APPENDIX 1

Useful Information

I. BASIC DATA

1. Basic Constants

$e = 2.7183$	$1/e = 0.3679$	$\log_e 2 = 0.6931$	$\log_e 3 = 1.0986$
$\pi = 3.1416$	$1/\pi = 0.3183$	$\log_e 10 = 2.3026$	$\log_{10} e = 0.4343$
$\sqrt{2} = 1.4142$	$\sqrt{3} = 1.732$	1 rad. $= 57° 17' 45''$	$1° = 0.0174$ rad.

2. Conversion Factors

1 ft. = 30.48 cm = 0.3048 m	1 m = 100 cm = 3.2804 ft.
1 ft^2 = 0.0929 m^2	1 acre = 4840 yd^2 = 4046.77 m^2
1 ft^3 = 0.0283 m^3	1 m^3 = 35.32 ft^3
1 m/sec = 3.2804 ft/sec.	1 mile/h = 1.609 km/h.

3. Systems of Units

Quantity	*F.P.S. system*	*C.G.S. system*	*M.K.S. system*
Length	foot (ft)	centrimetre (cm)	metre (m)
Mass	pound (lb)	gram (gm)	kilogram (kg)
Time	second (sec)	second (sec)	second (sec)
Force	lb. wt.	dyne	newton (nt)

Note. The M.K.S. system is also known as the *International system of units (SI system).*

4. Greek Letters Used

α	alpha	θ	theta	κ	kappa	τ	tau
β	beta	ϕ	phi	μ	mu	χ	chi
γ	gamma	ψ	psi	ν	nu	ω	omega
δ	delta	ξ	xi	π	pi	Γ	cap. gamma
ε	epsilon	η	eta	ρ	rho	Δ	cap. delta
ι	iota	ζ	zeta	σ	sigma	Σ	cap. sigma
		λ	lambda				

5. Some Notations

$\in$	belongs to	$\cup$	union
$\notin$	does not belong to	$\cap$	intersection
$\Rightarrow$	implies	$\ni$	such that
$\Leftrightarrow$	implies & implied by		

Factorial n i.e., $n! = n(n-1)(n-2) \ldots\ldots 3 \,.\, 2 \,.\, 1.$

Double factorials : $(2n)!! = 2n\,(2n-2)\,(2n-4) \ldots\ldots 6 \,.\, 4 \,.\, 2.$

$$(2n-1)!! = (2n-1)\,(2n-3)\,(2n-5) \ldots\ldots 5 \,.\, 3 \,.\, 1.$$

Stirling's approximation. When n is large $n! \sim \sqrt{2\pi n} \,.\, n^n e^{-n}$.

II. ALGEBRA

1. **Quadratic equation** : $ax^2 + bx + c = 0$ has roots

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac)}}{2a}, \beta = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}$$

$$\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}.$$

Roots are equal if $b^2 - 4ac = 0$

Roots are real and distinct if $b^2 - 4ac > 0$

Roots are imaginary if $b^2 - 4ac < 0$

2. **Cubic equation** : $x^3 + lx^2 + mx + n = 0$

Cardan's method :

(*i*) Remove x^2 term by putting $y = x - (-l/3)$

(*ii*) Equate coeffs. in the new equation and $y^3 - 3uvy - (u^3 + v^3) - 0$ [$\because \quad y = u + v$

(*iii*) Find u^3 and v^3. Then find u and v.

(*iv*) Get $y = u + v$ and $x = y - l/3$.

3. **Biquadratic equation** : $x^4 + kx^3 + lx^2 + mx + n = 0$

I. *Ferrari's method :*

(*i*) Combine x^4 and x^3 terms into a perfect square by adding a term in λ.

(*ii*) Make R.H.S. a perfect square to find λ.

(*iii*) Solve resulting quadratic equations.

II. *Descarte's method :*

(*i*) Remove x^3 term by putting $y = x - (-k/4)$

(*ii*) Equate transformed expression to $(y^2 + py + q)\,(y^2 - py + q')$

(*iii*) Equate coeffs. of like powers from both sides.

(*iv*) Find p, q and q' and solve resulting quadratics.

4. **Cross-multiplication :**

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$

Then

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

5. **Method of least squares :**

(*i*) *Straight line of best fit* $y = a + bx$.

Normal equations : $\Sigma y = na + b\Sigma x$, $\Sigma xy = a\Sigma x + b\Sigma x^2$.

To find a, b, solve these equations.

(*ii*) *Parabola of best fit* $y = a + bx + cx^2$

Normal equations : $\Sigma y = na + b\Sigma x + c\Sigma x^2$,

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3, \Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

To find a, b, c, solve these equations.

6. **Progressions :**

(*i*) Numbers $a, a + d, a + 2d, \ldots$ are said to be in *Arithmetic progression (A.P.)*

Its nth term $T_n = a + \overline{n-1}\,d$ and *sum* $S_n = \frac{n}{2}\,(2a + \overline{n-1}\,d)$

(*ii*) Numbers $a, ar, ar^2, \ldots\ldots$, are said to be in *Geometric progression (G.P.)*

Its nth term $T_n = ar^{n-1}$ and *sum* $S_n = \frac{a(1-r^n)}{1-r}$, $S_\infty = \frac{a}{1-r}$ $(r < 1)$

(*iii*) Numbers $1/a$, $1/(a + d)$, $1/(a + 2d)$, ... are said to be in *Harmonic progression (H.P.)* (*i.e.*, a sequence is said to be in H.P. if its reciprocals are in A.P.)

Its nth term $T_n = 1/(a + \overline{n-1}\, d)$

(*iv*) If a and b be two numbers then their

Arithmetic mean $= \frac{1}{2}(a + b)$, *Geometric mean* $= \sqrt{ab}$, *Harmonic mean* $= 2ab/(a + b)$

(*v*) *Natural numbers* are 1, 2, 3, ..., n.

$$\Sigma n = \frac{n(n+1)}{2},\ \Sigma n^2 = \frac{n(n+1)(2n+1)}{6},\ \Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

7. Permutations and Combinations

$${}^nP_r = \frac{n!}{(n-r)!};\ {}^nC_r = \frac{n!}{r!(n-r)!} = \frac{{}^nP_r}{r!};\ {}^nC_{n-r} = {}^nC_r;\quad {}^nC_0 = 1 = {}^nC_n.$$

8. Binomial theorem

(*i*) *When n is a positive integer*

$$(1 + x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots\ldots + {}^nC_n x^n.$$

(*ii*) *When n is a negative integer or a fraction*

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots\ldots \infty.$$

(*iii*) *Binomial coefficients* : ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

9. Logarithms

(*i*) *Naturals logarithm* $\log x$ has base e and is inverse of e^x.

Common logarithm $\log_{10} x = M \log x$ where $M = \log_{10} e = 0.4343$

(*ii*) $\log_a 1 = 0$; $\log_a 0 = -\infty\ (a > 1)$; $\log_a a = 1$.

(*iii*) $\log(mn) = \log m + \log n$; $\log(m/n) = \log m - \log n$; $\log(m^n) = n \log m$.

10. Partial Fractions

A fraction of the form $\dfrac{a_0x^m + a_1x^{m-1} + \ldots\ldots + a_m}{b_0x^n + b_1x^{n-1} + \ldots\ldots + b_n}$

in which m and n are positive integers, is called a *rational algebraic fraction*. When the numerator is of a lower degree than the denominator, it is called a *proper fraction*.

To resolve a given fraction into partial fractions, we first factorise the denominator into real factors. These will be either linear or quadratic, and some factors repeated. Then the proper fraction is resolved into a sum of partial fractions such that

(*i*) to a *non-repeated linear factor* $x - a$ in the denominator corresponds a partial fraction of the form $A/(x - a)$;

(*ii*) to a *repeated linear factor* $(x - a)^r$ in the denominator corresponds the sum of r partial fractions of the form $\dfrac{A_1}{x-a} + \dfrac{A_2}{(x-a)^2} + \dfrac{A_3}{(x-a)^3} + \ldots + \dfrac{A_r}{(x-a)^r}$;

(*iii*) to a *non-repeated quadratic factor* $(x^2 + ax + b)$ in the denominator, corresponds a partial fraction of the form $\dfrac{Ax + B}{x^2 + ax + b}$;

(*iv*) to a *repeated quadratic factor* $(x^2 + ax + b)^r$ in the denominator, corresponds the sum of r partial fractions of the form $\dfrac{A_1x + B_1}{x^2 + ax + b} + \dfrac{A_2x + B_2}{(x^2 + ax + b)^2} + \ldots\ldots + \dfrac{A_rx + B_r}{(x^2 + ax + b)^r}$.

Then we have to determine the unknown constants A, A_1, B_1 etc.

To obtain the partial fraction corresponding to the non-repeated linear factor $x - a$ in the denominator, put $x = a$ everywhere in the given fraction except in the fac tor $x - a$ itself.

In all other cases, equate the given fraction to a sum of suitable partial fractions in accordance with (*i*) to (*iv*) above, having found the partial fractions corresponding to the non-repeated linear factors by the above rule. Then multiply both sides by the denominator of the given fraction and equate the coefficients of like powers of x or substitute convenient numerical values of x on both sides. Finally solve the simplest of the resulting equations to find the unknown constants.

11. Matrices

(*i*) *A system of mn numbers arranged in a rectangular array of m rows and n columns is called a* **matrix** *of order* $m \times n$.

In particular if $m = n$, it is called a *square matrix of order n*.

(*ii*) *Two matrices of the same order can be added or subtracted* by adding or subtracting the corresponding elements.

(*iii*) *Product of a matrix A by a scalar k* is a matrix whose each element is k times the corresponding elements of A.

(*iv*) *Two matrices can be multiplied* only when the number of columns in the first is equal to the number of rows in the second. If A is of order $m \times n$ and B is of order $n \times p$, then the product AB is a matrix of order $m \times p$, obtained by multiplying and adding the row elements of A with the corresponding column elements of B.

(*v*) *Transpose of a matrix A* is the matrix obtained by interchanging its rows and columns and is denoted by A'.

A square matrix A is said to be *symmetric* if $A = A'$ and *skew symmetric* if $A = -A'$.

(*vi*) If A and B are two square matrices such that $AB = I$ (*i.e.*, a unit matrix), then B is called the *inverse* of A and is denoted by A^{-1}. Then $AA^{-1} = A^{-1}A = I$.

(*vii*) *Rank of a matrix* is the largest order of any non-vanishing minor of the matrix.

(*viii*) *Consistency of a system of equations in n unknowns.*

If the rank of the coefficient matrix A be r and that of the augmented matrix K be r', then

(*a*) the equations are inconsistent (*i.e.* there is no solution) when $r \neq r'$,

(*b*) the equations are consistent when $r = r'$.

(*c*) the equations are consistent and there are infinite number of solutions when $r = r' < n$.

(*ix*) *Eigen values:* If A is any square matrix of order n, then the determinant of the matrix $A - \lambda I_n$ equated to zero is called the *Characterist equation* of A and its roots are called the *eigen values of A*.

(*x*) *Cayley Hamilton theorem*: Every square matrix satisfies its own characteristic equation.

12. Determinants

(*i*) A **determinant** is defined for a square matrix A and is denoted by $|A|$. Unlike a matrix it has a single value *e.g.*,

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)$$

In this way, determinant can be expanded in terms of any row or column.

(*ii*) **Properties :**

I. A determinant remains unaltered if its rows and columns are interchanged.

II. A determinant vanishes if two of its rows (or columns) are identical or proportional.

III. If each elements of a row (or column) consists of m terms, the determinant can be expressed as the sum of m determinants.

IV. If to each elements of a row (or column) be added equi-multiples of the corresponding elements of two or more rows (or columns), the determinant remains unaltered.

V. If the elements of a determinant Δ are functions of x and two parallel lines become identical when $x = a$, then $x - a$ is a factor of Δ.

III. GEOMETRY

1. Coordinates of a point : *Cartesian* (x, y) *and polar* (r, θ).

$x = r\cos\theta, \qquad y = r\sin\theta$

or $\quad r = \sqrt{(x^2 + y^2)}, \qquad \theta = \tan^{-1}(y/x)$. (Fig. 0.1).

Distance between two points (x_1, y_1) and $(x_2, y_2) = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$

Point of division of the line joining (x_1, y_1) and (x_2, y_2) in the ratio $m_1 : m_2$ is

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

In a triangle having vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3)

(*i*) *Area* $= \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

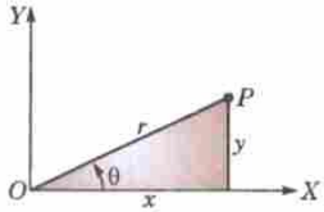


Fig. AP-1.1

(*ii*) *Centroid* (point of intersection of medians) is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

(*iii*) *Incentre* (point of intersection of the internal bisectors of the angles) is

$$\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$$

where a, b, c are the lengths of the sides of the triangle.

(*iv*) *Circumcentre* is the point of intersection of the right bisectors of the sides of the triangle.

(*v*) *Orthocentre* is the point of intersection of the perpendiculars drawn from the vertices to the opposite sides of the triangle.

2. Straight Line

(*i*) *Slope of the line* joining the points (x_1, y_1) and $(x_2, y_2) = \dfrac{y_2 - y_1}{x_2 - x_1}$

Slope of the line $ax + by + c = 0$ is $-\dfrac{a}{b}$ *i.e.*, $-\dfrac{\text{coeff. of } x}{\text{coeff. of } y}$

(*ii*) *Equation of a line*

(*a*) having slope m and cutting an intercept c on y-axis is $y = mx + c$.

(*b*) cutting intercepts a and b from the axes is $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

(*c*) passing through (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$

(*d*) passing through (x_1, y_1) and making an $\angle\theta$ with the x-axis is $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$

(*e*) through the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0$

(*iii*) *Angle between two lines* having slopes m_1 and m_2 is $\tan^{-1}\dfrac{m_1 - m_2}{1 + m_1m_2}$

Two lines are parallel if	$m_1 = m_2$		
Two lines are perpendicular if	$m_1m_2 = -1$		
Any line parallel to the line	$ax + by + c = 0$	is	$ax + by + k = 0$
Any line perpendicular to	$ax + by + c = 0$	is	$bx - ay + k = 0$

(*iv*) *Length of the perpendicular* from (x_1, y_1) to the line $ax + by + c = 0$, is $\dfrac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$.

3. Circle

(*i*) *Equation of the circle* having centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$

(*ii*) *Equation* $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle having centre $(-g, -f)$ and radius $= \sqrt{(g^2 + f^2 - c)}$.

(*iii*) *Equation of the tangent* at the point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

(*iv*) *Condition for the line* $y = mx + c$ *to touch the circle* $x^2 + y^2 = a^2$ is $c = a\sqrt{(1 + m^2)}$.

(*v*) *Length of the tangent* from the point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)}$.

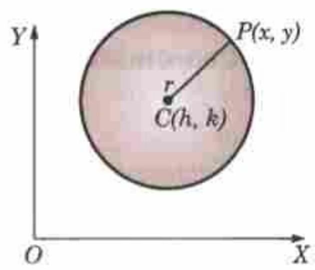


Fig. AP-1.2

4. Parabola

(*i*) *Standard equation of the parabola* $y^2 = 4ax$.

Its parametric equations are $x = at^2, y = 2at$.

Latus-rectum $LL' = 4a$, *Focus* is $S(a, 0)$

Directrix ZM is $x + a = 0$.

Focal distance of any point $P(x_1, y_1)$ on the parabola

$$y^2 = 4ax \text{ is } SP = x_1 + a$$

Equation of the tangent at (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1)$$

Condition for the line $y = mx + c$ *to touch the parabola*

$$y^2 = 4ax \text{ is } c = a/m.$$

Equation of the normal to the parabola $y^2 = 4ax$ *in terms of its slope* m is

$$y = mx - 2am - am^3.$$

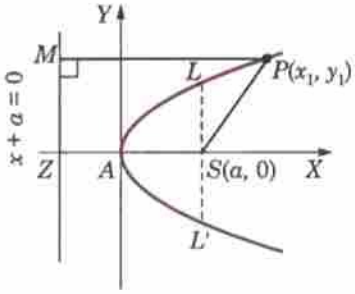
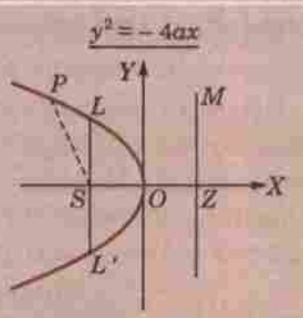
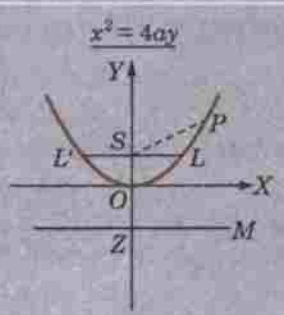
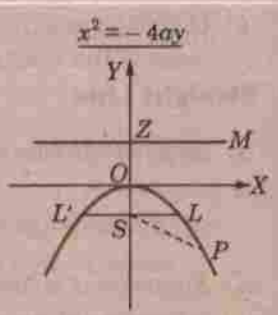


Fig. AP-1.3

(*ii*) *Other standard forms of parabola*

$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
		
Latus rectum : $LL' = 4a$	$LL' = 4a$	$LL' = 4a$
Focus : $S(-a, 0)$	$S(0, a)$	$S(0, -a)$
Directrix $ZM : x = a$	$ZM : y = -a$	$ZM : y = a$
Focal distance : $SP = a - x$	$SP = a + y$	$SP = a - y$

5. Ellipse

(*i*) *Standard equation of the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \ (a > b > 0)$.

It parametric equations are $x = a\cos\theta, y = b\sin\theta$.

Eccentricity $e = \sqrt{(1 - b^2/a^2)}$,

Latus-rectum $LSL' = 2b^2/a$.

Foci $S(-ae, 0)$ and $S'(ae, 0)$

Directrices ZM $(x = -\tilde{a}/e)$ and $Z'M'$ $(x = a/e)$.

Sum of the focal distances of any point on the ellipse is equal to the major axis i.e.,

$$SP + S'P = 2a.$$

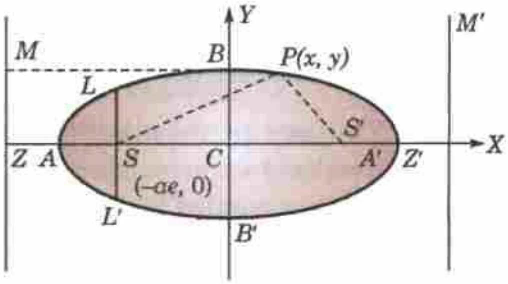


Fig. AP-1.4

Equation of the tangent at the point (x_1, y_1) to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

Condition for the line $y = mx + c$ *to touch the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c = \sqrt{(a^2 m^2 + b^2)}$

(*ii*) *Another standard form of ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b > 0)$

Vertices: $A\ (0o, a)$; $A'\ (0, -a)$

Foci : $S\ (0, ae)$; $S'\ (0, -ae)$

Directrices : $ZM : y = a/e$, $Z'M' : y = -a/e$

Latus rectum : $LSL' = 2b^2/a$

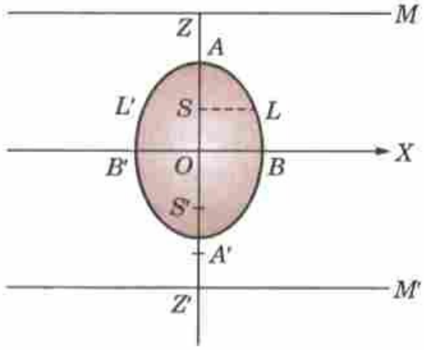


Fig. AP-1.4 (a)

6. Hyperbola

(*i*) *Standard equations of the hyperbola is*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Its parametric equations are

$x = a \sec \theta, \qquad y = b \tan \theta.$

Eccentricity $e = \sqrt{(1 + b^2/a^2)}$,

Latus-rectum $LSL' = 2b^2/a$.

Directrices

$ZM\ (x = a/e)$ and $Z'M'\ (x = -a/e)$.

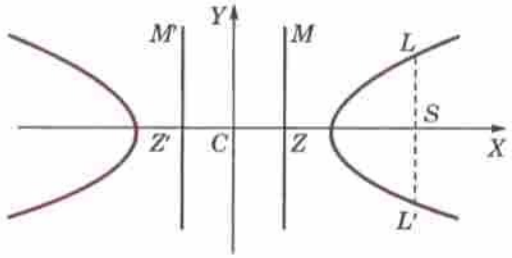


Fig. AP-1.5

(*ii*) *Equation of the tangent at the point* (x_1, y_1) to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

(*iii*) *Condition for the line* $y = mx + c$ *to touch the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } c = \sqrt{(a^2 m^2 - b^2)}$$

(*iv*) *Asymptotes of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$.

(*v*) *Equation of the rectangular hyperbola with asymptotes as axes is* $xy = c^2$.

Its parametric equations are $x = ct$, $y = c/t$.

7. Nature of a conic

The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *represents*

(*i*) *a pair of lines, if* $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ $(= \Delta) = 0$

(*ii*) *a circle, if* $a = b$, $h = 0$, $\Delta \neq 0$

(*iii*) *a parabola, if* $ab - h^2 = 0$, $\Delta \neq 0$.

(*iv*) *an ellipse, if* $ab - h^2 > 0$, $\Delta \neq 0$.

(*v*) *a hyperbola, if* $ab - h^2 < 0$, $\Delta \neq 0$,

and a rectangular hyperbola if in addition, $a + b = 0$.

IV. SOLID GEOMETRY

1. (*i*) If l, m, n be the *direction cosines* of a line then $l^2 + m^2 + n^2 = 1$.

If a, b, c be the *direction ratios* of a line then $l = \frac{a}{\sqrt{\Sigma a^2}}$; $m = \frac{b}{\sqrt{\Sigma a^2}}$; $n = \frac{c}{\sqrt{\Sigma a^2}}$

(*ii*) If θ be the *angle between the lines having* d.c.'s l, m, n and l', m', n', then

$$\cos \theta = ll' + mm' + nn'$$

Lines are perpendicular if, $ll' + mm' + nn' = 0$

Lines are parallel if $l = l'$, $m = m'$, $n = n'$

(*iii*) *Projection of the line joining the points* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *on a line having d.c.'s* l, m, n = $l(x_1 - x_2) + m(y_1 - y_2) + n(z_1 - z_2)$.

2. Plane

(*i*) *Different forms of equation of a plane*

— *General form* : $ax + by + cz = d$

where a, b, c are the d.r.s of a normal to the plane.

— *Normal form* : $lx + my + nz = p$

— *Intercept form* : $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

— *Any plane passing through the point* (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

(*ii*) *Angle* θ *between the planes* $ax + by + cz = d$ and $a'x + b'y + c'z = d'$ is given by

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(a'^2 + b'^2 + c'^2)}}$$

Planes are perpendicular if $aa' + bb' + cc' = 0$

Planes are parallel if $a/a' = b/b' = c/c'$

(*iii*) *Any plane parallel to the plane* $ax + by + cz = d$ is $ax + by + cz = k$.

3. Straight line

(*i*) *Equation of the line* through the point (x_1, y_1, z_1) having d.r.s a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \textit{(Symmetrical form)}$$

(*ii*) *Equation of the line* through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \qquad \textit{(Two point form)}$$

(*iii*) *Angle* θ *between the plane* $ax + by + cz = d$ *and the line*

$$\frac{x - x_1}{a'} = \frac{y - y_1}{b'} = \frac{z - z_1}{c'}$$

is $$\sin\theta = \frac{aa' + bb' + cc'}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(a'^2 + b'^2 + c'^2)}}$$

Line is parallel to the plane if $aa' + bb' + cc' = 0$

Line is perpendicular to the plane if $a/a' = b/b' = c/c'$

(*iv*) *Coplanar lines*

Two lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$

are coplanar if $$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and equation of the plane containing these lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

(*v*) *Shortest distance* between two skew lines

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \text{ and } \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$

is $$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

where l, m, n are given by $ll_1 + mm_1 + nn_1 = 0$ and $ll_2 + mm_2 + nn_2 = 0$

Equation of the line of S.D. is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \text{ and } \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0$$

4. Sphere

(i) *Equation of the sphere* having centre (a, b, c) and radius r is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

(ii) Equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere having *centre* $(-u, -v, -w)$ and *radius* $= \sqrt{(u^2 + v^2 + w^2 - d)}$

(iii) *Equation of the sphere* having the points (x_1, y_1, z_1) and (x_2, y_2, z_2) as the ends of a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

(iv) *Equation of a circle* (*i.e.*, section of a sphere $S = 0$ by the plane $U = 0$) is given by $S = 0$ and $U = 0$ taken together.

(v) *Equation of any sphere* through the circle of intersection of the sphere $S = 0$ and the plane $U = 0$ is $S + kU = 0$.

(vi) *Tangent plane* at any point (x_1, y_1, z_1) of the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \text{ is}$$
$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

(vii) Two spheres $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ and

$x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$ *cut orthogonally if* $2uu' + 2vv' + 2ww' = d + d'$.

5. Cone

(i) *Equation of a cone* with vertex at the origin is a homogeneous equation of the second degree in x, y, z.

(ii) *Enveloping cone* of the sphere $S = 0$ with vertex (x_1, y_1, z_1) is $SS_1 = T^2$ where $S = x^2 + y^2 + z^2 - a^2$, $S_1 = x_1^2 + y_1^2 + z_1^2 - a^2$, $T = xx_1 + yy_1 + zz_1 - a^2$

6. Quadric surfaces

(i) *Ellipoid:* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

(ii) *Hyperboloid* of one sheet: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

Hyperboloid of two sheets: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$

(iii) *Cone:* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$

(iv) *Elliptic paraboloid:* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{2z}{c}$

Hyperbolic paraboloid: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{2z}{c}$

7. Volumes and surface areas

Solid	*Volume*	*Curved surface area*	*Total surface area*
Cube (side a)	a^3	$4a^2$	$6a^2$
Cuboid (length l, breadth b, height h)	lbh	$2(l + b)\,h$	$2(lb + bh + hl)$
Sphere (radius r)	$\frac{4}{3}\pi r^3$	—	$4\pi r^2$
Cylinder (base radius r, height h)	$\pi r^2 h$	$2\pi rh$	$2\pi r\,(r + h)$
Cone (base radius r, height h)	$\frac{1}{3}\pi r^2 h$	πrl	$\pi r(r + l)$
where slant height l is given by $l = \sqrt{(r^2 + h^2)}$.			

V. TRIGONOMETRY

1.

$\theta°$ =	0	30	45	60	90	180	270	360
$\sin\theta$	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	– 1	0
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	– 1	0	1
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	0	$-\infty$	0

2. **Signs and variations of t-ratios**

Quadrant	*sin θ*	*cos θ*	*tan θ*
I	+ (0 to 1)	+ (1 to 0)	+ (0 to ∞)
II	+ (1 to 0)	– (0 to – 1)	– (– ∞ to 0)
III	– (0 to – 1)	– (– 1 to 0)	+ (0 to ∞)
IV	– (– 1 to 0)	+ (0 to 1)	– (– ∞ to 0)

3. *Any t-ratio of* $(n \,.\, 90° \pm \theta) = \pm$ *same ratio of* θ, *when n is even.*

$= \pm$ *co-ratio of* θ, *when n is odd.*

The sign + or – is to be decided from the quadrant in which $n \,.\, 90° \pm \theta$ *lies.*

e.g., $$\sin 570° = \sin(6 \times 90° + 30°) = -\sin 30° = -\frac{1}{2},$$

$$\tan 315° = \tan(3 \times 90° + 45°) = -\cot 45° = -1$$

4. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$\sin 2A = 2 \sin A \cos A = 2 \tan A/(1 + \tan^2 A)$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B};\ \tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

5. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

6. $\sin C + \sin D = 2 \sin\frac{C+D}{2} \cos\frac{C-D}{2}$

$\sin C - \sin D = 2 \cos\frac{C+D}{2} \sin\frac{C-D}{2}$

$\cos C + \cos D = 2 \cos\frac{C+D}{2} \cos\frac{C-D}{2}$

$\cos C - \cos D = -2 \sin\frac{C+D}{2} \sin\frac{C-D}{2}$

7. $\sin 3A = 3 \sin A - 4 \sin^3 A$, $\cos 3A = 4 \cos^3 A - 3 \cos A$; $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

8. $$a \sin x + b \cos x = r \sin (x + \theta)$$
$$a \cos x + b \sin x = r \cos (x - \theta)$$
where $a = r \cos \theta$, $b = r \sin \theta$ so that $r = \sqrt{(a^2 + b^2)}$, $\theta = \tan^{-1} (b/a)$

9. *In any* ΔABC :

(*i*) $a/\sin A = b/\sin B = c/\sin C$ (*Sine formula*)

(*ii*) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ (*Cosine formula*)

(*iii*) $a = b \cos C + c \cos B$ (*Projection formula*)

(*iv*) *Area of* $\Delta ABC = \dfrac{1}{2} bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{1}{2}(a + b + c)$.

10. **Series**

(*i*) *Exponential series* : $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots \infty$

(*ii*) *Sin, cos, sinh, cosh series*

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots \infty, \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots \infty$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots \infty, \qquad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots \infty$$

(*iii*) *Log series*

$$\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots \infty, \qquad \log (1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots \infty\right)$$

(*iv*) *Gregory series*

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots \infty, \qquad \tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots \infty$$

11. (*i*) **Complex number :** $z = x + iy = r(\cos \theta + i \sin \theta) = r e^{i\theta}$ [see Fig. AP-1.1]

(*ii*) **Euler's theorem :** $\cos \theta + i \sin \theta = e^{i\theta}$

(*iii*) **Demoivre's theorem :** $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

12. (*i*) **Hyperbolic functions :** (*i*) $\sinh x = \dfrac{e^x - e^{-x}}{2}$; $\cosh x = \dfrac{e^x + e^{-x}}{2}$;

$$\tanh x = \frac{\sinh x}{\cosh x};\ \coth x = \frac{\cosh x}{\sinh x};\ \text{sech}\, x = \frac{1}{\cosh x};\ \text{cosech}\, x = \frac{1}{\sinh x}$$

(*ii*) *Relations between hyperbolic and circular functions :*

$$\sin ix = i \sinh x;\ \cos ix = \cosh x;\ \tan ix = i \tanh x.$$

(*iii*) *Inverse hyperbolic functions:*

$$\sinh^{-1} x = \log [x + \sqrt{x^2 + 1}];\ \cosh^{-1} x = \log [x + \sqrt{(x^2 - 1)}];\ \tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}.$$

VI. DIFFERENTIAL CALCULUS

1. **Standard limits :**

(*i*) $\underset{x \to a}{Lt} \dfrac{x^n - a^n}{x - a} = na^{n-1}$, n any rational number

(*ii*) $\underset{x \to 0}{Lt} \dfrac{\sin x}{x} = 1$

(*iii*) $\underset{x \to 0}{Lt} (1 + x)^{1/x} = e$

(*iv*) $\underset{x \to \infty}{Lt} x^{1/x} = 1$

(*v*) $\underset{x \to 0}{Lt} \dfrac{a^x - 1}{x} = \log_e a$

2. Differentiation

(i) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du/dx - u\,dv/dx}{v^2}$

$\frac{du}{dx} = \frac{du}{dy}\cdot\frac{dy}{dx}$ (*Chain Rule*) $\qquad$ $\frac{d}{dx}(ax+b)^n = n\,(ax+b)^{n-1}\,.\,a$

(ii) $\frac{d}{dx}(e^x) = e^x$ $\qquad$ $\frac{d}{dx}(a^x) = a^x \log_e a$

$\frac{d}{dx}(\log_e x) = 1/x$ $\qquad$ $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$

(iii) $\frac{d}{dx}(\sin x) = \cos x$ $\qquad$ $\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\qquad$ $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\qquad$ $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(iv) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{(1-x^2)}}$ $\qquad$ $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{(1-x^2)}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\qquad$ $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{(x^2-1)}}$ $\qquad$ $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{(x^2-1)}}$

(v) $\frac{d}{dx}(\sinh x) = \cosh x$ $\qquad$ $\frac{d}{dx}(\cosh x) = \sinh x$.

$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$ $\qquad$ $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$

(vi) $D^n (ax+b)^m = m(m-1)(m-2) \ldots\ldots (m-n+1)(ax+b)^{m-n}\,.\,a^n$

$D^n \log(ax+b) = (-1)^{n-1}(n-1)!\,a^n/(ax+b)^n$

$D^n (e^{mx}) = m^n e^{mx}$ $\qquad$ $D^n (a^{mx}) = m^n (\log a)^n\,.\,a^{mx}$

$$D^n \begin{bmatrix} \sin(ax+b) \\ \cos(ax+b) \end{bmatrix} = a^n \begin{bmatrix} \sin(ax+b+n\pi/2) \\ \cos(ax+b+n\pi/2) \end{bmatrix}$$

$$D^n e^{ax} \begin{bmatrix} \sin(bx+c) \\ \cos(bx+c) \end{bmatrix} = (a^2+b^2)^{n/2} e^{ax} \begin{bmatrix} \sin(bx+c+n\tan^{-1} b/a) \\ \cos(bx+c+n\tan^{-1} b/a) \end{bmatrix}$$

(vii) *Leibnitz theorem:* $(uv)_n = u_n + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots\ldots + {}^nC_r u_{n-r} v_r + \ldots\ldots + {}^nC_n v_n$

3. (i) *Maclaurin's series* : $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$

(ii) *Taylor's series* : $f(x+a) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \ldots$

4. Curvature

(i) *Radius of curvature* $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$, $\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$; $\rho = r\frac{dr}{dp}$.

(ii) *Centre of curvature* : $\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$, $\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$.

(iii) **Evolute** *is the locus of the centre of curvature of a curve.* The curve is called the *involute* of the evolute.

(iv) **Envelope** *of a curve* $f(x, y, \alpha) = 0$ *is the 'α' eliminant from*

$$f(x, y, \alpha) = 0 \text{ and } \frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0.$$

The envelope of the normals to a curve is its ***evolute.***

5. Asymptotes

(*i*) *Asymptotes parallel to x-axis are obtained by equating to zero the coefficient of the highest power of x in the equation,* provided this is not merely a constant.

Asymptotes parallel to y-axis are obtained by equating to zero the coefficient of highest power of y in the equation, provided this is not merely a constant.

(*ii*) *Oblique asymptotes are obtained as follows:*

Put $x = 1, y = m$ in the highest degrees terms getting $\phi_n(m)$

Put $\phi_n(m) = 0$ and find the values of m.

Find c from $c = -\phi_{n-1}(m)/\phi_n'(m)$

If two values of m are equal, then find c from

$$\frac{c^2}{2}\phi_n''(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m) = 0$$

The asymptotes is $y = mx + c$.

(*iii*) *Asymptotes of polar curve* $1/r = f(\theta)$ is $r \sin(\theta - \alpha) = 1/f'(\alpha)$ where α is a root of $f(\theta) = 0$.

6. Curve tracing

(*i*) *A curve is symmetrical about x-axis,* if only even powers of y occur in its equation.

(*ii*) *A curve is symmetrical about y-axis,* if only even powers of x occur in its equation.

(*iii*) *A curve is symmetrical about the line* $y = x$, if on interchanging x and y, its equation remains unchanged.

(*iv*) *A curve passes through the origin,* if there is no constant term in its equation.

(*v*) *Tangents to curve at the origin* are found by equating to zero the lowest degree terms.

7. Partial Differentiation

(*i*) *Euler's theorem.* If u is a homogeneous function in x and y of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu\ ; \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)\,u$$

(*ii*) *Chain rule* : $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt}$, if $u = f(x, y), x = \phi(t), y = \psi(t)$.

(*iii*) $\dfrac{dy}{dx} = -\dfrac{\partial \phi}{\partial x}\Big/\dfrac{\partial \phi}{\partial y}$, if $\phi(x, y) = c$

(*iv*) *Jacobian* $J\left(\dfrac{u, v}{x, y}\right) = \dfrac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$

If $J = \partial(u, v)/\partial(x, y)$ and $J' = \partial(x, y)/\partial(u, v)$, then $JJ' = 1$

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}$$

(*v*) *Taylor's series* : $f(a + h, b + k) = f(a, b) + \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)f + \dfrac{1}{2!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^2 f + \ldots\ldots$

(*vi*) *Maxima Minima* (*a*) $\dfrac{\partial f}{\partial x} = 0, \dfrac{\partial f}{\partial y} = 0$

(*b*) $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} > \dfrac{\partial^2 f}{\partial x \partial y}$; $\dfrac{\partial^2 f}{\partial x^2} < 0$ for maximum $\dfrac{\partial^2 f}{\partial x^2} > 0$ for minimum.

(*vii*) *Leibnitz's Rule* $\dfrac{d}{d\alpha}\left\{\displaystyle\int_a^b f(x, \alpha)\, d\alpha\right\} = \displaystyle\int_a^b \dfrac{\partial f(x, \alpha)}{\partial \alpha}\, d\alpha$

where $f(x, \alpha)$ and $\dfrac{\partial f(x, \alpha)}{\partial \alpha}$ are continuous functions of x and α and a, b are constants.

VII. INTEGRAL CALCULUS

1. Integration

(i) $\int x^n \, dx = \frac{x^{n+1}}{n+1} \; (n \neq -1)$ $\qquad$ $\int \frac{1}{x} \, dx = \log_e x$

$\int e^x \, dx = e^x$ $\qquad$ $\int a^x \, dx = a^x / \log_e a$

(ii) $\int \sin x \, dx = -\cos x$ $\qquad$ $\int \cos x \, dx = \sin x$

$\int \tan x \, dx = -\log \cos x$ $\qquad$ $\int \cot x \, dx = \log \sin x$

$\int \sec x \, dx = \log (\sec x + \tan x)$ $\qquad$ $\int \text{cosec}\, x \, dx = \log (\text{cosec}\, x - \cot x)$

$\int \sec^2 x \, dx = \tan x$ $\qquad$ $\int \text{cosec}^2 x \, dx = -\cot x$

(iii) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$ $\qquad$ $\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1} \frac{x}{a}$

$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}$ $\qquad$ $\int \frac{dx}{\sqrt{(a^2 + x^2)}} = \sinh^{-1} \frac{x}{a}$

$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}$ $\qquad$ $\int \frac{dx}{\sqrt{(x^2 - a^2)}} = \cosh^{-1} \frac{x}{a}$

(iv) $\int \sqrt{(a^2 - x^2)} \, dx = \frac{x\sqrt{(a^2 - x^2)}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$

$\int \sqrt{(a^2 + x^2)} \, dx = \frac{x\sqrt{(a^2 + x^2)}}{2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a}$

$\int \sqrt{(x^2 - a^2)} \, dx = \frac{x\sqrt{(x^2 - a^2)}}{2} - \frac{a^2}{2} \cosh^{-1} \frac{x}{a}$

(v) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$

$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$

(vi) $\int \sinh x \, dx = \cosh x$ $\qquad$ $\int \cosh x \, dx = \sinh x$

$\int \tanh x \, dx = \log \cosh x$ $\qquad$ $\int \coth x \, dx = \log \sinh x$

$\int \text{sech}^2 x \, dx = \tanh x$ $\qquad$ $\int \text{cosech}^2 x \, dx = -\coth x$

(vii) $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \frac{(n-1)(n-3)(n-5)\ldots\ldots}{n(n-2)(n-4)\ldots\ldots} \times \left(\frac{\pi}{2}, \text{only if } n \text{ is even}\right)$

$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{(m-1)(m-3)\ldots\ldots \times (n-1)(n-3)\ldots\ldots}{(m+n)(m+n-2)(m+n-4)\ldots\ldots} \times \left(\frac{\pi}{2}, \text{only if both } m \text{ and } n \text{ are even}\right)$

(viii) $\int_0^a f(x) \, dx = \int_0^a f(a - x) \, dx$

$\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx,$ $\qquad$ if $f(x)$ is an *even function*

$= 0,$ $\qquad$ if $f(x)$ is an *odd function.*

$\int_0^{2a} f(x) \, dx = 2 \int_0^a f(x) \, dx,$ $\qquad$ if $f(2a - x) = f(x)$

$= 0,$ $\qquad$ if $f(2a - x) = -f(x).$

2. Lengths of curves

(*i*) *Lenth of curve* $y = f(x)$ *between* $x = a,\ x = b$ *is* $\int_a^b \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx$

(*ii*) *Length of curve* $x = f(y)$ *between* $y = a,\ y = b$ *is* $\int_a^b \sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\,dy$

(*iii*) *Length of curve* $x = f(t),\ y = \phi(t)$ *between* $t = t_1,\ t = t_2$ *is* $\int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}\,dt$

(*iv*) *Length of curve* $r = f(\theta)$ *between* $\theta = \alpha,\ \theta = \beta$ *is* $\int_\alpha^\beta \sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta$

3. Areas of curves

(*i*) *Area bounded by* $y = f(x)$, *x-axis and* $x = a,\ x = b$ *is* $\int_a^b y\,dx$

(*ii*) *Area bounded by* $x = f(y)$, *y-axis* and $y = a,\ y = b$ *is* $\int_a^b x\,dy$

(*iii*) *Area bounded by* $r = f(\theta)$ and *lines* $\theta = \alpha,\ \theta = \beta$ *is* $\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$

4. Volumes of revolution

(*i*) *Volume of revolution about x-axis of area bounded by* $y = f(x)$, *x-axis* and $x = a,\ x = b$ *is*

$$\int_a^b \pi y^2\,dx$$

(*ii*) *Volume of revolution about y-axis of area bounded by* $x = f(y)$, *y-axis and* $y = a,\ y = b$ *is*

$$\int_a^b \pi x^2\,dy$$

(*iii*) *Volume of revolution bounded by* $r = f(\theta)$ and $\theta = \alpha,\ \theta = \beta$

(*a*) *about OX* $= \int_\alpha^\beta \frac{2\pi}{3} r^3 \sin\theta\,d\theta$ (*b*) *about OY* $= \int_\alpha^\beta \frac{2\pi}{3} r^3 \cos\theta\,d\theta$

5. Surface areas of revolution

(*i*) *Surface area of revolution about x-axis of curve* $y = f(x)$ *from* $x = a$ *to* $x = b$ *is*

$$S = \int_{x=a}^{x=b} 2\pi y\,ds$$

Cartesian form : $S = \int_a^b 2\pi y \frac{ds}{dx}\,dx$ where $\frac{ds}{dx} = \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$

Parametric form : $S = \int 2\pi y \frac{ds}{dt}\,dt$ where $\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}$

Polar form : $S = \int 2\pi y \frac{ds}{d\theta}\,d\theta$ where $\frac{ds}{d\theta} = \sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}$

(*ii*) *Surface area of revolution about y-axis is* $\int 2\pi x\,ds$.

6. Multiple integrals

(*i*) *Area* $= \int_{x_1}^{x_2}\int_{y_1}^{y_2} dx\,dy$; *Volume* $= \int_{x_1}^{x_2}\int_{y_1}^{y_2} z\,dx\,dy$ *or* $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} dx\,dy\,dz$

(*ii*) *C.G. of a plane lamina:* $\bar{x} = \frac{\iint x\rho\,dx\,dy}{\iint \rho\,dx\,dy},\ \bar{y} = \frac{\iint y\rho\,dx\,dy}{\iint \rho\,dx\,dy}$

C.G. of a solid $\bar{x} = \dfrac{\iiint x\rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}, \bar{y} = \dfrac{\iiint y\rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}, \bar{z} = \dfrac{\iiint z\rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}$

(*iii*) *Centre of pressure* $\bar{x} = \dfrac{\iint_A px\, dx\, dy}{\iint_A p\, dx\, dy}, \bar{y} = \dfrac{\iint_A py\, dx\, dy}{\iint_A p\, dx\, dy}$

(*iv*) *M.I. about x-axis i.e.,* $I_x = \iiint_V \rho\,(y^2 + z^2)\, dx\, dy\, dz$

M.I. about y-axis i.e., $I_y = \iiint_V \rho\,(z^2 + x^2)\, dx\, dy\, dz$

M.I. about z-axis i.e., $I_z = \iiint_V \rho\,(x^2 + y^2)\, dx\, dy\, dz$

7. *Gamma function* $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx = (n-1)!, \Gamma(n+1) = n\,\Gamma(n) = n!, \Gamma(1/2) = \sqrt{\pi}$

Beta function $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx = \dfrac{\Gamma(m)\ \Gamma(n)}{\Gamma(m+n)}\ (m > 0, n > 0)$

VIII. VECTORS

1. (*i*) If $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ then $r = |\,\mathbf{R}\,| = \sqrt{(x^2 + y^2 + z^2)}$

(*ii*) $\vec{PQ}$ = Position vector of Q – position vector of P.

2. If $\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}, \mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$, then

(*i*) *Scalar product:* $\mathbf{A}\,.\,\mathbf{B} = ab\cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3$

(*ii*) *Vector product:* $\mathbf{A} \times \mathbf{B} = ab\sin\theta\ \hat{\mathbf{N}}$ = Area of the parallelogram having **A** and **B** as sides

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(*iii*) $\mathbf{B} \perp \mathbf{A}$ if $\mathbf{A}\,.\,\mathbf{B} = 0$ and **A** is parallel to **B** if $\mathbf{A} \times \mathbf{B} = \mathbf{0}$

3. (*i*) *Scalar triple product* $[\mathbf{A\ B\ C}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ = Volume of parallelopiped

(*ii*) If $[\mathbf{A\ B\ C}] = 0$, then **A, B, C** are coplanar.

(*iii*) *Vector triple product* $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\,.\,\mathbf{C})\,\mathbf{B} - (\mathbf{A}\,.\,\mathbf{B})\,\mathbf{C}$

$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C}\,.\,\mathbf{A})\,\mathbf{B} - (\mathbf{C}\,.\,\mathbf{B})\,\mathbf{A}$

4. (*i*) *grad* $f = \nabla f = \dfrac{\partial f}{\partial x}\mathbf{I} + \dfrac{\partial f}{\partial y}\mathbf{J} + \dfrac{\partial f}{\partial z}\mathbf{K}$

div $\mathbf{F} = \nabla F = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$

curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$ where $\mathbf{F} = f_1\mathbf{I} + f_2\mathbf{J} + f_3\mathbf{K}$

(*ii*) If *div* $\mathbf{F} = 0$, then **F** is called a *solenoidal vector.*

(*iii*) If *curl* $\mathbf{F} = \mathbf{0}$ then **F** is called an *irrotational vector*

5. *Velocity* = $d\mathbf{R}/dt$; *Acceleration* = $d^2\mathbf{R}/dt^2$; *Tangent vector* = $d\mathbf{R}/dt$; *Normal vector* = $\nabla\phi$

6. **Green's theorem:** $\int_C (\phi\, dx + \psi\, dy) = \iint_C \left(\dfrac{\partial\psi}{\partial x} - \dfrac{\partial\phi}{\partial y}\right) dxdy$

Stoke's theorem: $\int_C \mathbf{F} \cdot d\mathbf{R} = \int_S curl\, \mathbf{F} \cdot \mathbf{N}\, ds$

Gauss divergence theorem: $\int_S \mathbf{F} \cdot \mathbf{N}\, ds = \int_v div\, \mathbf{F}\, dv$

7. Coordinate systems

	Polar coordinates (r, θ)	*Cylindrical coordinates* (ρ, ϕ, z)	*Spherical polar coordinates* (r, θ, ϕ)
Coordinate transformations	$x = r\cos\theta$ $y = r\sin\theta$	$x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$
Jacobian	$\dfrac{\partial(x, y)}{\partial(r, \theta)} = r$	$\dfrac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho$	$\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$
(Arc-length)^2	$(ds)^2 = (dr)^2 + r^2(d\theta)^2$ $dxdy = r\, d\theta\, dr$	$(ds)^2 = (d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2$	$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2$
Volume-element		$dV = \rho\, d\rho\, d\phi\, dz$	$dV = r^2\sin\theta\, dr\, d\theta\, d\phi$

IX. DIFFERENTIAL EQUATIONS

1. Equations of first order

(*i*) *Variables separable* : $f(y)\, dy/dx = \phi(x)$, $\int f(y)\, dy = \int \phi(x)\, dx + c$.

(*ii*) *Homogeneous equation* $dy/dx = f(x, y)/\phi(x, y)$ where $f(x, y)$ and $\phi(x, y)$ are of the same degree.
Put $y = vx$ so that $dy/dx = v + x\, dv/dx$.

(*iii*) *Equations reducible to homogenous form* : $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a'x + b'y + c'}$

When $a/a' \neq b/b'$, put $x = X + h$, $y = Y + k$

When $a/a' = b/b'$, put $ax + by = t$.

(*iv*) *Leibnitz's linear equation* : $\dfrac{dy}{dx} + Py = Q$ where P, Q are functions of x.

$I.F. = e^{\int P\, dx}$, then *solution* is $y\,(I.F.) = \int Q\,(I.F.)\, dx + c$.

(*v*) *Bernoulli's equation* : $dy/dx + Py = Qy^n$, reducible to Leibnitz's equation by writing it as

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \text{ and putting } y^{1-n} = z.$$

(*vi*) *Exact equation* : $M(x, y)\, dx + N(x, y)\, dy = 0$

Solution is $\int_{(y\ cons.)} M\, dx + \int$ (terms of N not containing x) $dy = c$, provided $\partial M/\partial y = \partial N/\partial x$.

(*vii*) *Clairaut's equation* : $y = px + f(p)$ where $p = dy/dx$.
Solution is obtained on replacing p by c.

2. Linear equations with constant coefficients

$$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + k_{n-1}\frac{dy}{dx} + k_n y = X$$

Symbolic form : $(D^n + k_1 D^{n-1} + \ldots. + k_{n-1} D + k_n)\, y = X$.

I. To find C.F.

Roots of A.E.	C.F.
(i) $m_1, m_2, m_3, \ldots\ldots$	$c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \ldots.$
(ii) $m_1, m_1, m_3, \ldots\ldots$	$(c_1 + c_2x)\, e^{m_1x} + c_3e^{m_3x} + \ldots.$
(iii) $\alpha + i\beta, \alpha - i\beta, m_3, \ldots\ldots$	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3e^{m_3x} + \ldots.$
(iv) $\alpha \pm i\beta, \alpha \pm i\beta, m_5, \ldots\ldots$	$e^{\alpha x}\, [(c_1 + c_2x) \cos \beta x + (c_3 + c_4x) \sin \beta x] + c_5\, e^{m_5x} + \ldots.$

II. To find P.I.

(i) $X = e^{ax}$, $\quad$ P.I. $= \dfrac{1}{f(D)} e^{ax}$, put $D = a$, $\quad$ $[f(a) \neq 0$

$= x\, \dfrac{1}{f'(D)}\, e^{ax}$, put $D = a$, $\quad$ $[f(a) = 0, f'(a) \neq 0$

$= x^2\, \dfrac{1}{f''(D)}\, e^{ax}$, put $D = a$, $\quad$ $[f'(a) = 0, f''(a) \neq 0$

(ii) $X = \sin(ax + b)$ *or* $\cos(ax + b)$

P.I. $= \dfrac{1}{\phi(D^2)} \sin(ax + b)$ [or $\cos(ax + b)$], put $D^2 = -a^2$, $\quad$ $[\phi(-a)^2 \neq 0]$

$= x\, \dfrac{1}{\phi'(D^2)} \sin(ax + b)$ [or $(\cos ax + b)$], put $D^2 = -a^2$, $\quad$ $[\phi(-a^2) = 0, \phi'(-a^2) \neq 0]$

$= x^2\, \dfrac{1}{\phi''(D^2)} \sin(ax + b)$ [or $\cos(ax + b)$], put $D^2 = -a^2$, $\quad$ $[\phi'(-a^2) = 0, \phi''(-a^2) \neq 0$

(iii) $X = x^m$, P.I. $= \dfrac{1}{f(D)}\, x^m = [f(D)]^{-1} x^m$. Expand $[f(D)]^{-1}$ in ascending powers of D as for as D^m and operate on x^m term by term.

(iv) $X = e^{ax}\, V$, P.I. $= \dfrac{1}{f(D)}\, e^{ax}\, V = e^{ax}\, \dfrac{1}{f(D + a)}\, V.$

III. Complete Solution : C.S. is y = C.F. + P.I.

3. Homogeneous linear equation : $x^3 \dfrac{d^3y}{dx^3} + k_1x^2 \dfrac{d^2y}{dx^2} + k_2x \dfrac{dy}{dx} + k_3y = X$

reduces to linear equation with constant coefficients by putting

$$x = e^t,\ x \frac{dy}{dx} = Dy,\ x^2 \frac{d^2y}{dx^2} = D(D-1)\,y,\ x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)\,y$$

4. Lagrange's linear partial differential equation

$Pp + Qq = R$, P, Q, R being functions of x, y, z.

To solve it (i) form the subsidiary equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

(ii) solve these equations giving $u = a, v = b$.

(iii) Complete solution is $\phi(u, v) = 0$ or $u = f(v)$.

5. Homogeneous linear partial differential equations with constant coefficients

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\, \partial y} + \ldots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y)$$

Symbolic form : $(D^n + k_1D^{n-1}\, D' + \ldots + k_nD'^n)z = F(x, y)$

To find C.F.

Roots of A.E.	C.F.
(*i*) $m_1, m_2, m_3, \ldots\ldots$	$f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x) + \ldots\ldots$
(*ii*) $m_1, m_1, m_3, \ldots\ldots$	$f_1(y + m_1x) + xf_2(y + m_1x) + f_3(y + m_3x) + \ldots\ldots$
(*iii*) $m_1, m_1, m_1, \ldots\ldots$	$f_1(y + m_1x) + xf_2(y + m_1x) + x^2f_3(y + m_1x) + \ldots\ldots$

To find P.I.

(*i*) $F(x, y) = e^{ax+by}$, $P.I. = \dfrac{1}{f(D, D')} e^{ax+by}$, put $D = a$, $D' = b$.

(*ii*) $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

$$P.I. = \frac{1}{f(D^2, DD', D'^2)} \sin \text{ or } \cos(mx + ny), \text{ put } D^2 = -m^2, DD' = -mn, D^2 = -n^2$$

(*iii*) $F(x, y) = x^m y^n$, $P.I. = [f(D, D')]^{-1} x^m y^n$. Expand $[f(D, D')]^{-1}$ and operate on $x^m y^n$.]

(*iv*) $F(x, y)$ is any function of x and y, $P.I. = \dfrac{1}{f(D, D')}, F(x, y)$.

Resolve $1/f(D, D')$ into partial fractions considering $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that $\dfrac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx$.

Complete solution: C.S. is $y = C.F. + P.I.$

X. INFINITE SERIES

1. *Basic test:* If $\underset{n\to\infty}{Lt} u_n \neq 0$ then the series Σu_n diverges.
2. *G.P. Series:* $1 + r + r^2 + r^3 + \ldots\ldots \infty$ converge if $|r| < 1$; diverges if $r \geq 1$ and oscillates if $r \leq -1$.
3. *p-series:* $\dfrac{1}{1^p} + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \ldots\ldots \infty$ converge for $p > 1$; diverges for $p \leq 1$.
4. *Comparison test:* If two positive term series Σu_n and Σv_n be such that $\underset{n\to\infty}{Lt} \dfrac{u_n}{v_n}$ = *finite quantity* ($\neq 0$), then Σu_n and Σv_n converge or diverge together.
5. *Ratio test:* In the positive term series Σu_n, if $\underset{n\to\infty}{Lt} \dfrac{u_n}{u_{n+1}} = k$, then the series converges for $k > 1$, diverges for $k < 1$ and fails for $k = 1$.
6. *Raabes' test:* In the positive term series Σu_n, if $\underset{n\to\infty}{Lt}\, n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = k$, then the series converges for $k > 1$, diverges for $k < 1$ and fails for $k = 1$.
7. *Logarithmic test:* In the positive term series Σu_n, if $\underset{n\to\infty}{Lt}\left(n \log \dfrac{u_n}{u_{n+1}}\right) = k$, then the series converges for $k > 1$, diverges for $k < 1$ and fails for $k = 1$.
8. If u_n/u_{n+1} does not involve n as an exponent or a logarithm, then the series Σu_n diverges.
9. *Cauchy's root test:* In a positive term series Σu_n if $\underset{n\to\infty}{Lt} (u_n)^{1/n} = \lambda$, then the series converges for $\lambda < 1$, diverges for $\lambda > 1$ and fails for $\lambda = 1$.
10. *Integral test:* A positive term series $\Sigma f(n)$ converges or diverges according as $\int_1^\infty f(x)\, dx$ is finite or infinite where $f(n)$ is continuous in $1 < x < \infty$ and decreases as n increases.
11. *Leibnitz's test for alternating series:* An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots\ldots \infty$ converges if each term is numerically less than the previous term and $\underset{n\to\infty}{Lt} u_n = 0$.

    if $\underset{n\to\infty}{Lt} \neq 0$, then the given series is oscillatory.

12. *General Ratio test:* In an arbitrary term series Σu_n if $\underset{n \to \infty}{Lt} \frac{|u_{n+1}|}{|u_n|} = |k|$, then Σu_n is absolutely convergent if $|k| < 1$ and divergent if $|k| > 1$ and the test fails if $|k| = 1$.

XI. FOURIER SERIES

1. $f(x) = \frac{1}{2} a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots\ldots$ in $(0\ 2\pi)$.

   where $\alpha_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx\,, a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx,\, b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$

2. $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \cos nx$ in any interval $(0, 2c)$,

   where $a_0 = \frac{1}{c} \int_0^{2c} f(x)\, dx,\, a_n = \frac{1}{c} \int_0^{2c} f(x) \cos \frac{n\pi x}{c}\, dx\,, b_n = \frac{1}{c} \int_0^{c} f(x) \sin \frac{n\pi x}{c}\, dx$

3. *For even function $f(x)$, Fourier expansion contains only cosine terms.*

   i.e., $a_0 = \frac{2}{c} \int_0^{c} f(x)\, dx,\, a_n = \frac{2}{c} \int_0^{c} f(x) \cos \frac{n\pi x}{c}\, dx\,, b_n = 0.$

   For odd function $f(x)$, Fourier expansion contains only sine terms.

   i.e., $a_0 = 0, a_n = 0, b_n = \frac{2}{c} \int_0^{c} f(x) \sin \frac{n\pi x}{c}\, dx\,.$

XII. TRANSFORMS

1. **Laplace Transforms.** $L\,(f\,(t)) = \int_0^{\infty} e^{-st}\, f(t)\, dt$

   (*i*) $L\,(1) = \frac{1}{s}$

   (*ii*) $L(t^n) = \frac{n\,!}{s^{n+1}}$

   (*iii*) $L(e^{at}) = \frac{1}{s - a}$

   (*iv*) $L\,(\sin at) = \frac{a}{s^2 + a^2}$

   (*v*) $L\,(\cos at) = \frac{s}{s^2 + a^2}$

   (*vi*) $L(\sinh at) = \frac{a}{s^2 - a^2}$

   (*vii*) $L\,(\cosh at) = \frac{s}{s^2 - a^2}$

   (*viii*) $L\, e^{at} f(t) = F\,(s - a)$

   (*ix*) $L\, f'\,(t) = sL\, f\,(t) - f\,(0)$

   (*x*) $L\,[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\,[F(s)]$

   (*xi*) $L\left[\frac{1}{t} f(t)\right] = \int_s^{\infty} F(s)\, ds$

   (*xii*) $u(t - a) = \begin{cases} 0 \text{ when } t < a \\ 1 \text{ when } t > a \end{cases}$

   (*xiii*) $L[u(t - a)] = \frac{e^{-as}}{s}$

   (*xiv*) $L\delta\,(t - a) = e^{-as}$

   (*xv*) $L\, f\,(t) = \frac{\int_0^T e^{-st}\, f(t)\, dt}{1 - e^{-sT}}$ where $f\,(t)$ is a periodic function of period T.

2. **Inverse Laplace Transforms**

   (*i*) $L^{-1}\left(\frac{1}{s}\right) = 1$

   (*ii*) $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)\,!}$

   (*iii*) $L^{-1}\left(\frac{1}{s - a}\right) = e^{at}$

   (*iv*) $L^{-1}\left(\frac{1}{s^2 + a^2}\right) = \frac{1}{a} \sin at$

   (*v*) $L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at$

   (*vi*) $L^{-1}\left(\frac{s}{s^2 - a^2}\right) = \frac{1}{a} \sinh at$

(*vii*) $L^{-1}\left(\dfrac{s}{s^2-a^2}\right) = \cosh at.$ (*viii*) $L^{-1}\left(\dfrac{s}{(s^2+a^2)^2}\right) = \dfrac{1}{2a}\, t \sin at$

(*ix*) $L^{-1}\left[\dfrac{s^2-a^2}{(s^2+a^2)^2}\right] = t \cos at.$

3. **Fourier Transforms :** $F(s) = \int_{-\infty}^{\infty} f(t)\, e^{ist}\, .\, dt$

Fourier sine transform : $F_s(s) = \int_0^{\infty} f(t) \sin st\, dt$

Fourier cosine transform : $F_c(s) = \int_0^{\infty} f(t) \cos st\, dt$

$$F\left(\frac{\partial^2 u}{\partial x^2}\right) = -s^2 F(u).$$

4. **Z-Transforms :** $Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$

(*i*) $Z(1) = \dfrac{z}{z-1}$ (*ii*) $Z(n) = \dfrac{z}{(z-1)^2}$

(*iii*) $Z(n^2) = \dfrac{z^2+z}{(z-1)^3}$ (*iv*) $Z(a^n) = \dfrac{z}{z-a}$

(*v*) $Z(na^n) = \dfrac{az}{(z-a)^2}$ (*vi*) $Z(\sin n\theta) = \dfrac{z \sin \theta}{z^2 - 2z\cos\theta + 1}$

(*vii*) $Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$ (*viii*) $Z(\sinh n\theta) = \dfrac{z \sinh \theta}{z^2 - 2z\cosh\theta + 1}$

(*ix*) $Z(\cosh n\theta) = \dfrac{z(z-\cosh\theta)}{z^2 - 2z\cosh\theta + 1}$

XIII. STATISTICS AND PROBABILITY

1. *A.M.* $\bar{x} = \dfrac{\Sigma f_i x_i}{\Sigma f_i}$ 2. *S.D.* $\sigma = \sqrt{\dfrac{\Sigma f_i (x_i - \bar{x})^2}{\Sigma f_i}}$

3. *Moments about the mean :* $\mu_0 = 1, \mu_1 = 0, \mu_2 = \sigma^2, \mu_r = \dfrac{\Sigma f_i (x_i - \bar{x})^r}{\Sigma f_i}$

4. *Coeff. of skewness* = (mean – mode)/σ which lies between – 1 and 1.

5. Kurtosis : $\beta_2 = \mu_4/\mu_2^2$.

6. *Coeff. of correlation* $r = \dfrac{n\,\Sigma\, d_x d_y - \Sigma\, d_x \Sigma\, d_y}{\sqrt{[\{n\,\Sigma d_x^2 - (\Sigma d_x)^2\}\{n \Sigma d_y^2 - (\Sigma d_y)^2\}]}}; -1 < r < 1$

7. *Line of regression of y on x :* $y - \bar{y} = r\, \dfrac{\sigma_y}{\sigma_x}(x - \bar{x})$

Line of regression of x on y : $x - \bar{x} = \dfrac{r\,\sigma_x}{\sigma_y}(y - \bar{y})$

8. **Probability** $p(A) = \dfrac{\text{No. of ways favourable to } A}{\text{Total no. of equally likely ways}}, p + q = 1.$

(*i*) $p(A \text{ or } B) = p(A) + p(B)$, (*ii*) $p(A \text{ and } B) = p(A)\, .\, p(B)$

9. **Binomial distribution :** $p(r) = {}^nC_r\, p^r\, q^{n-r}$

Mean = np, *Variance* $(\sigma^2) = npq$

10. Poisson distribution : $p(r) = \frac{m^r}{r!} e^{-m}$

Mean $= m$, *Variance* $(\sigma^2) = m$.

11. Normal distribution : $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, *Standard variate* $= \frac{x-\mu}{\sigma}$

(*i*) *Probable error* $\lambda = 0.6745\ \sigma$.

(*ii*) 68% of values lie between $x = \mu - \sigma$ and $x = \mu + \sigma$.
95% of values lie between $x = \mu - 1.96\ \sigma$ and $x = \mu + 1.96\ \sigma$
99% of values lie between $x = \mu - 2.58\ \sigma$ and $x = \mu + 2.58\ \sigma$

XIV. NUMERICAL TECHNIQUES

1. Solution of equations

(*i*) *Bisection method* : $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$

(*ii*) *Method of False position* : $x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$.

(*iii*) *Newton-Raphson method* : $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

(*iv*) *Iterative formula to find* $1/N$ is $x_{n+1} = x_n (2 - Nx_n)$

(*v*) *Iterative formula to find* $\sqrt{N}$ is $x_{n+1} = \frac{1}{2}(x_n + N/x_n)$

2. Solution of Linear Simultaneous equations

(*i*) *Matrix inversion method.* For the equations

$$a_1x + b_1y + c_1 z = d_1,\ a_2x + b_2y + c_2 z = d_2,\ a_3x + b_3y + c_3z = d_3$$

if
$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

then
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|} \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

where A_1, B_1, etc., are the co-factors of a_1, b_1, etc., in the determinant $|A|$.

(*ii*) *Gauss-elimination method.* In this method the coefficient matrix is transformed to *upper triangular matrix.*

(*iii*) *Gauss-Jordan method.* In this method the coefficient matrix is transformed to *diagonal matrix.*

(*iv*) *Gauss-Jordan method of finding the inverse of a matrix A.* The matrices A and I are written side by side and the same row transformations are performed on both till A is reduced to I. Then the other matrix represents A^{-1}.

3. Finite differences and Interpolation

(*i*) *Forward differences:* $\Delta y_r = y_{r+1} - y_r$
Backward differences: $\nabla y_r = y_r - y_{r-1}$
Central differences: $\delta y_{n-1/2} = y_n - y_{n-1}$

(*ii*) *Relations between operations* :

$$\Delta = E - 1;\ \nabla = 1 - E^{-1};\ \delta = E^{1/2} - E^{-1/2}$$

$$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2});\ \Delta = E\nabla = \nabla E = \delta E^{1/2};\ E = e^{hD}$$

(*iii*) *Factorial notation* $|\,x\,|^r = x(x-1)(x-2) \ldots (x-r+1)$.
Factorial polynomial $[x]^n = x(x-h)(x-2h) \ldots (x-h-1h)$

(iv) *Newton's forward interpolation formula*

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\,\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \ldots \text{ where } p = (x - x_0)/h.$$

(v) *Newton's backward interpolation formula:*

$$y_p = y_n + p\,\nabla y_n + \frac{p(p+1)}{2!}\,\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\,\nabla^3 y_n + \ldots\ldots \text{ where } p = (x - x_n)/h.$$

(vi) *Stirling's formula:*

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\,\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\,\Delta^4 y_{-2} + \ldots\ldots$$

(vii) *Bessel's formula:*

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\,\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p - \frac{1}{2}\right) p\,(p-1)}{3!}\,\Delta^3 y_{-1}$$
$$+ \frac{(p+1)\,p(p-1)(p-2)}{4!}\,\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots$$

(viii) *Lagrange's interpolation formula:*

$$y = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)}\,y_0 + \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)}\,y_1 + \ldots\ldots +$$
$$\frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})}\,y_n$$

(ix) *Newton's divided difference formul*

$$y = f(x) = y_0 + (x-x_0)\,[x_0, x_1] + (x-x_0)(x-x_1)\,[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)\,[x_0, x_1, x_2, x_3] + \ldots$$

where $[x_0, x_1]\ \dfrac{y_1 - y_0}{x_1 - x_0}$, $[x_0, x_1, x_2] = \dfrac{[x_1, x_2] - [x_0, x_1]}{[x_2 - x_0]}$ and so on.

4. Numerical differentiation

(i) *Forward difference formulae:*

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\ldots\right] \text{ and so on.}$$

(ii) *Backward difference formulae:*

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\nabla y_0 + \frac{1}{2}\nabla^2 y_0 + \frac{1}{3}\nabla^3 y_0 + \frac{1}{4}\nabla^4 y_0 + \ldots\ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12}\nabla^4 y_0 + \ldots\ldots\right] \text{ and so on.}$$

(iii) *Central difference formulae:*

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\,\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\,\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \ldots\ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + \ldots\ldots\right]$$

5. Numerical integration

(i) *Trapezoidal rule:*

$$\int_{x_0}^{x_0 + nh} f(x)\,dx = \frac{h}{2}\,[(y_0 + y_n) + 2\,(y_1 + y_2 + \ldots\ldots + y_{n-1})]$$

(ii) *Simpon's 1/3 th rule:*

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots\ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots + y_{n-2})]$$

(Number of sub intervals should be taken as even)

(iii) *Simpson's 3/8 th rule:*

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{n-1}) + 2(y_3 + y_5 + \ldots\ldots + y_{n-2})]$$

(Number of sub-intervals should be taken as a multiple of 3)

(iv) *Weddle's rule:*

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + \ldots\ldots]$$

(Number of sub-intervals should be taken as multiple of (6)

6. Numerical solution of ordinary differential equations

(i) *Picard's method:* $y_1 = y_0 + \int_{x_0}^{x} (x, y_0)\,dx$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx \text{ etc.}$$

(ii) *Taylor's method:*

$$y = y_0 + (x - x_0)(y)_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \ldots$$

(iii) *Euler's method:* $y_2 = y_1 + h\,f(x_0 + h, y_1)$

Repeat this process till y_2 is stationary. Then calculate y_3 and so on.

(iv) *Modified Euler's method:* $y_2 = y_1 + \frac{1}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]$

Repeat this step till y_2 is stationary. Then calculate y_3 and so on.

(v) *Runge Kutta method:* $y_1 = y_0 + h$ where $h = \frac{1}{6}(k_1 + 2k_2, 2k_3 + k_4)$

such that $k_1 = h\,f(x_0, y_0)$; $k_2 = h\,f(x_0 + h/2, y_0 + k_1/2)$
$k_3 = h\,f(x_0 + h/2, y_0 + k_2/2)$; $k_4 = hf(x_0 + h, y_0 + k_3)$

(vi) *Milne's method*

Predictor formula: $y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

Corrector formula: $y_4 = y_2 + \frac{h}{3}(f_2 - 4f_3 + f_4)$

(Four prior values are required to find the next values)

(vii) *Adams-Bashforth method:*

Predictor formula: $y_1 = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$

Corrector formula: $y_1 = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} - f_{-2})$

(Four prior values are required to find the next values)

7. Numerical solution of partial differential equations

(i) *Classification of a second order equations:*

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + \left(F(x, y, u,)\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) = 0$$

is said to be *elliptic* if $B^2 - 4AC < 0$
parabolic if $B^2 - 4AC = 0$
hyperbolic if $B^2 - 4AC > 0$

(*ii*) *Laplace equation:* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Standard 5-point formula: $u_{i,j} = \dfrac{1}{4}\,[u_{i-1,j} + u_{i+1,j}\, u_{i,j+1} + u_{i,j-1}]$

Diagonal 5-point formula: $u_{i,j} = \dfrac{1}{4}\,[u_{i-1,j+1} + u_{i+1,j-1}\, u_{i+1,j+1} + u_{i-1,j-1}]$

(*iii*) *Poisson's equation:* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y).$

Standard 5-point formula:

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$

(*iv*) *One-dimensional heat equation:* $\dfrac{\partial u}{\partial t} = c^2\,\dfrac{\partial^2 u}{\partial x^2}$

Schmidt formula: $u_{i,j+1} = \alpha\, u_{i-1,j} + (1 - 2d)\, u_{i,j} + \alpha\, u_{i+1,j}$ where $\alpha = kc^2/h^2$
when $\alpha = 1/2$, it reduces to

Bendre-Schmidt relation: $u_{i,j+1} = \dfrac{1}{2}(u_{i-1,j} + u_{i+1,j})$

(*v*) *Wave equation:* $\dfrac{\partial^2 u}{\partial t^2} = c^2\,\dfrac{\partial^2 u}{\partial x^2}$

Explict formula for solution is

$$u_{i,j+1} = 2(1 - \alpha^2 c^2)\, u_{i,j} + \alpha^2 c^2\, (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \text{ where } \alpha = k/h$$

If α is so chosen that the coefficient of $u_{i,j}$ is zero *i.e.*, $k = h/c$ then the above explicit formula takes the *simplified form*

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$