (a) Prove that $[\mathbf{LMN}][\mathbf{ABC}]=\begin{vmatrix}\mathbf{L}\cdot\mathbf{A} & \mathbf{L}\cdot\mathbf{B} & \mathbf{L}\cdot\mathbf{C}\\ \mathbf{M}\cdot\mathbf{A} & \mathbf{M}\cdot\mathbf{B} & \mathbf{M}\cdot\mathbf{C}\\ \mathbf{N}\cdot\mathbf{A} & \mathbf{N}\cdot\mathbf{B} & \mathbf{N}\cdot\mathbf{C}\end{vmatrix}$
(b) The length of the edges OA, OB, OC of the tetrahedron $OABC$ are a, b, c and the angles BOC, COA, AOB are θ, ϕ, ψ, find its volume.

SOLID GEOMETRY

3.11 (1) EQUATION OF A PLANE

Let $P(x, y, z)$ be any point on the plane through $A(x_1, y_1 z_1)$ which is normal to the vector $\mathbf{N}=a\mathbf{I}+b\mathbf{J}+c\mathbf{K}$.

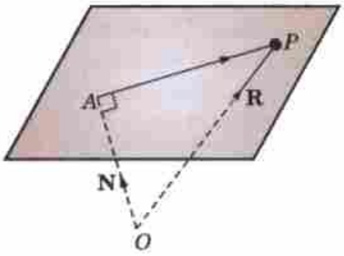


Fig. 3.35

Then $\quad \overrightarrow{OP}=x\mathbf{I}+y\mathbf{J}+z\mathbf{K}$ and $\overrightarrow{OA}=x_1\mathbf{I}+y_1\mathbf{J}+z_1\mathbf{K}$

Clearly the vectors $\overrightarrow{AP}=(x-x_1)\,\mathbf{I}+(y-y_1)\,\mathbf{J}+(z-z_1)\,\mathbf{K}$ and **N** are perpendicular to each other.

$$\therefore \qquad \overrightarrow{AP}\cdot\mathbf{N}=0 \qquad ...(i)$$

or $\quad [x-x_1)\,\mathbf{I}+(y-y_1)\,\mathbf{J}+(z-z_1)\,\mathbf{K}]\cdot(a\mathbf{I}+b\mathbf{J}+c\mathbf{K})=0$

or $\quad \mathbf{a(x-x_1)+b(y-y_1)+c(z-z_1)=0} \qquad ...(ii)$

which is *the equation of any plane through the point* (x_1, y_1, z_1).

Obs. Equation (ii) written as $ax+by+cz+d=0$ *is the general equation of a plane.*

Conversely, *every linear equation in x, y, z represents a plane and the coefficients of x, y, z are the direction ratios of the normal to the plane.*

Cor. IF l, m, n be the direction cosines of the normal to the plane, then

$$\mathbf{lx+my+nz=p} \qquad ...(iii)$$

which is called the **normal form** *of the equation of the plane* where p is the perpendicular distance from the origin.

(2) Angle between two planes. Def. *The angle between two planes is equal to the angle between their normals.*

Let the two planes be

$$ax+by+cz+d=0 \quad \text{and} \quad a'x+b'y+c'z+d'=0.$$

Now the direction ratio of their normals are a, b, c and a', b', c'.

Hence the angle θ between the planes is given by $\cos\theta = \dfrac{aa' + bb' + cc'}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{a'^2 + b'^2 + c'^2}}$

The planes will be perpendicular (if their normal are parallel), *i.e., if* $aa' + bb' + cc' = 0$

The planes will be parallel (if their normals are parallel), *i.e., if* $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$.

Cor. *Any plane parallel to the plane* $ax + by + cz + d = 0$

is $$\mathbf{ax + by + cz + k = 0} \qquad (k \text{ being any constant})$$

for the direction-ratios of their normals are the same.

(3) Perpendicular distance of the point (x_1, y_1, z_1) *from the plane*

$$ax + by + cz + d = 0 \qquad ...(i)$$

is $$\frac{\mathbf{ax_1 + by_1 + cz_1 + d}}{\sqrt{(\mathbf{a^2 + b^2 + c^2})}}$$

Let PL be the perpendicular distance of $P(x_1, y_1, z_1)$ from the plane (i) so that the direction cosines of $\vec{LP}$ are

$$\frac{a}{\sqrt{(\Sigma a^2)}}, \frac{b}{\sqrt{(\Sigma a^2)}}, \frac{c}{\sqrt{(\Sigma a^2)}}.$$

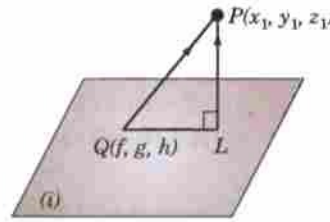


Fig. 3.36

If $Q(f, g, h)$ be a point on (i) then

$$af + bg + ch + d = 0 \qquad ...(ii)$$

$\therefore$ $$PL = \text{projection of } \vec{QP} \text{ on } \vec{LP} = \vec{QP} \,.\, \vec{LP}$$

$$= \frac{(x_1 - f)a + (y_1 - g)b + (z_1 - h)c}{\sqrt{(a^2 + b^2 + c^2)}} \qquad \text{[By } IX \text{ p. 82]}$$

$$= \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(a^2 + b^2 + c^2)}} \text{ by virtue of } (ii) \qquad ...(iii)$$

The sign of the radical in (iii) is taken to be positive or negative according as d is positive or negative.

Obs. The perpendicular to a plane from two points are taken to be of the same sign if the points lie on the same side and of different signs if they lie on the opposite sides of the plane.

$\therefore$ *The two points* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *lie on the same side or on opposite sides of the plane* $ax + by + cz + d = 0$, according as $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of the same sign or of opposite signs.

Cor. Planes bisecting the angles between two planes.

Let $$ax + by + cz + d = 0 \qquad ...(i)$$

and $$a'x + b'y + c'z + d' = 0 \qquad ...(ii)$$

be the given planes.

Let $P(x, y, z)$ be any point on either of the planes bisecting the angles between the planes (i) and (ii).

Then ⊥ distance of P from (i) = ⊥ distance of P from (ii),

$\therefore$ $$\frac{ax + by + cz + d}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{a'x + b'y + c'z + d'}{\sqrt{(a'^2 + b'^2 + c'^2)}}$$

which are the required equations of the bisecting planes.

Example 3.27. *Find the equation of the plane which*

(i) cuts off intercepts a, b, c from the axes.

(ii) passes through the points A(0, 1, 1), B(1, 1, 2) and C(–1, 2, –2).

Solution. (i) **Intercept form** *of the equation of the plane.* Let the required equation of the plane be

$$\alpha x + \beta y + \gamma z + \delta = 0 \qquad ...(1)$$

The plane cuts the axes at A, B, C such that $OA = a$, $OB = b$, $OC = c$,

i.e., it passes through the points $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$.

$\therefore$ $\alpha a + \delta = 0, \beta b + \delta = 0, \gamma c + \delta = 0$

whence $\alpha = -\delta/a, \beta = -\delta/b, \gamma = -\delta/c$

Substituting these values of α, β, γ in (1), $-\dfrac{\delta}{a}x - \dfrac{\delta}{b}y - \dfrac{\delta}{c}z + \delta = 0$ or $\dfrac{\mathbf{x}}{\mathbf{a}} + \dfrac{\mathbf{y}}{\mathbf{b}} + \dfrac{\mathbf{z}}{\mathbf{c}} = 1.$

(*ii*) **Three points form** *of the equation of the plane.*

Any plane through (0, 1, 1) is $a(x - 0) + b(y - 1) + c(z - 1) = 0$ ...(2)

It will pass through (1, 1, 2) and (–1, 2, – 2), if $a + c = 0$ and $-a + b - 3c = 0$.

By cross-multiplication, $\dfrac{a}{-1} = \dfrac{b}{2} = \dfrac{c}{1}$.

Substituting these values in (2), we get $-1 \cdot x + 2(y - 1) + 1(z - 1) = 0$

or $x - 2y - z + 3 = 0$, which is the required equation of the plane.

Example 3.28. *Find the equation of the plane which passes through the point (3, – 3, 1) and is*
(*i*) *parallel to the plane $2x + 3y + 5z + 6 = 0$.*
(*ii*) *normal to the line joining the points (3, 2, – 1) and (2, – 1, 5).* (V.T.V., 2006)
(*iii*) *Perpendicular to the planes $7x + y + 2z = 6$ and $3x + 5y - 6z = 8$.* (Cochin, 2005 ; V.T.U., 2005)

Solution. (*i*) Any plane parallel to the given plane is

$2x + 3y + 5z + k = 0$ which goes through (3, – 3, 1), if $k = -2$

Thus the required plane is $2x + 3y + 5z - 2 = 0$

(*ii*) Any plane through (3, – 3, 1) is $a(x - 3) + b(y + 3) + c(z - 1) = 0$

The direction cosines of the line joining the points (3, 2, – 1) and (2, – 1, 5) are proportional to 1, 3, – 6.

This line is normal to the plane (1). $\therefore$ a, b, c are proportional to 1, 3, – 6.

Substituting these values in (1), the required equation is

$1(x - 3) + 3(y + 3) - 6(z - 1) = 0$ or $x + 3y - 6z + 12 = 0$.

(*iii*) Any plane through (3, – 3, 1) is

$a(x - 3) + b(y + 3) + c(z - 1) = 0$ which will be $\perp$ to the planes

$7x + y + 2z = 6$ and $3x + 5y - 6z = 8$

if $7a + b + 2c = 0$ and $3a + 5b - 6c = 0$.

Solving these by cross-multiplication, we get $\dfrac{a}{1} = \dfrac{b}{-3} = \dfrac{c}{-2}$.

Hence the required equation is $1(x - 3) - 3(y + 3) - 2(z - 1) = 0$ or $x - 3y - 2z - 10 = 0$.

Example 3.29. *The plane $4x + 5y - z = 7$ is rotated through a right angle about its line of intersection with the plane $2x + 3y - 3z = 5$. Find the equation of this plane in its new position.*

Solution. Any plane through the line of intersection of

$4x + 5y - z = 7$ ...(*i*)

and $2x + 3y - 3z = 5$ ...(*ii*)

is $4x + 5y - z - 7 + k(2x + 3y - 3z - 5) = 0$

i.e., $(4 + 2k)x + (5 + 3k)y - (1 + 3k)z - (7 + 5k) = 0$ ...(*iii*)

Then new position of (*i*) when rotated through a right angle, is such that (*i*) and (*iii*) are perpendicular. This requires that

$4(4 + 2k) + 5(5 + 3k) + (1 + 3k) = 0$

i.e., $26k + 42 = 0$ or $k = -21/13$

Substituting $k = -21/13$ in (*iii*), we get $10x + 2y + 50z + 14 = 0$.

or $5x + y + 25x + 7 = 0$, which is the required plane.

Example 3.30. *Find the distance between the parallel planes $2x - 2y + z + 3 = 0$ and $4x - 4y + 2z + 9 = 0$. Find also the equation of the parallel plane that lies mid-way between the given planes.* (Madras, 2003)

Solution. The distance between the given planes is the perpendicular distance of any point on one of the planes from the other.

A point on the first plane is $(0, 0, -3)$.

$\therefore$ Required distance = $\perp$ distance of $(0, 0, -3)$ from $4x - 4y + 2z + 9 = 0$

$$= \frac{-6+9}{\sqrt{(16+16+4)}} = \frac{3}{6} = \frac{1}{2}$$

Let the equation of the parallel plane that lies mid-way between the given planes be

$$2x - 2y + z + k = 0 \qquad ...(i)$$

Now distance of any point $(0, 0, -3)$ on the first plane from (i) should be 1/4.

$$\therefore \qquad \pm \frac{-3+k}{\sqrt{(4+4+1)}} = 1/4 \quad i.e., \quad k = 15/4 \text{ or } 9/4.$$

Thus the required plane is $2x - 2y + z + 15/4 = 0$.

Assume that $k = 15/4$ and verify that the distance of a point on this plane $4x - 4y + 2z + 9 = 0$ is also 1/4.

A point on this plane is $(0, 0, -9/4)$. Its distance from the plane (i) = $\dfrac{-9/2 + 15/4}{3} = \dfrac{1}{4}$ (in magnitude)

Thus $k = 9/4$ is not admissible.

$\therefore$ The required plane is $2x - 2y + z + 15/4 = 0$.

Example 3.31. *A variable plane is at a constant distance p from the origin and meets the axes at A, B, C. Find the locus of the centroid of the tetrahedron OABC.*

Solution. As the given plane is at a $\perp$ distance p from the origin, therefore its equation is of the form

$$lx + my + nz = p \qquad ...(i) \quad \text{where } l, m, n \text{ are the d.c's of the } \perp \text{ from the origin.}$$

(i) may be rewritten as $\dfrac{x}{(p/l)} + \dfrac{y}{(p/m)} + \dfrac{z}{(p/n)} = 1$

so that $OA = p/l$, $OB = p/m$, $OC = p/n$.

$$\therefore \qquad A = (p/l, 0, 0), B = (0, p/m, 0), C = (0, 0, p/n).$$

Thus the coordinates of the centroid G of the tetrahedron $OABC$ are

$$(x_1, y_1, z_1) = (p/4l, p/4m, p/4n)$$ [See p. 81]

$$\therefore \qquad \frac{1}{x_1^2} + \frac{1}{y_1^2} + \frac{1}{z_1^2} = \frac{16}{p^2}(l^2 + m^2 + n^2) = \frac{16}{p^2}$$

Thus the locus of G is $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$.

Example 3.32. *A variable plane at a constant distance p from the origin meets the axes in A, B, C. Planes are drawn through A, B, C parallel to the coordinate planes, Show that the locus of their point of intersection is given by* $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

Solution. Let the variable plane be $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

Its distance from origin = $\dfrac{1}{\sqrt{(a^{-2} + b^{-2} + c^{-2})}} = p$ (given)

$$i.e., \qquad a^{-2} + b^{-2} + c^{-2} = p^{-2} \qquad ...(i)$$

Since $OA = a$, $OB = b$ and $OC = c$, therefore equations of the planes through A, B, C parallel to yz, zx and xy-planes are $x = a$, $y = b$, $z = c$

Let the point of intersection of these three planes be (x_1, y_1, z_1).

Then $$x_1 = a, y_1 = b, z_1 = c \qquad ...(ii)$$

Substituting (ii) in (i), we get $x_1^{-2} + y_1^{-2} + z_1^{-2} = p^{-2}$

Thus the locus of (x_1, y_1, z_1) is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

Example 3.33. *A variable plane passes through the fixed point (a, b, c) and meets the coordinate axes in A, B, C. Show that the locus of the point common to the planes through A, B, C parallel to the coordinate planes is*

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1.$$

Solution. Let ABC be any plane through the fixed point $H(a, b, c)$ such that $OA = x_1$, $OB = y_1$, $OC = z_1$. Then its equation is

$$\frac{x}{x_1}+\frac{y}{y_1}+\frac{z}{z_1}=1 \qquad \text{[See Ex. 3.27 (i)]}$$

Since H lies on it.

$$\therefore \qquad \frac{a}{x_1}+\frac{b}{y_1}+\frac{c}{z_1}=1. \qquad ...(1)$$

The planes through A, B, C parallel to the coordinate planes are $x = x_1$, $y = y_1$, $z = z_1$, which meet in $P(x_1, y_1, z_1)$.

Thus changing x_1 to x, y_1, to y and z_1 to z in the locus of the P is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1.$$

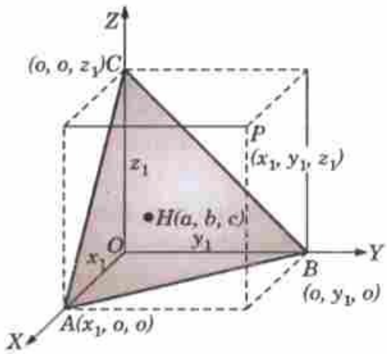


Fig. 3.37

Example 3.34. *Find the equations to the two planes which bisect the angles between the planes $3x - 4y + 5z = 3$, $5x + 3y - 4z = 9$.*

Also point out which of the planes bisects the acute angle. (V.T.U., 2007)

Solution. The equations of the planes bisecting the angles between the given planes are

$$\frac{3x-4y+5z-3}{\sqrt{[3^2+(-4)^2+5^2]}}=\pm\frac{5x+3y-4z-9}{\sqrt{[5^2+3^2+(-4)^2]}}$$

or $\qquad 2x + 7y - 9z - 6 = 0 \qquad ...(i)$

and $\qquad 8x - y + z - 12 = 0 \qquad ...(ii)$

which are the required planes.

Let θ be the angle between (*i*) and either of the given planes, say:

$$5x + 3y - 4z = 9.$$

Then, $$\cos\theta=\frac{2\times5+7\times3\,(-9)\times(-4)}{\sqrt{[2^2+7^2+(-9)^2]}\sqrt{[5^2+3^2+(-4)^2]}}=\frac{67}{5\sqrt{(268)}}$$

$$\therefore \qquad \tan\theta=\frac{\sqrt{2211}}{67} \text{ which is less than 1.}$$

i.e., $\qquad \theta < 45°.$

Now θ is half the angle between the given planes, so that (*i*) bisects that angle between the planes which is $2\theta < 90°$.

Hence the plane $2x + 7y - 9z = 6$, bisects the acute angle.

PROBLEMS 3.6

1. Find the equation of the plane passing through the point (1, 2, 3) and having the vector $\mathbf{N} = 2\mathbf{I} - \mathbf{J} + 3\mathbf{K}$ normal to it.
2. Find the equation of the plane through the points (3, – 1, 1), (1, 2, – 1) and (1, 1, 1).
3. Find a unit vector normal to the plane through the points (– 1, 2, 3), (1, 1, 1) and (2, – 1, 3).
4. Find the distance of the point (1, 4, 5) from the plane passing through the points (2, – 1, 5), (0, – 4, 1) and (2, – 6, 0). (*Rajasthan, 2006*)
5. Show that the four points (0, – 1, 0), (2, 1, – 1), (1, 1, 1) and (3, 3, 0) are coplanar. Find the equation of the plane through them. (*V.T.U., 2008*)

6. Show that the point (– 1/2, 2, 0) is the circumcentre of the triangle formed by the points (1, 1, 0), (1, 2, 1), (– 2, 2, – 1).
[**Hint.** Show that the point (– 1/2, 2, 0) lies in the plane of the triangle and is equidistant from its vertices.]
7. Find the equation of the plane through the point (2, 1, 0) and perpendicular to the planes $2x - y - z = 5$ and $x + 2y - 3z = 5$.
8. Find the equations of the plane through (0, 0, 0) parallel to the plane $x + 2y = 3z + 4$. *(Madras, 2006)*
9. Find the equation of the plane which bisects the join of the points (x_1, y_1, z_1) and (x_2, y_2, z_2) at right angles.
10. Find the equation of the plane through the points (– 1, 2, 1), (– 3, 2, – 3) and parallel to y-axis *(V.T.U., 2010)*
11. Find the equation of the plane through the points (2, 2, 1) and (9, 3, 6) and perpendicular to the plane $2x + 6y + 6z = 9$. *(V.T.U., 2004 ; Osmania, 1999)*
12. A plane contains the points $A(-4, 9, -9)$ and $B(5, -9, 6)$ and is perpendicular to the line which joins B and $C(4, -6, k)$. Evaluate k and find the equation of the plane.
13. Find the distance between the parallel planes
$$2x - 3y + 6z + 12 = 0 \text{ and } 6x - 9y + 18z - 6 = 0.$$
Also find the equation of the parallel plane that lies mid-way between the given planes.
14. Find the angle between the plane $x + y + z = 8$ and $2x + y - z = 3$. *(B.P.TU., 2006)*
15. Two planes are given by $x + 2y - 3z - 2 = 0$ and $2x + y + z + 3 = 0$, find
(*i*) direction cosines of their line of intersection,
(*ii*) acute angle between the planes, and
(*iii*) equation of the plane perpendicular to both of them through the point (2, 2, 1).
16. The plane $lx + my = 0$ is rotated about its line of intersection with the plane $z = 0$, through an angle α.
Prove that the equation of the plane is $lx + my + z\sqrt{(l^2 + m^2)} \tan \alpha = 0$. *(Anna, 2005 S)*
17. Find the equations of the two planes through the points (0, 4, – 3), (6, – 4, 3) other than the plane through the origin which cut off from the axes intercepts whose sum is zero.
18. A plane meets the coordinates axes at A, B, C, such that the centroid of the triangle ABC is the point (a, b, c), show that the equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$. *(Assam, 1999)*
19. A plane passes through a fixed point (a, b, c), show that the locus of the foot of the perpendicular from the origin on the plane is a sphere. *(P.T.U., 2005)*
20. A variable plane is at a constant distance p from the origin and meets the axes at A, B, C. Find the locus of the centroid of the triangle ABC. *(Rajasthan, 2005)*
21. A variable plane makes with the coordinate axes a tetrahedron of constant volume $64\ k^3$. Find the locus of the centroid of the tetrahedron. *(Rajasthan, 2006 ; Osmania, 2003)*
22. Find equations of the planes bisecting the angle between the planes
$$x + 2y + 2z = 9,\ 4x - 3y + 12z + 12 = 0$$
and specify the one which bisects the acute angle.

3.12 EQUATIONS OF A STRAIGHT LINE

(1) General form. Two linear equations in x, y, z

i.e., $$ax + by + cz + d = 0 \quad ...(i)$$
and $$a'x + b'y + c'z + d' = 0 \quad ...(ii)$$
taken together represent a straight line which is the line of intersection of the planes (*i*) and (*ii*). (Fig. 3.38).

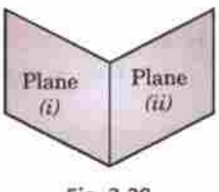


Fig. 3.38

(2) Symmetrical form. *Equations of the line through the point $A(x_1, y_1, z_1)$ and having direction cosines l, m, n are*
$$\frac{\mathbf{x} - \mathbf{x}_1}{\mathbf{l}} = \frac{\mathbf{y} - \mathbf{y}_1}{\mathbf{m}} = \frac{\mathbf{z} - \mathbf{z}_1}{\mathbf{n}}$$
Let $P(x, y, z)$ be any point on the given line through $A(x_1, y_1, z_1)$ and parallel to the unit vector $\mathbf{U} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$.

Since $\vec{AP}$ is parallel to $\mathbf{U}$, we can write $\vec{AP} = t\mathbf{U}$, where t is a parameter. ...(*i*) or $$(x - x_1)\,\mathbf{I} + (y - y_1)\,\mathbf{J} + (z - z_1)\,\mathbf{K} = t\,(l\mathbf{I} + m\mathbf{J} + n\mathbf{k})$$

$\therefore$ $$x - x_1 = tl,\ y - y_1 = tm,\ z - z_1 = tn \quad ...(ii)$$

Every point P on the line is given by (*ii*) for some value of t. Thus these are the parametric equations of the given line. Eliminating t, we get

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad ...(iii)$$

which are the *symmetrical form of the equations of the line.*

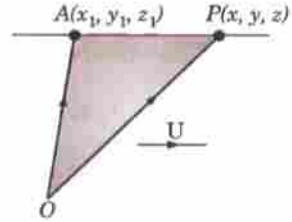


Fig. 3.39

Obs. Any point on the line (*iii*) is $(x_1 + lt, y_1 + mt, z_1 + nt)$.

Cor. *The equations of the line joining the points* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *are*

$$\frac{\mathbf{x} - \mathbf{x_1}}{\mathbf{x_2} - \mathbf{x_1}} = \frac{\mathbf{y} - \mathbf{y_1}}{\mathbf{y_2} - \mathbf{y_1}} = \frac{\mathbf{z} - \mathbf{z_1}}{\mathbf{z_2} - \mathbf{z_1}}$$

for the direction-ratios of the line joining the given points are

$$x_2 - x_1, y_2 - y_1, z_2 - z_1.$$

[**To reduce the general equation of a line of the symmetrical form:**

(*i*) *find a point on the line,* by putting $z = 0$ in the the given equations and solving the resulting equations for x and y.

(*ii*) *find the direction cosines of the line,* from the fact that it is perpendicular to the normals to the given planes.

(*iii*) *write the equations of the line in the symmetrical form.*]

Example 3.35. *Find in symmetrical form, the equations of the line*

$$x + y + z + 1 = 0,\ 4x + y - 2z + 2 = 0. \qquad \text{(Osmania, 1999)}$$

Solution. (*i*) To find a point on the line.

Putting $z = 0$ in the given equations, we have

$$x + y + 1 = 0;\ 4x + y + 2 = 0$$

Solving, $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{1}{-3}$ $\quad\therefore$ A point on the line is $(-1/3, -2/3, 0)$.

(*ii*) *To find the direction cosines l, m, n of the line.*

Since the line lies on both the given planes.

$\therefore$ It is perpendicular to their normals whose direction cosines are proportional to 1, 1, 1 and 4, 1, – 2.

i.e., $$l + m + n = 0;\ 4l + m - 2n = 0.$$

Solving, $$\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$$

$\therefore$ The direction cosines of the given line are proportional to – 1, 2, – 1.

(*iii*) Thus the equations of the line in the symmetrical form are

$$\frac{x + 1/3}{-1} = \frac{y + 2/3}{2} = \frac{z}{-1}.$$

Example 3.36. *Find the distance of the point (1, – 2, 3) from the plane $x - y + z = 5$ measured parallel to the line*

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{-6} \qquad \text{(Calicut, 1999)}$$

Solution. The line through $P(1, -2, 3)$ having direction ratios $(2, 3, -6)$ is

$$\frac{x - 1}{2} = \frac{y + 2}{3} = \frac{z - 3}{-6} = r.$$

Any point on this line is $(2r + 1, 3r - 2, 3 - 6r)$.

This point will lie on the plane $x - y + z = 5$

if $$2r + 1 - (3r - 2) + 3 - 6r = 5 \quad \text{or} \quad r = 1/7.$$

$\therefore$ The point of intersection is $Q(9/7, -11/7, 15/7)$

Thus the required distance = $PQ = \sqrt{\left(\frac{4}{49}+\frac{9}{49}+\frac{36}{49}\right)} = 1$

$x + 2y + 2z = 9$, $4x - 3y + 12z + 12 = 0$ and specify the one which bisects the acute angle.

Example 3.37. *(a) Find the image (reflection) of the point (p, q, r) in the plane $2x + y + z = 6$.*

(b) Find the image (reflection) of the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{4}$ in the same plane. (Delhi, 2002)

[If two points P, P′ be such that the line PP′ is bisected perpendicularly by a plane then either of the points is the image (or reflection) of the other in the plane.]

Solution. (a) Let $P'(p', q', r')$ be the image of $P(p, q, r)$. Then the mid-point of PP' must lie on the given plane.

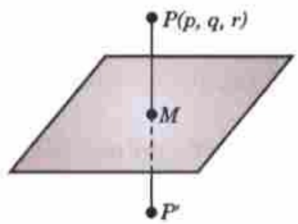


Fig. 3.40(a)

$$\therefore \qquad \frac{p+p'}{2}+\frac{q+q'}{2}+\frac{r+r'}{2}=6 \qquad ...(i)$$

Also the line PP' must be perpendicular to the plane. The direction ratios of PP' being $p - p', q - q', r - r'$, we therefore, have

$$\frac{p-p'}{2}=\frac{q-q'}{1}=\frac{r-r'}{1}=k \text{ (say)}$$

whence $p' = p - 2k$, $q' = q - k$, $r' = r - k$.

Substituting these in (*i*) and solving, we get

$$k=\frac{1}{3}(2p+q+r-6).$$

Hence P' is

$$\left[\frac{1}{3}(12-p-2q-2r), \frac{1}{3}(6-2p+2q-r), \frac{1}{3}(6-2p-q+2r)\right] \qquad ...(ii)$$

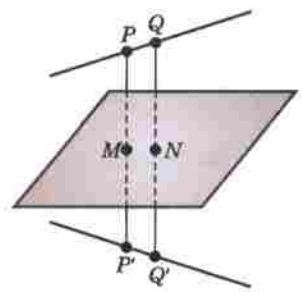


Fig. 3.40(b)

(*b*) Any two points on the given line are evidently $P(1, 2, 3)$ and (on putting $z = 7$) $Q(3, 3, 7)$. Their images are [by using (*ii*)] $P'\left(\frac{1}{3}, \frac{5}{3}, \frac{8}{3}\right)$ and $Q'\left(-\frac{11}{3}, -\frac{1}{3}, \frac{11}{3}\right)$. The line joining P' and Q' is, therefore

$$\frac{x-\frac{1}{3}}{-\frac{11}{3}-\frac{1}{3}}=\frac{y-\frac{5}{3}}{-\frac{1}{3}-\frac{5}{3}}=\frac{z-\frac{8}{3}}{\frac{11}{3}-\frac{8}{3}}, \text{ i.e., } \frac{3x-1}{4}=\frac{3y-5}{2}=\frac{3z-8}{-1}$$

which is the required image of the given line PQ [Fig. 3.40(*b*)].

Example 3.38. *Find the angle between the line*

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

and the plane $ax + by + cz + d = 0$.

Solution. If θ be the angle between the line and the plane, then $90° - \theta$ is the angle between the line and the normal to the plane (Fig. 3.41).

Now the direction ratios of the line are l, m, n and the direction ratios of the normal to the plane are a, b, c.

$$\therefore \qquad \cos(90° - \theta) = \frac{la+mb+nc}{\sqrt{(l^2+m^2+n^2)}\sqrt{(a^2+b^2+c^2)}}$$

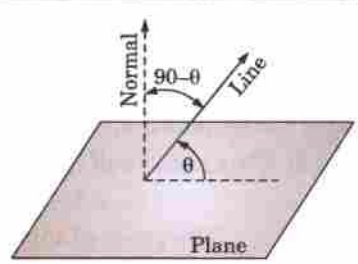


Fig. 3.41

or $$\sin\theta = \frac{la + mb + nc}{\sqrt{(\Sigma l^2)}\sqrt{(\Sigma a^2)}}$$

Hence the required angle $\theta = \sin^{-1}\left(\frac{al + bm + cn}{\sqrt{(\Sigma l^2)}\sqrt{(\Sigma a^2)}}\right)$

Cor. *If the line is parallel to the plane, sin θ = 0*

$\therefore$ $$\mathbf{al + bm + cn = 0}$$

If the line is perpendicular to the plane, it will be parallel to its normal.

$\therefore$ $$l/a = m/b = n/c.$$

Example 3.39. *Find the equations of the two straight lines through the origin, each of which intersects the straight line* $\frac{1}{2}(x-3) = y - 3 = z$ *and is inclined at an angle of 60° to it.*

Solution. Let AB be the given line so that any point A on it is $(2r+3, r+3, r)$. (Fig. 3.42)

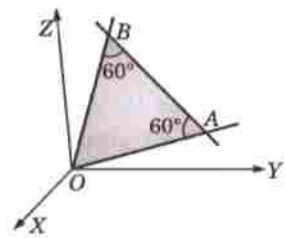


Fig. 3.42

$\therefore$ Direction ratios of OA are $2r+3-0, r+3-0, r-0$.

Angle between AO and AB has to be 60°,

$\therefore$ $$\cos 60° = \frac{2(2r+3) + 1(r+3) + 1.(r)}{\sqrt{(2^2+1^2+1^2)}\sqrt{[(2r+3)^2 + (r+3)^2 + r^2]}}$$

or $$\frac{1}{2} = \frac{6r+9}{\sqrt{[6(6r^2+18r+18)]}} \text{ or } r^2 + 3r + 2 = 0 \text{ i.e., } r = -1, -2$$

$\therefore$ Coordinates of A and B are $(1, 2, -1)$ and $(-1, 1, -2)$.

Hence the equations of the required lines OA and OB are $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ and $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$

PROBLEMS 3.7

1. Prove that the points (3, 2, 4), (4, 5, 2) and (5, 8, 0) are collinear. Find the equations of the line through them.
2. Find the angle between the line of intersection of the planes
   $2x + 2y - z + 15 = 0$, $4y + z + 29 = 0$ and the line $\frac{x+4}{4} = \frac{y-3}{-3} = \frac{z+2}{1}$. (V.T.U., 2003 S)
3. Find the angle between the line of intersection of the planes $3x + 2y + z = 5$ and $x + y - 2z = 3$ and the line of intersection of the plane $2x = y + z$ and $7x + 10y = 8z$.
4. Find the equation of the line through the point $(-2, 3, 4)$ and parallel to the planes $2x + 3y + 4z = 5$ and $4x + 3y + 5z = 6$.
5. Show that the line $\frac{x-1}{3} = \frac{y+2}{-2} = \frac{z-1}{2}$ is parallel to the plane $2x + 2y - z = 6$, and find the distance between them.
6. Find the equation of the line through $(1, 2, -1)$ perpendicular to each of the lines
   $$\frac{x}{1} = \frac{y}{0} = \frac{z}{-1} \text{ and } \frac{x}{3} = \frac{y}{4} = \frac{z}{5}.$$
7. Find the equation of the lines bisecting the angle between the lines
   $$\frac{x-1}{2} = \frac{y+2}{-2} = \frac{z-3}{1}; \frac{x-1}{12} = \frac{y+2}{4} = \frac{z-3}{-3}.$$
8. Find the foot of the perpendicular from (1, 1, 1) to the line joining the points (1, 4, 6) and (5, 4, 4). (V.T.U., 2010)
9. Find the perpendicular distance of the point (1, 1, 1) from the line
   $$\frac{x-2}{2} = \frac{y+3}{2} = \frac{z}{-1}.$$

10. Find the distance of the point $(3, -4, 5)$ from the plane $2x + 5y - 6z = 16$, measured parallel to the line $x/2 = y/1 = z/-2$. (V.T.U., 2002)
11. Find the reflection (image) of the point
    (i) $(1, 2, 3)$ in the plane $x + y + z = 9$. (V. T.U., 2010)
    (ii) $(2, -1, 3)$ in the plane $3x - 2y - z - 9 = 0$.
12. Find the angle between the line $\dfrac{x+1}{2} = \dfrac{y}{3} = \dfrac{z-3}{6}$ and the plane $3x + y + z = 7$.
13. Find the equation of the plane through the points $(1, 0, -1)$, $(3, 2, 2)$ and parallel to the line
    $$x - 1 = \frac{1}{2}(1 - y) = \frac{1}{3}(z - 2).$$ (V.T.U., 2000)
14. Find the equations of the straight line which passes through the point $(2, -1, -1)$, is parallel to the plane $4x + y + z + 2 = 0$ and is perpendicular to the line $2x + y = 0 = x - z + 5$.

3.13 CONDITIONS FOR A LINE TO LIE IN A PLANE

To find the conditions that the line $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$ ...(1)

may lie in the plane $ax + by + cz + d = 0$ ...(2)

Any point on the line (1) is $(lr + x_1, mr + y_1, nr + z_1)$ which will lie on the plane (2), if

$$a(lr + x_1) + b(mr + y_1) + c(nr + z_1) + d = 0.$$

or if
$$(al + bm + cn)\, r + (ax_1 + by_1 + cz_1 + d) = 0 \qquad ...(3)$$

The line (1) will lie in the plane (2), if every point of the line lies in the plane so that (3) is satisfied by all values of r.

$\therefore$ The coefficient of $r = 0$ and the constant term $= 0$.

i.e.,
$$\mathbf{al + bm + cn = 0} \qquad ...(4)$$

and
$$\mathbf{ax_1 + by_1 + cz_1 + d = 0} \qquad ...(5)$$

These are the required conditions which state that

(i) *the line should be parallel to the plane,* (ii) *a point of line should lie in the plane.*

Thus the equation of any plane through the line $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$

is
$$\mathbf{a(x - x_1) + b(y - y_1) + c(z - z_1) = 0} \textit{ where } \mathbf{al + bm + cn = 0.}$$

Obs. The equation of any plane through the line of intersection of the planes

$$ax + by + cz + d = 0 \qquad ...(i)$$

and
$$a'x + b'y + c'z + d' = 0. \qquad ...(ii)$$

is
$$\mathbf{ax + by + cz + d + k(a'x + b'y + c'z + d') = 0.}$$

For (i) is an equation of the first degree in x, y, z representing a plane and (ii) it is satisfied by the coordinates of the points which satisfy both the given planes, *i.e.*, it contains all the points common to these planes.

Example 3.40. *Obtain the equation of a plane passing through the line of intersection of the planes $7x - 4y + 7z + 16 = 0$ and $4x + 3y - 2z + 13 = 0$ and perpendicular to the plane $x - y - 2z + 5 = 0$.* (V.T.U., 2009)

Solution. The equation of any plane through the line of intersection of the two given planes is

$$7x - 4y + 7z + 16 + k(4x + 3y - 2z + 13) = 0$$

or
$$(7 + 4k)x + (-4 + 3k)y + (7 - 2k)z + (16 + 13k) = 0 \qquad ...(i)$$

The plane (i) will be perpendicular to the plane

$x - y - 2z + 5 = 0$ if their normals are perpendicular,

i.e., if $(7 + 4k) \,.\, 1 + (-4 + 3k) \,.\, (-1) + (7 - 2k) \,.\, (-2) = 0$ or if, $k = 3/5$.

Substituting this value of k in (i), we get

$$(7 + 12/5)x + (-4 + 9/5)y + (7 - 6/5)x + (16 + 39/5) = 0$$

or
$$47x - 11y + 29z + 119 = 0 \text{ which is the required equation.}$$

Example 3.41. *Find the equation in the symmetrical form of the projection of the line* $\frac{x-1}{2} = -(y+1) = \frac{z-3}{4}$ *on the plane* $x + 2y + z = 12$.

Solution. Any plane through the given line is

$$A(x-1) + B(y+1) + C(z-3) = 1 \qquad ...(i)$$

where
$$2A - B + 4C = 0 \qquad ...(ii)$$

The plane (*i*) will be perpendicular to the given plane, if

$$A + 2B + C = 0 \qquad ...(iii)$$

Solving (*ii*) and (*iii*), we get $\frac{A}{-9} = \frac{B}{2} = \frac{C}{5}$.

Substituting these values in (*i*), we get $9x - 2y - 5z + 4 = 0$ ...(*iv*)

which cuts the given plane $x + 2y + z = 12$ ...(*v*)

along the required line of projection.

One point on this line is got by putting $z = 0$ in (*iv*) and (*v*) and solving, it is (4/5, 28/5, 0).

The direction ratios of the line are found, by solving

$$l + 2m + n = 0 \quad \text{and} \quad 9l - 2m - 5n = 0$$

to be 4, – 7, 10.

Hence the required equations of the line of projection are

$$\frac{x - 4/5}{4} = \frac{y - 28/5}{-7} = \frac{z}{10}$$

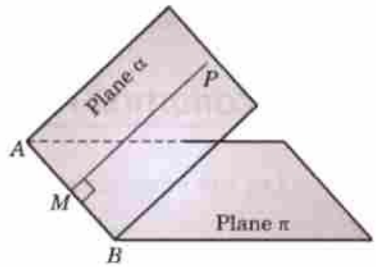


Fig. 3.43

[**The line of greatest slope** in a plane is a line which lies in the plane and is perpendicular to the line of intersection of the plane with the horizontal plane.

In Fig. 3.43, *AB* is the line of intersection of the given plane α with the horizontal plane π. Then *PM* drawn perpendicular to *AB*, is the line of greatest slope on the plane α through the point *P*.]

Example 3.42. *Assuming the line* $x/4 = y/-3 = z/7$ *as vertical, find the equations of the line of greatest slope in the plane* $2x + y - 5z = 12$ *and passing through the point* (2, 3, – 1).

Solution. The equation of the horizontal plane through the origin is $4x - 3y + 7z = 0$ ...(*i*)

[The direction ratios of the normal are those of the given vertical line.]

If l, m, n be the direction ratios of the line of intersection of the plane (*i*) and

$$2x + y - 5z = 12 \qquad ...(ii)$$

then solving, $4l - 3m + 7n = 0$ and $2l + m - 5n = 0$, we have $l/4 = m/17 = n/5$ ...(*iii*)

Let l', m', n' be the direction ratios of the line of greatest slope which lies in the plane (*ii*).

$$\therefore \qquad 2l' + m' - 5n' = 0 \qquad ...(iv)$$

Also the line of greatest slope is perpendicular to the line of intersection of the planes (*i*) and (*ii*).

$$\therefore \qquad 4l' + 17m' + 5n' = 0 \qquad ...(v)$$

Solving (*iv*) and (*v*), $\frac{l'}{3} = \frac{m'}{-1} = \frac{n'}{1}$.

Hence the equations of the line of greatest slope through (2, 3, – 1) and having direction ratios 3, – 1, 1 are

$$\frac{x-2}{3} = \frac{y-3}{-1} = \frac{z+1}{1}.$$

PROBLEMS 3.8

1. Find the equation of the plane which contains the line $\frac{x-1}{2} = y + 1 = \frac{z-3}{4}$ and is perpendicular to the plane $x + 2y + z = 12$. (*V.T.U., 2006*)

2. Find the equation of the plane through the line $\frac{x-1}{3}=\frac{y-4}{2}=\frac{z-4}{-2}$ and parallel to the line $\frac{x+1}{3}=\frac{y-1}{-4}=\frac{z+2}{1}$.
3. Find the equation of the plane passing through the line of intersection of the planes $x+y+z=1$ and $2x+3y-z+4=0$ and perpendicular to the plane $2y-3z=4$.
4. Find the equation of the plane which contains the line of intersection of the planes $x+y+z=3$ and $2x-y+3z=4$ and is parallel to the line joining the points (2, 1, 1) and (3, 2, 4). *(Madras, 2006)*
5. Find in symmetric form the equations of the line which lies in the plane $2x-y-3z=4$ and is perpendicular to the line
$$\frac{x+1}{3}=\frac{y-1}{3}=\frac{z+4}{2}$$
at the point where the line pierces the plane.
6. A plane is drawn through the line $x+y=1, z=0$ to make an angle $\sin^{-1}(1/3)$ with the plane $x+y+z=0$. Prove that two such planes can be drawn and find their equations. Prove also that the angle between the planes is $\cos^{-1}(7/9)$.
7. Find the equations of the projection of the line $3x-y+2z-1=x+2y-z-2=0$ on the plane $3x+2y+z=0$ in the symmetrical form.
8. Assuming the plane $4x-3y+7z=0$ to be horizontal, find the equations of the line of greatest slope through the point (2, 1, 1) in the plane $2x+y-5z=0$. *(Roorkee, 2000)*

3.14 CONDITION FOR THE TWO LINES TO INTERSECT (OR TO BE COPLANAR)

Let the equations of the lines be $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ ...(1)

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \quad ...(2)$$

The equation of any plane through the line (1) is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ ...(3)

where $al_1+bm_1+cn_1=0$ ...(4)

The line (2) will lie in the plane (3), if it is parallel to the plane and its point (x_2, y_2, z_2) lies on this plane.

$\therefore$ $al_2+bm_2+cn_2=0$ ...(5)

and $a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0$ ...(6)

Eliminating a, b, c from (6), (4) and (5), we get

$$\begin{vmatrix} \mathbf{x_2-x_1} & \mathbf{y_2-y_1} & \mathbf{z_2-z_1} \\ \mathbf{l_1} & \mathbf{m_1} & \mathbf{n_1} \\ \mathbf{l_2} & \mathbf{m_2} & \mathbf{n_2} \end{vmatrix} = 0 \text{ which is the required condition.}$$

Also eliminating a, b, c from (3), (4) and (5), we get $\begin{vmatrix} \mathbf{x-x_1} & \mathbf{y-y_1} & \mathbf{z-z_1} \\ \mathbf{l_1} & \mathbf{m_1} & \mathbf{n_1} \\ \mathbf{l_2} & \mathbf{m_2} & \mathbf{n_2} \end{vmatrix} = \mathbf{0}$

which is the equation of the plane containing the lines (1) and (2).

Example 3.43. *Show that the lines* $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$; $\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$ *are coplanar; find their common point and the equation of the plane in which they lie.* *(Madurai, 2002)*

Solution. Any point on the first line is $(5+4r, 7+4r, -3-5r)$ ...(*i*)

which lies on the second line if $\frac{-3+4r}{7}=\frac{3+4r}{7}=\frac{-8-5r}{3}$ ...(*ii*)

$\therefore$ From $\frac{-3+4r}{7}=3+4r$, we have $r=-1$.

This value clearly satisfies the equation $\frac{3+4r}{7}=\frac{-8-5r}{3}$

Hence the lines intersect, (*i.e.*, are coplanar) and from (*i*) their point of intersection is (1, 3, 2).

The equation of the plane in which they lie, is $\begin{vmatrix} x-5 & y-7 & z+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$

i.e., $17x - 47y - 24z + 172 = 0.$

Example 3.44. *Show that the lines*

$$\frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2} \text{ and } 3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$$

are coplanar. Find their point of intersection and the plane in which they lie.

Solution. Any point on the first line is $P(3r-4, 5r-6, -2r+1)$, which lie in the plane

$$3x - 2y + z + 5 = 0$$

if $3(3r-4) - 2(5r-6) + (-2r+1) + 5 = 0$ or $r = 2,$

The point P will also lie in the plane $2x + 3y + 4z - 4 = 0$

if $2(3r-4) + 3(5r-6) + 4(-2r+1) - 4 = 0$ or $r = 2.$

Since the two values of r are equal, the given lines intersect, *i.e.*, are coplanar.

Putting $r = 2$ in the coordinates of P, we get $(2, 4, -3)$ as their point of intersection.

The equation of a plane containing the second line is

$$3x - 2y + z + 5 + k(2x + 3y + 4z - 4) = 0$$

which will contain the first line if its point $(-4, -6, 1)$ lies on it.

$\therefore$ $-12 + 12 + 1 + 5 + k(-8 - 18 + 4 - 4) = 0$

i.e., $k = 3/13$

Substituting this value of k, (i) becomes $45x - 17y + 25z + 53 = 0$, which is the required plane.

Example 3.45. *Find the equations of the line drawn through the point $(1, 0, -1)$ and intersecting the lines $x = 2y = 2z$ and $3x + 4y = 1, 4x + 5z = 2.$* *(V.T.U., 2007)*

Solution. The required line will comprise of

(a) the plane containing the first line and the point $(1, 0, -1)$.

(b) the plane containing the second line and the point $(1, 0, -1)$.

The equation of any plane containing the first line

i.e., $\frac{x}{2} = \frac{y}{1} = \frac{z}{1}$

is $a(x-0) + b(y-0) + c(z-0) = 0$ ...(i)

where $2a + b + c = 0$ ...(ii)

Also $(1, 0, -1)$ lies on (i) $\therefore$ $a - c = 0$ ...(iii)

Solving (ii) and (iii), we have $\frac{a}{1} = \frac{b}{-3} = \frac{c}{1}.$

Substituting these values in (i), we get $x - 3y + z = 0$ ...(iv)

Again, the equation of any plane containing the second line is

$3x + 4y - 1 + k(4x + 5z - 2) = 0$. Also $(1, 0, -1)$ lies on it. ...(v)

$\therefore$ $3 + 0 - 1 + k(4 - 5 - 2) = 0,$ *i.e.,* $k = \frac{2}{3}.$

Substituting $k = 2/3$ in (v), we get $17x + 12y + 10z - 7 = 0$ ...(iv)

Hence (iv) and (vi) constitute the required line.

PROBLEMS 3.9

1. Prove that the lines
$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$$
are coplanar and find the equation of the plane containing them.
2. Prove that the lines $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{1+z}{7}$ and $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ intersect and find the coordinates of their point of intersection. (*V.T.U., 2000 S ; Andhra, 1999*)
3. Find the condition that the lines $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and $ax+by+cz+d=0=a'x+b'y+c'z+d'$ are coplanar.
4. Show that the lines $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}$ and $x+2y+3z-8=0=2x+3y+4z-11$ intersect. Find their point of intersection and the equation of the plane containing them. (*V.T.U., 2009*)
5. Show that the lines $x-3y+2z-4=0=2x+y+4z+1$ and
$3x+2y+5z-1=0=2y-z$, are coplanar. (*Andhra, 2000*)
6. Prove that the lines $x=ay+b=cz+d$ and $x=\alpha y+\beta=\gamma z+\delta$ are coplanar if
$(\gamma-c)(a\beta-b\alpha)-(\alpha-a)(c\delta-d\gamma)=0$ (*Rajasthan, 2006*)
7. Obtain the equations of the straight line lying in the plane.
$$x-2y+4z-51=0$$
and interesecting the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-6}{7}$ at right angles.
8. Find the equation of the straight line perpendicular to both the lines
$$\frac{x-1}{1}=\frac{y-1}{2}=\frac{z+2}{3} \quad \text{and} \quad \frac{x+2}{2}=\frac{y-5}{-1}=\frac{z+3}{2}$$
and passing through their point of intersection.
9. A line with direction cosines proportional to 2, 7, -5 is drawn to intersect the lines
$$\frac{x-8}{3}=\frac{y-6}{-1}=\frac{z+1}{1} \quad \text{and} \quad \frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}.$$
Find the coordinates of the point of intersection and the length intercepted.

3.15 SHORTEST DISTANCE BETWEEN TWO LINES

[Two straight lines which do not lie in one plane are called *skew lines*. Such lines possess a common perpendicular which is the *shortest distance* between them.

Let the given skew lines AB and CD be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \quad \text{and} \quad \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

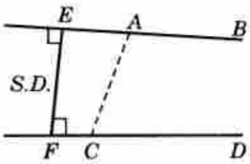


Fig. 3.44

so that $A\equiv(x_1,y_1,z_1)$ and $C\equiv(x_2,y_2,z_2)$.

Let l, m, n be the direction cosines of the shortest distance EF.

Since $EF \perp$ to both AB and CD.

$\therefore$ $ll_1+mm_1+nn_1=0$ and $ll_2+mm_2+nn_2=0$.

Solving,
$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}$$
$$=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{[\Sigma(m_1n_2-m_2n_1)^2]}}=\frac{1}{\sin\theta} \quad ...(1)$$

where θ is the angle between the lines AB and CD.

$\therefore$ *Length of S.D.* (EF) = projection of AC on EF

$=l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$ where l, m, n have the values as given by (1).

To find the equations of the line of shortest distance, we observe that it is coplanar with both AB and CD.

Plane containing the lines AB and EF is, $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0$ ...(2)

Plane containing the lines CD and EF is $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0$ ...(3)

Hence (2) and (3) are the equations of the line of shortest distance.

Obs. The condition for the given lines to be coplanar is also obtained by equating the shortest distance (EF) to zero.

Example 3.46. *Find the magnitude and the equations of the shortest distance between the lines*

$$\frac{x}{2}=\frac{y}{-3}=\frac{z}{1} \quad \text{and} \quad \frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}.$$ *(V.T.U., 2009 ; Cochin, 2005)*

Solution. Let l, m, n be the direction cosines of the shortest distance EF.

$\because$ $EF \perp$ to both AB and CD,

$\therefore$ $2l - 3m + n = 0, 3l - 5m + 2n = 0.$

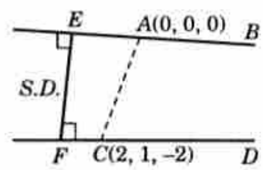


Fig. 3.45

Solving $$\frac{l}{1}=\frac{m}{1}=\frac{n}{1}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(1+1+1)}}=\frac{1}{\sqrt{3}}.$$

$\therefore$ *Length of S.D.* (EF) = projection of AC on EF

$$= (2-0)\frac{1}{\sqrt{3}}+(1-0)\frac{1}{\sqrt{3}}+(-2-0)\frac{1}{\sqrt{3}}=\frac{1}{\sqrt{3}}.$$

The equations of the line of shortest distance (EF) are

$$\begin{vmatrix} x & y & z \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

i.e., $4x + y - 5z = 0$ and $7x + y - 8z = 31.$

Example 3.47. *Find the points on the lines*

$$\frac{x-6}{3}=\frac{y-7}{-1}=\frac{z-4}{1} \quad ...(i)$$

$$\frac{x}{-3}=\frac{y+9}{2}=\frac{z-2}{4} \quad ...(ii)$$

which are nearest to each other. Hence find the shortest distance between the lines and its equations.

(V.T.U., 2004 ; Burdwan, 2003 ; Osmania, 2003)

Solution. Any point on the line (*i*) is $E(6 + 3r, 7 - r, 4 + r)$ ...(*iii*)

and any point on the line (*ii*) is $F(-3r', -9 + 2r', 2 + 4r')$ ...(*iv*)

Then the direction cosines of EF are proportional to $6 + 3r + 3r', 16 - r - 2r', 2 + r - 4r'$

Since $EF \perp$ both the lines (*i*) and (*ii*), $\therefore$ $3(6 + 3r + 3r') - (16 - r - 2r') + (2 + r - 4r') = 0$

and $-3(6 + 3r + 3r') + 2(16 - r - 2r') + 4(2 + r - 4r') = 0$

or $11r + 7r' + 4 = 0, 7r + 29r' - 22 = 0$, whence $r = -1, r' = 1.$

Substituting $r = -1$ in (*iii*) and $r' = 1$ in (*iv*), we get $E = (3, 8, 3)$ and $F = (-3, -7, 6)$ which are the points on (*i*) and (*ii*) nearest to each other.

$\therefore$ Length of the shortest distance (EF) = $\sqrt{[(3+3)^2+(8+7)^2+(3-6)^2]} = 3\sqrt{30}$

The equations of the shortest distance (EF) is $\dfrac{x-3}{2}=\dfrac{y-8}{5}=\dfrac{z-3}{-1}.$

Obs. This method is sometimes very convenient and is especially useful when the points of intersection of the line of shortest distance with the given lines are required.

Example 3.48. *Two control cables in the form of straight lines AB and CD are laid such that the coordinates of A, B, C and D are respectively (1, 2, 3), (2, 1, 1), (– 1, 1, 2) and (2, – 1, –3). Determine the amount of clearance between the cables.*

Solution. The direction ratios of AB are $1, -1, -2$ and those of CD are $3, -2, -5$.

The amount of clearance between AB and CD is nothing but the shortest distance PQ between the cables. If the direction cosines of PQ be l, m, n then

$$l - m - 2n = 0 \text{ and } 3l - 2m - 5n = 0$$

$$\therefore \qquad \frac{l}{1} = \frac{m}{-1} = \frac{n}{1} \qquad [\because \ PQ \perp \text{ to both } AB + CD].$$

Thus the clearance between the cables

$$= \text{shortest distance between } AB \text{ and } CD$$

$$= \text{projection of } AC \text{ (or } BD\text{) on } PQ$$

$$= \frac{1(-1-1) - 1(1-2) + 1(2-3)}{\sqrt{(1+1+1)}} = \frac{2}{\sqrt{3}} \text{ (in magnitude)}$$

Example 3.49. *Find the equation of the plane through the line*

$$\frac{x-1}{3} = \frac{y-4}{2} = \frac{z-4}{-2} \qquad ...(i)$$

and parallel to the line $\dfrac{x+1}{2} = \dfrac{y-1}{-4} = \dfrac{z+2}{1}$ ...(*ii*)

Hence find the shortest distance between them (*Hazaribagh, 2009*)

Solution. The equation of the plane containing the line (*i*) and parallel to (*ii*) is

$$\begin{vmatrix} x-1 & y-4 & z-4 \\ 3 & 2 & -2 \\ 2 & -4 & 1 \end{vmatrix} = 0$$

or $$6x + 7y + 16z = 98 \qquad ...(iii)$$

Now the shortest distance between the lines (*i*) and (*ii*)

= Length of the perpendicular drawn from the point (– 1, 1, – 2) of (*ii*) on the plane (*iii*)

$$= \frac{-6 + 7 - 32 - 98}{\sqrt{(6^2 + 7^2 + 16^2)}} = \frac{120}{\sqrt{(341)}}, \text{ numerically.}$$

Example 3.50. *Show that the shortest distance between z-axis and the line* $ax + by + cz + d = 0 = a'x + b'y + c'z + d'$ *is* $\dfrac{dc' - d'c}{\sqrt{[(ac' - a'c)^2 + (bc' - b'c)^2]}}$.

Solution. The plane containing the given line is

$$(ax + by + cz + d) + k(a'x + b'y + c'z + d') = 0 \qquad ...(i)$$

or $$(a + ka')x + (b + kb')y + (c + kc')z + (d + kd') = 0$$

This plane is parallel to the z-axis (d. c's, 0, 0, 1) if $c + kc' = 0$ or $k = -c/c'$. Then (*i*) becomes

$$(ac' - a'c)x + (bc' - b'c)y + (dc' - d'c) = 0 \qquad ...(ii)$$

A point on the z-axis is the origin.

$\therefore$ $\perp$ distance of the origin from the plane (*ii*)

$$= \frac{dc' - d'c}{\sqrt{[(ac' - a'c)^2 + (bc' - b'c)^2]}} \text{ which is the required S.D.}$$

Example 3.51. *A square ABCD of diagonal 2a is folded along the diagonal AC, so that the planes DAC and BAC are at right angles. Find the shortest distance between DC and AB.*

Solution. Let the diagonals AC and BD intersect at O the folded position of the square. Let OB, OC and OD be the axes. Then equations of DC are

$$\frac{x-0}{0-0}=\frac{y-a}{a-0}=\frac{z-0}{0-a} \quad \text{or} \quad \frac{x}{0}=\frac{y-a}{a}=\frac{z}{-a}$$

and those of AB are $\dfrac{x-a}{a}=\dfrac{y}{a}=\dfrac{z}{0}$

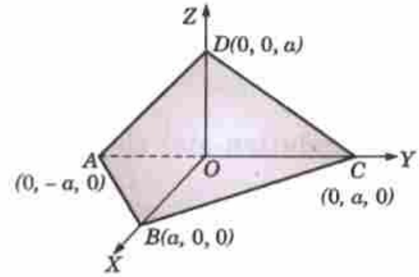


Fig. 3.46

The equation of the plane through DC and parallel to AB is

$$\begin{vmatrix} x & y-a & z \\ 0 & a & -a \\ a & a & 0 \end{vmatrix} = 0 \quad \text{or} \quad x-y-z+a=0 \qquad \ldots(i)$$

A point on the line AB is $(a, 0, 0)$.

Hence required S.D. = $\perp$ distance of $(a, 0, 0)$ from the plane (i)

$$= \frac{a+a}{\sqrt{1+1+1}} = \frac{2a}{\sqrt{3}}.$$

PROBLEMS 3.10

1. Find the magnitude and the equations of the shortest distance between the lines.
$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}.$$
(V.T.U., 2008 ; Rajasthan, 2005 ; Madras, 2003)
2. Find the magnitude and equations of the shortest distance between the lines
$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1} \text{ and } \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$$
(Anna, 2005 S ; Osmania, 2000 S)
Find also the points where it intersects the lines.
3. Find the shortest distance and the equation of the line of shortest distance between the line $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}$ and the y-axis. *(V.T.U., 2010)*
4. Show that the shortest distance between the lines $y-mx=0=z-c$ and $y+mx=0=z+c$ is c units.
5. If the shortest distance between the lines $\dfrac{y}{b}+\dfrac{z}{c}=1, x=0$ and $\dfrac{x}{a}-\dfrac{z}{c}=1, y=0$ be $2d$, then show that $d^{-2}=a^{-2}+b^{-2}+c^{-2}$.
6. Show that the shortest distance between x-axis and the line $ax+by+cz+d=0=a'x+b'y+c'z+d'$ is
$$\frac{da'-d'a}{\sqrt{[(ba'-b'a)^2+(ca'-c'a)^2]}}.$$
7. Show that the shortest distance between a diagonal of a rectangular parallelopiped whose edges are a, b, c and the edges not meeting it, are
$bc/(b^2+c^2)^{1/2}, ca/(c^2+a^2)^{1/2}, ab/(a^2+b^2)^{1/2}$.
8. Show that the shortest distance between two opposite edges of the tetrahedron formed by the planes $x+y=0$, $y+z=0, z+x=0$ and $x+y+z=a$ is $2a/\sqrt{6}$.

3.16 INTERSECTION OF THREE PLANES

Any three planes (no two of which are parallel) intersect in one of the following ways :

(1) *The planes may meet in a point,* if the line of section of two of them is not parallel to the third.

(2) *The planes may have a common line of section,* if the line of section of two of them lies on the third (Fig. 3.47).

(3) *The planes may form a triangular prism,* if the line of section of two of them is parallel to the third but does not lie on it. (See Fig. 3.48)

Example 3.52. *Prove that the planes*

(i) $12x - 15y + 16z - 28 = 0$, *(ii)* $6x + 6y - 7z - 8 = 0$. *and (iii)* $2x + 35y - 39z + 92 = 0$,

have a common line of intersection. Prove that the point in which the line $\frac{x-1}{3} = \frac{y}{-2} = \frac{z-3}{1}$ *meets the third plane is equidistant from other two planes.*

Solution. Any plane through the line of intersection of the planes (*i*) and (*ii*) is

$$12x - 15y + 16z - 28 + \lambda(6x + 6y - 7z - 8) = 0$$

or $$(12 + 6\lambda)x + (-15 + 6\lambda)y + (16 - 7\lambda)z - (28 + 8\lambda) = 0 \quad ...(iv)$$

Three planes will intersect in a common line if the planes (iii) and (iv) represent the same plane.

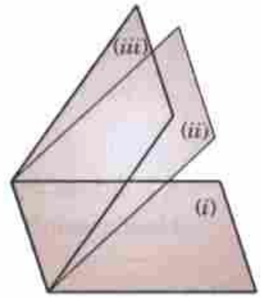


Fig. 3.47

$$\therefore \quad \frac{12 + 6\lambda}{2} = \frac{-15 + 6\lambda}{35} = \frac{16 - 7\lambda}{-39} = \frac{-28 - 8\lambda}{12} \quad ...(v)$$

From $\frac{12 + 6\lambda}{2} = \frac{-15 + 6\lambda}{35}$, we have $\lambda = \frac{-25}{11}$ which satisfies all the equations (*v*).

Hence the given planes intersect in a line.

Any point on the line $\frac{x-1}{3} = \frac{y}{-2} = \frac{z-3}{1} = r$ (say) ...(*vi*)

is $(3r + 1, -2r, r + 3)$ which lies in the plane (*iii*)

if $2(3r + 1) + 35(-2r) - 39(r + 3) + 12 = 0$, *i.e.* if $r = -1$.

$\therefore$ The coordinates of the point P in which (*vi*) meets (*iii*) are $(-2, 2, 2)$.

Distance of P from plane (i) $= \frac{12(-2) - 15(2) + 16(2) - 28}{\sqrt{144 + 225 + 256}} = \frac{-50}{\sqrt{(625)}} = 2$ (in magnitude)

Distance of P from plane (ii) $= \frac{6(-2) + 6(2) - 7(2) - 8}{\sqrt{36 + 36 + 49}} = 2$ (in magnitude)

Hence the point P is equidistant from the planes (*i*) and (*ii*).

Example 3.53. *Prove that the three planes*

(i) $2x + y + z = 3$, *(ii)* $x - y + 2z = 4$, *(iii)* $x + z = 2$,

form a triangular prism and find the area of the normal section of the prism.

Solution. Let l, m, n be the direction cosines of the line of intersection of the planes (*ii*) and (*iii*) so that $l - m + 2n = 0, l + n = 0$,

whence $$\frac{l}{1} = \frac{m}{-1} = \frac{n}{-1}.$$

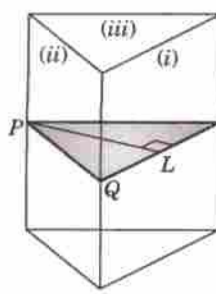


Fig. 3.48

To find a point P on this line, put $x = 0$ in (*ii*) and (*iii*), $-y + 2z = 4$ and $z = 2$. Thus the point P is $(0, 0, 2)$.

Now the line of intersection of (*ii*) and (*iii*) is parallel to the plane (*i*).

$$[\because \quad 2 \times 1 + 1 \times (-1) + 1 \times (-1) = 0]$$

Also the point P does not lie on the plane (*i*).

Hence the given planes form a triangular prism.

Let ΔPQR be its normal section through P.

The equation of the plane through P perpendicular to the line of intersection of the planes (*i*) and (*iii*) is,

$$1(x - 0) - 1(y - 0) - 1(z - 2) = 0$$

or $$x - y - z + 2 = 0 \quad ...(iv)$$

Solving the equations (*i*), (*ii*) and (*iv*), we get

$$Q \equiv \left(\frac{1}{3}, \frac{1}{3}, 2\right).$$

Solving the equation (*i*), (*iii*) and (*iv*), we get

$$R \equiv \left(\frac{1}{3}, \frac{2}{3}, \frac{5}{3}\right).$$

$\therefore$ $$QR = \sqrt{\left\{\left(\frac{1}{3}-\frac{1}{3}\right)^2 + \left(\frac{1}{3}-\frac{2}{3}\right)^2 + \left(2-\frac{5}{3}\right)^2\right\}} = \sqrt{\left(\frac{2}{9}\right)}$$

Also $PL \perp$ from P on the plane (*i*) $= \dfrac{3-2}{\sqrt{(4+1+1)}} = \dfrac{1}{\sqrt{6}}$.

Hence the area of $\Delta PQR = \dfrac{1}{2} QR \times PL = \dfrac{1}{2} \cdot \dfrac{\sqrt{2}}{3} \cdot \dfrac{1}{\sqrt{6}} = \dfrac{1}{6\sqrt{3}}$.

PROBLEMS 3.11

1. Prove that the three planes $2x - 3y - 7z = 0$, $3x - 14y - 13z = 0$, $8x - 31y - 33z = 0$ pass through one line.
2. Prove that the planes $x = cy + bz$, $y = az + cx$, $z = bx + ay$ intersect in a line if $a^2 + b^2 + c^2 + 2abc = 1$ and show that the equations of this line are
$$\frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{z}{\sqrt{1-c^2}}$$ (*Rajasthan, 2005*)
3. Show that the planes $x + 2y - 3 = 0$, $3x - 4y + z - 4 = 0$ and $4x + 3y - 2z - 24 = 0$ from a triangular prism.
4. Prove that the planes $2x + 3y + 4z = 6$, $3x + 4y + 5z = 20$, $x + 2y + 3z = 0$ form a prism : obtain the equation of one of its edges in the symmetrical form.

3.17 SPHERE

(1) Def. *A* ***sphere*** *is the locus of a point which remains at a constant distance from a fixed point.* The fixed point is called the *centre* and the constant distance the *radius* of the sphere

(2) The equation of the sphere whose centre is (a, b, c) and radius r, is

$$(\mathbf{x} - \mathbf{a})^2 + (\mathbf{y} - \mathbf{b})^2 + (\mathbf{z} - \mathbf{c})^2 = \mathbf{r}^2.$$

For the distance of any point $P(x, y, z)$ on the sphere from the centre $C(a, b, c)$ = the radius r.

In particular the *equation of the sphere whose centre is the origin and radius a, is*

$$\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 = \mathbf{a}^2$$

(3) The equation $\mathbf{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0}$ represents a sphere whose centre is $(-\mathbf{u}, -\mathbf{v}, -\mathbf{w})$ and radius

$$= \sqrt{\mathbf{u^2 + v^2 + w^2 - d}}.$$

For on writing it as $(x^2 + 2ux) + (y^2 + 2vy) + (z^2 + 2wz) = -d$

or as $$(x + u)^2 + (y + v)^2 + (z + w)^2 = u^2 + v^2 + w^2 - d$$

and comparing with $$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2,$$

it clearly represents a sphere whose centre is

$$(a, b, c) = (-u, -v, -w) \text{ and radius } r = \sqrt{(u^2 + v^2 + w^2 - d)}$$

Thus *the general equation of a sphere is such that*

(*i*) *it is the second degree in x, y, z,*

(*ii*) *the coefficient of x^2, y^2, z^2 are equal,*

and (*iii*) *there are no terms containing yz, zx or xy*

(4) Section of a sphere by a plane *is a circle and the section of a sphere by a plane through its centre is called a* **great circle.**

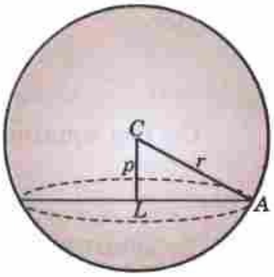


Fig. 3.49

Thus the equations $x^2 + y^2 + z^2 + 2ux\ 2vy + 2wz + d = 0$ [Sphere]

and $Ax + By + Cz + D = 0$ [Plane]

taken together represent a circle (Fig. 3.49) having centre L and radius $LA = \sqrt{(r^2 - p^2)}$.

(5) The equation of any sphere through the circle *of intersection of the sphere* $S = 0$
and the plane $U = 0$
is $S + kU = 0$

For the equation $S + kU = 0$
represents a sphere and the points of intersection of the sphere $S = 0$ and the plane $U = 0$ satisfy it.

Example 3.54. *Find the equation of the sphere through the points (0, 0, 0), (0, 1, – 1), (– 1, 2, 0) and (1, 2, 3). Locate its centre and find the radius.*

Solution. Let the required equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad ...(i)$$

It passes through (0, 0, 0), (0, 1, – 1), (– 1, 2, 0) and (1, 2, 3).

$\therefore$ $d = 0,$

$$1 + 1 + 2v - 2w + d = 0 \quad \text{or} \quad v - w + 1 = 0 \quad ...(ii)$$
$$1 + 4 - 2u + 4v + d = 0 \quad \text{or} \quad -2u + 4v + 5 = 0 \quad ...(iii)$$
$$1 + 4 + 9 + 2u + 4v + 6w + d = 0 \quad \text{or} \quad u + 2v + 3w + 7 = 0 \quad ...(iv)$$

Multiplying (*ii*) by (*iii*) and adding to (*iv*), we get

$$u + 5v + 10 = 0 \quad ...(v)$$

Solving (*iii*) and (*v*), we get $u = -\dfrac{15}{14}, v = -\dfrac{25}{14}$

From (*ii*), $w = v + 1 = \dfrac{-25}{14} + 1 = \dfrac{-11}{14}$

Substituting these values of u, v, w, d in (*i*), we get

$$x^2 + y^2 + z^2 - \frac{15}{7}x - \frac{25}{7}y - \frac{11}{7}z = 0 \quad ...(vi)$$

which is the required equaton of the sphere.

Its centre is (15/14, 25/14, 11/14) $[(-u, -v, -w)]$

and the radius = $[(-15/14)^2] + (-25/14)^2 + (-11/14)^2 - 0] = \sqrt{971/14}$.

Example 3.55. (*a*) *Find the equation of the sphere which has (x_1, y_1, z_1) and (x_2, y_2, z_2) as the extremities of a diameter.*

(*b*) *Deduce the equation of the sphere described on the line joining the points (2, – 1, 4) and (– 2, 2, – 2) as diameter. Find the area of the circle in which the sphere is intersected by the plane $2x + y - z = 3$.*

(*Anna, 2009 ; Hazaribagh, 2009*)

Solution. (*a*) Let $P(x, y, z)$ be any point on the sphere having $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as ends of diameter (Fig. 3.50. (*a*)). Then AP and BP are at right angles.

Now direction ratio of AP are $x - x_1, y - y_1, z - z_1$ and those of BP are $x - x_2, y - y_2, z - z_2$.

Hence

$$(\mathbf{x} - \mathbf{x_1})(\mathbf{x} - \mathbf{x_2}) + (\mathbf{y} - \mathbf{y_1})(\mathbf{y} - \mathbf{y_2}) + (\mathbf{z} - \mathbf{z_1})(\mathbf{z} - \mathbf{z_2}) = 0$$

which is the required equation.

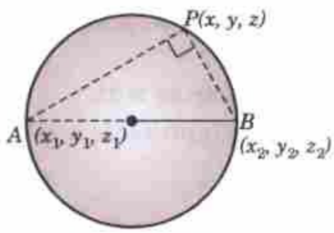


Fig. 3.50 (*a*)

(*b*) The equation of the required sphere is

$$(x - 2)(x + 2) + (y + 1)(y - 2) + (z - 4)(z + 2) = 0$$

or $$x^2 + y^2 + z^2 - y - 2z - 14 = 0 \quad ...(i)$$

Its centre is $C(0, 1/2, 1)$

and radius $(r) = \sqrt{(0, 1/4 + 1 + 14)} = \sqrt{(61/4)}$.

Let the given plane $2x + y - z - 3 = 0$ ...(*ii*)

cut the sphere (1) in the circle PP' having centre L.

$\therefore$ p = perpendicular CL from C on the plane (2)

$$= \frac{1/2 - 1 - 3}{\sqrt{4+1+1}} = \frac{7}{2\sqrt{6}} \text{ (in magnitude)}$$

If a be the radius of the circle PP', then

$$a^2 = r^2 - p^2 = \frac{61}{4} - \frac{49}{24} = \frac{317}{24}$$

Hence the area of circle $PP' = \pi a^2 = \frac{317}{24}\pi$.

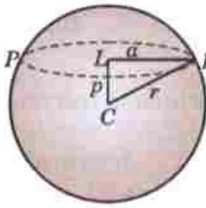


Fig. 3.50 (b)

Example 3.56. *A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere OABC is*

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2. \qquad \text{(P.T.U., 2010)}$$

Solution. Let the centre of the sphere $OABC$ be $P(f, g, h)$ so that its radius $OP = \sqrt{(f^2 + g^2 + h^2)}$.

$\therefore$ The equation of the sphere is

$$(x - f)^2 + (y - g)^2 + (z - h)^2 = f^2 + g^2 + h^2$$

or
$$x^2 + y^2 + z^2 - 2fx - 2gy - 2hz = 0 \qquad ...(i)$$

To find OA, putting $y = 0, z = 0$ in (i), we have

$x^2 - 2fx = 0$, *i.e.*, $OA = x = 2f$. Similarly, $OB = 2g$, $OC = 2h$.

Thus the equation of the plane ABC is $\frac{x}{2f} + \frac{y}{2g} + \frac{z}{2h} = 1$

Since the plane passes through (a, b, c) $\therefore$ $\frac{a}{2f} + \frac{b}{2g} + \frac{c}{2h} = 1$.

Hence the locus of the centre (f, g, h) of the sphere is,

$$\frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z} = 1 \quad \text{or} \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2.$$

Example 3.57. *Find the equation of the sphere having the circle*

$$x^2 + y^2 + z^2 + 10y - 4z - 8 = 0, x + y + z = 3$$

as a great circle. (Anna, 2009 ; Madras, 2001 S)

Solution. The equation of any sphere through the given circle is

$$x^2 + y^2 + z^2 + 10y - 4z - 8 + k(x + y + z - 3) = 0$$

i.e.,
$$x^2 + y^2 + z^2 + kx + (10 + k)y - (4 - k)z - (8 + 3k) = 0 \qquad ...(i)$$

In order that (i) may have the given circle as its great circle, its centre $[-k/2, -(10 + k)/2, (4 - k)/2]$ must lie on the plane $x + y + z = 3$

$$\therefore \qquad -\frac{k}{2} - \frac{10+k}{2} + \frac{4-k}{2} = 3, \text{ i.e., } k = -4$$

whence (i) becomes, $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$ which is the required equation.

Example 3.58. *Find the equation of the smallest sphere which contains the circle $x^2 + y^2 + z^2 + 2x + 6y + 4z - 11 = 0$ and $2x + 2y + z + 1 = 0$.*

Solution. Equation of any sphere containing the given circle is

$$x^2 + y^2 + z^2 + 2x + 6y + 4z - 11 + \lambda\,(2x + 2y + z + 1) = 0$$

or
$$x^2 + y^2 + z^2 + (2 + 2\lambda)\,x + (6 + 2\lambda)\,y + (4 + \lambda)\,z - 11 + \lambda = 0 \qquad ...(i)$$

Its radius r is given by

$$r^2 = (1 + \lambda)^2 + (3 + \lambda)^2 + (2 + \tfrac{1}{2}\lambda)^2 - (\lambda - 11) = \frac{9}{4}\left[\lambda^2 + 4\lambda + \frac{100}{9}\right] = \frac{9}{4}\left[(\lambda + 2)^2 + \frac{64}{9}\right]$$

Now r^2 has the least value when $\lambda = -2$.

$\therefore$ Substituting $\lambda = -2$ in (*i*), we get

$$x^2 + y^2 + z^2 - 2x + 2y + 2z - 13 = 0$$

which is the required smallest sphere.

Example 3.59. *Prove that the circles $x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0$, $5y + 6z + 1 = 0$ and $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0$, $x + 2y - 7z = 0$ lie on the same sphere and find its equation.*

Solution. Equation of any sphere containing the first circle is

$$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 + \lambda (5y + 6z + 1) = 0$$

or $$x^2 + y^2 + z^2 - 2x + (3 + 5\lambda)y + (4 + 6\lambda)z - 5 + \lambda = 0 \quad ...(i)$$

Similarly equation of any sphere containing the second given circle is

$$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 + \lambda' (x + 2y - 7z) = 0$$

or $$x^2 + y^2 + z^2 + (-3 + \lambda')x + (-4 + 2\lambda')y + (5 - 7\lambda') z - 6 = 0 \quad ...(ii)$$

(*i*) and (*ii*) will represent the same sphere when

$$-2 = -3 + \lambda' \quad ...(iii); \qquad 3 + 5\lambda = -4 + 2\lambda' \quad ...(iv)$$

$$4 + 6\lambda = 5 - 7\lambda' \quad ...(v); \qquad -5 + \lambda = -6 \quad ...(vi)$$

Now (*iii*) gives $\lambda' = 1$ and (*vi*) gives $\lambda = -1$.

Clearly $\lambda = -1$ and $\lambda' = 1$ also satisfy (*iv*) and (*v*). This shows that the given circles lie on the same sphere.

Substituting $\lambda = -1$ in (*i*) or $\lambda' = 1$ in (*ii*), we get

$$x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$$

which is the desired sphere.

PROBLEMS 3.12

1. Find the equation of the sphere through the points (2, 0, 1), (1, – 5, – 1), (0, –2, 3) and (4, – 1, 2). Also find its centre and radius.
2. Find the equation of the sphere whose diameter is the line joining the origin to the point (2, – 2, 4). Also find its centre and radius.
3. Obtain the equation of the sphere which passes through the points (1, 0, 0), (0, 1, 0), (0, 0, 1) and
   (*a*) has its centre on the plane $x + y + z = 6$.
   (*b*) has its radius as small as possible.
4. A sphere of constant radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the
   (*i*) triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) = 4k^2$. (*Assam, 1999*)
   (*ii*) tetrahedron $OABC$ lies on the sphere $x^2 + y^2 + z^2 = k^2/4$.
5. A plane passes through a fixed points (a, b, c), show that the locus of the foot of the perpendicular from the origin on the plane is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$.
6. A sphere of constant radius r passes through the origin O and cuts the axes in A, B, C. Prove that the locus of the foot of the perpendicular from O on the plane ABC is given by
   $$(x^2 + y^2 + z^2)^2 (x^{-2} + y^{-2} + z^{-2}) = 4r^2.$$
7. A plane cuts the coordinate axes at A, B, C. If $OA = a$, $OB = b$, $OC = c$, find the equation of the
   (*i*) circumsphere of the tetrahedron $OABC$, (*Assam, 1999*)
   (*ii*) circum-circle of the triangle ABC. Also obtain the coordinates of its centre.
8. Find the centre and radius of the circle $x^2 + y^2 + z^2 - 2y - 4z = 11$, $x + 2y + 2z = 15$. (*P.T.U., 2009 S ; Burdwan, 2003 ; Cochin, 2001*)
9. Show that the points (2, – 6, 0), (4, – 9, 6), (5, 0, 2), (7, – 3, 8) are concyclic.
10. Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 = 0$, $5x - 2y + 4z + 7 = 0$ is a great circle.
11. Find the equation of the sphere having its centre on the plane $4x - 5y - z = 3$ and passing through the circle $x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0$, $x - 2y + z = 8$. (*Delhi, 2001*)
12. Prove that the plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x - z - 2 = 0$ in a circle of radius unity. Find also the equation of the sphere which has this circle as one of its great circles. (*Nagpur, 2009*)
13. Find the equation of the sphere passing through the circle $x^2 + y^2 + z^2 + 2x + 3y + 6 = 0$, $x - 2y + 4z = 9$ and the centre of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$. (*Anna, 2009*)

3.18 EQUATION OF THE TANGENT PLANE

The equation of the tangent plane at any point (x_1, y_1, z_1) of the sphere

$$x^2 + y^2 + z^2 = a^2 \text{ is } \mathbf{xx_1 + yy_1 + zz_1 = a^2}.$$

If $P(x, y, z)$ be any point on the tangent plane at $P_1(x_1, y_1, z_1)$ to the given sphere, the direction ratios of P_1P are $x - x_1, y - y_1, z - z_1$. Also the direction ratios of radius OP_1 are $x_1 - 0, y_1 - 0, z_1 - 0$.

Since OP_1 is normal to the tangent plane at P_1, $OP_1 \perp P_1P$.

$\therefore$ $$x_1(x - x_1) + y_1(y - y_1) + z_1(z - z_1) = 0$$

or $$xx_1 + yy_1 + zz_1 = x_1^2 + y_1^2 + z_1^2 = a^2 \qquad [\because \ P_1(x_1, y_1, z_1) \text{ lies on the sphere.}$$

This is the desired equation of the tangent plane.

Similarly, *the tangent plane at (x_1, y_1, z_1) to the sphere*

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

is $$\mathbf{xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0}$$

Thus to write the equation of the tangent plane at (x_1, y_1, z_1) to a sphere, change x^2 to xx_1, y^2 to yy_1, z^2 to zz_1, $2x$ to $x + x_1$, $2y$ to $y + y_1$, $2z$ to $z + z_1$.

Obs. *The condition for a plane (or a line) to touch a sphere is that the perpendicular distance of the centre from the plane (or the line) = the radius.*

Example 3.60. *Find the equations of the spheres passing through the circle $x^2 + y^2 + z^2 - 6x - 2z + 5 = 0$, $y = 0$ and touching the plane $3y + 4z + 5 = 0$.*

Solution. The equation of any sphere through the given circle is

$$x^2 + y^2 + z^2 - 6x - 2z + 5 + ky = 0$$

or $$x^2 + y^2 + z^2 - 6x + ky - 2z + 5 = 0 \qquad ...(i)$$

$\therefore$ Its centre = $(3, -k/2, 1)$ and radius = $\sqrt{[9 + (k^2/4) + 1 - 5]} = \sqrt{(5 + k^2/4)}$.

The sphere (i) will touch the plane $3y + 4z + 5 = 0$, if $\perp$ distance of the centre $(3, -k/2, 1)$ from the plane = radius.

i.e., $$\frac{3(-k/2) + 4 + 5}{\sqrt{(9 + 16)}} = \sqrt{\left(5 + \frac{k^2}{4}\right)} \quad \text{or if,} \quad 4k^2 + 27k + 44 = 0$$

$\therefore$ $$k = \frac{-27 \pm \sqrt{[(27)^2 - 704]}}{8} = -\frac{11}{4} \text{ or } -4$$

Substituting the value of k in (1), we get

$$x^2 + y^2 + z^2 - 6x - \frac{11}{4}y + 2z + 5 = 0 \text{ and } x^2 + y^2 + z^2 - 6x - 4y - 2z + 5 = 0$$

as the two required spheres.

Example 3.61. *Find the equation of the sphere which touches the plane $x - 2y - 2z = 7$ at the point $L(3, -1, -1)$ and passes through the point $M(1, 1, -3)$.*

Solution. If C is the centre of the sphere, then CL is perpendicular to the given plane $x - 2y - 2z = 7$.

$\therefore$ The direction ratios of CL being $1, -2, -2$, the equation of CL is

$$\frac{x-3}{1} = \frac{y+1}{-2} = \frac{z+1}{-2} = k \text{ (say)}$$

Any point on CL is $(k + 3, -2k - 1, -2k - 1)$ which will represent C for some value of k.

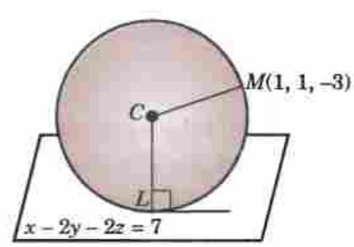


Fig. 3.51

Since M lies on the sphere, therefore its radius $CL = CM$ or $(CL)^2 = (CM)^2$

i.e. $$(k + 3 - 3)^2 + (-2k - 1 + 1)^2 + (-2k - 1 + 1)^2$$
$$= (k + 3 - 1)^2 + (-2k - 1 - 1)^2 + (-2k - 1 + 3)^2$$

or $$4k = -12 \quad \text{or} \quad k = -3.$$

$\therefore$ The centre C is $(0, 5, 5)$ and radius $CL = \sqrt{(9 + 36 + 36)} = 9$.

Hence the required equation of the sphere is

$$(x-0)^2 + (y-5)^2 + (2-5)^2 = (9)^2$$

or $$x^2 + y^2 + z^2 - 10y - 10z - 31 = 0$$

[**Orthogonal spheres.** Two spheres are said to cut orthogonally if the tangent planes at a point of intersection are at right angles (Fig. 3.52).

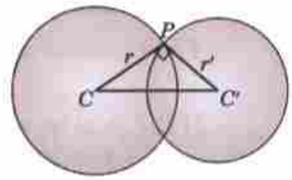


Fig. 3.52

The radii of such spheres through their point of intersection P, being $\perp$ to the tangent planes at P are also at right angles. *Thus two spheres cut orthogonally, if the square of the distance between their centre = sum of the squares of their radii*].

Example 3.62. *Show that the condition for spheres*

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

and $$x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$$

to cut orthogonally is $\mathbf{2uu' + 2vv' + 2ww' = d + d'}$ *(Anna, 2002 S)*

Solution. The centres of the spheres are

$C(-u, -v, -w)$, $C'(-u', -v', -w')$ and their radii are

$$r = \sqrt{(u^2 + v^2 + w^2 - d)},$$

$$r' = \sqrt{(u'^2 + v'^2 + w'^2 - d')}.$$

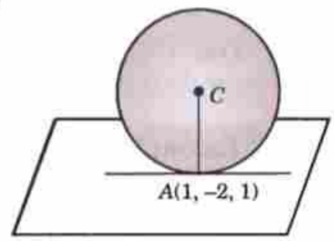


Fig. 3.53

Now these spheres will cut orthogonally, if $(CC')^2 = r^2 + r'^2$

i.e., $$(u-u')^2 + (v-v')^2 + (w-w')^2$$
$$= u^2 + v^2 + w^2 - d + u'^2 + v'^2 + w'^2 - d'$$

or $2uu' + 2vv' + 2ww' = d + d'$ which is the required condition.

Example 3.63. *Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and cuts the sphere $R^2 - 2(2I - 3J) . R + 4 = 0$ orthogonally.* *(Roorkee, 2000)*

Solution. The given plane $3x + 2y - z + 2 = 0$ ...(*i*)

will touch the required sphere at $A\,(1, -2, 1)$ if its centre lies on the normal to (*i*) at A (Fig. 3.53). The equations of the normal to (*i*) at A are $\dfrac{x-1}{3} = \dfrac{y+2}{2} = \dfrac{z-1}{-1}$

Any point on this line is $C\,(3r + 1, 2r - 2, \pi r + 1)$

Also radius (AC) of the required sphere.

$$= \sqrt{[(3r)^2 + (2r)^2 + (-r)^2]} = r\sqrt{14}.$$

Since the required sphere cuts the given sphere

$$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0 \quad [\text{Centre} = (2, -3, 0) \text{ and } + \text{radius} = 3$$

orthogonally, therefore (distance between their centres)$^2 = \Sigma$ of squares of their radii

i.e., $$(3r + 1 - 2)^2 + (2r - 2 + 3)^2 + (-r + 1)^2 = 14r^2 + 9 \text{ or } r = -3/2.$$

Thus centre C is $(-7/2, -5, 5/2)$ and radius $= \dfrac{3\sqrt{14}}{2}$.

Hence the required sphere is

$$(x + 7/2)^2 + (y + 5)^2 + (z - 5/2)^2 = (3\sqrt{14/2})^2$$

or $$x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0.$$

PROBLEMS 3.13

1. Find the equations of the tangent planes to the sphere
   (*i*) $x^2 + y^2 + z^2 - 4x + 2y - 6z + 11 = 0$ which are parallel to the plane $x = 0$. *(Anna, 2009)*
   (*ii*) $x^2 + y^2 + z^2 = 9$ which pass through the line $x + y = 6$, $x - 2z = 3$. *(Madras, 2006)*

2. Find the equations of the spheres which pass through the circle
$x^2 + y^2 + z^2 = 5x + 2y + 3z = 3$, and touch the plane $4x + 3y = 15$. (*Anna, 2009*)
3. Find the equation of the sphere which is tangential to the plane $x - 2y - 2z = 7$ at $(3, -1, -1)$ and passes through $(1, 1, -3)$.
4. (*i*) Prove that the equation of the sphere which lies in the first octant and touches the coordinates planes is of the form $(x^2 + y^2 + z^2) - 2\lambda(x + y + z) + 2\lambda^2 = 0$.
(*ii*) Find the equation of the sphere passing through (1, 4, 9) and touching the coordinate planes.
5. Tangent plane at any point of the sphere $x^2 + y^2 + z^2 = r^2$ meets the coordinates axes at A, B, C. Show that the locus of the point of intersection of the planes drawn parallel to the coordinate planes through A, B, C is the surface $x^{-2} + y^{-2} + z^{-2} = r^{-2}$. (*Rajasthan, 2006*)
6. Find the equation of the tangent line to the circle $x^2 + y^2 + z^2 = 3$, $3x - 2y + 4z + 3 = 0$ at the point $(1, 1, -1)$.
7. Show that the sphere $x^2 + y^2 + z^2 - 2x + 6y + 14z + 3 = 0$ divides the line joining the points $(2, -1, -4)$ and $(5, 5, 5)$ internally and externally in the ratio $1 : 2$.
8. Find the shortest and the longest distance from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$.
9. Show that the spheres $x^2 + y^2 + z^2 + 6y + 14z + 8 = 0$ and $x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$, intersect at right angles. Find their plane of intersection.
10. Show that the spheres $x^2 + y^2 + z^2 = 25$ and $x^2 + y^2 + z^2 - 24x - 40y - 18z + 225 = 0$ touch externally and find their point of contact.

3.19 (1) CONE

Def. *A cone is a surface generated by a straight line which passes through a fixed point and satisfies one more condition e.g.*, it may intersect a given curve (called the *guiding curve*).

The fixed point is called the **vertex** *and the straight line in any position is called a* **generator.**

The degree of the equation of a cone depends upon the nature of its guiding curve. In case the guiding curve is a conic, the equation of the cone shall be of the second degree. Such cones are called *Quadric cones.* In what follows, we shall be conncerned only with quadric cones.

Example 3.64. *Find the equation of the cone whose vertex is (3, 1, 2) and base the circle*
$$2x^2 + 3y^2 = 1, z = 1.$$

Solution. Any line through (3, 1, 2) is
$$\frac{x-3}{l} = \frac{y-1}{m} = \frac{z-2}{n} \quad ...(i)$$

It meets $z = 1$, where $\dfrac{x-3}{l} = \dfrac{y-1}{m} = \dfrac{-1}{n}$

whence $x = 3 - l/n$, $y = 1 - m/n$.

Substituting these values of x and y in $2x^2 + 3y^2 = 1$,
$$2(3 - l/n)^2 + 3(1 - m/n)^2 = 1 \quad ...(ii)$$

Eliminating l, m, n from (*i*) and (*ii*), the locus of the line (*i*) is
$$2\left(3 - \frac{x-3}{z-2}\right)^2 + 3\left(1 - \frac{y-1}{z-2}\right)^2 = 1$$

or $\quad 2x^2 + 3y^2 + 20z^2 - 6yz - 12xz + 12x + 6y - 38z + 17 = 0$ which is the required equation.

Example 3.65. *Find the equation of the cone whose vertex is at the origin and guiding curve is* $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{1} = 1, x + y + z = 1.$

Solution. Any line through (0, 0, 0) is $x/l = y/m = z/n$ ...(*i*)

Any point on it is $P(lr, mr, nr)$.

If (*i*) intersects the given curve, the coordinates of P should satisfy its equations.

$\therefore$ $$\frac{l^2r^2}{4}+\frac{m^2r^2}{9}+\frac{n^2r^2}{1}=1 \text{ and } lr+mr+nr=1.$$

Eliminating r, $$\left(\frac{l^2}{4}+\frac{m^2}{9}+n^2\right)\Big/(l+m+n)^2=1.$$

Simplifying, $27l^2+32m^2+72(lm+mn+nl)=0$ ...(*ii*)

Eliminating l, m, n from (*i*) and (*ii*), the locus of the line (*i*) is

$$27x^2+32y^2+72(xy+yz+zx)=0 \text{ which is the required equation.}$$

Obs. *The equation of a cone with vertex at the origin is a homogeneous equation of the second degree in x, y, z (i.e., all* terms are of the same degree). The reason is that every generator will have the equation of the form (*i*) above. So the point (lr, mr, nr) will satisfy the equation of the cone for every value of r. This is possible only if the equation is homogeneous.

Example 3.66. *A variable plane parallel to the plane* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ *meets the coordinate axes in A, B, C. Find the equation of the cone whose vertex is the origin and guiding curve the circle ABC.*

Solution. Let the plane $$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=k \quad \text{...(i)}$$

meet the axes at A, B, C, so that $A=(ka, 0, 0)$, $B=(0, kb, 0)$ and $C=(0, 0, kc)$.

$\therefore$ The equation of the sphere through $O\ (0, 0, 0)$ and A, B, C is

$$x^2+y^2+z^2-k(ax+by+cz)=0 \quad \text{...(ii)}$$

Since the equation of the cone with vertex at O is a homogeneous equation of the second degree, therefore, it must be satisfied by points lying on the circle ABC, *i.e.*, on (*i*) and (*ii*) both.

$\therefore$ Making (*ii*) homogeneous with the help of (*i*), we have

$$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

or $$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0 \text{ which is the required equation.}$$

Example 3.67. *Show that the general equation of the cone of the second degree which passes through the axes is of the form* $fyz+gzx+hxy=0$.

Solution. Any cone which passes through the axes will have origin V as its vertex. The general equation of a cone of the second degree having vertex at the origin is of the form

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \text{...(i)}$$

Since it passes through x-axis

$\therefore$ The direction cosines of x-axis (*i.e.*, 1, 0, 0) must satisfy (*i*). This gives $a=0$.

As the cone passes through y-axis, $b=0$.

Similarly, as the cone passes through z-axis, $c=0$.

Hence (*i*) reduces to $fyz+gzx+hxy=0$.

(2) **Right circular cone. Def.** *A right circular cone is a surface generated by a straight line which passes through a fixed point (vertex) and makes a constant angle with a fixed line* (Fig. 3.54).

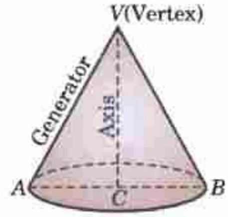


Fig. 3.54

The constant angle ($\angle AVC$) *is called its* **semi-vertical angle** *and the fixed line (VC) is called the* **axis.** The section of a right circular cone by a plane perpendicular to its axis is a circle.

Example 3.68. *Find the equation of the right circular cone whose vertex is the origin, whose axis is the line $x/1=y/2=z/3$ and which has semi-vertical angle of 30°.* (Anna, 2009)

Solution. Let $P(x, y, z)$ be any point on the cone with vertex O and axis (OC)

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}, \text{ so that } \angle POC=30°. \qquad \text{(Fig. 3.55)}$$

Now the direction ratios of OP are x, y, z and those of OC are 1, 2, 3.

$$\therefore \quad \cos 30° = \frac{x(1) + y(2) + z(3)}{\sqrt{(x^2 + y^2 + z^2)} \cdot \sqrt{(1 + 4 + 9)}}$$

or
$$\frac{\sqrt{3}}{2} = \frac{x + 2y + 3z}{\sqrt{[14\,(x^2 + y^2 + z^2)]}}$$

Squaring $3 \times 14\,(x^2 + y^2 + z^2) = 4(x + 2y + 3z)^2$

or
$$19x^2 + 13y^2 + 3z^2 - 8xy - 24yz - 12zx = 0$$

which is the required equation of the cone.

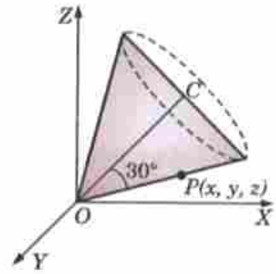


Fig. 3.55

Example 3.69. *Find the equation of the right circular cone generated when the straight line $2y + 3z = 6$, $x = 0$ revolves about z-axis.* *(Hazaribagh, 2009)*

Solution. The vertex is the point of intersection of the line $2y + 3z = 6$, $x = 0$ and the z-axis, *i.e.*, $x = 0$, $y = 0$ (Fig. 3.56).

$\therefore$ Vertex is $A(0, 0, 2)$. A generator of the cone is

$$\frac{x}{0} = \frac{y}{3} = \frac{z - 2}{-2}$$

$\therefore$ Direction ratios of the generator are $0, 3, -2$ and the axis (z-axis) are $0, 0, 1$. The semi-vertical angle α is, therefore, given by

$$\cos \alpha = \frac{0\,.\,0 + 3\,.\,0 + (-2)\,.\,1}{\sqrt{13}} = \frac{-2}{\sqrt{13}}$$

Let $P(x, y, z)$ be any point on the cone so that the direction ratios of AP are $x, y, z - 2$. Since AP makes an angle α with AZ, we have

$$\cos \alpha = \frac{x\,.\,0 + y\,.\,0 + (z - 2)\,.\,1}{\sqrt{[x^2 + y^2 + (z - 2)^2]}}$$

Thus
$$\frac{(z - 2)^2}{x^2 + y^2 + (z - 2)^2} = \cos^2 \alpha = \frac{4}{13}$$

or
$$4x^2 + 4y^2 - 9z^2 + 36z - 36 = 0$$

which is the required equation of the cone.

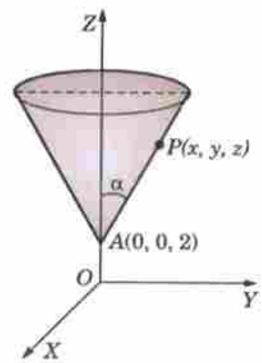


Fig. 3.56

Example 3.70. *Find the equations to the lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$.*

Solution. Let $x/l = y/m = z/n$ be one of the two lines in which the given plane $2x + y - z = 0$ ...(*i*)

cuts the given cone $4x^2 - y^2 + 3z^2 = 0$ ...(*ii*)

$\because$ This line lies on (*i*), $\therefore$ $2l + m - n = 0$ ...(*iii*)

and it lies on (*ii*), $\therefore$ $4l^2 - m^2 + 3n^2 = 0$ ...(*iv*)

To eliminate n from (*iii*) and (*iv*), put $n = 2l + m$ in (*iv*).

$$4l^2 - m^2 + 3(2l + m)^2 = 0 \quad \text{or} \quad (4l + m)(2l + m) = 0$$

$\therefore$	Either $4l + m = 0$	or $2l + m = 0$
From (*iii*)	$2l + m - n = 0$	and $2l + m - n = 0$
$\therefore$	$\dfrac{l}{-1} = \dfrac{m}{4} = \dfrac{n}{2}$	$\therefore$ $\dfrac{l}{-1} = \dfrac{m}{2} = \dfrac{n}{0}$

Hence the required lines are

$$\frac{x}{-1} = \frac{y}{4} = \frac{z}{2} \text{ and } \frac{x}{-1} = \frac{y}{2} = \frac{z}{0}.$$

Example 3.71. *Find the equation of the enveloping cone of the sphere $x^2 + y^2 + z^2 = a^2$ with vertex at the point (x_1, y_1, z_1).*

Solution. The equation of any generator through $V(x_1, y_1, z_1)$ having direction ratios l, m, n is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r \text{ (say)} \quad ...(i)$$

Any point on (*i*) is $P(x_1 + lr, y_1 + mr, z_1 + nr)$.

It lies on the given sphere if

$$(x_1 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 = a^2$$

or
$$(l^2 + m^2 + n^2) r^2 + 2(lx_1 + my_1 + nz_1) r + x_1^2 + y_1^2 + z_1^2 - a^2 = 0 \quad ...(ii)$$

The line (*i*) will touch the given sphere if (*ii*) has equal roots.

$$\therefore \quad (lx_1 + my_1 + nz_1)^2 = (l^2 + m^2 + n^2)(x_1^2 + y_1^2 + z_1^2 - a^2) \quad ...(iii)$$

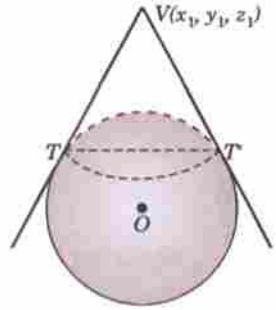


Fig. 3.57

The locus of all such lines is the enveloping cone of the given sphere which is obtained by eliminating l, m, n from (*i*) and (*iii*).

Thus $[(x - x_1)x_1 + (y - y_1)y_1 + (z - z_1)z_1]^2 = [(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2](x_1^2 + y_1^2 + z_1^2 - a^2)$

which is the equation of the enveloping cone. (Fig. 3.57)

Obs. It can be reduced to the form $SS_1 = T^2$

where $S = x^2 + y^2 + z^2 - a^2, S_1 = x_1^2 + y_1^2 + z_1^2 - a^2, T = xx_1 + yy_1 + zz_1 - a^2$.

Thus the enveloping cone of the surface $S = 0$ with vertex (x_1, y_1, z_1) is $SS_1 = T^2$

PROBLEMS 3.14

1. Find the equation of the cone with vertex (α, β, γ) and base $y^2 - 4ax = 0, z = 0$.
2. Find the equation of the cone whose vertex is (3, 4, 5) and base is the conic $3y^2 + 4z^2 = 16, z + 2x = 0$.
3. Find the equation of the cone whose vertex is (1, 2, 3) and whose guiding curve is the circle $x^2 + y^2 + z^2 = 4$, $x + y + z = 1$. (*P.T.U., 2010*)
4. The generators of a cone pass through the point (1, 1, 1) and their direction cosines l, m, n satisfy the relation $l^2 + m^2 = 3n^2$. Obtain the equation of the cone.
5. Find the equation of the right circular cone whose vertex is at the origin and semi-vertical angle is α and having axis of z as its axis. (*V.T.U., 2006 ; Rajasthan, 2005*)
6. Find the equation of the cone whose vertical angle is $\pi/2$, which has its vertex at the origin and its axis along the line $x = -2y = z$. (*V.T.U., 2005*)
   Also show that the plane $z = 0$ cuts the cone in two straight lines inclined at an angle $\cos^{-1} 4/5$.
7. Find the equation of the circular cone which passes through the point (1, 1, 2) and has its vertex at the origin and axis the line $x/2 = -y/4 = z/3$. (*Cochin, 2005 ; Rajasthan, 2005 ; V.T.U., 2004*)
8. Find the equation of the right circular cone generated by revoling the line $x = 0, y - z = 0$ about the axis $x = 0, x = 2$. (*Anna, 2009*)
9. Find the equation of the right circular cone passing through the coordinate axes having vertex at the origin. Obtain the semi-vertical angle and the equation of the axis.
10. Find the semi-vertical angle and the equation of the right circular cone having its vertex at the origin and passing through the circle $y^2 + z^2 = 25, x = 4$. (*Anna, 2009*)
11. Find the equation of the right circular cone which has its vertex at (0, 0, 10) whose intersection with the *XY*-plane is a circle of radius 5. (*Nagpur, 2009*)
12. Find the equations to the lines in which the plane $3x + y + 5z = 0$ cuts the cone $6yz - 2zx + 5xy = 0$.
13. Prove that the plane $ax + by + cz = 0$ meets the cone $yz + zx + xy = 0$ in perpendicular lines if $a^{-1} + b^{-1} + c^{-1} = 0$.
14. Find the equation of the enveloping cone of the sphere $x^2 + y^2 + z^2 + 2x - 4y + 2z - 1 = 0$ with vertex at (1, 1, 1).

3.20 (1) CYLINDER

Def. *A cylinder is a surface generated by a straight line which is parallel to a fixed line and satisfies one more condition e.g.,* it may intersect a given curve (called the *guiding curve*).

The straight line in any position is called the *generator* and the fixed line the *axis* of the cylinder.

Example 3.72. *Find the equation of a cylinder whose generating lines have the direction cosines l, m, n and which pass through the circumference of the fixed circle $x^2 + z^2 = a^2$ in the ZOX plane.*

Solution. Let $P(x_1, y_1, z_1)$ be any point of the cylinder so that the equation of the generator through P is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad ...(i)$$

Given guiding circle is $x^2 + z^2 = a^2, y = 0$ ...(*ii*)

The generator (*i*) cuts the plane $y = 0$, where

$$\frac{x - x_1}{l} = \frac{-y_1}{m} = \frac{z - z_1}{n}$$

i.e., where $x = x_1 - \dfrac{ly_1}{m}$ and $z = z_1 - \dfrac{ny_1}{m}$

But these values of x and z satisfy $x^2 + z^2 = a^2$

$$\therefore \qquad \left(x_1 - \frac{ly_1}{m}\right)^2 + \left(z_1 - \frac{ny_1}{m}\right)^2 = a^2$$

Hence the locus of (x_1, y_1, z_1) is

$(mx - ly)^2 + (mz - ny)^2 = a^2 m^2$, which is the required equation of the cylinder.

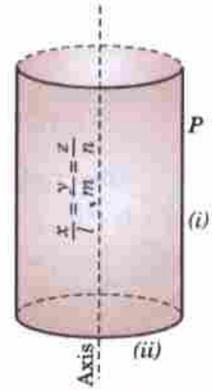


Fig. 3.58

(2) Right circular cylinder. Def. *A right circular cylinder is a surface generated by a straight line which is parallel to a fixed line and is at a constant distance from it.*

The constant distance is called the *radius of the cylinder.*

Example 3.73. *The radius of a normal section of a right circular cylinder is 2 units ; the axis lies along the straight line*

$$\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{5}, \text{ find its equation.} \qquad (P.T.U., 2005)$$

Solution. A point on the axis of the cylinder is $A(1, -3, 2)$ and its direction ratios are $2, -1, 5$.

$\therefore$ Its actual direction cosines are $\dfrac{2}{\sqrt{30}}, \dfrac{-1}{\sqrt{30}}, \dfrac{5}{\sqrt{30}}$.

Let $P(x, y, z)$ be any point on the cylinder. Draw $PM \perp$ to the axis AM. Then $MP = 2$. Now AM = Projection of AP on AM (axis)

$$= (x - 1)\frac{2}{\sqrt{30}} + (y + 3)\frac{-1}{\sqrt{30}} + (z - 2)\frac{5}{\sqrt{30}}$$

$$= \frac{2x - y + 5z - 15}{\sqrt{30}}$$

Also $\qquad AP = \sqrt{(x-1)^2 + (y+3)^2 + (z-2)^2}$

$\therefore$ From the rt. $\angle d \, \Delta AMP$, $(AM)^2 + (MP)^2 = (AP)^2$

or $\qquad \dfrac{1}{30}(2x - y + 5z - 15)^2 + 4 = (x-1)^2 + (y+3)^2 + (z-2)^2$

or $\qquad 26x^2 + 29y^2 + 5z^2 + 4xy + 10yz - 20zx + 150y + 30z + 75 = 0.$

This is the required equation of the right circular cylinder. (Fig. 3.59)

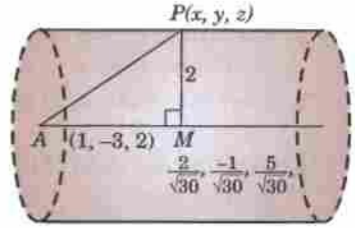


Fig. 3.59

Example 3.74. *Find the equation of the circular cylinder having for its base the circle $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$.* (*P.T.U., 2006 ; Cochin, 2005*)

Solution. The axis of the cylinder is the line through the centre L of the given circle (or through O (0, 0, 0) the centre of the sphere) (Fig. 3.60) and perpendicular to the plane of the circle.

i.e. $\qquad x - y + z = 3 \qquad ...(i)$

$\therefore$ Axis of the cylinder is $\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$

Also $OL \perp$ from $O(0, 0, 0)$ on (i)

$$= \frac{3}{\sqrt{(1+1+1)}} = \sqrt{3}.$$

$\therefore$ r, radius of the circle $= \sqrt{(OA^2 - OL^2)} = \sqrt{(9-3)} = \sqrt{6}$

Thus radius of the cylinder $(= r) = \sqrt{6}$

If $P(x, y, z)$ be any point on the cylinder, then

$$OP^2 = OM^2 + MP^2$$

i.e., $$x^2 + y^2 + z^2 = \left[\frac{1}{\sqrt{3}}(x-0) - \frac{1}{\sqrt{3}}(y-0) + \frac{1}{\sqrt{3}}(z-0)\right]^2 + 6$$

i.e., $x^2 + y^2 + z^2 + xy + yz - zx - 9 = 0$ which is the required equation.

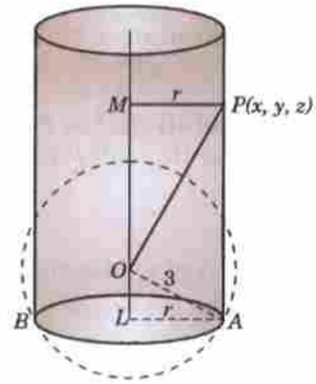


Fig. 3.60

Example 3.75. *Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 = 9$ having generator parallel to the line $x/3 = y/2 = z/1$.*

Solution. If $P(x_1, y_1, z_1)$ be a point on the enveloping cylinder, then the equation of the generator is

$$\frac{x - x_1}{3} = \frac{y - y_1}{2} = \frac{z - z_1}{1} = r(\text{say}). \quad ...(i)$$

Any point on (i) is $(x_1 + 3r, y_1 + 2r, z_1 + r)$. It lies on the sphere $x^2 + y^2 + z^2 = 9$. ...(ii)

Then $$(x_1 + 3r)^2 + (y_1 + 2r)^2 + (z_1 + r)^2 = 9$$

or $$14r^2 + 2(3x_1 + 2y_1 + z_1)r + x_1^2 + y_1^2 + z_1^2 - 9 = 0 \quad ...(iii)$$

In order that (i) touches (ii), the equation (iii) must have equal roots for which

$$4(3x_1 + 2y_1 + z_1)^2 = 4 \times 14(x_1^2 + y_1^2 + z_1^2 - 9) \quad [\because \ b^2 = 4ac]$$

or $$5x_1^2 + 10y_1^2 + 13z_1^2 + 12x_1y_1 + 4y_1z_1 + 6z_1x_1 = 126$$

$\therefore$ The locus of (x_1, y_1, z_1) is

$$5x^2 + 10y^2 + 13z^2 + 12xy + 4yz + 6zx = 126$$

which is the required equation of the enveloping cylinder.

PROBLEMS 3.15

1. Find the equation of the right circular cylinder whose axis is the line $x = 2y = -z$ and radius 4. *(Anna, 2009)*
2. Find the equation of the cylinder whose generators are parallel to the line $x = -y/2 = z/3$ and whose guiding curve is the ellipse $x^2 + 2y^2 = 1, z = 3$. *(Rajasthan, 2005 ; Roorkee, 2000)*
3. Find the equation of the right circular cylinder of radius 2 whose axis passes through (1, 2, 3) and has direction ratios (2, – 3, 6). *(V.T.U., 2006 ; Anna, 2005 S)*
4. Find the equation of the right circular cylinder describe on the circle through the points $(a, 0, 0)$, $(0, a, 0)$, $(0, 0, a)$ guiding curve.
5. Find the equation of the cylinder whose directing curve is $x^2 + z^2 - 4x - 2z + 4 = 0, y = 0$ and whose axis contains the point (0, 3, 0). Find also the area of the section of the cylinder by a plane parallel to xz-plane.
6. Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z - 2 = 0$ whose generators are perpendicular to the lines $\frac{x}{3} = \frac{y}{-1} = \frac{z}{0}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{0}$.
7. Find the equation to the cylinder whose generators intersect the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0$ and are parallel to the line $x/l = y/m = z/n$.

3.21 QUADRIC SURFACES

The surface represented by general equation of the second degree in x, y, z is called a **quadric surface** *or a* **conicoid.**

Thus the general equation of a *quadric surface* is of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

which can be reduced to any of the following standard forms so useful in engineering problems. We now, proceed to study their shapes.

(1) Ellipsoid : $\mathbf{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1}$.

(*i*) It is symmetrical about each of the coordinate planes for only even powers of x, y, z occur in its equation.

(*ii*) It meets the x-axis at $A(a, 0, 0), A'(-a, 0, 0)$;
the y-axis at $B(0, b, 0), B'(0, -b, 0)$;
and the z-axis at $C(0, 0, c), C'(0, 0, -c)$.

(*iii*) Its sections by the coordinate planes are ellipses. For the section by the yz-plane ($x = 0$) is the ellipse.

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ etc.}$$

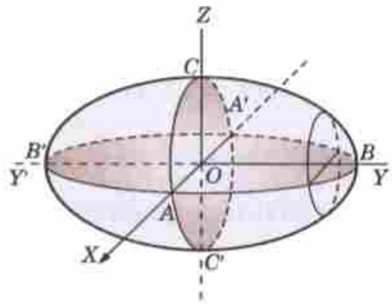


Fig. 3.61

(*iv*) The surface is generated by a variable ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}, z = k ;$$

(as k varies from $-c$ to c) and is limited in every direction.

Hence its shape is as shown in Fig. 3.61 which is like that of an *egg*.

(2) Hyperboloid of one sheet : $\mathbf{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1}$.

(*i*) It is symmetrical about each of the coordinate planes for only even powers of x, y, z occur in its equation.

(*ii*) It meets the x-axis at $A(a, 0, 0), A'(-a, 0, 0)$; the y-axis at $B(0, b, 0), B'(0, -b, 0)$; and the z-axis in imaginary points.

(*iii*) Its section by the yz-plane ($x = 0$) is the hyperbola $\dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$, (*i.e.*, $DE, D'E'$)

Its section by the zx-plane ($y = 0$) is the hyperbola $\dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 1$. (*i.e.*, $FG, F'G'$)

Its section by the xy-plane ($z = 0$) is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(*iv*) The surface is generated by a variable ellipse

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 + \dfrac{k^2}{c^2}, z = k$ (as k varies from $-\infty$ to ∞) and extends to infinity on both sides of the xy-plane.

Hence its shape is as shown in Fig. 3.62 which is like that of *juggler's dabru*.

(3) Hyperboloid of two sheets : $\mathbf{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1}$.

(*i*) It is symmetrical about each of the coordinate planes for only even powers of x, y, z occur in its equation.

(*ii*) It meets the z-axis at $C(0, 0, c), C'(0, 0, -c)$ and the x and y-axes in imaginary points.

(*iii*) Its section by the yz-plane ($x = 0$) is the hyperbola $\dfrac{z^2}{c^2} - \dfrac{y^2}{b^2} = 1$. (*i.e.*, $ACB, A'C'B'$)

Its section by the zx-plane ($y = 0$) is the hyperbola $\dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} = 1$. (*i.e.*, $DCE, D'C'E'$)

Its section by the xy-plane ($z = 0$), is the imaginary ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

Its section by the xy-plane ($z = 0$) is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(*iv*) The surface is generated by a variable ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2} - 1, z = k$,

(as k varies from $-\infty$ to $-c$ and c to $+\infty$) and extends to infinity on both sides of the xy-plane.

Hence its shape is as shown in Fig. 3.63.

(4) Cone : $\frac{\mathbf{x^2}}{\mathbf{a^2}} + \frac{\mathbf{y^2}}{\mathbf{b^2}} - \frac{\mathbf{z^2}}{\mathbf{c^2}} = \mathbf{0}$.

(*i*) It is symmetrical about each of the coordinate planes.

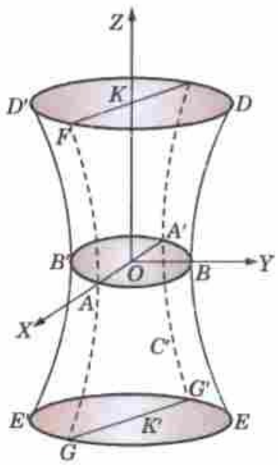


Fig. 3.62

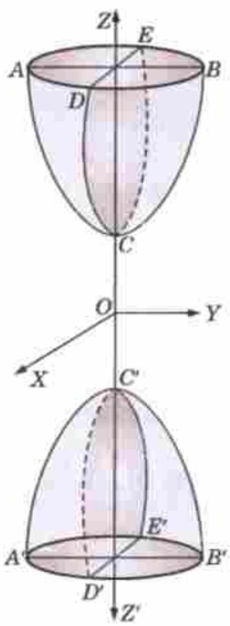


Fig. 3.63

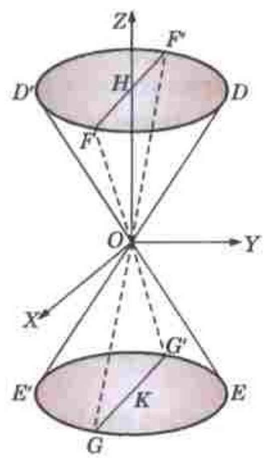


Fig. 3.64

(*ii*) It meets the axes only at the origin.

(*iii*) Its section by the yz-plane ($x = 0$) is the pair of straight lines

$$y = \pm \frac{b}{c} z \qquad (i.e., DOE' \text{ and } D'OE).$$

Its section by the zx-plane ($y = 0$) is the pair of straight lines

$$x = \pm \frac{a}{c} z \qquad (i.e., FOG' \text{ and } F'OG).$$

Its section by the zx-plane ($z = 0$) is the point ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$.

(*iv*) The surface is generated by a variable ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2}$, $z = k$ (k varies)

and extends to infinity on both sides of the xy-plane. Hence its shape is as shown in Fig. 3.64.

(5) Elliptic paraboloid : $\frac{\mathbf{x^2}}{\mathbf{a^2}} + \frac{\mathbf{y^2}}{\mathbf{b^2}} = \frac{\mathbf{2z}}{\mathbf{c}}$

(*i*) It is symmetrical about yz- and zx-planes for only even powers of x and y occur in its equation

(*ii*) It meets the axes at the origin only and touches the xy-plane threat.

(*iii*) Its section by the yz-plane ($x = 0$) is the parabola $y^2 = \frac{2b^2}{c} z$, (*i.e.*, DOD').

Its section by the zx-plane ($y = 0$) is the parabola $x^2 = \frac{2a^2}{c} z$ (*i.e.*, EOE').

Its section by the xy-plane ($z = 0$) is the point ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$

(*iv*) The surface is generated by a variable ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2k}{c}$, $z = k$ (as k varies from 0 to ∞) and it extends to infinity above the xy-plane.

Hence its shape is as shown in Fig. 3.65 and is like that of *tabla*.

(6) Hyperbolic paraboloid : $\frac{\mathbf{x^2}}{\mathbf{a^2}} - \frac{\mathbf{y^2}}{\mathbf{b^2}} = \frac{\mathbf{2z}}{\mathbf{c}}$.

(*i*) It is symmetrical about the yz and zx-planes for only even powers of x and y occur in its equation.

(*ii*) It meets the axes only at the origin and touches the xy-plane threat.

(*iii*) Its section by the yz-plane ($x = 0$) is the parabola $y^2 = -\frac{2b^2}{c} z$. (*i.e.*, DOD')

Its section by the zx-plane ($y = 0$) is the parabola $x^2 = \frac{2a^2}{c} z$ (*i.e.*, EOE').

Its section by the xy-plane ($z = 0$) is the part of lines $y = \pm \frac{b}{a} x$ (not shown in Fig. 3.66.)

(*iv*) The surface is generated by a variable hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2k}{c}$, $z = k$

and it extends to infinity on both sides of xy-plane. Hence its shape is as shown in Fig. 3.66.

(7) Cylinder. An equation of the form $f(x, y) = 0$ represents a cylinder generated by a straight line which is parallel to the z-axis and its section by the xy-plane is the curve $f(x, y) = 0$ (Fig. 3.67).

In particular (*i*) $y^2 = 4ax$ represents a *parabolic cylinder*,

(*ii*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ represents an *elliptic cylinder*, (*iii*) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ represents a *hyperbolic cylinder*.

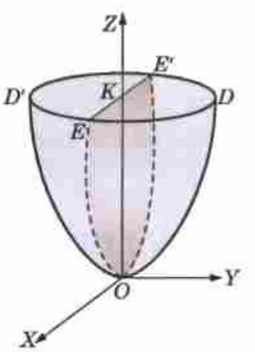


Fig. 3.65

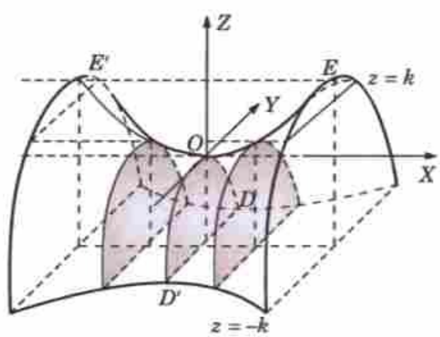


Fig. 3.66

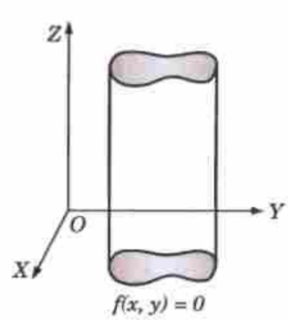


Fig. 3.67

3.22 SURFACES OF REVOLUTION

Let $P(x, y)$ be any point on the curve $y = f(x)$ in the xy-plane. Draw $PM \perp$ to x-axis so that $OM = x$ and $MP = y$. Thus the equation of this curve can be written as

$$MP = f(OM) \qquad ...(1)$$

As this curve revolves about the x-axis, the point P describe a circle with centre M and radius MP. Let $Q(x, y, z)$ be any other position of P. Draw $QN \perp$ to zx-plane and join MN so that $OM = x$, $MN = z$, $NQ = y$

and $\angle MNQ = 90°$. $\therefore$ $MP^2 = MQ^2 = MN^2 + NQ^2 = z^2 + y^2$.

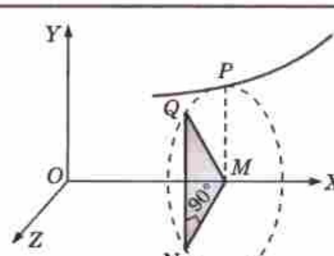


Fig. 3.68

Now substituting the values of MP and MO in (1), we have

$$\sqrt{(y^2+z^2)} = f(x) \quad \text{or} \quad \mathbf{y^2 + z^2 = [f(x)]^2}$$

which is *the equation of the surface generated by the revolution of the curve* $y = f(x)$ *about the x-axis* (Fig. 3.68)

Similarly, *the surface generated by the revolution of the curve*

(*i*) $x = f(y)$ *about y-axis is* $\mathbf{z^2 + x^2 = [f(y)]^2}$, (*ii*) $x = f(z)$ about z-axis is $\mathbf{x^2 + y^2 = [f(z)]^2}$

The given revolving curve is called the *generating curve.*

Some standard surfaces of revolution :

Let the generating curve be $y = f(x)$ in the xy-plane and the axis of rotation be the x-axis ; then the surface generated is $y^2 + z^2 = [f(x)]^2$.

(1) Right-circular cylinder. *When* $f(x) = a$, *the generating curve is a straight line* ($y = a$) *parallel to the x-axis.*

$\therefore$ The surface generated is $\mathbf{y^2 + z^2 = a^2}$

which represents a *right-circular cylinder of radius a and axis as x-axis* (Fig. 3.69).

(2) Right-circular cone. *When* $f(x) = mx$, the generating curve is a straight line ($y = mx$) passing through the origin.

$\therefore$ The surface generated is $y^2 + z^2 = m^2x^2$ or $\mathbf{y^2 + z^2 = x^2 \tan^2 \alpha}$

which represents *a right-circular cone of semi-vertical angle α and axis as the x-axis* (Fig. 3.70).

(3) Sphere. *When* $f(x) = \sqrt{(a^2 - x^2)}$, the generating curve is a circle ($x^2 + y^2 = a^2$).

$\therefore$ The surface generated is

$$y^2 + z^2 = a^2 - x^2 \quad i.e., \quad \mathbf{x^2 + y^2 + z^2 = a^2},$$

which is *a sphere of radius a and centre (0, 0, 0).*

(4) Ellipsoid of revolution. *When* $f(x) = b\sqrt{(1 - x^2/a^2)}$, the generating curve is an ellipse

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right). \qquad \therefore \text{ The surface generated is } y^2 + z^2 = b^2(1 - x^2/a^2)$$

or $$\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1},$$ which is called an *ellipsoid of revolution.*

If $a^2 > b^2$, the major axis of the generating ellipse is along the x-axis—the axis of revolution and the surface generated, in this case, is called a **prolate spheroid** (Fig. 3.71).

If $a^2 < b^2$, the minor axis of the ellipse lies along the x-axis—the axis of revolution and the surface thus generated is called an **oblate spheroid** (Fig. 3.72).

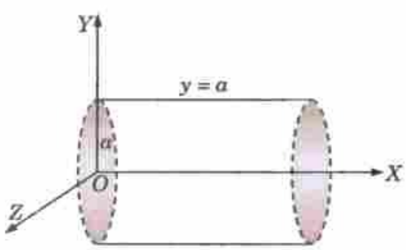


Fig. 3.69

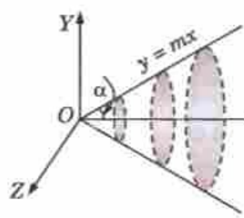
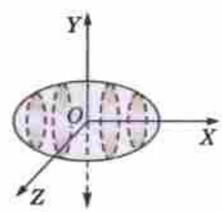
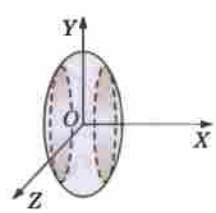


Fig. 3.70



Prolate spheroid

Fig. 3.71



Oblate spheroid

Fig. 3.72

(5) Hyperboloids of revolution

(*i*) *When* $f(x) = b\sqrt{(1 + x^2/a^2)}$, the generating curve is $\dfrac{y^2}{b^2} - \dfrac{z^2}{a^2} = 1$ which represents a hyperbola having conjugate axis along the x-axis.

$\therefore$ The surface generated is $y^2 + z^2 = b^2(1 + x^2/a^2)$

or $$\mathbf{\frac{y^2}{b^2} + \frac{z^2}{b^2} - \frac{x^2}{a^2} = 1}$$ which is called a *hyperboloid of revolution of one sheet* (Fig. 3.73).

(*ii*) When $f(x) = b\sqrt{(x^2/a^2 - 1)}$, the generating curve is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ which represents a hyperbola having transverse axis along the x-axis.

$\therefore$ The surface generated is $y^2 + z^2 = b^2(x^2/a^2 - 1)$

or $\quad \mathbf{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{b^2} = 1}$, which is called a *hyperboloid of revolution of two sheets* (Fig. 3.74).

(6) Paraboloid of revolution. When $f(x) = \sqrt{(ax)}$, the generating curve is a parabola ($y^2 = ax$). The surface generated is $\mathbf{y^2 + z^2 = ax}$. which is called a *paraboloid of revolution* (Fig. 3.75).

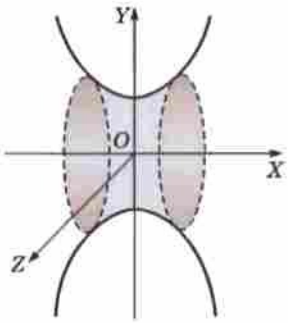


Fig. 3.73

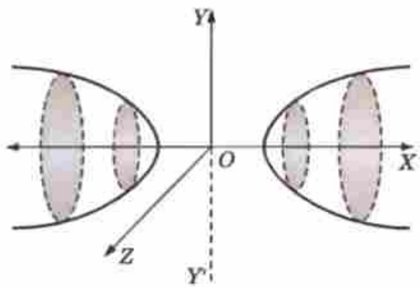


Fig. 3.74

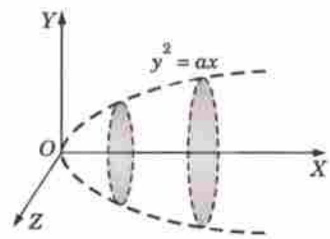


Fig. 3.75

PROBLEMS 3.16

1. What surface is represented by $4x^2 + 9y^2 + 16z^2 = 144$? Trace it roughly. Find the area of the plane curve in which $y = 2$ cuts it.
2. Sketch (roughly) the surface $5(x^2 + z^2) - y^2 = 6$.

   In what curve does the plane $z = 2$ intersect it ? Find the area of the curve of intersection ? What surfaces are represented by the following equations ? Draw diagrams to show their shapes.
3. $x^2 + y^2 = 16$.
4. $x^2/2 - y^2/3 = z$.
5. $z^2 = 4(1 + x^2 + y^2)$.
6. $y^2 = 4z - 8$ (*Andhra, 2000*)
7. $x^2 + y^2 = 5 - 2y$.
8. $x^2 + y^2 = 9z^2$.
9. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{2z}{c}$. (*P.T.U., 2009*)
10. $4x^2 - y^2 - 16z^2 = 36$.

Note. For the equations of the tangent plane and the normal line to *a* surface refer to § 5.8 (2).

3.23 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 3.17

Select the correct answer or fill up the blanks in each of the following questions :

1. The line $x = ay + b, z = cy + d$ and $x = a'y + b', z = c'y + d'$ are perpendicular if

   (*a*) $aa' + cc' = 1$ (*b*) $aa' + cc' = -1$ (*c*) $bb' + dd' = 1$ (*d*) $bb' + dd' = -1$.
2. The coordinates of the point of intersection of the line $\dfrac{x+1}{1} = \dfrac{y+3}{3} = \dfrac{z+2}{-2}$ with the plane $3x + 4y + 5z = 5$ is

   (*a*) $(5, 15, -14)$ (*b*) $(3, 4, 5)$ (*c*) $(1, 3, -2)$ (*d*) $(3, 12, -10)$.
3. The equation of a right circular cylinder, whose axis is the z-axis and radius a is

   (*a*) $x^2 + y^2 + z^2 = a^2$ (*b*) $z^2 + y^2 = a^2$ (*c*) $x^2 + y^2 = a^2$ (*d*) $z^2 + x^2 = a^2$.
4. The equation $\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$ represent a

   (*a*) sphere (*b*) cylinder (*c*) cone (*d*) pair of planes.

5. The plane $ax + by + cz = 0$ cuts the cone $xy + yz + zx = 0$ in perpendicular lines if
   (a) $a + b + c = 0$ (b) $1/a + 1/b + 1/c = 0$
   (c) $a^2 + b^2 + c^2 = 0$ (d) $abc = 0$.
6. The equation of the cylinder which intersects the curve $x^2 + y^2 + z^2 = 1, x + y + z = 1$ and whose generators are parallel to the axis of z, is
   (a) $x^2 + y^2 + xy - x - y = 0$ (b) $x^2 + y^2 + xy + x + y = 0$
   (c) $x^2 + y^2 - xy - x - y = 0$ (d) $x^2 + y^2 - xy + x + y = 0$.
7. The equation $x^2 + y^2 + z^2 + xy + yz - zx = 9$ represents
   (a) a sphere with $x^2 + y^2 + z^2 = 9, x - y + z = 3$ as a great circle
   (b) a cone with $x^2 + y^2 + z^2 = 9, x - y + z = 3$ as a guiding circle
   (c) a cylinder with $x^2 + y^2 + z^2 = 9, x - y + z = 3$ as a guiding circle
   (d) none of the above.
8. The sum of the direction cosines of a straight line is
   (a) zero (b) one (c) constant (d) n one of the above.
9. The angle between the planes $x - y + 2z - 9 = 0$ and $2x + y + z = 7$ is
   (a) 30° (b) 90° (c) 60° (d) 120° (V.T.U., 2010 S)
10. The equation of the right circular cone whose axis is $x = y = z$, vertex is the origin and the semi-vertical angle is 45° is given as
    (a) $x^2 + y^2 + z^2 = 0$ (b) $2(x^2 + y^2 + z^2) = 3(x + y + z)^2$
    (c) $3(x^2 + y^2 + z^2) = 2(x + y + z)^2$ (d) $x^2 + y^2 + z^2 + xy + yz + zx = 0$.
11. The equation of a straight line parallel to the x-axis is given by
    (a) $\frac{x-a}{1} = \frac{y-b}{1} = \frac{z-c}{1}$ (b) $\frac{x-a}{0} = \frac{y-b}{1} = \frac{z-c}{1}$
    (c) $\frac{x-a}{0} = \frac{y-b}{0} = \frac{z-c}{1}$ (d) $\frac{x-a}{1} = \frac{y-b}{0} = \frac{z-c}{0}$.
12. The equation of the plane passing through (4, – 2, 1) and perpendicular to the line with direction ratios 7, 2, – 3 is
    (a) $x + 3y - 4z - 8 = 0$ (b) $2x + 7y - 3z - 24 = 0$
    (c) $7x + 2y - 3z - 21 = 0$. (d) $7x + 3y - 2z + 21 = 0$. (V.T.U., 2009 S)
13. The equation to the axis of the right circular cylinder whose guiding circle is $x^2 + y^2 + z^2 = 9, x - y + z = 3$ is given by
    (a) $x = y = z$ (b) $x = -y = z$ (c) $x = y = -z$ (d) $x = -y = -z$.
14. In three dimensions, the equation $x^2 - y^2 = a^2$ represents
    (a) a pair of straight lines (b) a hyperbola
    (c) a cylinder (d) a cone.
15. Section of a sphere by a plane is
    (a) parabola (b) ellipse (c) circle.
16. A line makes angles α, β, γ with the coordinate axes, then $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$ is equal to
    (a) 1 (b) 2 (c) – 1 (d) – 2. (V.T.U., 2010 S)
17. Three lines are coplanar if
    (a) they are concurrent
    (b) a line is perpendicular to each of them
    (c) they are concurrent and a line is perpendicular to each of them.
18. The distance between the planes $2x + 2y + z - 6 = 0$ and $4x + 4y + 2z - 7 = 0$ is
    (a) 1/3 (b) 5/6 (c) 13/3.
19. The line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and the plane $x + y - z = 0$ are
    (a) parallel (b) perpendicular (c) such that the line lies in the plane.
20. The radius of a great circle of a sphere is
    (a) greater than the radius of the sphere (b) less than the radius of the sphere
    (c) equal to the radius of the sphere.
21. Which of the following lines are generators of the cone $yz + 4zx + 3xy = 0$?
    (a) $x = y = z$ (b) $x = -y = z$ (c) $x = 2y = -z$.

22. The semi-vertical angle of the cone generated by revolving the line $x + y = 0$, $z = 0$ about the x-axis is
(a) 90° (b) 45° (c) 30°.

23. All cones passing through the coordinate axes are given by the equation
(a) $x^2 + y^2 + z^2 - yz - zx - xy = 0$ (b) $ax^2 + by^2 + cz^2 - yz - zx - xy = 0$
(c) $ayz + bzx + cxy = 0$.

24. The line $\frac{x+1}{3} = \frac{y-2}{6} = \frac{z-3}{9}$ is perpendicular to the plane $ax + by + cz + d = 0$, if
(a) $a = 2b, b = 3c$ (b) $2a = b, b = 3c$ (c) $2a = b, 3b = 2c$ (d) $a = 3b, 2b = c$.

25. The equation $2(x^2 + y^2 + z^2) - 2xy + 2yz + 2zx = 3a^2$ represents a
(a) cone (b) right-circular cylinder
(c) sphere (d) pair of planes.

26. The equation of the plane through the point $(2, -3, 1)$ and parallel to the plane $3x - 4y + 2z = 5$ is
(a) $3x - 4y + 2z - 20 = 0$ (b) $3x + 4y - 2z + 20 = 0$
(c) $3x - 4y - 2z + 20 = 0$ (d) $3x + 4y + 2z - 20 = 0$.

27. The direction cosines of a line which is equally inclined to the coordinate axes are ....

28. The direction cosines of the line $x = 0 = y$ are ....

29. The equation of the axis of the cylinder $x^2 + y^2 = 25$ is ......

30. The image of the point $(3, 2, -1)$ in the YOZ plane is ......

31. The plane $x - 2y - 2z = k$ touches the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$ for $k =$ ...... (P.T.U., 2010)

32. The condition for the three concurrent lines to be coplanar is ......

33. The equation of the cone whose vertex is at the origin and base the circle $x = a$, $y^2 + z^2 = b^2$ is given by ....

34. The plane through points $(2, 2, 1)$, $(9, 3, 6)$ and perpendicular to the plane $2x + 6y + 6z = 9$ is ......

35. Volume of the sphere $x^2 + y^2 + z^2 + 2x - 4y + 8z - 2 = 0$ is ......

36. Angle between the planes $x - y + z = 1$ and $2x - 3y + z = 7$ is ......

37. The equation of the cone whose vertex is the origin and ginding curve is $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1$, $x + y + z = 1$, is ...... (Anna, 2009)

38. Any two points on the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ other than $(1, 2, 3)$ are ......

39. The equation of the line joining the points $(1, 2, 3)$ and $(2, 1, -3)$ is......

40. The equation of the sphere on the line joining $(1, 5, 6)$ and $(-2, 1, 1)$ as diameter is ......

41. The conditions for the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ to lie on the plane $ax + by + cz + d = 0$ are ......

42. The distance between the planes $4x + 3y + z + 4 = 0$ and $8x + 6y + 2z + 12 = 0$ is ....

43. The centre and radius of the sphere $2x^2 + 2y^2 + 2z^2 - 6x + 8y - 8z - 1 = 0$ are ......

44. The radius of the circle $x^2 + y^2 + z^2 - 2x - 4y - 11 = 0$, $x + 2y + 2z = 15$ is ......

45. The symmetric form of the line $x + y + z + 1 = 0 = 4x + y - 2z + 2$ is ......

46. The equation $y^2 = 4z - 8$ represents a ....

47. The equation $x^2 + y^2 = \frac{1}{4}z^2 - 1$ represents a ......

48. Angle between the lines whose d.r.s. are 1, 2, 3 and -1, 1, 2 is ......

49. The intercepts of the plane $2x - 3y + z = 12$ on the coordinate axes are ....

50. The radius of the sphere whose centre is $(4, 4, -2)$ and which passes through the origin is ......

51. The points $(0, 4, 1)$, $(2, 3, -1)$, $(4, 5, 0)$ and $(2, 6, 2)$ are the vertices of a square. (True or False)

52. The points $(3, -1, 1)$, $(5, -4, 2)$ and $(11, -13, 5)$ are collinear. (True or False)

53. The plane $5x + 6y + 7z = 110$, $2x + 3y - 4z = 29$ are perpendicular to each other. (True or False)

54. In three dimensional space, $9x^2 + 16y^2 = 144$ represents ....

55. Equation of the right circular cone with vertex at origin and passing through the curve $x^2 + y^2 + z^2 = 9$, $x + y + z = 1$ is ....

56. A unit vector perpendicular to the vectors $-2\mathbf{I} + 3\mathbf{J} + \mathbf{K}$ and $4\mathbf{I} + 2\mathbf{J}$ is ....