CHAPTER 14

Applications of Linear Differential Equations

1. Introduction. 2. Simple harmonic motion. 3. Simple Pendulum, Gain and Loss of Oscillations. 4. Oscillations of a spring. 5. Oscillatory electrical circuits. 6. Electro-mechanical analogy. 7. Deflection of Beams. 8. Whirling of Shafts. 9. Applications of simultaneous linear equations. 10. Objective Type of Questions.

14.1 INTRODUCTION

The linear differential equations with constant coefficients find their most important applications in the study of electrical, mechanical and other linear systems. In fact such equations play a dominant role in unifying the theory of electrical and mechanical oscillatory systems.

We shall begin by explaining the types of oscillations of the mechanical systems and the equivalent electrical circuits. Then we shall study at some length the slightly less striking applications such as deflection of beams and whirling of shafts. At the end, we'll take up some of the applications of simultaneous linear differential equations.

14.2 SIMPLE HARMONIC MOTION

When the acceleration of a particle is proportional to its displacement from a fixed point and is always directed towards it, then the motion is said to be *simple harmonic*.

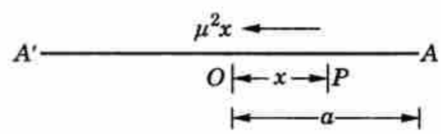

Fig. 14.1

If the displacement of the particle at any time t, from fixed point O is x (Fig. 14.1), then

$$\frac{d^2x}{dt^2} = -\mu^2 x \quad \text{or} \quad (D^2 + \mu^2)x = 0, \qquad ...(1)$$

$\therefore$ its solution is
$$x = c_1 \cos \mu t + c_2 \sin \mu t \qquad ...(2)$$

$\therefore$ its velocity at
$$P = \frac{dx}{dt} = \mu(-c_1 \sin \mu t + c_2 \cos \mu t) \qquad ...(3)$$

If the particle starts from rest at A, where $OA = a$,

then from (2), (at $t = 0, x = a$) $\quad a = c_1$

and from (3), (at $t = 0, dx/dt = 0$) $\quad 0 = c_2$.

Thus
$$x = a \cos \mu t \qquad ...(4)$$

and
$$\frac{dx}{dt} = -a\mu \sin \mu t = -\sqrt{(a^2 - x^2)} \qquad ...(5)$$

which give the displacement and the velocity of the particle at any time t.

Nature of motion. The particle starts from A towards O under acceleration directed towards O which gradually decreases until it vanishes at O, when the particle has acquired the maximum velocity. On passing through O, retardation begins and the particle comes to an instantaneous rest at A', where $OA' = OA$. It then retraces its path and goes on oscillating between A and A'.

The **amplitude** *or maximum displacement from the centre is a.*

The **periodic time,** *i.e., the time of complete oscillation is* $2\pi/\mu$, for when t is increased by $2\pi/\mu$, the values of x and dx/dt remain unaltered.

The **frequency** *or the number of oscillations per second is*

$$1/\text{periodic time, i.e., } \mu/2\pi.$$

Example 14.1. *In the case of a stretched elastic horizontal string which has one end fixed and a particle of mass m attached to the other, find the equation of motion of the particle given that l is the natural length of the string and e is its elongation due to weight mg. Also find the displacement s of particle when initially $s = 0$, $v = 0$.*

Solution. Let $OA(= l)$ be the elastic horizonta string with the end O fixed and having a particle of mass m attached to the end A. (Fig. 14.2)

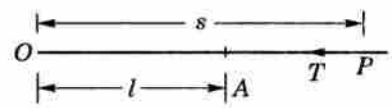

Fig. 14.2

At any time t, let the particle be at P where $OP = s$; so that the elongation $AP = s - l$.

Since for the elongation e, tension $= mg$

$\therefore$ for the elongation $s - l$, tension $T = \dfrac{mg(s-l)}{e}$

Tension being the only horizontal force, the equation of motion is

$$m\frac{d^2s}{dt^2} = -T \quad \text{or} \quad \frac{d^2s}{dt^2} = -\frac{T}{m} = -\frac{g(s-l)}{e} \qquad ...(i)$$

which is the required equation of motion.

Now (i) can be written as $(D^2 + g/e)s = gl/e$, where $D = d/dt$ ...(ii)

$\therefore$ the auxiliary equation is $D^2 + g/e = 0$ or $D = \pm i\sqrt{(g/e)}$

$$\therefore \qquad \text{C.F.} = c_1 \cos \sqrt{(g/e)}t + c_2 \sin \sqrt{(g/e)}t$$

and
$$\text{P.I.} = \frac{1}{D^2 + g/e} \cdot \frac{gl}{e} = \frac{gl}{e} \cdot \frac{l}{D^2 + g/e} e^{0t} = l$$

Thus the solution of (ii) is

$$s = c_1 \cos \sqrt{(g/e)}t + c_2 \sin \sqrt{(g/e)}t + l \qquad ...(iii)$$

When $t = 0$, $s = s_0$, $\quad \therefore \quad s_0 = c_1 + 0 + l \quad$ *i.e.,* $c_1 = s_0 - l$

Again from (iii), $\quad \dfrac{ds}{dt} = -c_1\sqrt{(g/e)} \sin \sqrt{(g/e)}t + c_2\sqrt{(g/e)} \cos \sqrt{(g/e)}t$

When $t = 0$, $ds/dt = 0$; $\quad \therefore \quad 0 = c_2$.

Substituting the values of c_1 and c_2 in (iii), we have

$$s = (s_0 - l) \cos \sqrt{(g/e)}t + l \text{ which is the required result.}$$

Example 14.2. *Two particles of masses m_1 and m_2 are tied to the ends of an elastic string of natural length a and modulus λ. They are placed on a smooth table so that the string is just taut and m_2 is projected with any velocity directly away from m_1. Show that the string will become slack after the lapse of time* $\pi\sqrt{[am_1m_2/\lambda(m_1 + m_2)]}$.

Solution. Taking O as fixed point of reference, let particle m_1 be at O and m_2 at a distance a from m_1 at time $t = 0$ Fig. 14.3. At any time t, let m_1 be of a distance x from O and m_2 be at a distance y from O. Then the equation of motion of m_1 is

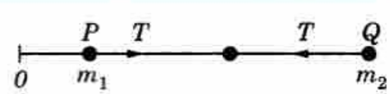

Fig. 14.3

$$m_1 d^2x/dt^2 = T \qquad ...(i)$$

and equation of motion of m_2 is $m_2 d^2y/dt^2 = -T$ ...(ii)

where $T = \lambda(y - x)/a$

From (*i*) and (*ii*) $d^2y/dt^2 - d^2x/dt^2 = -\dfrac{T}{m_2} - \dfrac{T}{m_1}$

or
$$\frac{d^2(y-x)}{dt^2} = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{\lambda(y-x)}{a} \quad \text{or} \quad \frac{d^2u}{dt^2} = -\frac{\lambda(m_1+m_2)u}{m_1 m_2 a} \text{ where } u = y - x$$

This is S.H.M. with periodic time $\tau = 2\pi\sqrt{\left\{\dfrac{a m_1 m_2}{\lambda(m_1+m_2)}\right\}}$

The string will acquire its original length (*i.e.* become slack) after time τ_1 of m_2 moving towards m_1 such that

$$\tau_1 = \frac{\tau}{4} + \frac{\tau}{4} = \frac{\tau}{2} = \pi\sqrt{\left\{\frac{a m_1 m_2}{\lambda(m_1+m_2)}\right\}}.$$

Example 14.3. *A particle of mass m executes S.H.M. in the line joining the points A and B, on a smooth table and is connected with these points by elastic strings whose tensions in equilibrium are each T. If l, l′ be the extensions of the strings beyond their natural lengths, find the time of an oscillation.*

Solution. In the equilibrium position, let the particle be at C so that $AC = a + l$ and $BC = a' + l'$, where a, a' are natural lengths of the strings (Fig. 14.4). Then the tensions (at this time) are given by

$$T = \lambda l/a = \lambda' l'/a' \quad \text{...(i)}$$

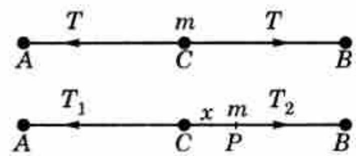

Fig. 14.4

At any time t, let the particle be at P, so that $CP = x$. Then

$$T_1 = \lambda\frac{l+x}{a} \text{ and } T_2 = \lambda'\frac{l'-x}{a'}$$

$\therefore$ the equation of motion is $m\dfrac{d^2x}{dt^2} = T_2 - T_1 = \lambda'\dfrac{l'-x}{a'} - \lambda\dfrac{l+x}{a}$

$$= \left(\frac{\lambda' l'}{a'} - \frac{\lambda l}{a}\right) - \left(\frac{\lambda'}{a'} + \frac{\lambda}{a}\right)x = -\left(\frac{T}{l'} + \frac{T}{l}\right)x \quad \text{[By (i)]}$$

or
$$\frac{d^2x}{dt^2} = -\mu x \quad \text{where } \mu = \frac{l+l'}{ll'}\cdot\frac{T}{m}$$

Hence the periodic time $= \dfrac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{\left\{\dfrac{mll'}{(l+l')T}\right\}}$.

14.3 (1) SIMPLE PENDULUM

A heavy particle attached by a light string to a fixed point and oscillating under gravity constitutes a *simple pendulum*.

Let O be the fixed point, l be the length of the string and A be the position of the bob initially (Fig. 14.5). If P be the position of the bob at any time t, such that arc $AP = s$ and $\angle AOP = \theta$, then $s = l\theta$.

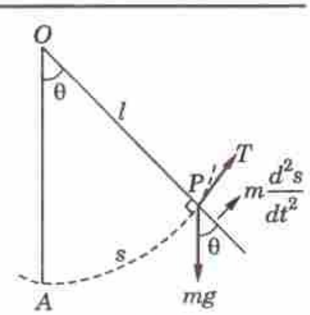

Fig. 14.5

$\therefore$ the equation of motion along PT is $m\dfrac{d^2s}{dt^2} = -mg\sin\theta$

i.e.,
$$\frac{d^2(l\theta)}{dt^2} = -g\sin\theta$$

or
$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta = -\frac{g}{l}\left(\theta - \frac{\theta^3}{3!} + \ldots\right) = -\frac{g\theta}{l} \text{ to a first approx.}$$

Here the auxiliary equation being $D^2 + g/l = 0$, we have $D = \pm\sqrt{(g/l)}\,i$

$\therefore$ its solution is $\theta = c_1\cos\sqrt{(g/l)}\,t + c_2\sin t$.

Thus the motion of the bob is simple harmonic and the time of an oscillation is $2\pi\sqrt{(l/g)}$.

Obs. The movement of the bob from one end to the other constitutes half an oscillation and is called a *beat* or *a swing*.

Thus the time of one beats = $\pi\sqrt{(l/g)}$.

A seconds pendulum beats 86400 times a day for there are 86,400 seconds in 24 hours.

(2) Gain or loss of oscillations. Let a pendulum of length l make n beats in time T, so that

$$T = \text{time of } n \text{ beats} = n\pi\sqrt{(l/g)} \quad \text{or} \quad n = \frac{T}{\pi}(g/l)^{1/2}$$

Taking logs, $\log n = \log (T/\pi) + \frac{1}{2}(\log g - \log l)$.

Taking differentials of both sides, we get $\frac{dn}{n} = \frac{1}{2}\left(\frac{dg}{g} - \frac{dl}{l}\right)$.

If only g changes, l remaining constant, $\frac{\mathbf{dn}}{\mathbf{n}} = \frac{\mathbf{dg}}{\mathbf{2g}}$ ...(1)

If only l changes, g remaining constant, $\frac{\mathbf{dn}}{\mathbf{n}} = -\frac{\mathbf{dl}}{\mathbf{2l}}$. ...(2)

Example 14.4. *Find how many seconds a clock would lose per day if the length of its pendulum were increased in the ratio 900 : 901.*

Solution. If the original length l of the string be increased to $l + dl$, then

$$\frac{l + dl}{dl} = \frac{901}{900}. \quad \therefore \quad \frac{dl}{l} = \frac{901}{900} - 1 = \frac{1}{900}.$$

$\therefore$ using (2) above, we have $\frac{dn}{n} = -\frac{dl}{2l} = -\frac{1}{1800}$

i.e.,
$$dn = -\frac{n}{1800} = -\frac{86400}{1800} = -48.$$

Since dn is negative, the clock will lose 4 seconds per day.

Example 14.5. *A simple pendulum of length l is oscillating through a small angle θ in a medium in which the resistance is proportional to the velocity. Find the differential equation of its motion. Discuss the motion and find the period of oscillation.*

Solution. Let the position of the bob (of mass m), at any time t be P and O be the point of suspension such that $OP = l$, $\angle AOP = \theta$ and therefore, arc $AP = s = l\theta$. (Fig. 14.6)

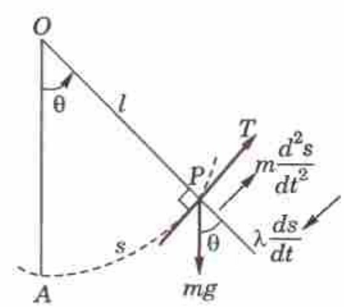

Fig. 14.6

$\therefore$ the equation of motion along the tangent PT is

$$m\frac{d^2s}{dt^2} = -mg \sin\theta - \lambda\frac{ds}{dt} \text{ where } \lambda \text{ is a constant.}$$

or
$$\frac{d^2(l\theta)}{dt^2} + \frac{\lambda}{m}\frac{d(l\theta)}{dt} + g\sin\theta = 0$$

Replacing $\sin\theta$ by θ since it is small and writing $\lambda/m = 2k$, we get

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \frac{g\theta}{l} = 0 \qquad ...(i)$$

which is the required differential equation.

Its auxiliary equation has roots $D = k \pm \sqrt{(k^2 - w^2)}$ where $w = g/l$.

The oscillatory motion of the bob is only possible when $k < w$.

Then the roots of the auxiliary equation are $-k \pm i\sqrt{(w^2 - k^2)}$.

$\therefore$ the solution of (i) is

$$\theta = e^{-kt}$$

which gives a vibratory motion of period $2\pi/\sqrt{(w^2 - k^2)}$.

Example 14.6. *A pendulum of length l has one end of the string fastened to a peg on a smooth plane inclined to the horizon at an angle α. With the string and the weight on the plane, its time of oscillation is t sec. If the pendulum of length l' oscillates in one sec. when suspended vertically, prove that $\alpha = \sin^{-1}\left(\frac{l}{l't^2}\right)$.*

(Kurukshetra, 2006)

Solution. At any time t, let the bob of mass m be at P and O be the point of suspension so that $OP = l$ and $\angle AOP = \theta$ (Fig. 14.7).

The component of weight along the plane being $mg \sin \alpha$, the equation of motion of the bob along the tangent at P is

$$m\frac{d^2s}{dt^2} = -mg \sin \alpha \sin \theta$$

or $$\frac{d^2(l\theta)}{dt^2} = -g \sin \alpha \sin \theta \qquad [\because \; s = l\theta]$$

or $$\frac{d^2\theta}{dt^2} = -g \sin \alpha\left(\theta - \frac{\theta^3}{3!} + \ldots\right)$$

or $$\frac{d^2\theta}{dt^2} = -\mu\theta \quad \text{where } \mu = \frac{g \sin \alpha}{l}, \text{ to a first approximation.}$$

$\therefore$ the motion being simple harmonic, the time of oscillation t.

$$= \frac{2\pi}{\mu} = 2\pi\sqrt{\frac{l}{g \sin \alpha}} \qquad ...(i)$$

We know that for a pendulum of length l' when suspended vertically, the time of oscillation

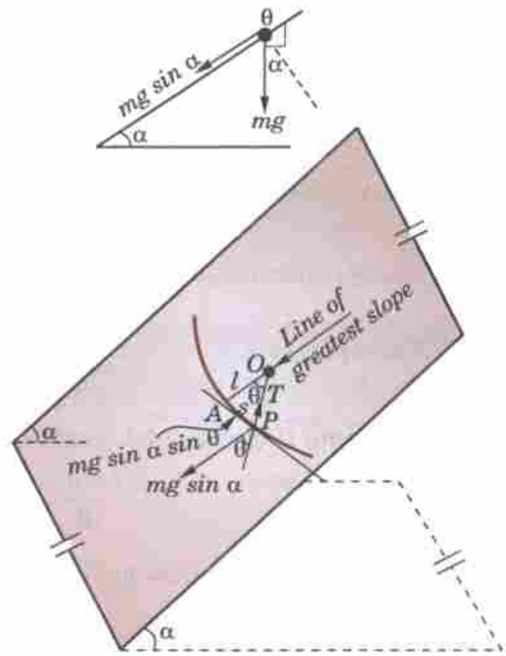

Fig. 14.7

$$1 = 2\pi\sqrt{l'/g} \qquad ...(ii)$$

$\therefore$ dividing (i) by (ii), we have $t = \sqrt{\left(\frac{l}{l' \sin \alpha}\right)}$

or $$t^2 = l/l' \sin \alpha \quad \text{or} \quad \alpha = \sin^{-1}(l/l't^2).$$

PROBLEMS 14.1

1. A particle is executive simple harmonic motion with amplitude 20 cm and time 4 seconds. Find the time required by the particle in passing between points which are at distances 15 cm and 5 cm from the centre of force and are on the same side of it.
2. At the ends of three successive seconds, the distances of a point moving with S.H.M. from its mean position are x_1, x_2, x_3. Show that the time of a complete oscillation is
$$2\pi / \cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right).$$
3. An elastic string of natural length $2a$ and modulus λ is stretched between two points A and B distant $4a$ apart on a smooth horizontal table. A particle of mass m is attached to the middle of the string. Show that it can vibrate in line AB with period $2\pi/\omega$, where $\omega^2 = 2\lambda/am$.
4. A particle of mass m moves in a straight line under the action of force $mn^2(OP)$, which is always directed towards fixed point O in the line. If the resistance to the motion is $2\lambda\, mnv$, where v is the speed and $0 < \lambda < 1$, find the displacement x in terms of the time t given that when $t = 0$, $x = 0$ and $dx/dt = u$ where $OP = x$.
5. A point moves in a straight line towards the centre of force μ/(distance^3) starting from rest at a distance a from the centre of force, show that the time of reaching a point b from the centre of force is $a\sqrt{(a^2 - b^2)}/\sqrt{\mu}$ and that its velocity then is $\frac{\sqrt{\mu}}{ab}\sqrt{(a^2 - b^2)}$. *(U.P.T.U., 2001)*

6. A clock loses five seconds a day, find the alteration required in the length of its pendulum in order that it may keep correct time.
7. A clock with a seconds pendulum loses 10 seconds per day at a place where $g = 32$ ft/sec^2. What change in the gravity is necessary to make it accurate ?
8. A seconds pendulum which gains 10 seconds per day at one place loses 10 seconds per day at another; compare the acceleration due to gravity at the two places. *(Kurukshetra, 2005)*
9. Show that the free oscillations of a galvanometer needle, as affected by the viscosity of the surrounding air which varies directly as the angular velocity of the needle, are determined by the equation $\dfrac{d^2\theta}{dt^2} + K\dfrac{d\theta}{dt} + \mu\theta = 0$, where k is the co-efficient of viscosity and θ is the angular deflection of the needle at time t. Obtain θ in terms of t and discuss the different cases that can arise.
10. If $I = \dfrac{d^2\theta}{dt^2} = -\,mgl \sin\theta$, where I, m, g, l are constant, given that at $t = 0$, $\theta = 0$ and $d\theta/dt = \omega_0 = m\sqrt{(mgl)}\,/\,I$, then show that $t = \dfrac{2}{\omega_0}\log\dfrac{\pi+\theta}{4}$. *(Nagpur, 2009)*

14.4 OSCILLATIONS OF A SPRING

(*i*) **Free oscillations.** Suppose a mass m is suspended from the end A of a light spring, the other end of which is fixed at O. (Fig. 14.8)

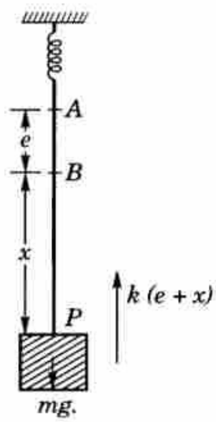

Fig. 14.8

Let e ($= AB$) be the elongation produced by the mass m hanging in equilibrium. If k be the restoring force per unit stretch of the spring due to elasticity, then for the equilibrium at B,

$$mg = T = ke \qquad \text{...(1)}$$

At any time t, after the motion ensues, let the mass be at P, where $BP = x$. Then the equation of motion of m is

$$m\frac{d^2x}{dt^2} = mg - k(e + x) = -\,kx \qquad \text{[By (1)]}$$

Or writing $k/m = \mu^2$, it becomes

$$\frac{d^2x}{dt^2} + \mu^2 x = 0 \qquad \text{...(2)}$$

This equation represents the free vibrations of the spring which are of the simple harmonic form having centre of oscillation at B—its equilibrium position and the *period of oscillation*

$$= \frac{2\pi}{\mu} = 2\pi\sqrt{\left(\frac{e}{g}\right)}. \qquad \left[\because\ \frac{1}{\mu} = \sqrt{\left(\frac{m}{k}\right)} = \sqrt{\left(\frac{e}{g}\right)}, \text{[By (1)]}\right]$$

(*ii*) **Damped oscillations.** If the mass m be subjected to do damping force proportional to velocity (say : $r\,dx/dt$) (Fig. 14.9), then the equation of motion becomes

$$m\frac{d^2x}{dt^2} = mg - k(e + x) - r\frac{dx}{dt}$$

$$= -\,kx - r\frac{dx}{dt} \qquad \text{[By (1)]}$$

Or writing $r/m = 2\lambda$ and $k/m = \mu^2$, it becomes

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \mu^2 x = 0 \qquad \text{...(3)}$$

$\therefore$ its auxiliary equation is

$$D^2 + 2\lambda D + \mu^2 = 0 \quad \text{whence} \quad D = -\,\lambda \pm .$$

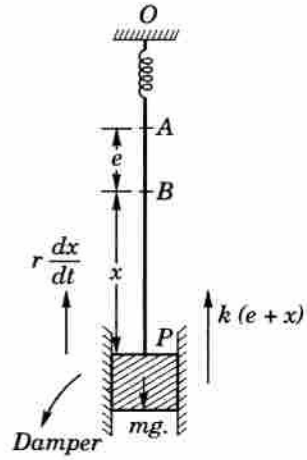

Fig. 14.9

Case I. *When* $\lambda > \mu$, the roots of the auxiliary equation are real and distinct (say γ_1, γ_2).

$\therefore$ the solution of (3) is $x = c_1 e^{\gamma_1 t} + c_2 e^{\gamma_2 t}$ ...(4)

To determine c_1, c_2 let the spring be stretched to a length $x = l$ and then released so that

$x = l$ and $dx/dt = 0$ at $t = 0$.

$\therefore$ from (4), $l = c_1 + c_2$

Also from $\dfrac{dx}{dt} = c_1\gamma_1 e^{\gamma_1 t} + c_2\gamma_2 e^{\gamma_2 t}$, we get

$0 = c_1\gamma_1 + c_2\gamma_2$

whence $c_1 = \dfrac{-l\gamma_2}{\gamma_1 - \gamma_2}$ and $c_2 = \dfrac{l\gamma_1}{\gamma_1 - \gamma_2}$

Hence the solution of (3) is

$$x = \frac{l}{\gamma_1 - \gamma_2}(\gamma_1 e^{\gamma_2 t} - \gamma_2 e^{\gamma_1 t}) \quad ...(5)$$

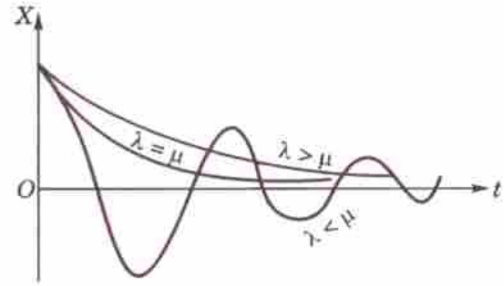

Fig. 14.10

which shows that x is always positive and decreases to zero as $t \to \infty$ (Fig. 14.10).

The restoring force, in this case, is so great that the motion is non-oscillatory and is, therefore, referred to as *over-damped* or *dead-beat* motion.

Case II. *When* $\lambda = \mu$, the roots of the auxiliary equation are real and equal, (each being $= -\lambda$).

$\therefore$ The general solution of (3) becomes $x = (c_1 + c_2 t)e^{-\lambda t}$.

As in case I, if $x = l$ and $dx/dt = 0$ at $t = 0$, then $c_1 = l$ and $c_2 = \lambda l$.

Hence the solution of (3) is $x = l\,(1 + \lambda t)e^{-\lambda t}$ which also shows that x is always positive and decreases to zero as $t \to \infty$ (Fig. 14.10).

The nature of motion is similar to that of the previous case and is called the *critically damped motion* for it separates the non-oscillatory motion of case I from the most interesting oscillatory motion of case III.

Case III. *When* $\lambda < \mu$, the roots of the auxiliary equation are imaginary, *i.e.* $D = -\lambda \pm i\alpha$, where $\alpha^2 = \mu^2 - \lambda^2$.

$\therefore$ the solution of (3) is $x = e^{-\lambda t}(c_1 \cos \alpha t + c_2 \sin \alpha t)$

As in case I, $x = l$, $dx/dt = 0$ at $t = 0$, then $c_1 = l$ and $c_2 = \lambda l/\alpha$

Thus the solution of (3) becomes $x = le^{-\lambda t}\left(\cos \alpha t + \dfrac{\lambda}{\alpha}\sin \alpha t\right)$.

which can be put in the form $x = l\sqrt{\left\{1 + \left(\dfrac{\lambda}{\alpha}\right)^2\right\}}\, e^{-\lambda t} \cos\left\{\alpha - \tan^{-1}\dfrac{\lambda}{\alpha}\right\}$ ...(7)

Here the presence of the trigonometric factor in (7) shows that the *motion is oscillatory*, having

(*a*) the variable amplitude $= l\sqrt{[1 + (\lambda/\alpha)^2]e^{-\lambda t}}$ which decreases with time,

(*b*) the periodic time $T = 2\pi/\alpha$.

But the periodic time of free oscillations is $T' = 2\pi/\mu$.

As $\alpha = \sqrt{(\mu^2 - \lambda^2)} < \mu$

$\therefore$ $\dfrac{2\pi}{\alpha} > \dfrac{2\pi}{\mu}$, *i.e.* $T > T'$.

This shows that the *effect of damping is to increase the period of oscillation and the motion ultimately dies away*. Such a motion is termed as *damped oscillatory motion*.

(*iii*) **Forced oscillations** (*without damping*). If the point of the support of the spring is also vibrating with some external periodic force, then the resulting motion is called the *forced oscillatory motion*.

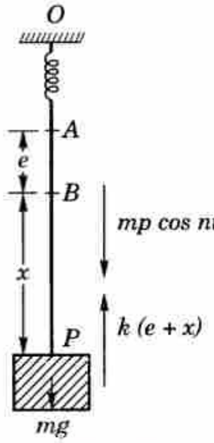

Fig. 14.11

Taking the external periodic force to be $mp \cos nt$, the equation of motion is

$$m\frac{d^2x}{dt^2} = mg - k(e + x) + mp \cos nt$$

$$= -kx + mp \cos nt \qquad [\because \; mg = ke] \quad ...(8)$$

Or writing $k/m = \mu^2$, (8) takes the form

$$\frac{d^2x}{dt^2} + \mu^2 x = p \cos nt \qquad ...(9)$$

Its C.F. $= c_1 \cos \mu t + c_2 \sin \mu t$ and P.I. $= p \dfrac{1}{D^2 + \mu^2} \cos nt.$

New two cases arise :

Case I. *When* $\mu \neq n$.

$$\text{P.I.} = \frac{p}{\mu^2 - n^2} \cos nt.$$

$\therefore$ the complete solution of (9) is $x = c_1 \cos \mu t + c_2 \sin \mu t + \dfrac{p}{\mu^2 - n^2} \cos nt.$

On writing $c_1 \cos \mu t + c_2 \sin \mu t$ as $r \cos (\mu t + \phi)$, we have

$$x = r \cos (\mu t + \phi) + \frac{p}{\mu^2 - n^2} \cos nt \qquad ...(10)$$

This shows that the motion is compounded of two oscillatory motions : the first (due to the C.F.) *gives free oscillations of period $2\pi/\mu$, and the second* (due to the P.I.) *gives forced oscillations of period $2\pi/n$.*

Also we observe that if the frequency of free oscillations is very high (*i.e.*, μ is large), then the amplitude of forced oscillations is small.

Case II. *When* $\mu = n$.

$$\text{P.I.} = pt \,.\, \frac{1}{2D} \cos \mu t = \frac{pt}{2} \int \cos \mu t \, dt = \frac{pt}{2\mu} \sin \mu t$$

$\therefore$ the complete solution of (9) is $x = c_1 \cos \mu t + c_2 \sin \mu t + \dfrac{pt}{2\mu} \sin \mu t$

$$= \left(c_2 + \frac{pt}{2\mu} \right) \sin \mu t + c_1 \cos \mu t.$$

Putting $c_2 + pt/2\mu = \rho \cos \psi$ and $c_1 = \rho \sin \psi$, we get

$$x = \rho \sin (\mu t + \psi) \qquad ...(11)$$

This shows that the oscillations are of period $2\pi/\mu$ and amplitude $\rho = \sqrt{[(c_2 + pt/2\mu)^2 + c_1^2]}$, which clearly increases with time (Fig. 14.12).

Thus the amplitude of the oscillations may become abnormally large causing over-strain and consequently breakdown of the system. In practice, however, collapse rarely occurs, though the amplitudes may become dangerously large since there is always some resistance present in the system.

This phenomenon of the impressed frequency becoming equal to the natural frequency of the system, is refered to as **resonance.**

Thus, while designing a machine or a structure, the occurrence of resonance should always be avoided to check the rupture of the system at any stage. That is why, the soldiers break step while marching over a bridge for the fear that their steps may not be in rhym with the natural frequency of the bridge causing its collapse due to 'resonance'.

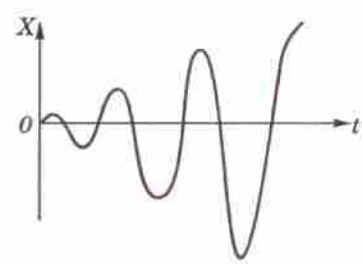

Fig. 14.12

(*iv*) **Forced oscillations** (*with damping*). If, in addition, there is a damping force proportional to velocity (say : $r\, dx/dt$) (Fig. 14.13), then the equation (8) becomes

$$m\frac{d^2x}{dt^2} = mg - k(e + x) + mp \cos nt - r\frac{dx}{dt}$$

$$= -kx + mp \cos nt - r\frac{dx}{dt} \qquad | \because \; mg = ke$$

On writing $r/m = 2\lambda$ and $k/m = \mu^2$, it takes the form

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \mu^2 x = p\cos nt \qquad ...(12)$$

Its auxiliary equation is $D^2 + 2\lambda D + \mu^2 = 0$ whence $D = -\lambda \pm \sqrt{\lambda^2 - \mu^2}$.

$\therefore$ $$\text{C.F.} = e^{-\lambda t}[c_1 e^{t\sqrt{(\lambda^2-\mu^2)}} + c_2 e^{-t\sqrt{(\lambda^2-\mu^2)}}].$$

It represents the free oscillations of the system which die out as $t \to \infty$.

Also the P.I.

$$= p\frac{1}{D^2 + 2\lambda D + \mu^2}\cos nt = p\frac{1}{-n^2 + 2\lambda D + \mu^2}\cos nt$$

$$= p\frac{(\mu^2 - n^2) - 2\lambda D}{(\mu^2 - n^2)^2 - 4\lambda^2 D^2}\cos nt = p\frac{(\mu^2 - n^2)^2\cos nt + 2\lambda n\sin nt}{(\mu^2 - n^2)^2 + 4\lambda^2 n^2}$$

Putting $\mu^2 - n^2 = R\cos\theta$ and $2\lambda n = R\sin\theta$, we get

$$\text{P.I.} = \frac{p}{\sqrt{[(\mu^2 - n^2)^2 + 4\lambda^2 n^2]}}\cos(nt - \theta)$$

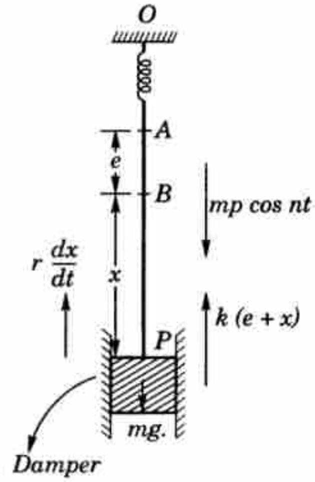

Fig. 14.13

which represents the forced oscillations of the system having

(*a*) a constant amplitude

$$= p/\sqrt{[(\mu^2 - n^2)^2 + 4\lambda^2 n^2]}$$

and (*b*) the period $= 2\pi/n$ which is the same as that of the impressed force.

Thus *with the increase of time, the free oscillations die away while the forced oscillations continue giving the steady state motion.*

Example 14.7. *A body weighing 10 kg is hung from a spring. A pull of 20 kg. wt. will stretch the spring to 10 cm. The body is pulled down to 20 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t sec., the maximum velocity and the period of oscillation.*

Solution. Let O be the fixed end and A, the lower end of the spring (Fig. 14.14).

Since a pull of 20 kg wt. at A stretches the spring by 0.1 m.

$\therefore$ $$20 = T_0 = k \times 0.1, \text{ i.e. } k = 200 \text{ kg/m}.$$

Let B be the equilibrium position when a body weighing $W = 10$ kg is hung from A ; then

$$10 = T_B = k \times AB$$

i.e., $$AB = \frac{10}{200} = 0.05 \text{ m}$$

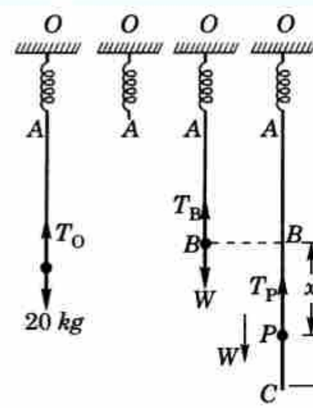

Fig. 14.14

Now the weight is pulled down to C, where $BC = 0.2$ m. After any time t sec. of its release from C, let the weight be at P where $BP = x$.

Then the tension $T_P = k \times AP = 200(0.05 + x) = 10 + 200x$.

$\therefore$ The equation of motion of the body is

$$\frac{W}{g}\frac{d^2x}{dt^2} = W - T_P, \text{ where } g = 9.8 \text{ m/sec}^2.$$

i.e., $$\frac{10}{9.8}\frac{d^2x}{dt^2} = 10 - (10 + 200x) \quad \text{or} \quad \frac{d^2x}{dt^2} = -\mu^2 x, \quad \text{where } \mu = 14.$$

This shows that the motion of the body is simple harmonic about B as centre and the period of oscillation $= 2\pi/\mu = 0.45$ sec.

Also the amplitude of motion being $BC = 0.2$ m., the displacement of the body from B at time t is given by $x = 0.2\cos\mu t = 0.2\cos 14t$ m

and the maximum velocity $= \mu$ (amplitude) $= 14 \times 0.2 = 2.8$ m/sec.

Example 14.8. *A spring fixed at the upper end supports a weight of 980 gm at its lower end. The spring stretches $\frac{1}{2}$ cm under a load of 10 gm and the resistance (in gm wt.) to the motion of the weigth is numerically equal to $\frac{1}{10}$ of the speed of the weight in cm/sec. The weight is pulled down $\frac{1}{4}$ cm. below its equilibrium position and then released. Find the expression for the distance of weight from its equilibrium position at time t during its first upward motion.*

Also find the time it takes the damping factor to drop to $\frac{1}{10}$ of its initial value.

Solution. Let O be the fixed end and A the other end of the spring (Fig. 14.15).

Since load of 10 gm attached to A stretches the spring by $\frac{1}{2}$ cm.

$$\therefore \qquad 10 = T_0 = k \cdot \frac{1}{2} \text{ i.e., } k = 20 \text{ gm/cm.}$$

Let B be the equilibrium position when 980 gm. weight is attached to A, then

$$980 = T_B = k \times AB, \quad i.e., \quad AB = \frac{980}{20} = 49 \text{ cm.}$$

Now the 980 gm weight is pulled down to C, where $BC = \frac{1}{4}$ cm.

After any time t of its release from C, let the weight be at P, where $BP = x$.

Then the tension

$$T = k \times AP = 20\,(49 + x) = 980 + 20x \text{ and the resistance to motion} = \frac{1}{10}\frac{dx}{dt}.$$

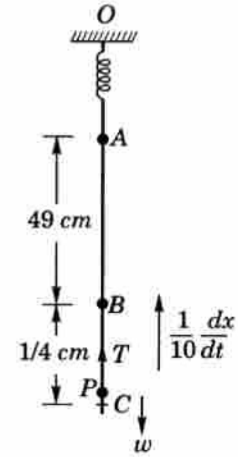

Fig. 14.15

$\therefore$ the equation of motion is

$$\frac{980}{g}\frac{d^2x}{dt^2} = w - T - \frac{1}{10}\frac{dx}{dt} \qquad [\because g = 980 \text{ cm/sec}^2 \text{ (p. 449)}$$

$$= 980 - (980 + 20x) - \frac{1}{10}\frac{dx}{dt} \quad i.e. \quad 10\frac{d^2x}{dt^2} + \frac{dx}{dt} + 200x = 0 \qquad ...(i)$$

Its auxiliary equation is $10D^2 + D + 200 = 0$,

whence $$D = \frac{-1 + \sqrt{[1 - 4 \times 10 \times 200]}}{20} = \frac{-1 + i(89.4)}{20} = -0.05 \pm i\,(4.5)$$

$\therefore$ the solution of (i) is $x = e^{-0.05t}[c_1 \cos(4.5)t + c_2 \sin(4.5)t]$ ...(ii)

Also $$\frac{dx}{dt} = e^{-0.05t}(-0.05)\,[c_1 \cos(4.5)t + c_2 \sin(4.5)t]$$
$$+ e^{-0.05t}[-c_1 \sin(4.5)t + c_2 \cos(4.5)t](4.5) \qquad ...(iii)$$

Initially when the mass is at C, $t = 0$, $x = \frac{1}{4}$ cm. and $dx/dt = 0$.

From (ii), $c_1 = \frac{1}{4}$, and from (iii) $0 = (-0.05)c_1 + c_2(4.5)$, *i.e.*, $c_2 = -0.003$.

Thus, substituting these value in (ii), we get

$$x = e^{-0.05t}[0.25 \cos(4.5)t + 0.003 \sin(4.5)t]$$

which gives the displacement of the weight from the equilibrium position at any time t.

Here damping factor $= re^{-0.05t}$, where r is a constant of proportionality.

Its initial value $= re^0 = r$.

Suppose after time t, the damping factor $= r/10$. $\therefore \quad r/10 = re^{-0.05t}$ or $e^{t/20} = 10$.

Thus $$t = 20 \log_e 10 = 20 \times 2.3 = 46 \text{ sec.}$$

Example 14.9. *A spring which stretches by an amount e under a force $m\lambda^2 e$ is suspended from a support O and has a mass m at the lower end. Initially the mass is at rest in its equilibrium position at a point A below O. A vertical oscillation is now given to the support O such that at any time $(t > 0)$ its displacement below its initial position is a sin nt. Show that the displacement x of the mass below A is given by*

$$d^2x/dt^2 + \lambda^2 x = \lambda^2 a \sin nt.$$

Hence show that if $n \neq \lambda$, the displacement is given by $x = \lambda a\,(\lambda \sin nt - n \sin \lambda t)/(\lambda^2 - n^2)$. What happens when $n = \lambda$?

Solution. If k be the stiffness of the spring then $m\lambda^2 e = ke$ *i.e.,* $k = m\lambda^2$.

Also in equilibrium $\quad mg = ke \quad$ ...(*i*)

Initially the mass is in equilibrium at A (Fig. 14.7). At time t, the support P is given a downward displacement $a \sin nt$. If the mass is displaced through a further distance x from A, then the equation of motion of the mass is given by

$$m\frac{d^2x}{dt^2} = mg - k(x + e) + ka \sin nt$$

$$= -kx + ka \sin nt \qquad \text{[By (i)]}$$

or
$$\frac{d^2x}{dt^2} + \lambda^2 x = \lambda^2 a \sin nt \qquad [\because \quad k = m\lambda^2]$$

or
$$(D^2 + \lambda^2)x = \lambda^2 a \sin nt \qquad \text{...(ii)}$$

Its A.E. $= c_1 \cos \lambda t + c_2 \sin \lambda t$

$$\text{P.I.} = \frac{1}{D^2 + \lambda^2}\lambda^2 a \sin nt.$$

Now two cases arise :

Case I. *When $n \neq \lambda$*

$$\text{P.I.} = \lambda^2 a \frac{1}{n^2 + \lambda^2}\sin nt$$

$\therefore$ the complete solution of (*ii*) is $x = c_1 \cos \lambda t + c_2 \sin \lambda t + \dfrac{\lambda^2 a}{\lambda^2 - n^2}\sin nt$ ...(*iii*)

$$\therefore \qquad \frac{dx}{dt} = -c_1\lambda \sin \lambda t + c_2\lambda \cos \lambda t + \frac{\lambda^2 an}{\lambda^2 - n^2}\cos nt$$

Initially when $t = 0$, $x = 0$ and $dx/dt = 0$.

$\therefore \qquad c_1 = 0$ and $0 = c_2\lambda + \lambda^2 an/(\lambda^2 - n^2)$ *i.e.,* $c_2 = \lambda an/(\lambda^2 - n^2)$

Thus, substituting the values of c_1 and c_2 in (*iii*), we have

$$x = -\frac{\lambda an}{\lambda^2 - n^2}\sin \lambda t + \frac{\lambda^2 a}{\lambda^2 - n^2}\sin nt = \frac{\lambda a}{\lambda^2 - n^2}(\lambda \sin nt - n \sin \lambda t)$$

Case II. *When $n = \lambda$*

$$\text{P.I.} = \lambda^2 a \frac{1}{D^2 + \lambda^2}\sin nt = \lambda^2 at \cdot \frac{1}{2D}\sin \lambda t = \frac{\lambda^2 at}{2}\int \sin \lambda t\, dt = -\frac{\lambda at}{2}\cos \lambda t$$

$\therefore$ the complete solution is

$$x = c_1 \cos \lambda t + c_2 \sin \lambda t - \frac{\lambda at}{2}\cos \lambda t \qquad \text{...(iv)}$$

$$\therefore \qquad \frac{dx}{dt} = -c_1\lambda \sin \lambda t + c_2\lambda \cos \lambda t + \frac{\lambda^2 at}{2}\sin \lambda t - \frac{\lambda a}{2}\cos \lambda t$$

When $t = 0$, $x = 0$ and $dx/dt = 0$

$\therefore \quad 0 = c_1$ and $0 = c_2\lambda - \lambda a/2$ *i.e.,* $c_2 = a/2$.

Thus, substituting the values of c_1 and c_2 in (*iv*), we get

$$x = \frac{a}{2}\sin \lambda t - \frac{\lambda at}{2}\cos \lambda t$$

$$= \frac{a}{2}(\sin \lambda t - \lambda t \cos \lambda t) \qquad [\text{Put } 1 = r\cos\phi \text{ and } \lambda t = r \sin \phi]$$

$$= \frac{ar}{2}\sin(\lambda t - \phi)$$

Its amplitude $\left(\frac{ar}{2}\right) = \frac{a}{2}\sqrt{(1+\lambda^2 t^2)}$, which increases with time. Hence the phenomenon of *resonance* occurs.

Example 14.10. *A spring of negligible weight which stretches 1 inch under tension of 2 lb is fixed at one end and is attached to a weight of w lb at the other. It is found that resonance occurs when an axial periodic force 2 cos 2t lb acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by x = ct sin 2t, and find the values of w and c.*

Solution. As a weight of 2 lb attached to the lower end A of the spring stretched it by $\frac{1}{12}$ ft.

$\therefore$ $$2 = T = k \cdot \frac{1}{12}, \quad i.e., \; k = 24 \text{ lb/ft}.$$

Let B be the equilibrium position of the weight w attached to A (Fig. 14.16), then

$$w = T_B = k \times AB = 24 \times AB$$

$\therefore$ $$AB = w/24 \text{ ft}.$$

At any time t, let the weight be at P, where $BP = x$.

Then the tension T at $P = k \times AP = 24\left(\frac{w}{24} + x\right) = w + 24x$

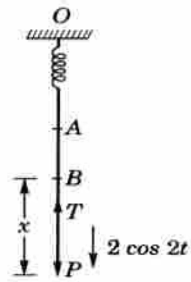

Fig. 14.16

$\therefore$ its equation of motion is

$$\frac{w}{g}\frac{d^2x}{dt^2} = -T + w + 2\cos 2t = -w - 24x + w + 2\cos 2t$$

or $$w\frac{d^2x}{dt^2} + 24gx = 2g\cos 2t \quad \text{...}(i)$$

The phenomenon of **resonance** *occurs when the period of free oscillations is equal to the period of forced oscillations.*

Writing (i) as $\frac{d^2x}{dt^2} + \mu^2 x = \frac{2g}{w}\cos 2t$, where $\mu^2 = 24g/w$, the period of free oscillations is found to be $2\pi/\mu$ and the period of the force $(2g/w)\cos 2t$ is π.

$\therefore$ $2\pi/\mu = \pi$ or $24g/w = \mu^2 = 4$. Thus the weight, $w = 6g$.

Taking this value of w, (i) takes the form

$$\frac{d^2x}{dt^2} + 4x = \frac{1}{3}\cos 2t \quad \text{...}(ii)$$

We know that the free oscillations are given by the C.F. and the forced oscillations by the P.I.

Thus, when the free oscillations have died out, the forced oscillations are given by the P.I. of (ii).

Now P.I. of $(ii) = \frac{1}{3}\cdot\frac{1}{D^2+4}\cos 2t = \frac{1}{3}t\cdot\frac{1}{2D}\cos 2t = \frac{1}{12}t\sin 2t.$

Hence $c = \frac{1}{12}.$

PROBLEMS 14.2

1. An elastic string of natural length a is fixed at one end and a particle of mass m hangs freely from the other end. The modulus of elasticity is mg. The particle is pulled down a further distance l below its equilibrium position and released from rest. Show that the motion of the particle is simple harmonic and find the pariodicity.
2. A mass of 4 lb suspended from a light elastic string of natural length 3 feet extends it to a distance 2 feet. One end of the string is fixed and a mass of 2 lb is attached to other. The mass is held so that the string is just unstretched and is then let go. Find the amplitude, the period and the maximum velocity of the ensuing simple harmonic motion.

3. A light elastic string of natural length l has one extremity fixed at a point A and the other end attached to a stone, the weight of which in equilibrium would extend the string to a depth l_1. Show that if the stone be dropped from rest at A, it will come to instantaneous rest at a depth $\sqrt{(l_1^2 - l^2)}$ below the equilibrium position.
4. A 4 lb weight on a string stretches it 6 in. Assuming that a damping force in lb wt. equal to λ times the instantaneous velocity in ft/sec. acts on the weight, show that the motion is over damped, critically damped or oscillatory according as $\lambda >=< 2$. Find the period of oscillation when $\lambda = 1.5$.
5. A mass of 200 gm is tied at the end of a spring which extends to 4 cm under a force 196,000 dynes. The spring is pulled 5 cm and released. Find the displacement t seconds after release if there be a damping force of 2000 dynes per cm per second.
6. A body weighing 16 lb is suspended by a spring in a fluid whose resistance in lb wt. is twice the speed of the body in ft/sec. A pull of 25 lb wt. would stretch the spring 3 inches. The body is drawn 3 inches below the equilibrium position in the fluid and then released. Find the period of oscillations and the time required for the damping factor to be reduced to one-tenth of its initial value. *(Sambhalpur, 1998)*
7. A mass M suspended from the end of a helical spring is subjected to a periodic force $f = F \sin \omega t$ in the direction of its length. The force f is measured positive vertically downwards and at zero time M is at rest. If the spring stiffness is S, prove that the displacement of M at time t from the commencement of motion is given by

$$x = \frac{F}{M(p^2 - \omega^2)}\left[\sin \omega t - \frac{\omega}{p}\sin pt\right]$$

where $p^2 = S/M$ and damping effects are neglected. *(U.P.T.U., 2002)*
8. A vertical spring having 4.5 lb/ft. has 16 lb wt. suspended from it. An external force of 24 sin $9t$ ($t \geq 0$) lb wt. is applied. A damping force given numerically in lb. wt. by four times its velocity in ft/sec, is assumed to act. Initially the weight is at rest at its equilibrium position. Determine the position of the weight at any time. Also find the amplitude, period and the frequency of the steady-state solution.
9. A body weighing 4 lb hangs at rest on a spring producing in the spring an extension of 1ft. The upper end of the spring in now made to execute a vertical simple harmonic oscillation $x = \sin 4t$, x being measured vertically downwards in feet. If the body is subject to a frictional resistance whose magnitude in lb wt. is one-quarter of its velocity in feet per second, obtain the differential equation for the motion of the body and find the expression for its displacement at time t, when t is large.
10. A body executes damped forced vibrations given by the equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + b^2x = e^{-kt}\sin nt.$$

Solve the equation for both the cases, when $n^2 \neq b^2 - k^2$ and $n^2 = b^2 - k^2$. *(U.P.T.U., 2004)*

14.5 OSCILLATORY ELECTRICAL CIRCUIT

(*i*) **L-C circuit**

Consider an electrical circuit containing an inductance L and capacitance C (Fig. 14.17).

Let i be the current and q the charge in the condenser plate at any time t, so that the voltage drop across

$$L = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$$

and the voltage drop across $C = q/C$.

As there is no applied e.m.f. in the circuit, therefore, by Kirchhoff's first law, we have

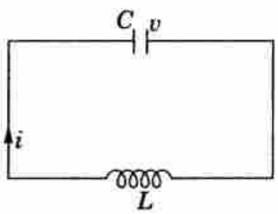

Fig. 14.17

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0.$$

Or dividing by L and writing $1/LC = \mu^2$, we get $\dfrac{d^2q}{dt^2} + \mu^2 q = 0$ ...(1)

This equation is precisely same as (2) on page 507 and, therefore, *it represents free electrical oscillations of the current having period* $2\pi/\mu = 2\pi\sqrt{LC}$.

Thus the discharging of a condenser through an inductance L is same as the motion of the mass m at the end of a spring.

(*ii*) **L–C–R circuit**

Now consider the discharge of a condenser C through an inductanc L and the resistance R (Fig. 14.18). Since the voltage drop across L, C and R are respectively

$$L\frac{d^2q}{dt^2}, \frac{q}{C} \text{ and } R\frac{dq}{dt}.$$

$\therefore$ by Kirchhoff's law, we have $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = 0$ ...(2)

Or writing $R/L = 2\lambda$ and $1/LC = \mu^2$, we have $\dfrac{d^2q}{dt^2} + 2\lambda\dfrac{dq}{dt} + \mu^2 q = 0$

This equation is same as (3) on page 507 and, therefore has the same solution as for the mass m on a spring with a damper.

Thus *the charging or discharging of a condenser through the resistance R and an inductance L is an electrical analogue of the damped oscillations of mass m on a spring.*

(*iii*) **L–C circuit with e.m.f.** = *p cos nt*.

The equation (1) for an L–C circuit (Fig. 14.19), now becomes $L\dfrac{d^2q}{dt^2} + \dfrac{q}{C} = p \cos nt.$

Or writing $1/LC = \mu^2$, we have $\dfrac{d^2q}{dt^2} + \mu^2 q = \dfrac{p}{L}\cos nt$ ...(3)

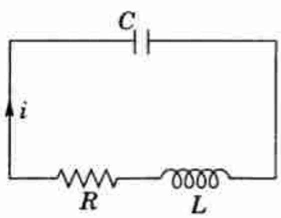

Fig. 14.18

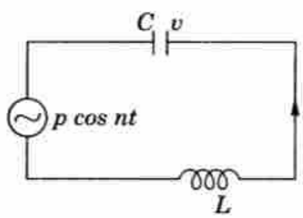

Fig. 14.19

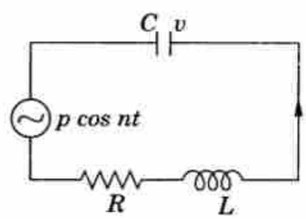

Fig. 14.20

This equation is of the same form as (9) on page 509 and, therefore, has the solution as for the motion of a mass m on a spring with external periodic force $p \cos nt$ acting on it.

Thus the condenser placed in series with source of e.m.f. (= p cos nt) and discharging through a coil containing inductance L is an electrical analogue of the forced oscillations of the mass m on a spring.

An electrical instance of *resonance* phenomena occurs while tuning a radio-station, for the natural frequency of the tuning of L–C circuit is made equal to the frequency of the desired radio-station, giving the maximum output of the receiver at the said receiving station.

(*iv*) **L–C–R circuit with e.m.f.** = *p cos nt*.

The equation of (2) above, now becomes $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = p \cos nt.$ (Fig. 14.20)

Or writing $R/L = 2\lambda$ and $1/LC = \mu^2$ as before, we have

$$\frac{d^2q}{dt^2} + 2\lambda\frac{dq}{dt} + \mu^2 q = \frac{p}{L}\cos nt \quad ...(4)$$

This equation is exactly same as (12) on page 510 and, therefore, its C.F. represents the free oscillations of the circuit whereas the P.I. represents the forced oscillations.

Here also as t increases, the free oscillations die out while the forced oscillations persist giving steady motion.

Thus the L–C–R circuit with a source of alternating e.m.f. is an electrical equivalent of the mechanical phenomena of forced oscillations with resistance.

14.6 ELECTRO-MECHANICAL ANALOGY

We have just seen, how merely by renaming the variables, the differential equation representing the oscillation of a weight on a spring represents an analogous electrical circuit. As electrical circuits are easy to assemble and the currents and voltages are accurately measured with ease, this affords a practical method of studying the oscillations of complicated mechanical systems which are expensive to make and unwieldy to handle by considering an equivalent electrical circuit. While making an electrical equivalent of a mechanical system, the following correspondences between the elements should be kept in mind, noting that the circuit may be in series or in parallels :

Mech. System	*Series circuit*	*Parallel circuit*
Displacement	Current i	Voltage E
Force or couple	Voltage E	Current i
Mass m or $M.I.$	Inductance L	Capacitance C
Damping force	Resistance R	Conductance $1/R$
Spring modulus	Elastance $1/C$	Susceptance $1/L$

Example 14.11. *An uncharged condenser of capacity C is charged by applying an e.m.f. $E \sin t/\sqrt{(LC)}$, through leads of self-inductance L and negligible resistance. Prove that at any time t, the charge on one of the plates is* $\frac{EC}{2}\left\{\sin\frac{t}{\sqrt{(LC)}} - \frac{t}{\sqrt{(LC)}}\cos\frac{t}{\sqrt{(LC)}}\right\}$ *(U.P.T.U., 2003)*

Solution. If q be the change on the condenser, the differential equation of the circuit is

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E\sin\frac{t}{\sqrt{(LC)}} \qquad \text{...}(i)$$

Its A.E. is $\quad LD^2 + 1/C = 0 \quad$ or $\quad D = \pm 1/\sqrt{(LC)}$

$\therefore \quad$ C.F. $= c_1 \cos t/\sqrt{LC} + c_2 \sin t/\sqrt{(LC)}$

and $\quad$ P.I. $= \dfrac{1}{LD^2 + \frac{1}{C}} E \sin\dfrac{t}{\sqrt{(LC)}}$ $\quad \left[\text{Putting } D^2 = -\dfrac{1}{LC}, \text{denom.} = 0\right]$

$$= Et\frac{1}{2LD}\sin\frac{t}{\sqrt{(LC)}} = \frac{Et}{2L}\int \sin\frac{t}{\sqrt{(LC)}}dt = -\frac{Et}{2L}\sqrt{(LC)}\cos\frac{t}{\sqrt{(LC)}} = -\frac{Et}{2}\sqrt{\frac{C}{L}}\cos\frac{t}{\sqrt{LC}}$$

Thus the C.S. of (i) is $q = c_1\cos\dfrac{t}{\sqrt{LC}} + c_2\sin\dfrac{t}{\sqrt{LC}} - \dfrac{Et}{2}\sqrt{\dfrac{C}{L}}\cos\dfrac{t}{\sqrt{LC}}$

When $t = 0, q = 0, c_1 = 0$

$$\therefore \quad q = c_2\sin\frac{t}{\sqrt{LC}} - \frac{Et}{2}\sqrt{\frac{C}{L}}\cos\frac{t}{\sqrt{LC}} \qquad \text{...}(ii)$$

Differentiating (ii) w.r.t. t, we get

$$\frac{dq}{dt} = \frac{c_2}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}} - \frac{E}{2}\sqrt{\frac{C}{L}}\left\{\cos\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}}\sin\frac{t}{\sqrt{LC}}\right\}$$

Also when $t = 0, dq/dt = i = 0,$

$$\therefore \quad \frac{c_2}{\sqrt{LC}} - \frac{E}{2}\sqrt{\frac{C}{L}} = 0 \quad \text{or} \quad c_2 = \frac{EC}{2}.$$

Substituting the value of c_2 in (ii), q at any time t is given by

$$q = \frac{EC}{2}\left\{\sin\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right\}.$$

Example 14.12. *In an L—C—R circuit, the charge q on a plate of a condenser is given by*

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E\sin pt.$$

The circuit is tuned to resonance so that $p^2 = 1/LC$. If initially the current i and the charge q be zero, show that, for small values of R/L, the current in the circuit at time t is given by

$$(Et/2L)\sin pt. \qquad \textit{(U.P.T.U., 2004)}$$

Solution. Given differential equation is $(LD^2 + RD + 1/C)q = E \sin pt$ ...(*i*)

Its auxiliary equation is $LD^2 + RD + 1/C = 0$,

which gives $$D = \frac{1}{2L}\left[-R \pm \sqrt{\left(R^2 - \frac{4L}{C}\right)}\right] = -\frac{R}{2L} + \sqrt{\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)}$$

As R/L is small, therefore, to the first order in R/L,

$$D = -\frac{R}{2L} \pm i\frac{1}{\sqrt{(LC)}} = -\frac{R}{2L} \pm ip \qquad \left[\because\ p^2 = \frac{1}{LC}\right]$$

$\therefore$ $\quad$ C.F. $= e^{-(Rt/2L)}(c_1 \cos pt + c_2 \sin pt)$

$= (1 - Rt/2L)(c_1 \cos pt + c_2 \sin pt)$ rejecting terms in $(R/L)^2$ etc.

and $$\text{P.I.} = \frac{1}{LD^2 + RD + 1/C} E \sin pt = E\frac{1}{-Lp^2 + RD + 1/C} \sin pt$$

$$= \frac{E}{R}\int \sin pt\, dt = -\frac{E}{Rp} \cos pt \qquad \left[\because p^2 = \frac{1}{LC}\right]$$

Thus the complete solution of (*i*) is $q = \left(1 - \dfrac{Rt}{2L}\right)(c_1 \cos pt + c_2 \sin pt) - \dfrac{E}{Rp} \cos pt$ ...(*ii*)

$$\therefore \quad i = \frac{dq}{dt} = \left(1 - \frac{Rt}{2L}\right)(-c_1 \sin pt + c_2 \cos pt)\, p - \frac{R}{2L}(c_1 \cos pt + c_2 \sin pt) + \frac{E}{R} \sin pt \qquad ...(iii)$$

Initially, when $t = 0, q = 0, i = 0 \therefore$ from (*ii*), $0 = c_1 - E/Rp \therefore c_1 = E/Rp$ and from (*iii*),

$0 = c_2 p - Rc_1/2L \therefore c_2 = Rc_1/2Lp = E/2Lp^2$

Thus, substituting these values of c_1 and c_2 in (*iii*), we get

$$i = \left(1 - \frac{Rt}{2L}\right)\left(-\frac{E}{Rp} \sin pt + \frac{E}{2Lp^2} \cos pt\right) p - \frac{R}{2L}\left(\frac{E}{Rp} \cos pt + \frac{E}{2Lp^2} \sin pt\right) + \frac{E}{R} \sin pt$$

$$= \frac{Et}{2L} \sin pt. \qquad [\because\ R/L \text{ is small}]$$

PROBLEMS 14.3

1. Show that the frequency of free vibrations in a closed electrical circuit with inductance L and capacity C in series is $\dfrac{30}{\pi\sqrt{(LC)}}$ per minute.
2. The differential equation for a circuit in which self-inductance and capacitance neutralize each other is $L\dfrac{d^2i}{dt^2} + \dfrac{i}{C} = 0$. Find the current i as a function of t given that I is the maximum current, and $i = 0$ when $t = 0$.
3. A constant e.m.f. E at $t = 0$ is applied to a circuit consisting of inductance L, resistance R and capacitance C in series. The initial values of the current and the charge being zero, find the current at any time t, if $CR^2 < 4L$. Show that the amplitudes of the successive vibrations are in geometrical progression.
4. The damped LCR circuit is governed by the equation $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = 0$ where, L, R, C are positive constants. Find the conditions under which the circuit is over damped, under damped and critically damped. Find also the critical resistance. (*U.P.T.U., 2005*)
5. A condenser of capacity C discharged through an inductance L and resistance R in series and the charge q at time t satisfies the equation $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = 0$. Given that $L = 0.25$ henries, $R = 250$ ohms, $C = 2 \times 10^{-6}$ farads, and that when $t = 0$, charge q is 0.002 coulombs and the current $dq/dt = 0$, obtain the value of q in terms of t.
6. An e.m.f. $E \sin pt$ is applied at $t = 0$ to a circuit containing a capacitance C and inductance L. The current i satisfies the equation $L\dfrac{di}{dt} + \dfrac{1}{C}\int i\, dt = E \sin pt$. If $p^2 = 1/LC$ and initially the current i and the charge q are zero, show that the current at time t is $(Et/2L) \sin pt$, where $i = dq/dt$.

7. For an L—R—C circuit, the charge q on a plate of the condenser is given by $L\dfrac{d^2q}{dt^2}+R\dfrac{dq}{dt}+\dfrac{q}{C}=E\sin\omega t$, where $i=\dfrac{dq}{dt}$. The circuit is tuned to resonance so that $\omega^2 = 1/LC$.

If $CR^2 < 4L$ and initially $q = 0$, $i = 0$, show that $q=\dfrac{E}{R\omega}\left[e^{-Rt/2L}\left(\cos pt+\dfrac{R}{2Lp}\sin pt\right)-\cos\omega t\right]$

where $$p^2=\frac{1}{LC}-\frac{R^2}{4L^2}.$$ (U.P.T.U., 2003)

8. An alternating E.M.F. $E \sin pt$ is applied to a circuit at $t = 0$. Given the equation for the current i as $L\dfrac{d^2i}{dt^2}+R\dfrac{di}{dt}+\dfrac{i}{C}=pE\cos pt$, find the current i when (i) $CR^2 > 4L$, (ii) $CR^2 < 4L$.

14.7 DEFLECTION OF BEAMS

Consider a uniform beam as made up of fibres running lengthwise. We have to find its deflection under given loadings.

In the bent form, the fibres of the lower half are stretched and those of upper half are compressed. In between these two, there is a layer of unstrained fibres called the *neutral surface*. The fibre which was initially along the x-axis (the central horizontal axis of the beam) now lies in the neutral surface, in the form of a curve called the *deflection curve* or the *elastic curve*. We shall encounter differential equations while finding the equation of this curve.

Consider a cross-section of the beam cutting the elastic curve in P and the neutral surface in the line AA'—called the neutral axis of this section (Fig. 14.21).

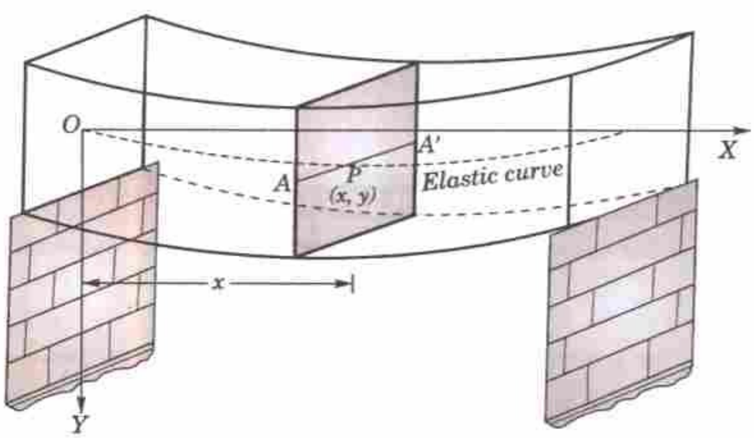

Fig. 14.21

It is well-known from mechanics that the bending moment M about AA', of all forces acting on either side of the two portions of the beam separated by this cross-section, is given by the *Bernoulli-Euler law*

$$M = EI/R$$

where E = modulus of elasticity of the beam,

I = moment of inertia of the cross-section about AA',

and R = radius of curvature of the elastic curve at $P(x, y)$.

If the deflection of the beam is small, the slope of the elastic curve is also small so that we may neglect $(dy/dx)^2$ in the formula,

$$R=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}\Bigg/\frac{d^2y}{dx^2}.$$ Thus for small deflections, $R = 1/(d^2y/dx^2)$.

Hence (1) *Bending moment* $\mathbf{M}=\mathbf{EI}\dfrac{\mathbf{d^2y}}{\mathbf{dx^2}}$

(2) *Shear force* $\left(=\dfrac{dM}{dx}\right) = EI\,\dfrac{d^3y}{dx^3}$;

(3) *Intensity of loading* $\left(=\dfrac{d^2M}{dx^2}\right) = EI\,\dfrac{d^4y}{dx^4}$

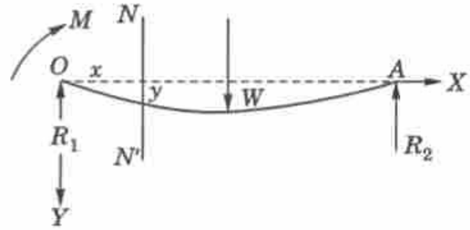

Fig. 14.22

(4) *Convention of signs.* The sum of the moments about a section NN' due to external forces on the left of the section, if anti-clockwise is taken as positive and if clockwise (as in Fig. 14.22) is taken as negative.

The deflection y downwards and length x to the right are taken as positive. The slope dy/dx will be positive if downwards in the direction of x-positive.

(5) *End conditions.* The arbitrary constants appearing in the solution of the differential equation (1) for a given problem are found from the following end conditions :

(*i*) *At a freely supported end* (Fig. 14.23), there being no deflection and no bending moment, we have $y = 0$ and $d^2y/dx^2 = 0$.

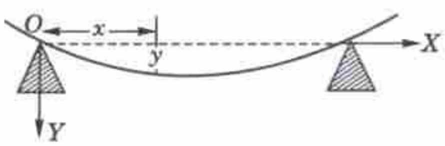

Fig. 14.23

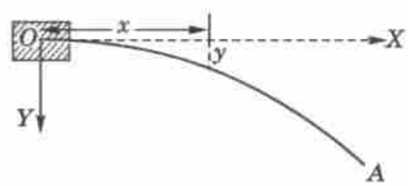

Fig. 14.24

(*ii*) *At a (horizontal) fixed end* (Fig. 14.24), the deflection and the slope of the beam being both zero, we have

$$y = 0 \text{ and } dy/dx = 0.$$

(*ii*) *At a perfectly free end* (*A* in Fig. 14.24), there being no bending moment or shear force, we have

$$\frac{d^2y}{dx^2} = 0 \quad \text{and} \quad \frac{d^3y}{dx^3} = 0$$

(6) *A member of a structure or a machine when subjected to end thrusts only is called a* **strut** *and a vertical strut is called a* **column**.

There are four possible ways of the end fixation of a strut:

(*i*) Both ends fixed, called a *built-in* or *encastre* strut.

(*ii*) One end fixed and the other freely supported, hinged or pin-jointed.

(*iii*) One end fixed and the other end free, called a *cantilever*.

(*iv*) Both ends freely supported or pin-jointed.

Example 14.13. *The deflection of a strut of length l with one end (x = 0) built-in and the other supported and subjected to end thrust P, satisfies the equation*

$$\frac{d^2y}{dx^2} + a^2y = \frac{a^2R}{P}(l - x).$$

Prove that the deflection curve is $y = \dfrac{R}{P}\left(\dfrac{\sin ax}{a} - l\cos ax + l - x\right)$, *where* $al = \tan al$.

(*U.P.T.U., 2001*)

Solution. Given differential equation is $(D^2 + a^2)y = \dfrac{a^2R}{P}(l - x)$ ...(*i*)

Its auxiliary equation is $D^2 + a^2 = 0$, whence $D = \pm\, ai$.

$$\therefore \qquad C.F. = \frac{1}{D^2 + a^2}\,\frac{a^2R}{P}(l - x) = \frac{R}{P}\left(1 + \frac{D^2}{a^2}\right)^{-1}(l - x)$$

$$= \frac{R}{P}\left(1 - \frac{D^2}{a^2} + \ldots\right)(l - x) = \frac{R}{P}(l - x)$$

Thus the complete solution of (*i*) is $y = c_1 \cos ax + c_2 \sin ax + \frac{R}{P}(l - x)$ ...(*ii*)

Also $\qquad \frac{dy}{dx} = -c_1 a \sin ax + c_2 a \cos ax - \frac{R}{P}$ ...(*iii*)

Now as the end O is built in (Fig. 14.25). $\therefore\ y = dy/dx = 0$ at $x = 0$.

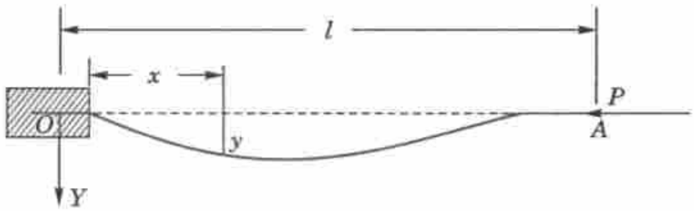

Fig. 14.25

$\therefore$ from (*ii*) and (*iii*), we have

$$0 = c_1 + Rl/P \text{ and } 0 = c_2 a - R/P$$

whence $\qquad c_1 = -Rl/P \quad \text{and} \quad c_2 = R/aP$

Thus (*ii*) becomes $\qquad y = \frac{R}{P}\left(\frac{\sin ax}{a} - l \cos ax + l - x\right)$ ...(*iv*)

which is the desired equation of the deflection curve.

The end A being freely supported $y = 0$ when $x = l$ (We don't need the other condition $d^2y/dx^2 = 0$].

$\therefore$ (*iv*) gives $\qquad 0 = \frac{R}{P}\left(\frac{\sin al}{a} - l \cos al\right)$ whence $al = \tan al$.

Example 14.14. *A horizontal tie-rod is freely pinned at each end. It carries a uniform load w lb per unit length and has a horizontal pull P. Find the central deflection and the maximum bending moment, taking the origin at one of its ends.*

Solution. Let OA be the given beam of length l (Fig.14.26).

At each end there is a vertical reaction $R = wl/2$.

The external forces acting to the left of the section NN' are : (*i*) the horizontal pull P, (*ii*) the reaction $R = wl/2$ and (*iii*) the weight of the portion $ON = wx$ acting mid-way.

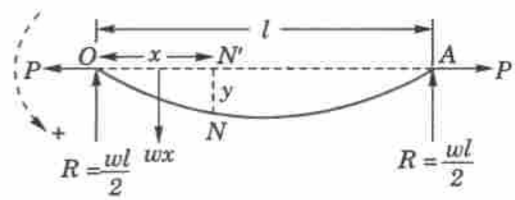

Fig. 14.26

Taking moments about, N, we have

$$EI\,\frac{d^2y}{dx^2} = Py - \frac{wl}{2}x + wx\,.\,\frac{x}{2}$$

or $\qquad EI\,\frac{d^2y}{dx^2} - Py = \frac{w}{2}(x^2 - lx) \quad \text{or} \quad \frac{d^2y}{dx^2} - a^2 y = \frac{w}{2EI}(x^2 - lx)$, where $a^2 = \frac{P}{EI}$ ...(*i*)

This is the differential equation of the elastic curve. Its auxiliary equation is $D^2 - a^2 = 0$, whence $D = \pm a$.

$\therefore \qquad$ C.F. $= c_1 \cosh ax + c_2 \sinh ax$

and $\qquad$ P.I. $= \frac{1}{D^2 - a^2}\,\frac{w}{2EI}(x^2 - lx) = \frac{-w}{2EIa^2}\left(1 - \frac{D^2}{a^2}\right)^{-1}(x^2 - lx)$

$$= -\frac{w}{2P}\left(1 + \frac{D^2}{a^2}\ldots\right)(x^2 - lx) = -\frac{w}{2P}\left(x^2 - lx + \frac{2}{a^2}\right).$$

Thus the complete solution of (*i*) is $y = c_1 \cosh ax + c_2 \sinh ax - \frac{W}{2P}\left(x^2 - lx + \frac{2}{a^2}\right)$ ...(*ii*)

At the end O, $y = 0$ when $x = 0$, $\qquad$ [We don't need the other condition $d^2y/dx^2 = 0$]

$\therefore$ (*ii*) gives $0 = c_1 - w/Pa^2$, or $c_1 = w/Pa^2$ ...(*iii*)

At the end A, $y = 0$ when $x = l$, [We don't need the other condition $d^2y/dx^2 = 0$]

$\therefore$ (*ii*) gives $0 = c_1 \cosh al + c_2 \sinh al - w/Pa^2$ or $c_2 \sinh al = \dfrac{W}{Pa^2}(1 - \cosh al)$

whence $$c_2 = -\frac{w}{Pa^2}\tanh\frac{al}{2} \quad ...(iv)$$

Substituting these values of c_1 and c_2 in (*ii*), we get

$$y = \frac{w}{Pa^2}\left(\cosh ax - \tanh\frac{al}{2}\sinh ax\right) - \frac{w}{2P}\left(x^2 - lx + \frac{2}{a^2}\right)$$

which gives the deflection of the beam at N.

Thus the central deflection $= y$ (at $x = l/2$)

$$= \frac{w}{Pa^2}\left(\cosh\frac{al}{2} - \tanh\frac{al}{2}\sinh\frac{al}{2} - 1\right) + \frac{wl^2}{8P} = \frac{w}{Pa^2}\left(\operatorname{sech}\frac{al}{2} - 1\right) + \frac{wl^2}{8P}$$

Also the bending moment is maximum at the point of maximum deflection ($x = l/2$).

$\therefore$ *The maximum bending moment*

$$= EI\frac{d^2y}{dx^2}(\text{at } x = l/2) = Py + \frac{w}{2}(x^2 - lx)\ (\text{at } x = l/2) = \frac{w}{a}\left(\operatorname{sech}\frac{al}{2} - 1\right)$$

Example 14.15. *A cantilever beam of length l and weighing w lb/unit is subjected to a horizontal compressive force P applied at the free end. Taking the origin at the free end and y-axis upwards, establish the differential equation of the beam and hence find the maximum deflection.*

Solution. Let $N(x, y)$ be any point of the beam referred to axes through the free end as shown (Fig. 14.27).

The external forces acting to the left of the section NN', are

(*i*) the compressive force P,

(*ii*) the weight of the portion $ON = wx$ acting midway.

$\therefore$ Taking moments about N, we get $EI\dfrac{d^2y}{dx^2} = -Py - wx \cdot \dfrac{x}{2}$

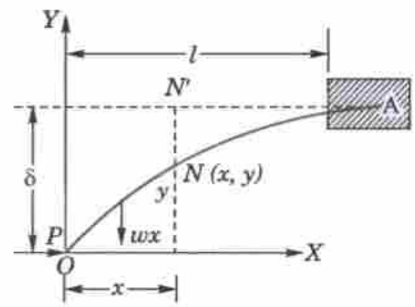

Fig. 14.27

or $$EI\frac{d^2y}{dx^2} + Py = -\frac{wx^2}{2} \quad ...(i)$$

which is the desired differential equation.

Dividing by EI and taking $P/EI = n^2$, we get

$$\frac{d^2y}{dx^2} + n^2y = -\frac{wn^2}{2P}\cdot x^2$$

Its auxiliary equation is $D^2 + n^2 = 0$, whence $D = \pm ni$.

$$\text{C.F.} = c_1 \cos nx + c_2 \sin nx$$

$\therefore$ $$\text{P.I.} = \frac{1}{D^2 + n^2}\left(-\frac{wn^2}{2P}x^2\right) = -\frac{w}{2P}\left(1 + \frac{D^2}{n^2}\right)^{-1}x^2 = -\frac{w}{2P}\left(1 - \frac{D^2}{n^2} + \dots\right)x^2 = \frac{w}{2P}\left(\frac{2}{n^2} - x^2\right)$$

Thus the complete solution of (*i*) is $y = c_1 \cos nx + c_2 \sin nx + \dfrac{w}{2P}\left(\dfrac{2}{n^2} - x^2\right)$ ...(*ii*)

The boundary conditions at the fixed end are

$x = l$, $y = \delta$, the maximum deflection and $dy/dx = 0$.

Using the first condition (*i.e.* $y = \delta$, when $x = l$), (*ii*) gives

$$\delta = c_1 \cos nl + c_2 \sin nl + \frac{w}{2P}\left(\frac{2}{n^2} - l^2\right) \quad ...(iii)$$

Differentiating (*ii*), we get $\frac{dy}{dx} = n(-c_1 \sin nx + c_2 \cos nx) - \frac{wx}{P}$.

Applying the second condition, it gives $0 = n(-c_1 \sin nl + c_2 \cos nl) - wl/P$ ...(*iv*)

Also imposing the boundary condition for the free end (*i.e.* $x = 0, d^2y/dx^2 = 0$) on

$$\frac{d^2y}{dx^2} = -n^2(c_1 \cos nx + c_2 \sin nx) - \frac{w}{P},$$

we get $$0 = -n^2c_1 - w/P, \text{ i.e., } c_1 = -w/Pn^2.$$

Substituting this value of c_1 in (*iv*), we get $c_2 = \frac{wl}{Pn} \sec nl - \frac{w}{Pn^2} \tan nl$

Thus, substituting the values of c_1 and c_2 in (*iii*), we get

the *maximum deflection* $\delta = \frac{w}{Pn^2}\left(1 - \frac{l^2n^2}{2} - \sec nl + nl \tan nl\right)$.

14.8 WHIRLING OF SHAFTS

(1) Critical or whirling speeds. A shaft seldom rotates about its geometrical axis for there is always some non-symmetrical crookedness in the shaft. In fact, the dead weight of the shaft causes some deflection which tends to become large at certain speeds. Such speeds at which the deflection of the shaft reaches a stage, where the shaft will fracture unless the speed is lowered are called the *critical or whirling speeds* of the shaft.

(2) Differential equation of the rotating shaft. Consider a shaft of weight W per unit length which is rotating with angular velocity ω.

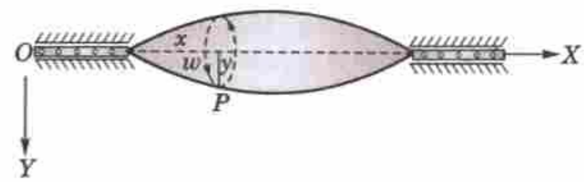

Fig. 14.28

Take its original horizontal position and the vertical downwards through the end O as the axes of x and y (Fig. 14.28). We know that for a uniformly loaded beam, the intensity of loading at $P(x, y) = EI\, d^4y/dx^4$.

$\therefore$ the restoring force (*i.e.* the internal action to oppose bending at $P(x, y) = EI\, d^4y/dx^4$.

Also the centrifugal force per unit length at $P = mr\omega^2$, *i.e.* $\frac{Wy}{g}\omega^2$.

As the restoring force arising out of the rigidity or stiffness of the shaft balances the centrifugal force which causes further deflection.

$$\therefore \qquad EI\frac{d^4y}{dx^4} = \frac{W}{g}y\omega^2 \quad \text{or} \quad \frac{d^4y}{dx^4} - a^4y = 0, \text{ where } a^4 = \frac{W\omega^2}{gEI}$$

which is the desired differential equation.

Its auxiliary equation being $D^4 - a^4 = 0$, we have

$$D = \pm a, \pm ai.$$

Hence its solution is $y = c_1e^{ax} + c_2e^{-ax} + c_3 \cos ax + c_4 \sin ax$ which may be put in the form

$$y = A \cosh ax + B \sinh ax + C \cos ax + D \sin ax.$$

(3) End conditions. To determine the arbitrary constants A, B, C, D we use the following end conditions :

(*i*) *At an end in a short or flexible bearings* (Fig. 14.29), there being no deflection and also no bending moment, we have

$$y = 0 \text{ and } \frac{d^2y}{dx^2} = 0.$$

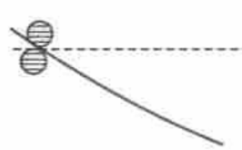

Fig. 14.29

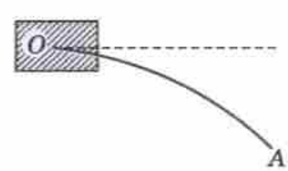

Fig. 14.30

(*ii*) *At an end in long or fixed bearings* (Fig 14.30), the deflection and the slope of the shaft being both zero, we have

$$y = 0 \text{ and } \frac{dy}{dx} = 0.$$

(*ii*) *At a perfectly free end* (such as *A* in Fig. 14.30), there being no bending moment and no shear force, we have

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} = 0.$$

Example 14.16. *The differential equation for the displacement y of a whirling shaft when the weight of the shaft is taken into account is*

$$EI\frac{d^4y}{dx^4} - \frac{W\omega^2}{g}y = W.$$

Taking the shaft of length 2l with the origin at the centre and short bearings at both ends, show that the maximum deflection of the shaft is

$$\frac{g}{2\omega^2}(\text{sech } al + \sec al - 2).$$

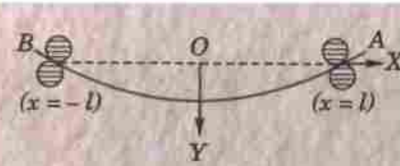

Fig. 14.31

Solution. Given differential equation can be written as

$$\frac{d^4y}{dx^4} - a^4y = \frac{W}{EI}, \text{ where } a^4 = \frac{W\omega^2}{EIg} \qquad ...(i)$$

Its C.F. $= A \cosh ax + B \sinh ax + C \cos ax + D \sin ax$

and P.I. $= \dfrac{1}{D^4 - a^4} \cdot \dfrac{W}{EI} = \dfrac{W}{EI} \cdot \dfrac{1}{D^4 - a^4} e^{0.x} = -\dfrac{W}{EIa^4} = -\dfrac{g}{\omega^2}$

Thus the complete solution of (*i*) is

$$y = A \cosh ax + B \sinh ax + C \cos ax + D \sin ax - \frac{g}{\omega^2} \qquad ...(ii)$$

Differentiating it twice, we get

$$\frac{1}{a}\frac{dy}{dx} = A \sinh ax + B \cosh ax - C \sin ax + D \cos ax$$

$$\frac{1}{a^2}\frac{d^2y}{dx^2} = A \cosh ax + B \sinh ax - C \cos ax - D \sin ax \qquad ...(iii)$$

As the end A of the shaft is in short bearings (Fig. 14.31)

$\therefore$ when $x = l$; $y = 0$, $d^2y/dx^2 = 0$

$\therefore$ from (*ii*) and (*iii*), we have

$$0 = A \cosh al + B \sinh al + C \cos al + D \sin al - \frac{g}{\omega^2} \qquad ...(iv)$$

$$0 = A \cosh al + B \sinh al - C \cos al - D \sin al \qquad ...(v)$$

Similarly *at the end B*, $x = -l$, $y = 0$, $d^2y/dx^2 = 0$.

$\therefore$ from (*ii*) and (*iii*), we get

$$0 = A \cosh al - B \sinh al + C \cos al - D \sin al - \frac{g}{\omega^2} \qquad ...(vi)$$

$$0 = A \cosh al - B \sinh al - C \cos al + D \sin al \qquad ...(vii)$$

Adding (*iv*) and (*vi*), and (*v*) and (*vii*), we get

$$A \cosh al + C \cos al = \frac{g}{\omega^2} \quad \text{and} \quad A \cosh al - C \cos al = 0.$$

whence $A = \dfrac{g}{2\omega^2 \cosh al}$ and $C = \dfrac{g}{2\omega^2 \cos al}$

Again subtracting (*vi*) from (*iv*) and (*vii*) from (*v*), we get

$$D \sinh al + D \sin al = 0 \text{ and } B \sinh al - D \sin al = 0, \text{ whence } B = 0 \text{ and } D = 0.$$

Substituting the values of A, B, C and D in (*ii*), we get

$$y = \frac{g}{2\omega^2}\left[\frac{\cosh ax}{\cosh al} + \frac{\cos ax}{\cos al} - 2\right]$$

Thus the maximum deflection = value of y at the centre (x = 0)

$$= \frac{g}{2\omega^2}(\text{sech } al + \sec al - 2).$$

Example 14.17. *The whirling speed of a shaft of length l is given by*

$$\frac{d^4y}{dx^4} - m^4y = 0 \text{ where } m^4 = \frac{W\omega^2}{gEI},$$

and y is the displacement at distance x from one end. If the ends of the shaft are constrained in long bearings, show that the shaft will whirl when cos ml cosh ml = 1.

Solution. The solution of the given differential equation is

$$y = A \cosh mx + B \sinh mx + C \cos mx + D \sin mx \quad ...(i)$$

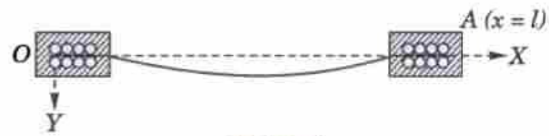

Fig. 14.32

which on differentiation gives,

$$\frac{1}{m}\frac{dy}{dx} = A \sinh mx + B \cosh mx - C \sin mx + D \cos mx \quad ...(ii)$$

As the end O of the shaft is fixed in long bearings (Fig. 14.32).

$\therefore$ when $x = 0$, $y = 0$, $dy/dx = 0$,

$\therefore$ from (*i*) and (*ii*), we have

$$0 = A + C \quad \text{or} \quad C = -A \quad ...(iii)$$

and

$$0 = B + D \quad \text{or} \quad D = -B \quad ...(iv)$$

Similarly, at the end A, $x = l$, $y = 0$, $dy/dx = 0$.

$\therefore$ From (*i*) and (*ii*), we have

$$0 = A \cosh ml + B \sinh ml + C \cos ml + D \sin ml \quad ...(v)$$

$$0 = A \sinh ml + B \cosh ml - C \sin ml + D \cos ml \quad ...(vi)$$

Substituting the values of C and D in (*v*) and (*vi*), we get

$$A(\cosh ml - \cos ml) + B(\sinh ml - \sin ml) = 0$$

and

$$A(\sinh ml + \sin ml) + B(\cosh ml - \cos ml) = 0$$

Eliminating A and B from these equations, we get

$$\frac{\cosh ml - \cos ml}{\sinh ml - \sin ml} = -\frac{B}{A} = \frac{\sinh ml + \sin ml}{\cosh ml - \cos ml}$$

or $\quad \cosh^2 ml - 2\cosh ml \cos ml + \cos^2 ml = \sinh^2 ml - \sin^2 ml$

or $\quad -2\cosh ml \cos ml + 2 = 0$ or $\cos ml \cosh ml = 1$

which must be satisfied when the shaft whirls.

The solution of this equation gives $ml = 4.73 = 3\pi/2$ radians approximately.

$$\therefore \quad \omega\sqrt{\left(\frac{W}{gEI}\right)}\, l^2 = m^2l^2 = \frac{9\pi^2}{4}$$

Thus the *whirling speed of a shaft with ends in long bearings.*

$$= \omega = \frac{9\pi^2}{4l^2}\sqrt{\left(\frac{gEI}{W}\right)} \textit{ approximately.}$$

Obs. 1. *When the shaft has one long bearing and the other short bearing*, the condition to be satisfied is $\tan ml = \tanh ml$, of which the solution is $ml = 3.927$

or $$\omega\sqrt{\left(\frac{W}{gEI}\right)}\cdot l^2 = m^2l^2 = (3.927)^2 = 15.4 \text{ nearly.}$$

Thus *the whirling speed* $= \omega = \dfrac{15.4}{l^2}\sqrt{\left(\dfrac{gEI}{W}\right)}$

Obs. 2. *When the shaft has both short bearings,* the condition to be satisfied is $\sin ml = 0$ *i.e.* $ml = \pi$ (least non-zero value).

$\therefore$ $\omega\sqrt{\left(\dfrac{W}{gEI}\right)}\cdot l^2 = m^2 l^2 = \pi^2$. Thus *the whirling speed* $= \omega = \dfrac{\pi^2}{l^2}\sqrt{\left(\dfrac{gEI}{W}\right)}$.

Obs. 3. *When the shaft has one long bearing,* the condition to the satisfied is $\cos ml \cosh ml = -1$.

Its solution gives $ml = 1.865$ [See Example 1.25]

$\therefore$ $\omega\sqrt{\left(\dfrac{W}{gEI}\right)}\cdot l^2 = m^2l^2 = (1.865)^2 = 3.5$ nearly. Thus *the whirling speed* $\omega = \dfrac{3.5}{l^2}\sqrt{\left(\dfrac{gEI}{W}\right)}$.

PROBLEMS 14.4

1. A horizontal tie-rod of length $2l$ with concentrated load W at the centre and ends freely hinged, satisfies the differential equation $EI\dfrac{d^2y}{dx^2} = Py - \dfrac{W}{2}x$. With conditions $x = 0, y = 0$ and $x = l, dy/dx = 0$, prove that the deflection δ and the bending moment M at the centre $(x = l)$ are given by $\delta = \dfrac{W}{2Pn}(nl - \tanh nl)$ and $M = -\dfrac{W}{2n}\tanh nl$, where $n^2EI = P$.
2. A light horizontal strut AB is freely pinned at A and B. It is under the action of equal and opposite compressive forces P at its ends and it carries a load W at its centre. Then for $0 < x < l/2$, $EI\dfrac{d^2y}{dx^2} + Py + \dfrac{1}{2}Wx = 0$. Also $y = 0$ at $x = 0$ and $dy/dx = 0$ at $x = l/2$.

   Prove that $y = \dfrac{W}{2P}\left(\dfrac{\sin nx}{n\cos nl/2} - x\right)$ where $n^2 = \dfrac{P}{EI}$.
3. A uniform horizontal strut of length l freely supported at both ends, carries a uniformly distributed load W per unit length. If the thrust at each end is, P, prove that the maximum deflection is $\dfrac{W}{Pa^2}\left(\sec\dfrac{al}{2} - 1\right) - \dfrac{Wl^2}{8P}$, where $\dfrac{P}{EI} = a^2$.

   Prove also that the maximum bending moment is of the magnitude $\dfrac{W}{a^2}\left(\sec\dfrac{al}{2} - 1\right)$.
4. The shape of a strut of length l subjected to an end thrust P and lateral load w per unit length, when the ends are built in, is given by $EI\dfrac{d^2y}{dx^2} + Py = \dfrac{wx^2}{2} - \dfrac{wlx}{2} + M$, where M is the moment at a fixed end. Find y in terms of x, given that $y = 0$, $dy/dx = 0$ at $x = 0$ and $dy/dx = 0$ at $x = l/2$.
5. A light horizontal strut of length l is clamped at one end carries a vertical load W at the free end. If the horizontally thrust at the free end is P, show that the strut satisfies the differential equation

   $EI\dfrac{d^2y}{dx^2} = (\delta - y)P + W(l - x)$, where y is the displacement of a point at a distance x from the fixed end and δ, the deflection at the free end.

   Prove that the deflection at the free end is given by $\dfrac{W}{nP}(\tan nl - nl)$, where $n^2EI = P$.
6. A long column fixed at one end $(x = 0)$ and hinged at the other $(x = l)$ is under the action of axial load P. If a force F is applied laterally at the hinge to prevent lateral movement, show that it satisfies the equation $\dfrac{d^2y}{dx^2} + n^2y = \dfrac{En^2}{P}(l - x)$, where $EIn^2 = P$. Hence determine the equation of the deflection curve.
7. A long column of length l is fixed at one end and is completely free at the other end. If y is the lateral deflection at a point distance x from the fixed end, when load P is axially applied, find the differential equation satisfied by x and y. Show that the deflection curve is given by $y = a\{1 - \cos\sqrt{P/EI}\,x\}$ and find the least value of the critical load (a is the lateral deflection of the free end).

8. The differential equation for the displacement y of a heavy whirling shaft is $\frac{d^4y}{dx^4} = a^4\left(y + \frac{g}{\omega^2}\right)$, where $a^4 = \frac{W\omega^2}{gEI}$. If both ends are in short bearings, the ends being $x = 0$ and $x = l$, find the bending moment of the centre of the shaft.

14.9 APPLICATIONS OF SIMULTANEOUS LINEAR EQUATIONS

So far we have considered engineering systems having only one degree of freedom. The analysis of a system having more than one degree of freedom depends on the solution of simultaneous linear equations. In fact such equations form the basis of the theory of projectiles and the coupled circuits having self and mutual inductance. The details of such applications are best explained through the following examples :

Example 14.18. Projectile with resistance. *Find the path of a particle projected with a velocity v at an angle α to the horizon in a medium whose resistance, apart from gravity, varies as velocity. Also find the greatest height attained.*

Solution. Let the axes of x and y be respectively horizontal and vertical with origin at the point of projection (Fig. 14.33).

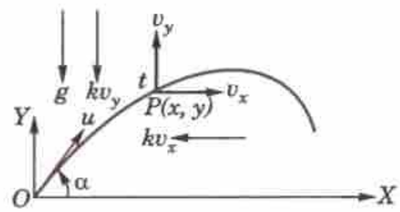

Fig. 14.33

Let $P(x, y)$ be the position of the projectile at the time t, where the velocity components parallel to the axes are

$$v_x = \frac{dx}{dt}, v_y = \frac{dy}{dt}$$

∴ the equations of motion are:

Parallel to x-axis	*Parallel to y-axis*
$m\frac{dv_x}{dt} = -mkv_x$	$m\frac{dv_y}{dt} = -mg - mkv_y$
or $\frac{dv_x}{dt} = -kv_x$	$\frac{dv_y}{dt} = -(g + kv_y)$

Separating the variables and integrating, we have

$\int \frac{dv_x}{v_x} = -k\int dt + c_1$	$\int \frac{dv_y}{g + kv_y} = -\int dt + c_2$
or $\log v_x = -kt + c_1$	$\frac{1}{k}\log(g + kv_y) = -t + c_2.$

Initially when $t = 0$, $v_x = u\cos\alpha$, $v_y = u\sin\alpha$.

$\log u\cos\alpha = c_1$	$\frac{1}{k}\log(g + ku\sin\alpha) = c_2$

Subtracting,

$\log\left(\frac{v_x}{u\cos\alpha}\right) = -kt$	$\frac{1}{k}\log\left(\frac{g + kv_y}{g + ku\sin\alpha}\right) = -t$
or $\frac{dx}{dt} = v_x = u\cos\alpha e^{-kt}$ ...(i)	$\frac{dy}{dt} = v_y = \frac{1}{k}[(g + ku\sin\alpha)e^{-kt} - g]$ ...(ii)

Again integrating, we get

$$x = \frac{u\cos\alpha}{-k}e^{-kt} + c_3, \quad y = -\frac{1}{k}\left(\frac{g}{k} + u\sin\alpha\right)e^{-kt} - \frac{g}{k}t + c_4$$

Initially when $t = 0, x = 0, y = 0$,

$$\therefore \quad 0 = \frac{u\cos\alpha}{k} + c_3, \quad 0 = -\frac{1}{k}\left(\frac{g}{k} + u\sin\alpha\right) + c_4$$

Subtracting, we get $\quad x = \frac{u\cos\alpha}{k}(1 - e^{-kt}) \quad$ ...(iii)

$$y = \frac{1}{k}\left(\frac{g}{k} + u \sin \alpha\right)\left(1 - e^{-kt}\right) - \frac{gt}{k} \qquad ...(iv)$$

Eliminating t from (iii) and (iv), we obtain $y = \left(\frac{g}{k} + u \sin \alpha\right)\frac{x}{u \cos \alpha} + \frac{g}{k^2} \log\left(1 - \frac{kx}{u \cos \alpha}\right)$

which is the required equation of the trajectory.

The projectile will attain the greatest height when $dy/dt = 0$.

i.e., when $e^{-kt} = g/(g + ku \sin \alpha)$, *i.e.*, at time $t = \frac{1}{k} \log\left(1 + \frac{ku \sin \alpha}{g}\right)$. [From (ii)]

Subtituting the value of t in (iv), we get the greatest height attained

$$(= y) = \frac{u \sin \alpha}{k} - \frac{g}{k^2} \log\left(1 + \frac{ku \sin \alpha}{g}\right).$$

Example 14.19. *Two particles each of mass m gm are suspended from two springs of same stiffness k as in Fig. 14.34. After the system comes to rest, the lower mass is pulled l cm downwards and released. Discuss their motion.*

Solution. Let x and y denote the displacement of the upper and lower masses at time t from their respective positions of equilibrium.

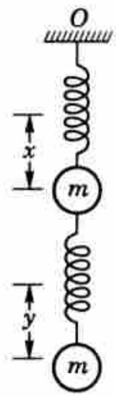

Fig. 14.34

Then the stretch of the upper spring is x and that of the lower spring is $y - x$.

$\therefore$ the restoring force acting on the upper mass

$$= -kx + k(y - x) = k(y - 2x)$$

and that on the lower mass $= -k(y - x)$.

Thus their equations of motion are

$$m \frac{d^2x}{dt^2} = k(y - 2x) \text{ and } m \frac{d^2y}{dt^2} = -k(y - x)$$

or $\quad (mD^2 + 2k)x - ky = 0 \qquad ...(i)$

and $\quad (mD^2 + k)y - kx = 0 \qquad ...(ii)$

Operating (i) by $(mD^2 + k)$ and adding to k times (ii), we get

$$[(mD^2 + k)(mD^2 + 2k) - k^2]\, x = 0 \text{ or } (D^4 + 3\lambda D^2 + \lambda^2)x = 0, \text{ where } \lambda^2 = k/m.$$

Its auxiliary equation is $D^4 + 3\lambda D^2 + \lambda^2 = 0$

which gives $\quad D^2 = \dfrac{-3\lambda \pm \sqrt{(9\lambda^2 - 4\lambda^2)}}{2} = -2.62\lambda \text{ or } -0.38\lambda = -\alpha^2, -\beta^2$ (say)

so that $D = \pm i\alpha, \pm i\beta$.

Thus $\quad x = c_1 \cos \alpha t + c_2 \sin \alpha t + c_3 \cos \beta t + c_4 \sin \beta t \qquad ...(iii)$

Also from (i), $y = \left(\frac{D^2}{\lambda} + 2\right)x = (2 - \alpha^2/\lambda)(c_1 \cos \alpha t + c_2 \sin \alpha t) + (2 - \beta^2/\lambda)(c_3 \cos \beta t + c_4 \sin \beta t) \qquad ...(iv)$

Initially when $\quad t = 0, x = y = l, dx/dt = dy/dt = 0$.

$\therefore$ from (iii), $\quad l = c_1 + c_3;\ 0 = \alpha c_2 + \beta c_4$

and from (iv) $\quad l = (2 - \alpha^2/\lambda)c_1 + (2 - \beta^2/\lambda)c_3$ and $0 = (2 - \alpha^2/\lambda)\alpha c_2 + (2 - \beta^2/\lambda)\beta c_4$

whence $\quad c_1 = \dfrac{l(\lambda - \beta^2)}{\alpha^2 - \beta^2},\ c_3 = \dfrac{l(\lambda - \alpha^2)}{\beta^2 - \alpha^2},\ c_3 = c_4 = 0.$

Substituting these values of constants in (iii) and (iv), we get x and y which show that the motion of the spring is a combination of two simple harmonic motions of periods $2\pi/\alpha$ and $2\pi/\beta$.

Example 14.20. *Two coils of a transformer are identical with resistance R, inductance L, mutual inductance M and a voltage E is impressed on the primary. Determine the currents in the coils at any instant, assuming that there is no current in either initially.*

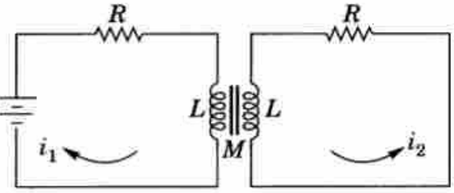

Fig. 14.35

Solution. Let i_1, i_2 ampere be the currents flowing through the primary and secondary coils at time t sec (Fig. 14.35). Then by Kirchhoff's law, we know that sum of the voltage drops across R, L and $M =$ applied voltage.

$\therefore$ for the primary circuit,

$$Ri_1 + L\frac{di_1}{dt} + M\frac{di_2}{dt} = E$$

and for the secondary circuit, $Ri_2 + L\frac{di_2}{dt} + M\frac{di_1}{dt} = 0.$

Replacing d/dt by D and rearranging the terms,

$$(LD + R)i_1 + MDi_2 = E \quad ...(i)$$

$$MDi_1 + (LD + R)i_2 = 0 \quad ...(ii)$$

Eliminating i_2, we get $[(LD + R)^2 - M^2D^2]i_1 = (LD + R)E$

i.e., $$[(L^2 - M^2)D^2 + 2LRD + R^2]i_1 = RE \quad ...(iii)$$

Its auxiliary equation is $(L^2 - M^2)D^2 + 2LRD + R^2 = 0$ whence $D = \dfrac{-R}{L+M}, \dfrac{-R}{L-M}$.

As L is usually $> M$, therefore, both values of D are negative and real.

$\therefore$ $$\text{C.F.} = c_1 e^{-\frac{Rt}{L+M}} + c_2 e^{-\frac{Rt}{L-M}} \text{ and P.I.} = RE \cdot \frac{1}{(L^2 - M^2)D^2 + 2LRD + R^2} e^{0.t} = E/R.$$

Thus the complete solution of (*iii*) is $i_1 = c_1 e^{-Rt/(L+M)} + c_2 e^{-Rt/(L-M)} + E/R$ ...(*iv*)

and from (*ii*), we have $i_2 = -\dfrac{MD}{LD+R} i_1$

$$= -\frac{MD}{LD+R}\left(c_1 e^{-Rt/(L+M)} + c_2 e^{-Rt/(L-M)}\right) - \frac{MD}{LD+R}\left(\frac{E}{R}\right)$$

$$= -\frac{Mc_1}{L\left(\frac{-R}{L+M}\right)+R} \cdot De^{-Rt/(L+M)} - \frac{Mc_2}{L\left(\frac{-R}{L-M}\right)+R} \cdot De^{-Rt/(L-M)}$$

$$= c_1 e^{-Rt/(L+M)} - c_2 e^{-Rt/(L-M)}$$

Initially, when $t = 0, i_1 = i_2 = 0.$

$\therefore$ $c_1 + c_2 = -E/R, c_1 - c_2 = 0$ $\quad \therefore c_1 = c_2 = -E/2R.$

Substituting the values of c_1, c_2 in (*iv*) and (*v*), we get

$$i_1 = \frac{E}{2R}\left[2 - e^{-Rt/(L+M)} - e^{-Rt/(L-M)}\right] \quad ...(vi)$$

and $$i_2 = \frac{E}{2R}\left[e^{-Rt/(L-M)} - e^{-Rt/(L+M)}\right] \quad ...(vii)$$

Thus (*vi*) and (*vii*) give the currents at any instant.

PROBLEMS 14.5

1. A particle is projected with velocity u, at an elevation α. Neglecting air resistance, show that the equation to its path is the parabola $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$. Also find the time of flight and range on the horizontal plane.
2. An inclined plane makes angle α with the horizontal. A projectile is launched from the bottom of the inclined plane with speed V in a direction making angle β with the horizontal. Set up the differential equations and find (*i*) the range on the incline, (*ii*) the maximum range up the incline.
3. A particle of unit mass is projected with velocity u at an inclination α above the horizon in a medium whose resistance is k times the velocity. Show that its direction will again make an angle α with the horizon after a time $\dfrac{1}{k}\log\left\{1 + \dfrac{2ku}{g}\sin\alpha\right\}$.

4. A particle moving in a plane is subjected to a force directed towards a fixed point O and proportional to the distance of the particle from O. Show that the differential equations of motion are of the form $\frac{d^2x}{dt^2} = -k^2x$, $\frac{d^2y}{dt^2} = -k^2y$. Find the cartesian equation of the path of the particle if $x = 1, y = 0, \frac{dx}{dt} = 0$ and $dy/dt = 2$, when $t = 0$.
5. The currents i_1 and i_2 in mesh are given by the differential equations
$\frac{di_1}{dt} - \omega i_2 = a \cos pt$, $\frac{di_2}{dt} + \omega i_1 = a \sin pt$. Find the currents i_1 and i_2 if $i_1 = i_2 = 0$ at $t = 0$.
6. The currents i_1 and i_2 in two coupled circuits are given by
$$L\frac{di_1}{dt} + Ri_1 + R(i_1 - i_2) = E;\ L\frac{di_2}{dt} + Ri_2 - R(i_1 - i_2) = 0,$$
where L, R, E are constants. Find i_1 and i_2 in terms of t given that $i_1 = i_2 = 0$ at $t = 0$.
7. The motion of a particle is governed by the equations
$\frac{d^2x}{dt^2} - n\frac{dy}{dt} = 0, \frac{d^2y}{dt^2} + n\frac{dx}{dt} = n^2a$, when $x = y = \frac{dx}{dt} = \frac{dy}{dt} = 0$ at $t = 0$. Find x and y in terms of t.
8. Under certain conditions, the motion of an electron is given by the equations $m\frac{d^2x}{dt^2} + eH\frac{dy}{dt} = eE$ and $m\frac{d^2y}{dt^2} - eH\frac{dx}{dt} = 0$. Find the path of the electron, if it started from rest at the origin.
9. The voltage V and the current i at a distance x from the source satisfy the equations $-dV/dt = Ri, -di/dx = GV$, where R, G are constants. If $V = V_0$ at $x = 0$ and $V = 0$ at the receiving end $x = l$, show that $V = V_0 \sinh n(l - x)/\sinh nl$, $i = V_0\sqrt{(G/R)}.\cosh n(l - x)/\sinh nl$, where $n^2 = RG$.

14.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 14.6

Fill up the blanks or choose the correct answer in the following problems:

1. A particle executing simple harmonic motion of amplitude 5 cm has a speed of 8 cm/sec when at a distance of 3 cm from the centre of the path. The period of the motion of the particle will be
(a) $\pi/2$ sec (b) π sec (c) 2π sec (c) 4π sec.
2. A ball of mass m is suspended from a fixed point O by a light string of natural length l and modulus of elasticity λ. If the ball is displaced vertically, its motion will be S.H.M. of period
(a) $2\pi\sqrt{(m/\lambda l)}$ (b) $2\pi\sqrt{(ml/\lambda)}$ (c) $2\pi\sqrt{(l/m\lambda)}$ (d) $2\pi\sqrt{(\lambda m/l)}$.
3. The periodic time of the motion described by the differential equation $\frac{d^2x}{dt^2} + 4x = 0$ is
(a) $\pi/2$ (b) π (c) 2π.
4. A particle is projected with a velocity u at an angle of 60° to the horizontal. The time of flight of the projectile is equal to
(a) $\sqrt{3u/2g}$ (b) $\sqrt{3u/g}$ (c) u/g (d) $u/2g$.
5. A body of 6.5 kg is suspended by two strings of lengths 5 and 12 metres attached to two points in the same horizontal line whose distance apart is 13 meters. The tension of the strings are
(a) 2 kg & 6.5 kg (b) 2.5 kg & 6 kg (c) 2.25 kg & 6.25 kg (d) 3 kg & 5.5 kg.
6. A particle is projected at an angle of 30° to the horizontal with a velocity of 1962 cm/sec then the time of flight is
(a) 1 sec (b) 2 sec (c) 2.5 sec (d) 3 sec.
7. A point moves with S.H.M. whose period is 4 seconds. If it starts from rest at a distance of 4 meters from the centre of its path, then the time it takes, before it has described 2 metres is
(a) $\frac{1}{3}$ second (b) $\frac{2}{3}$ second (c) $\frac{3}{4}$ second (d) $\frac{4}{5}$ second.

8. If the length of the pendulum of a clock be increased in the ratio 720 : 721, it would loose .... seconds per day.
9. The frequency of free vibrations in a closed circuit with inductance L and capacity C in series is .... per minute.
10. If a clock with a seconds pendulum loses 10 seconds per day at a place having $g = 32$ ft/sec^2, g should be increased by .... ft/sec^2, to keep correct time.
11. The soldiers break step while marching over a bridge for the fear that their steps may not be in rhym with the natural frequency of the bridge causing its collapse due to ....
12. A horizontal tie-rod is freely pinned at each end. If it carries a uniform load w lb per unit length and has a horizontal pull P, then the differential equation of the elastic curve is ....
13. The conditions for an end of a whirling shaft to be in fixed bearings are .... and ....