CHAPTER 34

Linear Programming

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34.1 INTRODUCTION

Linear programming deals with the optimization (maximization or minimization) of linear functions subject to linear constraints. This technique has found its applications to important areas of product mix, blending problems and diet problems. Oil refineries, chemical industries, steel industries and food processing industry are also using linear programming with considerable success.

In this chapter, our purpose is to present the principles of linear programming and the techniques of its application in a manner that will suit the engineering students. Beginning with the graphical method which provides a great deal of insight into the basic concepts, the simplex method of solving linear programming problems is developed. Then the reader is introduced to the Duality concept. Finally a special class of linear programming problems namely : Transportation and Assignment problems, is taken up. For a detailed study, the student should refer to author's book *'Numerical Methods in Engineering and Science'*.

34.2 FORMULATION OF THE PROBLEM

To begin with, a problem is to be presented in a linear programming form which requires defining the variables involved, establishing relationships between them and formulating the objective function and the constraints. We illustrate this through a few examples.

Example 34.1. *A manufacturer produces two types of models M_1 and M_2. Each M_1 model requires 4 hours of grinding and 2 hours of polishing ; whereas each M_2 model requires 2 hours of grinding and 5 hours of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works for 40 hours a week and each polisher works for 60 hours a week. Profit on an M_1 model is ₹ 3 and on an M_2 model is ₹ 4. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models so that he may make the maximum profit in a week.*

Solution. Let x_1 be the number of M_1 models and x_2, the number of M_2 models produced per week. Then the weekly profit (in ₹) is

$$Z = 3x_1 + 4x_2 \qquad ...(i)$$

To produce these number of models, the total number of grinding hours needed per week

$$= 4x_1 + 2x_2$$

and the total number of polishing hours required per week

$$= 2x_1 + 5x_2$$

Since the number of grinding hours available is not more than 80 and the number of polishing hours is not more than 180, therefore,

$$4x_1 + 2x_2 \le 80 \quad ...(ii)$$

$$2x_1 + 5x_2 \le 180 \quad ...(iii)$$

Also since the negative number of models are not produced, obviously we must have

$$x_1 \ge 0 \text{ and } x_2 \ge 0 \quad ...(iv)$$

Hence this allocation problem is, to find x_1, x_2 which

maximize $Z = 3x_1 + 4x_2$

subject to $4x_1 + 2x_2 \le 80,\ 2x_1 + 5x_2 \le 180,\ x_1, x_2 \ge 0.$

Obs. *The variables that enter into the problem are called* **decision variables.**

The expression (i) showing the relationship between the manufacturer's goal and the decision variables, is called the **objective function.**

The inequalities (ii), (iii) and (iv) are called the **constraints**.

The objective function and the constraints being all linear, it is a *linear programming problem* (*L.P.P*). This is an example of a real situation from industry.

Example 34.2. *A firm making castings uses electric furnace to melt iron with the following specifications :*

	Minimum	*Maximum*
Carbon	*3.20%*	*3.40%*
Silicon	*2.25%*	*2.35%*

Specifications and costs of various raw materials used for this purpose are given below :

Material	*Carbon %*	*Silicon %*	*Cost (₹)*
Steel scrap	*0.4*	*0.15*	*850/tonne*
Cast iron scrap	*3.80*	*2.40*	*900/tonne*
Remelt from foundry	*3.50*	*2.30*	*500/tonne*

If the total charge of iron metal required is 4 tonnes, find the weight in kg of each raw material that must be used in the optimal mix at minimum cost. (*J.N.T.U., 1999 S*)

Solution. Let x_1, x_2, x_3 be the amounts (in kg) of these raw materials. The objective is to minimize the cost *i.e.*,

minimize $$Z = \frac{850}{1000} x_1 + \frac{900}{1000} x_2 + \frac{500}{1000} x_3 \quad ...(i)$$

For iron melt to have a minimum of 3.2% carbon,

$$0.4x_1 + 3.8x_2 + 3.5x_3 \ge 3.2 \times 4{,}000 \quad ...(ii)$$

For iron melt to have a maximum of 3.4% carbon,

$$0.4x_1 + 3.8x_2 + 3.5x_3 \le 3.4 \times 4{,}000 \quad ...(iii)$$

For iron melt to have a minimum of 2.25% silicon,

$$0.15x_1 + 2.41x_2 + 2.35x_3 \ge 2.25 \times 4{,}000 \quad ...(iv)$$

For iron melt to have a maximum of 2.35% silicon,

$$0.15x_1 + 2.41x_2 + 2.35x_3 \le 2.35 \times 4{,}000 \quad ...(v)$$

Also, since the materials added up must be equal to the full charge weight of 4 tonnes.

$$\therefore \quad x_1 + x_2 + x_3 = 4{,}000 \quad ...(vi)$$

Finally since the amounts of raw material cannot be negative ...(*vii*)

$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 \quad ...(vii)$$

Thus the linear programming problem is to find x_1, x_2, x_3 which

minimize $Z = 0.85x_1 + 0.9x_2 + 0.5x_3$

subject to $0.4x_1 + 3.8x_2 + 3.5x_3 \ge 12{,}800$

$0.4x_1 + 3.8x_2 + 3.5x_3 \le 13{,}600$

$$0.15x_1 + 2.41x_2 + 2.35x_3 \geq 9{,}000$$
$$0.15x_1 + 2.41x_2 + 2.35x_3 \leq 9{,}400$$
$$x_1 + x_2 + x_3 = 4{,}000$$
$$x_1, x_2, x_3 \geq 0.$$

PROBLEMS 34.1

1. A firm manufactures two items. It purchases castings which are then machined, bored and polished. Castings for items A and B cost ₹ 3 and ₹ 4 each and are sold at ₹ 6 and ₹ 7 each respectively. Running costs of these machines are ₹ 20, ₹ 14 and ₹ 17.50 per hour respectively. Formulate the problem so that the product mix maximizes the profit? Capacities of the machines are

	Part A	*Part B*
Machining capacity	25 per hr.	40 per hr.
Boring capacity	28 per hr.	35 per hr.
Polishing capacity	35 per hr.	25 per hr.

2. A firm manufactures 3 products A, B and C. The profits are ₹ 3, ₹ 2 and ₹ 4 respectively. The firm has two machines M_1 and M_2 and below is the required processing time in minutes for each machine on each product.

		Product		
		A	B	C
Machine	M_1	4	3	5
	M_2	2	2	4

Machines M_1 and M_2 have 2000 and 2500 machine-minutes respectively. The firm must manufacture 100 A's, 200 B's and 50 C's but not more than 150 A's. Set up an L.P.P. to maximize profit. *(Kurukshetra, 2009 S)*

3. Three products are processed through three different operations. The time (in minutes) required per unit of each product, the daily capacity of the operations (in minutes per day) and the profit per unit sold for each product (in rupees) are as follows:

Operation	*Time per unit*			*Operation capacity*
	Product I	*Product II*	*Product III*	
1	3	4	3	42
2	5	0	3	45
3	3	6	2	41
Profit (₹)	3	2	1	

The zero time indicates that the product does not require the given operation. The problem is to determine the optimum daily production for three products that maximize the profit. Formulate this production planning problem as a linear programming problem assuming that all units produced are sold.

4. An aeroplane can carry a maximum of 200 passengers. A profit of ₹ 400 is made on each first class ticket and a profit of ₹ 300 is made on each economy class ticket. The airline reserves at least 20 seats for first class. However, at least 4 times as many passengers prefer to travel by economy class than by the first class. How many tickets of each class must be sold in order to maximize profit for the airline? Formulate the problem as an L.P. model. *(Rohtak, 2006)*

5. A firm manufactures headache pills in two sizes A and B. Size A contains 2 grains of asprin, 5 grains of bicarbonate and 1 grain of codeine. Size B contains 1 grain of asprin, 8 grains of bicarbonate and 6 grains of codeine. It is found by users that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard L.P.P.

6. Consider the following problem faced by a production planner in a soft-drink plant. He has two bottling machines A and B. A is designed for 8-ounce bottles and B for 16-ounce bottles. However, each be can used on both types with some loss of efficiency. The following data is available:

Machine	*8-ounce bottles*	*16-ounce bottles*
A	100/minute	40/minute
B	60/minute	75/minute

The machines can be run 8 hour per day, 5 days per week. Profit on a 8-ounce bottle is 15 paise and on a 16-ounce bottle is 25 paise. Weekly production of the drink cannot exceed 300,000 ounces and the market can absorb 25,000

8-ounce bottles and 7,000 16-ounce bottles per week. The planner wishes to maximise his profit subject, of course, to all the production and marketing restrictions. Formulate this as a L.P.P.

7. A dairy feed company may purchase and mix one or more of three types of grains containing different amounts of nutritional elements. The data is given in the table below. The production manager specifies that any feed mix for his live stock must meet at least minimum nutritional requirements and seeks the least costly among all three mixes.

	Item	*One unit weight of* Grain 1	Grain 2	Grain 3	*Minimum requirement*
Nutritional ingredients	*A*	2	3	7	1,250
	B	1	1	0	250
	C	5	3	0	900
	D	6	25	1	232.5
Cost per weight of		41	35	96	

Formulate the problem as a L.P. model.

8. A firm produces an alloy with the following specifications:

(*i*) specific gravity ≤ 0.97; (*ii*) chromium content $\geq 15\%$; (*iii*) melting temperature $\geq 494°C$

The alloy requires three raw materials *A*, *B* and *C* whose properties are as follows:

Property	*Properties of raw material* A	B	C
Sp. gravity	0.94	1.00	1.05
Chromium	10%	15%	17%
Melting pt.	470°C	500°C	520°C

Find the values of *A*, *B*, *C* to be used to make 1 tonne of alloy of desired properties, keeping the raw material costs at the minimum when they are ₹ 105/tonne for *A*, ₹ 245/tonne for *B* and ₹ 165/tonne for *C*. Formulate an L.P. model for the problem.

34.3 GRAPHICAL METHOD

Linear programming problems involving only two variables can be effectively solved by a graphical technique which provides a pictorial representation of the solution and one gets insight into the basic concepts used in solving large L.P.P.

Working procedure *to solve a linear programming problem graphically:*

Step 1. Formulate the given problem as a linear programming problem.

Step 2. Plot the given constraints as equalities on $x_1 - x_2$ coordinate plane and determine the convex region* formed by them.

Step 3. Determine the vertices of the convex region and find the value of the objective function at each vertex. The vertex which gives the optimal (maximum or minimum) value of the objective function gives the desired optimal solution to the problem.

Otherwise. Draw the dotted line through the origin representing the objective function with $Z = 0$. As Z is increased from zero, this line moves to the right remaining parallel to itself. We go on sliding this line (parallel to itself), till it is *farthest* away from the origin and passes through only one vertex of the convex region. This is the vertex where the maximum value of Z is attained.

* *A* **region** *or a* **set** *of points is said to be* **convex** *if the line joining any two of its points lies completely in the region (or the set).* Figs. 34.1 and 34.2 represent convex regions while Figs. 34.3 and 34.4 do not form convex sets.

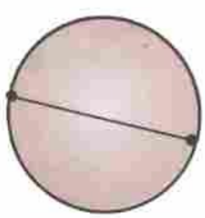

Fig. 34.1

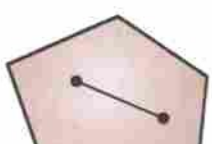

Fig. 34.2

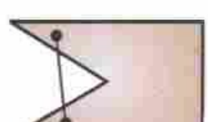

Fig. 34.3

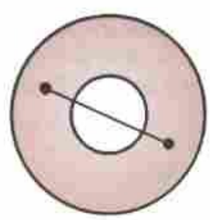

Fig. 34.4

When it is required to minimize Z, value of Z is increased till the dotted line passes through the *nearest* vertex of the convex region.

Example 34.3. *Solve the L.P.P. of Ex. 34.1 graphically.* (*V.T.U., 2003*)

Solution. The problem is:

Maximize $\quad Z = 3x_1 + 4x_2 \quad$ ...(*i*)

subject to $\quad 4x_1 + 2x_2 \le 80 \quad$ ...(*ii*)

$2x_1 + 5x_2 \le 180 \quad$ ...(*iii*)

$x_1, x_2 \ge 0 \quad$ ...(*iv*)

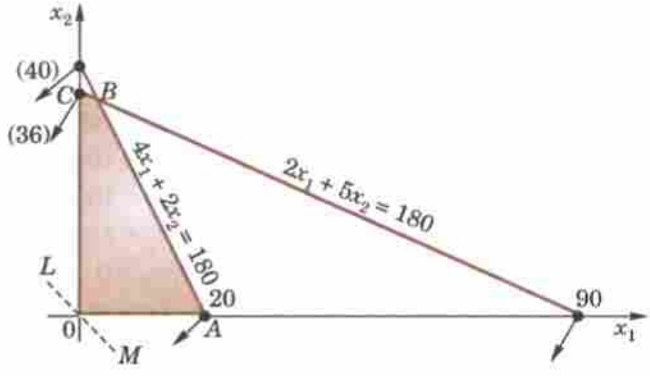

Fig. 34.5

Consider $x_1 - x_2$ coordinate system as shown in Fig. 34.5. The non-negativity restrictions (*iv*) imply that the values of x_1, x_2 lie in the first quadrant only.

We plot the lines $4x_1 + 2x_2 = 80$ and $2x_1 + 5x_2 = 180$.

Then any point on or below $4x_1 + 2x_2 = 80$ satisfies (*ii*) and any point on or below $2x_1 + 5x_2 = 180$ satisfies (*iii*). This shows that the desired point (x_1, x_2) must be somewhere in the shaded convex region $OABC$. This region is called the *solution space* or *region of feasible solutions* for the given problem. Its vertices are $O(0, 0)$, $A(20, 0)$ $B(2.5, 35)$ and $C(0, 36)$.

The values of the objective function (*i*) at these points are

$Z(O) = 0$, $Z(A) = 60$, $Z(B) = 147.5$, $Z(C) = 144$.

Thus the maximum value of Z is 147.5 and it occurs at B. Hence the optimal solution to the problem is

$$x_1 = 2.5, x_2 = 35 \text{ and } Z_{max} = 147.5.$$

Otherwise. Our aim is to find the point (or points) in the solution space which maximizes the profit function Z. To do this, we observe that on making $Z = 0$, (*i*) becomes $3x_1 + 4x_2 = 0$ which is represented by the dotted line LM through O. As the value of Z is increased, the line LM starts moving parallel to itself towards the right. Larger the value of Z, more will be the company's profit. In this way, we go on sliding LM till it is farthest away from the origin and passes through one of the corners of the convex region. This is the point where the maximum value of Z is attained. Just possible, such a line may be one of the edges of the solution space. In that case every point on that edge gives the same maximum value of Z.

Here Z_{max} is attained at $B(2.5, 35)$. Hence the optical solution is $x_1 = 2.5$, x_2 35 and $Z_{max} = 147.5$.

Example 34.4. *Find the maximum value of $Z = 2x + 3y$ subject to the constraints: $x + y \le 30$, $y \ge 3, 0 \le y \ge 12$, $x - y \ge 0$, and $0 \le x \le 20$.* (*Rohtak, 2006*)

Solution. Any point (x, y) satisfying the conditions $x \ge 0, y \ge 0$ lies in the first quadrant only. Also since $x + y \le 30$, $y \ge 3$, $y \le 12$, $x \ge y$ and $x \le 20$, the desired point (x, y) lies within the convex region $ABCDE$ (shown shaded in Fig. 34.6). Its vertices are $A(3, 3)$, $B(20, 3)$, $C(20, 10)$, $D(18, 12)$, and $E(12, 12)$.

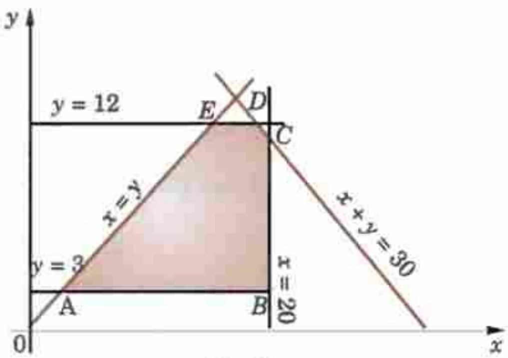

Fig. 34.6

The values of Z at these five vertices are $Z(A) = 15$, $Z(B) = 49$, $Z(C) = 70$, $Z(D) = 72$, and $Z(E) = 60$.

Since the maximum value of Z is 72 which occurs at the vertex D, the solution to the L.P.P. is $x = 18$, $y = 12$ and maximum $Z = 72$.

Example 34.5. *A company manufactures two types of cloth, using three different colours of wool. One yard length of type a cloth requires 4 oz of red wool, 5 oz of green wool and 3 oz of yellow wool. One yard length of type B cloth requires 5 oz of red wool, 2 oz of green wool and 8 oz of yellow wool. The wool available for manufacturer is 1000 oz of red wool, 1000 oz of green wool and 1200 oz of yellow wool. The manufacturer can make a profit of ₹ 5 on one yard of type A cloth and ₹ 3 on one yard of type B cloth. Find the best combination of the quantities of type A and type B cloth which gives him maximum profit by solving the L.P.P. graphically.*

Solution. Let the manufacturer decide to produce x_1 yards of type A cloth and x_2 yards of type B cloth. Then the total income in rupees, from these units of cloth is given by

$$Z = 5x_1 + 3x_2 \quad ...(i)$$

To produce these units of two types of cloth, he requires

$$\text{red wool} = 4x_1 + 5x_2 \text{ oz},$$

$$\text{green wool} = 5x_1 + 2x_2 \text{ oz},$$

and

$$\text{yellow wool} = 3x_1 + 8x_2 \text{ oz}.$$

Since the manufacturer does not have more than 1000 oz of red wool, 1000 oz of green wool and 1200 oz of yellow wool, therefore

$$4x_1 + 5x_2 \le 1000 \quad ...(ii)$$

$$5x_1 + 2x_2 \le 1000 \quad ...(iii)$$

$$3x_1 + 8x_2 \le 1200 \quad ...(iv)$$

Also $$x_1 \ge 0, x_2 \ge 0. \quad ...(v)$$

Thus the given problem is to maximize Z subject to the constraints (*ii*) to (*v*). (*V.T.U., 2004*)

Any point satisfying the condition (*v*) lies in the first quadrant only. Also the desired point satisfying the constraints (*ii*) to (*iv*) lies in the convex region $OABCD$ (Fig. 34.7). Its vertices are $O(0, 0)$ $A(200, 0)$, $B(3000/17, 1000/17)$, $C(2000/17, 1800/17)$ and $D(0, 150)$.

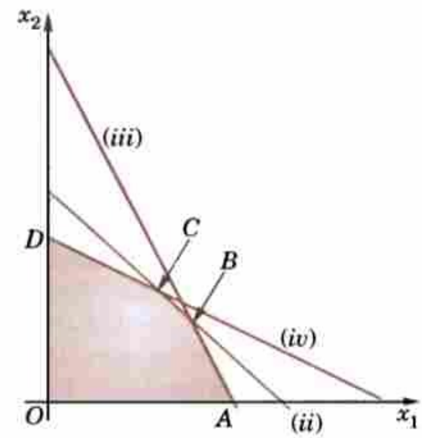

Fig. 34.7

The values of Z at these vertices are given by $Z(O) = 0$, $Z(A) = 1000$, $Z(B) = 1057.6$, $Z(C) = 905.8$ and $Z(D) = 450$.

Since the maximum value of Z is 1058.8 which occurs at the vertex B, the solution to the given problem is $x_1 = 3000/17$, $x_2 = 1000/17$ and max. $Z = 1058.8$.

Hence the manufacturer should produce 176.5 yards of type A cloth, 58.8 yards of type B cloth, so as to get the maximum profit of ₹ 1058.8.

Example 34.6. *A company making cold drinks has two bottling plants located at towns T_1 and T_2. Each plant produces three drinks A, B and C and their production capacity per day is shown below:*

Cold drinks	Plant at	
	T_1	T_2
A	6,000	2,000
B	1,000	2,500
C	3,000	3,000

The marketing department of the company forecasts a demand of 80,000 bottles of A, 22,000 bottles of B and 40,000 bottles of C during the month of June. The operating costs per day of plants at T_1 and T_2 are ₹6,000 and ₹ 4,000 respectively. Find (graphically) the number of days for which each plant must be run in June so as to minimize the operating costs while meeting the market demand.

Solution. Let the plants at T_1 and T_2 be run for x_1 and x_2 days. Then the objective is to minimize the operation costs *i.e.*,

$$\text{min. } Z = 6000x_1 + 4000x_2 \quad ...(i)$$

Constraints on the demand for the three cold drinks are:

for A, $6{,}000x_1 + 2{,}000x_2 \ge 80{,}000$ or $3x_1 + x_2 \ge 40$ ...(*ii*)

for B, $1{,}000x_1 + 2{,}500x_2 \ge 22{,}000$ or $x_1 + 2.5x_2 \ge 22$ ...(*iii*)

for C, $3{,}000x_1 + 3{,}000x_2 \ge 40{,}000$ or $x_1 + x_2 \ge 40/3$ ...(*iv*)

Also $x_1, x_2 \ge 0$ ...(*v*)

Thus the L.P.P. is to minimize (*i*) subject to constraints (*ii*) to (*v*). (*V.T.U., 2000 S*)

The solution space satisfying the constraints (*ii*) to (*v*) is shown shaded in Fig. 34.8. As seen from the direction of the arrows, the solution space is unbounded. The constraint (*iv*) is dominated by the constraints (*ii*) and (*iii*) and hence does not affect the solution space. Such a constraint as (*iv*) is called the *redundant constraint*.

The vertices of the convex region ABC are $A(22, 0)$, $B(12, 4)$ and $C(0, 40)$.

Values of the objective function (*i*) at these vertices are

$$Z(A) = 132{,}000,\ Z(B) = 88{,}000,\ Z(C) = 160{,}000.$$

Thus the minimum value of Z is ₹ 88,000 and it occurs at B. Hence the solution to the problem is $x_1 = 12$ days, $x_2 = 4$ days, Z_{min} = ₹ 88,000.

Otherwise. Making $Z = 0$, (*i*) becomes $3x_1 + 2x_2 = 0$ which is represented by the dotted line LM through O. As Z is increased, the line LM moves parallel to itself, to the right. Since we are interested in finding the minimum value of Z, value of Z is increased till LM passes through the vertex nearest to the origin of the shaded region, *i.e.* $B(12, 4)$.

Thus the operating cost will be minimum for $x_1 = 12$ days, x_2 = 4 days and

$$Z_{min} = 6000 \times 12 + 4000 \times 4 = ₹\ 88{,}000.$$

Obs. The dotted line parallel to the line LM is called the *iso-cost line* since it represents all possible combinations of x_1, x_2 which produce the same total cost.

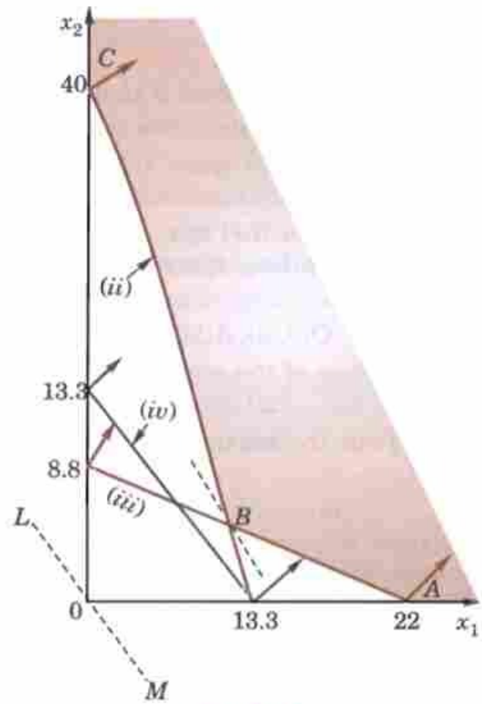

Fig. 34.8

34.4 SOME EXCEPTIONAL CASES

The constraints generally, give region of feasible solution which may be bounded or unbounded. In problems involving two variables and having a finite solution, we observed that the optimal solution existed at a vertex of the feasible region. In fact, this is true for all *L.P.* problems for which solutions exist. Thus it may be stated that *if there exists an optimal solution of an L.P.P., it will be at one of the vertices of the solution space.*

In each of the above exmples, the optimal solution was unique. But it is not always so. In fact, *L.P.P. may have*

(*i*) *a unique optimal solution*, or (*ii*) *an infinite number of optimal solutions*, or (*iii*) *an unbounded solution*, or (*iv*) *no solution.*

We now give below a few examples to illustrate the exceptional cases (*ii*) to (*iv*).

Example 34.7. *A firm uses milling machines, grinding machines and lathes to produce two motor parts. The machining times required for each part, the machining times available on different machines and the profit on each motor part are given below:*

Type of machine	*Machining time reqd. for the motor part (mts)*		*Max. time available per week (minutes)*
	I	*II*	
Milling machines	10	4	2,000
Grinding machines	3	2	900
Lathes	6	12	3,000
Profit/unit (₹)	100	40	

Determine the number of parts I and II to be manufactured per week to maximize the profit.

Solution. Let x_1, x_2 be the number of parts I and II manufactured per week. Then *objective* being to maximize the profit, we have maximize $Z = 100x_1 + 40x_2$ ...(*i*)

Constraints being on the time available on each machine, we obtain

for milling machines, $10x_1 + 4x_2 \le 2{,}000$ ...(*ii*)

for grinding machines, $3x_1 + 2x_2 \le 900$ ...(*iii*) for lathes, $6x_1 + 12x_2 \leq 3{,}000$ ...(*iv*)

Also $x_1, x_2 \geq 0$ ...(*v*)

Thus the problem is to determine x_1, x_2 which maximize (*i*) subject to the constraints (*ii*) to (*v*).

The solution space satisfying (*ii*), (*iii*), (*iv*) and meeting the non-negativity restrictions (*v*) is shown shaded in Fig. 34.9.

Note that (*iii*) is a redundant constraint as it does not affect the solution space. The vertices of the convex region *OABC* are

$$O(0, 0), A(200, 0), B(125, 187.5), C(0, 250).$$

Values of the objective function (*i*) at these vertices are $Z(O) = 0$, $Z(A) = 20{,}000$, $Z(B) = 20{,}000$ and $Z(C) = 10{,}000$.

Thus the maximum value of Z occurs at two vertices A and B.

$\therefore$ Any point on the line joining A and B will also give the same maximum value of Z *i.e.*, there are infinite number of feasible solutions which yield the same maximum value of Z.

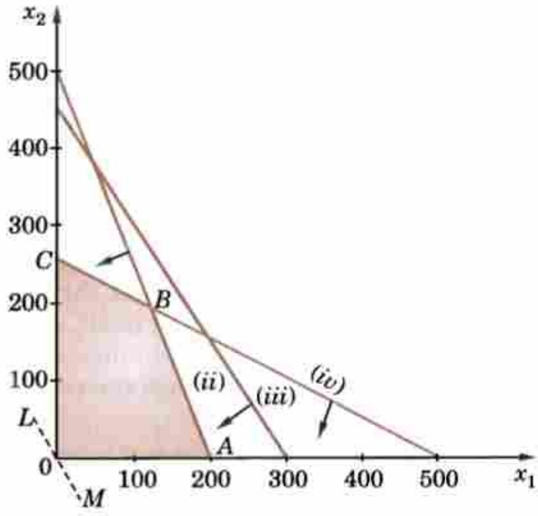

Fig. 34.9

Thus there is no unique optimal solution to the problem and any point on the line *AB* can be taken to give the profit of ₹ 20,000.

Obs. An L.P.P. having more than one optimal solution, is said to have alternative or multiple optimal solutions. It implies that the resources can be combined in more than one way to maximize the profit.

Example 34.8. *Using graphical method, solve the following L.P.P.:*

Maximize $Z = 2x_1 + 3x_2$ ...(*i*)

subject to $x_1 - x_2 \leq 2$ ...(*ii*)

$x_1 + x_2 \geq 4$ ...(*iii*)

$x_1, x_2 \geq 0.$ (*Kurukshetra, 2005; V.T.U., 2003 S*) ... (*iv*)

Solution. Consider $x_1 - x_2$ coordinate system. Any point (x_1, x_2) satisfying the restrictions (*iv*) lies in the first quadrant only. The solution space satisfying the constraints (*ii*) and (*iii*) is the convex region shown shaded in Fig. 34.10.

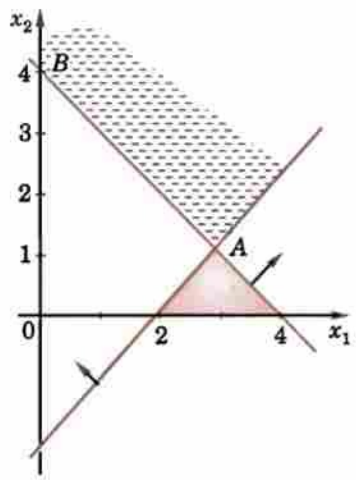

Fig. 34.10

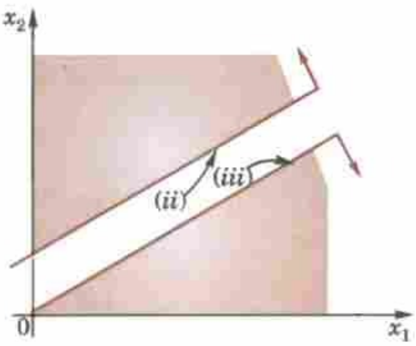

Fig. 34.11

Here the solution space is unbounded. The vertices of the feasible region (in the finite plane) are $A(3, 1)$ and $B(0, 4)$.

Values of the objective function (*i*) at these vertices are $Z(A) = 9$ and $Z(B) = 12$.

But there are points in this convex region for which Z will have much higher values. For instance, the point (5, 5) lies in the shaded region and the value of Z thereat is 25. In fact, the maximum value of Z occurs at infinity. Thus the problem has an unbounded solution.

Example 34.9. *Solve graphically the following L.P.P.:*

Maximize $Z = 4x_1 + 3x_2$ ...(*i*)

subject to $x_1 - x_2 \leq -1,$ ...(*ii*)

$-x_1 + x_2 \leq 0$ ...(*iii*)

and $x_1, x_2 \geq 0$ ...(*iv*)

Solution. Consider $x_1 - x_2$ coordinate system. Any point (x_1, x_2) satisfying (*iv*) lies in the first quadrant only. The two solution spaces, one satisfying (*ii*) and the other satisfying (*iii*) are shown shaded in Fig. 34.11.

There being no point (x_1, x_2) common to both the shaded regions, the problem cannot be solved. Hence the solution does not exist since the constraints are inconsistent.

Obs. The above problem had no solution because the constraints were incompatible. There may be cases in which the constraints are compatible but the problem may still have no feasible solution.

PROBLEMS 34.2

Using graphical method, solve the following L.P. problems:

1. Max. $Z = 3x_1 + 5x_2$
   subject to $x_1 + 2x_2 \leq 200$, $x_1 + x_2 \leq 150$, $x_1 \leq 60$, $x_1, x_2 \geq 0$ (*Rajasthan, 2003*)
2. Max. $Z = 5x_1 + 7x_2$
   subject to $x_1 + x_2 \leq 4$, $5x_1 + 8x_2 \leq 24$, $10x_1 + 7x_2 \leq 35$, and $x_1, x_2 \geq 0$.
3. Min. $Z = 20x_1 + 30x_2$
   subject to $x_1 + 2x_2 \leq 40$, $3x_1 + x_2 \geq 30$, $4x_1 + 3x_2 \geq 60$, $x_1, x_2 \geq 0$.
   (*Kurukshetra, 2009 S; Mumbai, 2004; V.T.U., 2004*)
4. Max. $z = 3x + 5x_2$ subject to $x_1 + 2x_2 \leq 2000$, $x_1 + x_2 \leq 1500$, $x_2 \leq 600$ and $x_1 \geq 0, x_2 \geq 0$. (*Rohtak, 2004*)
5. A firm manufactures two products *A* and *B* on which the profits earned per unit are ₹ 3 and ₹ 4 respectively. Each product is processed on two machines M_1 and M_2. Product *A* requires one minute of processing time on M_1 and 2 minutes on M_2 while *B* requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hours and 30 minutes while M_2 is available for 10 hours during any working day. Find the number of units of products *A* and *B* to be manufactured to get maximum profit.
6. Two spare parts *X* and *Y* are to be produced in a batch. Each one has to go through two processes *A* and *B*. The time required in hours per unit and total time available are given below:

	X	*Y*	*Total hours available*
Process A	3	4	24
Process B	9	4	36

   Profits per unit of *X* and *Y* are ₹ 5 and ₹ 6 respectively. Find how many number of spare parts of *X* and *Y* are to be produced in this batch to maximize the profit. (Each batch is complete in all respects and one cannot produce fractional units and stop the batch).
7. A manufacturer has two products I and II both of which are made in steps by machines *A* and *B*. The process times per hundred for the two products on the two machines are:

Product	*M/c. A*	*M/c. B*
I	4 hrs.	5 hrs.
II	5 hrs.	2 hrs.

   Set-up times are negligible. For the coming period machine *A* has 100 hrs. and *B* has 80 hrs. The contribution for product I is ₹ 10 per 100 units and for product II is ₹ 5 per 100 units. The manufacturer is in a market which can absorb both products as much as he can produce for the immediate period ahead. Determine graphically, how much of products I and II, he should produce to maximize his contribution.
8. A production manager wants to determine the quantity to be produced per month of products *A* and *B* manufactured by his firm. The data on resources required and availability of resources are given below:

Resources	*Requirements*		*Available per month*
	Product A	*Product B*	
Raw material (kg.)	60	120	12,000
Machine hrs/piece	8	5	600
Assembly man hrs.	3	4	500
Sale price/piece	₹ 30	₹ 40	

Formulate the problem as a standard L.P.P. Find product mix that would give maximum profit by graphical technique.

9. A pineapple firm produces two products: canned pineapple and canned juice. The specific amounts of material, labour and equipment required to produce each product and the availability of each of these resources are shown in the table given below:

	Canned Juice	*Pineapple*	*Available Resources*
Labour (man hrs)	3	2.0	12.0
Equipment (m/c hrs)	1	2.3	6.9
Material (unit)	1	1.4	4.9

Assuming one unit each of canned juice and canned pineapple has profit margins of ₹ 2 and ₹ 1 respectively. Formulate this as L.P. problem and solve it graphically.

Solve the following L.P. problems graphically:

10. Maximize $Z = 6x + 4y$ subject to $2x + y \geq 1$, $3x + 4y \geq 1.5$ and $x, y \geq 0$. *(Bombay, 2004)*

11. Minimize $Z = 8x_1 + 12x_2$ subject to $60x_1 + 30x_2 \geq 240$, $30x_1 + 60x_2 \geq 300$, $30x_1 + 180x_2 \geq 540$. and $x_1, x_2 \geq 0$.

12. G.J. Breweries Ltd. have two bottling plants one located at '*G*' and other at '*J*'. Each plant produces three drinks: whiskey, beer and brandy. The number of bottles produced per day are as follows:

Drink	*Plant at G*	*Plant at J*
Whiskey	1500	1500
Beer	3000	1000
Brandy	2000	5000

A market survey indicates that during the month of July, there will be a demand of 20,000 bottles of whiskey, 40,000 bottles of beer and 44,000 bottles of brandy. The operating cost per day for plants at *G* and *J* are ₹ 600 and ₹ 400. For how many days each plant be run in July so as to minimize the production cost, while still meeting the market demand. Solve graphically.

34.5 GENERAL LINEAR PROGRAMMING PROBLEM

Any L.P.P. problem involving more than two variables may be expressed as follows:

Find the values of the variables $x_1, x_2, ..., x_n$ which maximize (or minimize) the objective function

$$Z = c_1x_1 + c_2x_2 + ... + c_nx_n \qquad ...(i)$$

subject to the constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \leq b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \leq b_m \end{array}\right\} \qquad ...(ii)$$

and meet the non-negativity restrictions

$$x_1, x_2, ..., x_n \geq 0. \qquad ...(iii)$$

Def. 1. *A set of values $x_1, x_2, ..., x_n$ which satisfies the constraints of the L.P.P. is called its* **solution**.

Def. 2. *Any solution to a L.P.P. which satisfies the non-negativity restrictions of the problem is called its* **feasible solution**.

Def. 3. *Any feasible solution which maximizes (or minimizes) the objective function of the L.P.P. is called its* **optimal solution**.

Some of the constraints in (*ii*) may be equalities, some others may be inequalities of (≤) type and remaining ones inequalities of (≥) type. The inequality constraints are changed to equalities by adding (or subtracting) non-negative variables to (from) the left hand side of such constraints.

Def. 4. *If the constraints of a general L.P.P. be*

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$$ *(i = 1, 2, ... k) then the non-negative variables s_i which satisfy*

$$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i \ (i = 1, 2, \dots k), \textit{ are called } \textbf{slack variables.}$$

Def. 5. *If the constraints of a general L.P.P. be*

$$\sum_{j=1}^{n} a_{ij}x_i \geq b_i, \ (i = k, k+1, \dots) \textit{ then the non-negative variables } s_i \textit{ which satisfy}$$

$$\sum_{j=1}^{n} a_{ij}x_i - s_i = b_i, \ (i = k, k+1, \dots), \textit{ are called } \textbf{surplus variables.}$$

34.6 CANONICAL AND STANDARD FORMS OF L.P.P.

After the formulation of L.P.P., the next step is to obtain its solution. But before any method is used to find its solution, the problem must be presented in a suitable form. As such, we explain its following two forms:

(1) **Canonical form.** The general L.P.P. can always be expressed in the following form:

Maximize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

subject to the constraints $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i;\ i = 1, 2, \dots m$

$$x_1, x_2, \dots x_n \geq 0,$$

by making some elementary transformations. This form of the L.P.P. is called its **canonical form** and has the following characteristics:

(*i*) Objective function is of maximization type,

(*ii*) All constraints are of (≤) type,

(*iii*) All variables x_i are non-negative.

The canonical form is a format for a L.P.P. which finds its use in the Duality theory.

(2) **Standard form.** The general L.P.P. can also be put in the following form:

Maximize $Z = c_1x_1 + c_2x_2 + \dots c_nx_n$

subject to the constraints $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i;\ i = 1, 2, \dots m$

$$x_1, x_2, \dots x_n \geq 0,$$

This form of the L.P.P. is called its **standard form** and has the following characteristics:

(*i*) Objective function is of maximization type;

(*ii*) All constraints are expressed as equations;

(*iii*) Right hand side of each constraint is non-negative;

(*iv*) All variables are non-negative.

Obs. *Any L.P.P. can be expressed in the standard form.*

As minimize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

is equivalent to maximize $Z'(= -Z) = -c_1x_1 - c_2x_2 \dots - c_nx_n$,

the objective function can always be expressed in the maximization form.

The inequality constraints can always be converted to equalities by adding (or substracting) the slack (or surplus) variables to the left hand sides of such constraints.

So far, the decision variables $x_1, x_2, \dots, x_n$ have been assumed to be all non-negative. In actual practice, these variables could also be zero or negative. If a variable is negative, it can always be expressed as the difference of two non-negative variables *e.g.* a variable x_i can be written as

$$x_i = x_i' - x_i'' \qquad \text{where } x_i' \geq 0, x_i'' \geq 0.$$

Example 34.10. *Convert the following L.P.P. to the standard form:*

Maximize $Z = 3x_1 + 5x_2 + 7x_3$,

subject to $6x_1 - 4x_2 \leq 5,\ 3x_1 + 2x_2 + 5x_3 \geq 11,\ 4x_1 + 3x_3 \leq 2,\ x_1, x_2 \geq 0.$

Solution. As x_3 is unrestricted, let $x_3 = x_3' - x_3''$ where $x_3', x_3'' \geq 0$. Now the given constraints can be expressed as

$$6x_1 - 4x_2 \leq 5,$$

$$3x_1 + 2x_2 + 5x_3' - 5x_3'' \geq 11$$

$$4x_1 + 3x_3' - 3x_3'' \le 2$$
$$x_1, x_2, x_3', x_3'' \ge 0.$$

Introducing the slack/surplus variables, the problem in standard form becomes:

Maximize $\quad Z = 3x_1 + 5x_2 + 7x_3' - 7x_3''$

subject to
$$6x_1 - 4x_2 + s_1 = 5,$$
$$3x_1 + 2x_2 + 5x_3' - 5x_3'' - s_2 = 11,$$
$$4x_1 + 3x_3' - 3x_3'' + s_3 = 2,$$
$$x_1, x_2, x_3', x_3'', s_1, s_2, s_3 \ge 0.$$

Example 34.11. *Express the following problem in the standard form:*

Minimize $\quad Z = 3x_1 + 4x_2$

subject to $\quad 2x_1 - x_2 - 3x_3 = -4,\ 3x_1 + 5x_2 + x_4 = 10,\ x_1 - 4x_2 = 12,\ x_1, x_3, x_4 \ge 0.$

Solution. Here x_3, x_4 are the slack/surplus variables and x_1, x_2 are the decision variables. As x_2 is unrestricted, let $x_2 = x_2' - x_2''$ where $x_2', x_2'' \ge 0$.

$\therefore$ The problem in standard form is

Maximize $\quad Z' (= -Z) = -3x_1 - 4x_2' + 4x_2''$

subject to
$$-2x_1 + x_2' - x_2'' + 3x_3 = 4$$
$$3x_1 + 5x_2' - 5x_2'' + x_4 = 10$$
$$x_1 - 4x_2' - 4x_2'' = 12$$
$$x_1, x_2', x_2'', x_3, x_4 \ge 0.$$

34.7 SIMPLEX METHOD

(1) While solving an L.P.P. graphically, the region of feasible solutions was found to be convex, bounded by vertices and edges joining them. The optimal solution occurred at some vertex. If the optimal solution was not unique, the optimal points were on an edge. These observations also hold true for the general L.P.P. Essentially the problem is that of finding the particular vertex of the convex region which corresponds to the optimal solution. The most commonly used method for locating the optimal vertex is the **simplex method**. This method consists in moving step by step from one vertex to the adjacent one. Of all the adjacent vertices, the one giving better value of the objective function over that of the preceding vertex, is chosen. This method of jumping from one vertex to the other is then repeated. Since the number of vertices is finite, the simplex method leads to an optimal vertex in a finite number of steps.

(2) *In simplex method, an infinite number of solutions is reduced to a finite number of promising solutions by using the following facts:*

(*i*) When there are m constraints and $(m + n)$ variables (m being $\le n$), the starting solution is found by setting n variables equal to zero and then solving the remaining m equations, provided the solution exists and is unique. *The n zero variables are known as* **non-basic variables** *while the remaining m variables are called* **basic variables** *and they form a* **basic solution**.

(*ii*) In an L.P.P., the variables must always be non-negative. Some of the basic solutions may contain negative variables. Such solutions are called *basic infeasible solutions* and should not be considered. To achieve this, we start with a basic solution which is non-negative. The next basic solution must always be non-negative. This is ensured by feasibility condition. Such a solution is known as **basic feasible solution**.

If all the variables in the basic feasible solution are positive, then it is called **non-degenerate solution** *and if some of the variables are zero, it is called* **degenerate solution.**

(*iii*) A new basic feasible solution may be obtained from the previous one by equating one of the basic variables to zero and replacing it by a new non-basic variable. *The eliminated variable is called the* **outgoing variable** *while the new variable is known as the* **incoming variable.**

The incoming variable must improve the value of the objective function which is ensured by the optimality condition. *This process is repeated till no further improvement is possible. The resulting solution is called the* **optimal basic feasible solution** or simply **optimal solution.**

(3) *The simplex method is, therefore, based on the following two conditions:*

I. *Feasibility condition.* It ensures that if the starting solution is basic feasible, the subsequent will also be basic feasible.

II. *Optimality condition.* It ensures that only improved solutions will be obtained.

(4) *Now, we shall elaborate the above terms in relation to the general linear programming problem in standard form, i.e.,*

Maximize $$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \quad \dots(1)$$

subject to $$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i\ ,\ i = 1, 2, \dots m \quad \dots(2)$$

and $$x_j \geq 0,\ s_i \geq 0,\ j = 1, 2, \dots n \quad \dots(3)$$

(*i*) *Solution.* $x_1, x_2, \dots x_n$ is a solution of the general L.P.P. if it satisfies the constraints (2).

(*ii*) *Feasible solution,* $x_1, x_2, \dots x_n$ is a feasible solution of the general L.P.P. if it satisfies both the constraints (2) and the non-negativity restrictions (3). The set S of all feasible solutions is called the feasible region. A linear programme is said to be *infeasible* when the set S is empty.

(*iii*) *Basic solution* is the solution of the m basic variables when each of the n non-basic variables is equated to zero.

(*iv*) *Basic feasible solution* is that *basic solution* which also satisfies the non-negativity restrictions (3).

(*v*) *Optimal solution* is that basic feasible solution which also optimizes the objective function (1) while satisfying the conditions (2) and (3).

(*vi*) *Non-degenerate basic feasible solution* is that basic feasible solution which contains exactly m non-zero basic variables. If any of the basic variables becomes zero, it is called a *degenerate basic feasible solution.*

Example 34.12. *Find all the basic solutions of the following system of equations identifying in each case the basic and non-basic variables:* $2x_1 + x_2 + 4x_3 = 11,\ 3x_1 + x_2 + 5x_3 = 14.$ *(Mumbai, 2004 ; V.T.U., 2003 S)*

Investigate whether the basic solutions are degenerate basic solutions or not. Hence find the basic-feasible solution of the system.

Solution. Since there are $m + n = 3$ variables and there are $m = 2$ constraints in this problem, a basic solution can be obtained by setting any one variable equal to zero and then solving the resulting equations. Also the total number of basic solutions $= {}^{m+n}C_m = {}^3C_2 = 3.$

The characteristics of the various basic solutions are as given below:

No. of basic sol.	*Basic variables*	*Nonbasic variables*	*Values of basic variables*	*Is the sol. feasible ? (Are all $x_j > 0$?)*	*Is the sol. degenerate ?*
1.	x_1, x_2	x_3	$2x_1 + x_2 = 11$ $3x_1 + x_2 = 14$ $\therefore\ x_1 = 3, x_2 = 5$	Yes	No
2.	x_2, x_3	x_1	$x_2 + 4x_3 = 11$ $x_2 + 5x_3 = 14$ $\therefore\ x_2 = 3, x_3 = -1$	No	Yes
3.	x_1, x_3	x_2	$2x_1 + 4x_3 = 11$ $3x_1 + 5x_3 = 14$ $\therefore\ x_1 = 1/2, x_3 = 5/2$	Yes	No

The basic feasible solutions are:

(*i*) $x_1 = 3, x_2 = 5, x_3 = 0$; (*ii*) $x_1 = 1/2, x_2 = 0, x_3 = 5/2$

which are also non-degenerate basic solutions.

Example 34.13. *Find an optimal solution to the following L.P.P. by computing all basic solutions and then finding one that maximizes the objective function:*

$$2x_1 + 3x_2 - x_3 + 4x_4 = 8,\ x_1 - 2x_2 + 6x_3 - 7x_4 = -3,\ x_1, x_2, x_3, x_4 \geq 0,$$
$$\text{Max. } Z = 2x_1 + 3x_2 + 4x_3 + 7x_4.$$

Solution. Since there are four variables and two constraints, a basic solution can be obtained by setting any two variables equal to zero and then solving the resulting equations. Also the total number of basic solutions $= {}^4C_2 = 6$.

The characteristics of the various basic solutions are as given below:

No. of basic sol.	*Basic variables*	*Non-basic variables*	*Values of basic variables*	*Is the sol. feasible ? (Are all $x_j \geq 0$?)*	*Value of Z*	*Is the sol. optimal ?*
1.	x_1, x_2	$x_3, x_4 = 0$	$2x_1 + 3x_2 = 8$ $x_1 - 2x_2 = -3$ $\therefore\ x_1 = 1, x_2 = 2$	Yes	8	No
2.	x_1, x_3	$x_2, x_4 = 0$	$2x_1 - x_3 = 8$ $x_1 + 6x_3 = -3$ $\therefore\ x_1 = -14/13,$ $x_3 = -67/13$	No	—	—
3.	x_1, x_4	$x_2, x_3 = 0$	$2x_1 + 4x_4 = 8$ $x_1 - 7x_4 = -3$ $\therefore\ x_1 = 22/9,$ $x_4 = 7/9$	Yes	10.3	No
4.	x_2, x_3	$x_1, x_4 = 0$	$3x_2 - x_3 = 8$ $-2x_2 + 6x_3 = -3$ $\therefore\ x_2 = 45/16,$ $x_3 = 7/16$	Yes	10.2	No
5.	x_2, x_4	$x_1, x_3 = 0$	$3x_2 + 4x_4 = 8$ $-2x_2 - 7x_4 = -3$ $\therefore\ x_2 = 132/39,$ $x_4 = -7/13$	No	—	—
6.	x_3, x_4	$x_1, x_2 = 0$	$-x_3 + 4x_4 = 8$ $6x_3 - 7x_4 = -3$ $\therefore\ x_3 = 44/17,$ $x_4 = 45/17$	Yes	28.9	Yes

Hence the optimal basic feasible solution is

$$x_1 = 0, x_2 = 0, x_3 = 44/17, x_4 = 45/17 \text{ and the maximum value of } Z = 28.9.$$

PROBLEMS 34.3

1. Reduce the following problem to the standard form:
   Determine $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ so as to
   Maximize $Z = 3x_1 + 5x_2 + 8x_3$
   subject to the constraints $2x_1 - 5x_2 \leq 6, 3x_1 + 2x_2 + x_3 \geq 5, 3x_1 + 4x_3 \leq 3.$
2. Express the following L.P.P. in the standard form
   Maximize $Z = 3x_1 + 2x_2 + 5x_3$
   subject to $-5x_1 + 2x_2 \leq 5, 2x_1 + 3x_2 + 4x_3 \geq 7, 2x_1 + 5x_3 \leq 3, x_1, x_2, x_3 \geq 0.$ *(Kurukshetra, 2009)*
3. Convert the following L.P.P. to standard form:
   Maximize $Z = 3x_1 - 2x_2 + 4x_3$
   subject to $x_1 + 2x_2 + x_3 \leq 8, 2x_1 - x_2 + x_3 \geq 2, 4x_1 - 2x_2 - 3x_3 = -6, x_1, x_2 \geq 0.$ *(Kurukshetra, 2007 S)*
4. Obtain all the basic solutions to the following system of linear equations:
   $$x_1 + 2x_2 + x_3 = 4, 2x_1 + x_2 + 5x_3 = 5.$$
5. Show that the following system of linear equations has two degenerate feasible basic solutions and the non-degenerate basic solution is not feasible:
   $$2x_1 + x_2 - x_3 = 2, 3x_1 + 2x_2 + x_3 = 3.$$ *(Kurukshetra, 2007 S)*

6. Find all the basic solutions to the following problem:
Maximize $Z = x_1 + 3x_2 + 3x_3$,
subject to $x_1 + 2x_2 + 3x_3 = 4,\ 2x_1 + 3x_2 + 5x_3 = 7$ and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.
Which of the basic solutions are (*a*) non-degenerate basic feasible, (*b*) optimal basic feasible ?
(*Kurushetra, 2009 S; Mumbai, 2003*)

34.8 WORKING PROCEDURE OF THE SIMPLEX METHOD

Assuming the existence of an initial basic feasible solution, an optimal solution to any L.P.P. by simplex method is found as follows:

Step 1. (*i*) *Check whether the objective function is to be maximized or minimized.*

If $$Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n$$
is to be minimized, then convert it into a problem of maximization, by writing

Minimize Z = Maximize $(-Z)$

(*ii*) *Check whether all b's are positive.*

If any of the b_i's is negative, multiply both sides of that constraint by -1 so as to make its right hand side positive.

Step 2. Express the problem in the standard form.

Convert all inequalities of constraints into equations by introducings slack/surplus variables in the constraints giving equations of the form
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + s_1 + 0s_2 + 0s_3 + \ldots = b_1.$$

Step 3. Find an initial basic feasible solution.

If there are m equations involving n unknowns, then assign zero values to any $(n - m)$ of the variables for finding a solution. Starting with a basic solution for which $x_j : j = 1, 2, \ldots, (n - m)$ are each zero, find all s_i. If all s_i are ≥ 0, the basic solution is feasible and non-degenerate. If one or more of the s_i values are zero, then the solution is degenerate.

The above information is conveniently expressed in the following simplex table:

	c_j	c_1	c_2	$c_3 \ldots 0$	0	$0 \ldots$
c_B	Basis	x_1	x_2	$x_3 \ldots s_1$	s_2	$s_3 \ldots b$
0	s_1	a_{11}	a_{12}	$a_{13} \ldots 1$	0	$0 \ldots b_1$
0	s_2	a_{21}	a_{22}	$a_{23} \ldots 0$	1	$0 \ldots b_2$
0	s_3	a_{31}	a_{32}	$a_{33} \ldots 0$	0	$1 \ldots b_3$
:	:	:	:	: :	:	: :
			Body matrix		Unit matrix	

[The variables s_1, s_2, s_3 etc. are called *basic variables* and variables x_1, x_2, x_3 etc. are called non-basic variables. *Basis* refers to the basic variables $s_1, s_2, s_3 \ldots c_j$ row denotes the coefficients of the variables in the objective function, while c_B-column denotes the coefficients of the basic variables only in the objective function. b-column denotes the values of the basic variables while remaining variables will always be zero. The coefficients of x's (decision variables) in the constraint equations constitute the *body matrix* while coefficients of slack variables constitute the *unit matrix*].

Step 4. Apply optimality test.

Compute $C_j = c_j - Z_j$; where $Z_j = \Sigma\, c_B a_{ij}$

[C_j-row is called *net evaluation row* and indicates the per unit increase in the objective function if the variable heading the column is brought into the solution.]

If all C_j are negative, then the initial basic feasible solution is *optimal*.

If even one C_j is positive, then the current feasible solution is not optimal (*i.e.*, can be improved) and proceed to the next step.

Step 5. (*i*) *Identify the incoming and outgoing variables.*

If there are more than one positive C_j, then the *incoming variable* is the one that heads the column containing maximum C_j. The column containing it is known as the *key column* which is shown marked with an arrow at the bottom. If more than one variable has the same maximum C_j, any of these variables may be selected arbitrarily as the incoming variable.

Now divide the elements under b-column by the corresponding elements of key column and choose the row containing the minimum positive ratio θ. Then replace the corresponding basic variable (by making its value zero). It is termed as the *outgoing variable*. The corresponding row is called the *key row* which is shown marked with an arrow on its right end. The element at the intersection of the key row and key column is called the *key element* which is shown bracketted. If all these ratios are ≤ 0, the incoming variable can be made as large as we please without violating the feasibility condition. Hence the problem has an *unbounded solution* and no further iteration is required.

(*ii*) *Iterate towards an optimal solution.*

Drop the outgoing variable and introduce the incoming variable alongwith its associated value under c_B column. Convert the key element to unity by dividing the key row by the key element. Then make all other elements of the key column zero by subtracting proper multiples of key row from the other rows.

[This is nothing but the sweep-out process used to solve the linear equations. The operations performed are called *elementary row operations*.]

Step 6. Go to step 4 and repeat the computational procedure until either an optimal (or an unbounded) solution is obtained.

Example 34.14. *Using simplex method*

Maximize $Z = 5x_1 + 3x_2$

subject to $x_1 + x_2 \leq 2,\ 5x_1 + 2x_2 \leq 10,\ 3x_1 + 8x_2 \leq 12,\ x_1, x_2 \geq 0.$ (V.T.U., 2003 S)

Solution. Consists of the following steps :

Step 1. Check whether the objective function is to be maximized and all b's are positive.

The problem being of maximization type and all b's being ≥ 0, this step is not necessary.

Step 2. Express the problem in the standard form.

By introducing the slack variables s_1, s_2, s_3, the problem in standard form becomes

Max. $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$

subject to
$$x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 2 \quad ...(i)$$
$$5x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 10 \quad ...(ii)$$
$$3x_1 + 8x_2 + 0s_1 + 0s_2 + s_3 = 12 \quad ...(iii)$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0.$$

Step 3. Find an initial basic feasible solution.

There are three equations involving five unknowns and for obtaining a solution, we assign zero values to any two of the variables. We start with a basic solution for which we set $x_1 = 0$ and $x_2 = 0$. (This basic solution corresponds to the origin in the graphical method). Substituting $x_1 = x_2 = 0$ in (*i*), (*ii*) and (*iii*), we get the basic solution

$$s_1 = 2,\ s_2 = 10,\ s_3 = 12$$

Since all s_1, s_2, s_3 are positive, the basic solution is also feasible and non-degenerate.

$\therefore$ The basic feasible solution is

$$x_1 = x_2 = 0 \text{ (non-basic) and } s_1 = 2,\ s_2 = 10,\ s_3 = 12 \text{ (basic)}$$

$\therefore$ *Initial basic feasible solution* is given by the following table :

	c_j	5	3	0	0	0		
c_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	(1)	1	1	0	0	2	2/1←
0	s_2	5	2	0	1	0	10	10/5
0	s_3	3	8	0	0	1	12	12/3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	
	$C_j = c_j - Z_j$	5	3	0	0	0		
		↑						

[For x_1-column ($j = 1$), $Z_j = \Sigma c_B a_{i1} = 0(1) + 0(5) + 0(3) = 0$

and for x_2-column ($j = 2$), $Z_j = \Sigma c_B a_{i2} = 0(1) + 0(2) + 0(8) = 0$

Similarly $Z_j(b) = 0(2) + 0(10) + 0(12) = 0$.]

Step 4. Apply optimality test.

As C_j is positive under some columns, the initial basic feasible solution is not optimal (*i.e.* can be improved) and we proceed to the next step.

Step 5. (i) Identify the incoming and outgoing variables.

The above table shows that x_1 is the *incoming variable* as its incremental contribution C_j (= 5) is maximum and the column in which it appears is the *key column* (shown marked by an arrow at the bottom).

Dividing the elements under b-column by the corresponding elements of key-column, we find minimum positive ratio θ is 2 in two rows. We, therefore, arbitrarily choose the row containing s_1 as the *key row* (shown marked by an arrow on its right end). The element at the intersection of key row and the key column *i.e.*, (1), is the *key element*. s_1 is therefore, the *outgoing basic variable* which will now become non-basic.

Having decided that x_1 is to enter the solution, we have tried to find as to what maximum value x_1 could have without violating the constraints. So removing s_1, the new basis will contain x_1, s_2 and s_3 as the basic variables.

(ii) Iterate towards the optimal solution.

To transform the initial set of equations with a basic feasible solution into an equivalent set of equations with a different basic feasible solution, we make the key element unity. Here the key element being unity, we retain the key row as it is. Then to make all other elements in key column zero, we subtract proper multiples of key row from the other rows. Here we subtract 5 times the elements of key row from the second row and 3 times the elements of key row from the third row. These become the second and the third rows of the next table. We also change the corresponding value under c_B column from 0 to 5, while replacing s_1 by x_1 under the basis. Thus the *second basic feasible solution* is given by the following table :

	c_j	5	3	0	0	0		
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	b	θ
5	x_1	1	1	1	0	0	2	
0	s_2	0	– 3	– 5	1	0	0	
0	s_3	0	5	– 3	0	1	6	
	$Z_j = \Sigma c_B a_{ij}$	5	5	5	0	0	10	
	$C_j = c_j - Z_j$	0	– 2	– 5	0	0		

As C_j is either zero or negative under all columns, the above table gives the optimal basic feasible solution. This optimal solution is $x_1 = 2$, $x_2 = 0$ and maximum $Z = 10$.

Example 34.15. *A firm produces three products which are processed on three machines. The relevant data is given below :*

Machine	*Time per unit (minutes)*			*Machine capacity*
	Product A	*Product B*	*Product C*	*(minutes/day)*
M_1	2	3	2	440
M_2	4	—	3	470
M_3	2	5	—	430

The profit per unit for products A, B, and C is ₹ 4, ₹ 3 and ₹ 6 respectively. Determine the daily number of units to be manufactured for each product. Assume that all the units produced are consumed in the market.

Solution. Let the firm decide to produce x_1, x_2, x_3 units of products *A, B, C,* respectively. Then the L.P. model for this problem is :

Max. $Z = 4x_1 + 3x_2 + 6x_3$

subject to $2x_1 + 3x_2 + 2x_3 \leq 440,\ 4x_1 + 3x_3 \leq 470,\ 2x_1 + 5x_2 \leq 430,\ x_1, x_2, x_3 \geq 0.$ (*V.T.U., 2004*)

Step 1. Check whether the objective function is to be maximized and all b's are non-negative.

The problem being of maximization type and b's being ≥ 0, this step is not necessary.

Step 2. Express the problem in the standard form.

By introducing the slack variables s_1, s_2, s_3, the problem in standard form becomes :

Max. $Z = 4x_1 + 3x_2 + 6x_3 + 0s_1 + 0s_2 + 0s_3$

subject to $2x_1 + 3x_2 + 2x_3 + s_1 + 0s_2 + 0s_3 = 440$

$4x_1 + 0x_2 + 3x_3 + 0s_1 + s_2 + 0s_3 = 470$

$2x_1 + 5x_2 + 0x_3 + 0s_1 + 0s_2 + s_3 = 430$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$

Step 3. Find an initial basic feasible solution.

The basic (non-degenerate) feasible solution is

$x_1 = x_2 = x_3 = 0$ (non-basic)

$s_1 = 440, s_2 = 470, s_3 = 430$ (basic)

$\therefore$ *Initial basic feasible solution* is given by the following table :

	c_j	4	3	6	0	0	0		
c_B	*Basis*	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
0	s_1	2	3	2	1	0	0	440	440/2
0	s_2	4	0	(3)	0	1	0	470	470/3←
0	s_3	2	5	0	0	0	1	430	430/0
$Z_j = \Sigma c_B a_{ij}$		0	0	0	0	0	0		
$C_j = c_j - Z_j$		4	3	6	0	0	0		

Step 4. Apply optimality test.

As C_j is positive under some columns, the initial basic feasible solution is not optimal and we proceed to the next step.

Step 5. (i) Identify the incoming and outgoing variables.

The above table shows that x_3 is the incoming variable while s_2 is the outgoing variable and (3) is the key element.

(ii) Iterate towards the optimal solution.

Drop s_2 and introduce x_3 with its associated value 6 under c_B column. Convert the key element to unity and make all other elements of key column zero. Then the *second feasible solution* is given by the table below :

	c_j	4	3	6	0	0	0		
c_B	*Basis*	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
0	s_1	– 2/3	(3)	0	1	– 2/3	0	380/3	380/9←
6	s_2	4/3	0	1	0	1/3	0	470/3	∞
0	s_3	2	5	0	0	0	1	430	86
	Z_j	8	0	6	0	2	0	940	
	C_j	– 4	3	0	0	– 2	0		
			↑						

Step 6. As C_j is positive under the second column, the solution is not optimal and we proceed further. Now x_2 is the incoming variable and s_1 is the outgoing variable and (3) is the key element for the next iteration.

Drop s_1 and introduce x_2 with its associated value 3 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the *third basic feasible solution* is given by the following table :

	c_j	4	3	6	0	0	0		
c_B	*Basis*	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
3	x_2	– 2/9	1	0	1/3	– 2/9	0	380/9	
6	x_3	4/3	0	1	0	1/3	0	470/3	
0	s_3	28/9	0	0	– 5/3	10/9	0	1970/9	
	Z_i	22/3	3	6	1	4/3	0	3200/3	
	C_j	– 10/3	0	0	– 1	– 4/3	0		

Now since each $C_j \leq 0$, therefore it gives the optimal solution

$$x_1 = 0, x_2 = 380/9, x_3 = 470/3 \text{ and } Z_{max} = 3200/3 \text{ i.e., } 1066.67 \text{ rupees.}$$

Example 34.16. *Solve the following L.P.P. by simplex method :*

Minimize $Z = x_1 - 3x_2 + 3x_3$,

subject to $3x_1 - x_2 + 2x_3 \leq 7,\ 2x_1 + 4x_2 \geq -12,\ -4x_1 + 3x_2 + 8x_3 \leq 10,\ x_1, x_2, x_3 \geq 0.$

(Mumbai, 2004 ; V.T.U., 2003)

Solution. Consists of the following steps :

Step 1. Check whether objective function is to be maximized and all b's are non-negative.

As the problem is that of minimizing the objective function, converting it to the maximization type, we have Max. $Z' = -x_1 + 3x_2 - 3x_3$.

As the right hand side of the second constraint is negative, we write it as

$$-2x_1 - 4x_2 \leq 12$$

Step 2. Express the problem in the standard form.

By introducing the slack variables s_1, s_2, s_3, the problem in the standard form becomes

Max. $Z' = -x_1 + 3x_2 - 3x_3 + 0s_1 + 0s_2 + 0s_3$

subject to
$$3x_1 - x_2 + 2x_3 + s_1 + 0s_2 + 0s_3 = 7$$
$$-2x_1 - 4x_2 + 0x_3 + 0s_1 + s_2 + 0s_3 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + 0s_1 + 0s_2 + s_3 = 10$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

Step 3. Find initial basic feasible solution.

The basic (non-degenerate) feasible solution is

$$x_1 = x_2 = x_3 = 0 \text{ (non-basic) ; } s_1 = 7, s_2 = 12, s_3 = 10 \text{ (basic)}$$

$\therefore$ *Initial basic feasible solution* is given by the table below :

	c_j	-1	3	-3	0	0	0		
c_B	*Basis*	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
0	s_1	3	-1	2	1	0	0	7	$7/(-1)$
0	s_2	-2	-4	0	0	1	0	12	$12/(-4)$
0	s_3	-4	(3)	8	0	0	1	10	$10/3\leftarrow$
$Z_j = \Sigma c_B a_{ij}$		0	0	0	0	0	0	0	
$C_j = c_j - Z_j$		-1	3	-3	0	0	0		
			$\uparrow$						

Step 4. Apply optimality test.

As C_j is positive under second column, the initial basic feasible solution is not optimal and we proceed further.

Step 5. (i) Identify the incoming and outgoing variables.

The above table shows that x_2 is the incoming variable, s_3 is the outgoing variable and (3) is the key element.

(ii) Iterate towards the optimal solution.

$\therefore$ Drop s_3 and introduce x_2 with its associated value 3 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the *second basic feasible solution* is given by the following table :

	c_j	-1	3	-3	0	0	0		
c_B	*Basis*	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
0	s_1	(5/3)	0	14/3	1	0	1/3	31/3	$31/5\leftarrow$
0	s_2	$-22/3$	0	32/3	0	1	4/3	76/3	$-38/11$
3	x_2	$-4/3$	1	8.3	0	0	1/3	10/3	$-5/2$
	Z_j	-4	3	8	0	0	1	10	
	C_j	3	0	-11	0	0	-1		
		$\uparrow$							

Step 6. As C_j is positive under first column, the solution is not optimal and we proceed further x_1 is the incoming variable, s_1 is the outgoing variable and (5/3) is the key element.

$\therefore$ Drop s_1 and introduce x_1 with its associated value -1 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the *third basic feasible solution* is given by the table below :

c_B	c_j Basis	-1 x_1	3 x_2	-3 x_3	0 s_1	0 s_2	0 s_3	b
-1	x_1	1	0	14/5	3/5	0	1/5	31/5
0	s_2	0	0	156/5	22/5	1	14/5	354/5
3	x_2	0	1	32/5	4/5	0	3/5	58/5
	Z_j	-1	3	82/5	9/5	0	8/5	143/5
	C_j	0	0	$-97/5$	$-9/5$	0	$-8/5$	

Now since each $C_j \le 0$, therefore it gives the optimal solution

$$x_1 = 31/5, x_2 = 58/5, x_3 = 0 \text{ (non-basic) and } Z'_{\max} = 143/5$$

Hence $\qquad Z_{\min} = -143/5.$

Example 34.17. *Maximize* $Z = 107x_1 + x_2 + 2x_3$,
subject to the constraints : $14x_1 + x_2 - 6x_3 + 3x_4 = 7$,

$$16x_1 + \frac{1}{2}x_2 - 6x_3 \le 5,\ 3x_1 - x_2 - x_3 \le 0,\ x_1, x_2, x_3, x_4 \ge 0.$$

Solution. Consists of the following steps :

Step 1. Check whether objective function is to be maximized and all b's are non-negative.

This step is not necessary.

Step 2. Express the problem in the standard form.

Here x_4 is a slack variable. By introducing other slack variables s_1 and s_2 the problem in standard form becomes

Max. $\quad Z = 107x_1 + x_2 + 2x_3 + 0x_4 + 0s_1 + 0s_2$

subject to
$$\frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 + 0s_1 + 0s_2 = \frac{7}{3}$$
$$16x_1 + \frac{1}{2}x_2 - 6x_3 + 0x_4 + s_1 + 0s_2 = 5$$
$$3x_1 - x_2 - x_3 + 0x_4 + 0s_1 + s_2 = 0$$
$$x_1, x_2, x_3, x_4, s_1, s_2 \ge 0.$$

Step 3. Find initial basic feasible solution.

The basic feasible solution is

$$x_1 = x_2 = x_3 = 0 \text{ (non-basic)} ;\ x_4 = 7/3, s_1 = 5, s_2 = 0 \text{ (basic)}$$

$\therefore$ *Initial basic feasible solution* is given in the table below :

c_B	c_j Basis	107 x_1	1 x_2	2 x_3	0 x_4	0 s_1	0 s_2	b	θ
0	x_4	$\frac{14}{3}$	$\frac{1}{3}$	-2	1	0	0	$\frac{7}{3}$	$\frac{7}{3}/\frac{14}{3}$
0	s_1	16	$\frac{1}{2}$	-6	0	1	0	5	5/16
0	s_2	(3)	-1	-1	0	0	1	0	$0/3 \leftarrow$
$Z_j = \Sigma c_B a_{ij}$		0	0	0	0	0	0	0	
$C_j = c_j - Z_j$		107 $\uparrow$	1	2	0	0	0		

Step 4. Apply optimality test.

As C_j is positive under some columns, the initial basic feasible solution is not optimal and we proceed further.

Step 5. (i) Identify the incoming and outgoing variables.

The above table shows that x_1 is the incoming variable, s_2 is the outgoing variable and (3) is the key element.

(ii) Iterate towards the optimal solution.

Drop s_2 and introduce x_1 with its associated value 107 under c_B column. Convert key element to unity and make all other elements of the key column zeros. Then the *second basic feasible solution* is given by the following table :

	c_j	107	1	2	0	0	0		
c_B	Basis	x_1	x_2	x_3	x_4	s_1	s_2	b	θ
0	x_4	0	17/9	– 4/9	1	0	14/9	7/3	– 21/4
0	s_1	0	35/6	– 2/3	0	1	– 16/3	5	– 15/2
107	x_1	1	– 1/3	– 1/3	0	0	1/3	0	0
	Z_j	107	– 107/3	– 107/3	0	0	107/3		
	C_j	0	110/3	113/3	0	0	– 107/3		
				↑					

As C_j is positive under some columns, the solution is not optimal. Here 113/3 being the largest positive value of C_j, x_3 is the incoming variable. But all the values of θ being ≤ 0, x_3 will not enter the basis. This indicates that the solution to the problem is unbounded.

[**Remember that** (*i*) *the incoming variable is the non-basic variable corresponding to the largest positive value of* C_j *and*

(*ii*) *the outgoing variable is the basic-variable corresponding to the least positive ratio* θ, *obtained by dividing the b-column elements by the corresponding key-column elements.*]

PROBLEMS 34.4

Using simplex method, solve the following *L.P.P.* (1–8) :

1. Maximize $Z = x_1 + 3x_2$,
   subject to $x_1 + 2x_2 \le 10$, $0 \le x_1 \le 5$, $0 \le x_2 \le 4$. (*Kurushetra, 2009 ; V.T.U., 2003*)
2. Maximize $Z = 4x_1 + 10x_2$
   subject to $2x_1 + x_2 \le 50$, $2x_1 + 5x_2 \le 100$, $2x_1 + 3x_2 \le 90$, $x_1, x_2 \ge 0$. (*Kurushetra, 2006*)
3. Maximize $Z = 4x_1 + 5x_2$,
   subject to $x_1 - 2x_2 \le 2$, $2x_1 + x_2 \le 6$, $x_1 + 2x_2 \le 5$, $-x_1 + x_2 \le 2$, $x_1, x_2 \ge 0$.
4. Maximize $Z = 10x_1 + x_2 + 2x_3$
   subject to $x_1 + x_2 - 2x_3 \le 10$, $4x_1 + x_2 + x_3 \le 20$, $x_1, x_2, x_3 \ge 0$.
5. Maximize $Z = 3x_1 + 2x_2 + 5x_3$, subject to $x_1 + 2x_2 + x_3 \le 430$, $3x_1 + 2x_3 \le 460$, $x_1 + 4x_2 \le 420$, $x_1, x_2, x_3 \ge 0$. (*Mumbai, 2004*)
6. Minimize $Z = 3x_1 + 5x_2 + 4x_3$
   subject to $2x_1 + 3x_2 \le 8$, $2x_2 + 5x_3 \le 10$, $3x_1 + 2x_2 + 4x_3 \le 15$, $x_1, x_2, x_3 \ge 0$. (*Mumbai, 2004 S*)
7. Minimize $Z = x_1 - 3x_2 + 2x_3$,
   subject to $3x_1 - x_2 + 2x_3 \le 7$, $-2x_1 + 4x_2 \le 12$, $-4x_1 + 3x_2 + 8x_3 \le 10$, $x_1, x_2, x_3 \ge 0$. (*Madras, 2006*)
8. Maximize $Z = 4x_1 + 3x_2 + 4x_3 + 6x_4$
   subject to $x_1 + 2x_2 + 2x_3 + 4x_4 \le 80$, $2x_1 + 2x_3 + x_4 \le 60$, $3x_1 + 3x_2 + x_3 + x_4 \le 80$, $x_1, x_2, x_3, x_4 \ge 0$.
9. A firm produces products *A* and *B* and sells them at a profit of ₹ 2 and ₹ 3 each respectively. Each product is processed on machines *G* and *H*. Product *A* requires 1 minute on *G* and 2 minutes on *H* whereas product *B* requires 1 minute on each of the machines. Machine *G* is not available for more than 6 hrs. 40 min/day whereas the time constraint for machine *H* is 10 hrs. Solve this problem *via* simplex method for maximizing the profit.
10. A company makes two types of products. Each product of the first type requires twice as much labour time as the second type. If all products are of second type only, the company can produce a total of 500 units a day. The market limits daily sales of the first and the second type to 150 and 250 units respectively. Assuming that the profits per

unit are ₹ 8 for type I and ₹ 5 for type II, determine the number of units of each type to be produced to maximize profit ?

11. The owner of a dairy is trying to determine the correct blend of two types of feed. Both contain various percentages of four essential ingredients. With the following data determine the least cost blend ?

Ingredient	*% per kg of feed*		*Min requirement in kg.*
	Feed 1	*Feed 2*	
1	40	20	4
2	10	30	2
3	20	40	3
4	30	10	6
Cost (₹/kg.)	5	3	

12. A manufacturing firm has discontinued production of a certain unprofitable product line. This created considerable excess production capacity. Management is considering to devote their excess capacity to one or more of three product 1, 2, and 3. The available capacity on machines and the number of machine-hours required for each unit of the respective product, is given below :

Machine Type	*Available Time*	*Productivity (hrs/unit)*		
	(hrs/week)	*Product 1*	*Product 2*	*Product 3*
Milling machine	250	8	2	3
Lathe	150	4	3	[illegible]
Grinder	50	2	–	1

The unit profit would be ₹ 20, ₹ 6 and ₹ 8 respectively for products 1, 2 and 3. Find how much of each product the firm should produce in order to maximize profit.

13. The following table gives the various vitamin contents of three types of food and daily requirements of vitamins alongwith cost per unit. Find the combination of food for minimum cost.

Vitamin (mg)	*Food F*	*Food G*	*Food H*	*Minimum daily requirement (mg)*
A	1	1	10	1
C	100	10	10	50
D	10	100	10	10
Cost/unit (₹)	10	15	5	

14. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs ₹ 100 for preparation, requires 7 man-days of work and yields a profit of ₹ 30. An acre of wheat costs ₹ 120 to prepare, requires 10 man-days of work and yields a profit of ₹ 40. An acre of soyabeans costs ₹70 to prepare, requires 8 man-days of work and yields a profit of ₹ 20. If the farmer has ₹ 100,000 for preparation and can count on 8,000 man-days of work, how many acres should be allocated to each crop to maximize profits ?

34.9 ARTIFICIAL VARIABLE TECHNIQUES

So far we have seen that the introduction of slack/surplus variables provided the initial basic feasible solution. But there are many problems wherein at least one of the constraints is of (≥) or (=) type and slack variables fail to give such a solution. There are two similar methods for solving such problems which we explain below :

(1) M-method or Method of Penalties. This method is due to *A. Charnes* and consists of the following steps :

Step 1. Express the problem in standard form.

Step 2. Add non-negative variables to the left hand side of all those constraints which are of (≥) or (=) type. Such new variables are called *artificial variables* and the purpose of introducing these is just to obtain an initial basic feasible solution. But their addition causes violation of the corresponding constraints. As such, we would like to get rid of these variables and would not allow them to appear in the final solution. For this purpose, we assign a very large penalty ($-M$) to these artificial variables in the objective function.

Step 3. Solve the modified L.P.P. by simplex method.

At any iteration of simplex method, one of the following three cases may arise :

(*i*) There remains no artificial variable in the basis and the optimality condition is satisfied. Then the solution is an optimal basic feasible solution to the problem.

(*ii*) There is at least one artificial variable in the basis at zero level (with zero value in b-column) and the optimality condition is satisfied. Then the solution is a degenerate optimal basic feasible solution.

(*iii*) There is at least one artificial variable in the basis at non-zero level (with positive value in b-column) and the optimality condition is satisfied. Then the problem has no feasible solution. The final solution is not optimal, since the objective function contains an unknown quantity M. Such a solution satisfies the constraints but does not optimize the objective function and is therefore, called *pseudo optimal solution*.

Step 4. Continue the simplex method until either an optimal basic feasible solution is obtained or an unbounded solution is indicated.

Obs. The artificial variables are only a computational device for getting a starting solution. Once an artificial variable leaves the basis, it has served its purpose and we forget about it *i.e.*, the column for this variable is omitted from the next simplex table.

Example 34.18. *Use Charne's penalty method to*

Minimize $Z = 2x_1 + x_2$

subject to $3x_1 + x_2 = 3,\ 4x_1 + 3x_2 \geq 6,\ x_1 + 2x_2 \leq 3,\ x_1, x_2 \geq 0.$ *(Anna, M. Tech, 2006 ; V.T.U., 2000 S)*

Solution. Consists of the following steps :

Step 1. Express the problem in standard form.

The second and third inequalities are converted into equations by introducing the surplus and slack variables s_1, s_2 respectively.

Also the first and second constraints being of (=) and (≥) type, we introduce two artificial variables A_1, A_2.

Converting the minimization problem to the maximization form, the L.P.P. can be rewritten as

Max. $Z' = -2x_1 - x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$

subject to
$$3x_1 + x_2 + 0s_1 + 0s_2 + A_1 + 0A_2 = 3$$
$$4x_1 + 3x_2 - s_1 + 0s_2 + 0A_1 + A_2 = 6$$
$$x_1 + 2x_2 + 0s_1 + s_2 + 0A_1 + 0A_2 = 3$$
$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

Step 2. Obtain an initial basic feasible solution.

Surplus variable s_1 is not a basic variable since its value is -6. As negative quantities are not feasible, s_1 must be prevented from appearing in the initial solution. This is done by taking $s_1 = 0$. By setting the other non-basic variables x_1, x_2 each $= 0$, we obtain the initial basic feasible solution as

$$x_1 = x_2 = 0,\ s_1 = 0\ ;\ A_1 = 3, A_2 = 6, s_2 = 3$$

Thus the initial simplex table is

	c_j	-2	-1	0	0	$-M$	$-M$		
c_B	*Basis*	x_1	x_2	s_1	s_2	A_1	A_2	b	θ
$-M$	A_1	(3)	1	0	0	1	0	3	3/3←
$-M$	A_2	4	3	-1	0	0	1	6	6/4
0	s_2	1	2	0	1	0	0	3	3/1
$Z_j = \Sigma c_B a_{ij}$		$-7M$	$-4M$	M	0	$-M$	$-M$	$-9M$	
$C_j = c_j - Z_j$		$7M - 2$	$4M - 1$	$-M$	0	0	0		
		↑							

Since C_j is positive under x_1 and x_2 columns, this is not an optimal solution.

Step 3. Iterate towards optimal solution.

Introduce x_1, and drop A_1 from basis.

∴ The new simplex table is

c_B	c_j Basis	-2 x_1	-1 x_2	0 s_1	0 s_2	$-M$ A_2	b	θ
-2	x_1	1	1/3	0	0	0	1	3
$-M$	A_2	0	(5/3)	-1	0	1	2	6/5←
0	s_2	0	5/3	0	1	0	2	6/5
	Z_j	-2	$-\frac{2}{3}-\frac{5M}{3}$	M	0	$-M$	$-2-2M$	
	C_j	0	$-\frac{1}{3}+\frac{5M}{3}$ ↑	$-M$	0	0		

Since C_j is positive under x_2 column, this is not an optimal solution.

∴ Introduce x_2 and drop A_2.

Then the revised simplex table is

c_B	c_j Basis	-2 x_1	-1 x_2	0 s_1	0 s_2	b
-2	x_1	1	0	1/5	0	3/5
-1	x_2	0	1	$-3/5$	0	6/5
0	s_2	0	0	1	1	0
	Z_j	-2	-1	1/5	0	$-12/5$
	C_j	0	0	$-1/5$	0	

Since none of C_j is positive, this an optimal solution. Thus, an optimal basic feasible solution to the problem is

$$x_1 = 3/5, x_2 = 6/5, \text{Max. } Z' = -12/5.$$

Hence the optimal value of the objective function is

$$\text{Min. } Z = -\text{Max. } Z' = -(-12/5) = 12/5$$

Example 34.19. *Maximize $Z = 3x_1 + 2x_2$ subject to the constraints : $2x_1 + x_2 \le 2$, $3x_1 + 4x_2 \ge 12$, $x_1, x_2 \ge 0$.*

Solution. Consists of the following steps :

Step 1. Express the problem in standard form.

The inequalities are converted into equations by introducing the slack and surplus variables s_1, s_2 respectively. Also the second constraint being of (≥) type, we introduce the artificial variable A. Thus the L.P.P. can be rewritten as

Max. $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - MA$

subject to
$$2x_1 + x_2 + s_1 + 0s_2 + 0A = 2,$$
$$3x_1 + 4x_2 + 0s_1 - s_2 + A = 12,$$
$$x_1, x_2, s_1, A \ge 0$$

Step 2. Find an initial basic feasible solution.

Surplus variable s_2 is not a basic variable since its value is -12. Since a negative quantity is not feasible, s_2 must be prevented from appearing in the initial solution. This is done by letting $s_2 = 0$. By taking the other non-basic variables x_1 and x_2 each $= 0$, we obtain the initial basic feasible solution as

$$x_1 = x_2 = s_2 = 0, s_1 = 2, A = 12$$

∴ The initial simplex table is

c_B	c_j Basis	3 x_1	2 x_2	0 s_1	0 s_2	$-M$ A	b	θ
0	s_1	2	(1)	1	0	0	2	2←
$-M$	A	3	4	0	-1	1	12	3
$Z_j = \Sigma c_B a_{ij}$		$-3M$	$-4M$	0	M	$-M$	$-12M$	
$C_j = c_j - Z_j$		$3+3M$	$2+4M$ ↑	0	$-M$	0		

Since C_j is positive under some columns, this is not an optimal solution.

Step 3. Iterate towards optimal solution.

Introduce x_2 and drop s_1.

$\therefore$ The new simplex table is

	c_j	3	2	0	0	$-M$	
c_B	Basis	x_1	x_2	s_1	s_2	A	b
2	x_2	2	1	1	0	0	2
$-M$	A	-5	0	-4	-1	1	4
Z_j		$4+5M$	2	$2+4M$	M	$-M$	$4-4M$
C_j		$-(1+5M)$	0	$-(2+4M)$	$-M$	0	

Here each C_j is negative and an artificial variable appears in the basis at non-zero level. Thus there exists a *pseudo optimal solution* to the problem.

(2) Two-phase method. This is another method to deal with the artificial variables wherein the L.P.P. is solved in two phases.

Phase I. *Step 1.* Express the given problem in the standard form by introducing slack, surplus and artificial variables.

Step 2. Formulate an artificial objective function

$$Z^* = -A_1 - A_2 \dots - A_m$$

by assigning (-1) cost to each of the artificial variables A_i and zero cost to all other variables.

Step 3. Maximize Z^* subject to the constraints of the original problem using the simplex method. Then three cases arise :

(*a*) *Max. $Z^* < 0$ and at least one artificial variable appears in the optimal basis at a positive level*

In this case, the original problem dosn't possess any feasible solution and the procedure comes to an end.

(*b*) *Max. $Z^* = 0$ and no artificial variable appears in the optimal basis.*

In this case, a basic feasible solution is obtained and we proceed to phase II for finding the optimal basic feasible solution to the original problem.

(*c*) *Max. $Z^* = 0$ and at least one artificial variable appears in the optimal basis at zero level.*

Here a feasible solution to the auxiliary L.P.P. is also a feasible solution to the original problem with all artificial variables set = 0.

To obtain a basic feasible solution, we prolong phase I for pushing all the artificial variables out of the basis (without proceeding on to phase II).

Phase II. The basic feasible solution found at the end of phase I is used as the starting solution for the original problem in this phase *i.e.*, the final simplex table of phase I is taken as the initial simplex table of phase II and the artificial objective function is replaced by the original objective function. Then we find the optimal solution.

Example 34.20. *Use two-phase method to*

Minimize $Z = 7.5x_1 - 3x_2$

subject to the constraints $3x_1 - x_2 - x_3 \geq 3,\ x_1 - x_2 + x_3 \geq 2,$

$x_1, x_2, x_3 \geq 0.$

Phase I. *Step 1. Express the problem in standard form.*

Solution. Introducing surplus variables s_1, s_2 and artificial variables A_1, A_2, the phase I problem in standard form becomes

Max. $\quad Z^* = 0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 - A_1 - A_2$

subject to
$$3x_1 - x_2 - x_3 - s_1 + 0s_2 + A_1 + 0A_2 = 3$$
$$x_1 - x_2 + x_3 + 0s_1 - s_2 + 0A_1 + A_2 = 2$$
$$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0.$$

Step 2. Find an initial basic feasible solution.

Setting $x_1 = x_2 = x_3 = s_1 = s_2 = 0$,

we have $\quad A_1 = 3, A_2 = 2$ and $Z^* = -5$

$\therefore$ Initial simplex table is

	c_j	0	0	0	0	0	-1	-1		
c_B	*Basis*	x_1	x_2	x_3	s_1	s_2	A_1	A_2	b	θ
-1	A_1	(3)	-1	-1	-1	0	1	0	3	1←
-1	A_2	1	-1	1	0	-1	0	1	2	2
$Zj^* = \Sigma c_B a_{ij}$		-4	2	0	1	1	-1	-1	-5	
$C_j = c_j - Z_j^*$		4 ↑	-2	0	-1	-1	0	0		

As C_j is positive under x_1 column, this solution is not optimal.

Step 3. Iterate towards an optimal solution.

Making key element (3) unity and replacing A_1 by x_1, we have the new simplex table :

	c_j	0	0	0	0	0	-1	-1		
c_B	Basis	x_1	x_2	x_3	s_1	s_2	A_1	A_2	b	θ
0	x_1	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	1	-3
-1	A_2	0	$-\frac{2}{3}$	$\left(\frac{4}{3}\right)$	$\frac{1}{3}$	-1	$-\frac{1}{3}$	1	1	$\frac{3}{4}$←
	Z_j^*	0	$\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{1}{3}$	1	$\frac{1}{3}$	-1	-1	
	C_j	0	$-\frac{2}{3}$	$\frac{4}{3}$ ↑	$\frac{1}{3}$	-1	$-\frac{1}{3}$	0		

Since C_j is positive under x_3 and s_1 columns, this solution is not optimal.

Making key element (4/3) unity and replacing A_2 by x_3, we obtain the revised simplex table :

	c_j	0	0	0	0	0	-1	-1	
C_B	*Basis*	x_1	x_2	x_3	s_1	s_2	A_1	A_2	b
0	x_1	1	$-1/2$	0	$-1/4$	$-1/4$	1/4	1/4	5/4
0	x_2	0	$-1/2$	1	1/4	$-3/4$	$-1/4$	3/4	3/4
Z_j^*		0	0	0	0	0	0	0	0
C_j		0	0	0	0	0	-1	-1	

Since all $C_j \leq 0$, this table gives the optimal solution. Also $Z^*_{\max} = 0$ and no artificial variable appears in the basis. Thus an optimal basic feasible solution to the auxiliary problem and therefore to the original problem, has been attained.

Phase II. Considering the actual costs associated with the original variables, the objective function is

Max. $\quad Z' = -15/2x_1 + 3x_2 + 0x_3 + 0s_1 + 0s_2 - 0A_1 - 0A_2$

subject to
$$3x_1 - x_2 - x_3 - s_1 + 0s_2 + A_1 + 0A_2 = 3,$$
$$x_2 - x_2 + x_3 + 0s_1 - s_2 + 0A_1 + A_2 = 2,$$
$$x_2, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$$

The optimal initial feasible solution thus obtained, will be an optimal basic feasible solution to the original L.P.P.

Using final table of phase I, the initial simplex table of phase II is as follows :

	c_j	$-15/2$	3	0	0	0	
c_B	*Basis*	x_1	x_2	x_3	s_1	s_2	b
$-15/2$	x_1	1	$-1/2$	0	$-1/4$	$-1/4$	5/4
0	x_3	0	$-1/2$	1	1/4	$-3/4$	3/4
Z_j		$-15/2$	15/4	0	15/8	15/8	$-75/8$
C_j		0	$-3/4$	0	$-15/8$	$-15/8$	

Since all $C_j \leq 0$, this solution is optimal.

Hence an optimal basic feasible solution to the given problem is

$$x_1 = 5/4, x_2 = 0, x_3 = 3/4 \quad \text{and} \quad \text{min. } Z = 75/8.$$

34.10 EXCEPTIONAL CASES

(1) Tie for the incoming variable. When more than one variable has the same largest positive value in C_j row (in maximization problem), a tie for the choice of incoming variable occurs. As there is no method to break this tie, we choose any one of the prospective incoming variables arbitrarily. Such an arbitrary choice does not in any way affect the optimal solution.

(2) Tie for the outgoing variable. When more than one variable has the same least positive ratio under then θ-column, a tie for the choice of outgoing variable occurs. If the equal values of said ratio are > 1, choose any one of the prospective leaving variables arbitrarily. Such an arbitrary choice doesn't affect the optimal solution.

If the equal values of ratios are zero, the simplex method fails and we make use of the following degeneracy technique.

(3) Degeneracy. We know that a basic feasible solution is said to be degenerate if any of the basic variables vanishes. This phenomenon of getting a degenerate basic feasible solution is called *degeneracy* which may arise

(*i*) *at the initial state,* when atleast one basic variable is zero in the initial basic feasible solution

or (*ii*) *at any subsequent stage,* when the least positive ratios under θ-column are equal for two or more rows.

In this case, an arbitrary choice of one of these basic variables may result in one or more basic variables becoming zero in the next iteration. At times, the same sequence of simplex iterations is repeated endlessly without improving the solution. These are termed as *cycling* type of problems. Cycling occurs very rarely. In fact, cycling has seldom occurred in practical problems.

To avoid cycling, we apply the following **perturbation procedure :**

(*i*) Divide each element in the tied rows by the *positive coefficients* of the key column in that row.

(*ii*) Compare the resulting ratios (from left to right) first of unit matrix and then of the body matrix, column by column.

(*iii*) The outgoing variable lies in that row which first contains the smallest algebraic ratio.

Example 34.21. *Maximize* $Z = 5x_1 + 3x_2$
subject to $x_1 + x_2 \le 2$, $5x_1 + 2x_2 \le 10$, $3x_1 + 8x_2 \le 12$; $x_1, x_2 \ge 0$.

Solution. Consists of the following steps :

Step 1. Express the problem in the standard form.

Introducing the slack variables s_1, s_2, s_3, the problem in the standard form is

$$\text{Max. } Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$
$$x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 2$$
$$5x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 10$$
$$3x_1 + 8x_2 + 0s_1 + 0s_2 + s_3 = 12$$
$$x_1, x_2, s_1, s_2, s_3 \ge 0.$$

Step 2. Find the initial basic feasible solution.

The initial basic feasible solution is

$$x_1 = x_2 = 0 \text{ (non-basic)}$$
$$s_1 = 2, s_2 = 10, s_3 = 12 \text{ (basic) and } Z = 0.$$

∴ Initial simplex table is

	c_j	5	3	0	0	0		
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	1	1	1	0	0	2	2/1
0	s_2	(5)	2	0	1	0	10	10/5←
0	s_3	3	8	0	0	1	12	12/3
$Z_j = \Sigma c_B a_{ij}$		0	0	0	0	0	0	
$C_j = c_j - Z_j$		5	3	0	0	0		
		↑						

As C_j is positive under x_2 columns, this solution is not optimal.

Step 3. Iterate towards optimal solution.

x_1 is the incoming variable. But the first two rows have the same ratio under θ-column. Therefore we apply *perturbation* method.

First column of the unit matrix has 1 and 0 in the tied rows. Dividing these by the corresponding elements of the key columns, we get 1/1 and 0/5, s_2-row gives the smaller ratio and therefore s_2 is the first outgoing variable and (5) is the key element.

Thus the new simplex table is

	c_j	5	3	0	0	0		
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	0	(3/5)	1	− 1/5	0	0	0
5	x_1	1	2/5	0	1/5	0	2	5←
0	s_3	0	34/5	0	− 3/5	1	6	15/17
Z_j		5	2	0	1	0	10	
C_j		0	1	0	− 1	0		
			↑					

As C_j is positive under x_2 column, this solution is not optimal.

Making key element (3/5) unity and replacing s_1 by x_2, we obtain the revised simplex table :

	c_j	5	3	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	b
3	x_2	0	1	5/3	− 1/3	0	0
5	x_1	1	0	− 2/3	1/3	0	2
0	s_3	0	0	− 34/3	5/3	1	6
Z_j		5	3	5/3	2/3	0	10
C_j		0	0	− 5/3	− 2/3	0	

As $C_j \le 0$ under all columns, this table gives the optimal solution. Hence an optimal basic feasible solution is $x_1 = 2$, $x_2 = 0$ and $Z_{\max} = 10$.

PROBLEMS 34.5

Solve the following L.P. problems using *M*-method :

1. Maximize $Z = 3x_1 + 2x_2 + 3x_3$
   subject to : $2x_1 + x_2 + x_3 \le 2$, $3x_1 + 4x_2 + 2x_3 \ge 8$, $x_1, x_2, x_3 \ge 0$.
2. Maximize $Z = 2x_1 + x_2 + 3x_3$
   subject to : $x_1 + x_2 + 2x_3 \le 5$, $2x_1 + 3x_2 + 4x_3 = 12$, $x_1, x_2, x_3 \ge 0$.
3. Maximize $Z = 8x_2$,
   subject to : $x_1 - x_2 \ge 0$, $2x_1 + 3x_2 \le -6$, x_1, x_2 unrestricted.
4. Maximize $Z = 5x_1 - 2x_2 + 3x_3$
   subject to : $2x_1 + 2x_2 - x_3 \ge 2$, $3x_1 - 4x_2 \le 3$, $x_2 + 3x_3 \le 5$, $x_1, x_2, x_3 \ge 0$. *(Mumbai, 2004)*
5. Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$
   subject to : $x_1 + 2x_2 + 3x_3 = 15$, $2x_1 + x_2 + 5x_3 = 20$,
   $x_1 + 2x_2 + x_3 + x_4 = 10$, $x_1, x_2, x_3, x_4 \ge 0$. *(Madras, 2003)*

Use two phase method to solve the following L.P. problems :

6. Minimize $Z = x_1 + x_2$
   subject to : $2x_1 + x_2 \ge 4$, $x_1 + 7x_2 \ge 7$,
   $x_1, x_2 \ge 0$. *(Rajasthan, 2005)*
7. Maximize $Z = 5x_1 + 3x_2$
   subject to : $2x_1 + x_2 \le 1$, $x_1 + 4x_2 \ge 6$,
   $x_1, x_2 \ge 0$. *(Kottayam, 2005)*

8. Maximize $Z = 5x_1 - 4x_2 + 3x_3$,
subject to : $2x_1 + 2x_2 - x_3 \geq 2$,
$3x_1 - 4x_2 \leq 3, x_2 + x_3 \leq 5$,
$x_1, x_2, x_3 \geq 0$. *(Mumbai, 2009)*

9. Maximize $Z = 5x_1 - 4x_2 + 3x_3$,
subject to : $2x_1 + x_2 - 6x_3 = 20$,
$6x_1 + 5x_2 + 10x_3 \leq 76$,
$8x_1 - 3x_2 + 6x_3 \leq 50$,
$x_1, x_2, x_3 \geq 0$.

Solve the following degenerate L.P. problems :

10. Maximize $Z = 9x_1 + 3x_2$
subject to : $4x_1 + x_2 \leq 8, 2x_1 + x_2 \leq 4$,
$x_1, x_2 \geq 0$.

11. Maximize $Z = 2x_1 + 3x_2 + 10x_3$
subject to : $x_1 + 2x_3 = 0, x_2 + x_3 = 1$,
$x_1, x_2, x_3 \geq 0$.

34.11 (1) DUALITY CONCEPT

One of the most interesting concepts in linear programming is the *duality* theory. Every linear programming problem has associated with it, another linear programming problem involving the same data and closely related optimal solutions. Such two problems are said to be *duals* of each other. While one of these is called the *primal*, the other the *dual*.

The importance of the duality concept is due to two main reasons. Firstly, if the primal contains a large number of constraints and a smaller number of variables, the labour of computation can be considerably reduced by converting it into the dual problem and then solving it. Secondly, the interpretation of the dual variables from the cost or economic point of view proves extremely useful in making future decisions in the activities being programmed.

(2) Formulation of dual problem. Consider the following L.P.P. :

Maximize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$,

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\leq b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\leq b_2, \\ &\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\leq b_m \\ x_1, x_2, \ldots, x_n &\geq 0. \end{aligned}$$

To construct the dual problem, we adopt the following guidelines :

(*i*) The maximization problem in the primal becomes the minimization problem in the dual and *vice versa*.

(*ii*) ($\leq$) type of constraints in the primal become ($\geq$) type of constraints in the dual and *vice versa*.

(*iii*) The coefficients $c_1, c_2, \ldots, c_n$ in the objective function of the primal become $b_1, b_2, \ldots, b_m$ in the objective function of the dual.

(*iv*) The constants $b_1, b_2, \ldots, b_m$ in the constraints of the primal become $c_1, c_2, \ldots, c_n$ in the constraints of the dual.

(*v*) If the primal has n variables and m constraints, the dual will have m variables and n constraints *i.e.* the transpose of the body matrix of the primal problem gives the body matrix of the dual.

(*vi*) The variables in both the primal and dual are non-negative.

Then the dual problem will be

Minimize $W = b_1y_1 + b_2y_2 + \ldots + b_my_m$

subject to the constraints

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m &\geq c_1, \\ a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m &\geq c_2, \\ &\ldots\ldots \\ a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m &\geq c_n, \\ y_1, y_2, \ldots, y_m &\geq 0. \end{aligned}$$

Example 34.22. *Write the dual of the following L.P.P.:*

Minimize $Z = 3x_1 - 2x_2 + 4x_3$,

subject to $3x_1 + 5x_2 + 4x_3 \geq 7, 6x_1 + x_2 + 3x_3 \geq 4, 7x_1 - 2x_2 - x_3 \leq 10$,
$x_1 - 2x_2 + 5x_3 \geq 3, 4x_1 + 7x_2 - 2x_3 \geq 2, x_1, x_2, x_3 \geq 0$.

Solution. Since the problem is of minimization, all constraints should be of $\geq$ type. We multiply the third constraint throughout by -1 so that $-7x_1 + 2x_2 + x_3 \geq -10$.

Let y_1, y_2, y_3, y_4 and y_5 be the dual variables associated with the above five constraints. Then the dual problem is given by

Maximize $\quad W = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$

subject to $\quad 3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3,\ 5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2,$

$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4,\ y_1, y_2, y_3, y_4, y_5 \geq 0.$

(3) Formulation of dual problem when the primal has equality constraints. Consider the problem

Maximize $\quad Z = c_1x_1 + c_2x_2$

subject to $\quad a_{11}x_1 + a_{12}x_2 = b_1,\ a_{21}x_1 + a_{22}x_2 \leq b_2,\ x_1, x_2 \geq 0.$

The equality constraint can be written as

$$a_{11}x_1 + a_{12}x_2 \leq b_1 \text{ and } a_{11}x_1 + a_{12}x_2 \geq b_1$$

or
$$a_{11}x_1 + a_{12}x_2 \leq b_1 \text{ and } -a_{11}x_1 - a_{12}x_2 \leq -b_1,$$

Then the above problem can be restated as

Maximize $\quad Z = c_1x_1 + c_2x_2$

subject to $\quad a_{11}x_1 + a_{12}x_2 \leq b_1, -a_{11}x_1 - a_{12}x_2 \leq -b_1,$

$a_{21}x_1 + a_{22}x_2 \leq b_2,\ x_1, x_2 \geq 0.$

Now we form the dual using y_1', y_1'', y_2 as the dual variables. Then the dual problem is

Minimize $\quad W = b_1(y_1' - y_1'') + b_2y_2,$

subject to $\quad a_{11}(y_1' - y_1'') + a_{21}y_2 \geq c_1,\ a_{12}(y_1' - y_1'') + a_{22}y_2 \geq c_2,\ y_1' y_1'', y_2 \geq 0.$

The term $(y_1' - y_1'')$ appears in both the objective function and all the constraints of the dual. This will always happen whenever there is an equality constraint in the primal. Then the new variable $y_1' - y_1''\ (= y_1)$ becomes unrestricted in sign being the difference of two non-negative variables and the above dual problem takes the form.

Minimize $\quad W = b_1y_1 + b_2y_2,$

subject to $\quad a_{11}y_1 + a_{21}y_2 \geq c_1,\ a_{12}y_1 + a_{22}y_2 \geq c_2,\ y_1$ unrestricted in sign, $y_2 \geq 0.$

In general, if the primal problem is

Maximize $\quad Z = c_1x_1 + c_2x_2 + \dots + c_nx_n,$

subject to
$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$
$$x_1, x_2, \dots, x_n \geq 0,$$

then the dual problem is

Minimize $\quad W = b_1y_1 + b_2y_2 + \dots + b_my_m$

subject to
$$a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m \geq c_1,$$
$$a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m \geq c_2,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \geq c_n$$
$$y_1, y_2, \dots, y_m \text{ all unrestricted in sign.}$$

Thus *the dual variables corresponding to equality constraints are unrestricted in sign*. Conversely *when the primal variables are unrestricted in sign, corresponding dual constraints are equalities.*

Example 34.23. *Construct the dual of the L.P.P. :*

Maximize $\quad Z = 4x_1 + 9x_2 + 2x_3,$

subject to $\quad 2x_1 + 3x_2 + 2x_3 \leq 7,\ 3x_1 - 2x_2 + 4x_3 = 5,\ x_1, x_2, x_3 \geq 0.$

Solution. Let y_1 and y_2 by the dual variables associated with the first and second constraints. Then the dual problem is

Minimize $\quad W = 7y_1 + 5y_2,$

subject to $\quad 2y_1 + 3y_2 \leq 4,\ 3y_1 - 2y_2 \leq 9,\ 2y_1 + 4y_2 \leq 2,\ y_1 \geq 0,\ y_2$ is unrestricted in sign.

PROBLEMS 34.6

Write the duals of the following problems (1 – 4) :

1. Maximize $Z = 10x_1 + 13x_2 + 19x_3$
   subject to $6x_1 + 5x_2 + 3x_3 \le 26,\ 4x_1 + 2x_2 + 5x_3 \le 7,\ x_1, x_2, x_3 \ge 0.$
2. Minimize $Z = 2x_1 + 4x_2 + 3x_3$,
   subject to $3x_1 + 4x_2 + x_3 \ge 11,\ -2x_1 - 3x_2 + 2x_3 \le -7,\ x_1 - 2x_2 - 3x_3 \le -1$
   $3x_1 + 2x_2 + 2x_3 \ge 5,\ x_1, x_2, x_3 \ge 0.$
3. Maximize $Z = 3x_1 + 16x_2 + 7x_3$
   subject to $x_1 - x_2 + x_3 \ge 3,\ -3x_1 + 2x_3 \le 1,\ 2x_1 + x_2 - x_3 = 4,\ x_1, x_2, x_3 \ge 0.$
4. Minimize $Z = 3x_1 - 3x_2 + x_3$
   subject to $2x_1 - 3x_2 + x_3 \le 5,\ 4x_1 - 2x_2 \ge 9,\ -8x_1 + 4x_2 + 3x_3 = 8,$
   $x_1, x_2 \ge 0$ and x_3 is unrestricted. *(Mumbai, 2004)*
5. Obtain the dual problem of the following L.P.P.
   Maximize $f(x) = 2x_1 + 5x_2 + 6x_3$
   subject to $5x_1 + 6x_2 - x_3 \le 3,\ -2x_1 + x_2 + 4x_3 \le 4,\ x_1 - 5x_2 + 3x_3 \le 1,$
   $-3x_1 - 3x_2 + 7x_3 \le 6,\ x_1, x_2, x_3 \ge 0.$
   Also verify that the dual of the dual problem is the primal problem.

34.12 (1) DUALITY PRINCIPLE

If the primal and the dual problems have feasible solutions then both have optimal solutions and the optimal value of the primal objective function is equal to the optimal value of the dual objective function i.e.,

$$\text{Max. } Z = \text{Min. } W$$

This is the fundamental theorem of duality. It suggests that an optimal solution to the primal problem can directly be obtained from that of the dual problem and *vice-versa*.

(2) Working rules for obtaining an optimal solution to the primal (dual) problem from that of the dual (primal) :

Suppose we have already found an optimal solution to the dual (primal) problem by simplex method.

Rule I. If the primal variable corresponds to a slack starting variable in the dual problem, then its optimal value is directly given by the coefficient of the slack variable with changed sign, in the C_j row of the optimal dual simplex table and *vice-versa*.

Rule II. If the primal variable corresponds to an artificial variable in the dual problem, then its optimal value is directly given by the coefficient of the artificial variable, with changed sign, in the C_j row of the optimal dual simplex table, after deleting the constant M and *vice-versa*.

On the other hand, if the primal has an unbounded solution, then the dual problem will not have a feasible solution and *vice-versa*.

Now we shall workout two examples to demonstrate the primal dual relationships.

Example 34.24. *Construct the dual of the following problem and solve both the primal and the dual :*
Maximize $Z = 2x_1 + x_2$,
subject to $-x_1 + 2x_2 \le 2\ x_1 + x_2 \le 4,\ x_1 \le 3,\ x_1, x_2 \ge 0.$ *(Rohtak, 2005)*

Solution. *Using the primal problem.* Since only two variables are involved, it is convenient to solve the problem graphically.

In the x_1, x_2-plane, the five constraints show that the point (x_1, x_2) lies within the shaded region *OABCD* of Fig. 34.12. Values of the objective function $Z = 2x_1 + x_2$ at these corners are $Z(0) = 0$, $Z(A) = 6$, $Z(B) = 7$, $Z(C) = 6$ and $Z(D) = 1$. Hence the optimal solution is $x_1 = 3, x_2 = 1$ and max. $(Z) = 7$.

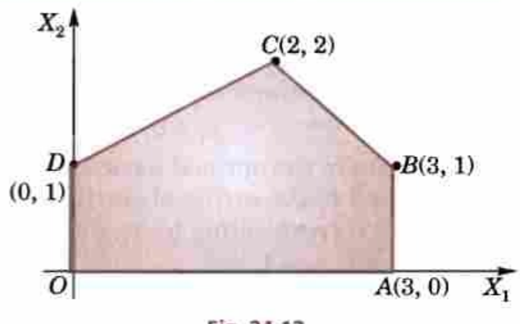

Fig. 34.12

Solution. *Using the dual problem.* The dual problem of the given primal is :

Minimize $W = 2y_1 + 4y_2 + 3y_3$

subject to $-y_1 + y_2 + y_3 \geq 2, 2y_1 + y_2 \geq 1, y_1, y_2 \geq 0.$

Step 1. Express the problem in the standard form.

Introducing the slack and the artificial variables, the dual problem in the standard form is

Max. $W' = -2y_1 - 4y_2 - 3y_3 + 0s_1 + 0s_2 - MA_1 - MA_2$

subject to $-y_1 + y_2 + y_3 - s_1 + 0s_2 + A_1 + 0A_2 = 2,$

$2y_1 + y_2 + 0y_3 + 0s_1 - s_2 + 0A_1 + A_2 = 1$

Step 2. Find an initial basic feasible solution.

Setting the non-basic variables y_1, y_2, y_3, s_1, s_2, each equal to zero, we get the initial basic feasible solution as

$$y_1 = y_2 = y_3 = s_1 = s_2 = 0 \text{ (non-basic)}; A_1 = 2, A_2 = 1. \quad \text{(basic)}$$

$\therefore$ Initial simplex table is

	c_j	-2	-4	-3	0	0	$-M$	$-M$		
c_B	*Basis*	y_1	y_2	y_3	s_1	s_2	A_1	A_2	b	θ
$-M$	A_1	-1	1	1	-1	0	1	0	2	2/1
$-M$	A_2	2	(1)	0	0	-1	0	1	1	1/1←
	Z_j	$-M$	$-2M$	$-M$	M	M	$-M$	$-M$	$-3M$	
	C_j	$M-2$	$2M-4$	$M-3$	$-M$	$-M$	0	0		
			↑							

As C_j is positive under some columns, the initial solution is not optimal.

Step 3. Iterate towards an optimal solution.

(*i*) Introduce y_2 and drop A_2. Then the new simplex table is

	c_j	-2	-4	-3	0	0	$-M$	$-M$		
c_B	*Basis*	y_1	y_2	y_3	s_1	s_2	A_1	A_2	b	θ
$-M$	A_1	-3	0	(1)	-1	1	1	-1	1	1/1←
-4	y_2	2	1	0	0	-1	0	1	1	1/0
	Z_j	$3M-8$	-4	$-M$	M	$4-M$	$-M$	$M-4$	$-M-4$	
	C_j	$6-3M$	0	$M-3$	$-M$	$M-4$	0	$4-2M$		
				↑						

As C_j is positive under some columns, this solution is not optimal.

(*ii*) Now introduce y_3 and drop A_1. Then the revised simplex table is

	c_j	-2	-4	-3	0	0	$-M$	$-M$	
c_B	*Basis*	y_1	y_2	y_3	s_1	s_2	A_1	A_2	b
-3	y_3	-3	0	1	-1	1	1	-1	1
-4	y_2	2	1	0	0	-1	0	1	1
	Z_j	1	-4	-3	3	1	-3	-1	-7
	C_j	-3	0	0	-3	-1	$3-M$	$1-M$	

As all $C_j \leq 0$, the optimal solution is attained.

Thus an optimal solution to the dual problem is

$$y_1 = 0, y_2 = 1, y_3 = 1, \text{ Min. } W = -\text{Max. } (W') = 7.$$

To derive the optimal basic feasible solution to the primal problem, we note that the primal variables x_1, x_2 correspond to the artificial starting dual variables A_1, A_2 respectively. In the final simplex table of the dual problem, C_j corresponding to A_1, and A_2 are 3 and 1 respectively after ignoring M. Thus by rule II, we get opt. $x_1 = 3$ and opt. $x_2 = 1$.

Hence an optimal basic feasible solution to the given primal is

$$x_1 = 3, x_2 = 1 \text{ ; max. } Z = 7.$$

Obs. The validity of the duality theorem is therefore, checked since max. Z = min. W = 7 from both the methods.

Example 34.25. *Using duality solve the following problem :*
Minimize $Z = 0.7x_1 + 0.5x_2$
subject to $x_1 \geq 4, x_2 \geq 6, x_1 + 2x_2 \geq 20, 2x_1 + x_2 \geq 18, x_1, x_2 \geq 0.$ (V.T.U., 2004)

Solution. The dual of the given problem is Max. $W = 4y_1 + 6y_2 + 20y_3 + 18y_4$,
subject to $y_1 + y_3 + 2y_4 \leq 0.7, y_2 + 2y_3 + y_4 \leq 0.5, y_1, y_2, y_3, y_4 \geq 0.$

Step 1. Express the problem in the standard form.

Introducing slack variables, the dual problem in the standard form becomes

Max. $W = 4y_1 + 6y_2 + 20y_3 + 18y_4 + 0s_1 + 0s_2$,
subject to $y_1 + 0y_2 + y_3 + 2y_4 + s_1 + 0s_2 = 0.7$,
$0y_1 + y_2 + 2y_3 + y_4 + 0s_1 + s_2 = 0.5, y_1, y_2, y_3, y_4 \geq 0.$

Step 2. Find an initial basic feasible solution.

Setting non-basic variables y_1, y_2, y_3, y_4 each equal to zero, the basic solution is

$y_1 = y_2 = y_3 = y_4 = 0$ (non-basic) ; $s_1 = 0.7, s_2 = 0.5$ (basic)

Since the basic variables $s_1, s_2 > 0$, the initial basic solution is feasible and non-degenerate.

Initial simplex table is

	c_j	4	6	20	18	0	0		
c_B	*Basis*	y_1	y_2	y_3	y_4	s_1	s_2	b	θ
0	s_1	1	0	1	2	1	0	0.7	0.7/1
0	s_2	0	1	(2)	1	0	1	0.5	0.5/2←
	Z_j	0	0	0	0	0	0	0	
	C_j	4	6	20	18	0	0		
				↑					

As C_j is positive in some columns, the initial basic solution is not optimal.

Step 3. Iterate towards an optimal solution.

(*i*) Introduce y_3 and drop s_2. Then the new simplex table is

	c_j	4	6	20	18	0	0		
c_B	*Basis*	y_1	y_2	y_3	y_4	s_1	s_2	b	θ
0	s_1	1	– 1/2	0	(3/2)	1	– 1/2	9/20	3/10←
20	y_3	0	1/2	1	$\frac{1}{2}$	0	1/2	1/4	1/2
	Z_j	0	10	20	10	0	10	5	
	C_j	4	– 4	0	8	0	– 10		
					↑				

As C_j, is positive under some of the columns, this solution is not optimal.

(*ii*) Introduce y_4 and drop s_1. Then the revised simplex table is

	c_j	4	6	20	18	0	0	
c_B	*Basis*	y_1	y_2	y_3	y_4	s_1	s_2	b
18	y_4	2/3	– 1/3	0	1	2/3	– 1/3	3/10
20	y_3	– 1/3	2/3	1	0	– 1/3	2/3	1/10
	Z_j	16/3	22/3	20	18	16/3	22/3	74/10
	C_j	– 4/3	– 4/3	0	0	– 16/13	– 22/3	

As all $C_j \le 0$, this table gives the optimal solution.

Thus the optimal basic feasible solution is $y_1 = 0, y_2 = 0, y_3 = 20, y_4 = 18$, max. $W = 7.4$

Step 4. Derive optimal solution to the primal.

We note that the primal variables x_1, x_2 correspond to the slack starting dual variables s_1, s_2 respectively. In the final simplex table of the dual problem, C_j values corresponding to s_1 and s_2 are $-16/3$ and $-22/3$ respectively.

Thus, by rule I, we conclude that opt. $x_1 = 16/3$ and opt. $x_2 = 22/3$.

Hence an optimal basic feasible solution to the given primal is

$$x_1 = 16/3, x_2 = 22/3; \text{ min. } Z = 7.4.$$

Obs. To check the validity of the duality theorem, the student is advised to solve the given L.P.P. directly by simplex method and see that min. Z = max. $W = 7.4$.

PROBLEMS 34.7

Using duality solve the following problems (1 – 4) :

1. Minimize $Z = 2x_1 + 9x_2 + x_3$,
   subject to $x_1 + 4x_2 + 2x_3 \ge 5, 3x_1 + x_2 + 2x_3 \ge 4$ and $x_1, x_2 \ge 0$. *(J.N.T.U., 2001)*
2. Maximize $Z = 2x_1 + x_2$,
   subject to $x_1 + 2x_2 \le 10, x_1 + x_2 \le 6, x_1 - x_2 \le 2, x_1 - 2x_2 \le 1, x_1, x_2 \ge 0$. *(Andhra M. Tech., 2006)*
3. Maximize $Z = 3x_1 + 2x_2$,
   subject to $x_1 + x_2 \ge 1, x_1 + x_2 \le 7, x_1 + 2x_2 \le 10, x_2 \le 3, x_1, x_2 \ge 0$.
4. Maximize $Z = 3x_1 + 2x_2 + 5x_3$
   subject to $x_1 + 2x_2 + x_3 \le 430, 3x_1 + 2x_3 \le 460, x_1 + 4x_2 \le 420, x_1, x_2, x_3 \ge 0$.

34.13 (1) DUAL SIMPLEX METHOD

In § 34.9, we have seen that a set of basic variables giving a feasible solution can be found by introducing artificial variables and using M-method or Two phase method. Using the primal-dual relationships for a problem, we have another method (known as *Dual simplex method*) for finding an initial feasible solution. Whereas the regular simplex method starts with a basic feasible (but non-optimal) solution and works towards optimality, the dual simplex method starts with a basic unfeasible (but optimal) solution and works towards feasibility. The dual simplex method is quite similar to the regular simplex method, the only difference lies in the criterion used for selecting the incoming and outgoing variables. In the dual simplex method, we first determine the outgoing variable and then the incoming variable while in the case of regular simplex method reverse is done.

(2) Working procedure for dual simplex method :

Step 1. (i) Convert the problem to maximization form, if it is not so.

(ii) Convert ($\ge$) *type constraints, if any to* ($\le$) *type* by multiplying such constraints by -1.

(iii) Express the problem in standard form by introducing slack variables.

Step 2. Find the initial basic solution and express this information in the form of dual simplex table.

Step 3. Test the nature of $C_j = c_j - Z_j$:

(*a*) If all $C_j \le 0$ and all $b_i \ge 0$, then optimal basic feasible solution has been attained.

(*b*) If all $C_j \le 0$ and at least one $b_i < 0$, then go to step 4.

(*c*) If any $C_j \ge 0$, the method fails.

Step 4. Mark the outgoing variable. Select the row that contains the most negative b_i. This will be the key row and the corresponding basic variable is the outgoing variable.

Step 5. Test the nature of key row elements :

(*a*) If all these elements are ≥ 0, the problem does not have a feasible solution.

(*b*) If at least one element < 0, find the ratios of the corresponding elements of C_j-row to these elements. Choose the smallest of these ratios. The corresponding column is the key column and the associated variable is the *incoming variable.*

Step 6. Iterate towards optimal feasible solution. Make the key element unity. Perform row operations as in the regular simplex method and repeat iterations until either an optimal feasible solution is attained or there is an indication of non-existence of a feasible solution.

Example 34.26. *Using dual simplex method :*
maximize $-3x_1 - 2x_2$
subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ *(Mumbai, 2004)*

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	$-10 \leftarrow$
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
			$\uparrow$					

Step 3. Test nature of C_j.

Since all C_j values are ≤ 0 and $b_1 = -1, b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

Step 4. Mark the outgoing variable.

Since b_3 is negative and numerically largest, the third row is the key row and s_3 is the outgoing variable.

Step 5. Calculate ratios of elements in C_j-row to the corresponding negative elements of the key row.

These ratios are $-3/-1 = 3, -2/-2 = 1$ (neglecting ratios corresponding to +ve or zero elements of key row). Since the smaller ratio is 1, therefore, x_2-column is the key column and (-2) is the key element.

Step 6. Iterate towards optimal feasible solution.

(i) Drop s_3 and introduce x_2 alongwith its associated value -2 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	0	4
0	s_2	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0	2
-2	x_2	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	0	5
0	s_4	$\left(-\frac{1}{2}\right)$	0	0	0	$\frac{1}{2}$	1	$-2 \leftarrow$
	$Z_j = \Sigma c_B a_{ij}$	-1	-2	0	0	1	0	-10
	$C_j = c_j - Z_j$	-2	0	0	0	-1	0	
		$\uparrow$						

Since all C_j values are ≤ 0 and $b_4 = -2$, this solution is optimal but infeasible. We therefore proceed further.

(ii) Mark the outgoing variable.

Since b_4 is negative, the fourth row is the key row and s_4 is the outgoing variable.

(iii) Calculate ratios of elements in C_j-row to the corresponding negative elements of the key row.

This ratios is $-2/-\frac{1}{2} = 4$ (neglecting other ratios corresponding to + ve or 0 elements of key row).

$\therefore$ x_1-column is the key column and $\left(-\frac{1}{2}\right)$ is the key element.

(iv) Drop s_4 and introduce x_1 with its associated value -3 under the c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the third solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	0	0	1	0	-1	-1	6
0	s_2	0	0	0	1	1	1	0
-2	x_2	0	1	0	0	0	1	3
-3	x_1	1	0	0	0	-10	-2	4
Z_j		-3	-2	0	0	3	4	-18
C_j		0	0	0	0	-3	-4	

Since all C_j values are ≤ 0 and all b's are ≥ 0, therefore this solution is optimal and feasible. Thus the *optimal solution* is $x_1 = 4, x_2 = 3$ and $Z_{max} = -18$.

Example 34.27. *Using dual simplex method, solve the following problem :*

Minimize $Z = 2x_1 + 2x_2 + 4x_3$

subject to $2x_1 + 3x_2 + 5x_3 \geq 2, 3x_1 + x_2 + 7x_3 \leq 3, x_1 + 4x_2 + 6x_3 \leq 5, x_1, x_2, x_3 \geq 0.$

(Kurukshetra, 2009 ; Kerala, 2005)

Solution. Consists of the following steps :

Step 1. (i) Convert the given problem to maximization form by writing

Maximize $Z' = -2x_1 - 2x_2 - 4x_3$.

(ii) Convert the first constraint into (≤) type. Thus it is equivalent to

$$-2x_1 - 3x_2 - 4x_3 \leq -2$$

(iii) Express the problem in standard form.

Introducing slack variables, s_1, s_2, s_3, the given problem becomes

Max. $Z' = -2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3$

subject to
$$-2x_1 - 3x_2 - 5x_3 + s_1 + 0s_2 + 0s_3 = -2,$$
$$3x_1 + x_2 + 7x_3 + 0s_1 + s_2 + 0s_3 = 3,$$
$$x_1 + 4x_2 + 6x_3 + 0s_1 + 0s_2 + s_3 = 5,$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

Step 2. Find the initial basic solution.

Setting the decision variables x_1, x_2, x_3 each equal to zero, we get the basic solution

$$x_1 = x_2 = x_3 = 0, s_1 = -2, s_2 = 3, s_3 = 5 \text{ and } Z' = 0.$$

$\therefore$ Initial solution is given by the table below :

	c_j	-2	-2	-4	0	0	0	
c_B	*Basis*	x_1	x_2	x_3	s_1	s_2	s_3	b
0	s_1	-2	(-3)	-5	1	0	0	$-2 \leftarrow$
0	s_2	3	1	7	0	1	0	3
0	s_3	1	4	6	0	0	1	5
	Z_j	0	0	0	0	0	0	0
	C_j	-2	-2	-4	0	0	0	
			$\uparrow$					

Step 3. Test nature of C_j.

Since all C_j values are ≤ 0 and $b_1 = -2$, the initial solution is optimal but infeasible.

Step 4. Mark the outgoing variable.

Since $b_1 < 0$, the first row is the key row and s_1 is the outgoing variable.

Step 5. Calculate the ratio of elements of C_j-row to the corresponding negative elements of the key row.

These ratios are $-2/-2 = 1$, $-2/-3 = 0.67$, $-4/-5 = 0.8$.

Since 0.67 is the smallest ratio, x_2-column is the key column and (-3) is the key element.

Step 6. Iterate towards optimal feasible solution.

Drop s_1 and introduce x_2 with its associated value -2 under c_B column. Then the revised dual simplex table is

	c_j	-2	-2	-4	0	0	0	
c_B	*Basis*	x_1	x_2	x_3	s_1	s_2	s_3	b
-2	x_2	2/3	1	5/3	$-1/3$	0	0	2/3
0	s_2	7/3	0	16/3	1/3	1	0	7/3
0	s_3	$-5/3$	0	$-2/3$	4/3	0	1	7/3
	Z_j	$-4/3$	-2	$-10/3$	2/3	0	0	$-4/3$
	C_j	$-2/3$	0	$-2/3$	$-2/3$	0	0	

Since all $C_j \le 0$ and all b_i's are > 0, this solution is optimal and feasible. Thus the optimal solution is

$$x_1 = 0,\ x_2 = 2/3,\ x_3 = 0 \text{ and max. } Z' = -4/3 \text{ i.e., min. } Z = 4/3.$$

PROBLEMS 34.8

Using dual simplex method, solve the following problems :

1. Maximize $Z = -3x_1 - x_2$
   subject to $x_1 + x_2 \ge 1,\ 2x_1 + 3x_2 \ge 2;\ x_1, x_2 \ge 0.$
2. Minimize $Z = 2x_1 + x_2$,
   subject to $3x_1 + x_2 \ge 3,\ 4x_1 + 3x_2 \ge 6,\ x_1 + 2x_2 \le 3,\ x_1, x_2 \ge 0.$ *(Kurukshetra, 2007 S)*
3. Minimize $Z = x_1 + 2x_2 + 3x_3$,
   subject to $2x_1 - x_2 + x_3 \ge 4,\ x_1 + x_2 + 2x_3 \le 8,\ x_2 - x_3 \ge 2;\ x_1, x_2, x_3 \ge 0.$
4. Minimize $Z = x_1 + 2x_2 + x_3 + 4x_4$
   subject to $2x_1 + 4x_2 + 5x_3 + x_4 \ge 10,\ 3x_1 - x_2 + 7x_3 - 2x_4 \ge 2$
   $5x_1 + 2x_2 + x_3 + 6x_4 \ge 15,\ x_1, x_2, x_3, x_4 \ge 0.$

34.14 (1) TRANSPORTATION PROBLEM

This is a special class of linear programming problems in which the objective is to transport a single commodity from various origins to different destinations at a minimum cost.

(2) Formulation of a transportation problem. *There are m plant locations (origins) and n distribution centres (destinations). The production capacity of the ith plant is a_i and the number of units required at the jth destination is b_j. The transportation cost of one unit from the ith plant to the jth destination is c_{ij}. Our objective is to determine the number of units to be transported from the ith plant to jth destination so that the total transportation cost is minimum.*

Let x_{ij} be the number of units shipped from ith plant to jth destination, then *the general transportation problem is :*

$$\text{Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, x_{ij}$$

subject to the constraints

$x_{i1} + x_{i2} + \dots + x_{in} = a_i$, for ith origin ($i = 1, 2, \dots m$)

$x_{1j} + x_{2j} + \dots + x_{mj} = b_j$, for jth destination ($j = 1, 2, \dots n$)

$x_{ij} \geq 0$.

Def. 1. The two sets of constraints will be consistent if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

which is the condition for a transportation problem to have a *feasible solution*. Problems satisfying this condition are called *balanced transportation problems.*

2. A feasible solution to a transportation problem is said to be a *basic feasible solution* if it contains at the most $(m + n - 1)$ strictly positive allocations, otherwise the solution will *degenerate.* If the total number of positive (non-zero) allocations is exactly $(m + n - 1)$, then the basic feasible solution is said to be *non-degenerate.*

3. A feasible solution which minimizes the transportation cost is called an *optimal solution.* This problem is explicitly represented in the following *transportation table :*

Distribution centres (Destinations)

		1	2		j		n	*Supply*
	1	c_{11}	c_{12}		c_{1j}		c_{1n}	a_1
Plants (origins)	2	c_{21}	c_{22}		c_{2j}		c_{2n}	a_2
								a_i
	i	c_{i1}	c_{i2}		c_{ij}		c_{in}	
								a_m
	m	c_{m1}	c_{m2}		c_{mj}		c_{mn}	
Demand		b_1	b_2		b_j		b_n	$\Sigma a_i = \Sigma b_j$

The mn squares are called *cells*. The per unit cost c_{ij} of transporting from the ith origin to the jth destination is displayed in the *lower right side of the (i, j)th cell.* Any feasible solution is shown in the table by entering the value of x_{ij} in the *small square at the upper left side of the (i, j)th cell.* The various a's and b's are called *rim requirements.* The feasibility of a solution can be verified by summing the values of x_{ij} along the rows and down the columns.

Obs. 1. The special features of a transportation problem are that

(*i*) the coefficients of all x_{ij} in the constraints are unity, and

(*ii*) the total supply Σa_i = total demand Σb_j.

Obs. 2. The objective function and the constraints being all linear, the problem can be solved be simplex method. But the number of variables being large, there will be too many calculations. However, the coefficients of all x_{ij} in the constraints being unity, we can look for some technique which would be simpler than the simplex method.

34.15 WORKING PROCEDURE FOR TRANSPORTATION PROBLEMS

Step 1. Construct transportation table. Express the supply from the origins a_i, demand at destinations b_j and the unit shipping cost c_{ij} in the form of a matrix, know as transportation table. If the supply and demand are equal, the problem is *balanced.*

Step 2. Find the initial basic feasible solution. We find an initial allocation which satisfies the demand at each project site without violating the capacities of the plants (origins) and also meeting the non-negativity restrictions. There are several methods for initial allocations *e.g.*, North-West corner rule, Row minima method, Least cost method, Vogel's approximation method. *The Vogel's approximation method (VAM) takes into account not only the least cost c_{ij} but also the costs that just exceed the least cost c_{ij} and therefore yields a better initial solution than obtained from other methods. As such we shall confine ourselves to VAM only which consists of the following steps :*

(*i*) Display the difference between the least and the next to least costs in each row, by enclosing them in brackets to the right of the row. Similarly display the differences for each column within brackets below that column.

(*ii*) Identify the row or column with the largest difference among all the rows and columns and allocate as much as possible under the rim requirements, to the lowest cost cell in that row or column. In case of a tie allocate to the cell associated with the lower cost.

If the greatest difference corresponds to ith row and c_{ij} is the lowest cost in the ith row, allocate as much as possible *i.e.*, min (a_i, b_j) in the (i, j)th cell and cross off the ith row or the jth column.

(*iii*) Recalculate the row and column differences for the reduced table and go to the previous step.

(*iv*) Repeat the procedure till all the rim requirements are satisfied. Note the solution in the upper left corner small squares of the basic cells.

Step 3. Apply optimality check

In the above solution, the number of allocations must be '$m + n - 1$', otherwise the basic solution degenerates.

Now to test for optimality, we apply the modified distribution (MODI) method and examine each unoccupied cell to determine whether making an allocation in it reduces the total transportation cost and then repeat this procedure until lowest possible transportation cost is obtained. This method consists of the following steps :

(*i*) Note the numbers u_i along the left and v_j along the top of the cost matrix such that their sums equal to the original costs of occupied cells *i.e.*, solve the equations $[u_i + v_j = c_{ij}]$ starting initially with some $u_i = 0$.

(*ii*) *Compute the net evaluations* $w_{ij} = u_i + v_j - c_{ij}$ for all the empty cells and enter them in upper right hand corners of the corresponding cells.

(*iii*) *Examine the sign of each* w_{ij}. If all $w_{ij} \leq 0$, then the current basic feasible solution is optimal. If even one $w_{ij} > 0$, this solution is not optimal and we proceed further.

Step 4. Iterate towards optimal solution

(*i*) Choose the unoccupied cell with the largest w_{ij} and mark θ in it.

(*ii*) Draw a closed path consisting of horizontal and vertical lines beginning and ending at θ-cell and having its other corners at the allocated cells.

(*iii*) Add and subtract θ alternately to and from the transition cells of the loop subject to rim requirements. Assign a maximum value to θ so that one basic variable becomes zero and the other basic variables remain non-negative. Now the basic cell whose allocation has been reduced to zero leaves the basis.

Step 5. Return to step 3 and repeat the process until an optimal basic feasible solution is obtained.

Example 34.28. *Solve the following transportation problem :*

		Destination				
		A	*B*	*C*	*D*	
	I	21	16	25	13	11
Source	*II*	17	18	14	23	13 *Availability*
	III	33	27	18	41	19
	Requirement	6	10	12	15	43

Solution. *Consists of the following steps :*

Step 1. Transportation table. Here the total availability and the total requirement being the same *i.e.* 43, the problem is balanced.

Step 2. Find the initial basic feasible solution. Following VAM, the differences between the smallest and next to the smallest costs in each row and each column are computed and displayed within brackets against the respective rows and columns (Table 1). The largest of these differences is (10) which is associated with the fourth column.

Table 1

21	16	25	[11] 13	11(3)
17	18	14	23	13(3)
32	27	18	41	19(9)
6	10	12	15	
(4)	(2)	(4)	(10)	

Table 2

			+	
17	18	14	[4] 23	13(3)
32	27	18	41	19(9)
6	10	12	4 +	
(15)	(9)	(4)	(18)	

Since c_{14} (= 13) is the minimum cost, we allocate x_{14} = min (11, 15) = 11. This exhausts the availability of first row and therefore we cross it.

Table 3

[6] 17	18	14	9(3)
32	27	18	19(9)
6	10	12	
(15)	(9)	(4)	

Table 4

[3] 18	14	3(4)
27	18	19(9)
10	12	
(9)	(4)	

Table 5

[7] 27	[12] 18	19
7	12 +	

The row and column differences are now computed for reduced table 2 and displayed within brackets. The largest of these is (18) which is against the fourth column. Since c_{14} (= 23) is the minimum cost, we allocate x_{14} = min (13, 4) = 4.

This exhausts the availability of fourth column which we cross off. Proceeding in this way, the subsequent reduced transportation tables and differences for the remaining rows and columns are shown in Tables 3, 4 and 5.

Finally the initial basic feasible solution is as shown in Table 6.

Table 6

21	16	25	[11] 13
[6] 17	[3] 18	14	[4] 23
32	[7] 27	[12] 18	41

Table 7

u_i \ v_j	17	18	9	23
− 10	(−) 21	(−) 16	(−) 25	[11] 13
0	[6] 17	[3] 18	(−) 14	[4] 23
9	(−) 32	[7] 27	[12] 18	(−) 41

Step 3. Apply optimality check

As the number of allocations = $m + n - 1$ (*i.e.*, 6), we can apply MODI method.

(*i*) We have $u_2 + v_1 = 17$, $u_2 + v_2 = 18$, $u_3 + v_2 = 27$

$$u_3 + v_3 = 18,\ u_1 + v_4 = 13,\ u_2 + v_4 = 23$$

Let $u_2 = 0$, then $v_1 = 17$, $v_2 = 18$, $u_3 = 9$, $v_3 = 9$, $v_4 = 23$, $u_1 = -10$.

(*ii*) Net evaluations $w_{ij} = (u_i + v_j) - c_{ij}$ for all empty cells are

$$w_{11} = -14,\ w_{12} = -8,\ w_{13} = -26,\ w_{23} = -5,\ w_{31} = -6,\ w_{34} = -9.$$

(*iii*) Since all the net evaluations are negative, the current solution is optimal. Hence the optimal allocation is given by

$$x_{14} = 11,\ x_{21} = 6,\ x_{22} = 3,\ x_{24} = 4,\ x_{32} = 7 \text{ and } x_{33} = 12.$$

∴ The optimal (minimum) transportation cost

$$= 11 \times 13 + 6 \times 17 + 3 \times 18 + 4 \times 23 + 7 \times 27 + 12 \times 18 = ₹\ 796.$$

Example 34.29. *A company has three cement factories located in cities 1, 2, 3 which supply cement to four projects located in towns 1, 2, 3, 4. Each plant can supply 6, 1, 10 truck loads of cement daily respectively and the daily cement requirements of the projects are respectively 7, 5, 3, 2 truck loads. The transportation costs per truck load of cement (in hundreds of rupees) from each plant to each project site are as follows :*

		Project sites			
		1	2	3	4
	1	2	3	11	7
Factories	2	1	0	6	1
	3	5	8	15	9

Determine the optimal distribution for the company so as to minimize the total transportation cost.

Solution. Consists of the following steps :

Step 1. Construct transportation table. Express the supply from the factories, demands at sites and the unit shipping cost in the form of the following transportation table (Table 1). Here the supply being equal to the demand, the problem is balanced.

Table 1

		Project sites				
		1	2	3	4	Supply
	1	2	3	11	7	6
Factories	2	1	0	6	1	1
	3	5	8	15	9	10
Demand		7	5	3	2	17

Step 2. Find the initial basic feasible solution.

Using VAM, the initial basic feasible solution is as shown in Table 2. The transportation cost according to this route is given by

$$Z = ₹\,(1 \times 2 + 5 \times 3 + 1 \times 1 + 6 \times 5 + 3 \times 15 + 1 \times 9)\ times\ 100 = ₹\ 10{,}200.$$

Step 3. Apply optimality check.

As the numbers of allocations = $(m + n - 1)$ *i.e.*, 6, we can apply MODI method.

We now compute the net evaluations $w_{ij} = (u_i + v_j) - c_{ij}$ which are exhibited in Table 3. Since the net evaluations in two cells are positive, a better solution can be found.

Table 2

[1] 2	[5] 3	11	7	6
1	0	6	[1] 1	1
[6] 5	8	[3] 15	[1] 9	10
7	5	3	2	

Table 3

$u_i \backslash v_j$	2	3	12	6
0	[1] 2	[5] 3	(+) 11	(–) 7
–5	(–) 1	(–) 0	(+) 6	[1] 1
3	[6] 5	(–) 8	[3] 15	[1] 9

Step 4. Iterate towards optimal solution.

First iteration :

(*a*) *Next basic feasible solution.*

(*i*) Choose the unoccupied cell with the maximum w_{ij}. In case of a tie, select the one with lower original cost. In Table 3, cells (1, 3) and (2, 3) each have $w_{ij} = 1$ and out of these all (2, 3) has lower original cost 6, therefore we take this as the next basic cell and note θ in it.

(*ii*) Draw a closed path beginning and ending at θ-cell. Add and subtract θ, alternately to and from the transition cells of the loop subject the rim requirements. Assign a maximum value to θ so that one basic variable becomes zero and the other basic variables remain ≥ 0. Now the basic cell whose allocation has been reduced to zero leaves the basis. This gives the *second basic feasible solution* (Table 5).

Table 4

$\boxed{1}$ 2	$\boxed{5}$ 3	11	7
1	0	$\boxed{\theta}$ 6	$\boxed{1}$ $-\theta$ 1
$\boxed{6}$ 5	8	$\boxed{3}$ $-\theta$ 15	$\boxed{1}$ $+\theta$ 9

Table 5

$\boxed{1}$ 2	$\boxed{5}$ 3	11	7
1	0	$\boxed{\theta = 1}$ 6	$\boxed{1 - 1}$ 1
$\boxed{6}$ 5	8	$\boxed{3 - 1}$ 15	$\boxed{1 + 1}$ 9

$\therefore$ Total transportation cost of this revised solution.

$= ₹\ (1 \times 2 + 5 \times 3 + 1 \times 6 + 6 \times 5 + 2 \times 15 + 2 \times 9)$ *times* $100 = ₹\ 10{,}100$.

(*b*) *Optimality check.* As the number of allocations in table 5 = $m + n - 1$ (*i.e.*, 6), we can apply MODI method. We compute the net evaluations which are shown in Table 6. Since the cell (1, 3) has a positive value, the second basic feasible solution is not optimal.

Table 6

$u_i \backslash v_j$	2	3	12	6
0	$\boxed{1}$ 2	$\boxed{5}$ 3	(+) 11	(–) 7
– 6	(–) 1	(–) 0	$\boxed{1}$ 6	(–) 1
3	$\boxed{6}$ 5	(–) 8	$\boxed{2}$ 15	$\boxed{2}$ 9

Table 7

$\boxed{1 - 1}$ 2	5 3	$\boxed{\theta = 1}$ 11	7
1	0	$\boxed{1}$ 6	1
$\boxed{6 + 1}$ 5	8	$\boxed{2 - 1}$ 15	$\boxed{2}$ 9

Second iteration :

(*a*) *Next basic feasible solution.* In the second basic feasible solution introduce the cell (1, 3) taking $\theta = 1$ and drop the cell (1, 1) giving Table 7. Thus we obtain the third basic feasible solution (Table 8).

Table 8

2	$\boxed{5}$ 3	$\boxed{1}$ 11	7
1	0	$\boxed{1}$ 6	1
$\boxed{7}$ 5	8	$\boxed{1}$ 15	$\boxed{2}$ 9

Table 9

$u_i \backslash v_j$	1	3	11	5
0	(–) 2	$\boxed{5}$ 3	$\boxed{1}$ 11	(–) 7
– 5	(–) 1	(–) 0	$\boxed{1}$ 6	(–) 1
4	$\boxed{7}$ 5	(–) 8	$\boxed{1}$ 15	$\boxed{2}$ 9

(*b*) *Optimality Check.* As the number of allocations in Table 8 = $m + n - 1$ (*i.e.*, 6), we can apply MODI method.

We compute the net evaluations which are shown in Table 9. Since all the net evaluations are ≤ 0, this basic feasible solution is optimal.

Thus the optimal transportation policy is as shown in Table 9 and the optimal transportation cost

$= ₹\ [5 \times 3 + 1 \times 11 + 1 \times 6 + 7 \times 5 + 1 \times 15 + 2 \times 9]$ *times* $100 = ₹\ 10{,}000$.

34.16 DEGENERACY IN TRANSPORTATION PROBLEMS

When the number of basic cells in a non-transportation table, is less than '$m + n - 1$' the basic solution degenerates. To remove the degeneracy, we assign a small positive value ε to as many zero-valued variables as may be necessary to complete '$m + n - 1$' basic variables. The cells containing ε are then treated like other basic cells and the problem is solved in the usual way. The ε's are kept till the optimum solution is attained. Then we let each $\varepsilon \to 0$.

Example 34.30. *Solve the following transportation problem :*

	To						
From	9	12	9	6	9	10	5
	7	3	7	7	5	5	6
	6	5	9	11	3	11	2
	6	8	11	2	2	10	9
	4	4	6	2	4	2	22

Solution. Consists of the following steps :

Step 1. Transporation table. The total supply and total demand being equal, the transportation problem is balanced.

Step 2. Find the initial basic feasible solution.

Using VAM, the initial basic feasible solution is as shown in Table 1.

Step 3. Apply optimality check. Since the number of basic cells is 8 which is less than $m + n - 1 = 9$, the basic solution degenerates. In order to complete the basis and thereby remove degeneracy, we require only one more positive basic variable. We select the variable x_{23} and allocate a small positive quantity ε to the cell (2, 3).

Table 1

						+
9	12	[5] 9	6	9	10	5
7	[4] 3	[ε] 7	7	5	[2] 5	$6 + \varepsilon = 6$
[1] 6	5	[1] 9	11	3	11	2
[3] 6	8	11	[2] 2	[4] 2	10	9
4	4	$6 + \varepsilon = 6$	2	4	2	+

We now compute the net evaluations $w_{ij} = (u_i + v_j) - c_{ij}$ which are exhibited in Table 2. Since all the net evaluations are ≤ 0, the current solutions is optimal. Hence the optimal allocation is

$$x_{13} = 5, x_{22} = 4, x_{26} = 2, x_{31} = 1, x_{33} = 1, x_{41} = 3, x_{44} = 2 \text{ and } x_{45} = 4.$$

$\therefore$ The minimum (optimal) transportation cost

$$= 5 \times 9 + 4 \times 3 + \varepsilon \times 7 + 2 \times 5 + 1 \times 6 + 1 \times 9 + 3 \times 6 + 2 \times 2 + 4 \times 2$$

$$= 112 + 7\varepsilon = ₹\ 112 \text{ as } \varepsilon \to 0.$$

Table 2

u_i \ v_j	4	3	7	0	0	5
2	(–) 9	(–) 12	[5] 9	(–) 6	(–) 9	(–) 10
0	(–) 7	[4] 3	[ε] 7	(–) 7	(–) 5	[2] 5
2	[1] 6	(0) 5	[1] 9	(–) 11	(–) 3	(–) 11
2	[3] 6	(–) 8	(–) 11	[2] 2	[4] 2	(–) 10

PROBLEMS 34.9

1. Obtain an initial basic feasible solution to the following transportation problem :

		To D	To E	To F	To G	
From	A	11	13	17	14	250
	B	16	18	14	10	300
	C	21	24	13	10	400
		200	225	275	250	

2. Solve the following transportation problem :

Suppliers / Consumers	A	B	C	Available
I	6	8	4	14
II	4	9	8	12
III	1	2	6	5
Required	6	10	15	31

3. Consider four bases of operations B_i and three targets T_j. The tons of bombs per aircraft from any base that can be delivered to any target are given in the following table :

B_i \ T_j	1	2	3
1	8	6	5
2	6	6	6
3	10	8	4
4	8	6	4

The daily sortie capability of each of the four bases is 150 sorties per day. The daily requirement in sorties over each target is 200. Find the allocation of sorties from each base to each target which maximizes the total tonnage over all the three targets.

4. A company has factories F_1, F_2, F_3 which supply warehouses at W_1, W_2 and W_3. Weekly factory capacities, weekly warehouse requirements and unit shipping costs (in rupees) are as follows :

Factories	*Warehouses*			*Supply*
	W_1	W_2	W_3	
F_1	16	20	12	200
F_2	14	8	18	160
F_3	26	24	16	90
Demand	180	120	150	450

Determine the optimal distribution for this company to minimize shipping costs.

5. A company is spending ₹ 1,000 on transportation of its units from plants to four distribution centres. The supply and demand of units, with unit cost of transportation are given below :

Plants	*Distribution centres*				*Availabilities*
	D_1	D_2	D_3	D_4	
P_1	19	30	50	12	7
P_2	70	30	40	60	10
P_3	40	10	60	20	18
Requirements	5	8	7	15	

What can be the maximum saving by optimal scheduling.

6. A departmental store wishes to stock the following quantities of a popular product in three types of containers :

Container type :	1	2	3
Quantity :	170	200	180

Tenders are submitted by four dealers who undertake to supply not more than the quantities shown below :

Dealer :	1	2	3	4
Quantity :	150	160	110	130

The store estimates that profit per unit will vary with the dealer as shown below :

Dealers → *Container type* ↓	1	2	3	4
1	8	9	6	3
2	6	11	5	10
3	3	8	7	9

Find the maximum profit of the store.

7. Obtain an optimum basic feasible solution to the following transportation problem :

	To			
From	7	3	4	2
	2	1	3	3 *Available*
	3	4	6	5
Demand	4	1	5	10

8. A company has three plants at locations A, B and C which supply to warehouses located as D, E, F, G and H. Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs in rupees are given below :

		To				
		D	E	F	G	H
	A	5	8	6	6	3
From	B	4	7	7	6	6
	C	8	4	6	6	3

Determine an optimum distribution for the company in order to minimize the total transportation cost.

34.17 (1) ASSIGNMENT PROBLEM

An assignment problem is a special type of transportation problem in which the objective is to assign a number of origins to an *equal* number of destinations at a minimum cost (or maximum profit).

(2) Formulation of an assignment problem. *There are n new machines M_i ($i = 1, 2, \ldots n$) which are to be installed in a machine shop. There are n vacant spaces S_j ($j = 1, 2, \ldots n$) available. The cost of installing the machine M_i at space S_j is c_{ij} rupees. Let us formulate the problem of assigning machines to spaces so as to minimize the overall cost.*

Let x_{ij} be the assignment of machine M_i to space S_j *i.e.*, let x_{ij} be a variable such that

$$x_{ij} = \begin{cases} 1, \text{ if } i\text{th machine is installed at } j\text{th space} \\ 0, \text{ otherwise} \end{cases}$$

Since one machine can only be installed at each space, we have

$$x_{i1} + x_{i2} + \ldots + x_{in} = 1, \text{ for machine } M_i \ (i = 1, 2, \ldots n)$$
$$x_{1j} + x_{2j} + \ldots + x_{nj} = 1, \text{ for space } S_j \ (j = 1, 2, \ldots n)$$

Also the total installation cost is $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$.

Thus the assignment problem can be stated as follows :

Determine $x_{ij} \geq 0$ ($j = 1, 2, \ldots n$) so as to

minimize $$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to the constraints $\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \ldots n$ and $\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, \ldots n$.

This problem is explicitly represented by the following $n \times n$ *cost matrix* :

		Spaces				
		S_1	S_2	S_3	...	S_n
	M_1	c_{11}	c_{12}	c_{13}	...	c_{1n}
	M_2	c_{21}	c_{22}	c_{23}	...	c_{2n}
Machines	M_3	c_{31}	c_{32}	c_{33}	...	c_{3n}
	:	:	:	:		:
	:	:	:	:		:
	M_n	c_{n1}	c_{n2}	c_{n3}		c_{nn}

Obs. This assignment problem constitutes n ! possible ways of installing n machines at n spaces. If we enumerate all these n ! alternatives and evaluate the cost of each one of them and select the one with the minimum cost, the problem would be solved. But this method would be very slow and time consuming, even for small value of n and hence it is not at all suitable. However, a much more efficient method of solving such problems is available. This is the **Hungarian method** for solution of assignment problems which we describe below.

34.18 WORKING PROCEDURE TO SOLVE AN ASSIGNMENT PROBLEM

Step 1. Reduce the matrix. Subtract the smallest element of each row (of the given cost matrix) from all elements of that row. See if each row contains at least one zero. If not, subtract the smallest element of each column (not containing zero) from all the elements of that column. This gives the *reduced matrix.*

Step 2. Assign the zeros

(*a*) Examine rows (of the reduced matrix) successively until a row with exactly one unmarked zero is found. Make an assignment to this single zero by encircling it. Cross all other zeros in the column of this encircled zero, as these will not be considered for any future assignment. Continue in this way until all the rows have been examined.

(*b*) Now examine columns successively until a column with exactly one unmarked zero is found. Encircle this zero and make an assignment there. Then cross any other zero in its row. Continue in this way until all the columns have been examined.

In case, some rows or columns contain mor than one unmarked zeros, encircle any unmarked zero arbitrarily and cross all other zeros in its row or column. Proceed in this way, till no zero is left unmarked.

Step 3. Apply optimality check.

Repeat step 2 (*a*) and (*b*) until one of the following occurs :

(*i*) If no row or no column is without assignment (encircled zero), then the current assignment is optimal.

(*ii*) If there is some row and/or column without an assignment, then the current assignment is not optimal and we go to next step.

Step 4. Find minimum number of lines crossing all zeros.

(*a*) Tick (✓) the rows which do not have assignments.

(*b*) Tick (✓) the columns (not already marked) which have zeros in the ticked row.

(*c*) Tick (✓) the rows (not already marked) which have assignments in ticked columns.

Repeat (*b*) and (*c*) until no more marking is required.

(*d*) Draw lines through all unticked rows and ticked columns. If the number of these lines is equal to the order of the matrix then it is an optimal solution otherwise not.

Step 5. Iterate towards optimal solution.

Select the smallest element and subtract it from all uncovered elements. Add this smallest element to every element lying at the intersection of two lines. The resulting matrix is the second basic feasible solution.

Step 6. Go to step 2 and repeat the procedure until the optimal solution is attained.

Example 34.31. *Four jobs are to be done on four different machines. The cost (in rupees) of producing ith job on the jth machine is given below :*

		Machines			
		M_1	M_2	M_3	M_4
	J_1	15	11	13	15
	J_2	17	12	12	13
Jobs	J_3	14	15	10	14
	J_4	16	13	11	17

Assign the jobs to different machines so as to minimize the total cost.

Solution. Consists of the following steps :

Step 1. Reduce the matrix. Subtract the smallest element 11 of row 1 from all its elements. Similarly subtract 12, 10 and 11 from rows 2, 3 and 4 respectively. The resulting matrix is as shown in Table 1. Columns 1 and 4 do not have any zero element. Subtract the smallest element 4 of Col. 1 from all its elements and element 1 from all elements of Col. 4. The *reduced matrix* is as given in Table 1.

Table 1

	M_1	M_2	M_3	M_4
J_1	4	0	2	4
J_2	5	0	0	1
J_3	4	5	0	4
J_4	5	2	0	6

Table 2

	M_1	M_2	M_3	M_4
J_1	0	0	2	3
J_2	1	~~0~~	~~0~~	0
J_3	0	5	~~0~~	3
J_4	1	2	0	5

Step 2. Assign the zeros. Row 4 has a single unmarked zero in Col. 3. Encircle it and cross all other zeros in Col. 3. Row 3 has a single unmarked zero in Col. 1. Encircle it and cross the other zero in col. 1. Row 1 has a single unmarked zero in Col. 2. Encircle it and cross the other zero in Col. 2. Finally row 2 has a single unmarked zero in Col. 4. Encircle it (Table 2).

Step 3. Apply optimality check. Since we have one encircled zero in each row and in each column, this gives the optimal solution.

∴ The optimal assignment policy is

Job 1 to machine 2, Job 2 to machine 4, Job 3 to machine 1, Job 4 to machine 3,
and the minimum assignment cost = ₹ (11 + 13 + 14 + 11) = ₹ 49.

Example 34.32. *A marketing manager has 5 salesmen and 5 sales districts. Considering the capabilities of the salesmen and the nature of districts, the marketing manager estimates that sales per month (in hundred rupees) for each salesman in each district would be as follows :*

		Sales districts				
		A	B	C	D	E
	1	32	38	40	28	40
	2	40	24	28	21	36
Salesman	3	41	27	33	30	37
	4	22	38	41	36	36
	5	29	33	40	35	39

Find the assignment of salesmen to districts that will result in maximum sales. (Madras, 2000)

Solution. Consists of the following steps :

Step 1. Reduce the matrix. Convert the given maximization problem into a minimization problem, by making all the profits negative, since max. $Z = \min. (-Z)$. Then subtract the smallest element of each row from the elements of that row. Now subtract the smallest element of each col. (not containing zero) from the elements of that column. This gives the *reduced matrix* (Table 1).

Table 1

8	0	$\cancel{0}$	7	0
0	14	12	14	4
0	12	8	6	4
19	1	0	$\cancel{0}$	5
11	5	0	$\cancel{0}$	1

Table 2

12	0	0	7	0
0	10	8	10	0
0	8	4	2	0
23	1	0	0	5
15	5	0	0	1

Step 2. Assign the zeros. Rows 2 and 3 have each a single unmarked zero in Col. 1. Encircle these. Columns 2 and 5 have each a single unmarked zero in row 1. Encircle these and cross the zero in row 1. Columns 3 and 4 have each unmarked zeros. Encircle the zeros in each of the rows 4 and 5 as shown in Table 1 and cross other zeros.

Step 3. Apply optimality check. As col. 4 is without assignment, this solution is not optimal. Therefore we go to next step.

Step 4. Find minimum number of lines crossing all zeros. Draw the least number of horizontal and vertical (dotted) lines which cover all the zeros. Since there are four dotted lines which are less than the order of the cost matrix (= 5), we got to step 5.

Step 5. Iterate towards optimal solution. Select the smallest element in the Table 1, not covered by the dotted lines. Such an element is 4 which lies at two different positions. Selecting the elements that lies at position (3, 5) arbitrarily, subtract it from all the uncovered elements of the cost matrix (Table 1) and add the same to the elements lying at the intersection of two dotted lines. Now draw more minimum number of dotted lines so as to cover the new zero. Here we draw such a line in Col. 5 (Table 2).

Table 3

	A	B	C	D	E
1		0	$\cancel{0}$		
2	0				$\cancel{0}$
3	$\cancel{0}$				0
4			0	$\cancel{0}$	
5			$\cancel{0}$	0	

Now, since the number of dotted lines is equal to the order for the cost matrix, the optimal solution is attained.

Finally, to determine this optimal assignment, we consider only the zero elements (Table 3) :

(*i*) Examine successively the rows with exactly one zero. There is no such row.

(*ii*) Examine successively the columns with exactly one zero. Col. 2 has one zero, encircle it and cross all zeros of row 1.

(*iii*) Encircle arbitrarily the zero in position (2, 1) and cross all zeros in row 2 and Col. 1. Then encircle the unmarked zero in row 3. Now encircle arbitrarily the zero in position (4, 3) and cross all zeros in row 4 and Col. 3. Finally encircle the remaining unmarked zero in row 5.

Now each row and each column has one encircled zero, therefore the optimal assignment policy is :

Salesman 1 to district B, 2 to A, 3 to E, 4 to C and 5 to D.

Hence the maximum sales = ₹ (38 + 40 + 37 + 41 + 35) × 100 = ₹ 19,100.

PROBLEMS 34.10

1. A firm plans to begin production of three new products on its three plants. The unit cost of producing i at plant j is as given below. Find the assignment that minimizes the total unit cost.

		Plant 1	Plant 2	Plant 3
Product	1	10	8	12
	2	18	6	14
	3	6	4	2

2. Solve the following assignment problem :

	1	2	3	4
A	10	12	19	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

3. A machine tool company decides to make four sub-assemblies through four contractors. Each contractor is to receive only one sub-assembly. The cost of each sub-assembly is determined by the bids submitted by each contractor and is shown in table below (in hundreds of rupees). Assign different assemblies to contractors so as to minimize the total cost.

		Contractor *A*	*B*	*C*	*D*
Sub-assembly	I	15	13	14	17
	II	11	12	15	13
	III	18	12	10	11
	IV	15	17	14	16

4. Four professors are each capable of teaching any one of the four different courses. Class preparations time in hours for different topics varies from professor to professor and is given in the table below. Each professor is assigned only one course. Find the assignment policy schedule so as to minimize the total course preparation time for all courses.

Prof.	*L.P.*	*Queuing Theory*	*Dynamic Programming*	*Regression analysis*
A	2	10	9	7
B	15	4	14	8
C	13	14	16	11
D	3	15	13	8

5. Consider the problem of assigning four working labour units to four jobs. The assignment costs in thousands of rupees are given by the following matrix.

Labour unit	*Job*			
↓	*I*	*II*	*III*	*IV*
L_1	42	35	28	21
L_2	30	25	20	15
L_3	30	25	20	15
L_4	24	20	16	12

Find the optimal assignment.

6. A company has six jobs to be processed by six mechanics. The following table gives the return in rupees when the ith job is assigned to the jth mechanic. How should the jobs be assigned to the mechanics so as to maximize the over all return ?

Mechanic ↓	Job I	II	III	IV	V	VI
1	9	22	58	11	19	27
2	43	78	72	50	63	48
3	41	28	91	37	45	33
4	74	42	27	49	39	32
5	36	11	57	22	25	18
6	13	56	53	31	17	28

34.19 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 34.11

Fill up the blanks in the following questions :

1. Infeasibility in a linear programming problem means ......
2. The significance of the $(Z_j - C_j)$ row in the simplex solution procedure is that ......
3. The duality principle states that ......
4. The difference between the transportation problem and the assignment problem is ......
5. The special features of a transportation problem are ......
6. The canonical form of an L.P.P. is such that ......
7. The dual problem of the L.P.P. :
   Max. $Z = 4x_1 + 9x_2 + 2x_3$,
   subject to $2x_1 + 3x_2 + 2x_3 \le 7$, $3x_1 - 2x_2 + 4x_3 = 5$, $x_1, x_2, x_3 \ge 0$, is ......
8. The optimality and feasibility conditions related with Dual simplex method are ......
9. Feasible and basic solutions related with a transportation problem are ......
10. A transportation problem is

					Supply
	2	3	11	4	15
	5	6	8	7	20
Demand	10	5	12	8	

    Its linear programming problem is ......
11. The basic feasible solutions of $2x_1 + x_2 + 4x_3 = 11$, $3x_1 + x_2 + 5x_3 = 14$ are ......
12. A slack variable is defined as ......
13. The advantage of dual simplex method is ......
14. If the total availability is equal to the total requirements, the transportation problem is called ......
15. An artificial variable is that ......
16. Two conditions on which the simplex method is based are ......
17. A feasible solution which minimizes the transportation cost is called an ...... solution.
18. The dual problem of : Maximize $5x_1 + 6x_2$ subject to $x_1 + 2x_2 = 5$, $-x_1 + 5x_2 \ge 3$, x_1 unrestricted and $x_2 \ge 0$, is ......
19. For a balanced transportation problem with 3 rows and 3 columns, the number of basic variables will be ......
20. Using graphical method, Max. $Z = 5x_1 + 3x_2$ subject to $5x_1 + 2x_2 \le 10$, $3x_1 + 5x_2 \le 15$, $x_1, x_2 \ge 0$, is ......
21. In a L.P. problem, unbounded solution is that ......
22. Degeneracy in a transportation problem is resolved by ....
23. A basic solution is said to be non-degenerate in L.P.P. when ......
24. The dual of the problem Max. $Z = 2x_1 + x_2$ subject to $-x_1 + 2x_2 \le 2$, $x_1 + x_2 \le 4$, $x_1 \le 3$, $x_1, x_2 \ge 0$ is ......
25. The two methods used to find the initial solution of a transportation problem are ....
26. Constraints involving 'equal to sign' do not require use of ...... or ...... variables.