CHAPTER 10

Fourier Series

10.1 INTRODUCTION

In many engineering problems, especially in the study of periodic phenomenae* in conduction of heat, electro-dynamics and acoustics, it is necessary to express a function in a series of sines and cosines. Most of the single-valued functions which occur in applied mathematics can be expressed in the form.

$$\tfrac{1}{2}a_0\dagger + a_1 \cos x + a_2 \cos 2x + \ldots\ldots\dagger$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots\ldots$$

within a desired range of values of the variable. Such a series is known as the **Fourier series$.**

10.2 EULER'S FORMULAE

The Fourier series for the function f(x) in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \qquad \ldots(1)$$

These values of a_0, a_n, b_n are known as *Euler's formulae***.

***Periodic functions.** If at equal intervals of abscissa x, the value of each ordinate $f(x)$ repeats itself, *i.e.*, $f(x) = f(x + a)$, for all x, then $y = f(x)$ is called a *periodic function* having **period** α, *e.g.*, $\sin x$, $\cos x$ are periodic functions having a period 2π.

† To write $a_0/2$ instead of a_0 is a conventional device to be able to get more symmetric formulae for the coefficients.

$ Named after the French mathematician and physicist *Jacques Fourier* (1768–1830) who was first to use Fourier series in his memorable work *'Theorie Analytique de la Chaleur'* in which he developed the theory of heat conduction. These series had a deep influence in the further development of mathematics and mathematical physics.

**See footnote p. 205.

To establish these formulae, the following definite integrals will be required :

1. $\int_{\alpha}^{\alpha+2\pi} \cos nx\, dx = \left| \frac{\sin nx}{n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(n \neq 0)$

2. $\int_{\alpha}^{\alpha+2\pi} \sin nx\, dx = -\left| \frac{\cos nx}{n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(n \neq 0)$

3. $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\, dx$ $(n \neq 0)$

$$= \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} [\cos(m+n)x + \cos(m-n)x]\, dx$$

$$= \frac{1}{2}\left| \frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right|_{\alpha}^{\alpha+2\pi} = 0 \qquad (m \neq n)$$

4. $\int_{\alpha}^{\alpha+2\pi} \cos^2 nx\, dx = \left| \frac{x}{2} + \frac{\sin 2nx}{4n} \right|_{\alpha}^{\alpha+2\pi} = \pi$ $(n \neq 0)$

5. $\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx\, dx = -\frac{1}{2}\left[\frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n} \right] = 0$ $(m \neq n)$

6. $\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx\, dx = \left| \frac{\sin^2 nx}{2n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(n \neq 0)$

7. $\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx\, dx = \frac{1}{2}\left| \frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(m \neq n)$

8. $\int_{\alpha}^{\alpha+2\pi} \sin^2 nx\, dx = \left| \frac{x}{2} - \frac{\sin 2nx}{4n} \right|_{\alpha}^{\alpha+2\pi} = \pi.$ $(n \neq 0)$

Proof. Let $f(x)$ be represented in the interval $(\alpha, \alpha + 2\pi)$ by the Fourier series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(i)$$

To find the coefficients a_0, a_n, b_n, we assume that the series (i) can be integrated term by term from $x = \alpha$ to $x = \alpha + 2\pi$.

To find a_0, integrate both sides of (i) from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_{\alpha}^{\alpha+2\pi} f(x)\, dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) dx$$

$$= \frac{1}{2} a_0(\alpha + 2\pi - \alpha) + 0 + 0 = a_0\pi \qquad \text{[By integrals (1) and (2) above]}$$

Hence $$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\, dx.$$

To find a_n, multiply each side of (i) by $\cos nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \cos nx\, dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) \cos nx\, dx$$

$$+ \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) \cos nx\, dx$$

$$= 0 + \pi a_n + 0 \qquad \text{[By integrals (1), (3), (4), (5) and (6)]}$$

Hence $$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx.$$

To find b_n, multiply each side of (*i*) by $\sin nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_\alpha^{\alpha+2\pi} f(x)\sin nx\,dx = \frac{1}{2}a_0\int_\alpha^{\alpha+2\pi}\sin nx\,dx + \int_\alpha^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)\sin nx\,dx + \int_\alpha^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + \pi b_n \qquad \text{[By integrals (2), (5), (6), (7) and (8)]}$$

Hence $$b_n = \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(x)\sin nx\,dx.$$

Cor. 1. Making $\alpha = 0$, *the interval becomes* $0 < x < 2\pi$, and the formulae (I) reduce to

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \qquad ...(II)$$

Cor. 2. Putting $\alpha = -\pi$, *the interval becomes* $-\pi < x < \pi$ and the formulae (I) take the form :

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}\right\} \qquad ...(III)$$

Example 10.1. *Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.* *(S.V.T.U., 2007)*

Solution. Let $$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \qquad ...(i)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left|-e^{-x}\right|_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}$$

and $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\left| e^{-x}(-\cos nx + n\sin nx)\right|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{n^2+1}$$

$\therefore$ $$a_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2},\ a_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{5}\text{ etc.}$$

Finally, $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\left| e^{-x}(-\sin nx - n\cos nx)\right|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{n}{n^2+1}$$

$\therefore$ $$b_1 = \frac{1-e^{-2\pi}}{\pi}\cdot\frac{1}{2},\ b_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{2}{5}\text{ etc.}$$

Substituting the values of a_0, a_n, b_n in (*i*), we get

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + ...\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + ...\right)\right\}.$$

Example 10.2. *Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$.*

(V.T.U., 2011 ; Madras, 2006)

Solution. Let $$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(i)$$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\, dx = \frac{1}{\pi}\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_{-\pi}^{\pi} = -\frac{2\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \cos nx\, dx^*$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x)\times\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-4(-1)^n}{n^2} \qquad [\because \cos n\pi = (-1)^n]$$

$\therefore$ $a_1 = 4/1^2, a_2 = -4/2^2, a_3 = 4/3^2, a_4 = -4/4^2$ etc.

Finally, $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \sin nx\, dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1 - 2x)\times\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi} = -2(-1)^n/n$$

$\therefore$ $b_1 = 2/1, b_2 = -2/2, b_3 = 2/3, b_4 = -2/4$ etc.

Substituting the values of a's and b's in (i), we get

$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + ...\right]$$
$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + ...\right]$$

Obs. Putting $x = 0$, we find another interesting series $0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...\right)$

i.e., $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... = \frac{\pi^2}{12}. \qquad \textit{(V.T.U., 2011)}$$

Note. In the above example, we have used the results $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$

Also $\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$ and $\cos\left(n + \frac{1}{2}\right)\pi = 0$. *The reader should remember these results.*

Example 10.3. *Expand $f(x) = x \sin x$ as a Fourier series in the interval $0 < x < 2\pi$.*

(S.V.T.U., 2009 ; Bhopal, 2009 ; Rohtak, 2006)

Solution. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(i)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} x \sin x\, dx = \frac{1}{\pi}\left| x(-\cos x) - 1.(-\sin x)\right|_0^{2\pi} = -2.$$

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* Apply the *general rule of integration by parts* which states that if u, v be two functions of x and dashes denote differentiations and suffixes integrations w.r.t. x, then

$$\int uv\ dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...$$

In other words : *Integral of the product of two functions*

= 1st function × integral of 2nd – go on differentiating 1st, integrating 2nd signs alternately +ve and –ve.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,(2\cos nx \sin x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,[\sin (n+1)x - \sin (n-1)x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{-\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right\} - 1\left\{-\frac{\sin (n+1)x}{(n+1)^2} + \frac{\sin (n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right] = \frac{2}{n^2-1}.\ (n \neq 1)$$

When $n = 1$, $a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} = -\frac{1}{2}.$$

Finally, $b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,[\cos (n-1)x - \cos (n+1)x]dx$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin (n-1)x}{n-1} - \frac{\sin (n+1)}{n+1}\right\} - 1.\left\{-\frac{\cos (n-1)x}{(n-1)^2} + \frac{\cos (n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0\ (n \neq 1)$$

When $n = 1$, $b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,(1 - \cos 2x)\, dx$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi} = \pi$$

Substituting the values of a's and b's, in (*i*), we get

$$x \sin x = -1 + \pi \sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + \ldots$$

Example 10.4. *Expand* $f(x) = \sqrt{(1-\cos x)}$, $0 < x < 2\pi$ *in a Fourier series. Hence evaluate*

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots$$ (*Mumbai, 2006 ; J.N.T.U., 2006*)

Solution. We have $f(x) = \sqrt{(1-\cos x)} = \sqrt{2}\sin x/2$.

Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(i)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin x/2 \, dx = \frac{\sqrt{2}}{\pi}\left| -2\cos\frac{\pi}{2}\right|_0^{2\pi} = \frac{4\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\cos nx \, dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\cos nx \sin x/2 \, dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi} \sin\left(n + \frac{1}{2}\right)x - \sin\left(n - \frac{1}{2}\right)x\,\Big]dx$$

$$= \frac{1}{\sqrt{2}\,\pi}\left| -\frac{2}{2n+1}\cos\left(\frac{2n+1}{2}\right) + \frac{2}{2n-1}\cos\frac{2n-1}{2}x\right|_0^{2\pi}$$

$$= \frac{2}{\sqrt{2}\,\pi}\left\{-\frac{1}{2n+1}[\cos (2n+1)\pi - 1] + \frac{1}{2n-1}[\cos (2n-1)\pi - 1]\right\}$$

$$= \frac{\sqrt{2}}{\pi}\left(\frac{2}{2n+1} - \frac{2}{2n-1}\right) = -\frac{4\sqrt{2}}{\pi(4n^2-1)} \qquad [\because \cos(2n+1)\pi = \cos(2n-1)\pi = -1]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\sin nx\,dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\sin nx \sin x/2\,dx$$

$$= \frac{1}{\sqrt{2}\,\pi}\int_0^{2\pi}\left[\cos\left(n-\frac{1}{2}\right)x - \cos\left(n+\frac{1}{2}\right)x\right]dx$$

$$= \frac{1}{\sqrt{2}\,\pi}\left|\frac{2}{2n-1}\sin\left(\frac{2n-1}{2}\right)x - \frac{2}{2n+1}\sin\left(\frac{2n+1}{2}\right)x\right|_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{2n-1}\{\sin(2n-1)\pi - 0\} - \frac{1}{2n+1}\{\sin(2n+1)\pi - 0\}\right] = 0$$

Substituting the values of a's and b's in (i), we get

$$\sqrt{(1-\cos x)} = \frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{(4n^2-1)\pi}\cos nx$$

When $x = 0$, we have

$$0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)} \qquad i.e., \quad \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots \infty = \frac{1}{2}.$$

PROBLEMS 10.1

1. Obtain a Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive series for $\pi/\sinh \pi$.

2. Prove that $x^2 = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\dfrac{\cos nx}{n^2}$, $-\pi < x < \pi$. *(P.T.U., 2009 ; Bhopal, 2008 ; B.P.T.U., 2006)*

   Hence show that (i) $\sum \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$. *(Anna, 2009 ; P.T.U., 2009 ; Osmania, 2003)*

   (ii) $\sum \dfrac{1}{(2n-1)^2} = \dfrac{\pi^2}{8}$ $\quad$ (iii) $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots = \dfrac{\pi^2}{12}$ *(S.V.T.U., 2008)*

   (iv) $\sum \dfrac{1}{n^4} = \dfrac{\pi^4}{90}$. *(Bhopal, 2008)*

3. If $f(x) = \left(\dfrac{\pi - x}{2}\right)^2$ in the range 0 to 2π, show that $f(x) = \dfrac{\pi^2}{12} + \sum_{n=1}^{\infty}\dfrac{\cos nx}{n^2}$. *(Delhi, 2002 ; Madras, 2000)*

4. Prove that in the range $-\pi < x < \pi$, $\cosh ax = \dfrac{2a^2}{\pi}\sinh a\pi\left[\dfrac{1}{2a^2} + \sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2+a^2}\cos nx\right]$.

5. $f(x) = x + x^2$ for $-\pi < x < \pi$ and $f(x) = \pi^2$ for $x = \pm\pi$. Expand $f(x)$ in Fourier series. *(Kurukshetra, 2005 ; U.P.T.U., 2003)*

   Hence show that $x + x^2 = \dfrac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left\{\dfrac{4}{n^2}\cos nx - \dfrac{2}{n}\sin nx\right\}$

   and $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots \dfrac{\pi^2}{8}$ *(V.T.U., 2008)*

10.3 CONDITIONS FOR A FOURIER EXPANSION

The reader must not be misled by the belief that the Fourier expansion of $f(x)$ in each case shall be valid. The above discussion has merely shown that if $f(x)$ has an expansion, then the coefficients are given by Euler's formulae. The problems concerning the possibility of expressing a function by Fourier series and convergence of this series are many and cumbersome. Such questions should be left to the curiosity of a pure-mathematician. However, almost all engineering applications are covered by the following well-known **Dirichlet's conditions***:

Any function f(x) can be developed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ *where* a_0, a_n, b_n *are constants, provided :*

(i) $f(x)$ *is periodic, single-valued and finite;*

(ii) $f(x)$ *has a finite number of discontinuties in any one period;*

(iii) $f(x)$ *has at the most a finite number of maxima and minima.* (*Anna, 2009 ; P.T.U., 2009*)

In fact the problem of expressing any function $f(x)$ as a Fourier series depends upon the evaluation of the integrals.

$$\frac{1}{\pi}\int f(x) \cos nx \, dx \,; \frac{1}{\pi}\int f(x) \sin nx \, dx$$

within the limits $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha + 2\pi)$ according as $f(x)$ is defined for every value of x in $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha + 2\pi)$.

PROBLEMS 10.2

State giving reasons whether the following functions can be expanded in Fourier series in the interval $-\pi \le x \le \pi$.

1. cosec x
2. sin $1/x$ (*P.T.U., 2002*)
3. $f(x) = (m + 1)/m, \pi/(m + 1) < | x | \le \pi/m, m = 1, 2, 3, \ldots \infty$.

10.4 FUNCTIONS HAVING POINTS OF DISCONTINUITY

In deriving the Euler's formulae for a_0, a_n, b_n, it was assumed that $f(x)$ was continuous. Instead a function may have a finite number of points of finite discontinuity *i.e.*, its graph may consist of a finite number of different curves given by different equations. Even then such a function is expressible as a Fourier series.

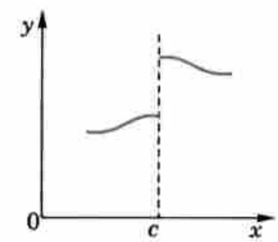


Fig. 10.1

For instance, if in the interval $(\alpha, \alpha + 2\pi)$, $f(x)$ is defined by

$$f(x) = \phi(x), \alpha < x < c.$$
$$= \psi(x), c < x < \alpha + 2\pi,$$ *i.e.*, c is the point of discontinuity, then

$$a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\, dx + \int_{c}^{\alpha+2\pi} \psi(x)\, dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x) \cos nx \, dx + \int_{c}^{\alpha+2\pi} \psi(x) \cos nx \, dx\right]$$

and $$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x) \sin nx \, dx + \int_{c}^{\alpha+2\pi} \psi(x) \sin nx \, dx\right]$$

At a point of finite discontinuity $x = c$, there is a finite jump in the graph of function (Fig. 10.1). Both the limit on the left [*i.e.*, $f(c - 0)$] and the limit on the right [*i.e.*, $f(c + 0)$] exist and are different. At such a point, Fourier series gives the value of $f(x)$ as the arithmetic mean of these two limits,

i.e., at $x = c$, $f(x) = \frac{1}{2}[f(c - 0) + f(c + 0)]$.

Example 10.5. *Find the Fourier series expansion for* $f(x)$, *if*

$$f(x) = -\pi, -\pi < x < 0$$
$$x, 0 < x < \pi.$$ (*Bhopal, 2008 S*)

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$. (*Kottayam, 2005*)

*See footnote p. 307.

Solution. Let $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ...(i)

Then $$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_0^{\pi} x\,dx\right] = \frac{1}{\pi}\left[-\pi \,\big| x \big|_{-\pi}^{0} + \big| x^2/2 \big|_0^{\pi}\right] = \frac{1}{\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right) = -\frac{\pi}{2};$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1)$$

$\therefore$ $$a_1 = \frac{-2}{\pi\,.\,1^2},\ a_2 = 0,\ a_3 = -\frac{2}{\pi\,.\,3^2},\ a_4 = 0,\ a_5 = -\frac{2}{\pi\,.\,5^2} \text{ etc.}$$

Finally, $$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{\pi\cos nx}{n}\right|_{-\pi}^{0} + \left|-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1 - 2\cos n\pi)$$

$\therefore$ $$b_1 = 3,\ b_2 = -\frac{1}{2},\ b_3 = 1,\ b_4 = -\frac{1}{4},\text{ etc.}$$

Hence substituting the values of a's and b's in (i), we get

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots \quad \text{...(ii)}$$

which is the required result.

Putting $x = 0$ in (ii), we obtain $f(0) = -\dfrac{\pi}{4} - \dfrac{2}{\pi}\left(1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots \infty\right)$ ...(iii)

Now $f(x)$ is discontinuous at $x = 0$. As a matter of fact

$f(0-0) = -\pi$ and $f(0+0) = 0$ $\therefore$ $f(0) = \dfrac{1}{2}[f(0-0) + f(0+0)] = -\pi/2.$

Hence (iii) takes the form $-\dfrac{\pi}{2} = -\dfrac{\pi}{4} - \dfrac{2}{\pi}\left[\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\right]$ whence follows the result.

Example 10.6. *If* $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$, *prove that* $f(x) = \dfrac{1}{\pi} + \dfrac{\sin x}{2} - \dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{\cos 2nx}{4n^2 - 1}$.

Hence show that $\dfrac{1}{1\,.\,3} - \dfrac{1}{3\,.\,5} + \dfrac{1}{5\,.\,7} - \ldots - \infty = \dfrac{1}{4}(\pi - 2)$ (Bhopal, 2008 ; Mumbai, 2005 S ; Rohtak, 2005)

Solution. Let $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}\sin x\,dx\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,.\,dx + \int_0^{\pi}\sin x\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx = \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \qquad (n \neq 1)$$

$$= \frac{1}{2\pi}\left\{\frac{1-(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}-1}{n-1}\right\} = 0, \text{ when } n \text{ is odd}$$

$$= -\frac{2}{\pi(n^2-1)}, \text{ when } n \text{ is even.}$$

When $n = 1$, $a_1 = \frac{1}{\pi}\int_0^\pi \sin x \cos x\, dx = \frac{1}{2\pi}\int_0^\pi \sin 2x\, dx = \frac{1}{2\pi}\left| -\frac{\cos 2x}{2}\right|_0^\pi = 0$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,.\,dx + \int_0^\pi \sin x \sin nx\, dx\right]$

$$= \frac{1}{2\pi}\int_0^\pi [\cos \overline{n-1}x - \cos \overline{n+1}x]\, dx = \frac{1}{2\pi}\left[\frac{\sin \overline{n-1}x}{n-1} - \frac{\sin \overline{n+1}x}{n+1}\right]_0^\pi = 0 \quad (n \neq 1)$$

When $n = 1$, $b_1 = \frac{1}{\pi}\int_0^\pi \sin x \sin x\, dx = \frac{1}{2\pi}\int_0^\pi (1-\cos 2x)\, dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{1}{2}$

Hence $f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \ldots\right] + \frac{1}{2}\sin x$ ...(i)

Putting $x = \frac{\pi}{2}$ in (i), we get $1 = \frac{1}{\pi} - \frac{2}{\pi}\left(-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \ldots \infty\right) + \frac{1}{2}$

Whence $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots \infty = \frac{1}{4}(\pi - 2).$

Example 10.7. *Find the Fourier series for the function*

$$f(t) = \begin{cases} -1 & \text{for } -\pi < t < -\pi/2 \\ 0 & \text{for } -\pi/2 < t < \pi/2 \\ 1 & \text{for } \pi/2 < t < \pi \end{cases}$$

Solution. Let $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$ ...(i)

Then $a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1)dt + \int_{-\pi/2}^{\pi/2} (0)dt + \int_{\pi/2}^{\pi} (1)\, dt\right\}$

$$= \frac{1}{\pi}\left\{\left| -x\right|_{-\pi}^{-\pi/2} + \left| x\right|_{\pi/2}^{\pi}\right\} = \frac{1}{\pi}(\pi/2 - \pi + \pi - \pi/2) = 0$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1)\cos nt\, dt + \int_{-\pi/2}^{\pi/2} (0)\cos nt\, dt + \int_{\pi/2}^{\pi} (1)\cos nt\, dt\right\}$$

$$= \frac{1}{\pi}\left\{\left| -\frac{\sin nt}{n}\right|_{-\pi}^{-\pi/2} + \left| \frac{\sin nt}{n}\right|_{\pi/2}^{\pi}\right\} = \frac{1}{n\pi}\left(\frac{\sin n\pi}{2} - \frac{\sin n\pi}{2}\right) = 0$$

and $b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1)\sin nt\, dt + \int_{-\pi/2}^{\pi/2} (0)\sin nt\, dt + \int_{\pi/2}^{\pi} (1)\sin nt\, dt\right\}$

$$= \frac{1}{\pi}\left\{\left| \frac{\cos nt}{n}\right|_{-\pi}^{-\pi/2} + \left| -\frac{\cos nt}{n}\right|_{\pi/2}^{\pi}\right\} = \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)$$

$\therefore \qquad b_1 = \frac{2}{\pi}, b_2 = -\frac{2}{\pi}, b_3 = \frac{2}{3\pi}$ etc.

Hence substituting the values of a's and b's in (i), we get $f(t) = \frac{2}{\pi}\left(\sin t - \sin 2t + \frac{1}{3}\sin 3t + \dots\right)$.

PROBLEMS 10.3

1. Find the Fourier series to represent the function $f(x)$ given by

$$f(x) = x \text{ for } 0 \le x \le \pi, \text{ and } = 2\pi - x \text{ for } \pi \le x \le 2\pi.$$
(S.V.T.U., 2008 ; B.P.T.U., 2005 S)

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8}$. (Madras 2000 S ; V.T.U., 2000 S)

2. An alternating current after passing through a rectifier has the form

$$i = I_0 \sin x \quad \text{for } 0 \le x \le \pi$$
$$= 0 \quad \text{for } \pi \le x \le 2\pi$$

where I_0 is the maximum current and the period is 2π (Fig. 10.2). Express i as a Fourier series and evaluate

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots \infty$$
(V.T.U., 2007 ; Calicut, 2005)

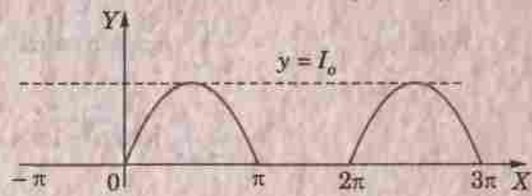


Fig. 10.2

3. Draw the graph of the function $f(x) = 0, -\pi < x < 0$

$$= x^2, 0 < x < \pi.$$

If $f(2\pi + x) = f(x)$, obtain Fourier series of $f(x)$.

4. Find the Fourier series of the following function :

$$f(x) = x^2, \quad 0 \le x \le \pi,$$
$$= -x^2, \quad -\pi \le x \le 0.$$
(Mumbai, 2009) (Hissar, 2007)

5. Find a Fourier series for the function defined by

$$f(x) = \begin{cases} -1, \text{ for } -\pi < x < 0 \\ 0, \text{ for } x = 0 \\ 1, \text{ for } 0 < x < \pi \end{cases}$$

Hence prove that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \infty = \frac{\pi}{4}$. (U.P.T.U., 2005)

10.5 CHANGE OF INTERVAL

In many engineering problems, the period of the function required to be expanded is not 2π but some other interval, say : $2c$. In order to apply the foregoing discussion to functions of period $2c$, this interval must be converted to the length 2π. This involves only a proportional change in the scale.

Consider the periodic function $f(x)$ defined in $(\alpha, \alpha + 2c)$. To change the problem to period 2π

put $\quad z = \pi x/c \quad$ or $\quad x = cz/\pi \quad$ ...(1)

so that when $\quad x = \alpha, \quad z = \alpha\pi/c = \beta$ (say)

when $\quad x = \alpha + 2c, \quad z = (\alpha + 2c)\pi/c = \beta + 2\pi.$

Thus the function $f(x)$ of period $2c$ in $(\alpha, \alpha + 2c)$ is transformed to the function $f(cz/\pi)$ [$= F(z)$ say] of period 2π in $(\beta, \beta + 2\pi)$. Hence $f(cz/\pi)$ can be expressed as the Fourier series

$$f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \qquad \dots(2)$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) dz \\ a_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \cos nz\, dz \\ b_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \sin nz\, dz \end{aligned}\right\} \qquad \dots(3)$$

Making the inverse substitutions $z = \pi x/c$, $dz = (\pi/c)\,dx$ in (2) and (3) *the Fourier expansion of $f(x)$ in the interval $(\alpha, \alpha + 2c)$ is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx \\ a_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx \\ b_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx \end{aligned}\right\} \qquad ...(4)$$

Cor. Putting $\alpha = 0$ in (4), we get the results for the interval $(0, 2c)$ and putting $\alpha = -c$ in (4), we get results for the interval $(-c, c)$.

Example 10.8. *Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.*

(Kerala, 2005 ; V.T.U., 2004)

Solution. The required series is of the form

$$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad ...(i)$$

Then $\quad a_0 = \dfrac{1}{l}\displaystyle\int_{-l}^{l} e^{-x}\,dx = \frac{1}{l}\Big|-e^{-x}\Big|_{-l}^{l} = \frac{1}{l}(e^{l} - e^{-l}) = \frac{2\sinh l}{l}$

and $\quad a_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} e^{x}\cos\frac{n\pi x}{l}\,dx \qquad \left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$

$$= \frac{1}{l}\left|\frac{e^{-x}}{1+(n\pi/l)^2}\left(-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right|_{-l}^{l} = \frac{2l(-1)^n\sinh l}{l^2+(n\pi)^2} \qquad [\because \cos n\pi = (-1)^n$$

$\therefore \quad a_1 = \dfrac{-2l\sinh l}{l^2+\pi^2},\ a_2 = \dfrac{2l\sinh l}{l^2+2^2\pi^2},\ a_3 = \dfrac{2l\sinh l}{l^2+3^2\pi^2}$ etc.

Finally, $\quad b_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}\,dx \qquad \left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$

$$= \frac{1}{l}\left|\frac{e^{-x}}{1+(n\pi/l)^2}\left(-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right|_{-l}^{l} = \frac{2n\pi(-1)^n\sinh l}{l^2+(n\pi)^2}$$

$\therefore \quad b_1 = \dfrac{-2\pi\sinh l}{l^2+\pi^2},\ b_2 = \dfrac{4\pi\sinh l}{l^2+2^2\pi^2},\ b_2 = \dfrac{-6\pi\sinh l}{l^2+3^2\pi^2}$ etc.

Substituting the values of a's and b's in (i), we get

$$e^{-x} = \sinh l\left\{\frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{l^2+3^2\pi^2}\cos\frac{3\pi x}{l} - ...\right)\right.$$
$$\left. - 2\pi\left(\frac{1}{l^2+\pi^2}\sin\frac{n\pi}{l} - \frac{2}{l^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \frac{3}{l^2+3^2\pi^2}\sin\frac{3\pi x}{l} - ...\right)\right\}$$

Example 10.9. *Find the Fourier series expansion of $f(x) = 2x - x^2$ in $(0, 3)$ and hence deduce that $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + ... - \infty = \dfrac{\pi}{12}$.*

(Mumbai, 2005)

Solution. The required series is of the form

$$2x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \quad \text{where } l = 3/2. \qquad ...(i)$$

Then $$a_0 = \frac{1}{l}\int_0^{2l} (2x - x^2)dx = \frac{2}{3}\left| x^2 - \frac{x^3}{3}\right|_0^3 = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} (2x - x^2)\cos\frac{n\pi x}{l}dx = \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{\sin 2n\pi x/3}{2n\pi/3} - (2 - 2x)\frac{-\cos 2n\pi x/3}{(2n\pi/3)^2} + (-2)\frac{-\sin 2n\pi x/3}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\cdot\frac{9}{4n^2\pi^2}[(2-6)\cos 2n\pi - 2] = -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} (2x - x^2)\sin\frac{n\pi x}{l}dx = \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{-\cos 2n\pi x/3}{2n\pi/3} - (2 - 2x)\frac{-\sin 2n\pi x/3}{(2n\pi/3)^2} + (-2)\frac{\cos 2n\pi x/3}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\left\{-\frac{6}{n^2\pi^2}\cos 2n\pi - \frac{27}{4n^3\pi^3}(\cos 2n\pi - 1)\right\} = \frac{3}{n\pi}$$

Substituting the values of a_0, a_n, b_n in (i), we get

$$2x - x^2 = -\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\frac{2n\pi x}{3}$$

Putting $x = 3/2$, we get

$$3 - \frac{9}{4} = -\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\cos n\pi \quad \text{or} \quad -\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2} = \frac{\pi^2}{9}\cdot\frac{3}{4}$$

or $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots \infty = \frac{\pi^2}{12}.$$

Example 10.10. *Obtain Fourier series for the function*

$$f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2 - x), & 1 \le x \le 2 \end{cases}$$ *(V.T.U., 2011 ; Bhopal, 2008 ; Mumbai, 2007)*

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{\pi^2}{8}$.

Solution. The required series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

Then $$a_0 = \int_0^1 \pi x dx + \int_1^2 \pi(2 - x)dx = \pi\left|\frac{x^2}{2}\right|_0^1 + \pi\left|2x - \frac{x^2}{2}\right|_1^2 = \pi\left(\frac{1}{2}\right) + \pi\left\{(4-2) - \left(2 - \frac{1}{2}\right)\right\} = \pi$$

$$a_n = \int_0^1 \pi x \cos n\pi x dx + \int_1^2 \pi(2 - x)\cos n\pi x dx$$

$$= \left|\pi x\cdot\frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|\pi(2 - x)\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$= \left(\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi^2}\right) - \left(\frac{\cos 2n\pi}{n^2\pi} - \frac{\cos n\pi}{n^2\pi}\right) = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$= 0 \text{ when } n \text{ is even} ; -\frac{4}{n^2\pi} \text{ when } n \text{ is odd.}$$

$$b_n = \int_0^1 \pi x \sin n\pi x\, dx + \int_1^2 \pi(2-x)\sin n\pi x\, dx$$

$$= \left| \pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left| \pi(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$= \left(-\frac{\cos n\pi}{n}\right) + \left(\frac{\cos n\pi}{n}\right) = 0$$

Hence $\quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots \infty\right)$

Putting $x = 2$, $0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos 2\pi}{1^2} + \frac{\cos 6\pi}{3^2} + \frac{\cos 10\pi}{5^2} + \dots \infty\right)$

Whence $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8}$.

Example 10.11. *Find the Fourier series for*

$$f(t) = 0, -2 < t < -1$$
$$= 1 + t, -1 < t < 0$$
$$= 1 - t, 0 < t < 1$$
$$= 0, \quad 1 < t < 2.$$

Solution. Let $\quad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{2} \quad$ ...(*i*)

[$\because \quad 2c = 2 - (-2)$ so that $c = 2$]

Then $\quad a_0 = \frac{1}{2}\left\{\int_{-2}^{-1}(0)dt + \int_{-1}^{0}(1+t)dt + \int_0^1 (1-t)dt + \int_1^2 (0)dt\right\} = \frac{1}{2}\left\{\left|t + \frac{t^2}{2}\right|_{-1}^{0} + \left|t - \frac{t^2}{2}\right|_0^1\right\}$

$$= \frac{1}{2}\left\{-\left(-1+\frac{1}{2}\right) + \left(1 - \frac{1}{2}\right)\right\} = \frac{1}{2}$$

$$a_n = \frac{1}{2}\left\{\int_{-1}^{0}(1+t)\cos\frac{n\pi t}{2}dt + \int_0^1 (1-t)\cos\frac{n\pi t}{2}dt\right\} \quad \text{[Integrate by parts]}$$

$$= \frac{1}{2}\left\{\left|(1+t)\left(\sin\frac{n\pi t}{2}\right)\frac{2}{n\pi} - (1)\left(-\cos\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_{-1}^{0}\right.$$
$$\left. + \left|(1-t)\left(\sin\frac{n\pi t}{2}\right)\frac{2}{n\pi} - (-1)\left(-\cos\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_0^1\right\}$$

$$= \frac{4}{n^2\pi^2}(1 - \cos n\pi/2)$$

$$b_n = \frac{1}{2}\left\{\int_{-1}^{0}(1+t)\sin\frac{n\pi t}{2}dt + \int_0^1 (1-t)\sin\frac{n\pi t}{2}dt\right\}$$

$$= \frac{1}{2}\left\{\left|(1+t)\left(-\cos\frac{n\pi t}{2}\right)\frac{2}{n\pi} - 1\left(-\sin\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_{-1}^{0}\right.$$
$$\left. + \left|(1-t)\left(-\cos\frac{n\pi t}{2}\right)\frac{2}{n\pi} - (-1)\left(-\sin\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_0^1\right\}$$

$$= \frac{1}{2}\left\{\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right) - \left(\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} = 0$$

Substituting the values of a's and b's in (i), we get

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(1 - \cos\frac{n\pi}{2}\right)\cos\frac{n\pi t}{2}.$$

PROBLEMS 10.4

1. Obtain the Fourier series for $f(x) = \pi x$ in $0 \le x \le 2$.
2. (i) Find the Fourier series to represent x^2 in the interval $(0, a)$. *(Mumbai, 2009)*
   (ii) Find a Fourier series for $f(t) = 1 - t^2$ when $-1 \le t \le 1$. *(Mumbai, 2006)*
3. If $f(x) = 2x - x^2$ in $0 \le x \le 2$, show that $f(x) = \frac{2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi x$. *(V.T.U., 2006)*
4. Find the Fourier series for $f(x) = \begin{cases} x & \text{in } 0 \le x \le 3 \\ 6 - x & \text{in } 3 \le x \le 6 \end{cases}$ *(Anna, 2008)*
5. A sinusoidal voltage $E \sin \omega t$ is passed through a half-wave rectifier which clips the negative portion of the wave. Develop the resulting periodic function

   $U(t) = 0$ when $-T/2 < t < 0$
   $= E \sin \omega t$ when $0 < t < T/2$,

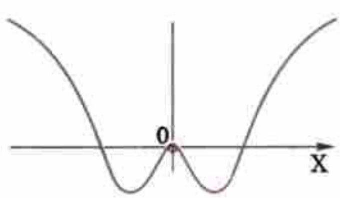
   and $T = 2\pi/\omega$, in a Fourier series. *(Calicut, 1999)*
6. Find the Fourier series of the function $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & x = 1 \\ \pi(x-2), & 1 < x < 2 \end{cases}$

   Hence show that $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ *(Mumbai, 2008)*

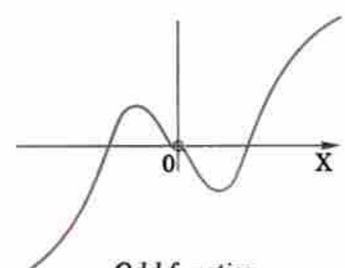
10.6 (1) EVEN AND ODD FUNCTIONS

A function $f(x)$ is said to be **even** *if $f(-x) = f(x)$,*
e.g., $\cos x$, $\sec x$, x^2 are all even functions. Graphically an even function is symmetrical about the y-axis.

A function $f(x)$ is said to be **odd** *if $f(-x) = -f(x)$,*



Even function



Odd function

Fig. 10.3

e.g. $\sin x$, $\tan x$, x^3 are odd functions. Graphically, an odd function is symmetrical about the origin.

We shall be using the following property of definite integrals in the next paragraph :

$$\int_{-c}^{c} \mathbf{f(x)\,dx} = \mathbf{2}\int_0^c \mathbf{f(x)\,dx}, \text{ when } f(x) \text{ is an even function.}$$
$$= \mathbf{0}, \text{ when } f(x) \text{ is an odd function.}$$

(2) Expansions of even or odd periodic functions. We know that a periodic function $f(x)$ defined in $(-c, c)$ can be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c},$$

where $$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx,\ a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx,\ b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx.$$

Case I. *When f(x) is an even function* $a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = \frac{2}{c}\int_{0}^{c} f(x)dx.$

Since $f(x)\cos\frac{n\pi x}{c}$ is also an even function,

$$\therefore \quad a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx = \frac{2}{c}\int_{0}^{c} f(x)\cos\frac{n\pi x}{c}dx$$

Again since $f(x)\sin\frac{n\pi x}{c}$ is an odd function, $\therefore \quad b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = 0.$

Hence, *if a periodic function f (x) is even, its Fourier expansion contains only cosine terms, and*

$$\left.\begin{aligned} a_0 &= \frac{2}{c}\int_{0}^{c} f(x)dx \\ a_n &= \frac{2}{c}\int_{0}^{c} f(x)\cos\frac{n\pi x}{c}dx \end{aligned}\right\} \quad \text{...(1)}$$

Case II. *When f(x) is an odd function,* $a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = 0,$

Since $\cos\frac{n\pi x}{c}$ is an even function, therefore, $f(x)\cos\frac{n\pi x}{c}$ is an odd function.

$$\therefore \quad a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c} = 0$$

Again since $\sin\frac{n\pi x}{c}$ is an odd function, therefore, $f(x)\sin\frac{n\pi x}{c}$ is an even function.

$$\therefore \quad b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = \frac{2}{c}\int_{0}^{c} f(x)\sin\frac{n\pi x}{c}dx$$

Thus, *if a periodic function f (x) is odd, its Fourier expansion contains only sine terms and*

$$b_n = \frac{2}{c}\int_{0}^{c} f(x)\sin\frac{n\pi x}{c}dx \quad \text{...(2)}$$

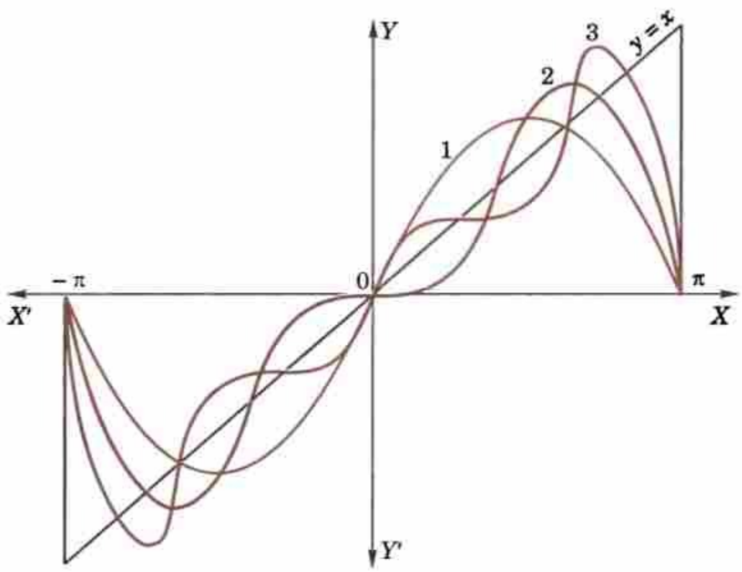


Fig. 10.4

Example 10.12. *Express f(x) = x/2 as a Fourier series in the interval – π < x < π.* (*J.N.T.U., 2006*)

Solution. Since $f(-x) = -x/2 = -f(x)$.

$\therefore$ $f(x)$ is an odd function and hence $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{x}{2}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = -\frac{\cos n\pi}{n}.$$

$\therefore$ $b_1 = 1/1, b_2 = -1/2, b_3 = 1/3, b_4 = -1/4$, etc.

Hence the series is $x/2 = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots$ ...(*i*)

Obs. The graphs of $y = 2\sin x$, $y = 2(\sin x - \frac{1}{2}\sin 2x)$ and $y = 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x)$ are shown in Fig. 10.4, by the curves 1, 2 and 3 respectively. These illustrate the manner in which the successive approximations to the series (*i*) approach more and more closely to $y = x$ for all values of x in $-\pi < x < \pi$, but not for $x = \pm\pi$.

As the series has a period 2π, it represents the discontinuous function, called *saw-toothed waveform*, shown in Fig. 10.5. It is important to note that the given function $y = x$ is continuous and each term of the series (*i*) is continuous, but the function represented by the series (*i*) has finite discontinuities at $x = \pm\pi, \pm 3\pi, \pm 5\pi$ etc.

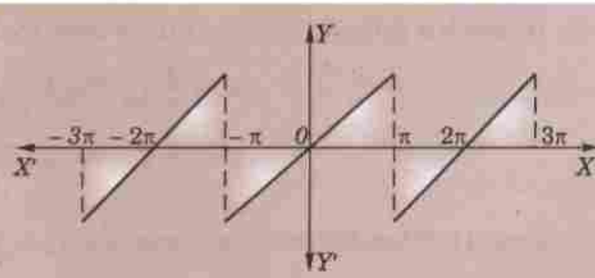


Fig. 10.5

Example 10.13. *Find a Fourier series to represent x^2 in the interval $(-l, l)$.* (S.V.T.U., 2008)

Solution. Since $f(x) = x^2$ is an even function in $(-l, l)$,

$\therefore$ $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \ldots(i)$$

Then $$a_0 = \frac{2}{l}\int_0^l x^2\,dx = \frac{2}{l}\left|\frac{x^3}{3}\right|_0^l = \frac{2l^2}{3}$$

$$a_n = \int_0^l x^2\cos\frac{n\pi x}{l}\,dx$$ [*See* footnote p. 398]

$$= \frac{2}{l}\left[x^2\left(\frac{\sin n\pi x/l}{n\pi/l}\right) - 2x\left(-\frac{\cos n\pi x/l}{n^2\pi^2/l^2}\right) + 2\left(-\frac{\sin n\pi x/l}{n^3\pi^3/l^3}\right)\right]_0^l$$

$$= 4l^2(-1)^n/n^2\pi^2 \qquad [\because \cos n\pi = (-1)^n]$$

$\therefore$ $a_1 = -4l^2/\pi^2, a_2 = 4l^2/2^2\pi^2, a_3 = -4l^2/3^2\pi^2, a_4 = 4l^2/4^2\pi^2$ etc.

Substituting these values in (*i*), we get

$$x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left(\frac{\cos \pi x/l}{1^2} - \frac{\cos 2\pi x/l}{2^2} + \frac{\cos 3\pi x/l}{3^2} - \frac{\cos 4\pi x/l}{4^2} + \ldots\right)$$

which is the required Fourier series.

Example 10.14. *If $f(x) = |\cos x|$, expand $f(x)$ as a Fourier series in the interval $(-\pi, \pi)$.*

Solution. As $f(-x) = |\cos(-x)| = |\cos x| = f(x)$, $|\cos x|$ is an even function.

$\therefore$ $$f(x) = \frac{a_0}{2} + \Sigma a_n \cos nx$$

where $$a_0 = \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx = \frac{2}{\pi}\int_0^{\pi/2}\cos x\,dx + \int_{\pi/2}^{\pi}(-\cos x)\,dx$$

[$\because$ $\cos x$ is $-$ve when $\pi/2 < x < \pi$]

$$= \frac{2}{\pi}\left\{\left| \sin x \right|_0^{\pi/2} - \left| \sin x \right|_{\pi/2}^{\pi}\right\} = \frac{2}{\pi}[(1-0)-(0-1)] = \frac{4}{\pi}$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx \, dx + \int_{\pi/2}^{\pi} (-\cos x) \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/2} [\cos (n+1)x + \cos (n-1)x] \, dx - \int_{\pi/2}^{\pi} [\cos (n+1)x + \cos (n-1)x] \, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left| \frac{\sin (n+1)x}{n+1} + \frac{\sin (n-1)x}{n-1}\right|_0^{\pi/2} - \left| \frac{\sin (n+1)x}{n+1} + \frac{\sin (n-1)x}{n-1}\right|_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin (n+1)\pi/2}{n+1} + \frac{\sin (n-1)\pi/2}{n-1}\right\} + \left\{\frac{\sin (n+1)\pi/2}{n+1} + \frac{\sin (n-1)\pi/2}{n-1}\right\}\right]$$

$$= \frac{2}{\pi}\left(\frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1}\right) = \frac{-4 \cos n\pi/2}{\pi(n^2-1)} \; (n \neq 1)$$

In particular $a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x \, dx - \int_{\pi/2}^{\pi} \cos^2 x \, dx\right] = 0$

Hence $|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left\{\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \ldots\right\}$.

Example 10.15. *Obtain Fourier series for the function f(x) given by*

$$f(x) = 1 + 2x/\pi, \qquad -\pi \le x \le 0,$$
$$= 1 - 2x/\pi, \qquad 0 \le x \le \pi.$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$. *(V.T.U., 2010 ; Mumbai, 2007)*

Solution. Since $f(-x) = 1 - \frac{2x}{\pi}$ in $(-\pi, 0) = f(x)$ in $(0, \pi)$

and $f(-x) = 1 + \frac{2x}{\pi}$ in $(0, \pi) = f(x)$ in $(-\pi, 0)$

$\therefore$ $f(x)$ is an even function in $(-\pi, \pi)$. This is also clear from its graph $A'BA$ (Fig. 10.6) which is symmetrical about the y-axis.

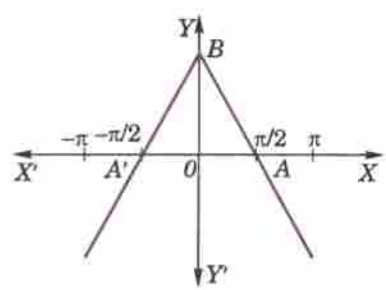


Fig. 10.6

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots(i)$$

where $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx = \frac{2}{\pi}\left(x - \frac{x^2}{\pi}\right)_0^{\pi} = 0$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx \, dx$$

$$= \frac{2}{\pi}\left|\left(1 - \frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} = \frac{2}{\pi}\left(-\frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right) = \frac{4}{n^2\pi^2}[1 - (-1)^n]$$

$\therefore$ $a_1 = 8/\pi^2, a^3 = 8/3^2\pi^2, a_5 = 8/5^2\pi^2, \ldots$

and $a_2 = a_4 = a_6 = \ldots = 0.$

Thus substituting the values of a's in (i), we get

$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] \qquad \ldots(ii)$$

as the required Fourier expansion

Putting $x = 0$ in (*ii*), we get $1 = f(0) = \dfrac{8}{\pi^2}\left(\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dots\right)$

whence follows the desired result.

PROBLEMS 10.5

1. Obtain the Fourier series expansion of $f(x) = x^2$ in $(0, a)$. Hence show that
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$ (*Mumbai, 2009 ; S.V.T.U., 2008*)

2. Show that for $-\pi < x < \pi$, $\sin ax = \dfrac{2 \sin a\pi}{\pi}\left(\dfrac{\sin x}{1^2 - a^2} - \dfrac{2 \sin 2x}{2^2 - a^2} + \dfrac{3 \sin 3x}{3^2 - a^2} - \dots\right)$

3. Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$. (*V.T.U., 2008 ; Anna, 2003*)
Deduce that $\dfrac{1}{1.3} - \dfrac{1}{3.5} + \dfrac{1}{5.7} - \dfrac{1}{7.9} + \dots = \dfrac{1}{4}(\pi - 2)$. (*U.P.T.U., 2005*)

4. Prove that in the interval $-\pi < x < \pi$, $x \cos x = -\dfrac{1}{2} \sin x + 2 \displaystyle\sum_{n=2}^{\infty} \dfrac{n(-1)^n}{n^2 - 1} \sin nx$. (*S.V.T.U., 2009*)

5. For a function $f(x)$ defined by $f(x) = |x|, -\pi < x < \pi$, obtain a Fourier series. (*Bhopal, 2007 ; V.T.U., 2004*)
Deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \dots = \dfrac{\pi^2}{8}$. (*S.V.T.U., 2009 ; Kerala, 2005 ; P.T.U., 2005*)

6. Find the Fourier series to represent the function
(*i*) $f(x) = |\sin x|, -\pi < x < \pi$. (*Mumbai, 2008*)
(*ii*) $f(x) = |\cos(\pi x/l)|$ in the interval $(-1, 1)$. (*P.T.U., 2009 S*)

7. Given $f(x) = \begin{cases} -x + 1 \text{ for } -\pi \le x \le 0, \\ x + 1 \text{ for } 0 \le x \le \pi. \end{cases}$
Is the function even or odd? Find the Fourier series for $f(x)$ and deduce the value of
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

8. Find the Fourier series of the periodic function $f(x)$: $f(x) = -k$ when $-\pi < x < 0$ and $f(x) = k$ when $0 < x < \pi$, and $f(x + 2\pi) = f(x)$. Sketch the graph of $f(x)$ and the two partial sums. (See Fig. 10.7)
Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n - 1} = \frac{\pi}{4}$. (*Rohtak, 2005*)

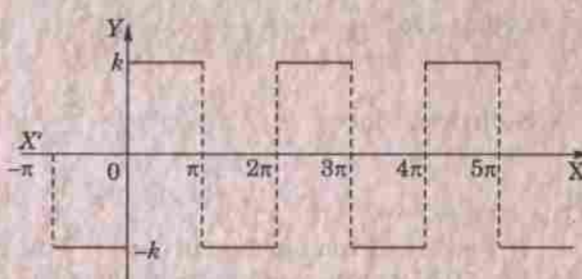


Fig. 10.7

9. A function is defined as follows :
$$f(x) = -x \text{ when } -\pi < x \le 0 = x \quad \text{when } 0 < x < \pi.$$
Show that $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left[\dfrac{1}{1^2} \cos x + \dfrac{1}{3^2} \cos 3x + \dfrac{1}{5^2} \cos 5x + \dots\right]$
Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

10.7 HALF RANGE SERIES

Many a time is is required to obtain a Fourier expansion of a function $f(x)$ for the range $(0, c)$ which is half the period of the Fourier series. As it is immaterial whatever the function may be outside the range $0 < x < c$, we extend the function to cover the range $-c < x < c$ so that the new function may be odd or even. The Fourier expansion of such a function of half the period, therefore, consists of sine or cosine terms only. In such cases the graphs for the values of x in $(0, c)$ are the same but outside $(0, c)$ are different for odd or even functions. That is why we get different forms of series for the same function as is clear from the examples 10.16 and 10.17.

Sine series. If it be required to expand $f(x)$ as a sine series in $0 < x < c$; then we extend the function reflecting it in the origin, so that $f(x) = -f(-x)$.

Then the extended function is odd in $(-c, c)$ and the expansion will give the desired Fourier sine series :

$$\left.\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} \\ \text{where} \quad b_n &= \frac{2}{c}\int_0^c f(x) \sin \frac{n\pi x}{c}\,dx \end{aligned}\right\} \quad ...(1)$$

Cosine series. If it be required to express $f(x)$ as a cosine series in $0 < x < c$, we extend the function reflecting it in the y-axis, so that $f(-x) = f(x)$.

Then the extended function is even in $(-c, c)$ and its expansion will give the required Fourier cosine series :

$$\left.\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} \\ \text{where} \quad a_0 &= \frac{2}{c}\int_0^c f(x)\,dx \\ \text{and} \quad a_n &= \frac{2}{c}\int_0^c f(x) \cos \frac{n\pi x}{c}\,dx \end{aligned}\right\} \quad ...(2)$$

Example 10.16. *Express $f(x) = x$ as a half-range sine series in $0 < x < 2$.* (U.P.T.U., 2004)

Solution. The graph of $f(x) = x$ in $0 < x < 2$ is the line OA. Let us extend the function $f(x)$ in the interval $-2 < x < 0$ (shown by the line BO) so that the new function is symmetrical about the origin and, therefore, *represents an odd function in* $(-2, 2)$ (*Fig. 10.8*)

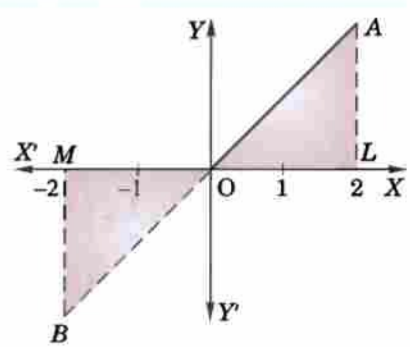


Fig. 10.8

Hence the Fourier series for $f(x)$ over the full period $(-2, 2)$ will contain only sine terms given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

where

$$b_n = \frac{2}{2}\int_0^2 f(x) \sin \frac{n\pi x}{2}\,dx = \int_0^2 x \sin \frac{n\pi x}{2}\,dx$$

$$= \left| -\frac{2x}{n\pi} \cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} \right|_0^2 = -\frac{4(-1)^n}{n\pi}$$

Thus $b_1 = 4/\pi$, $b_2 = -4/2\pi$, $b_3 = 4/3\pi$, $b_4 = -4/4\pi$ etc.

Hence the Fourier sine series for $f(x)$ over the half-range $(0, 2)$ is

$$f(x) = \frac{4}{\pi}\left(\sin \frac{\pi x}{2} - \frac{1}{2} \sin \frac{2\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} - \frac{1}{4} \sin \frac{4\pi x}{2} + ... \right).$$

Example 10.17. *Express $f(x) = x$ as a half-range cosine series in $0 < x < 2$.* (S.V.T.U., 2009 ; Bhopal, 2007 ; Mumbai, 2006)

Solution. The graph of $f(x) = x$ in $(0, 2)$ is the line OA. Let us extend the function $f(x)$ in the interval $(-2, 0)$ shown by the line OB' so that the new function is symmetrical about the y-axis and, therefore, *represents an even function in* $(-2, 2)$. (Fig. 10.9)

Hence the Fourier series for $f(x)$ over the full period $(-2, 2)$ will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

where $a_0 = \frac{2}{2}\int_0^2 f(x)dx = \int_0^2 x\,dx = 2$

and $a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^2 x\cos\frac{n\pi x}{2}\,dx$

$$= \left|\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right|_0^2 = \frac{4}{n^2\pi^2}[(-1)^n - 1]$$

Thus $a_1 = -8/\pi^2, a_2 = 0, a_3 = -8/3^2\pi^2, a_4 = 0, a_5 = -8/5^2\pi^2$ etc.

Hence the desired Fourier series for $f(x)$ over the half-range (0, 2) is

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos\pi x/2}{1^2} + \frac{\cos 3\pi x/2}{3^2} + \frac{\cos 5\pi x/2}{5^2} + \dots\right]$$

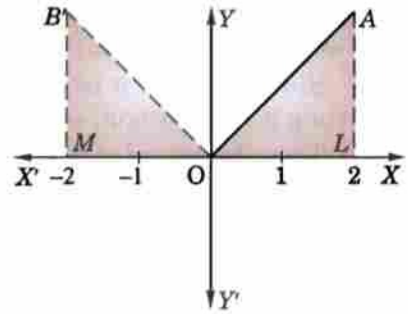


Fig. 10.9

Important Obs. It must be clearly understood that we expand a function in $0 < x < c$ as a series of sines or cosines, merely looking upon it as an odd or even function of period $2c$. It hardly matters whether the function is odd or even or neither.

Example 10.18. *Obtain the Fourier expansion of x sin x as a cosine series in (0, π).*

(V.T.U., 2003 ; U.P.T.U., 2002)

Hence show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\infty = \frac{\pi - 2}{4}$. *(Anna, 2001)*

Solution. Let $x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then $a_0 = \frac{2}{\pi}\int_0^\pi x\sin x\,dx = \frac{2}{\pi}|x(-\cos x) - 1(-\sin x)|_0^\pi = 2$

$$a_n = \frac{2}{\pi}\int_0^\pi x\sin x\cos nx\,dx = \frac{1}{\pi}\int_0^\pi x(\sin\overline{n+1}\,x - \sin\overline{n-1}\,x)\,dx$$

$$= \frac{1}{\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1.\left\{\frac{-\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^\pi$$

$$= \frac{1}{\pi}\pi\left\{\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}\right\}(n \neq 1).$$

When $n = 1$, $a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx$

$$= \frac{1}{\pi}\left|x\left(\frac{-\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{2}\right)\right|_0^\pi = \frac{1}{\pi}\left(-\frac{\pi\cos 2\pi}{2}\right) = -\frac{1}{2}.$$

Hence $x\sin x = 1 - \frac{1}{2}\cos x - 2\left\{\frac{\cos 2x}{1.3} - \frac{\cos 3x}{3.5} + \frac{\cos 4x}{5.7} - \dots\infty\right\}$

Putting $x = \pi/2$, we obtain $\pi/2 = 1 + 2\left\{\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\infty\right\}$

Hence $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\infty = \frac{\pi - 2}{4}$.

Example 10.19. *Obtain a half range cosine series for*

$$f(x) = \begin{cases} kx, & 0 \le x \le l/2 \\ k(l - x), & l/2 \le x \le l. \end{cases}$$ *(Bhopal, 2008 ; V.T.U., 2008)*

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\infty$ *(Rohtak, 2006 ; U.P.T.U., 2003)*

Solution. Let the half-range cosine series be

$$f(x) = \frac{a_0}{2} + \sum_{n+1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Then $$a_0 = \frac{2}{l}\left\{\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\right\}dx = \frac{2k}{l}\left\{\left|\frac{x^2}{2}\right|_0^{l/2} - \left|\frac{(l-x)^2}{2}\right|_{l/2}^{l}\right\}$$

$$= \frac{2k}{l}\cdot\frac{1}{2}\left\{\frac{l^2}{4} - \left(0 - \frac{l^2}{4}\right)\right\} = \frac{kl}{2}$$

$$a_n = \frac{2}{l}\left\{\int_0^{l/2} kx\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\cos\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{2k}{l}\left|x\left(\frac{\sin n\pi x/l}{n\pi/l}\right) - 1\left\{-\cos\frac{n\pi x/l}{(n\pi/l)^2}\right\}\right|_0^{l/2}$$

$$+ \frac{2k}{l}\left|\left\{\frac{(l-x)\sin n\pi x/l}{n\pi/l}\right\} - (-1)\left(\frac{-\cos n\pi x/l}{(n\pi/l)^2}\right)\right|_{l/2}^{l}$$

$$= \frac{2k}{l}\left[\left(\frac{l^2}{2n\pi}\cdot\sin\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - \cos 0\right)\right] + \frac{2k}{l}\left[\left(\frac{l}{n\pi}\left(-\frac{l}{2}\sin\frac{n\pi}{2}\right)\right.\right.$$
$$\left.\left. - \frac{l^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right)\right]$$

$$= \frac{2k}{l}\cdot\frac{l^2}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right] = \frac{2kl}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}$$

Hence the required Fourier series is

$$f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \frac{1}{10^2}\cos\frac{10\pi x}{l} + \dots\right]$$

Putting $x = l$, we get

$$0 = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + \dots\infty\right)$$

Thus $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\infty = \frac{\pi^2}{8}.$$

Example 10.20. *Expand* $f(x) = \frac{1}{4} - x$, *if* $0 < x < \frac{1}{2}$,

$= x - \frac{3}{4}$, *if* $\frac{1}{2} < x < 1$,

as the Fourier series of sine terms. *(V.T.U., 2011; Andhra, 2000)*

Solution. Let $f(x)$ represent an odd function in $(-1, 1)$ so that $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$

where $$b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx$$

$$= 2\left[\int_0^{\frac{1}{2}}\left(\frac{1}{4} - x\right)\sin n\pi x\,dx + \int_{\frac{1}{2}}^{1}\left(x - \frac{3}{4}\right)\sin n\pi x\,dx\right]$$

$$= 2\left|-\left(\frac{1}{4} - x\right)\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right|_0^{\frac{1}{2}} + 2\left|-\left(x - \frac{3}{4}\right)\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right|_{\frac{1}{2}}^{1}$$

$$= 2\left[\frac{1}{4n\pi}\cos\frac{n\pi}{2} + \frac{1}{4n\pi} - \frac{\sin n\pi/2}{n^2\pi^2}\right] + 2\left[-\frac{1}{4n\pi}\cos n\pi - \frac{1}{4n\pi}\cos\frac{n\pi}{2} - \frac{\sin n\pi/2}{n^2\pi^2}\right]$$

$$= \frac{1}{2n\pi}[1 - (-1)^n] - \frac{4\sin n\pi/2}{n^2\pi^2}$$

Thus
$$b_1 = \frac{1}{\pi} - \frac{4}{\pi^2};\ b_2 = 0$$
$$b_3 = \frac{1}{3\pi} + \frac{4}{3^2\pi^2};\ b_4 = 0$$
$$b_5 = \frac{1}{5\pi} - \frac{4}{5^2\pi^2};\ b_6 = 0 \text{ etc.}$$

Hence
$$f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} + \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \cdots$$

PROBLEMS 10.6

1. Show that a constant c can be expanded in an infinite series $\frac{4c}{\pi}\left\{\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right\}$ in the range $0 < x < \pi$. *(Marathwada, 2008 ; Kerala, 2005)*
2. Obtain cosine and sine series for $f(x) = x$ in the interval $0 \le x \le \pi$. Hence show that
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$
(Osmania, 2003 S)
3. Find the half-range cosine series for the function $f(x) = x^2$ in the range $0 \le x \le \pi$. *(B.P.T.U., 2005 ; Kottayam, 2005)*
4. Find the Fourier cosine series of the function $f(x) = \pi - x$ in $0 < x < \pi$. Hence show that
$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8}$$
(West Bengal, 2004)
5. Find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$. *(V.T.U., 2010 ; J.N.T.U., 2006)*

   Hence show that $\pi^2 = 8\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$ *(Anna, 2003)*
6. Find the half-range sine series for the function $f(t) = t - t^2$, $0 < t < 1$.
7. Represent $f(x) = \sin(\sin(\pi x/l))$, $0 < x < l$ by a half-range cosine series. *(Mumbai, 2009)*
8. Find the half range sine series for $f(x) = x \cos x$ in $(0, \pi)$. *(Anna, 2008 S)*
9. Obtain the half-range sine series for e^x in $0 < x < 1$.
10. Find the half range Fourier sine series of $f(x) = x(\pi - x)$, $0 \le x \le \pi$ and hence deduce that

    (*i*) $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$ *(Anna, 2009)* (*ii*) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$ *(Mumbai, 2005)*
11. If $f(x) = x$, $0 < x < \pi/2$

    $= \pi - x$, $\pi/2 < x < \pi$,

    show that (*i*) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \ldots\right]$ *(Mumbai, 2008 ; S.V.T.U., 2008 ; V.T.U., 2004)*

    (*ii*) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \ldots\right]$ *(V.T.U., 2011)*
12. Find the half-range cosine series expansion of the function $f(x) = \begin{cases} 0, & 0 \le x \le l/2 \\ l - x, & l/2 \le x \le l \end{cases}$. *(P.T.U., 2010)*
13. If $f(x) = \sin x$ for $0 \le x \le \pi/4$

    $= \cos x$ for $\pi/4 \le x \le \pi/2$, expand $f(x)$ in a series of sines.
14. For the function defined by the graph OAB in Fig. 10.10, find the half-range Fourier sine series.

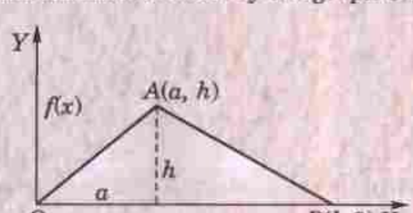


Fig. 10.10

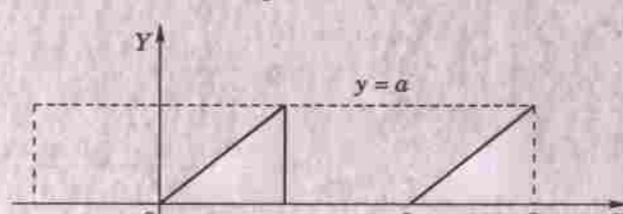


Fig 10.11

10.8 TYPICAL WAVEFORMS

We give below six typical waveforms usually met with in communication engineering :

(1) *Square waveform* (Fig. 10.7) is an extension of the function of Problem 8, page 412.

(2) *Saw-toothed waveform* (Fig. 10.5) is an extension of the function in Ex. 10.12, page 409.

(3) *Modified saw-toothed waveform* (Fig. 10.11) is extension of the function

$$f(x) = 0, \qquad -\pi < x \le 0$$
$$= x, \qquad 0 \le x < \pi,$$

Its Fourier expansion is

$$f(x) = \frac{a}{4} - \frac{2a}{\pi^2}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) + \frac{\alpha}{\pi}\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right)$$

(4) *Triangular waveform* (Fig. 10-6) is an extension of the function of Ex. 10.15, page 411.

(5) *Half-wave rectifier* (Fig. 10.2) is an extension of the function of Problem 2, page 412.

(6) *Full-wave rectifier* (Fig. 10.12) is an extension of the function $f(x) = a \sin x, 0 \le x \le \pi$. Its Fourier expansion is

$$f(x) = \frac{4a}{\pi}\left\{\frac{1}{2} - \frac{1}{1.3}\cos 2x - \frac{1}{3.5}\cos 4x - \frac{1}{5.7}\cos 6x - \ldots\right\}$$

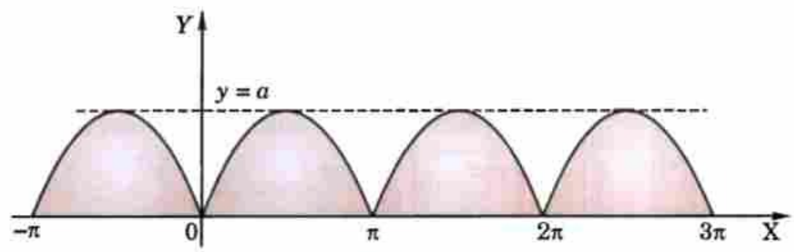


Fig. 10.12

10.9 (1) PARSEVAL'S FORMULA*

To prove that $\int_{-l}^{l} [f(x)]^2\, dx = l\left\{\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$,

provided the Fourier series for f(x) converges uniformly in (– l, l).

The Fourier series for $f(x)$ in $(-l, l)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$ ...(1)

Multiplying both sides of (1) by $f(x)$ and integrating term by term from $-l$ to l [which is justified as the series (1) is uniformly convergent –p. 389], we get

$$\int_{-l}^{l} [f(x)]^2\, dx = \frac{a_0}{2}\int_{-l}^{l} f(x)\, dx + \sum_{n=1}^{\infty}\left\{a_n \int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\, dx + b_n \int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\, dx\right\} \qquad \ldots(2)$$

Now $\int_{-l}^{l} f(x)\, dx = la_0,$

$\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\, dx = la_n$ and $\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\, dx = lb_n$, by (4) of p. 405

$\therefore$ (2) takes the form $\int_{-l}^{l} [f(x)]^2\, dx = l\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$ ...(3)

which is the desired *Parseval's formula.* *(Mumbai, 2005 S)*

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*Named after the French mathematician *Marc Antoine Parseval* (1755–1836).

Cor. 1. *If* $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$ *in* $(0, 2l)$, then

$$\int_0^{2l} [f(x)]^2 \, dx = l \left\{ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\} \quad ...(4)$$

Cor. 2. *If the half-range cosine series is* $(0, l)$ *for* $f(x)$ *is*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left(\frac{n\pi x}{l} \right), \text{ then}$$

$$\int_0^{l} [f(x)]^2 \, dx = \frac{l}{2} \left(\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \dots \infty \right) \quad ...(5)$$

Cor. 3. *If the half-range sine series in* $(0, l)$ *for* $f(x)$ *is* $f(x) = \sum_{n=1}^{\infty} b_n \sin \left(\frac{n\pi x}{l} \right)$, then

$$\int_0^{l} [f(x)]^2 = \frac{l}{2} (b_1^2 + b_2^2 + b_3^2 + \dots \infty) \quad ...(6)$$

(2) Root mean square (rms) value. *The root mean square value of the function* $f(x)$ *over an interval* (a, b) *is defined as*

$$[f(x)]_{\text{rms}} = \sqrt{\left\{ \frac{\int_a^b [f(x)]^2 \, dx}{b - a} \right\}} \quad ...(7)$$

The use of root mean square value of a periodic function is frequently made in the theory of mechanical vibrations and in electric circuit theory. The r.m.s. value is also known as the effective value of the function.

Example 10.21. *Obtain the Fourier series for* $y = x^2$ *in* $-\pi < x < \pi$. *Using the two values of* y, *show that*

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots = \frac{\pi^4}{90}$$

Solution. Let $y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

We have $a_0 = 2\frac{\pi^2}{3}, a_n = \frac{4}{n^2}(-1)^n, b_n = 0$ for all n (See problem 2, p. 400)

If $\bar{y}$ be the r.m.s. value of y in $(-\pi, \pi)$, then

$$(\bar{y})^2 = \frac{\pi}{2\pi} \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right] \quad \text{[By (3) and (7) §10.9]}$$

$$= \frac{1}{4} \left(\frac{2\pi^2}{3} \right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left[\frac{16}{n^4} (-1)^{2n} + 0 \right] = \frac{\pi^4}{9} + 8 \sum_{1}^{\infty} \frac{1}{n^4}$$

Also by definition,

$$(\bar{y})^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} y^2 \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 \, dx = \frac{\pi^4}{5}$$

Equating the two values of $(\bar{y})^2$, we get

$$\frac{\pi^4}{9} + 8 \sum_{1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{5} \text{ i.e., } \sum_{1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

PROBLEMS 10.7

1. By using the sine series for $f(x) = 1$ in $0 < x < \pi$, show that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$
2. Prove that in $0 < x < l$, $x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\right)$

   and deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots = \frac{\pi^4}{96}$.
3. If $\frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{l}$ is the half-range cosine series of $f(x)$ of period $2l$ in $(0, l)$, then show that the mean square value of $f(x)$ in $(0, l)$ is $\frac{l}{2}\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right\}$

   Use this result to evaluate $1^{-4} + 3^{-4} + 5^{-4} + \ldots\ldots$ from the half-range cosine series of the function $f(x)$ of period 4 defined in $(0, 2)$ by

$$f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}.$$

10.10 COMPLEX FORM OF FOURIER SERIES

The Fourier series of a periodic function $f(x)$ of period $2l$, is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \qquad \ldots(1)$$

Since $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$,

therefore, we can express (1) as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) + b_n\left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right)\right\}$$

$$= c_0 + \sum_{n=1}^{\infty}\left\{c_n e^{in\pi x/l} + c_{-n} e^{-in\pi x/l}\right\} \qquad \ldots(2)$$

where $c_0 = \frac{1}{2}a_0,\ c_n = \frac{1}{2}(a_n - ib_n),\ c_{-n} = \frac{1}{2}(a_n + ib_n)$

Now
$$c_n = \frac{1}{2l}\left\{\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx - i\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx\right\}$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx$$

and
$$c_{-n} = \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{in\pi x/l}\,dx$$

Combining these, we have $c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$ ...(3)

Then the series (2) can be compactly written as :

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

which is the *complex form of Fourier series* and its coefficients are given by (3).

Obs. The complex form of a Fourier series is especially useful in problems on electrical circuits having impressed periodic voltage.

Example 10.22. *Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 \le x \le 1$.*

(Mumbai, 2005 S ; Madras, 2000 S)

Solution. We have $f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x}$ $\qquad (\because \; l = 1)$

where
$$c_n = \frac{1}{2}\int_{-1}^{1} e^{-x} \cdot e^{-in\pi x}\, dx = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\, dx = \frac{1}{2}\left|\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right|_{-1}^{1} = \frac{e^{1+in\pi} - e^{-(1+in\pi)}}{2(1+in\pi)}$$

$$= \frac{e(\cos n\pi + i\sin n\pi) - e^{-1}(\cos n\pi - i \sin n\pi)}{2(1+in\pi)} = \frac{e - e^{-1}}{2}(-1)^n \cdot \frac{1 - in\pi}{1 + n^2\pi^2}$$

$$= \frac{(-1)^n (1 - in\pi)\sinh 1}{1 + n^2\pi^2}$$

Hence
$$e^{-x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1 - in\pi)}{1 + n^2\pi^2} \sinh 1 \cdot e^{in\pi x}.$$

PROBLEMS 10.8

Find the complex form of the Fourier series of the following periodic functions :

1. $f(x) = e^{ax}, -l < x < l.$ (Madras, 2003)
2. $f(t) = \sin t, 0 < t < \pi$
3. $f(x) = \cos ax, -\pi < x < \pi$ (Anna, 2009 ; Mumbai, 2009)
4. $f(x) = \cosh 3x + \sinh 3x$ in $(-3, 3)$. (Mumbai, 2008)
5. $f(x) = \begin{cases} 0 \text{ when } 0 < x < l \\ a \text{ when } l < x < 2l \end{cases}$

10.11 PRACTICAL HARMONIC ANALYSIS

We have discussed at length, the problem of expanding $y = f(x)$ as Fourier series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad ...(1)$$

where
$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx \end{aligned}\right\} \qquad ...(2)$$

So far, the function has always been defined by an explicit function of an independent variable. In practice, however, the function is often given not by a formula but by a graph or by a table of corresponding values. In such cases, the integrals in (2) cannot be evaluated and instead, the following alternative forms of (2) are employed.

Since the mean value of a function $y = f(x)$ over the range (a, b) is $\frac{1}{b-a}\int_a^b f(x)\, dx$.

$\therefore$ the equations (2) give,

$$\left.\begin{aligned} \mathbf{a_0} &= 2 \times \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx = \mathbf{2[mean\ value\ of\ f(x)\ in\ (0, 2\pi)]} \\ \mathbf{a}_n &= 2 \times \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \mathbf{2[mean\ value\ of\ f(x)\ cos\ nx\ in\ (0, 2\pi)]} \\ \mathbf{b}_n &= 2 \times \frac{1}{2\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \mathbf{2[mean\ value\ of\ f(x)\ sin\ nx\ in\ (0, 2\pi)]} \end{aligned}\right\} \qquad ...(3)$$

There are numerous other methods of finding the value of a_0, a_n, b_n which constitute the field of **harmonic analysis.**

In (1), *the term* $(a_1 \cos x + b_1 \sin x)$ *is called the* **fundamental or first harmonic,** *the term* $(a_2 \cos 2x + b_2 \sin 2x)$ **the second harmonic** *and so on.*

Example 10.23. *The displacement y of a part of a mechanism is tabulated with corresponding angular movement x° of the crank. Express y as a Fourier series neglecting the harmonic above the third :*

$x°$	0	30	60	90	120	150	180	210	240	270	300	330
y	1.80	1.10	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

Solution. Let the Fourier series upto the third harmonic representing y in $(0, 2\pi)$ be

$$y = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x \qquad ...(i)$$

To evaluate the coefficients, we form the following table.

$x°$	$\sin x$	$\cos x$	$\sin 2x$	$\cos 2x$	$\sin 3x$	$\cos 3x$	y	$y \sin x$	$y \cos x$	$y \sin 2x$	$y \cos 2x$	$y \sin 3x$	$y \cos 3x$
0	0	1	0	1	0	1	1.80	0.00	1.80	0.00	1.80	0.00	1.80
30	0.50	0.87	0.87	0.50	1	0	1.10	0.55	0.96	0.96	0.55	1.10	0.00
60	0.87	0.50	0.87	–0.50	0	–1	0.30	0.26	0.15	0.26	–0.15	0.00	–0.30
90	1.00	0	0	–1.00	–1	0	0.16	0.16	0.00	0.00	–0.16	–0.16	0.00
120	0.87	–0.50	–0.87	–0.50	0	1	0.50	0.43	–0.25	–0.43	–0.25	0.00	0.50
150	0.50	–0.87	–0.87	–0.50	1	0	1.30	0.65	–1.13	–1.13	0.65	1.30	0.00
180	0	–1.00	0	1.00	0	–1	2.16	0.00	–2.16	–0.00	2.16	0.00	–2.16
210	–0.50	–0.87	0.87	0.50	–1	0	1.25	–0.63	–1.09	1.09	0.63	–1.25	0.00
240	–0.87	–0.50	0.87	–0.50	0	1	1.30	–1.13	–0.65	1.13	–0.65	0.00	1.30
270	–1.00	0	0	–1.00	1	0	1.52	–1.52	0.00	0.00	–1.52	1.52	0.00
300	–0.87	0.50	–0.87	–0.50	0	–1	1.76	–1.53	0.88	–1.53	–0.88	0.00	–1.76
330	–0.50	0.87	–0.87	0.50	–1	0	2.00	–1.00	1.74	–1.74	1.00	–2.00	0.00
						Σ =	15.15	–3.76	0.25	–1.39	3.18	0.51	–0.62

$$\therefore \qquad a_0 = 2 \cdot \frac{\Sigma y}{12} = \frac{15.15}{6} = 2.53 \; ; \; a_1 = \frac{1}{6} \Sigma y \cos x = \frac{0.25}{6} = 0.04$$

$$a_2 = \frac{1}{6} \Sigma y \cos 2x = \frac{3.18}{6} = 0.53 \; ; \; a_3 = \frac{1}{6} \Sigma y \cos 3x = \frac{-0.62}{6} = -0.1$$

$$b_1 = \frac{1}{6} \Sigma y \sin x = \frac{-3.76}{6} = -0.63 \; ;$$

$$b_2 = \frac{1}{6} \Sigma y \sin 2x = \frac{-1.39}{6} = -0.23$$

$$b_3 = \frac{1}{6} \Sigma y \sin 3x = \frac{0.51}{6} = 0.085$$

Substituting the values of a's and b's in (i), we get

$$y = 1.26 + 0.04 \cos x + 0.53 \cos 2x - 0.1 \cos 3x - 0.63 \sin x - 0.23 \sin 2x + 0.085 \sin 3x.$$

Example 10.24. *The following table gives the variations of periodic current over a period.*

t sec	*0*	*T/6*	*T/3*	*T/2*	*2T/3*	*5T/6*	*T*
A amp.	*1.98*	*1.30*	*1.05*	*1.30*	*– 0.88*	*– 0.25*	*1.98*

Show that there is a direct current part of 0.75 amp in the variable current and obtain the amplitude of the first harmonic. (V.T.U., 2010 ; S.V.T.U., 2009)

Solution. Here length of the interval is T, *i.e.* $C = T/2$ (§ 10.5)

Then $$A = \frac{a_0}{2} + a_1 \cos \frac{2\pi t}{T} + b_1 \sin \frac{2\pi t}{T} + a_2 \cos \frac{4\pi t}{T} + b_2 \sin \frac{4\pi t}{T} + ...$$

The desired values are tabulated as follows :

t	$2\pi t/T$	$\cos 2\pi t/T$	$\sin 2\pi t/T$	A	$A \cos 2\pi t/T$	$A \sin 2\pi t/T$
0	0	1.0	0.000	1.98	1.980	0.000
$T/6$	$\pi/3$	0.5	0.866	1.30	0.650	1.126
$T/3$	$2\pi/3$	–0.5	0.866	1.05	–0.525	0.909
$T/2$	π	–1.0	0.000	1.30	–1.300	0.000
$2T/3$	$4\pi/3$	–0.5	–0.866	–0.88	0.440	0.762
$5T/6$	$5\pi/3$	0.5	–0.866	–0.25	–0.125	0.217
			$\Sigma =$	4.5	1.12	3.014

$\therefore$
$$a_0 = 2 \cdot \frac{1}{6} \Sigma A = \frac{1}{3}(4.5) = 1.5$$
$$a_1 = 2 \cdot \frac{1}{6} \Sigma A \cos \frac{2\pi t}{T} = \frac{1}{3}(1.12) = 0.373$$
$$b_1 = 2 \cdot \frac{1}{6} \Sigma A \sin \frac{2\pi t}{T} = \frac{1}{3}(3.014) = 1.005$$

Thus the direct current part in the variable current $= a_0/2 = 0.75$ and amplitude of the first harmonic
$$= \sqrt{(a_1^2 + b_1^2)} = \sqrt{[(0.373)^2 + (1.005)^2]} = 1.072$$

Example 10.25. *Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table :*

x :	*0*	*1*	*2*	*3*	*4*	*5*	
y :	*4*	*8*	*15*	*7*	*6*	*2*	*(V.T.U., 2009 ; V.T.U., 2006 ; J.N.T.U., 2004 S)*

Solution. Taking the interval as 60°, we have

$\theta =$	0°	60°	120°	180°	240°	300°
$x =$	0	1	2	3	4	5
$y =$	4	8	15	7	6	2

$\therefore$ Fourier cosine series in the intervals $(0, 2\pi)$ is

$$y = \frac{a_0}{2} + a_1 \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \ldots$$

$\theta°$	$\cos \theta$	$\cos 2\theta$	$\cos 3\theta$	y	$y \cos \theta$	$y \cos 2\theta$	$y \cos 3\theta$
0°	1	1	1	4	4	4	4
60°	$\frac{1}{2}$	$-\frac{1}{2}$	–1	8	4	–4	–8
120°	$-\frac{1}{2}$	$-\frac{1}{2}$	1	15	–7.5	–7.5	15
180°	– 1	1	–1	7	–7	7	–7
240°	$-\frac{1}{2}$	$-\frac{1}{2}$	1	6	–3	–3	6
300°	$\frac{1}{2}$	$-\frac{1}{2}$	–1	2	1	–1	–2
			$\Sigma =$	42	– 8.5	–4.5	8

Hence
$$a_0 = 2 \cdot \frac{42}{6} = 14, a_1 = 2\left(\frac{-8.5}{6}\right) = -2.8, a_2 = 2\left(\frac{-4.5}{6}\right) = -1.5,$$
$$a_3 = 2\left(\frac{8}{6}\right) = 2.7.$$

Example 10.26. *The turning moment T is given for a series of values of the crank angle θ° = 75°*

$\theta°$:	0	30	60	90	120	150	180
T :	0	5224	8097	7850	5499	2626	0

Obtain the first four terms in a series of sines to represent T and calculate T for θ = 75°.

Solution. Let the Fourier sine series to represent T in (0, 180) be

$T = b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta + ...$

To evaluate the coefficients, we form the following table :

$\theta°$	T	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$
0	0	0	0	0	0
30	5224	0.500	0.866	1	0.866
60	8097	0.866	0.866	0	–0.866
90	7850	1.000	0	–1	0
120	5499	0.866	–0.866	0	0.866
150	2626	0.500	–0.866	1	–0.866

$$\therefore \quad b_1 = \frac{2}{6}\Sigma y \sin\theta = \frac{1}{3}\{(5224 + 2626)\, 0.5 + (8097 + 5499)\, 0.866 + 7850\} = 7850$$

$$b_2 = \frac{2}{6}\Sigma y \sin 2\theta = \frac{1}{3}\{(5224 + 8097)\, 0.866 + (5499 + 2626)(-0.866)\} = 1500$$

$$b_3 = \frac{2}{6}\Sigma y \sin 3\theta = \frac{1}{3}\{5224 - 7850 + 2626\} = 0.$$

$$b_4 = \frac{2}{6}\Sigma y \sin 4\theta = \frac{1}{3}\{(5224 + 5499)(0.866) + (8097 + 2626)(-0.866)\} = 0$$

Hence $\quad T = 785° \sin\theta + 150° \sin 2\theta$

For $\theta = 75°$, $T = 7850 \sin 75° + 1500 \sin 150°$

$= 7850° (0.9659) + 1500 (0.5) = 8332.$

PROBLEMS 10.9

1. The following values of y give the displacement in inches of a certain machine part for the rotation x of the flywheel. Expand y in terms of a Fourier series :

x :	0	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$
y :	0	9.2	14.4	17.8	17.3	11.7

2. Compute the first two harmonics of the Fourier series of $f(x)$ given in the following table :

x :	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$:	1.0	1.4	1.9	1.7	1.5	1.2	1.0

(*Anna, 2009*)

3. Obtain the constant term and the coefficients of the first sine and cosine terms in the Fourier expansion of y as given in the following table :

x :	0	1	2	3	4	5
y :	9	18	24	28	26	20

(*V.T.U., 2011 ; Anna, 2005 S*)

4. In a machine the displacement y of a given point is given for a certain angle θ as follows :

$\theta°$:	0	30	60	90	120	150	180	210	240	270	300	330
y :	7.9	8.0	7.2	5.6	3.6	1.7	0.5	0.2	0.9	2.5	4.7	6.8

Find the coefficient of $\sin 2\theta$ in the Fourier series representing the above variation.

5. Determine the first two harmonics of the Fourier series for the following values :

$x°$:	30	60	90	120	150	180	210	240	270	300	330	360
y :	2.34	3.01	3.68	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

(*Madras, 2006 ; Cochin, 2005*)

6. The turning moment T on the crankshaft of a steam engine for the crank angle θ degrees is given as follows :

θ :	0	15	30	45	60	75	90	105	120	135	150	165	180
T :	0	2.7	5.2	7.0	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2	0

Expand T in a series of sines upto the fourth harmonics.

10.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 10.10

Fill up the blanks or choose the correct answer in each of the following problems :

1. The period of $\cos 3x$ is $x =$ ......
2. If $x = c$ is a point of discontinuity then the Fourier series of $f(x)$ at $x = c$ gives $f(x) =$ ......
3. A function $f(x)$ defined for $0 < x < 1$ can be extended to an odd periodic function in ......
4. The mathematical function representing the following graph is ......
5. Fourier expansion of an odd function has only ...... terms.
6. Formulae for evaluation of Fourier coefficients for a given set of points $(x_i, y_i) : i = 0, 1, 2, ... n$ are......
7. If $f(x) = x^4$ in $(-1, 1)$, then the Fourier coefficient $b_n =$ ......
8. The period of a constant function is ......
9. If $f(t) = \begin{cases} -1, & -1 < t < 0 \\ 1, & 0 < t < 1 \end{cases}$, then $f(t)$ is an ......
10. Fourier expansion of an even function $f(x)$ in $(-\pi, \pi)$ has only ...... terms.
11. If $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$, then $f(x)$ is an ...... function in $(-\pi, \pi)$.
12. The smallest period of the function $\sin\left(\dfrac{2n\pi x}{k}\right)$ is ......
13. In the Fourier series expansion of $f(x) = |\sin x|$ in $(-\pi, \pi)$, the value of $b_n =$ ......
14. In the Fourier series for $f(x) = x$ in $(-\pi \le x \le \pi)$, the ...... terms are absent.
15. If $f(x)$ is an even function in $(-l, l)$, then the value of $b_n =$ ......
16. If $f(x) = x^2$ in $-2 < x < 2$, $f(x + 4) = f(x)$, then a_n is ......
17. If $f(x)$ is a periodic function with period $2T$, then the value of the Fourier coefficient $b_n =$ ......
18. Dirichlet conditions for the expansion of a function as a Fourier series in the interval $c_1 \le x \le c_2$ are ......
19. If $f(x) = x \sin x$ in $(-\pi, \pi)$, then the value of $b_n =$ ......
20. The formulae for finding the half range cosine series for the function $f(x)$ in $(0, l)$ are ......
21. The half-range sine series for 1 in $(0, \pi)$, is ......
22. Period of $|\sin t|$ is ......
23. The value of bn in the Fourier series of $f(x) = |x|$ in $(-\pi, \pi) =$ ......
24. If $f(x)$ is defined in $(0, l)$ then the period of $f(x)$ to expand it as a half range sine series is ......
25. The complex form of Fourier series for e^{-x} in $(-1, 1)$ is ......
26. $f(x)$ is an odd function in $(-\pi, \pi)$, then the graph of $f(x)$ is symmetric about the x-axis. (True or False)
27. $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi, \end{cases}$ then $f(0) =$ ......
28. If $f(x) = \begin{cases} \pi x & \text{in } 0 \le x \le 1 \\ \pi(2 - x) & \text{in } 1 \le x \le 2, \end{cases}$ then it is ...... function. (odd or even)
29. If $f(x)$ is an odd function in $(-l, l)$, then the values of a_0 and a_n are......
30. The root mean square value of $f(t) = 3 \sin 2t + 4 \cos 2t$ over the range $0 \le t \le \pi$ is...... *(Nagpur, 2009)*
31. In the Fourier series expansion of the function

$$f(x) = \begin{cases} -(\pi + x), & -\pi < x < 0 \\ -(\pi - x), & 0 < x < \pi, \end{cases}$$ the value of b_n is ...... *(P.T.U., 2010)*

32. Let $f(x)$ be defined in $(0, 2\pi)$ by

$$f(t) = \begin{cases} \dfrac{1 + \cos x}{\pi - x}, & 0 < x < \pi \\ \cos x, & \pi < x < 2\pi, \end{cases}$$ $f(x) + 2\pi = f(x)$. The value of $f(\pi)$ is ...... *(Anna, 2009)*

33. The mean value of $f(x)\cos nx$ in $(0, 2\pi) =$ ......

34. Using sine series for $f(x) = 1$ in $0 < x < \pi$, show that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots \infty =$ .....

35. Fourier series representing $f(x) = |x|$ in $-\pi < x < \pi$, is ......

36. Fourier series of $f(x) = \cos^4 x$ in $(0, 2\pi)$ is ......

37. If $f(x) = x^2 + x$ in $(0, l)$, then the even extension of $f(x)$ in $(-l, 0)$ is ......

38. If $f(x) = x(l - x)$ in $(0, l)$, then the extension of $f(x)$ in $(l, 2l)$ so as to get sine series is ......

39. A function $f(x)$ defined in $(-\pi, \pi)$ can be expanded into Fourier series containing both sine and cosine terms. (True or False)

40. The function $f(x) = \begin{cases} 1 - x \text{ in } & -\pi < x < 0 \\ 1 + x \text{ in } & 0 < x < \pi, \end{cases}$ is an odd function. (True or False)

41. If $f(x) = x^2$ in $(-\pi, \pi)$, then the Fourier series of $f(x)$ contains only sine terms. (True or False)