CHAPTER 38

Tensor Analysis

38.1 INTRODUCTION

Some physical quantities are specified by their magnitude only while others by their magnitude and direction. But certain quantities are associated with two or more directions. Such a quantity is called a *tensor*. The stress at a point of an elastic solid is an example of a tensor which depends on two directions—one normal to the area and other that of the force on it.

The properties of tensors are independent of the frames of reference used to describe them. That is why *Einstein* found tensors as a convenient tool for formulation of his Relativity theory. Since then, the subject of tensor analysis shot into prominence and is of great use in the study of Riemannian geometry, mechanics, elasticity, electro-magnet theory and numerous other fields of science and engineering. The emergence of tensor calculus as a symmetric subject is due to *Ricci* and his student *Levi-Cita*.

38.2 SUMMATION CONVENTION

Consider a sum of the type

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \quad i.e., \quad \sum_{i=1}^{n} a_ix_i \qquad \dots(1)$$

In tensor analysis, the subscripts of the symbols $x_1, x_2, \dots x_n$ are replaced by superscripts and we write these as $x^1, x^2, \dots, x^n$. The superscripts do not stand for the various powers of x but act as labels to distinguish different symbols. The power of a symbol (say : x^i) will be indicated as $(x^i)^2$, $(x^i)^3$ etc. Hence (1) is written as

$$\sum_{i=1}^{n} a_ix^i \ . \qquad \dots(2)$$

A still simpler notation is to drop the Σ sign and write (2) as a_ix^i ...(3)

In this the repeated index i successively takes up the values 1, 2, ..., n and the expression (3) represents the sum of all such terms. The repeated index i over which the summation is to be done, is called a *dummy* index since it doesn't appear in the final result. This notation, known as *summation convention*, is due to Einstein. We shall adopt this convention throughout this chapter and take the sum whenever a letter appears in a term once as a subscript and once as superscript.

Example 38.1. *Write the terms contained in $S = a_{ij}x^ix^j$ taking $n = 3$.*

Solution. Since the index i occurs both as a subscript and as a superscript, we first sum on i from 1 to 3.

$\therefore$ $$S = a_{1j}x^1x^j + a_{2j}x^2x^j + a_{3j}x^3x^j$$

Now each term in S has to be summed up w.r.t. repeated index j from 1 to 3.

$\therefore$ $$S = a_{11}x^1x^1 + a_{12}x^1x^2 + a_{13}x^1x^3 + a_{21}x^2x^1 + a_{22}x^2x^2 + a_{23}x^2x^3 + a_{31}x^3x^1 + a_{32}x^3x^2 + a_{33}x^3x^3$$
$$= a_{11}(x^1)^2 + a_{22}(x^2)^2 + a_{33}(x^3)^2 + (a_{12} + a_{21})x^1x^2 + (a_{13} + a_{31})x^1x^3 + (a_{23} + a_{32})x^2x^3.$$

Example 38.2. *If f is a function of n variables x^i, write the differential of f.*

Solution. Since $f = f(x^1, x^2, ..., x^n)$

$\therefore$ From Calculus, we have

$$df = \frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial x^2}dx^2 + ... + \frac{\partial f}{\partial x^n}dx^n = \frac{\partial f}{\partial x^i}dx^i.$$

38.3 (1) TRANSFORMATION OF COORDINATES

In a 3-dimensional space, the coordinates of a point are (x^1, x^2, x^3) referred to a particular frame of reference. Similarly in an n-dimensional space, the coordinates of a point are n independent variables $(x^1, x^2, ..., x^n)$ with respect to a certain frame of reference. Let $(\bar{x}^1, \bar{x}^2, ..., \bar{x}^n)$ be the coordinates of the same point referred to another frame of reference. Suppose, $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$ are independent single-valued functions of $x^1, x^2, ..., x^n$ so that

$$\bar{x}^1 = \phi^1 (x^1, x^2, ..., x^n)$$
$$\bar{x}^2 = \phi^2 (x^1, x^2, ..., x^n)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\bar{x}^n = \phi^n (x^1, x^2, ..., x^n)$$

or more briefly $$\bar{x}^i = \phi^i (x^1, x^2, ..., x^n) \quad ...(1)$$

We can solve the equations (1) and express x^i as functions of $\bar{x}^i$ so that

$$x^i = \psi^i (\bar{x}^1, \bar{x}^2, ..., \bar{x}^n) \quad ...(2)$$

The equations (1) and (2) are said to define a transformation of the coordinates from one frame of reference to another.

(2) Scalars or invariants. *A function $\phi(x^1, x^2, x^3)$ is called a scalar or an invariant if its original value does not change upon transformation of coordinates from x^1, x^2, x^3 to $\bar{x}^1, \bar{x}^2, \bar{x}^3$.*

i.e., $$\phi(x^1, x^2, x^3) = \psi(\bar{x}^1, \bar{x}^2, \bar{x}^3)$$

A scalar or invariant is also called a *tensor of order (or rank) zero.*

38.4 KRONECKER DELTA*

The quantity δ_i^j defined by the relations

$$\delta_i^j = 0, \quad \text{when } j \neq i$$

and $$\delta_i^j = 1, \quad \text{when } j = i, \text{ is called } \textit{Kronecker delta.}$$

Evidently $$\delta_1^{\ 1} = \delta_2^{\ 2} = \delta_3^{\ 3} = ... = \delta_n^{\ n} = 1$$

while $$\delta_1^{\ 2} = \delta_2^{\ 3} = \delta_3^{\ 2} = ... = 0$$

*Called after the German mathematician *Leopold Kronecker* (1823–91) who made important contributions to number theory, algebra and group theory.

We note that by summing up w.r.t. the repeated index j,

$$\delta_{3j}\delta_2^j = a_{31}\delta_2^1 + a_{32}\delta_2^2 + a_{33}\delta_2^3 + a_{34}\delta_2^4 + \ldots$$
$$= 0 + a_{32} + 0 + 0 = a_{32}$$

In general,
$$a_{ij}\delta_k^j = a_{i1}\delta_k^1 + a_{i2}\delta_k^2 + \ldots + a_{ik}\delta_k^k + \ldots + a_{in}\delta_k^n$$
$$= 0 + 0 + \ldots + a_{ik} \cdot 1 + \ldots + 0 = a_{ik}.$$

Example 38.3. *Show that $a_{ij}A^{kj} = \Delta\delta_i^k$, where Δ is a determinant of order three and A^{ij} are cofactors of a^{ij}.* (*Delhi, 2002*)

Solution. By expansion of determinants, we have

$$a_{11}A^{11} + a_{12}A^{12} + a_{13}A^{13} = \Delta$$
$$a_{11}A^{21} + a_{12}A^{22} + a_{13}A^{23} = 0$$
$$a_{11}A^{31} + a_{12}A^{32} + a_{13}A^{33} = 0$$

which can be compactly written as

$$a_{1j}A^{1j} = \Delta,\ a_{1j}A^{2j} = 0,\ a_{1j}A^{3j} = 0$$

Using Kronecker delta notation, these can be combined into a single equation

$$a_{1j}A^{kj} = \Delta\delta_1^k$$

Similarly
$$a_{2j}A^{kj} = \Delta\delta_2^k,\ a_{3j}A^{kj} = \Delta\delta_3^k$$

All these nine equations are included in $a_{ij}A^{kj} = \Delta\delta_i^k$.

Example 38.4. *If x^i and $\bar{x}^i$ are independent coordinates of a point, show that*

$$\frac{\partial x^j}{\partial \bar{x}^k}\frac{\partial \bar{x}^k}{\partial x^i} = \delta_i^j.$$

Solution. The partial derivatives of ϕ in the two coordinate systems are different and are connected by the following formula of Differential Calculus :

$$\frac{\partial \phi}{\partial x^i} = \frac{\partial \phi}{\partial \bar{x}^1}\cdot\frac{\partial \bar{x}^1}{\partial x^i} + \frac{\partial \phi}{\partial \bar{x}^2}\cdot\frac{\partial \bar{x}^2}{\partial x^i} + \frac{\partial \phi}{\partial \bar{x}^3}\frac{\partial \bar{x}^3}{\partial x^i} = \frac{\partial \phi}{\partial \bar{x}^k}\frac{\partial \bar{x}^k}{\partial x^i}$$

In particular, when $\phi = x^j$, we have $\dfrac{\partial x^j}{\partial x^i} = \dfrac{\partial x^j}{\partial \bar{x}^k}\dfrac{\partial \bar{x}^k}{\partial x^i}$ ...(*i*)

Since x^j is independent of x^i, $\partial x^j/\partial x^i = 0$, when $j \neq i$
$= 1$, when $j = i$ ...(*ii*)

Hence the result follows from (*i*) and (*ii*).

38.5 (1) CONTRAVARIANT VECTORS

Let $A^1, A^2, \ldots, A^n$ (*i.e.*, A^i) *be a set of n functions of the coordinate system* $x^1, x^2, \ldots, x^n$ (*i.e.*, x^i). *If these transform in another system of coordinates* $\bar{x}^1, \bar{x}^2, \ldots\ldots \bar{x}^n$ (*i.e.*, $\bar{x}^i$) *according to the law*

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j}A^j \qquad \ldots(1)$$

then A^i are called components of a **contravariant vector** *or* **contravariant tensor of order one.**

An example of a contravariant vector. Let us transform the coordinates of a point x^i to $\bar{x}^i$ in a n-dimensional space.

Since x is a function of x^i (*i.e.*, $x_1, x_2, \ldots, x_n$), therefore,

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^1}dx^1 + \frac{\partial \bar{x}^i}{\partial x^2}dx^2 + \ldots + \frac{\partial \bar{x}^i}{\partial x^n}dx^n$$

$$= \frac{\partial \bar{x}^i}{\partial x^j}dx^j, \text{ using the summation convention.}$$

Comparing this with (1), it follows that *the set of differentials* $dx^1, dx^2, \ldots, dx^n$ *is an example of a contravariant vector.* That is why the coordinates of a point are numbered by superscripts and not by subscripts.

(2) Covariant vectors. *Let* A_i *be a set of n functions of the coordinate system* x^i. *If these transform in another system of coordinates* $\bar{x}^i$ *according to the law*

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j \qquad \ldots(2)$$

then A_i *are called the components of a* **covariant vector** *or* **covariant tensor of order one.**

An example of a covariant vector. Let ϕ be a function which has a fixed value at each point of space independent of the coordinate system employed. Therefore, ϕ is a function of the coordinates x^i in the first system and a function of the coordinates $\bar{x}^i$ in the second system. By the chain rule

$$\frac{\partial \phi}{\partial \bar{x}^i} = \frac{\partial \phi}{\partial x^1}\frac{\partial x^1}{\partial x^i} + \frac{\partial \phi}{\partial x^2}\frac{\partial x^2}{\partial x^i} + \ldots + \frac{\partial \phi}{\partial x^n}\frac{\partial x^n}{\partial x^i} = \frac{\partial \phi}{\partial x^1}\frac{\partial x^1}{\partial \bar{x}^i} + \frac{\partial \phi}{\partial x^2}\frac{\partial x^2}{\partial \bar{x}^i} + \ldots + \frac{\partial \phi}{\partial x^n}\frac{\partial x^n}{\partial \bar{x}^i}$$

Comparing this equation with (2), it follows that *the set of derivatives,*

$$\partial\phi/\partial x^1, \partial\phi/\partial x^2, \ldots, \partial\phi/\partial x^n$$

form a covariant vector.

Example 38.5. *A covariant tensor has components* xy, $2y - z^2$, xz *in rectangular coordinates. Find its covariant components in spherical coordinates.*

Solution. Here $\quad x^1 = x, x^2 = y, x^3 = z$

and $\quad \bar{x}^1 = r, \bar{x}^2 = \theta, \bar{x}^3 = \phi \qquad \ldots(i)$

Let $\quad A_1 = xy, A_2 = 2y - z^2, A_3 = xz \qquad \ldots(ii)$

According to the law of transformation, we have $\bar{A}_i = \dfrac{\partial x^j}{\partial \bar{x}^i} A_j \quad (i = 1, 2, 3)$

and we wish to evaluate $\bar{A}_1, \bar{A}_2, \bar{A}_3$ where A_1, A_2, A_3 are known.

We know that $x = r \sin\theta \cos\phi, y = r \sin\theta \sin\phi, z = r = \cos\theta \qquad \ldots(iii)$

Now
$$\bar{A}_1 = \frac{\partial x^1}{\partial \bar{x}^1}A_1 + \frac{\partial x^2}{\partial \bar{x}^1}A_2 + \frac{\partial x^3}{\partial \bar{x}^1}A_3 = \frac{\partial x}{\partial r}xy + \frac{\partial y}{\partial r}(2y - z^2) + \frac{\partial z}{\partial r}xz \qquad \text{[From (i) and (ii)]}$$
$$= \sin\theta\cos\phi \,.\, r\sin\theta\cos\phi \,.\, r\sin\theta\sin\phi + \sin\theta\sin\phi\,(2r\sin\theta\sin\phi - r^2\cos^2\theta)$$
$$+ \cos\theta \,.\, r\sin\theta\cos\phi \,.\, r\cos\theta \qquad \text{[From (iii)]}$$

Similarly
$$\bar{A}_2 = \frac{\partial x^1}{\partial \bar{x}^2}A_1 + \frac{\partial x^2}{\partial \bar{x}^2}A_2 + \frac{\partial x^3}{\partial \bar{x}^2}A_3 = r\cos\theta\cos\phi \,.\, r\sin\theta\cos\phi \,.\, r\sin\theta\sin\phi$$
$$+ r\cos\theta\sin\phi\,(2r\sin\theta\sin\phi - r^2\cos^2\theta) + (-r\sin\theta)\, r\sin\theta\cos\phi \,.\, r\cos\theta$$

and
$$\bar{A}_3 = \frac{\partial x^1}{\partial \bar{x}^3}A_1 + \frac{\partial x^2}{\partial \bar{x}^3}A_2 + \frac{\partial x^3}{\partial \bar{x}^3}A_3$$
$$= -r\sin\theta\sin\phi \,.\, r\sin\theta\cos\theta \,.\, r\sin\theta\sin\phi$$
$$+ r\sin\theta\cos\phi\,(2r\sin\theta\sin\phi - r^2\cos^2\theta) + 0$$

38.6 TENSORS OF HIGHER ORDER

Let i and j be each given values 1 to n, then the symbol A^{ij} will give rise to n^2 functions.

(1) *If* A^{ij} *be a set of* n^2 *functions of the coordinates* $x^1, x^2, \ldots, x^n$ *which transforms to* $\bar{A}^{ij}$ *in another system of coordinates* $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$, *according to the law*

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^i}{\partial x^l}A^{kl} \qquad \ldots(1)$$

then the functions $\bar{A}^{ij}$ *are said to be components of a* **contravariant tensor of the second order.**

(2) *If A_{ij} be a set of n^2 functions of $x^1, x^2, ..., x^n$ which transform to $\bar{A}_{ij}$ in another system of coordinates, $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$ according to the law*

$$\bar{A}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A_{kl}, \qquad ...(2)$$

then the functions $\bar{A}_{ij}$ are said to be the components of a **covariant tensor of the second order.**

(3) *If $A^i_{\ j}$ be a set of n^2 functions of $x^1, x^2, ..., x^n$ which transform to $\bar{A}^i_j$ in another system of coordinates, $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$, according to the law*

$$\bar{A}^i_j = \frac{\partial \bar{x}^i}{\partial x^k} \cdot \frac{\partial x^l}{\partial \bar{x}^j} A^k_{\ l}, \qquad ...(3)$$

then $\bar{A}^i_j$ are said to be the components of a **mixed tensor of the second order.** It transforms like a contravariant vector with respect to the index i and like a covariant vector with regard to the index j. That is why i is placed as a superscript and j as subscript.

We can similarly define tensors of the orders higher than two.

Obs. Each of the above laws of transformation (1) to (3), give rise to n^2 equations as i and j are each given the value 1 to n.

Example 38.6. *Show that the Kronecker delta is a mixed tensor of order two.*

Solution. If δ^j_i transforms to $\bar{\delta}^j_i$ in the coordinate system $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$ by the law for mixed tensors of order two, then

$$\bar{\delta}^j_i = \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^m} \delta^m_l = \frac{\partial \bar{x}^j}{\partial x^m} \cdot \frac{\partial x^m}{\partial \bar{x}^i} = \frac{\partial \bar{x}^j}{\partial \bar{x}^i} = \delta_i^{\ j} \qquad [\because \ \delta^m_l = 0 \text{ for } l \neq m]$$

Hence δ^j_i is a mixed tensor of order two, having the same components in every coordinate system.

Example 38.7. *Show that the velocity of a fluid at any point is a contravariant tensor of rank one.*

Solution. Let $dx^1/dt, dx^2/dt, dx^3/dt$ be the components of fluid velocity of the point (x^1, x^2, x^3), *i.e.*, dx^i/dt be the components of velocity in the coordinate system x^i. Suppose the corresponding components of velocity in the coordinate system $\bar{x}^j$ are $d\bar{x}^j/dt$. Then $\bar{x}^1, \bar{x}^2, \bar{x}^3$ being the functions of x^1, x^2, x^3 which in turn are functions of t, we can write

$$\frac{d\bar{x}^j}{dt} = \frac{\partial \bar{x}^j}{\partial x^1}\frac{dx^1}{dt} + \frac{\partial \bar{x}^j}{\partial x^2}\frac{dx^2}{dt} + \frac{\partial \bar{x}^j}{\partial x^3}\frac{dx^3}{dt}$$

or
$$\frac{d\bar{x}^j}{dt} = \frac{\partial \bar{x}^j}{\partial x^i}\frac{dx^i}{dt} \qquad ...(i)$$

Now according to the law of tensor transformation, (i) shows that the velocity of a fluid is a contravariant tensor of rank one.

Example 38.8. *Prove that there is no distribution between contravariant and covariant vectors if the transformation law is of the form $\bar{x}^i = a^i_m x^m + b^i$, where a's and b's are constants such that $a^i_{\ r} a^i_{\ m} = \delta^r_m$.*

(Bhopal, 2003)

Solution. Given transformation $\quad \bar{x}^i = a^i_m x^m + b^i \qquad ...(i)$

yields
$$\frac{\partial \bar{x}^i}{\partial x^m} = a^i_{\ m} \qquad ...(ii)$$

Also from (i),
$$a^i_{\ r}\bar{x}^i = a^i_{\ r} a^i_m x^m + a^i_{\ r} b^i$$
$$= \delta^r_m x^m + a^i_r b^i = x^r + a^i_{\ r} b^i.$$

$$\therefore \qquad \frac{\partial x^r}{\partial \bar{x}^i} = a^i_r \quad i.e., \quad \frac{\partial x^m}{\partial \bar{x}^i} = a^i_m \qquad ...(iii)$$

From (*ii*) and (*iii*), it is clear that any vector with components a, b, c will on transformation give the same components whether transformed as a contravariant vector or as a covariant vector. Thus in this case, there is no distinction between the two.

38.7 SYMMETRIC AND SKEW-SYMMETRIC TENSORS

(**1**) *A tensor is said to be* **symmetric** *with respect to two contravariant (or two covariant) indices if its components remain unchanged on an interchange of the two indices.*

Thus the tensor A^{ij} is symmetric if $A^{ij} = A^{ji}$, for every i and j.

(**2**) *A tensor is said to be* **skew-symmetric** *with respect to two contravariant (or covariant) indices, if its components change sign on interchange of the two indices.*

Thus the tensor A^{ij} is skew-symmetric if $A^{ji} = -A^{ij}$ for every i and j.

In general, the tensor A^{ijk}_{lm} is said to be symmetric or skew symmetric in i and j according as

$$A^{ijk}_{lm} = A^{jik}_{lm} \quad \text{or} \quad -A^{jik}_{lm}$$

Example 38.9. *Show that (i) a symmetric tensor of the second order has only $\frac{1}{2}n(n+1)$ different components.*

(ii) A skew-symmetric tensor of the second order has only $\frac{1}{2}n(n-1)$ different non-zero components.

Solution. (*i*) Let A^{ij} be a symmetric tensor of order two so that $A^{ij} = A^{ji}$.

If each of the indices i and j take the values 1 to n, then A^{ij} will have n^2 components. Out of these n^2 components, n components $A_{11}, A_{22}, \ldots A_{nn}$ are independent.

Thus the remaining components are $(n^2 - n)$ which can be taken in pairs ($\because$ $A_{12} = A_{21}, A_{31} = A_{13}$ etc.)

Hence the total number of independent components

$$= n + \frac{1}{2}(n^2 - n) = \frac{1}{2}n(n+1)$$

(*ii*) Let A^{ij} be a skew-symmetric tensor of order two so that $A^{ij} = -A^{ji}$. As above, A^{ij} will have n^2 components. Out of these, n components $A^{11}, A^{22}, \ldots A^{nn}$ are all zero. [$\because$ $A^{11} = -A^{11}$].

Omitting these, there are $(n^2 - n)$ components. Since $A^{12} = -A^{21}, A^{13} = -A^{31}$ etc., therefore ignoring the sign, $(n^2 - n)$ components can be taken in pairs.

Hence the total number of independent non-zero components

$$= \frac{1}{2}(n^2 - n) = \frac{1}{2}n(n-1).$$

PROBLEMS 38.1

1. Write the following using the summation convention :

   (*i*) $\dfrac{d\phi}{dt} = \dfrac{\partial\phi}{\partial x^1}\dfrac{dx^1}{dt} + \dfrac{\partial\phi}{\partial x^2}\dfrac{dx^2}{dt} + \ldots + \dfrac{\partial\phi}{\partial x^n}\dfrac{dx^n}{dt}$ (*ii*) $(x^1)^2 + (x^2)^2 + (x^3)^2 + \ldots + (x^n)^2$.

2. Write out in full the following :

   (*i*) $a_{ij}x^ix^j$ $(i, j = 1, 2, 3)$ (*ii*) $g_{ij}dx^idx^j (i, j = 1, 2, 3)$

   (*iii*) $g_{lm}g^{mp}$.

3. (*a*) Shows that δ^i_j is an invariant. (*Bhopal, 2003*)

   (*b*) Evaluate (*i*) $\delta^i_j \, \delta^j_k$ (*ii*) $\delta^p_q \, \delta^q_r \, \delta^r_s$.

4. Show that (*i*) $\dfrac{\partial x^p}{\partial x^q} = \delta^p_q$ (*ii*) $\dfrac{\partial x^p}{\partial \bar{x}^q}\dfrac{\partial \bar{x}^p}{\partial x^r} = \delta^p_r$.

5. If the $\bar{x}$'s are n independent functions of x's and i, j, k, l each take values from 1 to n, show that

$$\frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial x^j} \cdot \delta^k_l = \delta^i_j.$$

6. Write down the law of transformation for the tensors

(i) A_l^{jk} (ii) C_{mn}.

7. A quantity $A(i, j, k, l, m)$ which is a function of the coordinates x^p transforms to another coordinate system $\bar{x}^p$ according to the law :

$$\bar{A}(r, s, t, u, v) = \frac{\partial \bar{x}^r}{\partial x^i} \frac{\partial \bar{x}^s}{\partial x^j} \frac{\partial x^k}{\partial \bar{x}^t} \frac{\partial x^l}{\partial \bar{x}^u} \frac{\partial \bar{x}^v}{\partial x^m} A(i, j, k, l, m)$$

Is this quantity a tensor ? If so express it suitably and state its nature and rank ?

8. If the components of two tensors are equal in one coordinate system, show that they are equal in all coordinate systems.

9. A covariant tensor has components $2x - z$, $x^2 y$, yz in cartesian coordinate system. Find its components in (a) cylindrical coordinates (*Punjab, M.E., 1989*) (b) spherical coordinates.

10. If $g_{(ij)}$ denotes the components of a covariant tensor of rank two, show that the product $g_{ij}\, dx^i dx^j$ is an invariant scalar. (*Delhi, 2002*)

11. A contravariant tensor has components a, b, c in rectangular coordinates ; find the components in spherical coordinates.

12. Prove that $A_{ij} B^i C^j$ is an invariant, if B^i and C^j are contravariant vectors and A_{ij} is a covariant tensor. (*Madras, M.E., 2000*)

13. Show that $\partial A_p / \partial x^q$ is not a tensor even though A_p is a covariant tensor of rank one. (*Bhopal, 2003*)

14. If a tensor A^{pqr}_{st} is a skew-symmetric with respect to the indices p and q in one coordinate system, show that it remains skew-symmetric with respect to p and q in any coordinate system.

38.8 ADDITION OF TENSORS

The sum (or difference) of two tensors of the same order and type is another tensor of the same order and type.

Let A_{ij} and B_{ij} be two tensors of the same order and same type. Their components in the coordinates system $\bar{x}^1, \bar{x}^2, ..., \bar{x}^n$ are A_{ij} and B_{ij}, such that

$$\bar{A}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A_{kl} \qquad \text{and} \qquad \bar{B}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} B_{kl}$$

$$\therefore \qquad \bar{A}_{ij} \pm \bar{B}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} (A_{kl} \pm B_{kl}) \quad i.e., \quad \bar{C}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} C_{kl}$$

Thus C_{ij} transforms in exactly the same manner as A_{ij} and B_{ij} and is, therefore, a tensor of the same order and same type.

38.9 OUTER PRODUCT OF TWO TENSORS

If A^{ij} is a contravariant tensor of order two and B_{kl} is a covariant tensor of order two then their product is a mixed tensor $C^{ij}{}_{kl}$ of order four such that

$$C^{ij}_{kl} = \bar{A}^{ij} \bar{B}_{kl} = \left(\frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} A^{pq} \right) \left(\frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^l} B_{rs} \right) = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^l} A^{pq} B_{rs} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^l} C^{pq}_{rs}$$

But this is the law of transformation of a mixed tensor of order four. Therefore, $C^{ij}{}_{kl}$ is a mixed tensor of order four. Such products are called *outer products of two tensors.*

38.10 CONTRACTION OF A TENSOR

Consider a mixed tensor A_l^{ijk} of order four. By the law of transformation, we have

$$\bar{A}_l^{ijk} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^l} A_s^{pqr}$$

In this, put the covariant index l = a contravariant index i, so that

$$\bar{A}_i^{ijk} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^i} A_s^{pqr} = \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial x^s}{\partial x^p} A_s^{pqr}$$

$$= \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} \delta_p^s A_s^{pqr} = \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} A_p^{pqr}$$

This shows that A_i^{ijk} is a contravariant tensor of order two.

The process of getting a tensor of lower order (reduced by 2) by putting a covariant index equal to a contravariant index and performing the summation indicated is known as **contraction.**

The tensors A_i^{ijk} and A_j^{ijk} obtained from contraction of the same tensor A_l^{ijk} are generally different from each other unless the tensor A_l^{ijk} is symmetric with respect to i and j (*i.e.*, $A_j^{ijk} = A_i^{ijk}$).

38.11 INNER PRODUCT OF TWO TENSORS

Given the tensors A_k^{ij} and B_{qr}^p, if we first form their outer product $A_k^{ij} B_{qr}^p$ and contract this by putting $p = k$, then the result is $A^{ij}{}_k B_{qr}^k$ which is also a tensor, called the *inner product of the given tensors.*

Hence the inner product of two tensors is obtained by first taking their outer product and then by contracting it. We can get several inner products for the same two tensors by contracting in different ways.

Example 38.10. *Show that any inner product of the tensors A_r^p and B_t^{qs} is a tensor of rank three.*

Solution. The transformation laws for A_r^p and B_t^{qs} are

$$\bar{A}_r^p = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^k}{\partial \bar{x}^r} A_k^i \quad ...(i) \qquad \text{and} \qquad \bar{B}^{qs}{}_t = \frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial x^m}{\partial \bar{x}^t} B^{jl}{}_m \quad ...(ii)$$

$\therefore$ Inner product of $\bar{A}_q^p$ and $\bar{B}_t^{qs}$ is

$$\bar{A}_q^p \bar{B}_t^{qs} = \left(\frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^k}{\partial \bar{x}^q}\right)\left(\frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial x^m}{\partial \bar{x}^t}\right) A_k^i B_m^{jl}$$

$$= \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial x^m}{\partial \bar{x}^t} \delta_j^k A_k^l B_m^{jl} \qquad \left[\because \frac{\partial x^k}{\partial \bar{x}^q} \frac{\partial \bar{x}^q}{\partial x^j} = \frac{\partial x^k}{\partial x^j} = \delta_j^k\right.$$

$$= \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial x^m}{\partial \bar{x}^t} A_j^i B_m^{jl}$$

Hence the inner product of $\bar{A}_q^p$ and $\bar{B}_t^{qs}$ is a tensor of rank 3.

Similarly putting $p = t$ in the product of (*i*) and (*ii*) and noting that

$$\frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^m}{\partial \bar{x}^p} = \frac{\partial x^m}{\partial x^i} = \delta^m{}_i,$$

$A_r^p B_p^{qs}$ is found to be a tensor of rank 3.

Similarly, $A_r^p B_t^{qr}$ can also be shown to be a tensor of rank 3.

38.12 QUOTIENT LAW

To ascertain that a set of given functions forms the components of a tensor, we have to verify if the functions obey the tensor transformation laws. But this is a very tedious job. A simple test is provided by the quotient law which states that *if the inner product of a set of functions with an arbitrary tensor is a tensor, then these set of functions are the components of a tensor.*

The proof of this law is given below for a particular case.

Example 38.11. *Show that the expression $A(i, j, k)$ is a tensor if its inner product with an arbitrary tensor B_k^{jl} is a tensor.*

Solution. Let $$A(i, j, k)\, B_k^{jl} = C_i^l \qquad \text{...}(i)$$

where C_i^l is a tensor. In the coordinate system $\bar{x}^i$, let (i) transform to

$$\bar{A}(p, q, r)\bar{B}_r^{qs} = \bar{C}_p^s \qquad \text{...}(ii)$$

where $\bar{B}_r^{qs}$ and $\bar{C}_p^s$ are the components of the tensors $B_k^{\ jl}$ and $C_i^{\ l}$. Expressing $B_r^{\ qs}$ in terms of $\bar{B}_k^{\ jl}$ and $\bar{C}_p^{\ s}$ in terms of $\bar{C}_i^{\ l}$, (ii) takes the form

$$\bar{A}(p, q, r)\frac{\partial \bar{x}^q}{\partial x^i}\frac{\partial \bar{x}^s}{\partial \bar{x}^l}\frac{\partial x^k}{\partial \bar{x}^r}B_k^{\ jl} = \frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^l}{\partial \bar{x}^p}C_i^{\ l} \qquad \text{...}(iii)$$

Multiplying (i) by $\dfrac{\partial \bar{x}^s}{\partial x^l}\dfrac{\partial x^i}{\partial \bar{x}^p}$ and subtracting from (iii), we get

$$\left\{\bar{A}(p, q, r)\frac{\partial \bar{x}^q}{\partial x^i}\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^k}{\partial \bar{x}^r} - A(i, j, k)\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^i}{\partial \bar{x}^p}\right\} B_k^{jl} = 0$$

Now B_k^{jl} being an arbitrary tensor, the quantity within the brackets must be identically zero, *i.e.*,

$$\bar{A}(p, q, r)\frac{\partial \bar{x}^q}{\partial x^j}\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^k}{\partial \bar{x}^r} = A(i, j, k)\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial x^j}{\partial \bar{x}^p}$$

or $$\bar{A}(p, q, r) = \frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^q}\frac{\partial \bar{x}^r}{\partial x^k} = A(i, j, k)$$

But this is the law of tensor transformation. Hence $A(i, j, k)$ is a tensor of order three, with i, j as covariant indices and k as contravariant index.

PROBLEMS 38.2

1. Prove that if a tensor equation is true for one coordinate system, it is true for all coordinate systems.
2. Show that every tensor can be expressed as the sum of two tensors, one of which is symmetric and the other skew-symmetric.
3. If $A^{pq}{}_r$ and $B^{pq}{}_r$ are tensors, prove that their sum and differences are also tensors.
4. Show that A_{ij} is a tensor if its inner product with an arbitrary mixed tensor $B^j{}_k$ is a tensor.
5. Prove that (*a*) the contraction of the tensor $A^p{}_q$ is an invariant.
   (*b*) the contraction of the outer product of the tensors A^p and B_q is also an invariant.
6. Let $A^{pq}{}_{rst}$ be a tensor ; choose $p = t$ and $q = s$ and show that $A^{pq}{}_{rqp}$ is also a tensor. What is its rank ?

38.13 (1) RIEMANNIAN SPACE

The distance ds between two adjacent points whose rectangular Cartesian coordinates are (x, y, z) and $(x + dx, y + dy, z + dz)$ is given by $ds^2 = dx^2 + dy^2 + dz^2$.

Riemann extended the concept of distance to a space of n dimensions and defined the distance ds between two adjacent points x^i and $x^i + dx^i$ $(i = 1, 2, ..., n)$ by the relation

$$ds^2 = a_{11}(dx^1)^2 + a_{22}(dx^2)^2 + ... + a_{nn}(dx^n)^2 + a_{12}dx^1dx^2 + ... + a_{lm}dx^ldx^m + ...$$
$$= a_{ij}dx^idx^j, \text{ using summation convention.} \qquad \text{...}(1)$$

The coefficients a_{ij} are the functions of the coordinates x^i. The quadratic form (1) is called a Riemannian metric and any space in which the distance is given by a such a metric is called a *Riemannian space.**

If in a particular coordinate system X^i, the quadratic form (1) reduces to the form

$$ds^2 = (dX^1)^2 + (dX^2)^2 + ... (dX^n)^2,$$

then it is called a Euclidean metric and the corresponding space is called the *Euclidean space.*

* See footnote on p. 673

Obs. The geometry based on the Riemannian metric is called the *Riemannian geometry* and that based on the Euclidean metric is called the *Euclidean geometry*.

(2) Metric tensor. As in the physical space, the distance ds in the n-dimensional space is assumed to be independent of the coordinate system, *i.e.* a scalar invariant or a tensor of order zero. In the relation (1), dx^i and dx^j being displacements are components of a contravariant vector or a tensor of order one. Therefore, their outer product $dx^i dx^j$ is a contravariant tensor of order two. By the quotient law, the functions a_{ij} must be components of a covariant tensor of order two.

Let us write $\quad a_{ij} = g_{ij} + h_{ij}$ where $g_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ and $h_{ij} = \frac{1}{2}(a_{ij} - a_{ji})$.

Interchanging i and j, we have $g_{ji} = \frac{1}{2}(a_{ji} + a_{ij}) = g_{ij}$ and $h_{ji} = \frac{1}{2}(a_{ji} - a_{ij}) = -h_{ij}$

$\therefore$ g_{ij} is symmetric and h_{ij} is skew-symmetric. Thus (1) take the form

$$ds^2 = a_{ij}dx^i dx^j = (g_{ij} + h_{ij})\, dx^i dx^j$$

Now $h_{ij}dx^i dx^j$ is zero, since h_{ij} is skew-symmetric. Hence $ds^2 = g_{ij}dx^i dx^j$ where g_{ij} is a covariant symmetric tensor of order two. It is called the *metric tensor* or the *first fundamental tensor*.

38.14 CONJUGATE TENSOR

Let g be the determinant $|g_{ij}|$ and G_{ij} denote the cofactors of g_{ij} in g. Define the function of g^{ij} by the relation $g^{ij} = G_{ij}/g$ ...(1)

Since the functions g_{ij} and G_{ij} are symmetric in the subscripts, the functions g^{ij} will be symmetric in the superscripts. Now

$$g_{ij}g^{ij} = g_{ij}\frac{G_{ij}}{g} = \frac{g}{g} = 1 \quad \text{and} \quad g_{ij}g^{lj} = g_{ij}\frac{G_{lj}}{g}$$

$$= 1, \text{ if } l = i, \; = 0, \text{ if } l \neq i$$

Thus
$$g_{ij}\, g^{lj} = \delta_i^l \qquad ...(2)$$

If u^j be an arbitrary contravariant tensor, then its inner product with the tensor g_{ij} will be an arbitrary covariant tensor due to contraction, *i.e.*,

$$g_{ij}u^j = v_i \qquad ...(3)$$

$\therefore$
$$g^{lj}v_l = g^{lj}g_{lj}u^j = u^j,$$

which is a contravariant tensor of order one. Therefore by quotient law, g^{lj} are the components of a contravariant tensor of order two. Hence g^{ij} is a symmetric contravariant tensor which is called the *conjugate tensor* or the *second fundamental tensor*.

Obs. In view of (2), the relation between g_{ij} and g^{ij} is reciprocal. As such the *first and second fundamental tensors are also called* **reciprocal tensors.**

Example 38.12. *Find the components of the first and second fundamental tensors in spherical coordinates.*

Solution. Let (x^1, x^2, x^3) be the rectangular cartesian coordinates and $\bar{x}^1, \bar{x}^2, \bar{x}^3$ be the spherical coordinates of a point so that

$$x^1 = x, x^2 = y, x^3 = z, \text{ and } \bar{x}^1 = r, \bar{x}^2 = \theta, \bar{x}^3 = \phi \qquad ...(i)$$

and

We know that
$$x = r\sin\theta\cos\phi,\; y = r\sin\theta\sin\phi,\; z = r\cos\theta \qquad ...(ii)$$

Let g_{pq} and $\bar{g}_{ij}$ be the metric tensors in cartesian and spherical coordinates respectively.

Then
$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = g_{pq}dx^p dx^q$$

$\therefore$
$$g_{11} = 1 = g_{22} = g_{33}, \text{ and } g_{12} = 0 = g_{13} = g_{23} \text{ etc.} \qquad ...(iii)$$

On transformation

$$\bar{g}_{ij} = \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}g_{pq} = \frac{\partial x^1}{\partial \bar{x}^i}\frac{\partial x^1}{\partial \bar{x}^j}g_{11} + \frac{\partial x^2}{\partial \bar{x}^i}\frac{\partial x^2}{\partial \bar{x}^j}g_{22} + \frac{\partial x^3}{\partial \bar{x}^i}\frac{\partial x^3}{\partial \bar{x}^j}g_{33} \qquad ...(iv)$$

Putting $i = j = 1$ in (*iv*), we have

$$\bar{g}_{11} = \left(\frac{\partial x^1}{\partial \bar{x}^1}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^1}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^1}\right)^2 g_{33}$$

$$= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 \qquad \text{[By (i) and (iii)]}$$

$$= (\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2 = \sin^2\theta + \cos^2\theta = 1 \qquad \text{[By (ii)]}$$

Putting $i = j = 2$ in (*iv*), we have

$$\bar{g}_{22} = \left(\frac{\partial x^1}{\partial \bar{x}^2}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^2}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^2}\right)^2 g_{33}$$

$$= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \qquad \text{[By (i) and (iii)]}$$

$$= (r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

Similarly
$$\bar{g}_{33} = \left(\frac{\partial x^1}{\partial \bar{x}^3}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^3}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^3}\right)^2 g_{33} = r^2\sin^2\theta$$

and
$$\bar{g}_{12} = 0 = \bar{g}_{13} = \bar{g}_{21} = \bar{g}_{23} = \bar{g}_{31} = \bar{g}_{32}$$

Hence the first fundamental tensor, written in matrix form, is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

$\therefore$
$$g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = r^4\sin^2\theta$$

and the cofactors in g are given by

$$G_{11} = r^4\sin^2\theta,\ G_{22} = r^2\sin^2\theta,\ G_{33} = r^2\ ;\ G_{12} = 0 = G_{13} = G_{21} = G_{23} = G_{31} = G_{32}$$

The components of the second fundamental tensor are given by $g^{ij} = G_{ij}/g$. Hence the second fundamental tensor in matrix form, is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2\sin^2\theta \end{bmatrix}$.

Example 38.13. *Find the components of the metric tensor and the conjugate tensor in cylindrical coordinates.*

Solution. Let (x^1, x^2, x^3) be the cartesian coordinates and $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ be the cylindrical coordinates of a point so that

$$x^1 = x,\ x^2 = y,\ x^3 = z \quad \text{and} \quad \bar{x}^1 = \rho,\ \bar{x}^2 = \phi,\ \bar{x}^3 = z \qquad ...(i)$$

We know that
$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z \qquad ...(ii)$$

Let g_{pq} and $\bar{g}_{ij}$ be the metric tensors in cartesian and cylindrical coordinates respectively.

Then
$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = g_{pq}dx^p$$

$\therefore$
$$g_{11} = 1 = g_{22} = g_{33} \quad \text{and} \quad g_{12} = 0 = g_{13} = g_{23} \text{ etc.} \qquad ...(iii)$$

On transformation,

$$\bar{g}_{ij} = \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j} g_{pq} = \frac{\partial x^1}{\partial \bar{x}^i}\frac{\partial x^1}{\partial \bar{x}^j} g_{11} + \frac{\partial x^2}{\partial \bar{x}^i}\frac{\partial x^2}{\partial \bar{x}^j} g_{22} + \frac{\partial x^3}{\partial \bar{x}^i}\frac{\partial x^3}{\partial \bar{x}^j} g_{33} \qquad ...(iv)$$

Putting $i = j = 1$ in (*iv*), we have

$$\bar{g}_{11} = \left(\frac{\partial x^1}{\partial \bar{x}^1}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^1}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^1}\right)^2 g_{33} = \left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial y}{\partial \rho}\right)^2 + \left(\frac{\partial z}{\partial \rho}\right)^2$$

$$= \cos^2 \phi + \sin^2 \phi + 0 = 1 \qquad \text{[By } (i) \text{ and } (ii)]$$

Putting $i = j = 2$ in (*iv*), we have

$$\bar{g}_{22} = \left(\frac{\partial x^1}{\partial \bar{x}^2}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^2}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^2}\right)^2 g_{33} = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2$$

$$= (-\rho \sin \phi)^2 + (\rho \cos \phi)^2 + 0 = \rho^2 \qquad \text{[By } (i) \text{ and } (ii)]$$

Similarly $$\bar{g}_{33} = \left(\frac{\partial x^1}{\partial \bar{x}^3}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^3}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^3}\right)^2 g_{33} = 0 + 0 + 1 = 1$$

and $$\bar{g}_{12} = 0 = \bar{g}_{13} = \bar{g}_{21} = \bar{g}_{23} = \bar{g}_{31} = \bar{g}_{32}$$

Hence the metric tensor, written in matrix form, is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \therefore \qquad g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho^2$$

Also cofactors in g are given by

$$G_{11} = \rho^2, G_{22} = 1, G_{33} = \rho^2 ; G_{12} = 0 = G_{13} = G_{21} = G_{23} = G_{31} = G_{32}$$

The components of the conjugate tensor are given by $g^{ij} = G_{ij}/g$.

Hence the conjugate tensor in matrix form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

38.15 ASSOCIATED TENSORS

From (3) of § 38.14, we have $\quad u^j . g_{ij} = v_i \qquad$ ...(1)

i.e., the inner product of the tensor u^j with the fundamental tensor g_{ij} is another tensor v_i which is called the *associated tensor* of u^j.

Similarly, we have $\quad v_l . g^{lj} = u^j \qquad$ ...(2)

Hence u^j is the associated tensor of v_l.

Thus the indices of any tensor can be lowered or raised by forming its inner product with either of the fundamental tensors g_{ij} or g^{ij} as in (1) or (2) above.

38.16 (1) LENGTH OF A VECTOR

The vector **A** is given by

$$\mathbf{A} = A^i g_i \quad \text{or} \quad \mathbf{A} = A_i g^i \qquad \text{...(1)}$$

Also we have the associated vectors $A_i = g_{ij} A^j \qquad$ ...(2)

or $$A^i = g^{ij} A_j \qquad \text{...(3)}$$

$\therefore$ Length of vector $\mathbf{A} = (\mathbf{A} . \mathbf{A})^{1/2} = (A^i g_i . A^j g_j)^{1/2} \qquad$ [By (1)]

$= (g_{ij} A^i A^j)^{1/2} \qquad [\because \; g_i \cdot g_k = g_{ij}]$

$= (A_i A^i)^{1/2} \qquad$ [By (2)]

Also length of vector $\mathbf{A} = (\mathbf{A} . \mathbf{A})^{1/2} = (A_i g^i . A_j g^j)^{1/2} = (g^{ij} A_i A_j)^{1/2} = (A_i A^i)^{1/2} \qquad$ [By (3)]

Hence the magnitude or length of the vector $A = \sqrt{(g_{ij} A^i A^j)} = \sqrt{(g^{ij} A_i A_j)} = \sqrt{(A_i A^i)} \qquad$ ...(4)

which is an invariant.

Obs. The length of a vector $A^1, 0, 0$ (in 3-dimensions) is $\sqrt{(g_{11}A^1A^1)}$, *i.e.* $\sqrt{g_{11}}A^1$. Similarly the length of the vector $0, A^2, 0$ is $\sqrt{g_{22}}A^2$ and the length of the vector $0, 0, A^3$ is $\sqrt{g_{33}}A^3$. Hence the physical components of a vector A^i are $\sqrt{g_{11}}A^1$, $\sqrt{g_{22}}A^2$, $\sqrt{g_{33}}A^3$.

(2) Angle between two vectors. Let **A** and **B** be the given vectors such that

$$\mathbf{A} = A^i g_i \text{ and } B = B^j g_j \qquad \text{...(5)}$$

$\therefore$ $$\mathbf{A}\,.\,\mathbf{B} = |\mathbf{A}|\;|\mathbf{B}|\cos\theta$$

or $$\cos\theta = \frac{\mathbf{A}\,.\,\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{A^iB^jg_{ij}}{\sqrt{(g_{ij}A^iA^j)}\sqrt{(g_{ij}B^iB^j)}} \qquad \text{[Using (4) and (5)]}$$

In terms of associated vectors, we have

$$\cos\theta = \frac{A^iB_i}{\sqrt{(A^iA_i)}\sqrt{(B^iB_i)}}.$$

PROBLEMS 38.3

1. If $ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(x^3)^2 - 6dx^1dx^2 + 4dx^2dx^3$, find the values of g_{ij} and g^{ij}.
2. Find g and g^{ij} corresponding to the metric $ds^2 = \dfrac{dr^2}{1 - r^2/a^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$.
3. The contravariant components of a vector **A** in plane cartesian coordinates $x = x^1$, $y = x^2$ are (33, 56). Using the tensor law of transformation, obtain the new components in polar coordinates $r = \bar{x}^1$ and $\theta = \bar{x}^2$.
4. Prove that the angles θ_{12}, θ_{23} and θ_{31} between the coordinate curves in a 3-dimensional coordinate system are given by $\cos\theta_{12} = \dfrac{g_{12}}{\sqrt{(g_{11}g_{22})}}$, $\cos\theta_{23} = \dfrac{g_{23}}{\sqrt{(g_{22}g_{23})}}$, $\cos\theta_{31} = \dfrac{g_{31}}{\sqrt{(g_{33}g_{11})}}$.
5. Prove that for an orthogonal coordinate system
   (*i*) $g_{12} = g_{23} = g_{31} = 0$ (*ii*) $g^{11} = 1/g_{11}$, $g_{22} = 1/g^{22}$, $g_{33} = 1/g^{33}$.

38.17 CHRISTOFFEL SYMBOLS

Christoffel symbol of the first kind is denoted by $[ij, k]$ and is defined by

$$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) \qquad \text{...(1)}$$

where g_{ij} are the components of the metric tensor.

Christoffel symbol of the second kind is denoted by $\begin{Bmatrix} k \\ ij \end{Bmatrix}$ and is defined by

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = g^{kl}\,[ij, l] \qquad \text{...(2)}$$

Some authors write Christoffel symbol of the second kind as $\{ij, k\}$ or $\Gamma_{ij}^{\;k}$.

Obs. 1. No summation is indicated in the Christoffel symbol of the first kind, but summation is to be made over l in the Christoffel symbol of the second kind.

Obs. 2. It is evident from (1) and (2) that the Christoffel symbols of both kinds are symmetric in the indices i and j.

$$[ij, k] = [ji, k] \text{ and } \begin{Bmatrix} k \\ ij \end{Bmatrix} = \begin{Bmatrix} k \\ ji \end{Bmatrix}.$$

Example 38.14. *If* $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2$, *find the values of*

(*a*) $[22, 1]$ *and* $[13, 3]$ (*b*) $\begin{Bmatrix} 1 \\ 22 \end{Bmatrix}$ *and* $\begin{Bmatrix} 3 \\ 13 \end{Bmatrix}$.

Solution. It is a 3-dimensional space in spherical coordinates such that

$$x^1 = r,\ x^2 = \theta \text{ and } x^3 = \phi$$

Clearly $$g_{11} = 1, g_{22} = r^2, g_{33} = r^2 \sin^2 \theta \text{ and } g_{ij} = 0 \text{ for } i \neq j. \quad ...(i)$$

Also $g^{11} = 1, g^{22} = 1/r^2, g^{33} = 1/r^2 \sin^2 \theta$ (See Ex. 38.12) ...(ii)

(*a*) Christoffel symbols of the first kind are given by

$$[ij, k] = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right]_{i,\,j,\,k\,=\,1,\,2,\,3} \quad ...(iii)$$

Taking $i = 2, j = 2$, and $k = 1$ in (*iii*), we get

$$[22, 1] = \frac{1}{2}\left[\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right] = \frac{1}{2}\left[\frac{\partial(0)}{\partial \theta} + \frac{\partial(0)}{\partial \theta} - \frac{\partial(r^2)}{\partial r}\right] = -r \quad ...(iv)$$

Putting $i = 1, j = 3$ and $k = 3$ in (*iii*), we obtain

$$[13, 3] = \frac{1}{2}\left[\frac{\partial g_{33}}{\partial x^1} + \frac{\partial g_{13}}{\partial x^3} - \frac{\partial g_{13}}{\partial x^3}\right] = \frac{1}{2}\left[\frac{\partial(r^2 \sin^2 \theta)}{\partial r} + \frac{\partial(0)}{\partial \phi} - \frac{\partial(0)}{\partial \phi}\right] = r \sin^2 \theta \quad ...(v)$$

(*b*) Christoffel symbols of the second kind are defined by

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = g^{kl}\,[ij, l] = g^{k1}\,[ij, 1] + g^{k2}\,[ij, 2] + g^{k3}\,[ij, 3] \quad ...(vi)$$

$$\therefore \quad \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = g^{11}[22, 1] + g^{12}[22, 2] + g^{13}[22, 3] = [22, 1] + 0[22, 2] + 0[22, 3]$$

$$= -r \quad \text{[By (iv)]}$$

$$\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = g^{31}[13, 1] + g^{32}[13, 2] + g^{33}[13, 3]$$

$$= 0[13, 1] + 0[13, 2] + \frac{1}{r^2 \sin^2 \theta}[13, 3] \quad \text{[By (ii)]}$$

$$= \frac{1}{r^2 \sin^2 \theta} \cdot r \sin^2 \theta = \frac{1}{r}. \quad \text{[By (v)]}$$

Example 38.15. *Prove that*

(*a*) $\dfrac{\partial g_{ij}}{\partial x^k} = [ik, j] + [jk, i]$ (*b*) $\dfrac{\partial g^{ij}}{\partial x^k} = -g^{jl}\begin{Bmatrix} i \\ lk \end{Bmatrix} - g^{im}\begin{Bmatrix} j \\ mk \end{Bmatrix}.$

Solution. (*a*) By definition of Christoffel symbol of the first kind, we have

$$[ik, j] = \frac{1}{2}\left[\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j}\right] \quad ...(i)$$

and $$[jk, i] = \frac{1}{2}\left[\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right] \quad ...(ii)$$

Since g_{ij} is a symmetric tensor, $\therefore \quad g_{ij} = g_{ji}, g_{jk} = g_{kj}, g_{ki} = g_{ik}$

Adding (*i*) and (*ii*), we get the required result.

(*b*) We know that $g^{ij} g_{lj} = \delta^i_l$. [Refer to § 38.14 (2)

Differentiating w.r.t. x^k, we get

$$g^{ij}\frac{\partial g_{lj}}{\partial x^k} + g_{lj}\frac{\partial g^{ij}}{\partial x^k} = 0 \quad [\because \ \delta^i_l = 1 \text{ or } 0]$$

Multiplying by g^{lm} and transposing, we have

$$g^{lm} g_{lj}\frac{\partial g^{ij}}{\partial x^k} = -g^{lm} g^{ij}\frac{\partial g_{lj}}{\partial x^k} \quad \text{or} \quad \delta^m_j \frac{\partial g^{ij}}{\partial x^k} = -g^{lm} g^{ij}\{[lk, j] + [jk, l]\} \quad \text{[From (a)]}$$

or $$\frac{\partial g^{im}}{\partial x^k} = -\, g^{lm}\left(g^{ij}[lk, j]\right) - g^{ij}\left(g^{lm}[jk, l]\right) = -\, g^{lm}\begin{Bmatrix} i \\ lk \end{Bmatrix} - g^{ij}\begin{Bmatrix} m \\ jk \end{Bmatrix}$$

Interchanging m and j, we obtain the desired result.

38.18 TRANSFORMATION OF CHRISTOFFEL SYMBOLS

The fundamental tensors g_{ij}, g^{ij} and also $[ij, k]$ are functions of the coordinates x^i. Let these transform to $\bar{g}_{ij}, \bar{g}^{ij}$ and $[\overline{ij, k}]$ in another coordinate system $\bar{x}^i$.

(1) *Law of transformation of Christoffel symbol of first kind.*

Let $[ij, k]$ which is a function of x^i, transform to $[\overline{ij, k}]$ in another coordinate system $\bar{x}^i$. Then

$$[\overline{ij, k}] = \frac{1}{2}\left(\frac{\partial \bar{g}_{jk}}{\partial \bar{x}^i} + \frac{\partial \bar{g}_{ik}}{\partial \bar{x}^j} - \frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}\right) \qquad ...(1)$$

Since $\bar{g}_{ij}$ is a covariant tensor of order two, we have

$$\bar{g}_{ij} = \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j} g_{pq} \qquad ...(2)$$

Differentiating both sides w.r.t. $\bar{x}^k$, we get

$$\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k} = \left(\frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j} + \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial^2 x^q}{\partial \bar{x}^k \partial \bar{x}^j}\right) g_{pq} + \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial g_{pq}}{\partial x^r}\frac{\partial x^r}{\partial \bar{x}^k} \qquad ...(3)$$

[Note that g_{pq} is in terms of original coordinates x and to differentiate it w.r.t. $\bar{x}^k$, first we differentiate it w.r.t. x^r and then differentiate x^r w.r.t. $\bar{x}^k$.]

Interchanging i, k and also p, r in the last term of (3), we have

$$\frac{\partial \bar{g}_{ik}}{\partial \bar{x}^i} = \left(\frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^k}\frac{\partial x^q}{\partial \bar{x}^j} + \frac{\partial x^p}{\partial \bar{x}^k}\frac{\partial^2 x^q}{\partial \bar{x}^i \partial \bar{x}^j}\right) g_{pq} + \frac{\partial x^p}{\partial \bar{x}^k}\frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial g_{qr}}{\partial x^p} \qquad ...(4)$$

Similarly interchanging j, k and also q, r in the last term of (3), we get

$$\frac{\partial \bar{g}_{ik}}{\partial \bar{x}^j} = \left(\frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial^2 x^q}{\partial \bar{x}^k \partial \bar{x}^j}\right) g_{pq} + \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial g_{pr}}{\partial \bar{x}^q} \qquad ...(5)$$

Substituting the values from (3), (4) and (5) in (1), we obtain

$$[\overline{ij, k}] = \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k} g_{pq} + \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial x^r}{\partial \bar{x}^k}[pq, r] \qquad ...(6)$$

This is the desired law of transformation of Christoffel symbol of the first kind.

(2) *Law of transformation of Christoffel symbol of the second kind.*

Let $g^{kl}[ij, l]$ transform to $\bar{g}^{kl}\,[\overline{ij, l}]$.

Since $\bar{g}^{kl}$ is a contravariant tensor of order two.

$$\therefore \qquad \bar{g}^{kl} = \frac{\partial \bar{x}^k}{\partial x^s}\frac{\partial \bar{x}^l}{\partial x^t} g^{st} \qquad ...(7)$$

From (6), we have $$[\overline{ij, l}] = \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^l} g_{pq} + \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}\frac{\partial x^r}{\partial \bar{x}^l}[pq, r] \qquad ...(8)$$

Multiplying the respective sides of (7) and (8), we get

$$\overline{\begin{Bmatrix} l \\ ij \end{Bmatrix}} = \frac{\partial \bar{x}^t}{\partial \bar{x}^t}\delta^q{}_s \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j} g^{st} g_{pq} + \frac{\partial \bar{x}^l}{\partial x^t}\delta^r{}_s \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j} g^{st}\,[pq, r] \qquad ...(9)$$

Since $$\delta^q{}_s g^{st} g_{pq} = g^{qt} g_{pq} = \delta_p{}^t$$

and $$\delta_s^r g^{st}[pq, r] = g^{rt}[pq, r] = \begin{Bmatrix} t \\ pq \end{Bmatrix}$$

$$\therefore \quad \begin{Bmatrix} \bar{l} \\ ij \end{Bmatrix} = \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j} \delta^t{}_p \frac{\partial \bar{x}^l}{\partial x^t} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial \bar{x}^l}{\partial x^t} \begin{Bmatrix} t \\ pq \end{Bmatrix}$$

or $$\begin{Bmatrix} \bar{l} \\ ij \end{Bmatrix} = \frac{\partial^2 x^t}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial \bar{x}^l}{\partial x^t} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial \bar{x}^l}{\partial x^t} \begin{Bmatrix} t \\ pq \end{Bmatrix} \qquad ...(10)$$

This is the law of transformation of Christoffel symbol of the second kind.

Obs. 1. From (10), we obtain the following important relation :

$$\frac{\partial^2 x^t}{\partial \bar{x}^i \partial \bar{x}^j} = \frac{\partial x^t}{\partial \bar{x}^l} \begin{Bmatrix} \bar{l} \\ ij \end{Bmatrix} - \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \begin{Bmatrix} t \\ pq \end{Bmatrix} \qquad ...(11)$$

Obs. 2. It is evident from (6) and (10) that the Christoffel 3-index symbols are not tensors. These symbols transform like tensors only for linear transformation of coordinates.

Example 38.16. *Prove that* $\begin{Bmatrix} i \\ ij \end{Bmatrix} = \frac{\partial}{\partial x^j} (\log \sqrt{g})$.

Solution. Let G_{jk} be the co-factor of g_{ik} in g so that $g = g_{ik}\, G_{ik}$ (summation over k only)

$$\therefore \quad \frac{\partial g}{\partial g_{ik}} = G_{ik} \qquad [\because\ G_{ik} \text{ does not contain } g_{ik} \text{ implicitly}]$$

Also $$\frac{\partial g}{\partial g^j} = \frac{\partial g}{\partial g_{ik}} \frac{\partial g_{ik}}{\partial x^j} = G_{ik} \frac{\partial g_{ik}}{\partial x^j} \qquad \text{(summation over } i \text{ and } k) \qquad (i)$$

We know that $$g^{ik} = \frac{G_{ik}}{g} \qquad ...(ii)$$

Substituting the value of G_{ik} from (*ii*) in (*i*), we get

$$\frac{\partial g}{\partial x^j} = g g^{ik} \frac{\partial g_{ik}}{\partial x^j} \quad \text{or} \quad \frac{1}{g} \frac{\partial g}{\partial x^j} = g^{ik} \frac{\partial g_{ik}}{\partial x^j}$$

or $$\frac{\partial}{\partial x^j} (\log g) = g^{ik}([jk, i] + [ij, k]) \qquad \text{[By Ex. 38.15 (a)]}$$

$$= g^{ik}[jk, i] + g^{ik}[ji, k] = \begin{Bmatrix} k \\ jk \end{Bmatrix} + \begin{Bmatrix} i \\ ji \end{Bmatrix} = 2\begin{Bmatrix} i \\ ji \end{Bmatrix}$$

Hence $$\begin{Bmatrix} i \\ ij \end{Bmatrix} = \frac{1}{2} \frac{\partial}{\partial x^j} (\log g) = \frac{\partial}{\partial x^j} \log \sqrt{g}. \qquad ...(iii)$$

38.19 (1) COVARIANT DIFFERENTIATION OF A COVARIANT VECTOR

Let A_i and $\bar{A}_i$ be the components of a covariant vector (*i.e.*, a tensor of first order) in the coordinate system x^i and $\bar{x}^i$ respectively. Let us investigate the tensor character of the partial derivatives of A_i w.r.t. the variables x^j. From the law of transformation. $\bar{A}_i = \frac{\partial x^p}{\partial \bar{x}^i} A_p$

Differentiating w.r.t. $\bar{x}^j$, we have $$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} = \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^i} A_p + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial A_p}{\partial x^q} \frac{\partial x^q}{\partial \bar{x}^j} \qquad ...(1)$$

(Note that A_p is not directly a function of $\bar{x}^j$).

Due to presence of the first term on the R.H.S. of (1), it is evident that $\frac{\partial A_p}{\partial x^q}$ is not a tensor.

On replacing this term by $\dfrac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} A_s$ and substituting for the second derivative from (11) of § 35.18, we get

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} = \frac{\partial x^s}{\partial \bar{x}^l} A_s \left\{ \begin{matrix} \bar{l} \\ i \ j \end{matrix} \right\} - \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} A_s \left\{ \begin{matrix} s \\ pq \end{matrix} \right\} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial A_p}{\partial \bar{x}^q}$$

or
$$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} - \bar{A}_l \left\{ \begin{matrix} \bar{l} \\ i \ j \end{matrix} \right\} = \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \left[\frac{\partial A_p}{\partial \bar{x}^q} - A_s \left\{ \begin{matrix} s \\ pq \end{matrix} \right\} \right]$$

This shows that the expression $\dfrac{\partial \bar{A}_i}{\partial \bar{x}^j} - \bar{A}_l \left\{ \begin{matrix} \bar{l} \\ i \ j \end{matrix} \right\}$

is a covariant tensor of the second order. This is called the covariant derivative of $\bar{A}_i$ w.r.t $\bar{x}^j$ and is denoted by $\bar{A}_{i,\,j}$.

(2) Covariant differentiation of a contravariant vector. Let A^i and $\bar{A}^i$ be the components of a contravariant vector in the coordinate systems x^i and $\bar{x}^i$. From the law of transformation $A^s = \dfrac{\partial x^s}{\partial \bar{x}^i} \bar{A}^i$.

Differentiating w.r.t. $\bar{x}^j$, we have $\dfrac{\partial A^s}{\partial \bar{x}^j} = \dfrac{\partial^2 x^s}{\partial \bar{x}^j \partial \bar{x}^i} \bar{A}^i + \dfrac{\partial x^s}{\partial \bar{x}^i} \dfrac{\partial \bar{A}^i}{\partial \bar{x}^j}$

Substituting for the second derivative from (11) of § 35.18, we get

$$\frac{\partial A^s}{\partial x^q} \frac{\partial x^q}{\partial \bar{x}^j} = \frac{\partial x^s}{\partial x^l} \left\{ \begin{matrix} l \\ i \ j \end{matrix} \right\} - \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \left\{ \begin{matrix} s \\ p \ q \end{matrix} \right\} \bar{A}^i + \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{A}^i}{\partial \bar{x}^j}$$

Interchanging the dummy indices i, l in the first term on the R.H.S. and putting

$\dfrac{\partial x^p}{\partial \bar{x}^i} \bar{A}^i = A^p$ in the second term, we obtain

$$\frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial A^s}{\partial x^q} = \frac{\partial x^s}{\partial \bar{x}^i} \left\{ \begin{matrix} \bar{i} \\ l \ j \end{matrix} \right\} \bar{A}^l - \frac{\partial x^q}{\partial \bar{x}^j} A^p \left\{ \begin{matrix} s \\ p \ q \end{matrix} \right\} + \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{A}^i}{\partial \bar{x}^j}$$

Transposing the second term on the R.H.S. to the L.H.S., we get

$$\frac{\partial x^q}{\partial \bar{x}^j} \left[\frac{\partial A^s}{\partial x^q} + A^p \left\{ \begin{matrix} s \\ p \ q \end{matrix} \right\} \right] = \frac{\partial x^s}{\partial \bar{x}^i} \left[\frac{\partial \bar{A}^i}{\partial \bar{x}^j} + \bar{A}^l \left\{ \begin{matrix} \bar{i} \\ l \ j \end{matrix} \right\} \right]$$

or
$$\frac{\partial \bar{A}^i}{\partial \bar{x}^j} + \bar{A}^l \left\{ \begin{matrix} \bar{i} \\ l \ j \end{matrix} \right\} = \frac{\partial \bar{x}^i}{\partial x^s} \frac{\partial x^q}{\partial \bar{x}^j} \left[\frac{\partial A^s}{\partial x^q} + A^p \left\{ \begin{matrix} s \\ p \ q \end{matrix} \right\} \right]$$

This shows that $\dfrac{\partial A^i}{\partial x^j} + A^l \left\{ \begin{matrix} i \\ l \ j \end{matrix} \right\}$ is a mixed tensor of the second order. This is called the *covariant derivative of A^i w.r.t. x^j and is denoted by $A^i_{,j}$.*

Obs. *The following laws hold good for covariant differentiation :*

(i) Covariant derivative of the sum (or difference) of two tensors = sum (or difference) of their covariant derivatives.

(ii) Covariant derivative of the product of two tensors = covariant derivative of first tensor × second tensor + covariant derivative of second tensor × first tensor.

Example 38.17. *Prove that the covariant derivative of g^{ij} is zero.*

Solution. Let A_i denote a covariant vector which moves parallel to itself so that

$$A_{i,k} \text{ or } \frac{\partial A_i}{\partial x^k} - A_l \left\{ \begin{matrix} l \\ i \ k \end{matrix} \right\} = 0 \qquad ...(i)$$

Let $\phi = g^{ij} A_i A_j$ so that ϕ is a scalar invariant. Differentiating it w.r.t. x^k, we have

$$\frac{\partial \phi}{\partial x^k} = \frac{\partial g^{ij}}{\partial x^k} A_i A_j + g^{ij} \frac{\partial A_i}{\partial x^k} A_j + g^{ij} A_i \frac{\partial A_j}{\partial x^k}$$

$$= \frac{\partial g^{ij}}{\partial x^k} A_i A_j + g^{ij} \begin{Bmatrix} l \\ i\ k \end{Bmatrix} A_l A_j + g^{ij} \begin{Bmatrix} l \\ j\ k \end{Bmatrix} A_i A_l \qquad \text{[By (i)]}$$

Interchanging i and l in the second term and j and l in the last term on the right, we get

$$\frac{\partial \phi}{\partial x^k} = \left[\frac{\partial g^{ij}}{\partial x^k} + g^{lj} \begin{Bmatrix} i \\ l\ k \end{Bmatrix} + g^{il} \begin{Bmatrix} j \\ l\ k \end{Bmatrix} \right] A_i A_j$$

Since $\partial\phi/\partial x^k$ is a covariant vector, the expression

$$\frac{\partial g^{ij}}{\partial x^k} + g^{lj} \begin{Bmatrix} i \\ l\ k \end{Bmatrix} + g^{il} \begin{Bmatrix} j \\ l\ k \end{Bmatrix}$$

is a tensor of the third order by quotient law. Thus it is the covariant derivative $g^{ij}{}_{,k}$.

$$\therefore \qquad g^{ij}{}_{,k} = \frac{\partial g^{ij}}{\partial x^k} + g^{lj} \begin{Bmatrix} i \\ l\ k \end{Bmatrix} + g^{il} \begin{Bmatrix} j \\ l\ k \end{Bmatrix}$$

$$= - g^{jl} \begin{Bmatrix} i \\ l\ k \end{Bmatrix} - g^{im} \begin{Bmatrix} j \\ m\ k \end{Bmatrix} + g^{lj} \begin{Bmatrix} i \\ l\ k \end{Bmatrix} + g^{il} \begin{Bmatrix} j \\ l\ k \end{Bmatrix} \qquad \text{[By Ex. 38.15 (b)]}$$

$$= - g^{im} \begin{Bmatrix} j \\ m\ k \end{Bmatrix} + g^{il} \begin{Bmatrix} j \\ l\ k \end{Bmatrix} = - g^{il} \begin{Bmatrix} j \\ l\ k \end{Bmatrix} + g^{il} \begin{Bmatrix} j \\ l\ k \end{Bmatrix} = 0$$

[Changing the dummy index m to l]

38.20 (1) GRADIENT

If ϕ be a scalar function of the coordinates, then the gradient of ϕ is denoted by grad $\phi = \dfrac{\partial \phi}{\partial x^i}$ which is a *covariant vector.*

(2) Divergence. *The divergence of the contravariant vector A^i is defined by*

$$div\ A^i = \frac{\partial A^i}{\partial x^i} + A^k \begin{Bmatrix} l \\ k\ i \end{Bmatrix} \text{ which is sometimes written as } A^i{}_{,i}.$$

The divergence of the covariant vector A_i is defined by div $A_i = g^{ik} A_{ik}$.

(3) Curl. Let A_i be a covariant vector, then

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - A_k \begin{Bmatrix} k \\ i\ j \end{Bmatrix} \text{ and } A_{j,i} = \frac{\partial A_j}{\partial x^i} - A_k \begin{Bmatrix} k \\ j\ i \end{Bmatrix} \text{ are covariant tensors.}$$

$$\therefore \qquad A_{i,j} - A_{j,i} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$$

is a covariant tensor of second order, which is called curl of A_i.

Thus $\qquad curl\ A_i = A_{i,j} - A_{j,i}$.

Obs 1. *Curl A_i is a skew-symmetric tensor.*

Since $\qquad A_{j,i} - A_{i,j} = -(A_{i,j} - A_{j,i})$.

Obs. 2. *Curl is a tensor and not a vector.* In a 3-dimensional space, however, curl has only three independent non-zero components and it can, therefore, be taken as a vector.

Example 38.18. *Prove that*

(a) $div\ A_i = \dfrac{1}{\sqrt{g}} \cdot \dfrac{\partial}{\partial x^k} (\sqrt{g}\, A^k)$ $\qquad$ (b) $\nabla^2 \phi = \dfrac{1}{\sqrt{g}} \dfrac{\partial}{\partial x^k} \left(\sqrt{g}\, g^{kr} \dfrac{\partial \phi}{\partial x^r} \right)$.

Solution. (*a*) Using $\left\{\begin{matrix} i \\ k\ i \end{matrix}\right\} = \dfrac{\partial}{\partial x^k} \log \sqrt{g} = \dfrac{1}{\sqrt{g}} \dfrac{\partial \sqrt{g}}{\partial x^k}$ [By Ex. 38.16]

$$\text{div } A^i = \frac{\partial A^i}{\partial x^i} + A^k \left\{\begin{matrix} i \\ k\ i \end{matrix}\right\} = \frac{\partial A^i}{\partial x^i} + \frac{A^k}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k} = \frac{\partial A^k}{\partial x^k} + \frac{A^k}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k}$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} (\sqrt{g}\, A^k) \qquad \ldots(i)$$

(*ii*) We have $\nabla^2 \phi = \text{div grad } \phi$ ...(*ii*)

and grad $\phi = \dfrac{\partial \phi}{\partial x^r}$, which is a covariant vector.

The contravariant vector associated with $\partial\phi/\partial x^r$ is

$$A^k = g^{kr}\, \partial\phi/\partial x^r.$$

Then from (*i*) and (*ii*), $\nabla^2 \phi = \text{div} \left(g^{kr} \dfrac{\partial \phi}{\partial x^r} \right) = \dfrac{1}{\sqrt{g}} \dfrac{\partial}{\partial x^k} \left(\sqrt{g}\, g^{kr} \dfrac{\partial \phi}{\partial x^r} \right).$

PROBLEMS 38.4

1. Determine Christoffel symbols of the first and second kind for an orthogonal curvilinear coordinate system.
2. Determine the Christoffel symbols of the first kind in (*a*) rectangular, (*b*) cylindrical and (*c*) spherical coordinates.
3. Evaluate the Christoffel symbols of the second kind in (*a*) rectangular, (*b*) cylindrical and (*c*) spherical coordinates.
4. Prove that (*a*) $[pq, r] = [qp, r]$, (*b*) $[pq, r] = g_{rs} \left\{\begin{matrix} s \\ pq \end{matrix}\right\}$
5. If $(ds)^2 = r^2 (d\theta)^2 + r^2 \sin^2\theta\, (d\phi)^2$, find the values of

   (*a*) [22, 1] and [12, 2] (*b*) $\begin{bmatrix} 1 \\ 22 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 12 \end{bmatrix}$.
6. If $(ds)^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta\, (d\phi)^2$, find the values of

   (*a*) [33, 1] and [23, 3] (*b*) $\begin{bmatrix} 1 \\ 33 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 23 \end{bmatrix}$.
7. Show that the tensors g_{ij}, g^{ij} and δ^i_j are constants with respect to covariant differentiation.
8. Write the covariant derivative w.r.t. x^k of the tensors u^{ij} and A^k_{ij}.
9. Show that the covariant derivative of g_{ij} is zero. (*Madras M.E., 2000*)
10. Find the covariant derivative $A^j_k\, B^{lm}_n$ with respect to x^q.
11. Evaluate div A^i in (*a*) cylindrical, (*b*) spherical coordinates.
12. Obtain the Laplace's equation in (*a*) cylindrical, (*b*) spherical coordinates.
13. If $A_{ij,\,k}$ is the curl of a covariant vector, prove that

    $$A_{ij,\,k} + A_{jk,\,i} + A_{ki,\,j} = 0.$$ (*Madras M.E., 2000 S*)
14. Using tensor notation, show that

    (*a*) div curl $A^r = 0$ (*b*) curl grad $\phi = 0$.