CHAPTER 11

Differential Equations of First Order

11.1 DEFINITIONS

(1) *A* **differential equation** *is an equation which involves differential coefficients or differentials.*

Thus (*i*) $e^x\,dx + e^y\,dy = 0$ (*ii*) $\dfrac{d^2x}{dt^2} + n^2x = 0$

(*iii*) $y = x\dfrac{dy}{dx} + \dfrac{x}{dy/dx}$ (*iv*) $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2} \Big/ \dfrac{d^2y}{dx^2} = c$

(*v*) $\dfrac{dx}{dt} - wy = a\cos pt,\ \dfrac{dy}{dt} + wx = a\sin pt$ (*vi*) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$

(*vii*) $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$ are all examples of differential equations.

(2) *An* **ordinary differential equation** *is that in which all the differential coefficients have reference to a single independent variable.* Thus the equations (*i*) to (*v*) are all ordinary differential equations.

A **partial differential equation** *is that in which there are two or more independent variables and partial differential coefficients with respect to any of them.* Thus the equations (*vi*) and (*vii*) are partial differential equations.

(3) *The* **order** *of a differential equation is the order of the highest derivative appearing in it.*

The **degree** *of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.*

Thus, from the examples above,

(*i*) is of the first order and first degree ; (*ii*) is of the second order and first degree ;

(*iii*) written as $y\dfrac{dy}{dx} = x\left(\dfrac{dy}{dx}\right)^2 + x$ is clearly of the first order but of second degree ;

and (*iv*) written as $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^3 = c^2\left(\dfrac{d^2y}{dx^2}\right)^2$ is of the second order and second degree.

11.2 PRACTICAL APPROACH TO DIFFERENTIAL EQUATIONS

Differential equations arise from many problems in oscillations of mechanical and electrical systems, bending of beams, conduction of heat, velocity of chemical reactions etc., and as such play a very important role in all modern scientific and engineering studies.

The approach of an engineering student to the study of differential equations has got to be practical unlike that of a student of mathematics, who is only interested in solving the differential equations without knowing as to how the differential equations are formed and how their solutions are physically interpreted.

Thus for an applied mathematician, the study of a differential equation consists of three phases :

(*i*) *formulation of differential equation from the given physical situation, called* **modelling.**

(*ii*) *solutions of this differential equation, evaluating the arbitrary constants from the given conditions,* and

(*iii*) *physical interpretation of the solution.*

11.3 FORMATION OF A DIFFERENTIAL EQUATION

An ordinary differential equation is formed in an attempt to eliminate certain arbitrary constant from a relation in the variables and constants. It will, however, be seen later that the partial differential equations may be formed by the elimination of either arbitrary constants or arbitrary functions. In applied mathematics, every geometrical or physical problem when translated into mathematical symbols gives rise to a differential equation.

Example 11.1. *Form the differential equation of simple harmonic motion given by $x = A\cos(nt + \alpha)$.*

Solution. To eliminate the constants A and α differentiating it twice, we have

$$\frac{dx}{dt} = -nA\sin(nt+\alpha) \quad \text{and} \quad \frac{d^2x}{dt^2} = -n^2A\cos(nt+\alpha) = -n^2x$$

Thus $\quad \dfrac{d^2x}{dt^2} + n^2x = 0$

is the desired differential equation which states that the acceleration varies as the distance from the origin.

Example 11.2. *Obtain the differential equation of all circles of radius a and centre (h, k).* (*Andhra, 1999*)

Solution. Such a circle is $(x-h)^2 + (y-k)^2 = a^2$ ...(*i*)

where h and k, the coordinates of the centre, and a are the constants.

Differential it twice, we have

$$x - h + (y-k)\frac{dy}{dx} = 0 \quad \text{and} \quad 1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

Then $\quad y - k = -\dfrac{1+(dy/dx)^2}{d^2y/dx^2}$

and $\quad x - h = -(y-k)\,dy/dx = \dfrac{\dfrac{dy}{dx}\left[1+\left(\dfrac{dy}{dx}\right)^2\right]}{d^2y/dx^2}$

Substituting these in (*i*) and simplifying, we get $[1 + (dy/dx)^2]^3 = a^2(d^2y/dx^2)^2$ ...(*ii*)

as the required differential equation

Writing (*ii*) in the form $\dfrac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = a$,

it states that the radius of curvature of a circle at any point is constant.

Example 11.3. *Obtain the differential equation of the coaxial circles of the system $x^2 + y^2 + 2ax + c^2 = 0$ where c is a constant and a is a variable.* (*J.N.T.U., 2003*)

Solution. We have $x^2 + y^2 + 2ax + c^2 = 0$ ...(*i*)

Differentiating w.r.t. x, $2x + 2y\,dy/dx + 2a = 0$

or
$$2a = -2\left(x + y\frac{dy}{dx}\right)$$

Substituting in (*i*), $x^2 + y^2 - 2\,(x + y\,dy/dx)x + c^2 = 0$

or
$$2xy\,dy/dx = y^2 - x^2 + c^2$$

which is the required differential equation.

11.4 (1) SOLUTION OF A DIFFERENTIAL EQUATION

A solution (or integral) of a differential equation is a relation between the variables which satisfies the given differential equation.

For example,
$$x = A\cos(nt + \alpha) \qquad ...(1)$$

is a solution of
$$\frac{d^2x}{dt^2} + n^2x = 0 \text{ [Example 11.1]} \qquad ...(2)$$

The **general** (*or* **complete**) **solution** *of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.* Thus (1) is a general solution (2) as the number of arbitrary constants (A, α) is the same as the order of (2).

A **particular solution** *is that which can be obtained from the general solution by giving particular values to the arbitrary constants.*

For example,
$$x = A\cos(nt + \pi/4)$$

is the particular solution of the equation (2) as it can be derived from the general solution (1) by putting $\alpha = \pi/4$.

A differential equation may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a **singular solution** and is not of much engineering interest.

Linearly independent solution. *Two solutions $y_1(x)$ and $y_2(x)$ of the differential equation*
$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0 \qquad ...(3)$$

are said to be linearly independent if $c_1y_1 + c_2y_2 = 0$ such that $c_1 = 0$ and $c_2 = 0$

If c_1 and c_2 are not both zero, then the two solutions y_1 and y_2 are said to be *linearly dependent.*

If $y_1(x)$ and $y_2(x)$ any two solutions of (3), then their linear combination $c_1y_1 + c_2y_2$ where c_1 and c_2 are constants, is also a solution of (3).

Example 11.4. *Find the differential equation whose set of independent solutions is $[e^x, xe^x]$.*

Solution. Let the general solution of the required differential equation be $y = c_1e^x + c_2xe^x$ ...(*i*)

Differentiating (*i*) w.r.t. x, we get
$$y_1 = c_1e^x + c_2(e^x + xe^x)$$
$$\therefore \quad y - y_1 = c_2e^x \qquad ...(ii)$$

Again differentiating (*ii*) w.r.t. x, we obtain
$$y_1 - y_2 = c_2e^x \qquad ...(iii)$$

Subtracting (*iii*) from (*ii*), we get
$$y - y_1 - (y_1 - y_2) = 0 \quad \text{or} \quad y - 2y_1 + y_2 = 0$$

which is the desired differential equation.

(2) Geometrical meaning of a differential equation. Consider any differential equation of the first order and first degree
$$\frac{dy}{dx} = f(x, y) \qquad ...(1)$$

If $P(x, y)$ be any point, then (1) can be regarded as an equation giving the value of dy/dx $(= m)$ when the values of x and y are known (Fig. 11.1). Let the value of m at the point $A_0(x_0, y_0)$ derived from (1) be m_0. Take a neighbouring

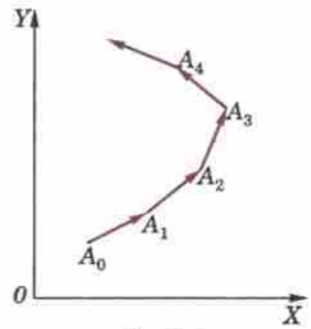

Fig. 11.1

point $A_1(x_1, y_1)$ such that the slope of A_0A_1 is m_0. Let the corresponding value of m at A_1 be m_1. Similarly take a neighbouring point $A_2(x_2, y_2)$ such that the slope of A_1A_2 is m_1 and so on.

If the successive points A_0, A_1, A_2, A_3 ... are chosen very near one another, the broken curve $A_0A_1A_2A_3$ ... approximates to a smooth curve $C[y = \phi(x)]$ which is a solution of (1) associated with the initial point A_0 (x_0, y_0). Clearly the slope of the tangent to C at any point and the coordinates of that point satisfy (1).

A different choice of the initial point will, in general, give a different curve with the same property. *The equation of each such curve is thus a* **particular solution** *of the differential equation (1). The equation of the whole family of such curves is the general solution of* (1). The slope of the tangent at any point of each member of this family and the co-ordinates of that point satisfy (1).

Such a simple geometric interpretation of the solutions of a second (or higher) order differential equation is not available.

PROBLEMS 11.1

Form the differential equations from the following equations :

1. $y = ax^3 + bx^2$.
2. $y = C_1 \cos 2x + C_2 \sin 2x$ *(Bhopal, 2008)*
3. $xy = Ae^x + Be^{-x} + x^2$. *(U.P.T.U., 2005)*
4. $y = e^x (A \cos x + B \sin x)$. *(P.T.U., 2003)*
5. $y = ae^{2x} + be^{-3x} + ce^x$.

Find the differential equations of :

6. A family of circles passing through the origin and having centres on the x-axis. *(J.N.T.U., 2006)*
7. All circles of radius 5, with their centres on the y-axis.
8. All parabolas with x-axis as the axis and $(a, 0)$ as focus.
9. If $y_1(x) = \sin 2x$ and $y_2(x) = \cos 2x$ are two solutions of $y'' + 4y = 0$, show that $y_1(x)$ and $y_2(x)$ are linearly independent solutions.
10. Determine the differential equation whose set of independent solutions is $[e^x, xe^x, x^2 e^x]$ *(U.P.T.U., 2002)*
11. Obtain the differential equation of the family of parabolas $y = x^2 + c$ and sketch those members of the family which pass through (0, 0), (1, 1), (0, 1) and (1, – 1) respectively.

11.5 EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

It is not possible to solve such equations in general. We shall, however, discuss some special methods of solution which are applied to the following types of equations :

(*i*) Equations where variables are separable, (*ii*) Homogeneous equations,
(*iii*) Linear equations, (*iv*) Exact equations.

In other cases, the particular solution may be determined numerically (Chapter 31).

11.6 VARIABLES SEPARABLE

If in an equation it is possible to collect all functions of x and dx on one side and all the functions of y and dy on the other side, then the *variables are said to be separable.* Thus the general form of such an equation is $f(y)\, dy = \phi(x)\, dx$

Integrating both sides, we get $\int f(y)\, dy = \int \phi(x)\, dx + c$ as its solution.

Example 11.5. *Solve* $dy/dx = \dfrac{x(2 \log x + 1)}{\sin y + y \cos y}$. *(V.T.U., 2008)*

Solution. Given equation is $x(2 \log x + 1)\, dx = (\sin y + y \cos y)\, dy$

Integrating both sides, $2 \int (\log x \,.\, x + x)\, dx = \int \sin y\, dy + \int y \cos y\, dy + c$

or
$$2\left[\left(\log x \,.\, \frac{x^2}{2} - \int \frac{1}{x} \,.\, \frac{x^2}{2}\, dx\right) + \frac{x^2}{2}\right] = -\cos y + \left[y \sin y - \int \sin y \,.\, 1\, dy + c\right]$$

or $$2x^2 \log x - \frac{x^2}{2} + \frac{x^2}{2} = -\cos y + y \sin y + \cos y + c$$

Hence the solution is $2x^2 \log x - y \sin y = c$.

Example 11.6. *Solve* $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$.

Solution. Given equation is $\frac{dy}{dx} = e^{-2y}(e^{3x} + x^2)$ or $e^{2y}\,dy = (e^{3x} + x^2)\,dx$

Integrating both sides, $\int e^{2y}\,dy = \int (e^{3x} + x^2)\,dx + c$

or $$\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + c \quad \text{or} \quad 3e^{2y} = 2(e^{3x} + x^3) + 6c.$$

Example 11.7. *Solve* $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$. *(V.T.U., 2005)*

Solution. Putting $x + y = t$ so that $dy/dx = dt/dx - 1$

The given equation becomes $\frac{dt}{dx} - 1 = \sin t + \cos t$

or $$dt/dx = 1 + \sin t + \cos t$$

Integrating both sides, we get $dx = \int \frac{dt}{1 + \sin t + \cos t} + c.$

or $$x = \int \frac{2d\theta}{1 + \sin 2\theta + \cos 2\theta} + c \qquad [\text{Putting } t = 2\theta]$$

$$= \int \frac{2d\theta}{2\cos^2\theta + 2\sin\theta\cos\theta} + c = \int \frac{\sec^2\theta}{1 + \tan\theta}\,d\theta + c$$

$$= \log(1 + \tan\theta) + c$$

Hence the solution is $x = \log\left[1 + \tan \tfrac{1}{2}(x+y)\right] + c$.

Example 11.8. *Solve* $dy/dx = (4x + y + 1)^2$, *if* $y(0) = 1$.

Solution. Putting $4x + y + 1 = t$, we get $\frac{dy}{dx} = \frac{dt}{dx} - 4$.

$\therefore$ the given equation becomes $\frac{dt}{dx} - 4 = t^2$ or $\frac{dt}{dx} = 4 + t^2$

Integrating both sides, we get $\int \frac{dt}{4 + t^2} = \int dx + c$

or $$\frac{1}{2}\tan^{-1}\frac{t}{2} = x + c \quad \text{or} \quad \frac{1}{2}\tan^{-1}\left[\frac{1}{2}(4x + y + 1)\right] = x + c.$$

or $$4x + y + 1 = 2\tan 2(x + c)$$

When $x = 0, y = 1 \quad \therefore \quad \frac{1}{2}\tan^{-1}(1) = c$ *i.e.* $c = \pi/8$.

Hence the solution is $4x + y + 1 = 2\tan(2x + \pi/4)$.

Example 11.9. *Solve* $\frac{y}{x}\frac{dy}{dx} + \frac{x^2 + y^2 - 1}{2(x^2 + y^2) + 1} = 0$. *(V.T.U., 2003)*

Solution. Putting $x^2 + y^2 = t$, we get $2x + 2y\frac{dy}{dx} = \frac{dt}{dx}$ or $\frac{y}{x}\frac{dy}{dx} = \frac{1}{2x}\frac{dt}{dx} - 1$.

Therefore the given equation becomes $\frac{1}{2x}\frac{dt}{dx} - 1 + \frac{t-1}{2t+1} = 0$

or
$$\frac{1}{2x}\frac{dt}{dx} = 1 - \frac{t-1}{2t+1} = \frac{t+2}{2t+1} \quad \text{or} \quad 2x\,dx = \frac{2t+1}{t+2}dt$$

or
$$2x\,dx = \left(2 - \frac{3}{t+2}\right)dt$$

Integrating, we get $x^2 = 2t - 3\log(t+2) + c$

or
$$x^2 + 2y^2 - 3\log(x^2 + y^2 + 2) + c = 0 \qquad [\because \; t = x^2 + y^2]$$

which is the required solution.

PROBLEMS 11.2

Solve the following differential equations :

1. $y\sqrt{(1-x^2)}\,dy + x\sqrt{(1-y^2)}\,dx = 0.$
2. $(x^2 - yx^2)\frac{dy}{dx} + y^2 + xy^2 = 0.$
3. $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0.$ (*P.T.U., 2003*)
4. $\frac{y}{x}\frac{dy}{dx} = \sqrt{(1 + x^2 + y^2 + x^2y^2)}.$ (*V.T.U., 2011*)
5. $e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0.$ (*V.T.U., 2009*)
6. $\frac{dy}{dx} = xe^{y-x^2}$, if $y = 0$ when $x = 0$. (*J.N.T.U., 2006*)
7. $x\frac{dy}{dx} + \cot y = 0$ if $y = \pi/4$ when $x = \sqrt{2}$.
8. $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0.$
9. $\frac{dy}{dx} = e^{2x-3y} + 4x^2e^{-3y}.$
10. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right).$
11. $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}.$ (*Madras, 2000 S*)
12. $(x-y)^2\frac{dy}{dx} = a^2.$
13. $(x+y+1)^2\frac{dy}{dx} = 1.$ (*Kurukshetra, 2005*)
14. $\sin^{-1}(dy/dx) = x + y$ (*V.T.U., 2010*)
15. $\frac{dy}{dx} = \cos(x+y+1)$ (*V.T.U., 2003*)
16. $\frac{dy}{dx} - x\tan(y-x) = 1.$
17. $x^4\frac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0.$

11.7 HOMOGENEOUS EQUATIONS

are of the form $\frac{dy}{dx} = \frac{f(x, y)}{\phi(x, y)}$

where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of the same degree in x and y (see page 205).

To solve a homogeneous equation (*i*) Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$,

(*ii*) Separate the variables v and x, and integrate.

Example 11.10. *Solve* $(x^2 - y^2)\,dx - xy\,dy = 0.$

Solution. Given equation is $\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$ which is homogeneous in x and y. ...(*i*)

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. $\therefore$ (*i*) becomes $v + x\frac{dv}{dx} = \frac{1 - v^2}{v}$

or
$$x\frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1-2v^2}{v}.$$

Separating the variables, $\dfrac{v}{1-2v^2}\,dv = \dfrac{dx}{x}$

Integrating both sides, $\displaystyle\int \frac{v\,dv}{1-2v^2} = \int \frac{dx}{x} + c$

or $$-\frac{1}{4}\int \frac{-4v}{1-2v^2}\,dv = \int \frac{dx}{x} + c \quad \text{or} \quad -\frac{1}{4}\log(1-2v^2) = \log x + c$$

or $$4\log x + \log(1-2v^2) = -4c \quad \text{or} \quad \log x^4(1-2v^2) = -4c \qquad [\text{Put } v = y/x]$$

or $$x^4(1-2y^2/x^2) = e^{-3c} = c'$$

Hence the required solution is $x^2(x^2-2y^2) = c'$.

Example 11.11. *Solve* $(x \tan y/x - y \sec^2 y/x)\,dx - x\sec^2 y/x\,dy = 0.$ *(V.T.U., 2006)*

Solution. The given equation may be rewritten as

$$\frac{dy}{dx} = \underset{vy}{\left(\frac{y}{x}\sec^2\frac{y}{x} - \tan\frac{y}{x}\right)}\cos^2 y/x \qquad ...(i)$$

which is a homogeneous equation. Putting $y = vx$, (i) becomes $v + x\dfrac{dv}{dx} = (v\sec^2 v - \tan v)\cos^2 v$

or $$x\frac{dv}{dx} = v - \tan v\cos^2 v - v$$

Separating the variables $\dfrac{\sec^2 v}{\tan v}\,dv = -\dfrac{dx}{x}$

Integrating both sides $\log\tan v = -\log x + \log c$

or $$x\tan v = c \quad \text{or} \quad x\tan y/x = c.$$

Example 11.12. *Solve* $(1+e^{x/y})\,dx + e^{x/y}(1-x/y)\,dy = 0.$ *(P.T.U., 2006 ; Rajasthan, 2005 ; V.T.U., 2003)*

Solution. The given equation may be rewritten as

$$\frac{dx}{dy} = -\frac{e^{x/y}(1-x/y)}{1+e^{x/y}} \qquad ...(i)$$

which is a homogeneous equation. Putting $x = vy$ so that (i) becomes

$$v + y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} \quad \text{or} \quad y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} - v = -\frac{v+e^v}{1+e^v}$$

Separating the variables, we get

$$-\frac{dy}{y} = \frac{1+e^v}{v+e^v}\,dv = \frac{d(v+e^v)}{v+e^v}$$

Integrating both sides, $-\log y = \log(v+e^v) + c$

or $$y(v+e^v) = e^{-c} \quad \text{or} \quad x + ye^{x/y} = c' \quad \text{(say)}$$

which is the required solution.

PROBLEMS 11.3

Solve the following differential equations :

1. $(x^2-y^2)\,dx = 2xy\,dy$
2. $(x^2y - 2xy^2)\,dx - (x^3-3x^2y)\,dy = 0.$ *(Bhopal, 2008)*
3. $x^2y\,dx - (x^3+y^3)\,dy = 0.$ *(V.T.U., 2010)*
4. $y\,dx - x\,dy = \sqrt{x^2+y^2}\,dx.$ *(Raipur, 2005)*
5. $y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}.$
6. $(3xy - 2ay^2)\,dx + (x^2-2axy)\,dy = 0.$ *(S.V.T.U., 2009)*

[Equations solvable like homogeneous equations : When a differential equation contains y/x a number of times, solve it like a homogeneous equation by putting $y/x = v$].

7. $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$. (V.T.U., 2000 S)
8. $ye^{x/y}\,dx = (xe^{x/y} + y^2)\,dy$. (V.T.U., 2006)
9. $xy\,(\log x/y)\,dx + [y^2 - x^2 \log (x/y)]\,dy = 0$.
10. $x\,dx + \sin^2 (y/x)\,(ydx - xdy) = 0$.
11. $x \cos \frac{y}{x}\,(ydx + xdy) = y \sin \frac{y}{x}\,(xdy - ydx)$.

11.8 EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

The equations of the form $$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'} \quad ...(1)$$

can be reduced to the homogeneous form as follows :

Case I. When $\frac{a}{a'} \neq \frac{b}{b'}$

Putting $x = X + h,\ y = Y + k$, (h, k being constants)

so that $dx = dX,\ dy = dY$, (1) becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \quad ...(2)$$

Choose h, k so that (2) may become homogeneous.

Put $ah + bk + c = 0$, and $a'h + b'k + c' = 0$

so that $$\frac{h}{bc' - b'c} = \frac{k}{ca' - c'a} = \frac{1}{ab' - ba'}$$

or $$h = \frac{bc' - b'c}{ab' - b'a},\ k = \frac{ca' - c'a}{ab' - ba'} \quad ...(3)$$

Thus when $ab' - ba' \neq 0$, (2) becomes $\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$ which is homogeneous in X, Y and can be solved by putting $Y = vX$.

Case II. When $\frac{a}{a'} = \frac{b}{b'}$.

i.e., $ab' - b'a = 0$, the above method fails as h and k become infinite or indeterminate.

Now $$\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m} \text{ (say)}$$

$\therefore$ $a' = am,\ b' = bm$ and (1) becomes

$$\frac{dy}{dx} = \frac{(ax + by) + c}{m(ax + by) + c'} \quad ...(4)$$

Put $ax + by = t$, so that $a + b\frac{dy}{dx} = \frac{dt}{dx}$

or $$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right) \quad \therefore \text{ (4) becomes } \frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t + c}{mt + c'}$$

or $$\frac{dt}{dx} = a + \frac{bt + bc}{mt + c'} = \frac{(am + b)t + ac' + bc}{mt + c'}$$

so that the variables are separable. In this solution, putting $t = ax + by$, we get the required solution of (1).

Example 11.13. *Solve* $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$. (Raipur, 2005)

Solution. Given equation is $$\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4} \quad \left[\text{Case } \frac{a}{a'} \neq \frac{b}{b'}\right] \quad ...(i)$$

Putting $x = X + h$, $y = Y + k$, (h, k being constants) so that $dx = dX$, $dy = dY$, (i) becomes

$$\frac{dY}{dX} = \frac{Y + X + (k + h - 2)}{Y - X + (k - h - 4)} \quad ...(ii)$$

Put $k + h - 2 = 0$ and $k - h - 4 = 0$ so that $h = -1$, $k = 3$.

$\therefore$ (ii) becomes $\dfrac{dY}{dX} = \dfrac{Y + X}{Y - X}$ which is homogeneous in X and Y. ...(iii)

$\therefore$ put $Y = vX$, then $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\therefore$ (iii) becomes $\quad v + X\dfrac{dv}{dX} = \dfrac{v+1}{v-1}$ or $X\dfrac{dv}{dX} = \dfrac{v+1}{v-1} - v = \dfrac{1 + 2v - v^2}{v - 1}$

or $$\frac{v - 1}{1 + 2v - v^2}\, dv = \frac{dX}{X}.$$

Integrating both sides, $-\dfrac{1}{2}\displaystyle\int \frac{2 - 2v}{1 + 2v - v^2}\, dv = \int \frac{dX}{X} + c.$

or $$-\frac{1}{2}\log(1 + 2v - v^2) = \log X + c$$

or $$\log\left(1 + \frac{2Y}{X} - \frac{Y^2}{X^2}\right) + \log X^2 = -2c$$

or $$\log(X^2 + 2XY - Y^2) = -2c \quad \text{or} \quad X^2 + 2XY - Y^2 = e^{-2c} = c' \quad ...(iv)$$

Putting $X = x - h = x + 1$, $Y = y - k = y - 3$, (iv) becomes

$$(x + 1)^2 + 2(x + 1)(y - 3) - (y - 3)^2 = c'$$

or $\quad x^2 + 2xy - y^2 - 4x + 8y - 14 = c'$ which is the required solution.

Example 11.14. *Solve* $(3y + 2x + 4)\, dx - (4x + 6y + 5)\, dy = 0.$ *(Madras, 2000 S)*

Solution. Given equation is $\dfrac{dy}{dx} = \dfrac{(2x + 3y) + 4}{2(2x + 3y) + 5}$ ...(i)

Putting $2x + 3y = t$ so that $2 + 3\dfrac{dy}{dx} = \dfrac{dt}{dx}$ $\quad\therefore$ (i) becomes $\dfrac{1}{3}\left(\dfrac{dt}{dx} - 2\right) = \dfrac{t + 4}{2t + 5}$

or $$\frac{dt}{dx} = 2 + \frac{3t + 12}{2t + 5} = \frac{7t + 22}{2t + 5} \quad \text{or} \quad \frac{2t + 5}{7t + 22}\, dt = dx$$

Integrating both sides, $\displaystyle\int \frac{2t + 5}{7t + 22}\, dt = \int dx + c$

or $$\int\left(\frac{2}{7} - \frac{9}{7}\cdot\frac{1}{7t + 22}\right) dt = x + c \quad \text{or} \quad \frac{2}{7}t - \frac{9}{49}\log(7t + 22) = x + c$$

Putting $t = 2x + 3y$, we have $14(2x + 3y) - 9\log(14x + 21y + 22) = 49x + 49c$

or $\quad 21x - 42y + 9\log(14x + 21y + 22) = c'$ which is the required solution.

PROBLEMS 11.4

Solve the following differential equations :

1. $(x - y - 2)\, dx + (x - 2y - 3)\, dy = 0.$ *(Rajasthan, 2006)*
2. $(2x + y - 3)\, dy = (x + 2y - 3)\, dx.$ *(V.T.U., 2009 S ; Madras, 2000)*
3. $(2x + 5y + 1)\, dx - (5x + 2y - 1)\, dy = 0.$ *(J.N.T.U., 2000)*
4. $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0.$
5. $\dfrac{dy}{dx} = \dfrac{x + y + 1}{2x + 2y + 3}.$
6. $(4x - 6y - 1)\, dx + (3y - 2x - 2)\, dy = 0.$
7. $(x + 2y)(dx - dy) = dx + dy.$ *(Bhopal, 2002 S ; V.T.U., 2001)*

11.9 LINEAR EQUATIONS

A differential equation is said to be linear if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together.

Thus **the standard form of a linear equation of the first order,** *commonly known as* **Leibnitz's linear equation,*** *is*

$$\frac{dy}{dx} + Py = Q \qquad \text{where, } P, Q \text{ are the functions of } x. \qquad \text{...(1)}$$

To solve the equation, multiply both sides by $e^{\int P dx}$ so that we get

$$\frac{dy}{dx} \cdot e^{\int P dx} + y\left(e^{\int P dx} P\right) = Qe^{\int P dx} \quad \text{i.e.,} \quad \frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx}$$

Integrating both sides, we get $ye^{\int P dx} = \int Q e^{\int P dx}\, dx + c$ as the required solution.

Obs. The factor $e^{\int P dx}$ on multiplying by which the left-hand side of (1) becomes the differential coefficient of a single function, is called the **integrating factor (I.F.)** of the linear equation (1).

It is important to remember that **I.F. =** $\mathbf{e^{\int P dx}}$

and the solution is **y (I.F.) =** $\int \mathbf{Q(I.F.)\, dx + c}$.

Example 11.15. *Solve* $(x + 1)\frac{dy}{dx} - y = e^{3x}(x + 1)^2$.

Solution. Dividing throughout by $(x + 1)$, given equation becomes

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x + 1) \text{ which is Leibnitz's equation.} \qquad \text{...}(i)$$

Here $\quad P = -\frac{1}{x+1}$ and $\int P dx = -\int \frac{dx}{x+1} = -\log(x + 1) = \log(x + 1)^{-1}$

$\therefore \quad$ I.F. $= e^{\int P dx} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$

Thus the solution of (1) is $y(\text{I.F.}) = \int [e^{3x}(x + 1)](\text{I.F.})\, dx + c$

or $\quad \frac{y}{x+1} = \int e^{3x}\, dx + c = \frac{1}{3}e^{3x} + c \quad$ or $\quad y = \left(\frac{1}{3}e^{3x} + c\right)(x + 1)$.

Example 11.16. *Solve* $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$.

Solution. Given equation can be written as $\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$ $\qquad$ ...(*i*)

$\therefore \quad$ I.F. $= e^{\int x^{-1/2} dx} = e^{2\sqrt{x}}$

Thus solution of (*i*) is $y\,(\text{I.F.}) = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}}(\text{I.F.})\, dx + c$

or $\quad ye^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}}\, dx + c$

or $\quad ye^{2\sqrt{x}} = \int x^{-1/2}\, dx + c \quad$ or $\quad ye^{2\sqrt{x}} = 2\sqrt{x} + c$.

* See footnote p. 139.

Example 11.17. *Solve* $3x(1-x^2)y^2\dfrac{dy}{dx}+(2x^2-1)y^3=ax^3$ *(Rajasthan, 2006)*

Solution. Putting $y^3 = z$ and $3y^2\dfrac{dy}{dx}=\dfrac{dz}{dx}$, the given equation becomes

$$x(1-x^2)\frac{dz}{dx}+(2x^2-1)z=ax^3, \quad \text{or} \quad \frac{dz}{dx}+\frac{2x^2-1}{x-x^3}z=\frac{ax^3}{x-x^3} \qquad ...(i)$$

which is Leibnitz's equation in z

$$\therefore \qquad \text{I.F.} = \exp\left(\int \frac{2x^2-1}{x-x^3}dx\right)$$

Now $$\int \frac{2x^2-1}{x-x^3}dx=\int\left(-\frac{1}{x}-\frac{1}{2}\frac{1}{1+x}+\frac{1}{2}\cdot\frac{1}{1-x}\right)dx=-\log x-\frac{1}{2}\log(1+x)-\frac{1}{2}\log(1-x)$$

$$=-\log[x\sqrt{(1-x^2)}]$$

$$\therefore \qquad \text{I.F.} = e^{-\log[x\sqrt{(1-x^2)}]}=[x\sqrt{(1-x^2)}]^{-1}$$

Thus the solution of (*i*) is

$$z(\text{I.F.})=\int\frac{ax^3}{x-x^3}(\text{I.F.})\,dx+c$$

or $$\frac{z}{[x\sqrt{(1-x^2)}]}=a\int\frac{x^3}{x(1-x^2)}\cdot\frac{1}{x\sqrt{(1-x^2)}}dx+c=a\int x(1-x^2)^{-3/2}dx$$

$$=-\frac{a}{2}\int(-2x)(1-x^2)^{-3/2}dx+c=a(1-x^2)^{-1/2}+c$$

Hence the solution of the given equation is

$$y^3=ax+cx\sqrt{(1-x^2)}. \qquad [\because \ z=y^3]$$

Example 11.18. *Solve* $y(\log y)\,dx+(x-\log y)\,dy=0.$ *(U.P.T.U., 2000)*

Solution. We have $\dfrac{dx}{dy}+\dfrac{x}{y\log y}=\dfrac{1}{y}$ ... (*i*)

which is a Leibnitz's equation in x

$$\therefore \qquad \text{I.F.} = e^{\int\frac{1}{y\log y}dy}=e^{\log(\log y)}=\log y$$

Thus the solution of (*i*) is $x\,(\text{I.F.})=\displaystyle\int\frac{1}{y}(\text{I.F.})\,dy+c$

$$x\log y=\int\frac{1}{y}\log y\,dy+c=\frac{1}{2}(\log y)^2+c$$

i.e., $$x=\frac{1}{2}\log y+c(\log y)^{-1}.$$

Example 11.19. *Solve* $(1+y^2)\,dx=(\tan^{-1}y-x)\,dy.$ *(Bhopal, 2008 ; V.T.U., 2008 ; U.P.T.U., 2005)*

Solution. This equation contains y^2 and $\tan^{-1}y$ and is, therefore, not a linear in y; but since only x occurs, it can be written as

$$(1+y^2)\frac{dx}{dy}=\tan^{-1}y-x \quad \text{or} \quad \frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$$

which is a Leibnitz's equation in x.

$$\therefore \qquad \text{I.F.} = e^{\int P dy}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

Thus the solution is $x\,(\text{I.F.})=\displaystyle\int\frac{\tan^{-1}y}{1+y^2}(\text{I.F.})\,dy+c$

or $$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} . e^{\tan^{-1} y} dy + c \qquad \left[\begin{array}{l} \text{Put } \tan^{-1} y = t \\ \therefore \ \dfrac{dy}{1+y^2} = dt \end{array}\right]$$

$$= \int te^t \, dt + c = t . e^t - \int 1 . e^t \, dt + c \qquad \text{(Integrating by parts)}$$

$$= t . e^t - e^t + c = (\tan^{-1} y - 1) e^{\tan^{-1} y} + c$$

or $$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}.$$

Example 11.20. *Solve $r \sin \theta \, d\theta + (r^3 - 2r^2 \cos \theta + \cos \theta) \, dr = 0$.*

Solution. Given equation can be rewritten as

$$\sin \theta \frac{d\theta}{dr} + \frac{1}{r}(1 - 2r^2) \cos \theta = - r^2 \qquad ...(i)$$

Put $\cos \theta = y$ so that $- \sin \theta \, d\theta/dr = dy/dr$

Then (i) becomes $-\dfrac{dy}{dr} + \left(\dfrac{1}{r} - 2r\right) y = - r^2$ or $\dfrac{dy}{dr} + \left(2r - \dfrac{1}{r}\right) y = r^2$

which is a Leibnitz's equation $\therefore$ I.F. $= e^{\int (2r - 1/r) dr} = e^{r^2 - \log r} = \dfrac{1}{r} e^{r^2}$

Thus its solution is $y \left(\dfrac{1}{r} e^{r^2}\right) = \int r^2 . e^{r^2} . \dfrac{1}{r} \, dr + c$

or $$y e^{r^2} / r = \frac{1}{2} \int e^{r^2} 2r \, dr + c = \frac{1}{2} e^{r^2} + c$$

or $$2e^{r^2} \cos \theta = re^{r^2} + 2cr \quad \text{or} \quad r(1 + 2ce^{-r^2}) = 2 \cos \theta.$$

PROBLEMS 11.5

Solve the following differential equations :

1. $\cos^2 x \dfrac{dy}{dx} + y = \tan x.$
2. $x \log x \dfrac{dy}{dx} + y = \log x^2.$ (*V.T.U., 2011*)
3. $2y' \cos x + 4y \sin x = \sin 2x$, given $y = 0$ when $x = \pi/3$. (*V.T.U., 2003*)
4. $\cosh x \dfrac{dy}{dx} + y \sinh x = 2 \cosh^2 x \sinh x$. (*J.N.T.U., 2003*)
5. $(1 - x^2) \dfrac{dy}{dx} - xy = 1$ (*V.T.U., 2010*)
6. $(1 - x^2) \dfrac{dy}{dx} + 2xy = x \sqrt{(1 - x^2)}$ (*Nagpur, 2009*)
7. $\dfrac{dy}{dx} = -\dfrac{x + y \cos x}{1 + \sin x}.$
8. $dr + (2r \cot \theta + \sin 2\theta) \, d\theta = 0.$ (*J.N.T.U., 2003*)
9. $\dfrac{dy}{dx} + 2xy = 2e^{-x^2}$ (*P.T.U., 2005*)
10. $(x + 2y^3) \dfrac{dy}{dx} = y.$ (*Marathwada, 2008*)
11. $\sqrt{(1 - y^2)} \, dx = (\sin^{-1} y - x) \, dy.$
12. $ye^y \, dx = (y^3 + 2xe^y) \, dy.$
13. $(1 + y^2) \, dx + (x - e^{-\tan^{-1} y}) \, dy = 0.$ (*V.T.U., 2006*)
14. $e^{-y} \sec^2 y \, dy = dx + x \, dy.$

11.10 BERNOULLI'S EQUATION

The equation $$\frac{dy}{dx} + Py = Qy^n \qquad ...(1)$$

where P, Q are functions of x, is reducible to the Leibnitz's linear equation and is usually called the Bernoulli's equation.*

*Named after the Swiss mathematician *Jacob Bernoulli* (1654–1705) who is known for his basic work in probability and elasticity theory. He was professor at Basel and had amongst his students his youngest brother *Johann Bernoulli* (1667–1748) and his nephew *Niklaus Bernoulli* (1687–1759). Johann is known for his basic contributions to Calculus while Niklaus had profound influence on the development of Infinite series and probability. His son *Daniel Bernoulli* (1700–1782) is known for his contributions to kinetic theory of gases and fluid flow.

To solve (1), divide both sides by y^n, so that $y^{-n}\dfrac{dy}{dx}+Py^{1-n}=Q$ ...(2)

Put $y^{1-n}=z$ so that $(1-n)y^{-n}\dfrac{dy}{dx}=\dfrac{dz}{dx}$.

$\therefore$ (2) becomes $\dfrac{1}{1-n}\dfrac{dz}{dx}+Pz=Q$ or $\dfrac{dz}{dx}+P(1-n)z=Q(1-n)$,

which is Leibnitz's linear in z and can be solved easily.

Example 11.21. *Solve* $x\dfrac{dy}{dx}+y=x^3y^6$.

Solution. Dividing throughout by xy^6, $y^{-6}\dfrac{dy}{dx}+\dfrac{y^{-5}}{x}=x^2$ ...(i)

Put $y^{-5}=z$, so that $-5y^{-6}\dfrac{dy}{dx}=\dfrac{dz}{dx}$ $\therefore$ (i) becomes $-\dfrac{1}{5}\dfrac{dz}{dx}+\dfrac{z}{x}=x^2$

or $$\frac{dz}{dx}-\frac{5}{x}z=-5x^2 \text{ which is Leibnitz's linear in } z. \quad ...(ii)$$

$$\text{I.F.}=e^{-\int(5/x)dx}=e^{-5\log x}=e^{\log x^{-5}}=x^{-5}$$

$\therefore$ the solution of (ii) is $z\,(\text{I.F.})=\int(-5x^2)(\text{I.F.})dx+c$ or $zx^{-5}=\int(-5x^2)\,x^{-5}\,dx+c$

or $$y^{-5}x^{-5}=-5.\frac{x^{-2}}{-2}+c \qquad [\because\ z=y^{-5}]$$

Dividing throughout by $y^{-5}x^{-5}$, $1=(2.5+cx^2)\,x^3y^5$ which is the required solution.

Example 11.22. *Solve* $xy\,(1+xy^2)\dfrac{dy}{dx}=1$. *(Nagpur, 2009)*

Solution. Rewriting the given equation as

$$\frac{dx}{dy}-yx=y^3x^2$$

and dividing by x^2, we have

$$x^{-2}\frac{dx}{dy}-yx^{-1}=y^3 \quad ...(i)$$

Putting $x^{-1}=z$ so that $-x^{-2}\dfrac{dx}{dy}=\dfrac{dz}{dy}$ (i) becomes

$$\frac{dz}{dy}+yz=-y^3 \text{ which is Leibnitz's linear in } z.$$

Here $\quad \text{I.F.}=e^{\int y\,dy}=e^{y^2/2}$

$\therefore$ the solution is $z\,(\text{I.F.})=\int(-y^3)(\text{I.F.})\,dy+c$

or $$ze^{y^2/2}=-\int y^2.e^{\frac{1}{2}y^2}.\,ydy+c$$ | Put $\frac{1}{2}y^2=t$ so that $y\,dy=dt$

$$=-2\int t.e^t\,dt+c \qquad \text{[Integrate by parts]}$$

$$=-2\,[t.e^t-\int 1.e^t\,dt]+c=-2\,[te^t-e^t]+c=(2-y^2)\,e^{y^2/2}+c$$

or $$z=(2-y^2)+ce^{-\frac{1}{2}y^2} \text{ or } 1/x=(2-y^2)+ce^{-\frac{1}{2}y^2}.$$

Note. *General equation reducible to Leibnitz's linear is* $f'(y)\dfrac{dy}{dx}+Pf(y)=Q$ ...(A)

where P, Q are functions of x. To solve it, put $f(y)=z$.

Example 11.23. *Solve* $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$. (*V.T.U., 2011 ; Marathwada, 2008 ; J.N.T.U., 2005*)

Solution. Dividing throughout by $\cos^2 y$, $\sec^2 y \frac{dy}{dx} + 2x \frac{\sin y \cos y}{\cos^2 y} = x^3$

or $$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \text{ which is of the form } (A) \text{ above.} \qquad ...(i)$$

$\therefore$ put $\tan y = z$ so that $\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$ $\quad\therefore\quad$ (*i*) becomes $\frac{dz}{dx} + 2xz = x^3$.

This is Leibnitz's linear equation in z. $\therefore$ I.F. $= e^{\int 2x\,dx} = e^{x^2}$

$\therefore$ the solution is $ze^{x^2} = \int e^{x^2} x^3 dx + c = \frac{1}{2}(x^2 - 1)e^{x^2} + c$.

Replacing z by $\tan y$, we get $\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$ which is the required solution.

Example 11.24. *Solve* $\frac{dz}{dx} + \left(\frac{z}{x}\right) \log z = \frac{z}{x} (\log z)^2$.

Solution. Dividing by z, the given equation becomes

$$\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\log z = \frac{1}{x}(\log z)^2 \qquad ...(i)$$

Put $\log z = t$ so that $\frac{1}{z}\frac{dz}{dx} = \frac{dt}{dx}$. $\quad\therefore\quad$ (*i*) becomes

$$\frac{dt}{dx} + \frac{t}{x} = \frac{t^2}{x} \quad \text{or} \quad \frac{1}{t^2}\frac{dt}{dx} + \frac{1}{x}\cdot\frac{1}{t} = \frac{1}{x} \qquad ...(ii)$$

This being Bernoulli's equation, put $1/t = v$ so that (*ii*) reduces to

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x} \quad \text{or} \quad \frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}$$

This is Leibnitz's linear in v. $\therefore$ I.F. $= e^{-\int 1/x\,dx} = 1/x$

$\therefore$ the solution is $\qquad v \cdot \frac{1}{x} = -\int \frac{1}{x}\cdot\frac{1}{x}dx + c = \frac{1}{x} + c$

Replacing v by $1/\log z$, we get $(x \log z)^{-1} = x^{-1} + c$ or $(\log z)^{-1} = 1 + cx$ which is the required solution.

PROBLEMS 11.6

Solve the following equations :

1. $\frac{dy}{dx} = y \tan x - y^2 \sec x$. (*P.T.U., 2005*)
2. $r \sin\theta - \cos\theta \frac{dr}{d\theta} = r^2$. (*V.T.U., 2005*)
3. $2xy' = 10x^3y^5 + y$.
4. $(x^3y^2 + xy)\,dx = dy$. (*B.P.T.U., 2005*)
5. $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$. (*Bhillai, 2005*)
6. $x(x - y)\,dy + y^2\,dx = 0$. (*I.S.M., 2001*)
7. $\frac{dy}{dx} - \frac{\tan y}{1 + x} = (1 + x)e^x \sec y$. (*Bhopal, 2009*)
8. $e^y\left(\frac{dy}{dx} + 1\right) = e^x$. (*V.T.U., 2009*)
9. $\sec^2 y \frac{dy}{dx} + x \tan y = x^3$.
10. $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$. (*Sambalpur, 2002*)
11. $\frac{dy}{dx} = \frac{y}{x - \sqrt{(xy)}}$. (*V.T.U., 2011*)
12. $(y \log x - 2)\,y dx - x dy = 0$. (*V.T.U., 2006*)

11.11 EXACT DIFFERENTIAL EQUATIONS

(1) Def. *A differential equation of the form* $\mathbf{M}(x, y)\, \mathbf{dx} + \mathbf{N}(x, y)\, \mathbf{dy} = 0$ *is said to be* ***exact*** *if its left hand member is the exact differential of some function* $u(x, y)$ *i.e.,* $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.

(2) Theorem. The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial \mathbf{M}}{\partial \mathbf{y}} = \frac{\partial \mathbf{N}}{\partial \mathbf{x}}$$

Condition is necessary :

The equation $Mdx + Ndy = 0$ will be exact, if

$$Mdx + Ndy \equiv du \qquad ...(1)$$

where u is some function of x any y.

But $$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \qquad ...(2)$$

$\therefore$ equating coefficients of dx and dy in (1) and (2), we get $M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$

$\therefore$ $$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

But $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \qquad \text{(Assumption)}$$

$\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is the necessary condition for exactness.

Condition is sufficient : i.e., if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then $Mdx + Ndy = 0$ is exact.

Let $\int Mdx = u$, where y is supposed constant while performing integration.

Then $$\frac{\partial}{\partial x}\left(\int Mdx\right) = \frac{\partial u}{\partial x}, \quad i.e., \quad M = \frac{\partial u}{\partial x} \qquad \left\{ \begin{array}{l} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ (given)} \\ \text{and } \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \end{array} \right. \qquad ...(3)$$

$\therefore$ $$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{or} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

Interating both sides w.r.t. x (taking y as constant).

$$N = \frac{\partial u}{\partial y} + f(y), \text{ where } f(y) \text{ is a function of } y \text{ alone.} \qquad ...(4)$$

$\therefore$ $$Mdx + Ndy = \frac{\partial u}{\partial x} dx + \left\{\frac{\partial u}{\partial y} + f(y)\right\} dy \qquad \text{[By (3) and (4)]}$$

$$= \left\{\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right\} + f(y)\, dy = du + f(y)\, dy = d\left[u + \int f(y) dy\right] \qquad ...(5)$$

which shows that $Mdx + Ndy = 0$ is exact.

(3) Method of solution. By (5), the equation $Mdx + Ndy = 0$ becomes $d[u + \int f(y) dy] = 0$

Integrating $\quad u + \int f(y) dy = 0.$

But $\quad u = \int_{y \text{ constant}} Mdx$ and $f(y)$ = terms of N not containing x.

$\therefore$ *The solution of* $Mdx + Ndy = 0$ *is*

$$\int_{(y \text{ cons.})} \mathbf{M}\, \mathbf{dx} + \int (\textbf{terms of N not containing x})\, \mathbf{dy} = \mathbf{c}$$

provided $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example 11.25. *Solve* $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0.$ *(V.T.U., 2006)*

Solution. Here $M = y^2 e^{xy^2} + 4x^3$ and $N = 2xy\,e^{xy^2} - 3y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2y\,e^{xy^2} + y^2 e^{xy^2} \,.\, 2xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y\text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

i.e., $$\int_{(y\text{ const.})} (y^2 e^{y^2 x} + 4x^3)dx + \int (-3y^2)\,dy = c \quad \text{or} \quad e^{xy^2} + x^4 - y^3 = c.$$

Example 11.26. *Solve* $\left\{y\left(1+\frac{1}{x}\right) + \cos y\right\} dx + (x + \log x - x \sin y)\,dy = 0.$

(Marathwada, 2008 S ; V.T.U., 2006)

Solution. Here $M = y\,(1 + 1/x) + \cos y$ and $N = x + \log x - x \sin y$

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + 1/x - \sin y = \frac{\partial N}{\partial x}$$

Then the equation is exact and its solution is

$$\int_{(y\text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

$$\int_{(y\text{ const})} \left\{\left(1+\frac{1}{x}\right) y + \cos y\right\} dx = c \quad \text{or} \quad (x + \log x)\,y + x \cos y = c.$$

Example 11.27. *Solve* $(1 + 2xy \cos x^2 - 2xy)\,dx + (\sin x^2 - x^2)\,dy = 0.$

Solution. Here $M = 1 + 2xy \cos x^2 - 2xy$ and $N = \sin x^2 - x^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x \cos x^2 - 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y\text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x) = c$$

i.e., $$\int_{(y\text{ const})} (1 + 2xy \cos x^2 - 2xy)\,dx = c \quad \text{or} \quad x + y\left[\int \cos x^2 \,.\, 2xdx - \int 2x\,dx\right] = c$$

or $$x + y \sin x^2 - yx^2 = c.$$

Example 11.28. *Solve* $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$ *(Kurukshetra, 2005)*

Solution. Given equation can be written as

$$(y \cos x + \sin y + y)\,dx + (\sin x + x \cos y + x)\,dy = 0.$$

Here $M = y \cos x + \sin y + y$ and $N = \sin x + x \cos y + x.$

$$\therefore \quad \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y\text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

i.e., $$\int_{(y\text{ const.})} (y \cos x + \sin y + y)\,dx + \int (0)\,dx = c \quad \text{or} \quad y \sin x + (\sin y + y)\,x = c.$$

Example 11.29. *Solve* $(2x^2 + 3y^2 - 7)\, xdx - (3x^2 + 2y^2 - 8)\, ydy = 0.$ (*U.P.T.U., 2005*)

Solution. Given equation can be written as

$$\frac{ydy}{xdx} = \frac{2x^2 + 3y^2 - 7}{3x^2 + 2y^2 - 8}$$

or $$\frac{ydy + xdx}{ydy - xdx} = \frac{5(x^2 + y^2 - 3)}{-x^2 + y^2 + 1}$$ [By componendo & dividendo]

or $$\frac{xdx + ydy}{x^2 + y^2 - 3} = 5 \cdot \frac{xdx - ydy}{x^2 - y^2 - 1}$$

Integrating both sides, we get

$$\int \frac{2xdx + 2ydy}{x^2 + y^2 - 3} = 5 \int \frac{2xdx - 2ydy}{x^2 - y^2 - 1} + c$$

or $$\log (x^2 + y^2 - 3) = 5 \log (x^2 - y^2 - 1) + \log c'$$ [Writing $c = \log c'$]

or $$x^2 + y^2 - 3 = c' (x^2 - y^2 - 1)^5$$

which is the required solution.

PROBLEMS 11.7

Solve the following equations :

1. $(x^2 - ay)\, dx = (ax - y^2)\, dy.$
2. $(x^2 + y^2 - a^2)\, xdx + (x^2 - y^2 - b^2)\, ydy = 0$ (*Kurukshetra, 2005*)
3. $(x^2 - 4xy - 2y^2)\, dx + (y^2 - 4xy - 2x^2)\, dy = 0.$
4. $(x^4 - 2xy^2 + y^4)\, dx - (2x^2y - 4xy^3 + \sin y)\, dy = 0$
5. $ye^{xy} dx + (xe^{xy} + 2y)\, dy = 0$
6. $(5x^4 + 3x^2y^2 - 2xy^3)\, dx + (2x^3y - 3x^2y^2 - 5y^4)\, dy = 0$ (*V.T.U., 2008*)
7. $(3x^2 + 6xy^2)\, dx + (6x^2y + 4y^3)\, dy = 0$
8. $\dfrac{2x}{y^3}\, dx + \dfrac{y^2 - 3x^2}{y^4}\, dy = 0$
9. $y \sin 2x\, dx - (1 + y^2 + \cos^2 x)\, dy = 0$
10. $(\sec x \tan x \tan y - e^x)\, dx + \sec x \sec^2 y\, dy = 0$ (*Marathwada, 2008*)
11. $(2xy + y - \tan y)\, dx + x^2 - x \tan^2 y + \sec^2 y)\, dy = 0.$ (*Nagpur, 2009*)

11.12 EQUATIONS REDUCIBLE TO EXACT EQUATIONS

Sometimes a differential equation which is not exact, can be made so on multiplication by a suitable factor called an *integrating factor*. The rules for finding integrating factors of the equation $Mdx + Ndy = 0$ are as follows :

(1) I.F. found by inspection. In a number of cases, the integrating factor can be found after regrouping the terms of the equation and recognizing each group as being a part of an exact differential. In this connection the following integrable combinations prove quite useful :

$$xdy + ydx = d\,(xy)$$

$$\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right); \quad \frac{xdy - ydx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$$

$$\frac{xdy - ydx}{y^2} = -\,d\left(\frac{x}{y}\right); \quad \frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$$

$$\frac{xdy - ydx}{x^2 - y^2} = d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right).$$

Example 11.30. *Solve* $y\,(2xy + e^x)\, dx = e^x\, dy.$ (*Kurukshetra, 2005*)

Solution. It is easy to note that the terms $ye^x dx$ and $e^x dy$ should be put together.

$\therefore$ $$(ye^x dx - e^x dy) + 2xy^2\, dx = 0$$

Now we observe that the term $2xy^2\,dx$ should not involve y^2. This suggests that $1/y^2$ may be I.F. Multiplying throughout by $1/y^2$, it follows

$$\frac{ye^x\,dx - e^x\,dy}{y^2} + 2xdx = 0 \quad \text{or} \quad d\left(\frac{e^x}{y}\right) + 2xdx = 0$$

Integrating, we get $\dfrac{e^x}{y} + x^2 = c$ which is the required solution.

(2) I.F. of a homogeneous equation. *If $Mdx + Ndy = 0$ be a homogeneous equation in x and y, then $1/(Mx + Ny)$ is an integrating factor $(Mx + Ny \neq 0)$.*

Example 11.31. *Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$.* *(Osmania, 2003 S)*

Solution. This equation is homogeneous in x and y.

$$\therefore \quad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)\,x - (x^3 - 3x^2y)y} = \frac{1}{x^2y^2}$$

Multiplying throughout by $1/x^2y^2$, the equation becomes

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \text{ which is exact.}$$

$\therefore$ the solution is $\displaystyle\int_{(y\ \text{const})} Mdx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$ or $\dfrac{x}{y} - 2\log x + 3\log y = c$.

(3) I.F. for an equation of the type $f_1(xy)ydx + f_2(xy)xdy = 0$.

If the equation $Mdx + Ndy = 0$ be of this form, then $1/(Mx - Ny)$ is an integrating factor $(Mx - Ny \neq 0)$.

Example 11.32. *Solve $(1 + xy)\,ydx + (1 - xy)\,xdy = 0$.* *(S.V.T.U., 2008)*

Solution. The given equation is of the form $f_1(xy)\,ydx + f_2(xy)\,xdy = 0$

Here $\qquad M = (1 + xy)\,y,\ N = (1 - xy)\,x.$

$$\therefore \quad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{(1 + xy)yx - (1 - xy)xy} = \frac{1}{2x^2y^2}$$

Multiplying throughout by $1/2x^2y^2$, it becomes

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0, \text{ which is an exact equation.}$$

$$\therefore \text{ the solution is } \int_{(y\ \text{const})} Mdx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

or $$\frac{1}{2y}\left(-\frac{1}{x}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c \quad \text{or} \quad \log\frac{x}{y} - \frac{1}{xy} = c'.$$

(4) In the equation $Mdx + Ndy = 0$,

(a) if $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ be a function of x only $= f(x)$ say, then $e^{\int f(x)dx}$ is an integrating factor.

(b) if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ be a function of y only $= F(y)$ say, then $e^{\int F(y)dy}$ is an integrating factor.

Example 11.33. *Solve $(xy^2 - e^{1/x^3})\,dx - x^2y\,dy = 0$.* *(S.V.T.U., 2009 ; Mumbai, 2007)*

Solution. Here $M = xy^2 - e^{1/x^3}$ and $N = -x^2y$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x} \text{ which is a function of } x \text{ only.}$$

$\therefore$ $$\text{I.F.} = e^{\int \frac{-4}{x} dx} = e^{-4 \log x} = x^{-4}$$

Multiplying throughout by x^{-4}, we get $\left(\dfrac{y^2}{x^3} - \dfrac{1}{4^4} e^{1/x^3}\right) dx - \dfrac{y}{x^2}\, dy = 0$

which is an exact equation.

$\therefore$ the solution is $\int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c.$

or $$\int \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^3} \right) dx + 0 = c$$

or $$-\frac{y^2 x^{-2}}{2} + \frac{1}{3} \int e^{x^{-3}} (-3x^{-4})\, dx = c \text{ or } \frac{1}{3} e^{x^{-3}} - \frac{1}{2} \frac{y^2}{x^2} = c.$$

Otherwise it can be solved as a Bernoulli's equation (§ 11.10)

Example 11.34. *Solve* $(xy^3 + y)\, dx + 2\,(x^2y^2 + x + y^4)\, dy = 0.$

Solution. Here $M = xy^3 + y$, $N = 2(x^2y^2 + x + y^4)$

$\therefore$ $$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(xy^2+1)} (4xy^2 + 2 - 3xy^2 - 1) = \frac{1}{y},$$ which is a function of y alone.

$\therefore$ $$\text{I.F.} = e^{\int 1/y\, dy} = e^{\log y} = y$$

Multiplying throughout by y, it becomes $(xy^4 + y^2)\, dx + (2x^2y^3 + 2xy + 2y^5)\, dy = 0$, which is an exact equation.

$\therefore$ its solution is $\int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = 0$

or $$\int_{(y \text{ const})} (xy^4 + y^2) dx + \int 2y^5\, dy = c \quad \text{or} \quad \frac{1}{2} x^2y^4 + xy^2 + \frac{1}{3} y^6 = c.$$

Example 11.35. *Solve* $(y \log y)\, dx + (x - \log y)\, dy = 0$ (U.P.T.U., 2004)

Solution. Here $M = y \log y$ and $N = x - \log y$

$\therefore$ $$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y \log y} (1 - \log y - 1) = -\frac{1}{y},$$ which is a function of y alone.

$\therefore$ $$\text{I.F.} = e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying the given equation throughout by $1/y$, it becomes

$$\log y\, dx + \frac{1}{y}\, (x - \log y)\, dy = 0$$

which is an exact equation $\left[\because \dfrac{\partial}{\partial y} (\log y) = \dfrac{\partial}{\partial x}\left(\dfrac{x - \log y}{y}\right)\right.$

$\therefore$ its solution is $\int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c$

or $$\log y \int dx + \int \left(\frac{-\log y}{y}\right) dy = c \qquad \text{or} \quad x \log y - \frac{1}{2} (\log y)^2 = c.$$

(5) For the equation of the type

$$x^a y^b\, (mydx + nxdy) + x^{a'} y^{b'}\, (m'ydx + n'xdy) = 0,$$

an integrating factor is $x^h y^k$

where $$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}.$$

Example 11.36. *Solve* $y(xy + 2x^2y^3)\, dx + x\,(xy - x^2y^2)\, dy = 0.$ *(Hissar, 2005 ; Kurukshetra, 2005)*

Solution. Rewriting the equation as $xy\,(ydx + xdy) + x^2y^2\,(2ydx - xdy) = 0$ and comparing with

$$x^a y^b\,(mydx + nxdy) + x^{a'}y^{b'}\,(m'ydx + n'xdy) = 0,$$

we have $a = b = 1, m = n = 1; a' = b' = 2, m' = 2, n' = -1.$

$$\therefore \qquad \text{I.F.} = x^h y^k.$$

where $$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

i.e. $$\frac{1+h+1}{1} = \frac{1+k+1}{1}, \frac{2+h+1}{2} = \frac{2+k+1}{-1}$$

or $$h - k = 0,\ h + 2k + 9 = 0$$

Solving these, we get $h = k = -3$. $\therefore$ I.F. $= 1/x^3y^3$.

Multiplying throughout by $1/x^3y^3$, it becomes

$$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dx = 0, \text{ which is an exact equation.}$$

$\therefore$ The solution is $\displaystyle\int_{(y\ const)} Mdx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$

or $$\frac{1}{y}\left(-\frac{1}{x}\right) + 2\log x - \log y = c \quad \text{or} \quad 2\log x - \log y - 1/xy = c.$$

PROBLEMS 11.8

Solve the following equations :

1. $xdy - ydx + a(x^2 + y^2)\, dx = 0.$
2. $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}.$ *(U.P.T.U., 2005)*
3. $ydx - xdy + \log x\, dx = 0.$
4. $\dfrac{dy}{dx} = \dfrac{x^3 + y^3}{xy^2}.$
5. $(x^3y^2 + x)\, dy + (x^2y^3 - y)\, dx = 0.$
6. $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)\, xdy = 0.$
7. $(y^4 + 2y)\, dx + (xy^3 + 2y^4 - 4x)\, dy = 0.$
8. $(4xy + 3y^2 - x)\, dx + x\,(x + 2y)dy = 0$ *(Mumbai, 2006)*
9. $x^4 \dfrac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0.$
10. $(y - xy^2)\, dx - (x + x^2y)\, dy = 0$ *(Mumbai, 2006)*
11. $ydx - xdy + 3x^2y^2 e^{x^3}\, dx = 0.$ *(Kurukshetra, 2006)*
12. $(y^2 + 2x^2y)\, dx + (2x^3 - xy)dy = 0.$ *(Rajasthan, 2005)*
13. $2ydx + x(2\log x - y)\, dy = 0.$ *(P.T.U., 2005)*

11.13 EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

As dy/dx will occur in higher degrees, it is convenient to denote dy/dx by p. Such equations are of the form $f(x, y, p) = 0$. Three cases arise for discussion :

Case. I. Equation solvable for p. A differential equation of the first order but of the nth degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots + P_n = 0 \qquad \ldots(1)$$

where $P_1, P_2, \ldots, P_n$ are functions of x and y.

Splitting up the left hand side of (1) into n linear factors, we have

$$[p - f_1(x, y)]\,[p - f_2(x, y)] \ldots [p - f_n(x, y)] = 0.$$

Equating each of the factors to zero,

$$p = f_1(x, y), p = f_2(x, y), \ldots, p = f_n(x, y)$$

Solving each of these equations of the first order and first degree, we get the solutions

$$F_1(x, y, c) = 0, F_2(x, y, c) = 0, \ldots, F_n\,(x, y, c) = 0.$$

These n solutions constitute the general solution of (1).

Otherwise, the general solution of (1) may be written as

$$F_1(x, y, c) \,.\, F_2(x, y, c) \,.......\, F_n(x, y, c) = 0.$$

Example 11.37. *Solve* $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$.

Solution. Given equation is $p - \dfrac{1}{p} = \dfrac{x}{y} - \dfrac{y}{x}$ where $p = \dfrac{dy}{dx}$ or $p^2 + p\left(\dfrac{y}{x} - \dfrac{x}{y}\right) - 1 = 0.$

Factorising $(p + y/x)(p - x/y) = 0.$

Thus we have $\quad p + y/x = 0 \quad ...(i) \quad$ and $\quad p - x/y = 0 \quad ...(ii)$

From (i), $\quad \dfrac{dy}{dx} + \dfrac{y}{x} = 0$ or $xdy + ydx = 0$

i.e., $\quad d(xy) = 0$. Integrating, $xy = c$.

From (ii), $\quad \dfrac{dy}{dx} - \dfrac{x}{y} = 0$ or $xdx - ydx = 0$

Integrating, $x^2 - y^2 = c$. Thus $xy = c$ or $x^2 - y^2 = c$, constitute the required solution.

Otherwise, combining these into one, the required solution can be written as

$$(xy - c)(x^2 - y^2 - c) = 0.$$

Example 11.38. *Solve* $p^2 + 2py \cot x = y^2$. *(Bhopal, 2008 ; Kerala, 2005)*

Solution. We have $p^2 + 2py \cot x + (y \cot x)^2 = y^2 + y^2 \cot^2 x$

or $\quad p + y \cot x = \pm\, y \operatorname{cosec} x$

i.e., $\quad p = y\,(-\cot x + \operatorname{cosec} x) \quad ...(i)$

or $\quad p = y\,(-\cot x - \operatorname{cosec} x) \quad ...(ii)$

From (i), $\dfrac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$ or $\dfrac{dy}{y} = (\operatorname{cosec}\ x - \cot x)\, dx$

Integrating, $\log y = \log \tan \dfrac{x}{2} - \log \sin x + \log c = \log \dfrac{c \tan x/2}{\sin x}$

or $\quad y = \dfrac{c}{2 \cos x^2/2}$ or $y\,(1 + \cos x) = c \quad ...(iii)$

From (ii), $\quad \dfrac{dy}{dx} = -\,y\,(\cot x + \operatorname{cosec} x)$ or $\dfrac{dy}{y} = -\,(\cot x + \operatorname{cosec} x)\, dx$

Integrating, $\log y = -\log \sin x - \log \tan \dfrac{x}{2} + \log c = \log \dfrac{c}{\sin x \tan \dfrac{x}{2}}$

or $\quad y = \dfrac{c}{2 \sin^2 \dfrac{x}{2}}$ or $y(1 - \cos x) = c \quad ...(iv)$

Thus combining (iii) and (iv), the required general solution is

$$y\,(1 \pm \cos x) = c.$$

PROBLEMS 11.9

Solve the following equations :

1. $y\left(\dfrac{dy}{dx}\right)^2 + (x - y)\dfrac{dy}{dx} - x = 0.$
2. $p(p + y) = x\,(x + y)$. *(V.T.U., 2011)*
3. $y = x\,[p + \sqrt{(1 + p^2)}]$.
4. $xy\left(\dfrac{dy}{dx}\right)^2 - (x^2 + y^2)\dfrac{dy}{dx} + xy = 0.$
5. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0.$ *(Madras, 2003)*

Case II. Equations solvable for y. If the given equation, on solving for y, takes the form

$$y = f(x, p). \qquad \text{...(1)}$$

then differentiation with respect to x gives an equation of the form

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right).$$

Now it may be possible to solve this new differential equation in x and p.

Let its solution be $\quad F(x, p, c) = 0. \qquad$ ...(2)

The ellimination of p from (1) and (2) gives the required solution.

In case elimination of p is not possible, then we may solve (1) and (2) for x and y and obtain

$$x = F_1(p, c),\ y = F_2(p, c)$$

as the required solution, where p is the parameter.

Obs. This method is especially useful for equations which do not contain x.

Example 11.39. *Solve* $y - 2px = \tan^{-1}(xp^2)$.

Solution. Given equation is $y = 2px + \tan^{-1}(xp^2) \qquad$ ...(*i*)

Differentiating both sides with respect to x, $\dfrac{dy}{dx} = p = 2\left(p + x\dfrac{dp}{dx}\right) + \dfrac{p^2 + 2xp\dfrac{dp}{dx}}{1 + x^2p^4}$

or $\quad p + 2x\dfrac{dp}{dx} + \left(p + 2x\dfrac{dp}{dx}\right) \cdot \dfrac{p}{1 + x^2p^4} = 0$ or $\left(p + 2x\dfrac{dp}{dx}\right)\left(1 + \dfrac{p}{1 + x^2p^4}\right) = 0$

This gives $p + 2x\, dp/dx = 0$.

Separating the variables and integrating, we have $\displaystyle\int \frac{dx}{x} + 2\int \frac{dp}{p}$ = a constant

or $\quad \log x + 2\log p = \log c \quad$ or $\quad \log xp^2 = \log c$

whence $\quad xp^2 = c \quad$ or $\quad p = \sqrt{(c/x)} \qquad$ ...(*ii*)

Eliminating p from (*i*) and (*ii*), we get $y = 2\sqrt{(c/x)}\,x + \tan^{-1} c$

or $\quad y = 2\sqrt{(cx)} + \tan^{-1} c$ which is the general solution of (*i*).

Obs. The significance of the factor $1 + p/(1 + x^2p^4) = 0$ which we didn't consider, will not be considered here as it concerns 'singular solution' of (*i*) whereas *we are interested only in finding general solution.*

Caution. Sometimes one is tempted to write (*ii*) as

$$\frac{dy}{dx} = \sqrt{\left(\frac{c}{x}\right)}$$

and integrating it to say that the required solution is $y = 2\sqrt{(cx)} + c'$. Such a reasoning is *incorrect*.

Example 11.40. *Solve* $y = 2px + p^n$. (*Bhopal, 2009*)

Solution. Given equation is $\quad y = 2px + p^n \qquad$ ...(*i*)

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \quad \text{or} \quad p\frac{dx}{dp} + 2x = -np^{n-1}$$

or $$\frac{dx}{dp} + \frac{2x}{p} = -np^{n-2} \qquad \text{...(ii)}$$

This is Leibnitz's linear equation in x and p. Here I.F. $= e^{\int \frac{2}{p} dp} = e^{\log p^2} = p^2$

$\therefore$ the solution of (*ii*) is

$$x(\text{I.F.}) = \int(-np^{n-2}) \cdot (\text{I.F.})\, dp + c \quad \text{or} \quad xp^2 = -n\int p^n\, dp + c = -\frac{np^{n+1}}{n+1} + c$$

or $$x = cp^{-2} - \frac{np^{n-1}}{n+1} \qquad ...(iii)$$

Substituting this value of x in (*i*), we get $y = \dfrac{2c}{p} + \dfrac{1-n}{1+n}p^n$ ...(*iv*)

The equations (*iii*) and (*iv*) taken together, with parameter p, constitute the general solution (*i*).

Obs. In general, the equations of the form $y = xf(p) + \phi(p)$, known as *Lagrange's* equation, are solvable for y and lead to Leibnitz's equation in dx/dp.

PROBLEMS 11.10

Solve the following equations :

1. $y = x + a\tan^{-1} p$.
2. $y + px = x^4p^2$. (*S.V.T.U., 2007*)
3. $x^2\left(\dfrac{dy}{dx}\right)^4 + 2x\dfrac{dy}{dx} - y = 0$.
4. $xp^2 + x = 2yp$.
5. $y = xp^2 + p$.
6. $y = p\sin p + \cos p$.

Case III. **Equations solvable for x.** If the given equation on solving for x, takes the form

$$x = f(y, p) \qquad ...(1)$$

then differentiation with respect to y gives an equation of the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

Now it may be possible to solve the new differential equation in y and p. Let its solution be $F(y, p, c) = 0$.

The elimination of p from (1) and (2) gives the required solution. In case the elimination is not feasible, (1) and (2) may be expressed in terms of p and p may be regarded as a parameter.

Obs. This method is especially useful for equations which do not contain y.

Example 11.41. *Solve* $y = 2px + y^2p^3$. (*Bhopal, 2008*)

Solution. Given equation, on solving for x, takes the form $x = \dfrac{y - y^2p^3}{2p}$

Differentiating with respect to y, $\dfrac{dx}{dy}\left(=\dfrac{1}{p}\right) = \dfrac{1}{2} \cdot \dfrac{p\left(1 - 2y \cdot p^3 - y^2 3p^2\dfrac{dp}{dy}\right) - (y - y^2p^3)\dfrac{dp}{dy}}{p^2}$

or $$2p = p - 2yp^4 - 3y^2p^3\frac{dp}{dy} - y\frac{dp}{dy} + y^2p^3\frac{dp}{dy}$$

or $$p + 2yp^4 + 2y^2p^3\frac{dp}{dy} + y\frac{dp}{dy} = 0 \text{ or } p(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0.$$

or $$\left(p + y\frac{dp}{dy}\right)(1 + 2py^3) = 0 \text{ This gives } p + y\frac{dp}{dy} = 0. \text{ or } \frac{d}{dy}(py) = 0.$$

Integrating $py = c$. ...(*i*)

Thus eliminating from the given equation and (*i*), we get $y = 2\dfrac{c}{y}x + \dfrac{c^3}{y^3}y^2$ or $y^2 = 2cx + c^3$

which is the required solution.

PROBLEMS 11.11

Solve the following equations :

1. $p^3 - 4xyp + 8y^2 = 0$. (*Kanpur, 1996*)
2. $p^3y + 2px = y$.
3. $x - yp = ap^2$. (*Andhra, 2000*)
4. $p = \tan\left(x - \frac{p}{1+p^2}\right)$. (*S.V.T.U., 2008*)

11.14 CLAIRAUT'S EQUATION*

An equation of the form $y = px + f(p)$ is known as Clairaut's equation ...(1)

Differentiating with respect to x, we have $p = p + x\,\frac{dp}{dx} + f'(p)\,\frac{dp}{dx}$

or $$[x + f'(p)]\,\frac{dp}{dx} = 0 \qquad \therefore\ \frac{dp}{dx} = 0, \text{ or } x + f'(p) = 0$$

$$\frac{dp}{dx} = 0, \text{ gives } p = c \qquad ...(2)$$

Thus eliminating p from (1) and (2), we get $y = cx + f(c)$ ...(3)

as the general solution of (1).

Hence the solution of the Clairaut's equation is obtained on replacing p *by* c.

Obs. If we eliminate p from $x + f'(p) = 0$ and (1), we get an equation involving no constant. *This is the* **singular solution** of (1) *which gives the envelope of the family of straight lines* (3).

To obtain the singular solution, we proceed as follows :

(*i*) Find the general solution by replacing p by c *i.e.*, (3)

(*ii*) Differentiate this w.r.t. c giving $x + f'(c) = 0$. ...(4)

(*iii*) Eliminate c from (3) and (4) which will be the *singular solution*.

Example 11.42. *Solve $p = \sin(y - xp)$. Also find its singular solutions.*

Solution. Given equation can be written as

$$\sin^{-1} p = y - xp \text{ or } y = px + \sin^{-1} p \text{ which is the } \textit{Clairaut's equation.}$$

$\therefore$ its solution is $y = cx + \sin^{-1} c$.

To find the singular solution, differentiate (*i*) w.r.t. c giving

$$0 = x + \frac{1}{\sqrt{1-c^2}} \qquad ...(ii)$$

To eliminate c from (*i*) and (*ii*), we rewrite (*ii*) as

$$c = N(x^2 - 1)/x$$

Now substituting this value of c in (*i*), we get

$$y = N(x^2 - 1) + \sin^{-1}\{N(x^2 - 1)/x\}$$

which is the desired singular solution.

Obs. Equations reducible to Clairaut's form. Many equations of the first order but of higher degree can be easily reduced to the Clairaut's form by making suitable substitutions.

Example 11.43. *Solve $(px - y)(py + x) = a^2p$.* (*V.T.U., 2011 ; J.N.T.U., 2006*)

Solution. Put $x^2 = u$ and $y^2 = v$ so that $2xdx = du$ and $2ydy = dv$

$$\therefore \qquad p = \frac{dy}{dx} = \frac{dv}{y}\Big/\frac{du}{x} = \frac{x}{y}P, \text{ where } P = \frac{dv}{du}$$

*After the name of a youthful prodigy *Alexis Claude Clairaut* (1713–65) who first solved this equation. A French mathematician who is also known for his work in astronomy and geodesy.

Then the given equation becomes $\left(\frac{xP}{y}.x-y\right)\left(\frac{xP}{y}.y+x\right)=a^2\frac{xP}{y}$

or $(uP-v)(P+1)=a^2P$ or $uP-v=\frac{a^2P}{P+1}$

or $v=uP-a^2P/(P+1)$, which is Clairaut's form.

$\therefore$ its solution is $v=uc-a^2c/(c+1)$, *i.e.*, $y^2=cx^2-a^2c/(c+1)$.

PROBLEMS 11.12

1. Find the general and singular solution of the equations :
   (*i*) $xp^2-yp+a=0$. (*J.N.T.U., 2006*) (*ii*) $p=\log(px-y)$.
   (*iii*) $y=px+\sqrt{(a^2p^2+b^2)}$ (*W.B.T.U., 2005*) (*iv*) $\sin px\cos y=\cos px\sin y+p$ (*P.T.U., 2006*)

Solve the following equations :

2. $y+2\left(\frac{dy}{dx}\right)^2=(x+1)\frac{dy}{dx}$.
3. $(y-px)(p-1)=p$.
4. $(x-a)\left(\frac{dy}{dx}\right)^2+(x-y)\frac{dy}{dx}-y=0$.
5. $x^2(y-px)=yp^2$.
6. $(px+y)^2=py^2$.
7. $(px-y)(x+py)=2p$.

11.15 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 11.13

Fill up the blanks or choose the correct answer in the following problems :

1. $y=cx-c^2$, is the general solution of the differential equation
   (*i*) $(y')^2-xy'+y=0$ (*ii*) $y''=0$ (*iii*) $y'=c$ (*iv*) $(y')^2+xy'+y=0$.
2. The differential equation having a basis for its solution as sinh $6x$ and cosh $6x$ is
   (*i*) $y''+36y=0$ (*ii*) $y''-36y=0$ (*iii*) $y''+6y=0$ (*iv*) none of these.
3. The differential equation $(dx/dy)^2+5y^{1/3}=x$ is
   (*i*) linear of degree 3 (*ii*) non-linear of order 1 and degree 6
   (*iii*) non-linear of order 1 and degree 2.
4. The differential equation $y\,dx/dy+1=y$, $y(0)=1$, has
   (*i*) a unique solution (*ii*) two solutions
   (*iii*) infinite number of solutions (*iv*) no solution
5. Solution of $(x^2+y^2)\,dy=xy\,dx$ is ....
6. Solution of $(3x-2y)\,dx=x\,dy$ is ....
7. Solution of $dy/dx-y=2xy^2e^{-x}$ is ....
8. The differential equation $(y^2e^{xy^2}+6x)\,dx+(2xye^{xy^2}-4y)\,dy=0$ is
   (*i*) linear, homogeneous and exact (*ii*) non-linear, homogeneous and exact
   (*iv*) non-linear, non-homogeneous and exact (*iv*) non-linear, non-homogeneous and inexact.
9. Solution of $x\,dx+y\,dy+\frac{x\,dy-y\,dx}{x^2+y^2}$ is ....
10. Solution of $dy/dx=\frac{x^3+y^3}{xy^2}$ is ......
11. The differential equation $(x+x^8+ay^2)\,dx+(y^8-y+bxy)\,dy=0$ is exact if
    (*i*) $b=2a$ (*ii*) $a=b$ (*iii*) $a\neq 2b$ (*iv*) $a=1, b=3$.
12. Solution of $xy\,(1+xy^2)\,dy=dx$ is ....
13. Solution of $xp^2-yp+a=0$ is ....
14. The differential equation $p=\log(px-y)$ has the solution ....
15. Solution of $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$ is ....

16. The order of the differential equation $(1 + y_1^2)^{3/2}/y_2 = c$ is ......

17. The general solution of $\frac{1}{x^2y^2}(xdy + ydx) = 0$ is ....

18. Integrating factor of the differential equation $\frac{dx}{dy} + \frac{3x}{y} = \frac{1}{y^2}$ is

(a) e^{y^3} (b) y^3 (c) x^3 (d) $-y^3$. (V.T.U., 2009)

19. Solution of the equation $\frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\frac{y}{x}$ is

(a) $\cos(y/x) - \log x = c$ (b) $\cos(y/x) + \log x = c$
(c) $\cos^2(y/x) + \log x = c$ (d) $\cos^2(y/x) - \log x = c$. (V.T.U., 2010)

20. Solution of $x\sqrt{(1+x^2)} + y\sqrt{(1+y^2)}\, dy/dx = 0$ is ....

21. Solution of $dy/dx + y = 0$ given $y(0) = 5$ is ....

22. The substitution that transforms the equation $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$ to homogeneous form is ....

23. Integrating factor of $xy' + y = x^3y^6$ is ....

24. Solution of the exact differential equation $Mdx + Ndy = 0$ is ....

25. Solution of $(2x^3y^2 + x^4)\, dx + (x^4y + y^4)\, dy = 0$ is ....

26. The general solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = \tan 2x$ is ....

27. Degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^5 x^2y = 0$ is

(a) 2 (b) 0 (c) 3 (d) 5 (Bhopal, 2008)

28. Integrating factor of the differential equation $\frac{dy}{dx} + y\cos x = \frac{\sin 2x}{2}$ is

(a) $e^{\sin^2 x}$ (b) $e^{\sin^3 x}$ (c) $e^{\sin x}$ (d) $\sin x$

29. The differential equation of the family of circles with centre as origin is ...... (Nagarjuna, 2008)

30. Solution of $xe^{-x^2}\, dx + \sin y\, dy = 0$ is .... (Nagarjuna, 2008)

31. Solution of $p = \sin(y - xp)$ is

(a) $y = \frac{c}{x} + \sin^{-1} c$ (b) $y = cx + \sin c$ (c) $y = cx + \sin^{-1} c$ (d) $y = x + \sin^{-1} c$ (V.T.U., 2011)

32. Differential equation obtained by eliminating A and B form $y = A\cos x + B\sin x$ is $d^2y/dx^2 - y = 0$ (True or False)

33. $(x^3 - 3xy^2)\, dx + (y^3 - 2x^2y)\, dy = 0$ is an exact differential equation. (True or False)