CHAPTER 4

Differential Calculus & Its Applications

1. Successive differentiation ; Standard results. 2. Leibnitz's theorem. 3. Fundamental theorems : Rolle's theorem, Lagrange's Mean-value theorem, Cauchy's mean value theorem, Taylor's theorem. 4. Expansions of functions : Maclaurin's series, Taylor's series. 5. Indeterminate forms. 6. Tangents & Normals—Cartesian curves, Angle of intersection of two curves. 7. Polar curves. 8. Pedal equation. 9. Derivative of arc. 10. Curvature. 11. Radius of curvature. 12. Centre of curvature, Evolute, Chord of curvature. 13. Envelope. 14. Increasing and decreasing functions : Concavity, convexity & Point on inflexion. 15. Maxima & Minima, Practical problems. 16. Asymptotes. 17. Curve tracing. 18. Objective Type of Questions.

4.1 (1) SUCCESSIVE DIFFERENTIATION

The reader is already familiar with the process of differentiating a function $y = f(x)$. For ready reference, a list of derivatives of some standard functions is given in the beginning.

The derivative dy/dx is, in general, another function of x which can be differentiated. The derivative of dy/dx is called the *second derivative* of y and is denoted by d^2y/dx^2. Similarly, the derivative of d^2y/dx^2 is called the *third derivative* of y and is denoted by d^3y/dx^3. In general, the nth derivative of y is denoted by d^ny/dx^n.

Alternative notations for the successive derivatives of $y = f(x)$ are

$$Dy, D^2y, D^3y, \ldots, D^ny\ ;$$

or $$y_1, y_2, y_3, \ldots, y_n\ ;$$

or $$f'(x), f''(x), f'''(x) \ldots, f^n(x).$$

The nth derivative of $y = f(x)$ at $x = a$ is denoted by $(d^ny/dx^n)_a$, $(y_n)_a$ or $f^n(a)$.

Example 4.1. *If $y = e^{ax} \sin bx$, prove that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$.* (Cochin, 2005)

Solution. We have $\quad y = e^{ax} \sin bx \quad$ ...(*i*)

$\therefore \quad y_1 = e^{ax} (\cos bx \,.\, b) + \sin bx\, (e^{ax} \,.\, a) = be^{ax} \cos bx + ay \quad$ [By (*i*)]

or $\quad y_1 - ay = be^{ax} \cos bx \quad$ ...(*ii*)

Again differentiating both sides,

$$y_2 - ay_1 = be^{ax} (-\sin bx \,.\, b) + b \cos bx\, (e^{ax} \,.\, a) = -b^2y + a(y_1 - ay)$$

or $$y_2 - 2ay_1 + (a^2 + b^2)y = 0.$$

Example 4.2. *If $x = a (\cos t + t \sin t)$, $y = a (\sin t - t \cos t)$, find d^2y/dx^2.*

Solution. We have $\quad \dfrac{dx}{dt} = a(-\sin t + t \cos t + \sin t) = at \cos t$

and $\quad \dfrac{dy}{dt} = a (\cos t + t \sin t - \cos t) = at \sin t$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{at \sin t}{at \cos t} = \tan t$$

$$\therefore \qquad \frac{d^2y}{dx^2} = \frac{d}{dt}(\tan t) \cdot \frac{dt}{dx} = \sec^2 t \cdot \frac{1}{at \cos t} = 1/at \cos^3 t.$$

Example 4.3. *Given $y^2 = f(x)$, a polynomial of third degree, then evaluate* $\frac{d}{dx}\left(y^3 \frac{d^2y}{dx^2}\right)$.

Solution. Differentiating $y^2 = f(x)$ w.r.t. x, we get

$$2y \frac{dy}{dx} = f'(x) \qquad \ldots(i)$$

Differentiating (i) w.r.t. x again, we obtain

$$2\left(\frac{dy}{dx} \cdot \frac{dy}{dx} + y \frac{d^2y}{dx^2}\right) = f''(x) \quad \text{or} \quad 2\left(\frac{dy}{dx}\right)^2 + 2y \frac{d^2y}{dx^2} = f''(x)$$

Again differentiating, we get

$$4 \cdot \frac{dy}{dx} \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} \frac{d^2y}{dx^2} + 2y \frac{d^3y}{dx^3} = f'''(x)$$

or
$$3y^2 \frac{dy}{dx} \frac{d^2y}{dx^2} + y^3 \frac{d^3y}{dx^3} = \frac{1}{2} y^2 f'''(x) \qquad \text{[Multiplying by } y^2\text{]}$$

Hence
$$\frac{d}{dx}\left(y^3 \frac{d^2y}{dx^2}\right) = \frac{1}{2} f(x) f'''(x). \qquad [\because \ y^2 = f(x)]$$

Example 4.4. *If $ax^2 + 2hxy + by^2 = 1$, prove that* $\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}$.

Solution. Differentiating the given equation w.r.t. x,

$$2ax + 2h\left(x \frac{dy}{dx} + y\right) + 2by \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{ax + hy}{hx + by} \qquad \ldots(i)$$

Differentiating both sides of (i) w.r.t. x,

$$\frac{d^2y}{dx^2} = -\frac{(hx + by)(a + h\,dy/dx) - (ax + hy)(h + b\,dy/dx)}{(hx + by)^2}$$

[Substituting the value of dy/dx from (i)]

$$= -\frac{(hx + by)\left(a - h \cdot \dfrac{ax + hy}{hx + by}\right) - (ax + hy)\left(h - b \cdot \dfrac{ax + hy}{hx + by}\right)}{(hx + by)^2}$$

$$= \frac{(h^2 - ab)(ax^2 + 2hxy + by^2)}{(hx + by)^3}$$

$$= (h^2 - ab)/(hx + by)^3 \qquad [\because \ ax^2 + 2hxy + by^2 = 1]$$

PROBLEMS 4.1

1. If $y = (ax + b)/(cx + d)$, show that $2y_1 y_3 = 3y_2^2$.
2. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y \cos^2 x = 0$.
3. If $y = e^{-kt} \cos(lt + c)$, show that $\frac{d^2y}{dx^2} + 2k \frac{dy}{dx} + n^2 y = 0$, where $n^2 = k^2 + l^2$.

4. If $y = \sinh [m \log \{x + \sqrt{(x^2+1)}\}]$, show that $(x^2+1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = m^2 y$.

5. If $y = \sin^{-1} x$, show that $(1-x^2)\,y_5 - 7xy_4 - 9y_3 = 0$. (*Madras, 2000 S*)

6. If $x = \dfrac{1}{2}\left(t + \dfrac{1}{t}\right), y = \dfrac{1}{2}\left(t - \dfrac{1}{t}\right)$, find $\dfrac{d^2y}{dx^2}$. (*Cochin, 2005*)

7. If $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$, find the value of d^2y/dx^2 when $t = \pi/2$.

8. If $x = a(\cos t + \log \tan t/2)$, $y = a \sin t$, find d^2y/dx^2.

9. If $x = \sin t$, $y = \sin pt$, prove that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2 y = 0$.

10. If $x^3 + y^3 = 3axy$, prove that $\dfrac{d^2y}{dx^2} = -\dfrac{2a^2xy}{(y^2-ax)^3}$.

(2) Standard Results

We have (*1*) $\mathbf{D^n (ax+b)^m = m(m-1)(m-2) \ldots (m-n+1)\, a^n (ax+b)^{m-n}}$

(*2*) $\mathbf{D^n \left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n (n!)\, a^n}{(ax+b)^{n+1}}}$

(*3*) $\mathbf{D^n \log (ax+b) = \dfrac{(-1)^{n-1} (n-1)!\, a^n}{(ax+b)^n}}$

(*4*) $\mathbf{D^n (a^{mx}) = m^n (\log a)^n . a^{mx}}$

(*5*) $\mathbf{D^n (e^{mx}) = m^n e^{mx}}$

(*6*) $\mathbf{D^n \sin (ax+b) = a^n \sin (ax + b + n\pi/2)}$

(*7*) $\mathbf{D^n \cos (ax+b) = a^n \cos (ax + b + n\pi/2)}$

(*8*) $\mathbf{D^n [e^{ax} \sin (bx+c)] = (a^2+b^2)^{n/2} e^{ax} \sin (bx + c + n \tan^{-1} b/a)}$

(*9*) $\mathbf{D^n [e^{ax} \cos (bx+c)] = (a^2+b^2)^{n/2} e^{ax} \cos (bx + c + n \tan^{-1} b/a)}$

To prove (1), let $y = (ax+b)^m$

$$y_1 = m \,.\, a(ax+b)^{m-1}$$
$$y_2 = m(m-1)a^2 (ax+b)^{m-2}$$
$$y_3 = m(m-1)(m-2)\, a^3 (ax+b)^{m-3}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Hence $y_n = m(m-1)(m-2) \ldots (m-n+1)\, a^n (ax+b)^{m-n}$

In particular, $\mathbf{D^n (x^n) = n\,!}$.

(2) follows from (1) by taking $m = -1$. The proof of (3) is left as an exercise for the student.

To prove (4), let $y = a^{mx}$

$$y_1 = m \log a \,.\, a^{mx},\; y_2 = (m \log a)^2 a^{mx}, \text{ etc.}$$

In general $y_n = (m \log a)^n a^{mx}$.

(5) follows from (4) by taking $a = e$.

To prove (6), let $y = \sin (ax+b)$

$\therefore$
$$y_1 = a \cos (ax+b) = a \sin (ax + b + \pi/2)$$
$$y_2 = a^2 \cos (ax + b + \pi/2) = a^2 \sin (ax + b + 2\pi/2)$$
$$y_3 = a^3 \cos (ax + b + 2\pi/2) = a^3 \sin (ax + b + 3\pi/2)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

In general, $y_n = a^n \sin (ax + b + n\pi/2)$.

The proof of (7), is left as an exercise for the reader.

To prove (8), let $y = e^{ax} \sin (bx+c)$

$\therefore$
$$y_1 = e^{ax} \cos (bx+c) \,.\, b + ae^{ax} \sin (bx+c)$$
$$= e^{ax} [a \sin (bx+c) + b \cos (bx+c)]$$

Put $a = r\cos\alpha$, $b = r \sin \alpha$ so that $r = \sqrt{(a^2+b^2)}$, $\alpha = \tan^{-1} b/a$

$\therefore$
$$y_1 = re^{ax} [\sin (bx+c) \cos \alpha + \cos (bx+c) \sin \alpha]$$
$$= re^{ax} \sin (bx + c + \alpha)$$

Similarly,
$$y_2 = r^2 e^{ax} \sin (bx + c + 2\alpha)$$
$$y_3 = r^3 e^{ax} \sin (bx + c + 3\alpha)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

In general, $y_n = r^n e^{ax} \sin(bx + c + n\alpha)$ (V.T.U., 2000)

where $r = \sqrt{(a^2 + b^2)}$ and $\alpha = \tan^{-1} b/a$.

Proceeding as in (8), the student should prove (9) himself.

(3) **Preliminary transformations.** Quite often preliminary simplification reduces the given function to one of the above standard forms and then the nth derivative can be written easily.

To find the nth derivative of the powers of sines or cosines or their products, we first express each of these as a series of sines or cosines of multiple angles and then use the above formulae (6) and (7).

Example 4.5. *If* $y = x \log \frac{x-1}{x+1}$, *show that* $y_n = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$. (U.P.T.U., 2003)

Solution. Differentiating y w.r.t. x, we have

$$y_1 = \log \frac{x-1}{x+1} + x\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$$

$$= \log(x-1) - \log(x+1) + \frac{1}{x-1} + \frac{1}{x+1} \qquad ...(i)$$

Now differentiating (i) $(n-1)$ times w.r.t. x,

$$y_n = \frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} - \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}} + \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} + \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}$$

$$= (-1)^{n-2}(n-2)!\left\{\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n} + \frac{-(n-1)}{(x-1)^n} + \frac{-(n-1)}{(x+1)^n}\right\}$$

$$= (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right].$$

Example 4.6. *Find the nth derivative of (i)* $\cos x \cos 2x \cos 3x$ (S.V.T.U., 2009)
(ii) $e^{2x} \cos^2 x \sin x$.

Solution. (i) $y = \cos x \cos 2x \cos 3x = \frac{1}{2}\cos x(\cos 5x + \cos x)$

$$= \tfrac{1}{4}(2\cos x \cos 5x + 2\cos^2 x) = \tfrac{1}{4}[(\cos 6x + \cos 4x) + (1 + \cos 2x)]$$

$$= \tfrac{1}{4}(1 + \cos 2x + \cos 4x + \cos 6x)$$

$\therefore$ $$y_n = \tfrac{1}{4}[2^n \cos(2x + n\pi/2) + 4^n \cos(4x + n\pi/2 + 6^n \cos(6x + n\pi/2)]$$

(ii) $\cos^2 x \sin x = \cos x(\sin x \cos x) = \cos x \cdot \frac{1}{2}\sin 2x$

$$= \tfrac{1}{4}(2\sin 2x \cos x) = \tfrac{1}{4}(\sin 3x + \sin x)$$

$\therefore$ $D^n(e^{2x}\cos^2 x \sin x) = \frac{1}{4}[D^n(e^{2x}\sin 3x) + D^n(e^{2x}\sin x)]$

$$= \tfrac{1}{4}[(2^2 + 3^2)^{n/2}\sin(3x + n\tan^{-1} 3/2) + (2^2 + 1^2)^{n/2}\sin(x + n\tan^{-1}\tfrac{1}{2})]$$

$$= \tfrac{1}{4}[(13)^{n/2}\sin(3x + n\tan^{-1} 3/2) + (5)^{n/2}\sin(x + n\tan^{-1}\tfrac{1}{2})].$$

(4) Use of partial fractions. To find the nth derivative of any rational algebraic fraction, we first split it up into partial fractions. Even when the denominator cannot be resolved into real factors, the method of partial fractions can still be used after breaking the denominator into complex linear factors. Then to put the result back in a real form, we apply De Moivre's theorem (p. 647).

Example 4.7. *Find the nth derivative of* $\frac{x}{(x-1)(2x+3)}$.

Solution. $$\frac{x}{(x-1)(2x+3)} = \frac{1}{(x-1)(2.1+3)} + \frac{-3/2}{(-3/2-1)(2x+3)}$$

$$= \frac{1}{5} \cdot \frac{1}{x-1} + \frac{3}{5} \cdot \frac{1}{2x+3}$$

Hence $$D^n \left[\frac{x}{(x-1)(2x+3)}\right] = \frac{1}{5} \cdot \frac{(-1)^n\, n!}{(x-1)^{n+1}} + \frac{3}{5} \cdot \frac{(-1)^n\,(n!)2^n}{(2x+3)^{n+1}}$$

$$= \frac{(-1)^n\, n!}{5} \left\{\frac{1}{(x-1)^{n+1}} + \frac{3 \,.\, 2^n}{(2x+3)^{n+1}}\right\}.$$

Example 4.8. *Find the nth derivative of* $\dfrac{1}{x^2+a^2}$.

Solution. We have $$y = \frac{1}{x^2+a^2} = \frac{1}{(x+ia)(x-ia)} = \frac{1}{2ia}\left(\frac{1}{x-ia} - \frac{1}{x+ia}\right)$$

$\therefore$ $$y_n = \frac{1}{2ia}\left\{\frac{(-1)^n\, n!}{(x-ia)^{n+1}} - \frac{(-1)^n\, n!}{(x+ia)^{n+1}}\right\}$$

[Put $x = r\cos\theta$, $a = r\sin\theta$ so that $r = \sqrt{(x^2+a^2)}$, $\theta = \tan^{-1}(a/x)$]

$$= \frac{(-1)^n\, n!}{2ia}\left\{\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right\}$$

$$= \frac{(-1)^n\, n!}{2iar^{n+1}} \{(\cos\theta - i\sin\theta)^{-(n+1)} - (\cos\theta + i\sin\theta)^{-(n+1)}\}$$

$$= \frac{(-1)^n\, n!}{2iar^{n+1}} [\cos(n+1)\theta + i\sin(n+1)\theta - [\cos(n+1)\theta - i\sin(n+1)\theta]]$$

[By De Moivre's theorem]

$$= \frac{(-1)^n\, n!}{2iar^{n+1}} \,.\, 2i\sin(n+1)\theta \qquad \left[\text{Put } \frac{1}{r} = \frac{\sin\theta}{a}\right]$$

$$= \frac{(-1)^n\, n!}{a^{n+2}} \sin(n+1)\theta \sin^{n+1}\theta.$$

PROBLEMS 4.2

Find the nth derivative of (1 to 11) :

1. $\log(4x^2-1)$ *(V.T.U., 2010)*
2. $\dfrac{x+2}{x+1} + \log\dfrac{x+2}{x+1}$
3. $\sin^3 x \cos^2 x$ *(V.T.U., 2006)*
4. $\cos^9 x$ *(Mumbai, 2008)*
5. $\sinh 2x \sin 4x$ *(V.T.U., 2010 S)*
6. $e^{5x}\cos x \cos 3x$ *(Mumbai, 2007)*
7. $\dfrac{x+3}{(x-1)(x+2)}$ *(V.T.U., 2009)*
8. $\dfrac{x^2}{2x^2+7x+6}$ *(V.T.U., 2005)*
9. $\dfrac{1}{1+x+x^2+x^3}$ *(Mumbai, 2009)*
10. $\dfrac{x}{x^2+a^2}$ *(Mumbai, 2007)*
11. Find the nth derivative of $\tan^{-1}\dfrac{2x}{1-x^2}$ in terms of r and θ. *(U.P.T.U., 2002)*

4.2 LEIBNITZ'S THEOREM for the *n*th Derivative of the product of two functions*

If u, v be two function of x possessing derivatives of the nth order, then

$$(\mathbf{uv})_n = \mathbf{u}_n \mathbf{v} + {}^n\mathbf{C}_1 \mathbf{u}_{n-1} \mathbf{v}_1 + {}^n\mathbf{C}_2 \mathbf{u}_{n-2} \mathbf{v}_2 + \ldots + {}^n\mathbf{C}_r \mathbf{u}_{n-r} \mathbf{v}_r + \ldots + {}^n\mathbf{C}_n \mathbf{u}\mathbf{v}_n$$

We shall prove this theorem by mathematical induction.

Step I. By actual differentiation,

$$(uv)_1 = u_1 v + uv_1$$

$$(uv)_2 = (u_2 v + u_1 v_1) + (u_1 v_1 + uv_2)$$

$$= u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 uv_2 \qquad [\because \; 2 = {}^2C_1, 1 = {}^2C_2]$$

Thus we see that the theorem is true for $n = 1, 2$.

Step II. Assume the theorem to be true for $n = m$ (say) so that

$$(uv)_m = u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \ldots + {}^mC_{r-1} u_{m-r+1} v_{r-1} + {}^mC_r u_{m-r} v_r + \ldots + {}^mC_m uv_m$$

Differentiating both sides,

$$(uv)_{m+1} = (u_{m+1} v + u_m v_1) + {}^mC_1 (u_m v_1 + u_{m-1} v_2) + {}^mC_2 (u_{m-1} v_2 + u_{m-2} v_3) + \ldots$$
$$+ {}^mC_{r-1} u_{m-r+2} v_{r-1} + u_{m-r+1} v_r) + {}^mC_r (u_{m-r+1} v_r + u_{m-r} v_{r+1}) + \ldots$$
$$+ {}^mC_m (u_1 v_m + uv_{m+1})$$
$$= u_{m+1} v + (1 + {}^mC_1) u_m v_1 + ({}^mC_1 + {}^mC_2) u_{m-1} v_2 + \ldots$$
$$+ ({}^mC_{r-1} + {}^mC_r) u_{m-r+1} v_r + \ldots + {}^mC_m uv_{m+1}$$

But $\quad 1 + {}^mC_1 = {}^mC_0 + {}^mC_1 = {}^{m+1}C_1, {}^mC_1 + {}^mC_2 = {}^{m+1}C_2 \ldots$

$${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r, \ldots \quad \text{and} \quad {}^mC_m = 1 = {}^{m+1}C_{m+1}$$

$$\therefore \qquad (uv)_{m+1} = u_{m+1} v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + \ldots + {}^{m+1}C_r u_{m+1-r} v_r + \ldots + {}^{m+1}C_{m+1} uv_{m+1}$$

which is of exactly the same form as the given formula with n replaced by $m + 1$. Hence if the theorem is true for $n = m$, it is also true for $n = m + 1$.

Step III. In step I, the theorem has been seen to be true for $n = 2$, and by step II, it must be true for $n = 2 + 1$ *i.e.*, 3 and so for $n = 3 + 1$ *i.e.*, 4 and so on.

Hence the theorem is true for all positive integral values of n.

Example 4.9. *Find the nth derivative of $e^x (2x + 3)^3$.*

Solution. Take $u = e^x$ and $v = (2x + 3)^3$, so that $u_n = e^x$ for all integral values of n, and $v_1 = 6(2x + 3)^2$, $v_2 = 24(2x + 3)$, $v_3 = 48$, v_4, v_5 etc. are all zero.

$\therefore$ By Leibnitz's theorem,

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + {}^nC_3 u_{n-3} v_3$$

i.e.,

$$[e^x (2x + 3)^3]_n = e^x (2x + 3)^3 + ne^x[6(2x + 3)^2]$$
$$+ \frac{n(n-1)}{1.2} e^x [24(2x + 3)] + \frac{n(n-1)(n-2)}{1.2.3} e^x [48]$$
$$= e^x \{(2x + 3)^3 + 6n(2x + 3)^2 + 12n(n-1)(2x + 3) + 8n(n-1)(n-2)\}.$$

Example 4.10. *If $y = (\sin^{-1} x)^2$, show that $(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} - n^2 y_n = 0$. Hence find $(y_n)_0$.* (U.P.T.U., 2005)

Solution. We have $\qquad y = (\sin^{-1} x)^2$

Differentiating, $\qquad y_1 = \dfrac{2 \sin^{-1} x}{\sqrt{(1 - x^2)}}$ or $(1 - x^2) y_1^2 = 4(\sin^{-1} x)^2 = 4y \qquad \ldots(i)$

Again differentiating, $\qquad (1 - x^2) 2y_1 y_2 + (-2x) y_1^2 = 4y_1 \qquad \ldots(ii)$

Dividing by $2y_1$, $(1 - x^2) y_2 - xy_1 - 2 = 0$

Differentiating it n times by Leibnitz's theorem,

*Named after the German mathematician and philosopher *Gottfried Wilhelm Leibnitz* (1646–1716) who invented the differential and integral calculus independent of Sir Issac Newton.

$$(1-x^2)\,y_{n+2}+n(-2x)\,y_{n+1}+\frac{n(n-1)}{2}\,(-2)y_n-[xy_{n+1}+n(1)\,y_n]=0$$

or $$(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-n^2\,y_n=0$$

which is the required result.

Putting $x=0$, $\quad (y_{n+2})_0=n^2(y_n)_0 \quad$ ...(*iii*)

From (*i*), $\quad (y_1)_0=0$. From (*ii*), $(y_2)_0=2$.

Putting $n=1,3,5,7,\ldots$ in (*iii*), $\quad 0=y_1=y_3=y_5=y_7=\ldots$

i.e., if n is odd, $\quad (y_n)_0=0$

Again putting $n=2,4,6,\ldots$ in (*iii*)

$$(y_4)_0=2^2\,(y_2)_0=2\,.\,2^2$$
$$(y_6)_0=4^2\,(y_6)_0=2.\,2^2\,.\,4^2$$
$$(y_8)_0=6^2\,(y_6)_0=2.\,2^2\,.\,4^2\,.\,6^2$$

In general, if n is even, $\quad (y_n)_0=2\,.\,2^2\,.\,4^2\,.\,6^2\ldots(n-2)^2,\ (n\neq 2).$

Example 4.11. *If* $y=e^{a\sin^{-1}x}$, *prove that* $(1-x^2)\,y_{n+2}-(2n+1)\,x\,y_{n+1}-(n^2+a^2)y_n=0$. *Hence find the value of* y_n *when* $x=0$. (V.T.U., 2003)

Solution. We have $\quad y=e^{a\sin^{-1}x} \quad$ ...(*i*)

Differentiating, $\quad y_1=e^{a\sin^{-1}x}\,\dfrac{a}{\sqrt{(1-x^2)}}=\dfrac{ay}{\sqrt{(1-x^2)}} \quad$ ...(*ii*)

or $$(1-x^2)\,y_1^{\,2}=a^2\,y^2.$$

Again differentiating, $(1-x^2)\,2y_1y_2+(-2x)y_1^{\,2}=2a^2\,yy_1$.

Dividing by $2y_1$, $(1-x^2)y_2-xy_1-a^2y=0 \quad$ ...(*iii*)

Differentiating it n times by Leibnitz's theorem,

$$(1-x^2)\,y_{n+2}+n\,.\,(-2x)\,y_{n+1}+\frac{n(n-1)}{2}\,.\,(-2)\,y_n-[xy_{n+1}+n\,.\,1\,.\,y_n]-a^2\,y_n=0$$

or $$(1-x^2)y_{n+2}-(2n+1)\,xy_{n+1}-(n^2+a^2)\,y_n=0$$

which is the required result.

Putting $x=0$, $\quad (y_{n+2})_0=(n^2+a^2)\,(y_n)_0 \quad$ ...(*iv*)

From (*i*), (*ii*), (*iii*) : $\quad (y)_0=1,\ (y_1)_0=a,\ (y_2)_0=a^2$

Putting $n=1,2,3,4\ldots$ in (*iv*),

$$(y_3)_0=(1^2+a^2)\,(y_1)_0=a(1^2+a^2)$$
$$(y_4)_0=(2^2+a^2)\,(y_2)_0=a^2(2^2+a^2)$$
$$(y_5)_0=(3^2+a^2)\,(y_3)_0=a(1^2+a^2)\,(3^2+a^2)$$
$$(y_6)_0=(4^2+a^2)\,(y_4)_0=a^2(2^2+a^2)\,(4^2+a^2).$$

Hence in general, $\quad (y_n)_0=a(1^2+a^2)\,(3^2+a^2)\ldots[(n-2)^2+a^2],\quad$ when n is odd.

$\quad =a^2(2^2+a^2)\,(4^2+a^2)\ldots\ldots[(n-2)^2+a^2],\quad$ when n is even.

Example 4.12. *If* $y^{1/m}+y^{-1/m}=2x$, *prove that*

$$(x^2-1)y_{n+2}+(2n+1)\,xy_{n+1}+(n^2-m^2)\,y_n=0.$$

(V.T.U., 2008 S ; Mumbai, 2007 ; S.V.T.U., 2007)

Solution. We have $\quad y^{1/m}+\dfrac{1}{y^{1/m}}=2x$

or $$(y^{1/m})^2-2x(y^{1/m})+1=0$$

$$\therefore \quad y^{1/m}=\frac{2x\pm\sqrt{(4x^2-4)}}{2}=x\pm\sqrt{(x^2-1)}$$

Hence $$y=[x\pm\sqrt{(x^2-1)}\,]^m$$

Taking logarithm, $\quad \log y=m\log[x\pm\sqrt{(x^2-1)}\,]$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}y_1 = m \cdot \frac{1}{x \pm \sqrt{(x^2-1)}} \cdot \left\{1 \pm \frac{x}{\sqrt{(x^2-1)}}\right\} = \pm \frac{m}{\sqrt{(x^2-1)}}$$

Squaring, $\quad y_1^2 (x^2 - 1) = m^2 y^2$

Again differentiating, $\quad (x^2 - 1)\, 2y_1 y_2 + y_1^2 (2x) = m^2 \cdot 2y \cdot y_1$

Dividing by $2y_1$, $\quad (x^2 - 1)\, y_2 + xy_1 - m^2 y = 0$

Differentiating it n times by Leibnitz's theorem,

$$(x^2-1)\, y_{n+2} + ny_{n+1}(2x) + \frac{n(n-1)}{2} y_n (2) + xy_{n+1} + n \cdot y_n (1) - m^2 y_n = 0$$

or
$$(x^2-1)\, y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$$

PROBLEMS 4.3

1. Find the nth derivative of (*i*) $x^2 \log 3x$. (*ii*) $2^x \cos^9 x$. *(Mumbai, 2009)*
2. If $y = a \cos (\log x) + b \sin (\log x)$, show that $x^2 y_2 + xy_1 + y = 0$ and $x^2 y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)\, y_n = 0$. *(U.P.T.U., 2004 ; Madras, 2000)*
3. If $y = \sin^{-1} x$, prove that $(1 - x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - n^2 y_n = 0$. Also find (y_n). *(S.V.T.U., 2009)*
4. If $\cos^{-1} (y/b) = \log (x/n)^n$, prove that $x^2 y_{n+2} + (2n+1)\, xy_{n+1} + 2n^2 y_n = 0$. *(U.P.T.U., 2006)*
5. If $y = \tan^{-1} x$, prove that $(1 + x^2)\, y_{n+1} + 2nxy_n + n(n-1)\, y_{n-1} = 0$. Find $y_{n(0)}$.
6. If $y = \cos (m \sin^{-1} x)$, prove that $(1 - x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} + (m^2 - n^2)\, y_n = 0$. *(Mumbai, 2008 S)*
7. If $y = \sin (m \sin^{-1} x)$, prove that $(1 - x^2)\, y_2 - xy_1 + m^2 y = 0$
   and $\quad (1 - x^2)\, y_{n+2} - 2(n+1)\, xy_{n+1} + (m^2 - n^2)\, y_n = 0$. *(V.T.U., 2009 ; Cochin, 2005)*
   Also find $(y_n)_0$. *(U.P.T.U., 2005)*
8. If $y = e^{m \cos^{-1} x}$, prove that (*i*) $(1 - x^2)\, y_2 - xy_1 = m^2 y$
   (*ii*) $(1 - x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - (n^2 + m^2)\, y_n = 0$. Also find $(y_n)_0$.
9. If $y = (x^2 - 1)^n$, prove that $(x^2 - 1)\, y_{n+2} + 2xy_{n+1} - n(n+1)\, y_n = 0$. *(U.T.U., 2010)*
10. If $\sin^{-1} y = 2 \log (x + 1)$, prove that $(x + 1)^2 y_{n+2} + (2n+1)(x+1)\, y_{n+1} + (x^2 + 4)\, y_n = 0$. *(V.T.U., 2003)*
11. If $y = x^n \log x$, prove that $y_{n+1} = n\,!/x$. *(Mumbai, 2008)*
12. If $V_n = \dfrac{d^n}{dx^n} (x^n \log x)$, show that $V_n = nV_{n-1} + (n-1)\,!$

    Hence, show that $V_n = n\,!\left(\log x + 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}\right)$. *(V.T.U., 2001)*
13. Show that $\dfrac{d^n}{dx^n}\left(\dfrac{\log x}{x}\right) = \dfrac{(-1)^n\, n\,!}{x^{n+1}}\left\{\log x - 1 - \dfrac{1}{2} - \dfrac{1}{3} - \ldots - \dfrac{1}{n}\right\}$. *(V.T.U., 2006)*
14. If $y = x \log \left(\dfrac{x-1}{x+1}\right)$, show that $y_n = (-1)^{n-2} (n-2)\,! \left[\dfrac{x-n}{(x-1)^n} - \dfrac{x+n}{(x+1)^n}\right]$. *(U.P.T.U., 2003)*
15. If $x = \sin t$, $y = \cos pt$, show that $(1 - x^2)\, y_2 - xy_1 + p^2 y = 0$. Hence prove that
    $$(1 - x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - (n^2 - p^2)\, y_n = 0.$$ *(Raipur, 2005 ; V.T.U., 2005)*
16. If $y = \log [x + \sqrt{(1 + x^2)}\,]^2$, prove that $(1 + x^2)\, y_{n+2} + (2n+1)\, xy_{n+1} + n^2 y_n = 0$. *(V.T.U., 2007 ; Bhillai, 2005)*
    Hence show that $(y_{2k})_0 = (-1)^{k-1} \cdot 2^k \cdot [(k-1)\,!]^2$, where k is positive integer.
17. If $y = [x + \sqrt{(x^2 + 1)}\,]^m$, prove that (*i*) $(x^2 + 1)\, y_2 + xy_1 - m^2 y = 0$, (*ii*) $y_{n+2} + (n^2 - m^2)\, y_n = 0$ at $x = 0$. *(V.T.U., 2009 S)*
    Hence find $y_n (0)$. *(Madras, 2000)*
18. If $y = \sin \log (x^2 + 2x + 1)$, prove that (*i*) $(x + 1)^2 y_2 + (x + 1)\, y_1 + 4y = 0$
    (*ii*) $(x + 1)^2 y_{n+2} + (2n+1)(x+1)\, y_{n+1} + (n^2 + 4)\, y_n = 0$. *(U.P.T.U., 2006)*

19. If $y = \dfrac{\sinh^{-1} x}{\sqrt{(1+x^2)}}$, show that $(1 + x^2)\, y_{n+2} + (2n + 3)\, xy_{n+1} + (n + 1)^2\, y_n = 0.$ (V.T.U., 2010)

20. If $y = \sinh [m \log (x + \sqrt{x^2 + 1})]$, prove that $(x^2 + 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$ (V.T.U., 2010 S)

4.3 FUNDAMENTAL THEOREMS

(1) Rolle's Theorem

If (*i*) *f(x) is continuous in the closed interval [a, b], (ii) f ′(x) exists for every value of x in the open interval (a, b) and (iii) f (a) = f (b), then there is at least one value c of x in (a, b) such that f ′(c) = 0.*

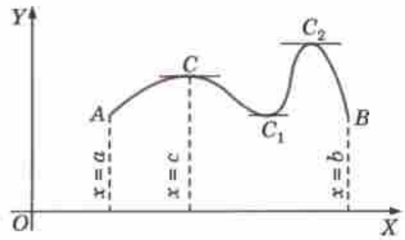

Fig. 4.1

Consider the portion AB of the curve $y = f(x)$, lying between $x = a$ and $x = b$, such that

(*i*) it goes continuously from A to B,

(*ii*) it has a tangent at every point between A and B, and

(*iii*) ordinate of A = ordinate of B.

From the Fig. 4.1, it is self-evident that there is at least one point C (may be more) of the curve at which the tangent is parallel, to the x-axis.

i.e., slope of the tangent at C $(x = c) = 0$

But the slope of the tangent at C is the value of the differential coefficient of $f(x)$ w.r.t. x thereat, therefore $f'(c) = 0$.

Hence the theorem is proved.

Example 4.13. *Verify Rolle's theorem for (i) sin x / e^x in (0, π).* (J.N.T.U., 2003)
(ii) $(x - a)^m (x - b)^n$ where m, n are positive integers in [a, b]. (V.T.U., 2010 ; Nagarjuna, 2008)

Solution. (*i*) Let $f(x) = \sin x / e^x$.

$f(x)$ is derivable in $(0, \pi)$.

Also $f(0) = f(\pi) = 0.$

Hence the conditions of Rolle's theorem are satisfied.

$\therefore$ $$f'(x) = \frac{e^x \cos x - e^x \sin x}{e^{2x}}$$ vanishes where $e^x (\cos x - \sin x) = 0$

or $\tan x = 1$ *i.e.*, $x = \pi/4$.

The value $x = \pi/4$ lies in $(0, \pi)$, so that Rolle's theorem is verified.

(*ii*) Let $f(x) = (x - a)^m (x - b)^n$.

Since every polynomial is continuous for all values, $f(x)$ is also continuous in $[a, b]$.

$$f'(x) = m(x - a)^{m-1} (x - b)^n + (x - a)^m \,.\, n(x - b)^{n-1}$$
$$= (x - a)^{m-1} (x - b)^{n-1} [(m + n)\, x - (mb + na)]$$

which exists, *i.e.*, $f(x)$ is derivable in (a, b).

Also $f(a) = 0 = f(b)$.

Thus all the conditions of Rolle's theorem are satisfied and there exists c in (a, b) such that $f'(c) = 0$.

$\therefore$ $(c - a)^{m-1} (c - b)^{n-1} [(m + n)\, c - (mb + na)] = 0$ or $c = (mb + na)/(m + n)$.

Hence, Rolle's theorem is verified.

(2) Lagrange's Mean-Value Theorem*

First form. *If (i) f(x) is continuous in the closed interval [a, b], and*

(ii) f′(x) exists in the open interval (a, b), then there is at least one value c of x in (a, b), such that

$$\mathbf{\frac{f(b) - f(a)}{b - a} = f'(c).}$$

*Named after the great French mathematician *Joseph Louis Lagrange* (1736–1813) who became professor at Military Academy, Turin when he was just 19 and director of Berlin Academy in 1766. His important contribution are to algebra, number theory, differential equations, mechanics, approximation theory and calculus of variations.

Consider the function $\phi(x) = f(x) - \dfrac{f(b)-f(a)}{b-a}\, x$

Since $f(x)$ is continuous in $[a, b]$; $\therefore$ $\phi(x)$ is also continuous in $[a, b]$.

Since $f'(x)$ exists in (a, b) ;

$$\therefore \quad \phi'(x) \text{ also exists in } (a, b) \text{ and } = f'(x) - \frac{f(b)-f(a)}{b-a} \qquad ...(i)$$

Clearly, $$\phi(a) = \frac{b\,f(a) - a\,f(b)}{b-a} = \phi(b).$$

Thus $\phi(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore$ There is at least one value c of x between a and b such that $\phi'(c) = 0$. Substituting $x = c$ in (1), we get

$$f'(c) = \frac{f(b)-f(a)}{b-a} \qquad ...(2)$$

which proves the theorem.

Second form. If we write $b = a + h$, then since $a < c < b$,

$$c = a + \theta h \text{ where } 0 < \theta < 1.$$

Thus the mean value theorem may be stated as follows :

If (i) f(x) is continuous in the closed interval [a, a + h] and (ii) f′ (x) exists in the open interval (a, a + h), then there is at least one number θ(0 < θ < 1) such that

$$\mathbf{f(a + h) = f(a) + hf'(a + \theta h)}$$

Geometrical Interpretation. Let A, B be the points on the curve $y = f(x)$ corresponding to $x = a$ and $x = b$ so that $A = [a, f(a)]$ and $B = [b, f(b)]$. (Fig. 4.2)

$\therefore$ Slope of chord $AB = \dfrac{f(b)-f(a)}{b-a}$

By (2), the slope of the chord $AB = f'(c)$, the slope of the tangent of the curve at $C(x = c)$.

Hence the Lagrange's mean value theorem asserts that *if a curve AB has a tangent at each of its points, then there exists at least one point C on this curve, the tangent at which is parallel to the chord AB.*

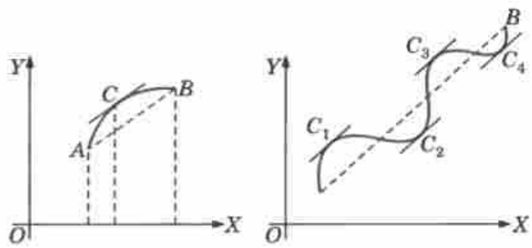

Fig. 4.2

Cor. *If f′(x) = 0 in the interval (a, b) then f (x) is constant in [a, b].* For, if x_1, x_2 be any two values of x in (a, b), then by (2),

$$f(x_2) - f(x_1) = (x_2 - x_1)\, f'(c) = 0 \; (x_1 < c < x_2)$$

Thus, $f(x_1) = f(x_2)$ *i.e.*, $f(x)$ has the same value for every value of x in (a, b).

Example 4.14. *In the Mean value theorem $f(b) - f(a) = (b - a) f'(c)$,* ...(i)
determine c lying between a and b, if $f(x) = x(x - 1)(x - 2)$, $a = 0$ and $b = 1/2$. (Gorakhpur, 1999)

Solution. $f(a) = 0$, $\qquad f(b) = \dfrac{1}{2}\left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right) = \dfrac{3}{8}$

$f'(x) = 3x^2 - 6x + 2$, $\qquad f'(c) = 3c^2 - 6c + 2$

Substituting in (*i*), $$\frac{3}{8} - 0 = \left(\frac{1}{2} - 0\right)(3c^2 - 6c + 2)$$

or $$12c^2 - 24c + 5 = 0$$

whence $$c = \frac{24 \pm \sqrt{(24)^2 - 12 \times 5 \times 4}}{24} = 1 \pm 0.764 = 1.764 \; ; 0.236.$$

Hence $c = 0.236$, since it only lies between 0 and 1/2.

Example 4.15. *Prove that (if $0 < a < b < 1$), $\dfrac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b-a}{1+a^2}$.*

Hence show that $\dfrac{\pi}{4} + \dfrac{3}{25} < \tan^{-1}\dfrac{4}{3} < \dfrac{\pi}{4} + \dfrac{1}{6}$. (Mumbai, 2009 ; V.T.U., 2006)

Solution. Let $f(x) = \tan^{-1} x$, so that $f'(x) = \dfrac{1}{1+x^2}$.

By Mean value theorem, $\dfrac{\tan^{-1} b - \tan^{-1} a}{b-a} = \dfrac{1}{1+c^2}, a < c < b$ ...(*i*)

Now $a < c < b$, $\therefore$ $1 + a^2 < 1 + c^2 < 1 + b^2$.

$$\therefore \quad \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2} \text{ i.e., } \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

i.e.,
$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2} \quad \text{[By (i)]}$$

Hence
$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

Now let $a = 1, b = 4/3$.

Then
$$\frac{1/3}{1+16/9} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1/3}{1+1}$$

i.e.,
$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

Example 4.16. *Prove that* $\log(1+x) = x/(1+\theta x)$, *where* $0 < \theta < 1$ *and hence deduce that*

$\dfrac{x}{1+x} < \log(1+x) < x, x > a$ *(Mumbai, 2008)*

Solution. Let $f(x) = \log(1+x)$, then by second form of Lagrange's mean value theorem

$$f(a+h) = f(a) + h f'(a+\theta h), \quad (0 < \theta < 1)$$

we have
$$f(x) = f(0) + x f'(\theta x) \quad [\text{Taking } a = 0, h = x]$$

or
$$\log(1+x) = \log(1) + x \,.\, 1/(1+\theta x) \quad [\because \ f'(x) = 1/1+x)$$

Hence
$$\log(1+x) = x/(1+\theta x) \quad \text{...(i)} \ [\because \ \log(1) = 0$$

Since $0 < \theta < 1$, $\therefore$ $0 < \theta x < x$ for $x > 0$.

or
$$1 < 1 + \theta x < 1 + x \quad \text{or} \quad 1 > \frac{1}{1+\theta x} > \frac{1}{1+x}$$

or
$$x > \frac{x}{1+\theta x} > \frac{x}{1+x}$$

or
$$\frac{x}{1+x} < \log(1+x) < x, x > 0. \quad \text{[By (i)]}$$

(3) Cauchy's Mean-value Theorem*

If (i) $f(x)$ and $g(x)$ *be continuous in* $[a, b]$

(*ii*) $f'(x)$ and $g'(x)$ exist in (a, b)

and (*iii*) $g'(x) \neq 0$ *for any value of* x *in* (a, b),

then there is at least one value c of x in (a, b), such that $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{f'(c)}$

Consider the function $\phi(x) = f(x) - \dfrac{f(b)-f(a)}{g(b)-g(a)}\, g(x)$

Since $f(x)$ and $g(x)$ are continuous in $[a, b]$

$\therefore$ $\phi(x)$ is also continuous in $[a, b]$.

Again since $f'(x)$ and $g'(x)$ exist in (a, b).

*Named after the great French mathematician *Augustin-Louis Cauchy* (1789–1857) who is considered as the father of modern analysis and creator of complex analysis. He published nearly 800 research papers of basic importance. Cauchy is also well known for his contributions to differential equations, infinite series, optics and elasticity.

$\therefore$ $\phi'(x)$ also exists in (a, b) and $= f'(x) - \dfrac{f(b)-f(a)}{g(b)-g(a)} g'(x)$

Clearly, $\phi(a) = \phi(b)$.

Thus, $\phi(x)$ satisfies all the conditions of Rolle's theorem. There is therefore, at least one value c of x between a and b, such that $\phi'(c) = 0$

i.e.,
$$0 = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) \text{ whence follows the result.}$$

(*P.T.U., 2007 S ; V.T.U., 2006*)

Obs. Cauchy's mean value theorem is a generalisation of Lagrange's mean value theorem, where $g(x) = x$.

Example 4.17. *Verify Cauchy's Mean-value theorem for the functions e^x and e^{-x} in the interval (a, b).*

Solution. $f(x) = e^x$ and $g(x) = e^{-x}$ are both continuous in $[a, b]$ and both functions are differentiable in (a, b).

$\therefore$
$$f'(x) = e^x, g'(x) = -e^{-x}$$

By Cauchy's mean value theorem.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$\therefore$
$$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}} \quad i.e., c = \frac{1}{2}(a+b)$$

Thus c lies in (a, b) which verifies the Cauchy's Mean value theorem.

(4) Taylor's Theorem* (Generalised mean value theorem)

If (i) $f(x)$ and its first $(n-1)$ derivatives be continuous in $[a, a+h]$, and (ii) $f^n(x)$ exists for every value of x in $(a, a+h)$, then there is at least one number θ $(0 < \theta < 1)$, such that

$$\mathbf{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^n}{n!} f^n(a+\theta h)} \quad ...(1)$$

which is called Taylor's theorem with Lagrange's form remainder, the remainder R_n being $\dfrac{h^n}{n!} f^n(a+\theta h)$.

Proof. Consider the function

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + ... + \frac{(a+h-x)^n}{n!} K$$

where K is defined by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^n}{n!} K \quad ...(2)$$

(*i*) Since $f(x), f'(x), ..., f^{n-1}(x)$ are continuous in $[a, a+h]$, therefore $\phi(x)$ is also continuous in $[a, a+h]$,

(*ii*) $\phi'(x)$ exists and $= \dfrac{(a+h-x)^{n-1}}{(n-1)!} [f^n(x) - K]$

(*iii*) Also $\phi(a) = \phi(a+h)$. [By (2)]

Hence $\phi(x)$ satisfies all the conditions of Rolle's theorem, and therefore, there exists at least one number $\theta (0 < \theta < 1)$, such that $\phi'(a+\theta h) = 0$ *i.e.*, $K = f^n(a+\theta h)$ $(0 < \theta < 1)$

Substituting this value of K in (2), we get (1).

Cor. 1. Taking $n = 1$ in (1), *Taylor's theorem reduces to Lagrange's Mean-value theorem.*

Cor. 2. Putting $a = 0$ and $h = x$ in (1), we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^n}{n!} f^n(\theta x). \quad ...(3)$$

which is known as *Maclaurin's theorem with Lagrange's form of remainder.*

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*Named after an English mathematician, *Brooke Taylor* (1685–1731).

Example 4.18. *Find the Maclaurin's theorem with Lagrange's form of remainder for $f(x) = \cos x$.* (J.N.T.U., 2003)

Solution. $f^n(x) = \dfrac{d^n}{dx^n}(\cos x) = \cos\left(\dfrac{n\pi}{2} + x\right)$ so that $f^n_{(0)} = \cos(n\pi/2)$

Thus $f(0) = 1$,

$$f^{2n}(0) = \cos(2n\pi/2) = (-1)^n$$
$$f^{2n+1}(0) = \cos[(2n+1)\pi/2] = 0$$

Substituting these values in the Maclaurin's theorem with Lagrange's form of remainder *i.e.*,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^{2n}}{(2n)!}f^{2n}(0) + \frac{x^{2n+1}}{(2n+1)!}f^{2n+1}(\theta x)$$

We get $$\cos x = 1 + 0 + \frac{x^2}{2!}(-1) + 0 + ... + \frac{x^{2n}}{(2n)!}(-1)^n + \frac{x^{2n+1}}{(2n+1)!}(-1)^n(-1)\cos(\theta x)$$

i.e., $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^n\frac{x^{2n}}{(2n)!} + \frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!}\cos(\theta x)$$

Example 4.19. *If $f(x) = \log(1+x)$, $x > 0$, using Maclaurin's theorem, show that for $0 < \theta < 1$,*

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}.$$

Deduce that $\log(1+x) < x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$ for $x > 0$. (J.N.T.U., 2005)

Solution. By Maclaurin's theorem with remainder R_3, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(\theta x) \quad ...(i)$$

Here $f(x) = \log(1+x)$, $\quad f(0) = 0$

$\therefore$ $f'(x) = \dfrac{1}{1+x}$, $\quad f'(0) = 1$

$f''(x) = \dfrac{-1}{(1+x)^2}$, $\quad f''(0) = -1$

and $f'''(x) = \dfrac{2}{(1+x)^3}$, $\quad f'''(\theta x) = \dfrac{2}{(1+\theta x)^3}$

Substituting in (*i*), we get $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3(1+\theta x)^3}$ ...(*ii*)

Since $x > 0$ and $\theta > 0$, $\theta x > 0$

or $(1+\theta x)^3 > 1$ *i.e.,* $\dfrac{1}{(1+\theta x)^3} < 1$

$$\therefore \quad x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} < x - \frac{x^2}{2} + \frac{x^3}{3}$$

Hence $\log(1+x) < x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$ [By (*ii*)]

PROBLEMS 4.4

1. Verify Rolle's theorem for (*i*) $f(x) = (x+2)^3(x-3)^4$ in $(-2, 3)$.
   (*ii*) $y = e^x(\sin x - \cos x)$ in $(\pi/4, 5\pi/4)$. (*iii*) $f(x) = x(x+3)e^{-1/2x}$ in $(-3, 0)$.
   (*iv*) $f(x) = \log\left\{\dfrac{x^2+ab}{x(a+b)}\right\}$ in (a, b). (V.T.U., 2005)

2. Using Rolle's theorem for $f(x) = x^{2m-1}(a-x)^{2n}$, find the value of x between o and a where $f'(x) = 0$.
3. Verify Lagrange's Mean value theorem for the following functions and find the appropriate value of c in each case :
   (*i*) $f(x) = (x-1)(x-2)(x-3)$ in $(0, 4)$ (*V.T.U., 2009*)
   (*ii*) $f(x) = \sin x$ in $[0, \pi]$ (*Nagpur, 2008*)
   (*iii*) $f(x) = \log_e x$ in $[1, e]$. (*Burdwan, 2003*)
   (*iv*) $f(x) = e^x$ in $[0, 1]$. (*V.T.U., 2007*)
4. By applying Mean value theorem to

$$f(x) = \log 2 \,.\, \sin \frac{\pi x}{2} + \log x, \text{ prove that } \frac{\pi}{2} \log 2 \,.\, \cos \frac{\pi x}{2} + \frac{1}{x} = 0 \text{ for some } x \text{ between 1 and 2.}$$

5. In the Mean value theorem : $f(x+h) = f(x) + h f'(x + \theta h)$,
   show that $\theta = 1/2$ for $f(x) = ax^2 + bx + c$ in $(0, 1)$.
6. If $f(h) = f(0) + h f'(0) + \dfrac{h^2}{2!} f''(\theta h)$, $0 < \theta < 1$, find θ when $h = 1$ and $f(x) = (1-x)^{5/2}$.
7. If x is positive, show that $x > \log(1+x) > x - \dfrac{1}{2}x^2$. (*V.T.U., 2000*)
8. If $f(x) = \sin^{-1} x$, $0 < a < b < 1$, use Mean value theorem to prove that

$$\frac{b-a}{\sqrt{(1-a^2)}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{(1-b^2)}}$$

9. Prove that $\dfrac{b-a}{b} < \log\left(\dfrac{b}{a}\right) < \dfrac{b-a}{b}$ for $0 < a < b$.

   Hence show that $\dfrac{1}{4} < \log \dfrac{4}{3} < \dfrac{1}{3}$. (*Mumbai, 2008*)
10. Verify the result of Cauchy's mean value theorem for the functions
    (*i*) $\sin x$ and $\cos x$ in the interval $[a, b]$. (*J.N.T.U., 2006 S*)
    (*ii*) $\log_e x$ and $1/x$ in the interval $[1, e]$.
11. If $f(x)$ and $g(x)$ are respectively e^x and e^{-x}, prove that 'c' of Cauchy's mean value theorem is the arithmetic mean between a and b. (*Mumbai, 2008*)
12. Verify Maclaurin's theorem $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder upto 3 terms where $x = 1$.
13. Using Taylor's theorem, prove that

$$x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}, \quad \text{for } x > 0.$$

4.4 EXPANSIONS OF FUNCTIONS

(1) Maclaurin's series. *If $f(x)$ can be expanded as an infinite series, then*

$$\mathbf{f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots \infty} \qquad \ldots(1)$$

If $f(x)$ possess derivatives of all orders and the remainder R_n in (3) on page 145 tends to zero as $n \to \infty$, then the Maclaurin's theorem becomes the Maclaurin's series (1).

Example 4.20. *Using Maclaurin's series, expand tan x upto the term containing x^5.* (V.T.U., 2006)

Solution. Let $f(x) = \tan x$ $\qquad f(0) = 0$

$\therefore$ $f'(x) = \sec^2 x = 1 + \tan^2 x$ $\qquad f'(0) = 1$

$$f''(x) = 2 \tan x \sec^2 x = 2 \tan x (1 + \tan^2 x)$$
$$= 2 \tan x + 2 \tan^3 x \qquad f''(0) = 0$$
$$f'''(0) = 2 \sec^2 x + 6 \tan^2 x \sec^2 x$$
$$= 2(1 + \tan^2 x) + 6 \tan^2 x (1 + \tan^2 x)$$
$$= 2 + 8 \tan^2 x + 6 \tan^4 x \qquad f'''(0) = 2$$
$$f^{iv}(0) = 16 \tan x \sec^2 x + 24 \tan^3 x \sec^2 x$$

$$= 16 \tan x\,(1 + \tan^2 x) + 24 \tan^3 x\,(1 + \tan^2 x)$$

$$= 16 \tan x + 40 \tan^3 x + 24 \tan^5 x \qquad f^{iv}(0) = 0$$

$$f^{v}(0) = 16 \sec^2 x + 120 \tan^2 x \sec^2 x + 120 \tan^4 x \sec^2 x. \qquad f^{v}(0) = 16$$

and so on.

Substituting the values of $f(0)$, $f'(0)$, etc. in the Maclaurin's series, we get

$$\tan x = 0 + x\,.\,1 + \frac{x^2}{2!} + \frac{x^3}{3!}\,.\,2 + \frac{x^4}{4!}\,.\,0 + \frac{x^5}{5!}\,.\,16 + \ldots = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

(2) Expansion by use of known series. When the expansion of a function is required only upto first few terms, it is often convenient to employ the following well-known series :

1. $\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$
2. $\sinh \theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \ldots$
3. $\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$
4. $\cosh \theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \ldots$
5. $\tan \theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \ldots$
6. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$
7. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$
8. $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$
9. $\log(1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\right)$
10. $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

Example 4.21. *Expand $e^{\sin x}$ by Maclaurin's series or otherwise upto the term containing x^4.*

(Bhopal, 2009; V.T.U., 2011)

Solution. We have $e^{\sin x} = 1 + \sin x + \frac{(\sin x)^2}{2!} + \frac{(\sin x)^3}{3!} + \frac{(\sin x)^4}{4!} + \ldots$

$$= 1 + \left(x - \frac{x^3}{3!} + \ldots\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!} + \ldots\right)^2 + \frac{1}{3!}\left(x - \frac{x^3}{3!} + \ldots\right)^3 + \frac{1}{4!}(x - \ldots)^4 + \ldots$$

$$= 1 + \left(x - \frac{x^3}{6} + \ldots\right) + \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \ldots\right) + \frac{1}{6}\left(x^3 - \ldots\right) + \frac{1}{24}(x^4 + \ldots) + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots$$

Otherwise, let $f(x) = e^{\sin x}$ $\qquad f(0) = 1$

$\therefore$ $f'(x) = e^{\sin x} \cos x = f(x)\,.\cos x$ $\qquad f'(0) = 1$

$f''(x) = f'(x) \cos x - f(x) \sin x,$ $\qquad f''(0) = 1$

$f'''(x) = f''(x) \cos x - 2f'(x) \sin x - f(x) \cos x,$ $\qquad f'''(0) = 0$

$f^{iv}(x) = f'''(x) \cos x - 3f''(x) \sin x - 3f'(x) \cos x + f(x) \sin x,$ $\qquad f^{iv}(0) = -3$

and so on.

Substituting the values of $f(0)$, $f'(0)$ etc., in the Maclaurin's series, we obtain

$$e^{\sin x} = 1 + x\,.\,1 + \frac{x^2}{2!}\,.\,1 + \frac{x^3}{3!}\,.\,0 + \frac{x^4}{4!}\,.\,(-3) + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots$$

Example 4.22. *Expand log $(1 + \sin^2 x)$ in powers of x as far as the term in x^6.* (*Hissar, 2005 S*)

Solution. We have $\sin^2 x = \left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots\right)^2 = \left[x - \left(\dfrac{x^3}{6} - \dfrac{x^5}{120} + \ldots\right)\right]^2$

$$= x^2 - 2x\left(\frac{x^3}{6} - \frac{x^5}{120} + \ldots\right) + \left(\frac{x^3}{6} - \frac{x^5}{120} + \ldots\right)^2$$

$$= x^2 - \frac{x^4}{3} + \frac{x^6}{60} + \frac{x^6}{36} + \ldots = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \ldots = t, \text{ say.}$$

Now $\quad \log(1 + \sin^2 x) = \log(1 + t) = t - \dfrac{t^2}{2} + \dfrac{t^3}{3} - \dfrac{t^4}{4} + \ldots$

Substituting the value of t, we get

$$\log(1 + \sin^2 x) = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \ldots - \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \ldots\right)^2 - \frac{1}{3}(x^2 - \ldots)^3 - \ldots$$

$$= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{1}{2}\left(x^4 - \frac{2x^6}{3} + \ldots\right) + \frac{1}{3}(x^6 + \ldots) + \ldots$$

$$= x^2 - \frac{5}{6}x^4 + \frac{32}{45}x^6 + \ldots$$

Obs. As it is very cumbersome to find the successive derivatives of $\log(1 + \sin^2 x)$, therefore the above method is preferable to Maclaurin's series method.

Example 4.23. *Expand $e^{a \sin^{-1} x}$ in ascending powers of x.*

Solution. Let $\quad y = e^{a\sin^{-1} x}$. In Ex. 4.9, we have shown that

$(y)_0 = 1, (y_1)_0 = a, (y_2)_0 = a^2, (y_3)_0 = a(1 + a^2), (y_4)_0 = a^2(2^2 + a^2)$

and so on.

Substituting these values in the Maclaurin's series

$$y = (y)_0 + \frac{(y_1)_0}{1!}x + \frac{(y_2)_0}{2!}x^2 + \frac{(y_3)_0}{3!}x^3 + \frac{(y_4)_0}{4!}x^4 + \ldots$$

we get $\quad e^{a\sin^{-1} x} = 1 + ax + \dfrac{a^2}{2!}x^2 + \dfrac{a(1^2 + a^2)}{3!}x^3 + \dfrac{a^2(2^2 + a^2)}{4!}x^4 + \ldots$

(3) Taylor's series. *If $f(x + h)$ can be expanded as an infinite series, then*

$$f(x + h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots \infty \qquad \ldots(1)$$

If $f(x)$ possesses derivatives of all orders and the remainder R_n in (1) on page 147, tends to zero as $n \to \infty$, then the Taylor's theorem becomes the *Taylor's series* (1).

Cor. Replacing x by a and h by $(x - a)$ in (1), we get

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x - a)^2}{2!}f''(a) + \frac{(x - a)^3}{3!}f'''(a) + \ldots \infty$$

Taking $a = 0$, we get *Maclaurin's series.*

Example 4.24. *Expand $\log_e x$ in powers of $(x - 1)$ and hence evaluate $\log_e 1{\cdot}1$ correct to 4 decimal places.* (*Bhopal, 2007 ; Kurukshetra 2006*)

Solution. Let $\quad f(x) = \log_e x \qquad f(1) = 0$

$\therefore \quad f'(x) = \dfrac{1}{x}, \qquad f'(1) = 1$

$\quad f''(x) = -\dfrac{1}{x^2}, \qquad f''(1) = -1$

$$f'''(x) = \frac{2}{x^3}, \qquad f'''(1) = 2$$

$$f^{iv}(x) = -\frac{6}{x^4}, \qquad f^{iv}(0) = -6$$

etc. etc.

Substituting these values in the Taylor's series

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \ldots,$$

we get
$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

Now putting $x = 1.1$, so that $x - 1 = 0.1$, we have

$$\log(1.1) = 1.1 - \tfrac{1}{2}(0.1)^2 + \tfrac{1}{3}(0.1)^3 - \tfrac{1}{4}(0.1)^4 + \ldots$$
$$= 0.1 - 0.005 + 0.0003 - 0.00002 + \ldots = 0.0953.$$

Example 4.25. *Use Taylor's series, to prove that*

$$\tan^{-1}(x+h) = \tan^{-1}x + (h\sin z)\cdot\frac{\sin z}{1} - (h\sin z)^2\cdot\frac{\sin 2z}{2} + (h\sin z)^3\cdot\frac{\sin 3z}{3} - \ldots$$

where $z = \cot^{-1}x$. *(Bhillai, 2005)*

Solution. We have $\cot z = x$ ...(*i*)

$\therefore$ $-\operatorname{cosec}^2 z.\, dz/dx = 1$ or $dz/dx = -\sin^2 z$ ...(*ii*)

Now let $f(x+h) = \tan^{-1}(x+h)$, so that $f(x) = \tan^{-1}x$

$$\therefore \quad f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2 z} = \sin^2 z \qquad \text{[By } (i)\text{]}$$

$$f''(x) = 2\sin z\cos z\,\frac{dz}{dx} = \sin 2z\cdot(-\sin^2 z) \qquad \text{[By } (ii)\text{]}$$

$$f'''(x) = -[2\cos 2z\cdot\sin^2 z + \sin 2z\cdot 2\sin z\cos z]\,\frac{dz}{dx}$$
$$= -2\sin z[\sin z\cos 2z + \sin 2z\cos z](-\sin^2 z) = 2\sin^3 z\sin 3z$$

and so on.

Substituting these values in the Taylor's series

$$f(x+h) = f(x) + hf(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots,$$

we get the required result.

PROBLEMS 4.5

Using Maclaurin's series, expand the following functions :

1. $\log(1+x)$. Hence deduce that $\log\sqrt{\dfrac{1+x}{1-x}} = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots$
2. $\sin x$ *(P.T.U., 2005)*
3. $\sqrt{(1+\sin 2x)}$ *(V.T.U., 2010)*
4. $\sin^{-1}x$ *(Mumbai, 2007)*
5. $\tan^{-1}x$
6. $\log\sec x$ *(Mumbai, 2009 S ; V.T.U., 2009)*

Prove that :

7. $\sec x = 1 + \dfrac{x^2}{2!} + \dfrac{5x^4}{4!} + \ldots$
8. $x\operatorname{cosec} x = 1 + \dfrac{x^2}{6} + \dfrac{7x^4}{360} + \ldots$ *(Mumbai, 2007)*
9. $\sin^{-1}\dfrac{2x}{1+x^2} = 2\left\{x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots\right\}$
10. $\tan^{-1}\dfrac{\sqrt{(1+x^2)}-1}{x} = \dfrac{1}{2}\left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots\right)$

11. $\sin^{-1}(3x-4x^3) = 3\left(x+\frac{x^3}{3}+\frac{3x^5}{40}+\ldots\right)$ 12. $e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4!} \ldots$ (*Raipur, 2005*)

13. $e^{x\sin x} = 1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120} + \ldots$ (*Kurukshetra, 2009*)

14. $e^{\cos^{-1}x} = e^{\pi/2}\left(1 - x + \frac{x^2}{3} - \frac{x^3}{3} + \ldots\right)$ (*Mumbai, 2008*) 15. $\log \frac{\sin x}{x} = -\left(\frac{x^2}{6} + \frac{x^4}{180} + \frac{x^6}{2835} + \ldots\right)$

16. $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \ldots$ (*S.V.T.U. 2009 ; J.N.T.U., 2006 S*)

17. $\sqrt{(1+\sin x)} = 1 + \frac{x}{2} - \frac{x^2}{2} - \frac{x^3}{48} + \frac{x^4}{384} + \ldots$ (*V.T.U., 2006*)

18. $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \ldots$ (*Bhopal, 2008*)

19. $\frac{e^x}{e^x+1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \ldots$ (*Bhopal, 2008 S*) 20. $\frac{x}{2}\left(\frac{e^x+1}{e^x-1}\right) = 1 + \frac{1}{6}\cdot\frac{x^2}{2!} - \frac{1}{30}\cdot\frac{x^4}{4!} + \ldots$ (*Mumbai, 2007*)

21. $\sin x \cosh x = x + \frac{x^3}{3} - \frac{x^5}{30} + \ldots$

By forming a differential equation, show that

22. $(\sin^{-1}x)^2 = 2\frac{x^2}{2!} + 2\cdot2^2\frac{x^4}{4!} + 2\cdot2^2\cdot4^2\cdot\frac{x^6}{6!} + \ldots$

23. $\log[1+\sqrt{(1+x^2)}] = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\frac{x^5}{5} - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^7}{7} + \ldots$

24. If $y = \sin(m\sin^{-1}x)$, show that $(1-x^2)y_2 - xy_1 + m^2y = 0$
Hence expand $\sin m\theta$ in powers of $\sin\theta$. (*S.V.T.U., 2008*)

25. Using Taylor's theorem, express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of $(x-1)$ (*Burdwan, 2003*)

26. Expand (*i*) e^x (*Cochin., 2005*) (*ii*) $\tan^{-1}x$, in powers of $(x-1)$ upto four terms.

27. Expand $\sin x$ in powers of $(x-\pi/2)$. Hence find the value of $\sin 91°$ correct to 4 decimal places. (*Rohtak, 2003*)

28. Prove that $\log \sin x = \log \sin a + (x-a)\cot a - \frac{1}{2}(x-a)^2 \operatorname{cosec}^2 a + \ldots$

29. Find the Taylor's series expansion for $\log \cos x$ about the point $\pi/3$.

30. Compute to four decimal places, the value of $\cos 32°$, by the use of Taylor's series. (*Kurukshetra, 2006*)

31. Calculate approximately (*i*) $\log_{10} 404$, given $\log 4 = 0.6021$. (*Rohtak, 2005 S*)
(*ii*) $(1.04)^{3.01}$ (*Mumbai, 2007*)

4.5 INDETERMINATE FORMS

In general $\underset{x\to a}{\text{Lt}}[f(x)/\phi(x)] = \underset{x\to a}{\text{Lt}} f(x)/\underset{x\to a}{\text{Lt}}\phi(x)$. But when $\underset{x\to a}{\text{Lt}} f(x)$ and $\underset{x\to a}{\text{Lt}}\phi(x)$ are both zero, then the quotient reduces to the indeterminate form 0/0. This does not imply that $\underset{x\to a}{\text{Lt}}[f(x)/\phi(x)]$ is meaningless or it does not exist. In fact, in many cases, it has a finite value. We shall now, study the methods of evaluating the limits in such and similar other cases :

(1) Form 0/0. *If $f(a) = \phi(a) = 0$, then*

$$\underset{x\to a}{\text{Lt}}\frac{f(x)}{\phi(x)} = \underset{x\to a}{\text{Lt}}\frac{f'(x)}{\phi'(x)}$$

By Taylor's series,

$$\underset{x\to a}{\text{Lt}}\frac{f(x)}{\phi(x)} = \underset{x\to a}{\text{Lt}}\frac{f(a)+(x-a)f'(a)+\frac{1}{2!}(x-a)^2f''(a)+\ldots}{\phi(a)+(x-a)\phi'(a)+\frac{1}{2!}(x-a)^2\phi''(a)+\ldots}$$

$$= \underset{x \to a}{\text{Lt}} \frac{f'(a) + \frac{1}{2}(x-a) f'(a) + \dots}{\phi'(a) + \frac{1}{2}(x-a)\phi'(a) + \dots}$$

$$= \frac{f'(a)}{\phi'(a)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{\phi'(x)} \qquad \dots(1)$$

This is known as *L'Hospital's rule.*

In general, if

$$f(a) = f'(a) = f''(a) = \dots = f^{n-1}(a) = 0, \text{ but } f^n(a) \neq 0,$$

and
$$\phi(a) = \phi'(a) = \phi''(a) = \dots = \phi^{n-1}(a) = 0, \text{ but } \phi^n(a) \neq 0,$$

then from (1),

$$\underset{x \to a}{\textbf{Lt}} \frac{\mathbf{f(x)}}{\boldsymbol{\phi}\mathbf{(x)}} = \frac{\mathbf{f^n(a)}}{\boldsymbol{\phi}^\mathbf{n}\mathbf{(a)}} = \underset{x \to a}{\textbf{Lt}} \frac{\mathbf{f^n(x)}}{\boldsymbol{\phi}^\mathbf{n}\mathbf{(x)}}$$

[Rule to evaluate Lt $[f(x)/\phi(x)]$ in 0/0 form :

Differentiating the numerator and denominator separately as many times as would be necessary to arrive a determinate form].

Example 4.26. *Evaluate (i)* $\underset{x \to 0}{Lt} \frac{xe^x - \log(1+x)}{x^2}$. *(V.T.U., 2004 ; Osmania, 2000 S)*

(ii) $\underset{x \to 1}{Lt} \frac{x^x - x}{x - 1 - \log x}$

Solution. *(i)*

$$\underset{x \to 0}{\text{Lt}} \frac{xe^x - \log(1+x)}{x^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(xe^x + e^x \cdot 1) - 1/(1+x)}{2x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{xe^x + e^x + e^x + 1/(1+x)^2}{2} = \frac{0+1+1+1}{2} = 1\frac{1}{2}.$$

(ii)

$$\underset{x \to 1}{\text{Lt}} \frac{x^x - x}{x - 1 - \log x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 1}{\text{Lt}} \frac{d(x^x)/dx - 1}{1 - 0 - 1/x}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{x^x(1 + \log x) - 1}{1 - 1/x} \qquad \left(\text{form } \frac{0}{0}\right)$$

Let $y = x^x$ so that

$\log y = x \log x$

$\therefore \quad \frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + 1 \,.\, \log x$

or $\quad \frac{d}{dx}(x^x) = x^x(1 + \log x) \qquad \dots(i)$

$$= \underset{x \to 1}{\text{Lt}} \frac{d(x^x)/dx \,.\, (1 + \log x) + x^x(1/x) - 0}{1/x^2}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{x^x(1 + \log x)^2 + x^x(1/x)}{x^{-2}} \qquad [\text{By } (i)]$$

$$= \frac{1(1+0)^2 + 1 \cdot 1}{1} = 2.$$

Example 4.27. *Find the values of a and b such that* $\underset{x \to 0}{Lt} \frac{x(a + b\cos x) - c\sin x}{x^5} = 1.$ *(Mumbai., 2007)*

Solution. $$\underset{x\to 0}{\text{Lt}} \frac{x(a+b\cos x)-c\sin x}{x^5} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{a+b\cos x - bx\sin x - c\cos x}{5x^4} \qquad ...(i)$$

As the denominator is 0 for $x = 0$, (i) will tend to a finite limit if and only if the numerator also becomes 0 for $x = 0$. This requires $a + b - c = 0$ ...(ii)

With this condition, (i) assumes the form 0/0.

$$\therefore \qquad (i) = \underset{x\to 0}{\text{Lt}} \frac{-b\sin x - b(\sin x + x\cos x) + c\sin x}{20x^3}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{(c-2b)\sin x - bx\cos x}{20x^3} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{(c-2b)\cos x - b(\cos x - x\sin x)}{60x^2} \qquad ...(iii)$$

$$= \frac{c-2b-b}{0} = \frac{c-3b}{0} = 1 \qquad \text{(Given)}$$

$$\therefore \qquad c - 3b = 0 \quad i.e., \quad c = 3b.$$

Now (iii) $$= \underset{x\to 0}{\text{Lt}} \frac{b\cos x - b\cos x + bx\sin x}{60x^2}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{b\sin x}{60x} = \frac{b}{60} \underset{x\to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right) = \frac{b}{60} = 1.$$

i.e., $b = 60$, and $\therefore \quad c = 180$.

Form (ii), $a = 120$.

(2) Form ∞/∞. It can be shown that L'Hospital's rule can also be applied to this case by differentiating the numerator and denominator separately as many times as would be necessary.

Example 4.28. *Evaluate* $\underset{x\to 0}{Lt} \dfrac{\log x}{\cot x}$.

Solution. $$\underset{x\to 0}{\text{Lt}} \frac{\log x}{\cot x} = \underset{x\to 0}{\text{Lt}} \frac{1/x}{-\text{cosec}^2 x} = -\underset{x\to 0}{\text{Lt}} \frac{\sin^2 x}{x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= -\underset{x\to 0}{\text{Lt}} \frac{2\sin x\cos x}{1} = 0$$

Obs. Use of known series and standard limits. In many cases, it would be found more convenient to use expansions of known functions and standard limits for evaluating the indeterminate forms. For this purpose, remember the series of § 4.4 (2) and the following limits :

$$\underset{x\to 0}{\text{Lt}} \frac{\sin x}{x} = 1, \qquad \underset{x\to 0}{\text{Lt}} (1+x)^{1/x} = e$$

Example 4.29. *Evaluate* $\underset{x\to 0}{Lt} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$.

Solution. Using the expansions of e^x, $\sin x$ and $\log(1-x)$, we get

$$\underset{x\to 0}{\text{Lt}} \frac{e^x\sin x - x - x^2}{x^2 + x\log(1-x)}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...\right)\left(x - \frac{1}{3!}x^3 + ...\right) - x - x^2}{x^2 + x\left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - ...\right)}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\left(x + x^2 + \frac{1}{3}x^3 - 0 \cdot x^4 + \ldots\right) - x - x^2}{x^2 - \left(x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + \ldots\right)} = \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{3}x^3 - 0 \cdot x^4 + \ldots}{-\frac{1}{2}x^3 - \frac{1}{3}x^4 - \ldots} = \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{3} + \ldots}{-\frac{1}{2} - \frac{1}{3}x - \ldots} = -\frac{2}{3}.$$

Example 4.30. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{(1+x)^{1/x} - e}{x}$.

Solution. Let $y = (1 + x)^{1/x}$

$$\therefore \quad \log y = \frac{1}{x} \log (1 + x) = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right) = 1 - \frac{x}{2} + \frac{x^2}{3} - \ldots$$

or

$$y = e^{1 - \frac{1}{2}x + \frac{1}{3}x^2 - \ldots} = e \cdot e^{-\frac{1}{2}x + \frac{1}{3}x^2 - \ldots}$$

$$= e\left[1 + \left(-\frac{1}{2}x + \frac{1}{3}x^2 - \ldots\right) + \frac{1}{2!}\left(-\frac{1}{2}x + \frac{1}{3}x^2 - \ldots\right)^2 + \ldots\right] = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 + \ldots\right)$$

$$\therefore \quad \underset{x \to 0}{\text{Lt}} \frac{(1+x)^{1/x} - e}{x} = \underset{x \to 0}{\text{Lt}} \frac{e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 + \ldots\right) - e}{x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e\left(-\frac{1}{2}x + \frac{11}{24}x^2 + \ldots\right)}{x} = \underset{x \to 0}{\text{Lt}} \left(\frac{-e}{2} + \frac{11}{24}ex + \ldots\right) = -\frac{e}{2}.$$

PROBLEMS 4.6

Evaluate the following limits :

1. $\underset{x \to 0}{\text{Lt}} \dfrac{a^x - b^x}{x}$ *(V.T.U., 2008)*
2. $\underset{x \to 0}{\text{Lt}} \dfrac{x \cos x - \sin x}{x^2 \sin x}$ *(J.N.T.U., 2006 S)*
3. $\underset{\theta \to 0}{\text{Lt}} \dfrac{\theta - \sin \theta}{\sin \theta (1 - \cos \theta)}$
4. $\underset{x \to \pi/2}{\text{Lt}} \dfrac{a^{\sin x} - a}{\log_e \sin x}$
5. $\underset{x \to 0}{\text{Lt}} \dfrac{2 \sin x - \sin 2x}{x^3}$
6. $\underset{x \to 0}{\text{Lt}} \dfrac{\sin x \sin^{-1} x - x^2}{x^6}$
7. $\underset{x \to 0}{\text{Lt}} \dfrac{x \cos x - \log (1 + x)}{x^2}$
8. $\underset{x \to 0}{\text{Lt}} \dfrac{\log \sec x - \frac{1}{2} x^2}{x^4}$
9. $\underset{x \to 0}{\text{Lt}} \dfrac{x e^x - \log (1 + x)}{\cosh x - \cos x}$
10. $\underset{x \to 0}{\text{Lt}} \dfrac{\cos x - \log (1 + x) - 1 + x}{\sin^2 x}$
11. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x + 2 \sin x - e^{-x} - 4x}{x^5}$
12. $\underset{x \to 0}{\text{Lt}} \dfrac{\log (x - a)}{\log (e^x - e^a)}$
13. $\underset{x \to 0}{\text{Lt}} \dfrac{\tan x - x}{x^2 \tan x}$
14. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$
15. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{\sin x}}{x - \sin x}$
16. $\underset{x \to 0}{\text{Lt}} \dfrac{\sin \{(\log (1 + x)\}}{\log (1 + \sin x)}$
17. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x + \sin x - 1}{\log (1 + x)}$
18. $\underset{x \to 0}{\text{Lt}} \dfrac{(1 + x)^{1/x} - e + \frac{1}{2} ex}{x^2}$
19. If $\underset{x \to 0}{\text{Lt}} \dfrac{\sin 2x + a \sin x}{x^3}$ is finite, find the value of a and the limit. *(Nagpur, 2009)*
20. Find a, b if $\underset{x \to 0}{\text{Lt}} \dfrac{a \sinh x + b \sin x}{x^3} = \dfrac{5}{3}$. *(Mumbai, 2009)*
21. Find a, b, c so that $\underset{x \to 0}{\text{Lt}} \dfrac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$. *(Mumbai, 2008)*

(3) Forms reducible to 0/0 form. Each of the following indeterminate forms can be easily reduced to the form 0/0 (or ∞/∞) by suitable transformation and then the limits can be found as usual.

I. Form $0 \times \infty$. If $\underset{x \to 0}{Lt} f(x) = 0$ and $\underset{x \to \infty}{Lt} \phi(x) = \infty$, then

$$\underset{x \to a}{\text{Lt}} [f(x) \cdot \phi(x)] \text{ assumes the form } 0 \times \infty.$$

To evaluate this limit, we write

$$f(x) \cdot \phi(x) = f(x)/[1/\phi(x)] \text{ to take the form } 0/0.$$
$$= \phi(x)/[1/f(x)] \text{ to take the form } \infty/\infty.$$

Example 4.31. *Evaluate* $\underset{x \to 0}{Lt} (\tan x \log x)$ *(V.T.U., 2009)*

Solution.
$$\underset{x \to 0}{\text{Lt}} (\tan x \log x) = \underset{x \to 0}{\text{Lt}} \left(\frac{\log x}{\cot x}\right) \quad \left(\text{form } \frac{\infty}{\infty}\right)$$
$$= \underset{x \to 0}{\text{Lt}} \left(\frac{1/x}{-\operatorname{cosec}^2 x}\right) = -\underset{x \to 0}{\text{Lt}} \left(\frac{\sin^2 x}{x}\right) \quad \left(\text{form } \frac{0}{0}\right)$$
$$= \underset{x \to 0}{\text{Lt}} \frac{2 \sin x \cos x}{1} = 0.$$

II. Form $\infty - \infty$. If $\underset{x \to a}{Lt} f(x) = \infty = \underset{x \to a}{Lt} \phi(x)$, then $\underset{x \to a}{\text{Lt}} [f(x) - \phi(x)]$ assumes the form $\infty - \infty$.

It can be reduced to the from 0/0 by writing

$$f(x) - \phi(x) = \left[\frac{1}{\phi(x)} - \frac{1}{f(x)}\right] \Big/ \frac{1}{f(x)\phi(x)}$$

Example 4.32. *Evaluate* $\underset{x \to 0}{Lt} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Solution.
$$\underset{x \to 0}{\text{Lt}} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \underset{x \to 0}{\text{Lt}} \frac{x - \sin x}{x \sin x} = \underset{x \to 0}{\text{Lt}} \frac{1 - \cos x}{x \cos x + \sin x} \quad \left(\text{form } \frac{0}{0}\right)$$
$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x(-\sin x) + \cos x + \cos x} = \frac{0}{0 + 1 + 1} = 0.$$

III. Forms $0^0, 1^\infty, \infty^0$. If $y = \underset{x \to a}{\text{Lt}} [f(x)]^{\phi(x)}$ assumes one of these forms, then $\log y = \underset{x \to a}{\text{Lt}} \phi(x) \log f(x)$ takes the form $0 \times \infty$, which can be evaluated by the method given in I above. If $\log y = l$, then $y = e^l$.

Example 4.33. *Evaluate (i)* $\underset{x \to \pi/2}{Lt} (\sin x)^{\tan x}$ *(ii)* $\underset{x \to 0}{Lt} \left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$ *(V.T.U., 2011)*

(iii) $\underset{x \to 0}{Lt} \left(\frac{\tan x}{3}\right)^{1/x^2}$

Solution. (*i*) Let
$$y = \underset{x \to \pi/2}{\text{Lt}} (\sin x)^{\tan x}.$$
$$\therefore \quad \log y = \underset{x \to \pi/2}{\text{Lt}} \tan x \log \sin x = \underset{x \to \pi/2}{\text{Lt}} \frac{\log \sin x}{\cot x} \quad \left(\text{form } \frac{0}{0}\right)$$
$$= \underset{x \to \pi/2}{\text{Lt}} \frac{(1/\sin x) \cos x}{-\operatorname{cosec}^2 x} = -\underset{x \to \pi/2}{\text{Lt}} (\sin x \cos x) = 0$$

Hence $y = e^0 = 1.$

(*ii*) Let $$y = \underset{x \to 0}{\text{Lt}} \left(\frac{a^x + b^x + c^x}{3} \right)^{1/x}$$

so that $$\log y = \underset{x \to 0}{\text{Lt}} \frac{\log (a^x + b^x + c^x) - \log 3}{x} \qquad \text{form } \left(\frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(a^x + b^x + c^x)^{-1} (a^x \log a + b^x \log b + c^x \log c)}{1}$$

$$= (1 + 1 + 1)^{-1} (\log a + \log b + \log c) = \frac{1}{3} \log (abc) = \log (abc)^{1/3}.$$

$\therefore$ $$y = (abc)^{1/3}$$

(*iii*) $$\underset{x \to 0}{\text{Lt}} \left(\frac{\tan x}{x} \right)^{1/x^2} = \underset{x \to 0}{\text{Lt}} \left(\frac{x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots}{x} \right)^{1/x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \dots \right)^{1/x^2}$$

$$= \underset{x \to 0}{\text{Lt}} (1 + tx^2)^{1/x^2} \qquad \text{where } t = \frac{1}{3} + \frac{2}{15}x^2 + \dots$$

$$= \underset{x \to 0}{\text{Lt}} [(1 + tx^2)^{1/tx^2}]^t = \underset{x \to 0}{\text{Lt}} e^t = e^{1/3}. \qquad \left[\because \underset{z \to 0}{\text{Lt}} (1 + z)^{1/z} = e\right]$$

PROBLEMS 4.7

Evaluate the following limits :

1. $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$

2. $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$ (*Burdwan, 2003*)

3. $\underset{x \to 1}{\text{Lt}} (2x \tan x - \pi \sec x)$ (*V.T.U., 2008*)

4. $\underset{x \to 0}{\text{Lt}} \left(\frac{\cot x - 1/x}{x} \right)$

5. $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2} - \cot^2 x \right)$

6. $\underset{x \to 1}{\text{Lt}} (x)^{1/(1-x)}$

7. $\underset{x \to 0}{\text{Lt}} (a^x + x)^{1/x}$ (*V.T.U., 2007*)

8. $\underset{x \to \pi/2}{\text{Lt}} (\sec x)^{\cot x}$

9. $\underset{x \to 0}{\text{Lt}} (1 + \sin x)^{\cot x}$

10. $\underset{x \to 0}{\text{Lt}} (\cos x)^{1/x^2}$

11. $\underset{x \to \pi/2}{\text{Lt}} (\tan x)^{\tan 2x}$ (*V.T.U., 2004*)

12. $\underset{x \to 0}{\text{Lt}} (\cot x)^{1/\log x}$

13. $\underset{x \to \pi/2}{\text{Lt}} (\cos x)^{\frac{\pi}{2} - x}$

14. $\underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x} \right)^{1/x}$

15. $\underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x} \right)^{1/x^2}$ (*V.T.U., 2001*)

16. $\underset{x \to 1}{\text{Lt}} (1 - x^2)^{1/\log(1-x)}$

17. $\underset{x \to a}{\text{Lt}} \left(2 - \frac{x}{a} \right)^{\tan (\pi x/2a)}$ (*V.T.U., 2010 ; Nagpur, 2009*)

18. $\underset{x \to 0}{\text{Lt}} \left\{ \frac{2 (\cosh x - 1)^{1/x^2}}{x^2} \right\}$

19. $\underset{x \to 2}{\text{Lt}} \left\{ \frac{1}{x - 2} - \frac{1}{\log (x - 1)} \right\}$ (*Osmania, 2000 S*)

20. $\underset{x \to 0}{\text{Lt}} \left(\frac{1^x + 2^x + 3^x}{3} \right)^{1/x}$. (*V.T.U., 2008*)

4.6 TANGENTS AND NORMALS – CARTESIAN CURVES

(1) Equation of the tangent *at the point* (x, y) *of the curve* $y = f(x)$ *is*

$$\mathbf{Y - y = \frac{dy}{dx}(X - x).}$$

The equation of any line through $P(x, y)$ is

$$Y - y = m(X - x)$$

where X, Y are the current coordinates of any point on the line (Fig. 4.3).

If this line is the tangent PT, then

$$m = \tan \psi = dy/dx$$

Hence the equation of the tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x) \quad ...(2)$$

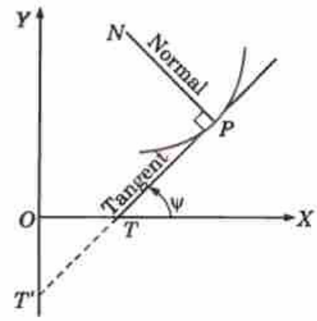

Fig. 4.3

Cor. Intercepts. Putting $Y = 0$ in (2)

$$-y = \frac{dy}{dx}(X - x) \quad \text{or} \quad X = x - y/\frac{dy}{dx}$$

∴ *Intercept which the tangent cuts off from x-axis* $(= OT) = \mathbf{x - y\, dx/dy}$

Similarly putting $X = 0$ *in (2), we see that*

the intercept which the tangent cuts off from the y-axis

$$(= OT') = \mathbf{y - x\frac{dy}{dx}}$$

(2) Equation of the normal *at the point* (x, y) *of the curve* $y = f(x)$ is

$$\mathbf{Y - y = -\frac{dx}{dy}(X - x)}$$

A normal *to the curve* $y = f(x)$ *at* $P(x, y)$ *is a line through P perpendicular to the tangent there at.*

∴ Its equation is $Y - y = m'(X - x)$

where $$m' \cdot dy/dx = -1 \quad \text{or} \quad m' = -1/\frac{dy}{dx} = -dx/dy$$

Hence the equation of the normal at (x, y) is $Y - y = -\dfrac{dx}{dy}(X - x)$.

Example 4.34. *Find the equation of the tangent at any point* (x, y) *to the curve* $x^{2/3} + y^{2/3} = a^{2/3}$. *Show that the portion of the tangent intercepted between the axes is of constant length.*

Solution. Equation of the curve is $x^{2/3} + y^{2/3} = a^{2/3}$. ...(*i*)

Differentiating (*i*) w.r.t. x,

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

∴ Slope of the tangent at $(x, y) = \dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{1/3}$

∴ Equation of the tangent at (x, y) is

$$Y - y = -(y/x)^{1/3}(X - x) \quad ...(ii)$$

Put $Y = 0$ in (*ii*). Then $X = x + x^{1/3} \cdot y^{2/3}$

i.e., Intercept on x-axis $= (x^{2/3} + y^{2/3})x^{1/3} = a^{2/3} \cdot x^{1/3}$ [By (*i*)]

Put $X = 0$ in (*ii*). Then $Y = y + y^{1/3} \cdot x^{2/3}$

i.e., Intercept on y-axis $= (x^{2/3} + y^{2/3})\, y^{1/3} = a^{2/3} \cdot y^{1/3}$ [By (*i*)]

Thus the portion of the tangent intercepted between the axes

$$= \sqrt{[(\text{Intercept on } x\text{-axis})^2 + (\text{Intercept on } y\text{-axis})^2]}$$

$$= \sqrt{[(a^{2/3} \cdot x^{1/3})^2 + (a^{2/3} \cdot y^{1/3})^2]}$$

$$= \sqrt{[a^{4/3}(x^{2/3} + y^{2/3})]} = a^{2/3}\sqrt{(a)^{2/3}} \qquad \text{[By } (i)]$$

$= a$, which is a constant length.

Example 4.35. *Show that the conditions for the line $x \cos \alpha + y \sin \alpha = p$ to touch the curve $(x/a)^m + (y/b)^m = 1$ is $(a \cos \alpha)^{m/(m-1)} + (b \sin \alpha)^{m/(m-1)} = p^{m/(m-1)}$.*

Solution. Equation of the curve is $\dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} = 1$ ...(i)

Differentiating (i) w.r.t. x, $\dfrac{mx^{m-1}}{a^m} + \dfrac{my^{m-1}}{b^m}\dfrac{dy}{dx} = 0$

$\therefore$ Slope of the tangent at $(x, y) = \dfrac{dy}{dx} = -\left(\dfrac{b}{a}\right)^m \left(\dfrac{x}{y}\right)^{m-1}$

$\therefore$ Equation of the tangent at (x, y) is

$$Y - y = -\left(\frac{b}{a}\right)^m \left(\frac{x}{y}\right)^{m-1} (X - x)$$

or
$$\frac{x^{m-1}X}{a^m} + \frac{y^{m-1}Y}{b^m} = \frac{x^m}{a^m} + \frac{y^m}{b^m} = 1 \qquad \text{...(ii) [By } (i)]$$

If the given line touches (i) at (x, y) then (ii) must be same as $X \cos \alpha + Y \sin \alpha = p$ ...(iii)

Comparing coefficients in (ii) and (iii),

$$\frac{x^{m-1}}{a^m}\Big/\cos\alpha = \frac{y^{m-1}}{b^m}\Big/\sin\alpha = \frac{1}{p}$$

or
$$\left(\frac{x}{a}\right)^{m-1} = \frac{a\cos\alpha}{p}, \left(\frac{y}{b}\right)^{m-1} = \frac{b \sin\alpha}{p}$$

or
$$\left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}} + \left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}} = \left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1 \qquad \text{[By } (i)]$$

whence follows the required condition.

Example 4.36. *Find the equation of the normal at any point θ to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$. Verify that these normals touch a circle with its centre at the origin and whose radius is constant.*

Solution. We have $\dfrac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta$

$$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta$$

$\therefore$
$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{\sin\theta}{\cos\theta}$$

$\therefore$ Slope of the normal at $\theta = -\dfrac{\cos\theta}{\sin\theta}$

Hence the equation of the normal at θ

$$y - a(\sin\theta - \theta\cos\theta) = -\frac{\cos\theta}{\sin\theta}[x - a(\cos\theta + \theta\sin\theta)]$$

i.e., $y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$

i.e., $x\cos\theta + y\sin\theta = a(\cos^2\theta + \sin^2\theta) = a.$

Now the perpendicular distance of this normal from $(0, 0) = a$, which is a constant. Hence it touches a circle of radius a having its centre at $(0, 0)$.

(3) Angle of intersection of two curves *is the angle between the tangents to the curves at their point of intersection.*

To find this angle θ, proceed as follows :

(*i*) Find P, the point of intersection of the curves by solving their equations simultaneously.

(*ii*) Find the values of dy/dx at P for the two curves (say : m_1, m_2).

(*iii*) Find $\angle\theta$, using the $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$.

When $m_1 m_2 = -1$, $\theta = 90°$ *i.e.,* the curves cut orthogonally.

Example 4.37. *Find the angle of intersection of the curves* $x^2 = 4y$ ...(*i*)

and $y^2 = 4x$. ...(*ii*)

Solution. We have $x^4 = 16y^2 = 16.4\,x = 64x$

or $x(x^3 - 64) = 0$ whence $x = 0$ and 4.

Substituting these values in (*i*), $y = 0$ and 4.

∴ The curves intersect at (0, 0) and (4, 4).

For the curve (*i*), $dy/dx = x/2$. For the curve (*ii*), $dy/dx = 2/y$

At (0, 0), slope of tangent to (*i*) ($= m_1$) $= 0/2 = 0$ and slope of tangent to (*ii*) ($= m_2$) $= 2/0 = \infty$.

Evidently the curves intersect at right angles.

At (4, 4), slope of tangent to (*i*) ($= m_1$) $= 4/2 = 2$ and slope of tangent to (*ii*) ($= m_2$) $= 2/4 = \frac{1}{2}$

∴ Angle of intersection of the curves

$$= \tan^{-1}\frac{m_1 - m_2}{1 + m_1 m_2} = \tan^{-1}\frac{2 - \frac{1}{2}}{1 + 2\cdot\frac{1}{2}} = \tan^{-1}\frac{3}{4}.$$

Example 4.38. *Show that the condition that the curves* $ax^2 + by^2 = 1$ *and* $a'x^2 + b'y^2 = 1$ *should intersect orthogonally is that*

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}.$$

Solution. Given curves are $ax^2 + by^2 = 1$ ...(*i*) and $a'x^2 + b'y^2 = 1$ ...(*ii*)

Let $P(h, k)$ be a point of intersection of (*i*) and (*ii*) so that

$$ah^2 + bk^2 = 1 \quad \text{and} \quad a'h^2 + b'k^2 = 1$$

$$\therefore \quad \frac{h^2}{-b + b'} = \frac{k^2}{-a' + a} = \frac{1}{ab' - a'b}$$

or $$h^2 = (b' - b)/(ab' - a'b),\ k^2 = (a - a')/(ab' - a'b) \qquad ...(iii)$$

Differentiating (*i*) w.r.t. x,

$$2ax + 2by\,dy/dx = 0 \quad \text{or} \quad dy/dx = -ax/by.$$

Similarly for (*ii*), $dy/dx = -a'x/b'y$

∴ m_1 = slope of tangent to (*i*) at $P = -ah/bk$; m_2 = slope of tangent to (*ii*) at $P = -a'h/b'k$

For orthogonal intersection, we should have $m_1 m_2 = -1$.

i.e., $$\frac{-ah}{bk} \times \frac{-a'h}{b'k} = 1 \text{ i.e., } aa'h^2 + bb'k^2 = 0$$

Substituting the values of h^2 and k^2 from (*iii*),

$$\frac{aa'(b' - b)}{ab' - a'b} + \frac{bb'(a - a')}{ab' - a'b} = 0 \quad \text{or} \quad \frac{b' - b}{bb'} + \frac{a - a'}{aa'} = 0$$

i.e., $$\frac{1}{b} - \frac{1}{b'} = \frac{1}{a} - \frac{1}{a'} \text{ which leads to the required condition.}$$

(4) Lengths of tangent, normal, subtangent and subnormal. Let the tangent and the normal at any point $P(x, y)$ of the curve meet the x-axis at T and N respectively. (Fig. 4.4). Draw the ordinate PM. Then PT *and* PN *are called the lengths of the tangent and the normal respectively. Also* TM *and* MN *are called the subtangent and subnormal respectively.*

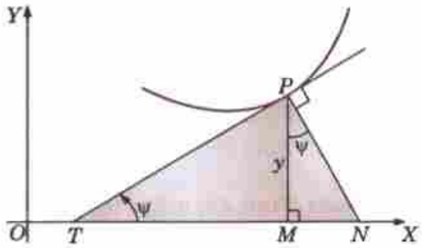

Fig. 4.4

Let $\angle MTP = \psi$ so that $\tan \psi = dy/dx$.

Clearly, $\angle MPN = \psi$.

(1) Tangent $= TP = MP \operatorname{cosec} \psi = y\sqrt{(1+\cot^2\psi)} = \mathbf{y}\sqrt{[1+(dx/dy)^2]}$

(2) Normal $= NP = MP \sec \psi = y\sqrt{(1+\tan^2\psi)} = \mathbf{y}\sqrt{[1+(dy/dx)^2]}$

(3) Subtangent $= TM = y \cot \psi = \mathbf{y\ dx/dy}$

(4) Subnormal $= MN = y \tan \psi = \mathbf{y\ dy/dx}$.

Example 4.39. *For the curve* $x = a(\cos t + \log \tan t/2)$, $y = a \sin t$, *prove that the portion of the tangent between the curve and x-axis is constant.*

Also find its subtangent.

Solution. Differentiating with respect to t,

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan t/2}\cdot \sec^2\frac{t}{2}\cdot\frac{1}{2}\right) = a\left(-\sin t + \frac{\cos t/2}{2\sin t/2}\cdot\frac{1}{\cos^2 t/2}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = \frac{a(1-\sin^2 t)}{\sin t} = a\cos^2 t/\sin t\ ; \quad \frac{dy}{dt} = a\cos t\ .$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt} = a\cos t\cdot\frac{\sin t}{a\cos^2 t} = \tan t.$$

Thus *length of the tangent between the curve and x-axis*

$$= y\sqrt{[1+(dx/dy)^2]} = a\sin t\cdot\sqrt{(1+\cot^2 t)} = a \sin t\ .\ \operatorname{cosec} t = a \text{ which is a constant.}$$

Also *subtangent* $= y\dfrac{dx}{dy} = a\sin t\cdot\cot t = a\cos t$.

PROBLEMS 4.8

1. Find the equation of the tangent and the normal to the curve $y(x-2)(x-3) - x + 7 = 0$ at the point where it cuts the x-axis.
2. The straight line $x/a + y/b = 2$ touches the curve $(x/a)^n + (y/b)^n = 2$ for all values of n. Find the point of contact. *(Bhopal, 2008)*
3. Prove that $\dfrac{x}{a} + \dfrac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the axis of y. *(Bhopal, 2009)*
4. If $p = x\cos\alpha + y\sin\alpha$, touches the curve $(x/a)^{n/(n-1)} + (y/b)^{n/(n-1)} = 1$, prove that
$$p^n = (a\cos\alpha)^n + (b\sin\alpha)^n.$$
5. Prove that the condition for the line $x\cos\alpha + y\sin\alpha = p$ to touch the curve $x^m y^n = a^{m+n}$, is
$$p^{m+n}\cdot m^m\cdot n^n = (m+n)^{m+n}\, a^{m+n}\cos^m\alpha\,\sin^n\alpha.$$
6. Show that the sum of the intercepts on the axes of any tangent to the curve $\sqrt{x} + \sqrt{y} = a$ is a constant.
7. If x, y be the parts of the axes of x and y intercepted by the tangent at any point (x, y) on the curve $(x/a)^{2/3} + (x/b)^{2/3} = 1$, then show that $(x_1/a)^2 + (y_1/b)^2 = 1$. *(Bhopal, 2008)*
8. If the tangent at (x_1, y_1) to the curve $x^3 + y^3 = a^3$ meets the curve again in (x_2, y_2), show that
$$\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1.$$

9. If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the axis of x, show that its equation is $y \cos \phi - x \sin \phi = a \cos 2\phi$.

10. Find the angle of intersection of the curves $x^2 - y^2 = a^2$ are $x^2 + y^2 = a^2\sqrt{2}$.

11. Show that the parabolas $y^2 = 4ax$ and $2x^2 = ay$ intersect at an angle $\tan^{-1}(3/5)$.

12. Prove that the curves $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 1$ and $\dfrac{x^2}{a'} + \dfrac{y^2}{b'} = 1$ will cut orthogonally if $a - b = a' - b'$.

13. Show that in the exponential curve $y = be^{x/a}$, the subtangent is of constant length and that the subnormal varies as the square of the ordinate. *(Madras, 2000 S)*

14. Find the lengths of the tangent, normal, subtangent and subnormal for the cycloid:
$$x = a(t + \sin t),\ y = a(1 - \cos t),$$

15. For the curve $x = a \cos^3 \theta, y = a \sin^3 \theta$, show that the portion of the tangent intercepted between the point of contact and the x-axis is y cosec θ. Also find the length of the subnormal.

4.7 POLAR CURVES

(1) Angle between radius vector and tangent. *If ϕ be the angle between the radius vector and the tangent at any point of the curve $r = f(\theta)$,* $\tan \theta = r\dfrac{d\theta}{dr}$.

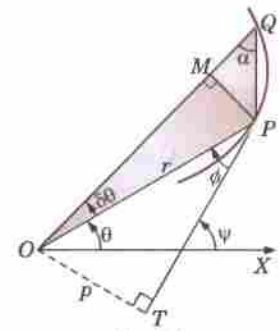

Fig. 4.5

Let $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points on the curve (Fig. 4.5). Join PQ and draw $PM \perp OQ$. Then from the rt. angled ΔOMP, $MP = r \sin \delta\theta$, $OM = r \cos \delta\theta$.

$$\therefore \qquad MQ = OQ - OM = r + \delta r - r \cos \delta\theta$$
$$= \delta r + r(1 - \cos \delta\theta) = \delta r + 2r \sin^2 \delta\theta/2.$$

If $\angle MQP = \alpha$, then
$$\tan \alpha = \frac{MP}{MQ} = \frac{r \sin \delta\theta}{\delta r + 2r \sin^2 \delta\theta/2}$$

In the limit as $Q \to P$ (*i.e.*, $\delta\theta \to 0$), the chord PQ turns about P and becomes the tangent at P and $\alpha \to \phi$.

$$\therefore \qquad \tan \phi = \underset{Q \to P}{\text{Lt}} (\tan \alpha) = \underset{\delta\theta \to 0}{\text{Lt}} \frac{r \sin \delta\theta}{\delta r + 2r \sin^2 \delta\theta/2}$$
$$= \underset{\delta\theta \to 0}{\text{Lt}} \frac{r(\sin \delta\theta/\delta\theta)}{(\delta r/\delta\theta) + r \sin \delta\theta/2 \cdot (\sin \delta\theta/2 \div \delta\theta/2)}$$
$$= \frac{r \cdot 1}{(dr/d\theta) + r \cdot 0 \cdot 1} = r\frac{d\theta}{dr}$$

Cor. Angle of intersection of two curves. *If ϕ_1, ϕ_2 be the angles between the common radius vector and the tangents to the two curves at their point of intersection, then the angle of intersection of these curves is $\phi_1 \sim \phi_2$.*

(2) Length of the perpendicular from pole on the tangent. *If p be the perpendicular from the pole on the tangent, then*

(*i*) $\mathbf{p = r \sin \phi}$ (*ii*) $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

From the rt. $\angle$ed ΔOTP, $p = r \sin \phi$

$$\therefore \qquad \frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \phi = \frac{1}{r^2}(1 + \cot^2 \phi)$$
$$= \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right] = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \qquad \text{[By (1)]}$$

(3) Polar subtangent and subnormal. Let the tangent and the normal at any point $P(r, \theta)$ of a curve meet the line through the pole perpendicular to the radius vector OP in T and N respectively (Fig. 4.6). Then OT is called the *polar subtangent* and ON the *polar subnormal*.

Let $\angle OTP = \phi$ so that $\tan\phi = r d\theta/dr$

Clearly, $\angle PNO = \phi$.

$\therefore$ (*i*) **Polar subtangent**

$$= OT = r\tan\phi = r\cdot r d\theta/dr = \mathbf{r^2\frac{d\theta}{dr}}$$

(*ii*) **Polar subnormal**

$$= ON = r\cot\phi = r\cdot\frac{1}{r}\frac{dr}{d\theta} = \mathbf{\frac{dr}{d\theta}}$$

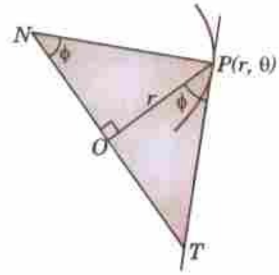

Fig. 4.6

Example 4.40. *For the cardioid $r = a(1 - \cos\theta)$, prove that*

(*i*) $\phi = \theta/2$ (*ii*) $p = 2a\sin^3\theta/2$

(*iii*) *polar subtangent* $= 2a\sin^2\dfrac{\theta}{2}\tan\dfrac{\theta}{2}$.

Solution. We have $\dfrac{dr}{d\theta} = a\sin\theta$

$$\therefore \qquad \tan\phi = r\frac{d\theta}{dr} = a(1-\cos\theta)\cdot\frac{1}{a\sin\theta}$$

$$= 2\sin^2\theta/2 \div 2\sin\theta/2\cos\theta/2 = \tan\theta/2. \text{ Thus } \phi = \theta/2 \qquad ...(i)$$

Also $p = r\sin\phi = a(1-\cos\theta)\cdot\sin\theta/2 = a\cdot 2\sin^2\theta/2\cdot\sin\theta/2$

$$= 2a\sin^3\theta/2 \qquad ...(ii)$$

Polar subtangent $= r^2 d\theta/dr = [a(1-\cos\theta)]^2 \div a\sin\theta$

$$= 4a\sin^4\theta/2 \div 2\sin\theta/2\cos\theta/2 = 2a\sin^2\theta/2\tan\theta/2. \qquad ...(iii)$$

Example 4.41. *Find the angle of intersection of the curves $r = \sin\theta + \cos\theta$, $r = 2\sin\theta$.*

Solution. To find the point of intersection of the curves $r = \sin\theta + \cos\theta$

and $r = 2\sin\theta$, ...(*ii*), we eliminate r.

Then $2\sin\theta = \sin\theta + \cos\theta$ or $\tan\theta = 1$ *i.e.*, $\theta = \pi/4$.

For (*i*), $\dfrac{dr}{d\theta} = \cos\theta - \sin\theta$

$$\therefore \qquad \tan\phi = r\frac{d\theta}{dr} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta} \text{ which} \to \infty \text{ at } \theta = \pi/4. \text{ Thus } \phi = \pi/2.$$

For (*ii*), $dr/d\theta = 2\cos\theta$ $\therefore$ $\tan\phi' = r\dfrac{d\theta}{dr} = \dfrac{2\sin\theta}{2\cos\theta} = 1$ at $\theta = \pi/4$. Thus $\phi' = \pi/4$

Hence the angle of intersection of (*i*) and (*ii*) $= \phi - \phi' = \pi/4$.

PROBLEMS 4.9

1. For a curve in Cartesian form, show that $\tan\phi = \dfrac{xy' - y}{x + yy'}$.
2. Show that in the equiangular spiral $r = ae^{\theta\cot\alpha}$, the tangent is inclined at a constant angle to the radius vector.
3. Show that the tangent to the cardioid $r = a(1 + \cos\theta)$ at the points $\theta = \pi/3$ and $\theta = 2\pi/3$ are respectively parallel and perpendicular to the initial line. (*V.T.U., 2006*)
4. Prove that, in the parabola $2a/r = 1 - \cos\theta$,

   (*i*) $\phi = \pi - \theta/2$ (*ii*) $\pi = a\operatorname{cosec}\theta/2$, and (*iii*) polar subtangent $= 2a\operatorname{cosec}\theta$.
5. Show that the angle between the tangent at any point P and the line joining P to the origin is the same at all points of the curve

$$\log(x^2 + y^2) = k\tan^{-1}(y/x).$$

6. Show that in the curve $r = a\theta$, the polar subnormal is constant and in the curve $r\,\theta = a$ the polar subtangent is constant.
7. Find the angle of intersection of the curves
   (*i*) $r = 2\sin\theta$, and $r = 2\cos\theta$ (*Bhopal, 1991*)
   (*ii*) $r = a/(1+\cos\theta)$ and $r = b/(1-\cos\theta)$. (*V.T.U., 2008 S*)
8. Prove that the curves $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta)$ intersect at right angles. (*V.T.U., 2011 S*)
9. Show that the curves $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ cut each other orthogonally.
10. Show that the angle of intersection of the curves $r = a\log\theta$ and $r = a/\log\theta$ is $\tan^{-1}[2e/(1-e^2)]$. (*V.T.U., 2005*)

4.8 PEDAL EQUATION

If r be the radius vector of any point on the curve and p, the length of the perpendicular from the pole on the tangent at that point, them *the relation between* p *and* r *is called pedal equation of the curve.*

Given the cartesian or polar equation of a curve, we can derive its pedal equation. The method is explained through the following examples.

Example 4.42. *Find the pedal equation of the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$ ...(*i*)

Solution. Equation of the tangent at (x, y) is $\dfrac{Xx}{a^2}+\dfrac{Yy}{b^2}=1$ ...(*ii*)

p, length of $\perp$ from (0, 0) on (*ii*) $= \dfrac{-1}{\sqrt{[(x/a^2)^2+(y/b^2)^2]}}$

or
$$\frac{1}{p^2}=\frac{x^2}{a^4}+\frac{y^2}{b^4} \qquad ...(iii)$$

Also
$$r^2 = x^2 + y^2 \qquad ...(iv)$$

Substituting the value of y^2 from (*iv*) in (*i*),

$$\frac{x^2}{a^2}=\frac{r^2-b^2}{a^2-b^2}.$$

Then from (*i*),
$$\frac{y^2}{b^2}=\frac{a^2-r^2}{a^2-b^2}$$

Now substituting these values of x^2/a^2 and y^2/b^2 in (*iii*),

$$\frac{1}{p^2}=\frac{1}{a^2}\left(\frac{r^2-b^2}{a^2-b^2}\right)+\frac{1}{b^2}\left(\frac{a^2-r^2}{a^2-b^2}\right)$$

or
$$\frac{a^2b^2}{p^2}=\frac{r^2b^2-b^4+a^4-a^2r^2}{a^2-b^2}=a^2+b^2-r^2$$

He[illegible] [illegible]e required pedal equation.

Example 4.43. [illegible] *of the curves*
(*i*) $2a/r =$ [illegible] $r^n = a^n \cos n\,\theta$ (*V.T.U., 2010*)

Solution.
Takin[illegible]

$\log 2a -$ [illegible]

Differentiating both sides with respect to θ, we get

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-\cos\theta}\cdot\sin\theta = \cot\frac{\theta}{2}$$

$\therefore$ $$\tan\phi = r\frac{d\theta}{dr} = -\tan\theta/2 = \tan(\pi-\theta/2) \text{ i.e., } \phi = \pi - \theta/2$$

Also $$p = r\sin\phi = r\sin(\pi-\theta/2) \text{ i.e., } p = r\sin\theta/2$$

or $$p^2 = r^2\sin^2\theta/2 = r^2\left(\frac{1-\cos\theta}{2}\right) = r^2.\, a/r \qquad \text{[By } (i)]$$

Hence $p^2 = ar$, which is the required pedal equation.

(*ii*) From the given equation, $nr^{n-1}\dfrac{dr}{d\theta} = -na^n\sin n\theta$

so that $$\tan\phi = r\,d\theta/dr = r\frac{nr^{n-1}}{-na^n\sin n\theta} = -\cot n\theta = \tan\left(\frac{\pi}{2}+n\theta\right)$$

i.e., $$\phi = \pi/2 + n\theta$$

$\therefore$ $$p = r\sin\phi = r\sin\left(\frac{\pi}{2}+n\theta\right) = r\cos n\theta = r\,.\,(r^n/a^n) = r^{n+1}/a^n.$$

Hence $p\,a^n = r^{n+1}$, which is the required pedal equation.

4.9 DERIVATIVE OF ARC

(1) *For the curve $y = f(x)$, we have*

$$\mathbf{\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}}$$

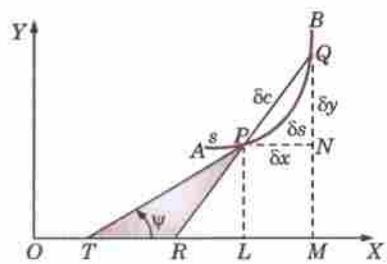

Fig. 4.7

Let $P(x, y)$, $Q(x+\delta x, y+\delta y)$ be two neighbouring points on the curve AB (Fig. 4.7). Let arc $AP = s$, arc $PQ = \delta s$ and chord $PQ = \delta c$.

Draw PL, QM ⊥s on the x-axis and $PN \perp QM$.

$\therefore$ From the rt. ∠ed ΔPNQ,

$$PQ^2 = PN^2 + NQ^2$$

i.e., $$\delta c^2 = \delta x^2 + \delta y^2$$

or $$\left(\frac{\delta c}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

$\therefore$ $$\left(\frac{\delta s}{\delta x}\right)^2 = \left(\frac{\delta s}{\delta c}\cdot\frac{\delta c}{\delta x}\right)^2$$

$$= \left(\frac{\delta s}{\delta c}\right)^2 = \left[1+\left(\frac{\delta y}{\delta x}\right)^2\right]$$

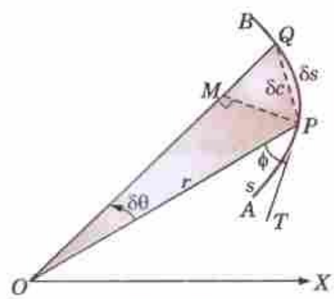

Fig. 4.8

Taking limits as $Q \to P$ (*i.e.*, $\delta c \to 0$),

$$\left(\frac{ds}{dx}\right)^2 = 1\cdot\left[1+\left(\frac{dy}{dx}\right)^2\right] \qquad \left[\because \frac{\delta s}{\delta c} = 1\right]$$

If s increases with x as in Fig. 4.7, dy/dx is positive.

Thus $$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}, \text{ taking positive sign before the radical.} \qquad ...(1)$$

Cor. 1. *If the equation of the curve is $x = f(y)$, then*

$$\frac{ds}{dy} = \frac{ds}{dx}\cdot\frac{dx}{dy} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\cdot\frac{dx}{dy}$$

$$\therefore \qquad \frac{\mathbf{ds}}{\mathbf{dy}} = \sqrt{\left[1 + \left(\frac{\mathbf{dx}}{\mathbf{dy}}\right)^2\right]} \qquad ...(2)$$

Cor. 2. *If the equation of the curve is in parametric form $x = f(t)$, $y = \phi(t)$, then*

$$\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} \cdot \frac{dx}{dt}$$

$$= \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx} \cdot \frac{dx}{dt}\right)^2\right]}$$

$$\therefore \qquad \frac{\mathbf{ds}}{\mathbf{dt}} = \sqrt{\left[\left(\frac{\mathbf{dx}}{\mathbf{dt}}\right)^2 + \left(\frac{\mathbf{dy}}{\mathbf{dt}}\right)^2\right]} \qquad ...(3)$$

Cor. 3. We have
$$\frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} = \sqrt{(1 + \tan^2 \psi)} = \sec \psi$$

$$\therefore \qquad \cos \psi = \frac{\mathbf{dx}}{\mathbf{ds}}. \qquad ...(4)$$

Also
$$\sin \psi = \tan \psi \cos \psi = \frac{dy}{dx} \cdot \frac{dx}{ds}$$

$$\therefore \qquad \sin \psi = \frac{\mathbf{dy}}{\mathbf{ds}} \qquad ...(5)$$

(2) *For the curve $r = f(\theta)$, we have* $\dfrac{\mathbf{ds}}{\mathbf{d\theta}} = \sqrt{\left[\mathbf{r}^2 + \left(\dfrac{\mathbf{dr}}{\mathbf{d\theta}}\right)^2\right]}$.

Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points on the curve AB (Fig. 4.8). Let arc $AP = s$, arc $PQ = \delta s$ and chord $PQ = \delta c$.

Draw $PM \perp OQ$, then

$$PM = r \sin \delta\theta \text{ and } MQ = OQ - OM = r + \delta r - r \cos \delta\theta = \delta r + 2r \sin^2 \delta\theta/2$$

From the rt. $\angle$ed ΔPMQ,

$$PQ^2 = PM^2 + MQ^2$$

or
$$\delta c^2 = (r \sin \delta\theta)^2 + (\delta r + 2r \sin^2 \delta\theta/2)^2$$

$$\therefore \qquad \left(\frac{\delta s}{\delta\theta}\right)^2 = \left(\frac{\delta s}{\delta c} \cdot \frac{\delta c}{\delta\theta}\right)^2 = \left(\frac{\delta s}{\delta c}\right)^2 \left[\left(\frac{r \sin \delta\theta}{\delta\theta}\right)^2 + \left(\frac{\delta r}{\delta\theta} + \frac{2r \sin^2 \delta\theta/2}{\delta\theta}\right)^2\right]$$

$$= \left(\frac{\delta s}{\delta c}\right)^2 \left[r^2 \left(\frac{\sin \delta\theta}{\delta\theta}\right)^2 + \left(\frac{\delta r}{\delta\theta} + r \sin \frac{\delta\theta}{2} \cdot \frac{\sin \delta\theta/2}{\delta\theta/2}\right)^2\right]$$

Taking limits as $Q \to P$

$$\left(\frac{ds}{d\theta}\right)^2 = 1^2 . \left[r^2 \cdot 1^2 + \left(\frac{dr}{d\theta} + r \cdot 0 \cdot 1\right)^2\right] = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

As s increases with the increase of θ, $ds/d\theta$ is positive. Thus

$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} \qquad ...(1)$$

Cor. 1. If the equation of the curve is $\theta = f(r)$, then

$$\frac{ds}{dr} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dr} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} \cdot \frac{d\theta}{dr}$$

$$\therefore \qquad \frac{\mathbf{ds}}{\mathbf{dr}} = \sqrt{\left[1 + \mathbf{r}^2\left(\frac{\mathbf{d\theta}}{\mathbf{dr}}\right)^2\right]} \qquad ...(2)$$

Cor. 2. We have $\frac{ds}{dr} = \sqrt{\left[1 + \left(r\frac{d\theta}{dr}\right)^2\right]} = \sqrt{[1+\tan^2\phi]}\ \frac{ds}{dr} = \sqrt{\left[1 + \left(r\frac{d\theta}{dr}\right)^2\right]} = \sqrt{[1+\tan^2\phi]} = \sec\phi$

$$\therefore \qquad \cos\phi = \frac{\mathbf{dr}}{\mathbf{ds}} \qquad ...(3)$$

Also $\sin\phi = \tan\phi \cdot \cos\phi = r\frac{d\theta}{dr}\cdot\frac{dr}{ds}$

$$\therefore \qquad \sin\phi = \mathbf{r}\frac{\mathbf{d\theta}}{\mathbf{ds}} \qquad ...(4)$$

PROBLEMS 4.10

Prove that the pedal equation of :

1. the parabola $y^2 = 4a(x + a)$ is $p^2 = ar$.
2. the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $a^2b^2/p^2 = r^2 - a^2 + b^2$.
3. the astroid $x = a\cos^3 t, y = a\sin^3 t$ is $r^2 = a^2 - 3p^2$.

Find the pedal equations of the following curves :

4. $r = a(1 + \cos\theta)$ (V.T.U., 2009)
5. $r^2 = a^2\sin^2\theta$
6. $r^m \cos m\theta = a^m$. (V.T.U., 2004)
7. $r^m = a^m(\cos m\theta + \sin m\theta)$ (V.T.U., 2010)
8. $r = ae^{m\theta}$. (V.T.U., 2007)
9. Calculate ds/dx for the following curves :
   (i) $ay^2 = x^3$. (ii) $y = c\cosh x/c$.
10. Find $ds/d\theta$ for the curve $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$ (V.T.U., 2007)
11. Find $ds/d\theta$ for the following curves :
    (i) $r = a(1 - \cos\theta)$ (V.T.U., 2004) (ii) $r^2 = a^2\cos^2 2\theta$
    (iii) $r = \frac{1}{2}\sec^2\theta$ (V.T.U., 2007)
12. For the curves $\theta = \cos^{-1}(r/k) - \sqrt{(k^2 - r^2)}/r$, prove that $r\frac{ds}{dr}$ = constant. (V.T.U., 2005)
13. With the usual meanings for r, s, θ and ϕ for the polar curve $r = f(\theta)$, show that $\frac{d\phi}{d\theta} + r\,\mathrm{cosec}^2\theta\,\frac{d^2r}{ds^2} = 0$. (V.T.U., 2000)

4.10 CURVATURE

Let P be any point on a given curve and Q a neighbouring point. Let arc $AP = s$ and arc $PQ = \delta s$. Let the tangents at P and Q make angle ψ and $\psi + \delta\psi$ with the x-axis, so that the angle between the tangents at P and $Q = \delta\psi$ (Fig. 4.9).

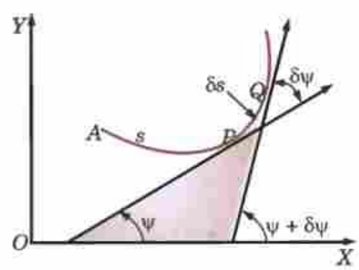

Fig. 4.9

In moving from P to Q through a distance δs, the tangent has turned through the angle $\delta\psi$. This is called the *total bending or total curvature* of the arc PQ.

$\therefore$ The *average curvature* of arc $PQ = \frac{\delta\psi}{\delta s}$

The limiting value of average curvature when Q approaches P (i.e., $\delta s \to 0$) is defined as the curvature of the curve at P.

Thus **curvature** K (at P) $= \frac{\mathbf{d\psi}}{\mathbf{ds}}$

Obs. Since $\delta\psi$ is measured in radians, the *unit of curvature is radians per unit length e.g.*, radians per centimetre.

(2) Radius of curvature. *The reciprocal of the curvature of a curve at any point P is called the* **radius of curvature** *at P and is denoted by* ρ, so the $\rho = \mathbf{ds/d\psi}$.

(3) Centre of curvature. *A point C on the normal at any point P of a curve distant* ρ *from it, is called the* **centre of curvature** *at P.*

(4) Circle of curvature. *A circle with centre C (centre of curvature at P) and radius* ρ *is called the* **circle of curvature** *at P.*

4.11 (1) RADIUS OF CURVATURE FOR CARTESIAN CURVE y = f (x), is given by

$$\rho = \frac{(1+\mathbf{y_1}^2)^{3/2}}{\mathbf{y_2}}$$

We know that $\tan\psi = dy/dx = y_1$ or $\psi = \tan^{-1}(y_1)$

Differentiating both sides w.r.t. x,

$$\frac{d\psi}{dx} = \frac{1}{1+y_1^2}\cdot\frac{d(y_1)}{dx} = \frac{y_2}{1+y_1^2}$$

$$\therefore \qquad \rho = \frac{ds}{d\psi} = \frac{ds}{dx}\cdot\frac{dx}{d\psi} = \sqrt{(1+y_1^2)}\cdot\frac{1+y_1^2}{y_2} = \frac{(1+y_1^2)^{3/2}}{y_2} \qquad \text{...(1)}$$

(2) Radius of curvature for parametric equations

$$x = f(t), \qquad y = \phi(t).$$

Denoting differentiations with respect to t by dashes,

$$y_1 = \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = y'/x'.$$

$$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\cdot\frac{dt}{dx} = \frac{x'y'' - y'x''x}{(x')^2}\cdot\frac{1}{x'}$$

Substituting the values of y_1 and y_2 in (1)

$$\rho = \left[1+\left(\frac{y'}{x'}\right)^2\right]^{3/2}\Big/\left[\frac{x'y''-y'x''}{(x')^3}\right] = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''} \qquad \textit{(Rajasthan, 2005)}$$

(3) Radius of curvature at the origin. Newton's formulae*

(*i*) *If x-axis is tangent to a curve at the origin, then*

$$\rho \text{ at } (0, 0) = \mathop{\mathbf{Lt}}_{\mathbf{x\to0}}\left(\frac{\mathbf{x^2}}{\mathbf{2y}}\right)$$

Since x-axis is a tangent at (0, 0), $(dy/dx)_0$ or $(y_1)_0 = 0$

$$\text{Also} \qquad \mathop{\text{Lt}}_{x\to0}\left(\frac{x^2}{2y}\right) = \mathop{\text{Lt}}_{x\to0}\left(\frac{2x}{2dy/dx}\right) = \mathop{\text{Lt}}_{x\to0}\frac{1}{d^2y/dx^2} = \frac{1}{(y_2)_0} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$\therefore \qquad \rho \text{ at } (0,0) = \frac{\{1+(y_1^2)_0\}^{3/2}}{(y_2)_0} = \frac{1}{(y_2)_0} = \mathop{\text{Lt}}_{x\to0}\frac{x^2}{2y} \qquad \text{[From (1)]}$$

(*ii*) Similarly, *if y-axis is tangent to a curve at the origin, then*

$$\rho \text{ at } (0, 0) = \mathop{\text{Lt}}_{x\to0}\left(\frac{\mathbf{y^2}}{\mathbf{2x}}\right)$$

---

* Named after the great English mathematician and physicist *Sir Issac Newton* (1642–1727) whose contributions are of utmost importance. He discovered many physical laws, invented Calculus alongwith *Leibnitz* (see footnote p. 139) and created analytical methods of investigating physical problems. He became professor at Cambridge in 1699, but his '*Mathematical Principles of Natural Philosophy*' containing development of classical mechanics had been completed in 1687.

(*iii*) In case the curve passes through the origin but neither x-axis nor y-axis is tangent at the origin, we write the equation of the curve as

$$y = f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots \qquad \text{[By Maclaurin's series]}$$

$$= px + qx^2/2 + \ldots \qquad [\because \; f(0) = 0]$$

where $p = f'(0)$ and $q = f''(0)$

Substituting this in the equation $y = f(x)$, we find the values of p and q by equating coefficients of like powers of x. Then $\rho\,(0, 0) = (1 + p^2)^{3/2}/q$.

Obs. *Tangents at the origin to a curve are found by equating to zero the lowest degree terms in its equation.*

Example 4.44. *Find the radius of curvature at the point* (*i*) $(3a/2, 3a/2)$ *of the Folium* $x^3 + y^3 = 3axy$.
(*Anna, 2009 ; Kurukshetra, 2009 S ; V.T.U., 2008*)
(*ii*) $(a, 0)$ *on the curve* $xy^2 = a^3 - x^3$. (*Anna, 2009 ; Kerala, 2005*)

Solution. (*i*) Differentiating with respect to x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a \left(y + x\frac{dy}{dx} \right)$$

or $$(y^2 - ax) \frac{dy}{dx} = ay - x^2 \qquad \ldots(i) \qquad \therefore \; \frac{dy}{dx} \text{ at } (3a/2, 3a/2) = -1$$

Differentiating (*i*),

$$\left(2y\frac{dy}{dx} - a \right)\frac{dy}{dx} + (y^2 - ax) \frac{d^2y}{dx^2} = a\frac{dy}{dx} - 2x \quad \therefore \quad \frac{d^2y}{dx^2} \text{ at } (3a/2, 3a/2) = -32/3a$$

Hence ρ at $(3a/2, 3a/2)$ $$= \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1 + (-1)^2]^{3/2}}{-32/3a} = \frac{3a}{8\sqrt{2}} \text{ (in magnitude).}$$

(*ii*) We have $$y^2 = a^3 x^{-1} - x^2$$

$$\therefore \quad 2y\, dy/dx = -a^3x^{-2} - 2x \quad \text{or} \quad dy/dx = -a^3/(2x^2y) - x/y$$

At $(a, 0)$, $dy/dx \to \infty$, so we find dx/dy from $xy^2 = a^3 - x^3$

$$\therefore \quad x - 2y + y^2 \frac{dx}{dy} = -3x^2 \frac{dx}{dy}$$

or $$\frac{dx}{dy} = \frac{-2xy}{3x^2 + y^2} \quad \text{or} \quad \frac{dx}{dy} \text{ at } (a, 0) = 0.$$

$$\therefore \quad \frac{d^2x}{dy^2} = \frac{(3x^2 + y^2)\left(-2y\frac{dx}{dy} - 2x \right) - (-2xy)\left(6x\frac{dx}{dy} + 2y \right)}{(3x^2 + y^2)^2}$$

or $$\frac{d^2x}{dy^2} \text{ at } (a, 0) = \frac{(3a^2 + 0)(0 - 2a) - 0}{(3a^2 + 0)^2} = \frac{-2}{3a}$$

Hence $$\rho \text{ at } (a, 0) = \frac{\left[1 + \left(\frac{dx}{dy} \right)_{(a,0)} \right]^{3/2}}{\left(\frac{d^2x}{dy^2} \right)_{(a,0)}} = \frac{(1 + 0)^{3/2}}{(-2/3a)} = -\frac{3a}{2}.$$

Example 4.45. *Show that the radius of curvature at any point of the cycloid* $x = a\,(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ *is* $4a \cos \theta/2$. (*V.T.U., 2011 ; P.T.U., 2006*)

Solution. We have $$\frac{dx}{d\theta} = a\,(1 + \cos\theta), \; \frac{dy}{d\theta} = a \sin\theta.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a \sin \theta}{a(1+\cos\theta)} = \frac{2 \sin \theta/2 \cos \theta/2}{2\cos^2 \theta/2} = \tan \theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx} = \frac{1}{2} \sec^2 \frac{\theta}{2} \cdot \frac{1}{a(1+\cos\theta)}$$

$$= \frac{1}{2} \sec^2 \frac{\theta}{2} \cdot \frac{1}{2a \cos^2 \theta/2} = \frac{1}{4a} \sec^4 \frac{\theta}{2}.$$

$$\therefore \quad \rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{4a(1+\tan^2\theta/2)^{3/2}}{\sec^4 \theta/2}$$

$$= 4a \cdot (\sec^2 \theta/2)^{3/2} \cdot \cos^4 \theta/2 = 4a \cos \theta/2.$$

Example 4.46. *Prove that the radius of curvature at any point of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, is three times the length of the perpendicular from the origin to the tangent at that point.* *(J.N.T.U., 2005 ; Bhopal, 2002 S)*

Solution. The parametric equation of the curve is

$$x = a \cos^3 t, y = a \sin^3 t.$$

$$\therefore \quad x' (= dx/dt) = -3a \cos^2 t \sin t, y' = 3a \sin^2 t \cos t.$$

$$x'' = -3a(\cos^3 t - 2 \cos t \sin^2 t) = 3a \cos t (2 \sin^2 t - \cos^2 t)$$

$$y'' = 3a(2 \sin t \cos^2 t - \sin^3 t) = 3a \sin t (2 \cos^2 t - \sin^2 t)$$

$$x'^2 + y'^2 = 9a^2(\cos^4 t \sin^2 t + \sin^4 t \cos^2 t) = 9a^2 \sin^2 t \cos^2 t$$

$$x'y'' - y'x'' = -9a^2 \cos^2 t \sin^2 t(2\cos^2 t - \sin^2 t)$$
$$- 9a^2 \cos^2 t \sin^2 t (2 \sin^2 t - \cos^2 t) = -9a^2 \sin^2 t \cos^2 t$$

$$\therefore \quad \rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{27a^3 \sin^3 t \cos^3 t}{-9a^2 \sin^2 t \cos^2 t} = -3a \sin t \cos t.$$

Since $\quad dy/dx = y'/x' = -\tan t,$

$\therefore$ Equation of the tangent at $(a \cos^3 t, a \sin^3 t)$ is $y - a \sin^3 t = -\tan t (x - a \cos^3 t)$

i.e., $\quad x \tan t + y - a \sin t = 0 \quad$ ...(*i*)

p, length of $\perp$ from $(0, 0)$ on $(i) = \dfrac{0 + 0 - a \sin t}{\sqrt{(\tan^2 t + 1)}} = -a \sin t \cos t$. Thus $\rho = 3p$.

EExample 4.47. *If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then show that $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$.* *(Rohtak, 2006 S ; Kurukshetra, 2005)*

Solution. Given parabola is $y^2 = 4ax$ or $x = at^2$, $y = 2at$. If dashes denote differentiation w.r.t. t, then

$$x' = 2at, y' = 2a ; x'' = 2a, y'' = 0.$$

$$\therefore \quad \rho \text{ at } (at^2, 2at) = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} = \frac{(4a^2t^2 + 4a^2)^{3/2}}{0 - 4a^2} = 2a(1+t^2)^{3/2} \quad \text{(Numerically)}$$

If $P(t_1)$ and $Q(t_2)$ be the extremities of the focal chord of the parabola, then

$$t_1 t_2 = -1 \quad i.e., \quad t_2 = -1/t_1 \quad \text{...(}i\text{)}$$

$$\therefore \quad \rho_1 \text{ at } P(t_1) = 2a(1+t_1^2)^{3/2} ; \rho_2 \text{ at } Q(t_2) = 2a(1+t_2^2)^{3/2}$$

Thus $\quad \rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3} = [(1+t_1^2)^{-1} + (1+t_2^2)^{-1}]$

$$= (2a)^{-2/3}\left[\frac{1}{1+t_1^2} + \frac{t_1^2}{1+t_1^2}\right] \quad \text{[By (}i\text{)]}$$

$$= (2a)^{-2/3}$$

Example 4.48. *Show that the radius of curvature of P on an ellipse $x^2/a^2 + y^2/b^2 = 1$ is CD^3/ab where CD is the semi-diameter conjugate to CP.* *(J.N.T.U., 2002)*

Solution. Two diameters of an ellipse are said to be conjugate if each bisects chords parallel to the other. If CP and CD are two semi-conjugate diameters and P is $(a \cos \theta, b \sin \theta)$ then D is $a \cos\left(\theta+\frac{\pi}{2}\right), b \sin\left(\theta+\frac{\pi}{2}\right)$ *i.e.,* $(-a \sin \theta, b \cos \theta)$.

Also $C(0, 0)$ is the centre of the ellipse.

$$\therefore \quad CD = \sqrt{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)}$$

At P, we have $x = a \cos \theta, y = b \sin \theta$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b \cos \theta}{-a \sin \theta} = \frac{-b}{a} \cot \theta \,; \frac{d^2y}{dx^2} = \frac{b}{a} \operatorname{cosec}^2 \theta \,.\, \frac{d\theta}{dx} = \frac{-b}{a^2} \operatorname{cosec}^3 \theta.$$

$$\therefore \quad \rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{\left(1+\dfrac{b^2}{a^2}\cot^2\theta\right)^{3/2}}{-\dfrac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$= \frac{a^2}{b \operatorname{cosec}^3 \theta} \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{a^3 \sin^3 \theta} \qquad \text{(Numerically)}$$

$$= \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab} = \frac{CD^3}{ab}.$$

Example 4.49. *Find ρ at the origin for the curves*

(*i*) $y^4 + x^3 + a(x^2 + y^2) - a^2 y = 0$ (*ii*) $y - x = x^2 + 2xy + y^2$

Solution. (*i*) Equating to zero the lowest degree terms, we get $y = 0$.

$\therefore$ x-axis is the tangent at the origin. Dividing throughout by y, we have

$$y^3 + x \,.\, \frac{x^2}{y} + a\left(\frac{x^2}{y} + y\right) - a^2 = 0$$

Let $x \to 0$, so that $\quad \underset{x\to\infty}{\text{Lt}} (x^2/2y) = \rho$.

$$\therefore \quad 0 + 0.2\rho + a(2\rho + 0) - a^2 = 0 \quad \text{or} \quad \rho = a/2.$$

(*ii*) Equating to zero the lowest degree terms, we get $y = x$, as the tangent at the origin, which is neither of the coordinates axes.

$\therefore$ Putting $y = px + qx^2/2 + \ldots$ in the given equation, we get

$$px + qx^2/2 + \ldots - x = x^2 + 2x\,(px + qx^2/2 + \ldots) + (px + qx^2/2 + \ldots)^2$$

Equating coefficients of x and x^2,

$$p - 1 = 0, q/2 = 1 + 2p + p^2 \quad i.e., \quad p = 1 \text{ and } q = 2 + 4 \cdot 1 + 2 \cdot 1^2 = 8.$$

$$\therefore \quad \rho\,(0, 0) = (1 + p^2)^{3/2}/q = (1 + 1)^{3/2}/8 = 1/2\sqrt{2}.$$

(4) Radius of curvature for polar curve $r = f(\theta)$ is given by

$$\rho = \frac{(\mathbf{r}^2 + \mathbf{r}_1^2)^{3/2}}{\mathbf{r}^2 + 2\mathbf{r}_1^2 - \mathbf{r}\mathbf{r}_2}$$

With the usual notations, we have from Fig. 4.10.

$$\psi = \theta + \phi$$

Differentiating w.r.t. s,

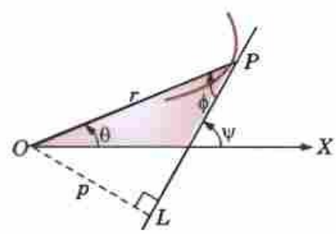

Fig. 4.10

$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds}$$

$$= \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right) \qquad \ldots(1)$$

Also we know that

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{r_1} \quad \text{or} \quad \phi = \tan^{-1}\left(\frac{r}{r_1}\right) \qquad \text{where } r_1 = \frac{dr}{d\theta}$$

Differentiating w.r.t. θ,

$$\frac{d\phi}{d\theta} = \frac{1}{1+(r/r_1)^2}\cdot\frac{r_1\cdot r_1 - rr_2}{r_1^2} = \frac{r_1^2 - rr_2}{r^2+r_1^2} \qquad ...(2)$$

Also, $$\frac{ds}{d\theta} = \sqrt{(r^2+r_1^2)} \qquad ...(3)$$

Substituting the value from (2) and (3) in (1),

$$\frac{1}{\rho} = \frac{1}{\sqrt{r^2+r_1^2}}\cdot\left(1+\frac{r_1^2-rr_2}{r^2+r_1^2}\right)$$

Hence $$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

(5) Radius of curvature for pedal curve $p = f(r)$ *is given by*

$$\rho = \mathbf{r}\frac{\mathbf{dr}}{\mathbf{dp}}$$

With the usual notation (Fig. 4.10), we have $\psi = \theta + \phi$

Differentiating w.r.t. s,

$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \qquad ...(1)$$

Also we know that $p = r\sin\phi$

$$\therefore \quad \frac{dp}{dr} = \sin\phi + r\cos\phi\,\frac{d\phi}{ds}$$

$$= r\frac{d\theta}{ds} + r\frac{dr}{ds}\cdot\frac{d\phi}{dr} \qquad \text{[By (3) and (4) of § 4.9 (2)]}$$

$$= r\left(\frac{d\theta}{ds}+\frac{d\phi}{ds}\right) = \frac{r}{\rho} \qquad \text{[By (1)]}$$

Hence $$\rho = r\frac{dr}{dp}.$$

Example 4.50. *Show that the radius of curvature at any point of the cardioid $r = a(1-\cos\theta)$ varies as $\sqrt{r}$.* *(V.T.U., 2003)*

Solution. Differentiating w.r.t. θ, we get

$$r_1 = a\sin\theta,\ r_2 = a\cos\theta$$

$$\therefore \quad (r^2+r_1^2)^{3/2} = [a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{3/2} = a^3[2(1-\cos\theta)]^{3/2}$$

$$r^2 - rr_2 + 2r_1^2 = a^2(1-\cos\theta)^2 - a^2(1-\cos\theta)\cos\theta + 2a^2\sin^2\theta = 3a^2(1-\cos\theta)$$

Thus $$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2-rr_2+2r_1^2} = \frac{a^3\,2\sqrt{2}(1-\cos\theta)^{3/2}}{3a^2(1-\cos\theta)}$$

$$= \frac{2\sqrt{2}}{3}a(1-\cos\theta)^{1/2} = \frac{2\sqrt{2}a}{3}\left(\frac{r}{a}\right)^{1/2} \propto \sqrt{r}.$$

Otherwise. The pedal equation of this cardioid is $2ap^2 = r^3$ ...(*i*)

Differentiating w.r.t. p, we get

that $$4ap = 3r^2\frac{dr}{dp} \text{ whence } \rho = r\frac{dr}{dp} = \frac{4ap}{3r} = \frac{4ar^{3/2}}{3r\cdot\sqrt{(2a)}} \propto \sqrt{r}.$$

$$[\because \quad p = r^{3/2}/\sqrt{(2a)} \text{ from } (i)]$$

PROBLEMS 4.11

1. Find the radius of curvature at any point
   (*i*) $(at^2, 2at)$ of the parabola $y^2 = 4ax$.
   (*ii*) (o, c) of the catenary $y = c \cosh x/c$.
   (*iii*) $(a, 0)$ of the curve $y = x^3 (x - a)$. (*V.T.U., 2010*)
2. Show that for (*i*) the rectangular hyperbola $xy = c^2$, $\rho = \dfrac{(x^2 + y^2)^{3/2}}{2c^2}$. (*Rohtak, 2005 ; Madras, 2000*)
   (*ii*) the curve $y = ae^{x/a}$, $\rho = a \sec^2 \theta \operatorname{cosec} \theta$ where $\theta = \tan^{-1} (y/a)$. (*Rajasthan, 2006*)
3. Show that the radius of curvature at
   (*i*) $(a, 0)$ on the curve $y^2 = a^2 (a - x)/x$ is $a/2$. (*V.T.U., 2000 S*)
   (*ii*) $(a/4, a/4)$ on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is $a/\sqrt{2}$. (*J.N.T.U., 2006 S*)
   (*iii*) $x = \pi/2$ of the curve $y = 4 \sin x - \sin 2x$ is $5\sqrt{5}/4$. (*V.T.U., 2009 S*)
4. For the curve $y = \dfrac{ax}{a + x}$, show that $\left(\dfrac{2\rho}{a}\right)^{2/3} = \left(\dfrac{x}{y}\right)^2 + \left(\dfrac{y}{x}\right)^2$. (*V.T.U., 2008*)
5. Find the radius of curvature at any point on the
   (*i*) *ellipse* : $x = a \cos \theta, y = b \sin \theta$. (*V.T.U., 2003*)
   (*ii*) *cycloid* : $x = a (\theta - \sin \theta), y = a (1 - \cos \theta)$.
   (*iii*) *curve* : $x = a (\cos t + t \sin t), y = a (\sin t - t \cos t)$.
6. Show that the radius of curvature (*i*) at the point $(a \cos^3 \theta, a \sin^3 \theta)$ on the curve
   $x^{2/3} + y^{2/3} = a^{2/3}$ is $3a \sin \theta \cos \theta$. (*Anna, 2009*)
   (*ii*) at the point t on the curve $x = e^t \cos t, y = e^t \sin t$ is $\sqrt{2}e^t$. (*Calicut, 2005*)
7. If ρ be the radius of curvature at any point P on the parabola, $y^2 = 4ax$ and S be its focus, then show that ρ^2 varies as $(SP)^3$. (*Kurukshetra, 2006*)
8. Prove that for the ellipse in pedal form $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{1}{a^2b^2}$, the radius of curvature at the point (p, r) is $\rho = a^2 b^2/p^3$. (*V.T.U., 2010 S*)
9. Show that the radius of curvature at an end of the major axis of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is equal to the semi-latus rectum. (*Osmania, 2000 S*)
10. Show that the radius of curvature at each point of the curve $x = a (\cos t + \log \tan t/2)$, $y = a \sin t$, is inversely proportional to the length of the normal intercepted between the point on the curve and the x-axis. (*J.N.T.U., 2003*)
11. Find the radius of curvature at the origin for
    (*i*) $x^3 + y^3 - 2x^2 + 6y = 0$ (*Burdwan, 2003*)
    (*ii*) $2x^4 + 3y^4 + 4x^2y + xy - y^2 + 2x = 0$
    (*iii*) $y^2 = x^2 (a + x)/(a - x)$.
12. Find the radius of the curvature at the point (r, θ) on each of the curves :
    (*i*) $r = a (1 - \cos \theta)$ (*Kurukshetra, 2005*)
    (*ii*) $r^n = a^n \cos n \theta$. (*P.T.U., 2010; J.N.T.U., 2006*)
13. For the cardioid $r = a(1 + \cos \theta)$, show that ρ^2/r is constant. (*P.T.U., 2005*)
14. Find the radius of curvature for the parabola $2a/r = 1 + \cos \theta$. (*Kurukshetra, 2006*)
15. If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $r = a (1 + \cos \theta)$ which passes through the pole, show that $\rho_1^2 + \rho_2^2 = 16 a^2/9$.
16. For any curve $r = f(\theta)$, prove that $\dfrac{r}{\rho} = \sin \phi \left(1 + \dfrac{d\phi}{d\theta}\right)$.

4.12 (1) CENTRE OF CURVATURE at any point P (x, y) on the curve y = f (x) is given by

$$\bar{x} = x - \frac{y_1(1 + y_1^2)}{y_2}, \quad \bar{y} = y + \frac{1 + y_1^2}{y_2}.$$

Let $C(x, y)$ be the centre of curvature and ρ the radius of curvature of the curve at $P(x, y)$ (Fig. 4.11). Draw PL and CM $\perp$s to OX and $PN \perp CM$. Let the tangent at P make an $\angle\psi$ with the x-axis. Then $\angle NCP = 90° - \angle NPC = \angle NPT = \psi$

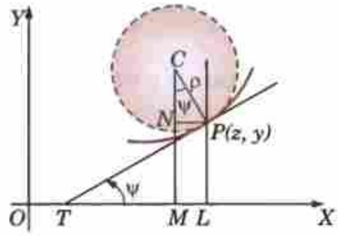

Fig. 4.11

$$\therefore \quad \bar{x} = OM = OL - ML = OL - NP$$

$$= x - \rho \sin\psi = x - \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{y_1}{\sqrt{(1+y_1^2)}}$$

$$\left[\because \tan\psi = y_1, \quad \therefore \sin\psi = \frac{y_1}{\sqrt{(1+y_1^2)}}\right]$$

$$= x - \frac{y_1(1+y_1^2)}{y_2}$$

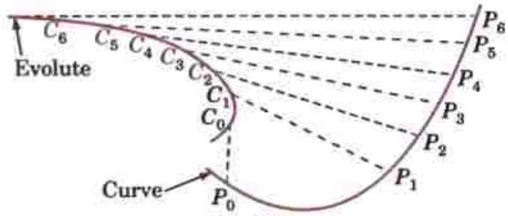

Fig. 4.12

and $$\bar{y} = MC = MN + NC = LP + \rho\cos\psi$$

$$\left[\because \sec\psi = \sqrt{(1+\tan^2\psi)} = \sqrt{(1+y_1^2)}\right]$$

$$= y + \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{1}{\sqrt{(1+y_1^2)}} = y + \frac{1+y_1^2}{y_2}$$

Cor. Equation of the circle of curvature at P is $(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$.

(2) Evolute. *The locus of the centre of curvature for a curve is called its* **evolute** *and the curve is called an* **involute** *of its evolute.* (*Fig. 4.12*)

Example 4.51. *Find the coordinates of the centre of curvature at any point of the parabola $y^2 = 4ax$. Hence show that its evolute is*

$$27ay^2 = 4(x - 2a)^3.$$

(*V.T.U., 2000*)

Solution. We have $\quad 2yy_1 = 4a \quad$ *i.e.,* $\quad y_1 = 2a/y$

and $$y_2 = -\frac{2a}{y^2} \cdot y_1 = -\frac{4a^2}{y^3}$$

If $(\bar{x}, \bar{y})$ be the centre of curvature, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x - \frac{2a/y(1+4a^2/y^2)}{-4a^2/y^3}$$

$$= x + \frac{y^2+4a^2}{2a} = x + \frac{4ax+4a^2}{2a} = 3x + 2a \qquad [\because y^2 = 4ax] \qquad ...(i)$$

and $$\bar{y} = y + \frac{1+y_1^2}{y_2} = y + \frac{1+4a^2/y^2}{-4a^2/y^2}$$

$$= y - \frac{y(y^2+4a^2)}{4a^2} = \frac{-y^3}{4a^2} = -\frac{2x^{3/2}}{\sqrt{a}} \qquad ...(ii)$$

To find the evolute, we have to eliminate x from (i) and (ii)

$$\therefore \quad (\bar{y})^2 = \frac{4x^3}{a} = \frac{4}{a}\left(\frac{\bar{x}-2a}{3}\right)^3 \quad \text{or} \quad 27a(\bar{y})^2 = 4(\bar{x} - 2a)^3.$$

Thus the locus of $(\bar{x}, \bar{y})$ *i.e.,* evolute, is $27ay^2 = 4(x - 2a)^3$.

Example 4.52. *Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is another equal cycloid.* (*Madras, 2006*)

Solution. We have $y_1 = \dfrac{dy}{d\theta} \div \dfrac{dx}{d\theta} = \dfrac{a\sin\theta}{a(1-\cos\theta)} = \cot\dfrac{\theta}{2}$.

$$y^2 = \frac{d}{dx}(y_1) = \frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right)\cdot\frac{d\theta}{dx}$$

$$= -\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{1}{2}\cdot\frac{1}{a(1-\cos\theta)} = -\frac{1}{4a\sin^4\theta/2}$$

If $(\bar{x}, \bar{y})$ be the centre of curvature, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = a(\theta - \sin\theta) + \cot\frac{\theta}{2}\left(-4a\sin^4\frac{\theta}{2}\right)\left(1+\cot^2\frac{\theta}{2}\right)$$

$$= a(\theta - \sin\theta) + \frac{\cos\theta/2}{\sin\theta/2}\cdot 4a\sin^4\frac{\theta}{2}\cdot\operatorname{cosec}^2\frac{\theta}{2}$$

$$= a(\theta - \sin\theta) + 4a\sin\theta/2\cos\theta/2 = a(\theta - \sin\theta) + 2a\sin\theta = a(\theta + \sin\theta)$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = a(1-\cos\theta) + \left(1+\cot^2\frac{\theta}{2}\right)\left(-4a\sin^4\frac{\theta}{2}\right)$$

$$= a(1-\cos\theta) - 4a\sin^4\theta/2\,.\operatorname{cosec}^2\theta/2$$

$$= a(1-\cos\theta) - 4a\sin^2\theta/2$$

$$= a(1-\cos\theta) - 2a(1-\cos\theta) = -a(1-\cos\theta)$$

Hence the locus of $(\bar{x}, \bar{y})$ *i.e.*, the evolute, is given by

$x = a(\theta + \sin\theta)$, $y = -a(1-\cos\theta)$ which is another equal cycloid.

(3) Chord or curvature *at a given point of a curve*

(*i*) parallel to x-axis $= 2\rho\sin\psi$

(*ii*) parallel to y-axis $= 2\rho\cos\psi$

Consider the circle of curvature at a given point P on a curve. Let C be the centre and ρ the radius of curvature at P so that $PQ = 2\rho$. (Fig. 4.13)

Let PL, PM be the chords of curvature parallel to the axes of x and y respectively. Let the tangent PT make an $\angle\psi$ with the x-axis so that $\angle LQP = \angle QPM = \psi$.

Then from the rt. $\angle$ed ΔPLQ,

$$PL = 2\rho\sin\psi\sigma$$

and $$PM = 2\rho\cos\psi.$$

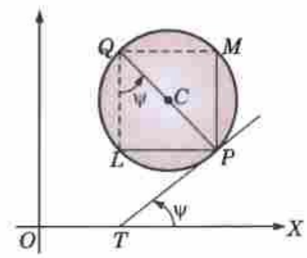

Fig. 4.13

4.13 (1) ENVELOPE

The equation $x\cos\alpha + y\sin\alpha = 1$ ...(1)

represents a straight line for a given value of α. If different values are given to α, we get different straight lines. All these straight lines thus obtained are said to constitute a family of straight lines.

In general, the *curves corresponding to the equation $f(x, y, \alpha) = 0$ for different values of α, constitute a* **family of curves** *and α is called the* **parameter** *of the family.*

The envelope of a family of curves is the curve which touches each member of the family. For example, we know that all the straight lines of the family (1) touch the circle

$$x^2 + y^2 = 1 \qquad ...(2)$$

i.e., the envelope of the family of lines (1) is the circle (2)—Fig. 4.14, which may also be seen as the locus of the ultimate points of intersection of the consecutive members of the family of lines (1). This leads to the following :

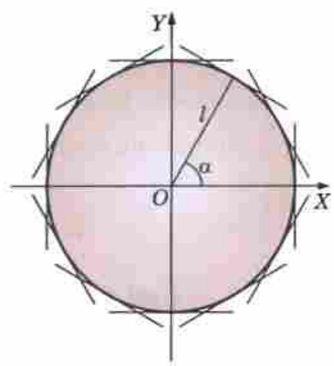

Fig. 4.14

Def. *If $f(x, y, \alpha) = 0$ and $f(x, y, \alpha + \delta\alpha) = 0$ be two consecutive members of a family of curves, then the locus of their ultimate points of intersection is called the* **envelope** *of that family.*

(2) Rule to find the envelope of the family of curves $f(x, y, \alpha) = 0$:

Eliminate α from $f(x, y, \alpha) = 0$ and $\dfrac{\partial f(x, y, \alpha)}{\partial \alpha} = 0$.

Example 4.53. *Find the envelope of the family of lines $y = mx + \sqrt{(1+m^2)}$, m being the parameter.*

Solution. We have $(y - mx)^2 = 1 + m^2$ ...(*i*)

Differentiating (*i*) partially with respect to m,

$$2(y - mx)(-x) = 2m \quad \text{or} \quad m = xy/(x^2 - 1) \qquad ...(ii)$$

Now eliminating m from (*i*) and (*ii*)

Substituting the value of m in (*i*), we get

$$\left(y - \frac{x^2 y}{x^2 - 1}\right)^2 = 1 + \left(\frac{xy}{x^2 - 1}\right)^2 \quad \text{or} \quad y^2 = (x^2 - 1)^2 + x^2y^2$$

or $x^2 + y^2 = 1$ which is the required equation of the envelope.

Obs. Sometimes the equation to the family of curves contains two parameters which are connected by a relation. In such cases, we eliminate one of the parameters by means of the given relation, then proceed to find the envelope.

Example 4.54. *Find the envelope of a system of concentric and coaxial ellipses of constant area.*

Solution. Taking the common axes of the system of ellipses as the coordinate axes, the equation to an ellipse of the family is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ where } a \text{ and } b \text{ are the parameters.} \qquad ...(i)$$

The area of the ellipse $= \pi ab$ which is given to be constant, say $= \pi c^2$.

$$\therefore \qquad ab = c^2 \quad \text{or} \quad b = c^2/a. \qquad ...(ii)$$

Substituting in (*i*), $\dfrac{x^2}{a^2} + \dfrac{y^2}{(c^2/a^2)} = 1$ or $x^2 a^{-2} + (y^2/c^4)\, a^2 = 0$ ...(*iii*)

which is the given family of ellipses with a as the only parameter.

Differentiating partially (*iii*) with respect to a,

$$-2x^2 a^{-3} + 2(y^2/c^4)\, a = 0 \quad \text{or} \quad a^2 = c^2\, x/y \qquad ...(iv)$$

Eliminate a from (*iii*) and (*iv*).

Substituting the value of a^2 in (*iii*), we get

$$x^2(y/c^2x) + (y^2/c^4)(c^2x/y) = 1 \text{ or } 2xy = c^2$$

which is the required equation of the envelope. P

(3) Evolute *of a curve is the envelope of the normals to that curve* (*Fig. 4.12*)

Example 4.55. *Find the evolute of the parabola $y^2 = 4ax$.* (*Madras, 2003*)

Solution. Any normal to the parabola is $y = mx - 2am - am^3$ ...(*i*)

Differentiating it with respect to m partially,

$$0 = x - 2a - 3am^2 \text{ or } m = [(x - 2a)/3a]^{1/2}$$

Substituting this value of m in (*i*),

$$y = \left(\frac{x - 2a}{3a}\right)^{1/2}\left[x - 2a - a \cdot \frac{x - 2a}{3a}\right]$$

Squaring both sides, we have

$$27ay^2 = 4(x - 2a)^3$$

which is the evolute of the parabola. (*cf.* Example 4.51).

PROBLEMS 4.12

1. Find the coordinates of the centre of curvature at $(at^2, 2at)$ on the parabola $y^2 = 4ax$. *(V.T.U., 2000 S)*
2. If the centre of curvature of the ellipse $x^2/a^2 + y^2/b^2 = 1$ at one end of the minor axis lies at the other end, then show that the eccentricity of the ellipse is $1/\sqrt{2}$. *(Anna, 2005 S ; Madras, 2003)*
3. Show that the equation of the evolute of the
   (i) *parabola* $x^2 = 4ay$ is $4(y-2a)^3 = 27ax^2$. *(Anna, 2009)*
   (ii) *ellipse* $x = a\cos\theta, y = b\sin\theta$ (*i.e.*, $x^2/a^2 + y^2/b^2 = 1$) is
   $$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}.$$
   (iii) *rectangular hyperbola* $xy = c^2$, (*i.e.*, $x = ct, y = c/t$) is
   $$(x+y)^{2/3} - (x-y)^{2/3} = (4c)^{2/3}.$$ *(Anna, 2003)*
4. Find the evolute of (i) cycloid $x = a(t+\sin t), y = a(1-\cos t)$
   (ii) the curve $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$. *(Anna, 2009 S)*
5. Find the evolute of the curve $x = a\cos^3\theta, y = a\sin^3\theta$ *i.e.*, $x^{2/3} + y^{2/3} = a^{2/3}$. *(Osmania, 2002)*
6. Show that the evolute of the curve $x = a(\cos t + \log\tan t/2), y = a\sin t$ is $y = a\cosh x/a$. *(Anna, 2005 S)*
7. Find the circle of curvature at the point (i) $(a/4, a/4)$ of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$.
   (ii) $(3/2, 3/2)$ of the curve $x^3 + y^3 = 3xy$ *(Anna, 2009 ; Madras, 2006 ; Calicut, 2005)*
8. Show that the circle of curvature at the origin for the curve $x + y = ax^2 + by^2 + cx^3$ is
   $$(a+b)(x^2+y^2) = 2(x+y).$$ *(Nagpur, 2009)*
9. If C_x, C_y be the chords of curvature parallel to the axes at any point on the curve $y = ae^{x/a}$, prove that
   $$\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{2aC_x}.$$
10. In the curve $y = a\cosh x/a$, prove that the chord of curvature parallel to y-axis is the double the ordinate.

Find the envelope of the following family of lines :

11. $y = mx + a/m$, m being the parameter. *(Madras, 2006)*
12. $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$, α being the parameter.
13. $y = mx - 2am - am^3$.
14. $y = mx + \sqrt{(a^2m^2 + b^2)}$, m being the parameter. *(Anna, 2009)*
15. Find the envelope of the family of parabolas $y = x\tan\alpha - \frac{gx^2}{2u^2\cos\alpha}$, α being the parameter.
16. Find the envelope of the straight line $x/a + y/b = 1$, where the parameters a and b are connected by the relation :
   (i) $a + b = c$. (ii) $ab = c^2$
   (iii) $a^2 + b^2 = c^2$.
17. Find the envelope of the family of ellipses $x^2/a^2 + y^2/b^2 = 1$ for which $a + b = c$. *(Madras, 2006)*

Prove that the evolute of the

18. *ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$. *(J.N.T.U., 2006 ; Anna, 2005)*
19. *hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$. *(Anna, 2009)*
20. *parabola* $x^2 = 4by$ is $27bx^2 = 4(y-2b)^3$.

4.14 (1) INCREASING AND DECREASING FUNCTIONS

In the function $y = f(x)$, *if y increases as x increases* (as at A), *it is called an* **increasing function of** x. On the contrary, *if y decreases as x increases* (as at C), *it is called a* **decreasing function** *of* x.

Let the tangent at any point on the graph of the function make an $\angle\psi$ with the x-axis (Fig. 4.15) so that

$$dy/dx = \tan\psi$$

At any point such as A, where the function is increasing $\angle\psi$ is acute *i.e.*, dy/dx is positive. At a point such as C, where the function is decreasing $\angle\psi$ is obtuse *i.e.*, dy/dx is negative.

Hence the *derivative of an increasing function is + ve*, and *the derivative of a decreasing function is – ve.*

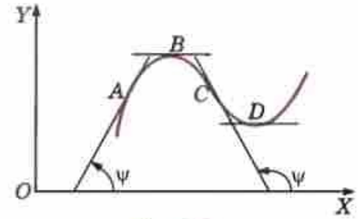

Fig. 4.15

Obs. *If the derivative is zero* (as at B or D), *then y is neither increasing nor decreasing.* In such cases, we say *that the function is* **stationary.**

(2) Concavity, Convexity and Point of Inflexion

(*i*) *If a portion of the curve on both sides of a point, however small it may be, lies above the tangent* (*as at D*), *then the curve is said to be* **concave upwards** at D where d^2y/dx^2 is positive.

(*ii*) *If a portion of the curve on both sides of a point lies below the tangent* (as at B), *then the curve is said to be* **Convex upwards** at B where d^2y/dx^2 is negative.

(*iii*) *If the two portions of the curve lie on different sides of the tangent thereat* (*i.e.*, the curve crosses the tangent (as at C), *then the point C* is said to be a **point of inflexion** of the curve.

At a point of inflexion $\dfrac{d^2y}{dx^2} = 0$ and $\dfrac{d^3y}{dx^3} \neq 0.$

4.15 (1) MAXIMA AND MINIMA

Consider the graph of the continuous function $y = f(x)$ in the interval (x_1, x_2) (Fig. 4.16). Clearly the point P_1 is the highest in its own immediate neighbourhood. So also is P_3. At each of these points P_1, P_3 the function is said to have a *maximum* value.

On the other hand, the point P_2 is the lowest in its own immediate neighbourhood. So also is P_4. At each of these points P_2, P_4 the function is said to have a *minimum* value.

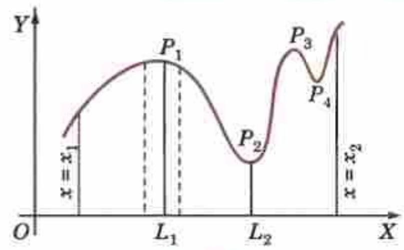

Fig. 4.16

Thus, we have

Def. *A function f(x) is said to have a* **maximum** *value at x = a, if there exists a small number h, however small, such that f(a) > both f(a – h) and f(a + h).*

A function f(x) is said to have a **minimum** *value at x = a, if there exists a small number h, however small, such that f(a) < both f(a – h) and f(a + h).*

Obs. 1. *The maximum and minimum values of a function taken together are called its* **extreme** *values and the points at which the function attains the extreme values are called the* **turning points** *of the function.*

Obs. 2. *A maximum or minimum value of a function is not necessarily the greatest or least value of the function in any finite interval.* The maximum value is simply the greatest value in the immediate neighbourhood of the maxima point or the minimum value is the least value in the immediate neighbourhood of the minima point. In fact, *there may be several maximum and minimum values of a function in an interval and a minimum value may be even greater than a maximum value.*

Obs. 3. It is seen from the Fig. 4.16 that *maxima and minima values occur alternately.*

(2) Conditions for maxima and minima. At each point of extreme value, it is seen from Fig. 4.16 that the tangent to the curve is parallel to the x-axis, *i.e.*, its slope (= dy/dx) is zero. Thus if *the function is maximum or minimum at* $x = a$, *then* $(dy/dx)_a = 0$.

Around a maximum point say, P_1 $(x = a)$, the curve is increasing in a small interval $(a - h, a)$ before L_1 and decreasing in $(a, a + h)$ after L_1 where h is positive and small.

i.e., in $(a - h, a)$, $dy/dx \geq 0$; at $x = a$, $dy/dx = 0$ and in $(a, a + h)$, $dy/dx \leq 0$.

Thus dy/dx (which is a function of x) changes sign from positive to negative in passing through P_1, *i.e.*, it is a decreasing function in the interval $(a - h, a + h)$ and therefore, its derivative d^2y/dx^2 is negative at $P_1 (x = a)$.

Similarly, around a minimum point say P_2, dy/dx changes sign from negative to positive in passing through P_2, *i.e.*, it is an increasing function in the small interval around L_2 and therefore its derivative d^2y/dx^2 is positive at P_2.

Hence (*i*) *f(x) is maximum at x = a if f′(a) = 0 and f″(a) is – ve [i.e., f′(a) changes sign from + ve to – ve]*
(*ii*) *f(x) is minimum at x = a, if f′(a) = 0 and f″(a) is + ve [i.e., f′(a) changes sign from – ve to + ve]*

Obs. A maximum or a minimum value is a stationary value but a stationary value may neither be a maximum nor a minimum value.

(3) Procedure for finding maxima and minima

(*i*) *Put the given function = f(x)*

(*ii*) *Find f′(x) and equate it to zero. Solve this equation and let its roots be a, b, c, ...*

(*iii*) *Find f″(x) and substitute in it by turns x = a, b, c, ...*

If f″(a) is – ve, f(x) is maximum at x = a.

If f″(a) is + ve, f″(x) is minima at x = a.

(*iv*) Sometimes $f''(x)$ may be difficult to find out or $f''(x)$ may be zero at $x = a$. In such cases, see if $f'(x)$ *changes sign from + ve to – ve as x passes through a, then f(x) is maximum at x = a.*

If f′(x) changes sign from – ve to + ve as x passes through a, f(x) is minimum at x = a.

If f′(x) does not change sign while passing through x = a, f(x) is neither maximum nor minimum at x = a.

Example 4.56. *Find the maximum and minimum values of $3x^4 - 2x^3 - 6x^2 + 6x + 1$ in the interval (0, 2).*

Solution. Let $f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

Then $f'(x) = 12x^3 - 6x^2 - 12x + 6 = 6(x^2 - 1)(2x - 1)$

$\therefore$ $f'(x) = 0$ when $x = \pm 1, \frac{1}{2}$.

So in the interval (0, 2) $f(x)$ can have maximum or minimum at $x = \frac{1}{2}$ or 1.

Now $f''(x) = 36x^2 - 12x - 12 = 12(3x^2 - x - 1)$ so that $f''\left(\frac{1}{2}\right) = -9$ and $f''(1) = 12$.

$\therefore$ $f(x)$ has a maximum at $x = \frac{1}{2}$ and a minimum at $x = 1$.

Thus the maximum value $= f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) + 1 = 2\frac{7}{16}$

and the minimum value $= f(1) = 3(1)^4 - 2(1)^3 - 6(1)^2 + 6(1) + 1 = 2$.

Example 4.57. *Show that sin x (1 + cos x) is a maximum when x = π/3.* *(Bhopal, 2009 ; Rajasthan, 2005)*

Solution. Let $f(x) = \sin x\,(1 + \cos x)$

Then $f'(x) = \cos x\,(1 + \cos x) + \sin x\,(-\sin x)$

$= \cos x\,(1 + \cos x) - (1 - \cos^2 x) = (1 + \cos x)(2\cos x - 1)$

$\therefore$ $f'(x) = 0$ when $\cos x = \frac{1}{2}$ or -1 *i.e.*, when $x = \pi/3$ or π.

Now $f''(x) = -\sin x\,(2\cos x - 1) + (1 + \cos x)(-2\sin x) = -\sin x(4\cos x + 1)$

so that $f''(\pi/3) = -3\sqrt{2}/2$ and $f''(\pi) = 0$.

Thus $f(x)$ has a maximum at $x = \pi/3$.

Since $f''(\pi)$ is 0, let us see whether $f'(x)$ changes sign or not.

When x is slightly $< \pi$, $f'(x)$ is – ve, then when x is slightly $> \pi$, $f'(x)$ is again – ve *i.e.*, $f'(x)$ does not change sign as x passes through π. So $f(x)$ is neither maximum nor minimum at $x = \pi$.

(4) Practical Problems

In many problems, the function (whose maximum or minimum value is required) is not directly given. It has to be formed from the given data. If the function contains two variables, one of them has to be eliminated with the help of the other conditions of the problem. A number of problems deal with triangles, rectangles, circles, spheres, cones, cylinders etc. The student is therefore, advised to remember the formulae for areas, volumes, surfaces etc. of such figures.

Example 4.58. *A window has the form of a rectangle surmounted by a semi-circle. If the perimeter is 40 ft., find its dimensions so that the greatest amount of light may be admitted.* *(Madras, 2000 S)*

Solution. The greatest amount of light may be admitted means that the area of the window may be maximum.

Let x ft. be the radius of the semi-circle so that one side of the rectangle is $2x$ ft. (Fig. 4.17). Let the other side of the rectangle y ft. Then the perimeter of the whole figure

$$= \pi x + 2x + 2y = 40 \text{ (given) and the area } A = \tfrac{1}{2}\pi x^2 + 2xy. \quad ...(i)$$

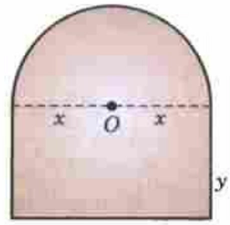

Fig. 4.17

Here A is a function of two variables x and y. To express A in terms of one variable x (say), we substitute the value of y from (i) in it.

$$\therefore \quad A = \tfrac{1}{2}\pi x^2 + x[40 - (\pi + 2)x] = 40\,x - \left(\frac{\pi}{2} + 2\right)x^2$$

Then $$\frac{dA}{dx} = 40 - (\pi + 4)x$$

For A to be maximum or minimum, we must have $dA/dx = 0$ *i.e.*, $40 - (\pi + 4)x = 0$

or $$x = 40/(\pi + 4)$$

$\therefore$ From (i), $$y = \tfrac{1}{2}[40 - (\pi + 2)x] = \tfrac{1}{2}[40 - (\pi + 2)\,40/(\pi + 4)] = 40/(\pi + 4) \text{ i.e., } x = y$$

Also $$\frac{d^2A}{dx^2} = -(\pi + 4), \text{ which is negative.}$$

Thus the area of the window is maximum when the radius of the semi-circle is equal to the height of the rectangle.

Example 4.59. *A rectangular sheet of metal of length 6 metres and width 2 metres is given. Four equal squares are removed from the corners. The sides of this sheet are now turned up to form an open rectangular box. Find approximately, the height of the box, such that the volume of the box is maximum.*

Solution. Let the side of each of the squares cut off be x m so that the height of the box is x m and the sides of the base are $6 - 2x$, $2 - 2x$ m (Fig. 4.18).

$\therefore$ Volume V of the box

$$= x(6 - 2x)(2 - 2x) = 4(x^3 - 4x^2 + 3x)$$

Fig. 4.18

Then $$\frac{dV}{dx} = 4(3x^2 - 8x + 3)$$

For V to be maximum or minimum, we must have

$$dV/dx = 0 \text{ i.e., } 3x^2 - 8x + 3 = 0$$

$$\therefore \quad x = \frac{8 \pm \sqrt{[64 - 4 \times 3 \times 3]}}{6} = 2.2 \text{ or } 0.45 \text{ m.}$$

The value $x = 2.2$ m is inadmissible, as no box is possible for this value.

Also $\dfrac{d^2V}{dx^2} = 4(6x - 8)$, which is – ve for $x = 0.45$ m.

Hence the volume of the box is maximum when its height is 45 cm.

Example 4.60. *Show that the right circular cylinder of given surface (including the ends) and maximum volume is such that its height is equal to the diameter of the base.*

Solution. Let r be the radius of the base and h, the height of the cylinder.

Then given surface $S = 2\pi rh + 2\pi r^2$ ...(i) and the volume $V = \pi r^2 h$ ...(ii)

Hence V is a function of two variables r and h. To express V in terms of one variable only (say r), we substitute the value of h from (i) in (ii).

Then $$V = \pi r^2 \left(\frac{S - 2\pi r^2}{2\pi r}\right) = \frac{1}{2}Sr - \pi r^3 \quad \therefore \quad \frac{dV}{dr} = \frac{1}{2}S - 3\pi r^2.$$

For V to be maximum or minimum, we must have $dV/dr = 0$,

i.e., $$\tfrac{1}{2}S - 3\pi r^2 = 0 \quad \text{or} \quad r = \sqrt{(S/6\pi)}.$$

Also $\dfrac{d^2V}{dr^2} = -6\pi r$, which is negative for $r = \sqrt{(S/6\pi)}$.

Hence V is maximum for $r = \sqrt{(S/6\pi)}$.

i.e., for $6\pi r^2 = S = 2\pi rh + 2\pi r^2$ *i.e.*, for $h = 2r$, which proves the required result. [By (*i*)]

Example 4.61. *Show that the diameter of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is equal to the radius of the cone.*

Solution. Let r be the radius OA of the base and α the semi-vertical angle of the given cone (Fig. 4.19). Inscribe a cylinder in it with base-radius $OL = x$.

Then the height of the cylinder LP

$$= LA \cot\alpha = (r - x)\cot\alpha$$

$\therefore$ The curved surface S of the cylinder

$$= 2\pi x . LP = 2\pi x(r - x)\cot\alpha$$
$$= 2\pi \cot\alpha\,(rx - x^2)$$

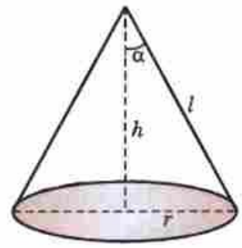

Fig. 4.19

$\therefore$ $$\frac{dS}{dx} = 2\pi\cot\alpha\,(r - 2x) = 0 \text{ for } x = r/2.$$

and $$\frac{d^2S}{dx^2} = -4\pi\cot\alpha$$

Hence S is maximum when $x = r/2$.

Example 4.62. *Find the altitude and the semi-vertical angle of a cone of least volume which can be circumscribed to a sphere of radius a.*

Solution. Let h be the height and α the semi-vertical angle of the cone so that its radius $BD = h\tan\alpha$ (Fig. 4.20).

$\therefore$ The volume V of the cone is given by

$$V = \frac{1}{3}\pi(h\tan\alpha)^2 h = \frac{1}{3}\pi h^3\tan^2\alpha.$$

Now we must express $\tan\alpha$ in terms of h.

In the rt. $\angle d\ \Delta AEO$,

$$EA = \sqrt{(OA^2 - a^2)} = \sqrt{[(h-a)^2 - a^2]} = \sqrt{(h^2 - 2ha)}$$

$\therefore$ $$\tan\alpha = \frac{EO}{EA} = \frac{a}{\sqrt{(h^2 - 2ha)}}$$

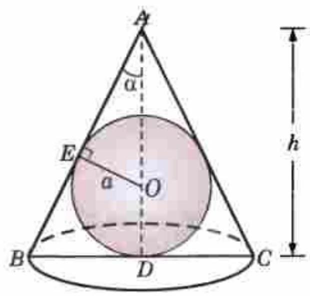

Fig. 4.20

Thus $$V = \frac{1}{3}\pi h^3 \cdot \frac{a^2}{h^2 - 2ha} = \frac{1}{3}\pi a^3 \cdot \frac{h^2}{h - 2a}$$

$\therefore$ $$\frac{dV}{dh} = \frac{1}{3}\pi a^2 \cdot \frac{(h-2a)2h - h^2\cdot 1}{(h-2a)^2} = \frac{1}{3}\pi a^2 \cdot \frac{h(h-4a)}{(h-2a)^2}$$

Thus $\dfrac{dV}{dh} = 0$ for $h = 4a$, the other value $(h = 0)$ being not possible.

Also dV/dh is – ve when h is slightly $< 4a$, and it is + ve when h is slightly $> 4a$.

Hence V is minimum (*i.e.* least) when $h = 4a$

and $$\alpha = \sin^{-1}\left(\frac{a}{OA}\right) = \sin^{-1}\left(\frac{a}{3a}\right) = \sin^{-1}\frac{1}{3}.$$

Example 4.63. *Find the volume of the largest possible right-circular cylinder that can be inscribed in a sphere of radius a.*

Solution. Let O be the centre of the sphere of radius a. Construct a cylinder as shown in Fig. 4.21. Let $OA = r$.

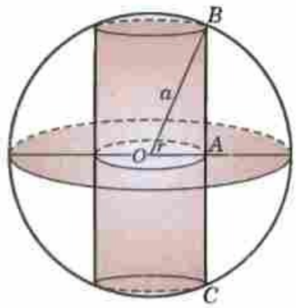

Fig. 4.21

Then $$AB = \sqrt{(OB^2 - OA^2)} = \sqrt{(a^2 - r^2)}$$

$\therefore$ Height h of the cylinder $= 2 \cdot AB = 2\sqrt{(a^2 - r^2)}$.

Thus volume V of the cylinder

$$= \pi r^2 h = 2\pi r^2 \sqrt{(a^2 - r^2)}$$

$$\therefore \quad \frac{dV}{dr} = 2\pi \{2r\sqrt{(a^2 - r^2)} + r^2 \cdot \frac{1}{2}(a^2 - r^2)^{-1/2}(-2r)\}$$

$$= \frac{2\pi r(2a^2 - 3r^2)}{\sqrt{(a^2 - r^2)}}$$

The $dV/dr = 0$ when $r^2 = 2a^2/3$, the other value ($r = 0$) being not admissible.

Now $$\frac{d^2V}{dr^2} = 2\pi \frac{\sqrt{(a^2 - r^2)}(2a^2 - 9r^2) - r(2a^2 - 3r^2) \times \frac{1}{2}(a^2 - r^2)^{-1/2} . (-2r)}{(a^2 - r^2)}$$

$$= 2\pi \frac{(a^2 - r^2)(2a^2 - 9r^2) + r^2(2a^2 - 3r^2)}{(a^2 - r^2)^{3/2}}$$ which is $-$ ve for $r^2 = 2a^2/3$.

Hence V is maximum for $r^2 = 2a^2/3$ and maximum volume

$$= 2\pi r^2 \sqrt{(a^2 - r^2)} = 4\pi a^3/3\sqrt{3}.$$

Example 4.64. *Assuming that the petrol burnt (per hour) in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c miles per hour is $\frac{3}{2}c$ miles per hour.*

Solution. Let v m.p.h. be the velocity of the boat so that its velocity relative to water (when going against the current) is $(v - c)$ m.p.h.

$\therefore$ Time required to cover a distance of s miles $= \dfrac{s}{v - c}$ hours.

Since the petrol burnt per hour $= kv^3$, k being a constant.

$\therefore$ The total petrol burnt, y, is given by

$$y = k\frac{v^3 s}{v - c} = ks\frac{v^3}{v - c} \quad \therefore \quad \frac{dy}{dv} = ks \cdot \frac{(v - c)3v^2 - v^3 \cdot 1}{(v - c)^2}$$

$$= ks \cdot \frac{v^2(2v - 3c)}{(v - c)^2}$$

Thus $dy/dv = 0$ for $v = 3c/2$, the other value ($v = 0$) is inadmissible.

Also dy/dv is $-$ ve, when v is slightly $< 3c/2$ and it is $+$ ve, when v is slightly $> 3c/2$.

Hence y is minimum for $v = 3c/2$.

PROBLEMS 4.13

1. (*i*) Test the curve $y = x^4$ for points of inflexion ? (*Burdwan, 2003*)

   (*ii*) Show that the points of inflexion of the curve $y^2 = (x - a)^2(x - b)$ lie on the straight line $3x + a = 4b$. (*Rajasthan, 2005*)

2. The function $f(x)$ defined by $f(x) = a/x + bx$, $f(2) = 1$, has an extremum at $x = 2$. Determine a and b. Is this point (2, 1), a point of maximum or minimum on the graph of $f(x)$?
3. Show that $\sin^p\theta \cos^q\theta$ attains a maximum when $\theta = \tan^{-1}(p/q)$. (*Rajasthan, 2006*)
4. If a beam of weight w per unit length is built-in horizontally at one end A and rests on a support O at the other end, the deflection y at a distance x from O is given by

$$EIy = \frac{w}{48}(2x^4 - 3lx^3 + l^3x),$$

where l is the distance between the ends. Find x for y to be maximum.

5. The horse-power developed by an aircraft travelling horizontally with velocity v feet per second is given by

$$H = \frac{aw^2}{v} + bv,$$

where a, b and w are constants. Find for what value of v the horse-power is maximum.

6. The velocity of waves of wave-length λ on deep water is proportional to $\sqrt{(\lambda/a + a/\lambda)}$, where a is a certain constant, prove that the velocity is minimum when $\lambda = a$.
7. In a submarine telegraph cable, the speed of signalling varies as $x^2 \log_e(1/x)$, where x is the ratio of the radius of the core to that of the covering. Show that the greatest speed is attained when this ratio is $1/\sqrt{e}$.
8. The efficiency e of a screw-jack is given by $e = \tan\theta/\tan(\theta + \alpha)$, where α is a constant. Find θ if this efficiency is to be maximum. Also find the maximum efficiency.
9. Show that of all rectangles of given area, the square has the least parameter.
10. Find the rectangle of greatest perimeter that can be inscribed in a circle of radius a.
11. A gutter of rectangular section (open at the top) is to be made by bending into shape of a rectangular strip of metal. Show that the capacity of the gutter will be greatest if its width is twice its depth.
12. Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.
13. An open box is to be made from a rectangular piece of sheet metal 12 cms × 18 cms, by cutting out equal squares from each corner and folding up the sides. Find the dimensions of the box of largest volume that can be made in this manner.
14. An open tank is to be constructed with a square base and vertical sides to hold a given quantity of water. Find the ratio of its depth to the width so that the cost of lining the tank with lead is least.
15. A corridor of width b runs perpendicular to a passageway of width a. Find the longest beam which can be moved in a horizontal plane along the passageway into the corridor ?
16. One corner of a rectangular sheet of paper of width a is folded so as to reach the opposite edge of the sheet. Find the minimum length of the crease.
17. Show that the height of closed cylinder of given volume and least surface is equal to its diameter.
18. Prove that a conical vessel of a given storage capacity requires the least material when its height is $\sqrt{2}$ times the radius of the base. (*Warangal, 1996*)
19. Show that the semi-vertical angle of a cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$.
20. The shape of a hole bored by a drill is cone surmounting a cylinder. If the cylinder be of height h and radius r and the semi-vertical angle of the cone be α where $\tan\alpha = h/r$, show that for a total fixed depth H of the hole, the volume removed is maximum if $h = \frac{H}{6}(1 + \sqrt{7})$. (*Raipur, 2005*)
21. A cylinder is inscribed in a cone of height h. If the volume of the cylinder is maximum, show that its height is $h/3$.
22. Show that the volume of the biggest right circular cone that can be inscribed in a sphere of given radius is 8/27 times that of the sphere.
23. A given quantity of metal is to be cast into a half-cylinder with a rectangular base and semi-circular ends. Show that in order that the total surface area may be a minimum, the ratio of the length of the cylinder to the diameter of its semi-circular ends is $\pi/(\pi + 2)$.
24. A person being in a boat a miles from the nearest point of the beach, wishes to reach as quickly as possible a point b miles from that point along the shore. The ratio of his rate of walking to his rate of rowing is $\sec\alpha$. Prove that he should land at a distance $b - a\cot\alpha$ from the place to be reached.
25. The cost per hour of propelling a steamer is proportional to the cube of her speed through water. Find the relative speed at which the steamer should be run against a current of 5 km per hour to make a given trip at the least cost.

4.16 ASYMPTOTES

(1) Def. *An asymptote of a curve is a straight line at a finite distance from the origin, to which a tangent to the curve tends as the point of contact recedes to infinity.*

In other words, an asymptote is a straight line which cuts a curve on two points, at an infinite distance from the origin and yet is not itself wholly at infinity.

(2) Asymptotes parallel to axes. Let the equation of the curve arranged according to powers of x be

$$a_0x^n + (a_1y + b_1)x^{n-1} + (a_2y^2 + b_2y + c_2)x^{n-2} + \dots 0 \qquad \dots(1)$$

If $a_0 = 0$ and y be so chosen that $a_1y + b_1 = 0$, then the coefficients of two highest powers of x in (1) vanish and therefore, two of its roots are infinite. Hence $a_1y + b_1 = 0$ is an asymptote of (1) which is parallel to x-axis.

Again if a_0, a_1, b_1 are all zero and if y be so chosen that $a_2y^2 + b_2y + c_2 = 0$, then three roots of (1) become infinite. Therefore, the two lines represented by $a_2y^2 + b_2y + c_2 = 0$ are the asymptotes of (1) which are parallel to x-axis, and so on.

Similarly, for asymptotes parallel to y-axis.

Thus we have the following **rules :**

I. *To find the asymptotes parallel to x-axis, equate to zero the coefficient of the highest power of x in the equation,* provided this is not merely a constant.

II. *To find the asymptotes parallel to y-axis, equate to zero the coefficient of the highest power of y in the equation,* provided this is not merely a constant.

Example 4.65. *Find the asymptotes of the curve*

$$x^2y^2 - x^2y - xy^2 + x + y + 1 = 0.$$

Solution. The highest power of x is x^2 and its coefficient is $y^2 - y$.

$\therefore$ The asymptotes parallel to the x-axis are given by

$$y(y-1) = 0 \text{ i.e., by } y = 0 \text{ and } y = 1.$$

The highest power of y is y^2 and its coefficient is $x^2 - x$.

$\therefore$ The asymptotes parallel to the y-axis are given by

$$x(x-1) = 0 \text{ i.e., by } x = 0 \text{ and } x = 1.$$

Hence the asymptotes are $x = 0$, $x = 1$, $y = 0$ and $y = 1$.

(3) Inclined asymptotes. Let the equation of the curve be of the form

$$x^n\phi_n(y/x) + x^{n-1}\phi_{n-1}(y/x) + x^{n-2}\phi_{n-2}(y/x) + \dots = 0 \qquad \dots(1)$$

where $\phi_r(y/x)$ is an expression of degree r is y/x.

To find where this curve is cut by the line $y = mx + c$, $\qquad \dots(2)$

put $y/x = m + c/x$ in (1). The resulting equation is

$$x^n\phi_n(m + c/x) + x^{n-1}\phi_{n-1}(m + c/x) + x^{n-2}\phi_{n-2}(m + c/x) + \dots = 0$$

which gives the abscissae of the points of intersection.

Expanding each of the ϕ-functions by Taylor's series,

$$x^n\left\{\phi_n(m) + \frac{c}{x}\phi_n'(m) + \frac{c^2}{2!\,x^2}\phi_n''(m) + \dots\right\} + x^{n-1}\left\{\phi_{n-1}(m) + \frac{c}{x}\phi'_{n-1}(m) + \dots\right\}$$
$$+ x^{n-2}\{\phi_{n-2}(m) + \dots\} = 0$$

or

$$x^n\phi_n(m) + x^{n-1}\{c\phi'_n(m) + \phi_{n-1}(m)\}$$
$$+ x^{n-2}\left\{\frac{c^2}{2!}\phi_n''(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m)\right\} + \dots = 0 \qquad \dots(3)$$

If the line (2) is an asymptote to the curve, it cuts the curve in two points at infinity *i.e.*, the equation (3) has two infinite roots for which the coefficients of two highest terms should be zero.

i.e., $\qquad \phi_n(m) = 0 \quad \dots(4) \quad$ and $\quad c\phi'_n(m) + \phi_{n-1}(m) = 0 \qquad \dots(5)$

If the roots of (4) be $m_1, m_2, \dots m_n$, then the corresponding values of c (*i.e.* $c_1, c_2, \dots\dots c_n$) are given by (5). Hence the asymptotes are

$$y = m_1x + c_1,\ y = m_2x + c_2,\ \dots\dots,\ y = m_nx + c_n.$$

Obs. If (4) gives two equal values of m, then the corresponding values of c cannot be found from (5). Then c is determined by equating to zero the coefficient of x^{n-2} *i.e.*, from

$$\frac{c^2}{2!}\phi''_n(m) + c\,\phi'_{n-1}(m) + \phi_{n-2}(m) = 0 \qquad \text{...(6)}$$

In this case, *there will be two parallel asymptotes.*

Working rule :

1. Put $x = 1, y = m$ in the highest degree terms, thus getting $\phi_n(m)$. Equate it to zero and solve for m. Let its roots be $m_1, m_2, \ldots\ldots$
2. Form $\phi_{n-1}(m)$ by putting $x = 1$ and $y = m$ in the $(n-1)$th degree terms.
3. Find the values of c (*i.e.* $c_1, c_2, \ldots\ldots$) by substituting $m = m_1, m_2, \ldots\ldots$ in turn in the formula

$$c = -\phi_{n-1}(m)/\phi'_n(m)$$

[Sometimes it take (0/0) form, then find c from (6).]

4. Substitute the values of m and c in $y = mx + c$ in turn.

Example 4.66. *Find the asymptotes of the curve*

(i) $y^3 - 2xy^2 - x^2y + 2x^3 + 3y^2 - 7xy + 2y^2 + 2y + 2x + 1 = 0.$

(ii) $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0.$

(iii) $(x + y)^2 (x + y + 2) = x + 9y - 2.$ *(Rohtak, 2005)*

Solution. (*i*) Putting $x = 1$ and $y = m$ in the third degree terms,

$$\phi_3(m) = m^3 - 2m^2 - m + 2, \quad \therefore \quad \phi_3(m) = 0 \text{ gives } m^3 - 2m^2 - m + 2 = 0$$

or $\quad (m^2 - 1)(m - 2) = 0$ whence $m = 1, -1, 2.$

Also putting $x = 1$ and $y = m$ in the 2nd degree terms, $\phi_2(m) = 3m^2 - 7m + 2$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{3m^2 - 7m + 2}{3m^2 - 4m - 1}$$

$$= -1 \text{ when } m = 1, = -2 \text{ when } m = -1, = 0 \text{ when } m = 2.$$

Hence the asymptotes are $y = x - 1, y = -x - 2$ and $y = 2x$.

(*ii*) Putting $x = 1$ and $y = m$ in the third degree terms,

$$\phi_3(m) = 1 + 3m - 4m^3$$

$$\therefore \quad \phi_3(m) = 0 \text{ gives } 4m^3 - 3m - 1 = 0, \quad \text{or} \quad (m - 1)(2m + 1)^2 = 0$$

whence $\quad m = 1, -1/2, -1/2.$

Similarly, $\quad \phi_2(m) = 0$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{0}{3 - 12m^2}$$

$$= 0 \text{ when } m = 1, = \frac{0}{0} \text{ form when } m = -\frac{1}{2}.$$

Thus $\left(\text{when } m = -\frac{1}{2}\right)$ c is to be obtained from

$$\frac{c^2}{2!}\phi''_3(m) + c\,\phi'_2(m) + \phi_1(m) = 0$$

or
$$\frac{c^2}{2}(-24m) + c\,.\,0 + (-1 + m) = 0$$

Putting $m = -1/2$, $6c^2 - 3/2 = 0$ whence $c = \pm 1/2$.

Hence the asymptotes are $y = x, y = -\frac{1}{2}x + \frac{1}{2}, y = -\frac{1}{2}x - \frac{1}{2}$.

(*iii*) Putting $x = 1$ and $y = m$ in the third degree terms, $\phi_3(m) = (1 + m)^3$.

$$\therefore \quad \phi_3(m) = 0 \text{ gives } (m + 1)^3 = 0 \text{ whence } m = -1, -1, -1.$$

Similarly, $\quad \phi_2(m) = 2(1 + m)^2, \phi_1(m) = -1 - 9m, \phi_0(m) = 2.$

For these three equal values of $m = -1$, values of c are obtained from

$$\frac{c^3}{3!}\phi_3'''(m) + \frac{c^2}{2!}\phi_2''(m) + c\,\phi_1'(m) + \phi_0(m) = 0$$

or
$$\frac{c^3}{6}(6) + \frac{c^2}{2}(4) + c(-9) + 2 = 0 \quad \text{or} \quad c^3 + 2c^2 - 9c + 2 = 0.$$

Solving for c, we have $c = 2, -2 \pm \sqrt{5}$.

Hence the three asymptotes are

$$y = -x + 2,\ y = -x - 2 + \sqrt{5},\ y = -x - 2 - \sqrt{5}.$$

4. Asymptotes of polar curves. It can be shown that an *asymptote of the curve* $1/r = f(\theta)$ *is*

$$r \sin(\theta - \alpha) = 1/f'(\alpha),$$

where α *is a root of the equation* $f(\theta) = 0$
and $f'(\alpha)$ *is the derivative of* $f(\theta)$ *w.r.t.* θ *at* $\theta = \alpha$.

Example 4.67. *Find the asymptote of the spiral* $r = a/\theta$.

Equation of the curve can be written as $1/r = \theta/a = f(\theta)$, say.

$$f(\theta) = 0, \text{ if } \theta = 0\ (= \alpha). \text{ Also } f'(\theta) = 1/a \quad \therefore \quad f'(\alpha) = 1/a.$$

$\therefore$ The asymptote is $r \sin(\theta - 0) = 1/f'(0)$ or $r \sin\theta = a$.

PROBLEMS 4.14

Find the asymptotes of

1. $x^3 + y^3 = 3axy$ *(Agra, 2002)*
2. $(x^2 - a^2)(y^2 - b^2) = a^2 b^2$ *(Osmania, 2002)*
3. $(a/x)^2 + (b/y)^2 = 1$ *(Burdwan, 2003)*
4. $x^2y + xy^2 + xy + y^2 + 3x = 0$.
5. $4x^3 + 2x^2 - 3xy^2 - y^3 - 1 - xy - y^2 = 0$. *(U.P.T.U., 2001)*
6. $x^2(x - y)^2 - a^2(x^2 + y^2) = 0$ *(Kurukshetra, 2006)*
7. $(x + y)^2(x + 2y + 2) = (x + 9y - 2)$ *(Rajasthan, 2006)*
8. Show that the asymptotes of the curve $x^2y^2 = a^2(x^2 + y^2)$ form a square of side $2a$.
9. Find the asymptotes of the curve $x^2y - xy^2 + xy + y^2 + x - y = 0$ and show that they cut the curve again in three points which lie on the line $x + y = 0$. *(Kurukshetra, 2006)*

Find the asymptotes of the following curves :

10. $r = a \tan\theta$. *(Rohtak, 2006 S)*
11. $r = a(\sec\theta + \tan\theta)$
12. $r \sin\theta = 2\cos 2\theta$. *(Kurukshetra, 2009 S)*
13. $r \sin n\theta = a$.

4.17 (1) CURVE TRACING

In many practical applications, a knowledge about the shapes of given equations is desirable. On drawing a sketch of the given equation, we can easily study the behaviour of the curve as regards its symmetry asymptotes, the number of branches passing through a point etc.

A point through which two branches of a curve pass is called a **double point.** At such a point P, the curve has two tangents, one for each branch.

If the tangents are real and distinct, the double point is called a **node** [Fig. 4.22 (a)].

If the tangents are real and coincident, the double point is called a **cusp** [Fig. 4.22 (b)].

If the tangents are imaginary, the double point is called a *conjugate point* (or an *isolated point*). Such a point cannot be shown in the figure.

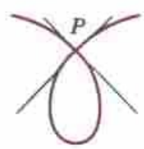

Fig. 4.22 (a)

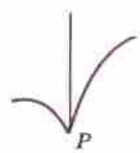

Fig. 4.22 (b)

(2) Procedure for tracing cartesian curves.

1. Symmetry. *See if the curve is symmetrical about any line.*

(*i*) *A curve is symmetrical about the x-axis, if only even powers of y occur in its equation.* (*e.g.*, $y^2 = 4ax$ is symmetrical about x-axis).

(ii) *A curve is symmetrical about the y-axis, if only even powers of x occur in its equation.* (*e.g.*, $x^2 = 4ay$ is symmetrical about y-axis).

(iii) *A curve is symmetrical about the line $y = x$, if on interchanging x* and *y its equation remains unchanged,* (*e.g.*, $x^3 + y^3 = 3axy$ *is symmetrical about the line* $y = x$).

2. Origin. (*i*) *See if the curve passes through the origin.*

(A curve passes through the origin if there is no constant term in its equation).

(*ii*) *If it does, find the equation of the tangents thereat,* by equating to zero the lowest degree terms.

(*iii*) *If the origin is a double point, find whether the origin is a node, cusp or conjugate point.*

3. Asymptotes. (*i*) *See if the curve has any asymptote parallel to the axes* (p. 183).

(*ii*) *Then find the inclined asymptotes,* if need be. (p. 183).

4. Points. (*i*) *Find the points where the curve crosses the axes and the asymptotes.*

(*ii*) *Find the points where the tangent is parallel or perpendicular to the x-axis,* (*i.e.* the points where $dy/dx = 0$ or ∞).

(*iii*) *Find the region (or regions) in which no portion of the curve exists.*

Example 4.68. *Trace the curve* $y^2(2a - x) = x^3$.
(*P.T.U., 2010, V.T.U., 2008 ; Rajasthan, 2006 ; U.P.T.U., 2005*)

Solution. (*i*) *Symmetry*: The curve is symmetrical about the x-axis.

[$\because$ only even powers of y occur in the equation.

(*ii*) *Origin :* The curve passes through the origin

[$\because$ there is no constant term in its equation.

The tangents at the origin are $y = 0, y = 0$ [Equating to zero the lowest degree terms.]

$\therefore$ Origin is a *cusp*

(*iii*) *Asymptotes :* The curve has an asymptote $x = 2a$.

[$\because$ co-eff. of y^3 is absent, co-eff. of y^2 is an asymptote.

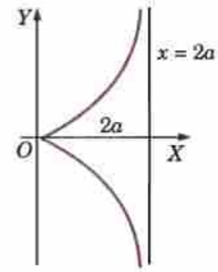

Fig. 4.23

(*iv*) *Points* : (*a*) curve meets the axes at (0, 0) only. (*b*) $y^2 = x^3/(2a - x)$

When x is –ve, y^2 is –ve (*i.e.* y is imaginary) so that no portion of the cuve lies to the left of the y-axis. Also when $x > 2a$, y^2 is again –ve, so that no portion of the curve lies to the right of the line $3x = 2a$.

Hence, the shape of the curve is as shown in Fig. 4.23. This curve is known as *Cissoid.*

Example 4.69. *Trace the curve* $y^2(a - x) = x^2(a + x)$. (*V.T.U., 2010 ; B.P.T.U., 2005*)

Solution. (*i*) *Symmetry* : The curve is symmetrical about the x-axis.

(*ii*) *Origin* : The curve passes through the origin and the tangents at the origin are $y^2 = x^2$,

i.e. $y = x$ and $y = -x$. $\therefore$ Origin is a *node.*

(*iii*) *Asymptotes* : The curve has an asymptote $x = a$

(*iv*) *Points* : (*a*) When $x = 0, y = 0$; when $y = 0, x = 0$ or $-a$.

$\therefore$ The curve crosses the axes at (0, 0) and $(-a, 0)$.

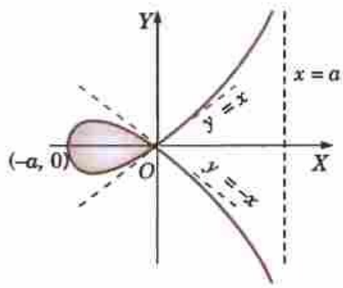

Fig. 4.24

We have $y = \pm x\sqrt{\left(\dfrac{a + x}{a - x}\right)}$

When $x > a$ or $< -a$, y is imaginary.

$\therefore$ No portion of the curve lies to the right of the line $x = a$ or to the left of the line $x = -a$.

Hence the shape of the curve is as shown in Fig. 4.24. This curve is known as *Strophoid.*

Example 4.70. *Trace the curve* $y = x^2/(1 - x^2)$.

Solution. (*i*) *Symmetry* : The curve is symmetrical about y-axis.

(*ii*) *Origin* : It passes through the origin and the tangent at the origin is $y = 0$ (*i.e.*, x-axis).

(*iii*) *Asymptotes* : The asymptotes are given by $1 - x^2 = 0$ or $x = \pm 1$ and $y = -1$.

(*iv*) *Points* : (*a*) The curve crosses the axes at the origin only. (*b*) When $x \to 1$ from left, $y \to \infty$

When $x \to 1$ from right $y \to -\infty$

When $x > 1$, y is – ve

Hence the curve is as shown in Fig. 4.25.

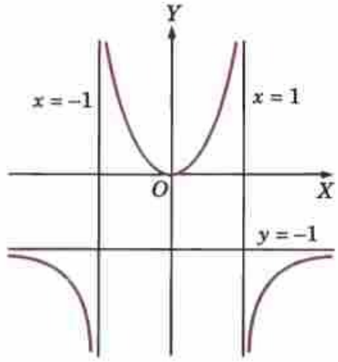

Fig. 4.25

Example 4.71. *Trace the curve* $a^2y^2 = x^2(a^2 - x^2)$. (*P.T.U., 2009 ; V.T.U., 2008 S*)

Solution. (*i*) *Symmetry*. The curve is symmetrical about x-axis, y-axis and origin.

(*ii*) *Origin*. The curve passes through the origin and the tangents at the origin are $a^2y^2 = a^2x^2$ *i.e.*, $y = \pm x$.

(*iii*) *Asymptotes*. The curve has no asymptote.

(*iv*) *Points*. (*a*) The curve cuts x-axis ($y = 0$) at $x = 0, \pm a$. and cuts y-axis ($x = 0$) at $y = 0$ *i.e.*, (0, 0) only.

(*b*) $\dfrac{dy}{dx} = \dfrac{x(a^2 - 2x^2)}{a^2 y} \to \infty$ at $(a, 0)$

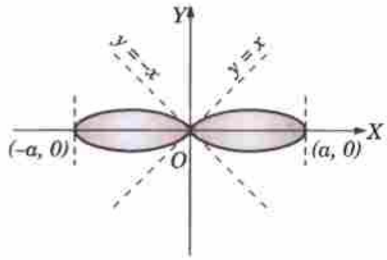

Fig. 4.26

i.e., tangent to the curve at $(a, 0)$ is parallel to y-axis. Similarly the tangent at $(-a, 0)$ is parallel to y-axis.

(*c*) We have $y = \dfrac{x}{a}\sqrt{a^2 - x^2}$ which is real for $x^2 < a^2$ *i.e.*, $-a < x < a$.

$\therefore$ The curve lies between $x = a$ and $x = -a$

Hence the shape of the curve is as shown is Fig. 4.26.

Example 4.72. *Trace the curve* $y = x^3 - 12x - 16$. (*P.T.U., 2008*)

Solution. (*i*) *Symmetry*. The curve has no symmetry.

(*ii*) *Origin*. It doesn't pass through the origin.

(*iii*) *Asymptotes* : The curve has no asymptote.

(*iv*) *Points*. (*a*) The curve cuts x-axis ($y = 0$) at $(-2, 0)$, $(4, 0)$ and cuts y-axis ($x = 0$) at $(0, -16)$.

(*b*) $\dfrac{dy}{dx} = 3x^2 - 12$

At (– 2, 0), $\dfrac{dy}{dx} = 0$ *i.e.*, tangent is parallel to x-axis at $(-2, 0)$.

At (4, 0), $\dfrac{dy}{dx} = 36$ *i.e.*, $\tan\theta = 36$ *i.e.*, tangent makes an acute angle $\tan^{-1} 36$ with x-axis at $(4, 0)$.

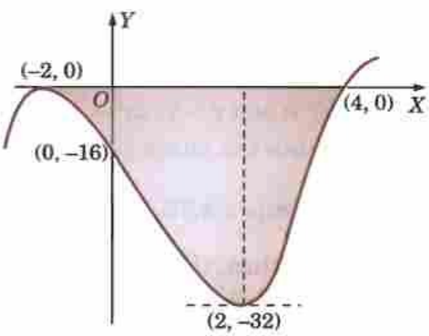

Fig. 4.27

Also $\dfrac{dy}{dx} = 0$ at $3x^2 - 12 = 0$ or $x = \pm 2$ *i.e.*, tangent is also parallel to x-axis at $(2, -32)$.

(*c*) $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$; y is + *ve* for $x > 4$ and y is – *ve* for $x < 4$.

Hence the shape of the curve is as shown in Fig. 4.27.

Example 4.73. *Trace the curve* $9ay^2 = (x - 2a)(x - 5a)^2$ (*J.N.T.U., 2008*)

Solution. (*i*) *Symmetry*. The curve is symmetrical about the x-axis.

(*ii*) *Origin*. The curve does'nt pass through the origin.

(*iii*) *Asymptotes*. It has no asymptotes.

(*iv*) *Points*. (*a*) The curve cuts the x-axis ($y = 0$) at $x = 2a$, and $x = 5a$. *i.e.*, at A $(2a, 0)$ and $B(5a, 0)$.

It cuts the y-axis ($x = 0$) at $y^2 = -50\,a^2/9$, *i.e.*, y is imaginary.

So the curve doesn't cut the y-axis.

(*b*) $y = \dfrac{(x-5a)\sqrt{(x-2a)}}{3\sqrt{a}}$ *i.e.*, y is imaginary for $x < 2a$. So the curve exists only for $x \geq 2a$.

(*c*) $\dfrac{dy}{dx} = \pm \dfrac{x-3a}{2\sqrt{a}\,\sqrt{(x-2a)}}$

At A $(2a, 0)$, $\dfrac{dy}{dx} \to \infty$ *i.e.*, tangent is parallel to y-axis.

At B $(5a, 0)$, $\dfrac{dy}{dx} = \pm \dfrac{1}{\sqrt{3}}$ *i.e.*, there are two distinct tangents.

So there is a node at B $(5a, 0)$.

Hence the shape of the curve is as shown in Fig. 4.28.

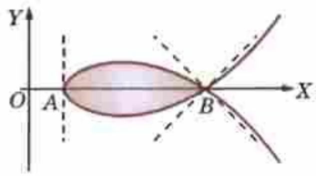

Fig. 4.28

Example 4.74. *Trace the curve* $x^3 + y^3 = 3axy$ *(Kurukshetra, 2005 ; U.P.T.U., 2003)*

or
$$r = \frac{3a \sin\theta \cos\theta}{\sin^3\theta + \cos^3\theta}.$$

Solution. (*i*) *Symmetry* : The curve is symmetrical about the line $y = x$.

(*ii*) *Origin* : It passes through the origin and tangents at the origin are $xy = 0$, *i.e.*, $x = 0$, $y = 0$.

$\therefore$ Origin is a *node*.

(*iii*) *Asymptotes* : (*a*) It has no asymptote parallel to the axes.

(*b*) Putting $y = m$ and $x = 1$ in the third degree terms,

$$\phi_3(m) = 1 + m^3,\ \phi_3(m) = 0 \text{ gives } m = -1.$$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\left(\frac{-3am}{3m^2}\right) = \frac{a}{m}.$$
$$= -a, \text{ when } m = -1.$$

Hence $y = -x - a$ $\left(\text{i.e., } \dfrac{x}{-a} + \dfrac{y}{-a} = 1\right)$ is an asymoptote.

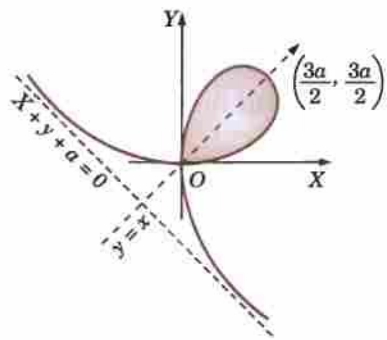

Fig. 4.29

(*iv*) *Points* : (*a*) It meets the axes at the origin only.

(*b*) When $y = x$, $2x^3 = 3ax^2$, *i.e.* $x = 0$ or $3a/2$. *i.e.*, the curve crosses the line $y = x$ at $(3a/2, 3a/2)$.

Hence the shape of the curve is as shown in Fig. 4.29. This curve is known as *Folium of Descartes*.

Example 4.75. *Trace the curve* $x^3 + y^3 = 3ax^2$.

Solution. (*i*) *Symmetry* : The curve has no symmetry.

(*ii*) *Origin* : The curve passes through the origin and the tangents at the origin are $x = 0$ and $x = 0$.

$\therefore$ The origin is a *cusp*.

(*iii*) *Asymptotes* : (*a*) The curve has no asymptote parallel to the axes.

(*b*) Putting $x = 1$, $y = m$ in the third degree terms, we get

$$\phi_3(m) = m^3 + 1\ ; \quad \therefore \quad \phi_3(m) = 0, \text{ gives } m = -1.$$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-3a}{3m^2} = a \text{ for } m = 1.$$

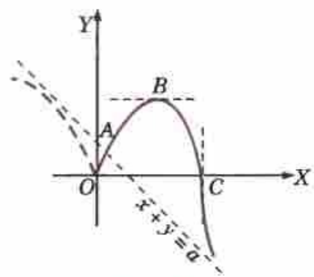

Fig. 4.30

Thus $x + y = a$ is the only asymptote.

The curve lies above the asymptote when x is positive and large and it lies below the asymptote when x is negative.

(iv) *Points.* (a) The curve crosses the axes at $O(0, 0)$ and $C(3a, 0)$. It crosses the asymptote at $A(a/3, 2a/3)$.

(b) Since $y^2\, dy/dx = x(2a - x)$. $\therefore$ $dy/dx = 0$ for $x = 2a$.

(c) Now $y = [x^2(3a - x)]^{1/3}$.

When $0 < x < 3a$, y is positive. As x increases from 0, y also increases till $x = 2a$ where the tangent is parallel to the x-axis. As x increases from $2a$ to $3a$, y constantly decreases to zero.

When $x > 3a$, y is negative.

When $x < 0$, y is positive and constantly increases as x varies from 0 to $-\infty$.

Combining all these facts we see that the shape of the curve is as shown in Fig. 4.30.

Example 4.76. *Trace the curve* $y^2(x - a) = x^2(x + a)$.

Solution. (i) *Symmetry* : The curve is symmetrical about the x-axis.

(ii) *Origin* : The curve passes through the origin and the tangents at the origin are $y^2 = -x^2$ *i.e.*, $y = \pm ix$, which are imaginary lines. $\therefore$ The origin is an *isolated point.*

(iii) *Asymptotes* : (a) $x = a$ is the only asymptote parallel to the y-axis.

(b) Putting $x = 1$ and $y = m$ in the third degree terms, we get

$$\phi_3(m) = m^2 - 1.$$

$\therefore$ $$\phi_3(m) = 0 \text{ gives } m = \pm 1$$

$\therefore$ $$c = \frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-a(m^2+1)}{2m} = \pm a \text{ for } m = \pm 1.$$

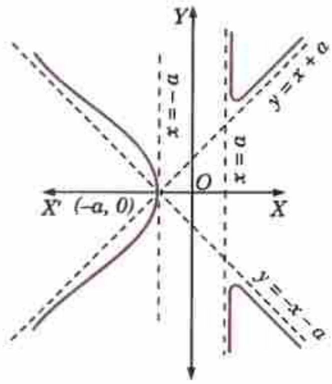

Fig. 4.31

Thus the other two asymptotes are $y = x + a$; $y = -x - a$.

(iv) *Points* : (a) The curve crosses the axes at $(-a, 0)$ and $(0, 0)$.

It crosses the asymptotes $y = x + a$ and $y = -x - a$ at $(-a, 0)$.

(b) $y = \pm x\sqrt{\left(\dfrac{x+a}{x-a}\right)}$

When $x < a$ and $x > -a$, y is imaginary.

$\therefore$ no portion of the curve lies between the lines $x = a$ and $x = -a$. Thus the vertical asymptote must be approached from the right.

(c) $$\frac{dy}{dx} = \pm\frac{x^2 - ax + a^2}{(x-a)^{3/2}(x+a)^{1/2}}$$

$\therefore$ $$dy/dx = 0, \text{ when } x = \tfrac{1}{2}(1 + \sqrt{5})a = 1.6a \text{ approx.}$$

[rejecting the value $\frac{1}{2}(1 - \sqrt{5})a$ which lies between $-a$ and a]

and $$dy/dx \to \infty, \text{ when } x = \pm a.$$

Thus the tangent is parallel to the x-axis at $x = 1.6a$ and perpendicular to the x-axis at $x = \pm a$.

Hence the shape of the curve is as shown in Fig. 4.31.

4.17 (3) PROCEDURE FOR TRACING CURVES IN PARAMETRIC FORM : $x = f(t)$ and $y = \phi(t)$

1. **Symmetry.** *See if the curve has any symmetry.*

(i) *A curve is symmetrical about the x-axis, if on replacing t by – t, f(t) remains unchanged and φ(t) changes to – φ(t).*

(ii) *A curve is symmetrical about the y-axis if on replacing t by – t, f(t) changes to – f(t) and φ(t) remains unchanged.*

(iii) *A curve is symmetrical in the opposite quadrants, if on replacing t by – t, both f(t) and φ(t) remains unchanged.*

2. **Limits.** *Find the greatest and least values of x and y so as to determine the strips, parallel to the axes, within or outside which the curve lies.*

3. **Points.** (*a*) *Determine the points where the curve crosses the axes.*

The points of intersection of the curve with the x-axis given by the roots of $\phi(t) = 0$, while those with the y-axis are given by the roots of $f(t) = 0$.

(*b*) *Giving t a series of value, plot the corresponding values of x and y, noting whether x and y increase or decrease for the intermediates values of t.* For this purpose, we consider the sign of dx/dt and dy/dt for the different values of t.

(*c*) *Determine the points where the tangent is parallel or perpendicular to the x-axis,* (*i.e.*, where $dy/dx = 0$ or $\to \infty$).

(*d*) *When x and y are periodic functions of t with a common period, we need to study the curve only for one period,* because the other values of t will repeat the same curve over and over again.

Obs. Sometimes it is convenient to eliminate t between the given equations and use the resulting cartesian equation to trace the curve.

Example 4.77. *Trace the curve* $x = a\cos^3 t,\ y = a\sin^3 t$ *or* $x^{2/3} + y^{2/3} = a^{2/3}$.
(*P.T.U., 2009 S ; U.P.T.U., 2005 ; V.T.U., 2003*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the x-axis.

[$\because$ On changing t to $-t$, x remains unchanged but y changed to $-y$]

(*ii*) *Limits.* $\because$ $|x| \le a$ and $|y| \le a$.

$\therefore$ The curve lies entirely within the square bounded by the lines $x = \pm a$, $y = \pm a$.

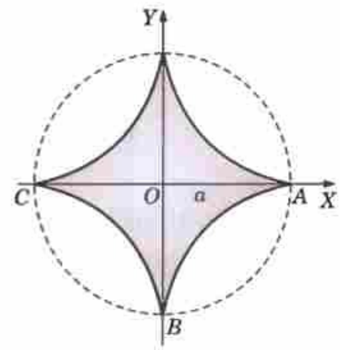

Fig. 4.32

(*iii*) *Points* : We have

$$\frac{dx}{dt} = -3a\cos^2 t \sin t,$$

$$\frac{dy}{dt} = 3a\sin^2 t\cos t,\ \frac{dy}{dx} = -\tan t.$$

$\therefore$ $dy/dx = 0$ when $t = 0$ or π

and $dy/dx \to \infty$, when $t = \pi/2$.

The following table gives the corresponding values of t, x, y and dy/dx.

As t increases	*x*	*y*	*dy/dx varies*	*Portion traced*
from 0 to $\pi/2$	+ve and decreases from a to 0	+ve and increases from 0 to a	from 0 to ∞	*A* to *B*
from $\pi/2$ to π	+ ve and increases numerically from 0 to $-a$	+ve and decreases from a to 0	from ∞ to 0	*B* to *C*

As t increases from π to 2π, we get the reflection of the curve ABC in the x-axis. The values of $t > 2\pi$ give no new points.

Hence the shape of the curve is as shown in Fig. 4.32 and is known as **Astroid.**

Example 4.78. *Trace the curve* $x = a(\theta + \sin\theta),\ y = a(1 + \cos\theta)$. (*J.N.T.U., 2009 S*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the y-axis.

[$\because$ On changing θ to $-\theta$, x changes to $-x$ and y remains unchanged]

Thus we may consider the curve only for positive value of x, *i.e.*, for $\theta > 0$.

(*ii*) *Limits.* The greatest value of y is $2a$ and the least value is zero.

Hence the curve lies entirely between the lines $y = 2a$ and $y = 0$.

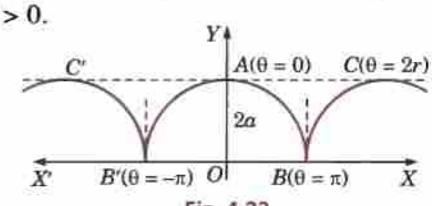

Fig. 4.33

(*iii*) *Points.* We have

$$\frac{dx}{d\theta} = a(1 + \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta \text{ and } \frac{dy}{dx} = -\tan\theta/2.$$

$\therefore$ $dy/dx = 0$ when $\theta = 0$ or 2π and $dy/dx \to \infty$ when $\theta = \pi$.

The following table gives the corresponding values of θ, x, y and $\dfrac{dy}{dx}$:

As θ increases	*x*	*y*	*dy / dx varies*	*Portion traced*
from 0 to π	increases from 0 to $a\pi$	decreases from $2a$ to 0	from 0 to ∞	A to B
from π to 2π	increases from $a\pi$ to $2a\pi$	increases from 0 to $2a$	from ∞ to 0	B to C

As θ decreases from 0 to -2π, we get the reflection of the curve ABC in the y-axis.

The curve consists of congruent arches extending to infinity in both the directions of the x-axis in the intervals ... $(-3\pi, -\pi)$ $(-\pi, \pi)$ $(\pi, 3\pi)$, ...

Hence the shape of the curve is as shown in Fig. 4.33 and is known as **Cycloid.**

Obs. 1. Cycloid is the curve described by a point on the circumference of a circle which rolls without sliding on a fixed straight line. This fixed line (x-axis) is called the *base* and the farthest point (A) from it the *vertex* of the cycloid.

The complete cycloid consists of the arch $B'AB$ and its endless repetitions on both sides.

2. Inverted cycloid: $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

The complete inverted cycloid consists of the arch BOA and an endless repetitions of the same on both sides. Here AB is the base and O the *vertex* of this cycloid. (Fig. 4.34).

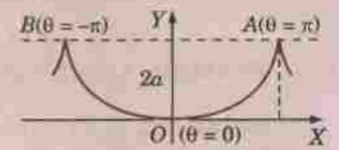

Fig. 4.34

4.17 (4) PROCEDURE FOR TRACING POLAR CURVES

1. **Symmetry.** *See if the curve is symmetrical about any line.*

(*i*) *A curve is symmetrical about the initial line OX, if only* cos θ *(or* sec θ*) occur in its equation.* (*i.e.*, it remains unchanged when θ is changed to $-\theta$) *e.g.*, $r = a(1 + \cos\theta)$ is symmetrical about the initial line.

(*ii*) *A curve is symmetrical about the line through the pole ⊥ to the initial line (i.e., OY), if only* sin θ *(or* cosec θ*) occur in its equation.* (*i.e.*, it remains unchanged when θ is changed to $\pi - \theta$) *e.g.*, $r = a \sin 3\theta$ is symmetrical about OY.

(*iii*) *A curve is symmetrical about the pole, if only even powers of r occur in the equation* (*i.e.*, it remains unchanged when r is changed to $-r$) *e.g.*, $r^2 = a^2 \cos 2\theta$ is symmetrical about the pole.

2. **Limits.** See if r and θ are confined between certain limits.

(*i*) *Determine the numerically greatest value of r, so as to notice whether the curve lies within a circle or not e.g.*, $r = a \sin 3\theta$ lies wholly within the circle $r = a$.

(*ii*) *Determine the region in which no portion of the curve lies by finding those values of θ for which r is imaginary e.g.*, $r^2 = a^2 \cos 2\theta$ does not lie between the lines $\theta = \pi/4$ and $\theta = 3\pi/4$.

3. **Asymptotes.** *If the curve possesses an infinite branch, find the asymptotes (p. 183).*

4. **Points.** (*i*) *Giving successive values to θ, find the corresponding values of r.*

(*ii*) *Determine the points where the tangent coincides with the radius vector or is perpendicular to it* (*i.e.*, the points where $\tan\phi = r\, d\theta/dr = 0$ or ∞).

Example 4.79. *Trace the curve* $r = a \sin 3\theta$. *(U.P.T.U., 2002)*

Solution. (*i*) *Symmetry.* The curve is symmetrical about the line through the pole ⊥ to the initial line.

(*ii*) *Limits.* The curve wholly lies within the curve $r = a$. ($\because$ r is never $> a$)

(*iii*) *Asymptotes.* It has no asymptotes.

(*iv*) *Points.* (*a*) $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a \sin 3\theta}{3a \cos 3\theta} = \dfrac{1}{3} \tan 3\theta$

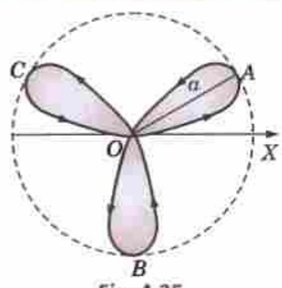

Fig. 4.35

$\therefore$ $\phi = 0$, when $\theta = 0, \pi/3, ...$ Hence the curve of the curve

$\phi = \pi/2$, when $\theta = \pi/6, \pi/2, ...$

(*b*) The following table gives the variations of r, θ and ϕ :

As θ varies from	*r varies from*	*ϕ varies from*	*Portion traced from*
0 to $\pi/6$	0 to a	0 to $\pi/2$	O to A
$\pi/6$ to $\pi/3$	a to 0	$\pi/2$ to 0	A to O
$\pi/3$ to $\pi/2$	0 to $-a$	0 to $\pi/2$	O to B

As θ increases from $\pi/2$ to π, portions of the curve from B to O, O to C and C to O are traced by symmetry about the line $\theta = \pi/2$.

Hence the curve consists of three loops as shown in Fig. 4.35 and is known as *three-leaved rose.*

Obs. *The curves of the form $r = a \sin n\theta$ or $r = a \cos n\theta$ are called **Roses** having*
(*i*) n leaves (loops) when n is odd,
(*ii*) $2n$ leaves (loops) when n is even.

Example 4.80. *Trace the curve $r = a \sin 2\theta$.(Four Leaved Rose)* (*V.T.U., 2009*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the line through the pole,$\perp$ to the initial line.

(*ii*) *Limits* : The curve lies wholly within the circle $r = a$ ($\because$ r is never $> a$)

(*iii*) *Points* : (*a*)

As θ increases from	*r varies from*	*Loop*	
0 to $\frac{\pi}{4}$	0 to a	no : 1,	
$\frac{\pi}{4}$ to $\frac{\pi}{2}$	a to 0		
$\frac{\pi}{2}$ to $\frac{3\pi}{4}$	0 to $-a$	no : 2,	
$\frac{3\pi}{4}$ to $\frac{\pi}{2}$	$-a$ to 0		etc. etc.

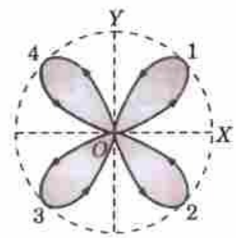

Fig. 4.36

(*b*) $$\tan \phi = r \frac{d\theta}{dr} = \frac{1}{2} \tan 2\theta ;$$

$\therefore$ $$\phi = 0, \text{ when } \theta = 0, \frac{\pi}{2}, \pi, 3\frac{\pi}{2}, 2\pi \,...$$

$$\phi = \frac{\pi}{2}, \text{ when } \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \,...$$

Hence, the shape of the curve is as shown in Fig. 4.36.

Example 4.81. *Trace the curve $r^2 = a^2 \cos 2\theta$.* (*V.T.U., 2007 ; Kurukshetra, 2006 ; B.P.T.U., 2005*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the pole.

(*ii*) *Limits* : (*a*) The curve lies wholly within the circle $r = a$.

(*b*) No portion of the curve lies between the lines $\theta = \pi/4$ and $\theta = 3\pi/4$.

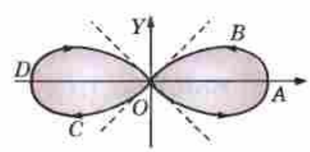

Fig. 4.37

(*iii*) *Points* : (*a*) $\tan \phi = r \dfrac{d\theta}{dr} = -\cot 2\theta = \tan \left(\dfrac{\pi}{2} + 2\theta\right)$

i.e., $$\phi = \frac{\pi}{2} + 2\theta \quad \therefore \quad \phi = 0, \text{ when } \theta = -\pi/4 \,;\, \phi = \pi/2 \text{ when } \theta = 0.$$

Thus, the tangent at O is $\theta = -\pi/4$ and the tangent at A is $\perp$ to the intial line.

(*b*) The variations of r and θ are given below :

As θ varies from	*r varies from*	*Portion traced*
0 to $\pi/4$	a to 0	ABO
$3\pi/4$ to π	0 to a	OCD

As θ increase from π to 2π, we get the reflection of the arc $ABOCD$ in the initial line. Hence the shape of the curve is as shown in Fig. 4.37. This curve is known as *Lemniscate of Bernoulli.*

Example 4.82. *Trace the curve $r = a + b \cos\theta$ (Limacon)*

Solution. (*i*) *Symmetry.* It is symmetrical about the initial line.

(*ii*) *Limits* : The curve wholly lies within the circle $r = a + b$

($\because$ r is never $> a + b$)

(*iii*) *Points* : (α) when $a > b$.

As θ increases from 0 to $\pi/2$; r decreases from $a + b$ to a

As θ increases from $\pi/2$ to π ; r decreases from a to $a - b$

The shape of the curve is as shown in Fig. 4.38 (*i*).

(β) when $a < b$.

As θ increases from 0 to $\pi/2$; r decreases from $a + b$ to a

As θ increases from $\pi/2$ to α ; r decreases from a to 0

As θ increases from α to π ; r decreases from 0 to $a - b$

when $\alpha = \cos^{-1}\left(-\frac{a}{b}\right)$

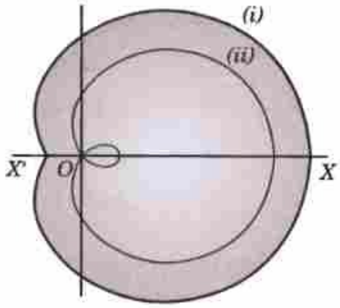

Fig. 4.38

In this case, the curve consists of two parts, one of which forms a loop within the other and the shape is as shown in Fig. 4.38 (*ii*).

Example 4.83. *Trace the curve $r\theta = a$.* *(Spiral)*

Solution. (*i*) *Symmetry.* There is no symmetry.

(*ii*) *Limits* : There are no limits to the values of r.

The curve does not pass through the pole for r does not become zero for any real value of θ.

(*iii*) *Asymptotes* : $\frac{1}{r} = \frac{\theta}{a} = f(\theta)$

$f(\theta) = 0$ for $\theta = 0$; $f'(\theta) = 1/a$, $f'(0) = 1/a$.

$\therefore$ Asymptote is $r \sin(\theta - 0) = 1/f'(0)$

i.e., $y = r \sin\theta = a$ is an asymptote.

(*iv*) *Points* : As θ increases from 0 to ∞, r to positive and decreases from ∞ to 0.

Hence the space of the curve is as shown in Fig. 4.39.

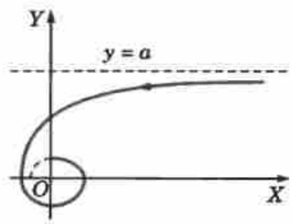

Fig. 4.39

Example 4.84. *Trace the curve $x^5 + y^5 = 5ax^2y^2$.*

Solution. (*i*) *Symmetry.* The curve is symmetrical about the line $y = x$.

$\because$ On interchanging x and y, it remains unchanged.]

(*ii*) *Origin* : It passes through the origin and the tangents at the origin are given by

$$x^2 y^2 = 0, i.e., x = 0, x = 0 ; y = 0, y = 0.$$

Hence the curve has both *node* and the *cusp* at the origin.

(*iii*) *Asymptotes* : (*a*) It has no asymptotes parallel to the axes.

(*b*) Putting $x = 1, y = m$ in the fifth degree terms, we get

$$\phi_5(m) = 1 + m^5. \quad \therefore \quad \phi_5(m) = 0 \text{ gives } m = -1.$$

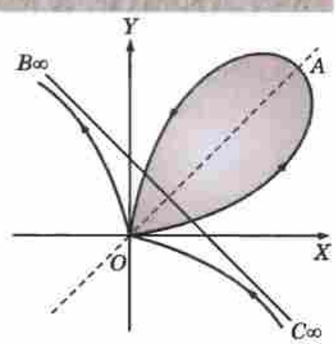

Fig. 4.40

$$\therefore \quad c = -\frac{\phi_4(m)}{\phi_5'(m)} = -\frac{-5am^2}{5m^4} = a \text{ for } m = -1.$$

Hence $y = -x + a$ or $x + y = a$ is an asymptote.

(*iv*) *Points* : Since it is not convenient to express y as a function of x or *vice versa*, hence we change the equation into polar coordinates by putting, $x = r\cos\theta$ and $y = r\sin\theta$. The equation of the curve becomes :

$$r = \frac{5a\sin^2\theta\cos^2\theta}{\cos^5\theta + \sin^5\theta} = \frac{5a}{4}\cdot\frac{\sin^5 2\theta}{\cos^5\theta + \sin^5\theta}$$

As θ increases from	*r*	*Portion traced from*
0 to $\pi/4$	is +ve and increases from 0 to $\frac{5\sqrt{2}}{2}a$	0 to A
$\pi/4$ to $\pi/2$	is +ve and decreases from $\frac{5\sqrt{2}}{2}a$ to 0	A to 0
$\pi/2$ to $3\pi/4$	is +ve and increases from 0 to ∞	0 to B_∞
$3\pi/4$ to π	is –ve and decreases from ∞ to 0	C_∞ to 0

As θ increases from π to 2π, the curve will retraced.

Hence the shape of the curve is as shown in Fig. 4.40.

PROBLEMS 4.15

Trace the following curves :

1. $y^2(a+x) = x^2(a-x)$. *(S.V.T.U., 2008 ; U.P.T.U., 2006 ; Rajasthan, 2005)*
2. $y^2(a^2+x^2) = x^2(a^2-x^2)$ *(V.T.U., 2010)*
3. $y = (x^2+1)/(x^2-1)$
4. $ay^2 = x^2(a-x)$ *(Kuruksehtra, 2009 S ; V.T.U., 2004)*
5. $x^2y^2 = a^2(y^2-x^2)$
6. $x = a\cos^3\theta, y = b\sin^3\theta$
7. $x = a(\theta - \sin\theta), y = a(1-\cos\theta)\ (0 < \theta < 2\pi)$
8. $x = (a\cos t + \log\tan t/2), y = a\sin t$.
9. $r = a\cos 2\theta$
10. $r = a\cos 3\theta$
11. $r = a(1-\cos\theta)$
12. $r = 2 + 3\cos\theta$
13. $r^2\cos 2\theta = a^2$. *(S.V.T.U., 2009)*

[**Hint.** Changing to Cartesian form $x^2 - y^2 = a^2$. This is a rectangular hyperbola with asymptotes $x + y = 0$ and $x - y = 0$]

4.18 OBJECTIVE TYPES OF QUESTIONS

PROBLEMS 4.16

Select the correct answer or fill up the blanks in each of the following questions :

1. The radius of curvature of the catenary $y = c\cosh x/c$ at the point where it crosses the y-axis is ......
2. The envelope of the family of straight lines $y = mx + am^2$, (m being the parameter) is ......
3. The curvature of the circle $x^2 + y^2 = 25$ at the point (3, 4) is ......
4. The value of $\underset{x\to\pi/2}{\text{Lt}} \frac{\log\sin x}{(\pi/2 - x)^2}$ is

   (*a*) zero (*b*) 1/2 (*c*) – 1/2 (*d*) – 2. *(V.T.U., 2010)*
5. Taylor's expansion of the function $f(x) = \frac{1}{1+x^2}$ is

(a) $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $-1 < x < 1$ (b) $\sum_{n=0}^{\infty} x^{2n}$ for $-1 < x < 1$

(c) $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ for any real x (d) $\sum_{n=0}^{\infty} (-1)^n x^n$ for $-1 < x \le 1$.

6. A triangle of maximum area inscribed in a circle of radius r
   (a) is a right angled triangle with hypotenuse measuring $2r$
   (b) is an equilateral triangle
   (c) is an isosceles triangle of height r
   (d) does *not* exist.
7. The extreme value of $(x)^{1/x}$ is
   (a) e (b) $(1/e)^e$ (c) $(e)^{1/e}$ (d) 1.
8. The percentage error in computing the area of an ellipse when an error of 1 per cent is made in measuring the major and minor axes is
   (a) 0.2% (b) 2% (c) 0.02%.
9. The length of subtangent of the rectangular hyperbola $x^2 - y^2 = a^2$ at the point $(a, \sqrt{2}a)$ is
   (a) $\sqrt{2}\,a$ (b) $2a$ (c) $\frac{1}{2a}$ (d) $\frac{a^{3/2}}{\sqrt{2}}$.
10. The length of subnormal to the curve $y = x^2$ at $(2, 8)$ is
   (a) 2/3 (b) 32 (c) 96 (d) 64.
11. If the normal to the curve $y^2 = 5x - 1$ at the point $(1, -2)$ is of the form $ax - 5y + b = 0$, then a and b are
   (a) 4, 14 (b) 4, – 14 (c) – 4, 14 (d) – 4, – 14.
12. The radius of curvature of the curve $y = e^x$ at the point where it crosses the y-axis is
   (a) 2 (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) $\frac{1}{2}\sqrt{2}$.
13. The equation of the asymptotes of $x^3 + y^3 = 3axy$, is
   (a) $x + y - a = 0$ (b) $x - y + a = 0$ (c) $x + y + a = 0$ (d) $x - y - a = 0$.
14. If ϕ be the angle between the tangent and radius vector at any point on the curve $r = f(\theta)$, then $\sin\phi$ equals to
   (a) $\frac{dr}{ds}$ (b) $r\frac{d\theta}{ds}$ (c) $r\frac{d\theta}{dr}$.
15. Envelope of the family of lines $x = my + 1/m$ is ...
16. The chord of curvature parallel to y-axis for the curve $y = a \log \sec x/a$ is ....
17. $\sinh x = \ldots x + \ldots x^3 + \ldots x^5 + \ldots$
18. The nth derivative of $(\cos x \cos 2x \cos 3x)$ = ...
19. If $x^3 + y^3 - 3axy = 0$, then d^2y/dx^2 at $(3a/2, 3a/2)$ = ...
20. When the tangent at a point on a curve is parallel to x-axis, then the curvature at that point is same as the second derivative at that point. *(True or False)*
21. If $x = at^2, y = 2at$, t being the parameter, then $xy\, d^2y/dx^2$ = ...
22. The radius of curvature for the parabola $x = a, y = 2at$ at any point t = ...
23. If (a, b) are the coordinates of the centre of curvature whose curvature is k, then the equation of the circle of curvature is ....
24. Evolute is defined as the .... of the normals for a given curve.
25. Envelope of the family of lines $\frac{x}{t} + yt = 2c$ (where t is the parameter) is ....
26. The angle between the radius vector and tangent for the curve $r = ae^{\theta \cot \alpha}$ is ....
27. The subnormal of the parabola $y^2 = 4ax$ is ....
28. The fourth derivative of $(e^{-x} x^3)$ is ....

29. If $y^2 = P(x)$, a polynomial of degree 3, then $\frac{2d}{dx}\left(y^3 \frac{d^2y}{dx^2}\right)$ equals ....

(a) $P'''(x) + P'(x)$ (b) $P''(x) + P'''(x)$ (c) $P(x)\,P'''(x)$.

30. The envelope of the family of straight line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$.

31. Curvature of a straight line is

(A) ∞ (B) zero (C) Both (A) and (B) (D) None of these.

32. The value of 'c' of the Cauchy's Mean value theorem for $f(x) = e^x$ and $g(x) = e^{-x}$ in $[2, 3]$ is ....

33. If the equation of a curve remains unchanged when x and y are interchanged, then the curve is symmetrical about ....

34. For the curve $y^2(1 + x) = x^2(1 - x)$, the origin is a .... (node/cusp/conjugate point).

35. The number of loops of $r = a\sin 2\theta$ are .... and these of $r = a\cos 3\theta$ are ....

36. Tangents at the origin for the curve $y^2(x^2 + y^2) + a^2(x^2 - y^2) = 0$ are ....

37. The asymptote to the curve $y^2(4 - x) = x^3$ is ....

38. The points of inflexion of the curve $y^2 = (x - a)^2(x - b)$ lie on the line $3x + a =$ ....

39. The curve $r = a/(1 + \cos\theta)$ intersects orthogonally with the curve

(A) $r = b/(1 - \cos\theta)$ (B) $r = b/(1 + \sin\theta)$ (C) $r = b/(1 + \sin^2\theta)$ (D) $r = b/(1 + \cos^2\theta)$. (V.T.U., 2010)

40. The region where the curve $r = a\sin\theta$ does not lie is ....

41. If $f(x)$ is continuous in the closed interval $[a, b]$, differentiable in (a, b) and $f(a) = f(b)$, then there exists at least one value c of x in (a, b) such that $f'(c)$ is equal to

(A) 1 (B) -1 (C) 2 (D) 0. (V.T.U., 2009)

42. If two curves intersect orthogonally in cartesian form, then the angle between the same two curves in polar form is

(A) $\pi/4$ (B) Zero (C) 1 radian (D) None of these.

43. If the angle between the radius vector and the tangent is constant, then the curve is,

(A) $r = a\cos\theta$ (B) $r^2 = a^2\cos^2\theta$ (C) $r = ae^{b\theta}$. (V.T.U., 2009)