CHAPTER 15

Differential Equations of Other Types

1. Introduction. 2. Equations of the form $d^2y/dx^2 = f(x)$. 3. Equations of the form $d^2y/dx^2 = f(y)$. 4. Equations which do not contain y. 5. Equations which do not contain x. 6. Equations whose one solution is known. 7. Equations which can be solved by changing the independent variable. 8. Total differential equation : $Pdx + Qdy + Rdz = 0$. 9. Simultaneous total differential equations. 10. Equations of the form $dx/P = dy/Q = dz/R$.

15.1 INTRODUCTION

In this chapter, we propose to study some other important types of ordinary differential equations which require special methods for their solution and have varied applications as illustrated side by side.

15.2 EQUATIONS OF THE FORM $d^2y/dx^2 = f(x)$

Integrating with respect to x, we have $\dfrac{dy}{dx} = \int f(x)dx + c = F(x)$. (say)

Again integrating, we get $y = \int F(x)dx + c'$ as the required solution.

In general, the solution of the equations of the form $\dfrac{d^n y}{dx^n} = f(x)$ is obtained by integrating it n times successively.

Example 15.1. *Solve* $\dfrac{d^2y}{dx^2} = xe^x$.

Solution. Integrating, we get $\dfrac{dy}{dx} = xe^x - \int e^x\, dx + c_1 = (x-1)\, e^x + c_1$

Again integrating, we get

$$y = (x-1)e^x - \int e^x dx + c_1 x + c_2 = (x-2)e^x + c_1 x + c_2.$$

PROBLEMS 15.1

Solve :

1. $\dfrac{d^2y}{dx^2} = x^2 \sin x$.

2. $\dfrac{d^3y}{dx^3} = x + \log x$.

3. A beam of length $2l$ with uniform load w per unit length is freely supported at both ends. Prove that the maximum deflection of the beam is $\frac{5wl^4}{24EI}$.

[**Hint.** Taking the origin at the left end, we have $EI\, d^4y/dx^4 = w$. At each end, $y = 0$ and $d^2y/dx^2 = 0$.]

4. For a cantilever beam of length l with a uniform load of w per unit length, show that the maximum deflection at the free end is wl^4/EI, where the symbols have the usual meaning.

15.3 EQUATIONS OF THE FORM $d^2y/dx^2 = f(y)$

Multiplying both sides by $2dy/dx$, we have $2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 2f(y)\frac{dy}{dx}$

Integrating with respect to x, $\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\, dy + c = F(y)$ (say)

or
$$\frac{dy}{dx} = \sqrt{[F(y)]}$$

Separating the variables and integrating, we get $\int \frac{dy}{\sqrt{[F(y)]}} = x + c$, whence follows the desired solution.

Such equations occur quite frequently in Dynamics.

Example 15.2. *Solve $d^2y/dx^2 = 2(y^3 + y)$ under the conditions $y = 0$, $dy/dx = 1$, when $x = 0$.* *(U.P.T.U., 2003)*

Solution. Multiplying by $2\, dy/dx$, the given equation becomes

$$2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 4(y^3 + y)\frac{dy}{dx}$$

Integrating w.r.t. x, $\left(\frac{dy}{dx}\right)^2 = 4\left(\frac{y^4}{4} + \frac{y^2}{2}\right) + c = y^4 + 2y^2 + c$ ...(*i*)

As $dy/dx = 1$ for $y = 0$, $\therefore\ c = 1$

$\therefore$ (*i*) takes the form $(dy/dx)^2 = y^4 + 2y^2 + 1 = (y^2 + 1)^2$ or $dy/dx = y^2 + 1$

Separating the variables and integrating, we have $\int \frac{dy}{1 + y^2} = \int dx + c'$

or
$$\tan^{-1} y = x + c' \quad ...(ii)$$

Thus (*ii*) becomes $\tan^{-1} y = x$ or $y = \tan x$ which is the required solution.

Example 15.3. *A point moves in a straight line towards a centre of force $\mu/(\text{distance})^3$, starting from rest at a distance 'a' from the centre of force, show that the time of reaching a point distant 'b' from the centre of force is $\frac{a}{\sqrt{\mu}}\sqrt{(a^2 - b^2)}$ and that its velocity is $\frac{\sqrt{\mu}}{ab}\sqrt{(a^2 - b^2)}$.* *(U.P.T.U., 2001)*

Solution. Let O be the centre of force and A the point of start so that $OA = a$. At any time t, let the point be at P where $OP = x$ so that

$$\frac{d^2x}{dt^2} = \frac{-\mu}{x^3} \quad ...(i)$$

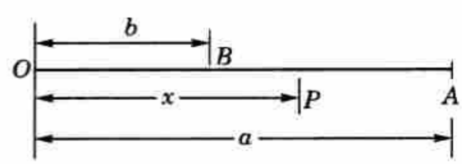

Fig. 15.1

Multiplying both sides by $2\, dx/dt$, we get

$$\frac{2dx}{dt}\cdot\frac{d^2x}{dt^2} = -\frac{\mu}{x^3}\cdot\frac{2dx}{dt}$$

Integrating both sides, we obtain

$$\left(\frac{dx}{dt}\right)^2 = -\mu \int \frac{2}{x^3}\frac{dx}{dt}\,.\,dt + c = +\frac{\mu}{x^2} + c$$

When $x = a$, velocity $dx/dt = 0$. $\therefore$ $c = -\mu/a^2$.

$$\therefore \qquad \left(\frac{dx}{dt}\right)^2 = \mu\left(\frac{1}{x^2} - \frac{1}{a^2}\right) = \frac{\mu(a^2 - x^2)}{a^2x^2} \qquad ...(ii)$$

At B $(x = b)$, velocity towards $O = \dfrac{\sqrt{\mu(a^2 - b^2)}}{ab}$

Again (ii) can be rewritten as $\dfrac{-ax\,dx}{\sqrt{(a^2 - x^2)}} = \sqrt{\mu}\,dt$ [– ve is taken since point is moving towards O]

Integrating both sides, we get

$$\sqrt{\mu}\int dt = -\int \frac{ax\,dx}{\sqrt{(a^2 - x^2)}} + c' \quad \text{or} \quad \sqrt{\mu}t = a\sqrt{(a^2 - x^2)} + c' \qquad ...(iii)$$

Since $t = 0$ at $x = a$, $\therefore$ $c' = 0$

Thus (iii) gives $t = \dfrac{a}{\sqrt{\mu}}\sqrt{(a^2 - x^2)}$

Hence at B $(x = b)$ $t = \dfrac{a}{\sqrt{\mu}}\sqrt{(a^2 - b^2)}$.

PROBLEMS 15.2

Solve :

1. $d^2y/dx^2 = 3\sqrt{y}$ given that $y = 1$, $dy/dx = 2$ when $x = 0$.
2. $\dfrac{d^2y}{dx^2} = \dfrac{36}{y^2}$, given that when $x = 0$, $\dfrac{dy}{dx} = 0$, $y = 8$.
3. If $d^2r/dt^2 = \omega^2 r$, find the value of r in terms of t, if $r = a$ and $dr/dt = v$, when $t = 0$.
4. The motion of a particle let fall from a point outside the earth is given by $d^2x/dt^2 = -ga^2/x^2$. Given that $x = h$ and $dx/dt = 0$, when $t = 0$, find t in terms of x.
5. A particle is acted upon by a force $\mu\,(x + a^4/x^3)$ per unit mass towards the origin, where x is the distance from the origin at time t. If it starts from rest at a distance a, show that it will arrive at the origin in time $\pi/(4\sqrt{\mu})$.

15.4 EQUATIONS WHICH DO NOT CONTAIN y

A second order equation of this form is

$$f(d^2y/dx^2, dy/dx, x) = 0$$

On putting $dy/dx = p$ and $d^2y/dx^2 = dp/dx$, it becomes

$$f(dp/dx, p, x) = 0.$$

This is an equation of the first order in x and p and can, therefore, be solved easily.

If its solution is $(p =)\ dy/dx = \phi(x)$, then $y = \int \phi(x)dx + c$ is the required solution.

Obs. This method may be used to reduce any such equation of the nth order to one of the $(n-1)$th order. If, however, the lowest derivative in such an equation is d^ry/dx^r

(i) put $d^ry/dx^r = p$; (ii) find p and therefrom find y, (*See* Ex. 15.5).

Example 15.4. *Solve* $x\dfrac{d^2y}{dx^2} = \sqrt{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}$.

Solution. Putting $dy/dx = p$ and $d^2y/dx^2 = dp/dx$, the given equation becomes

$$x\,dp/dx = \sqrt{(1+p^2)}.$$

Separating the variables and integrating, we get

$$\int \frac{dp}{\sqrt{(1+p^2)}} = \int \frac{dx}{x} + \text{constant}$$

or $$\log\left[p + \sqrt{(1+p^2)}\right] = \log x + \log c = \log cx.$$

$\therefore$ $$p + \sqrt{(1+p^2)} = cx \quad \text{or} \quad 1 + p^2 = (cx - p)^2$$

or $$(p =)\ \frac{dy}{dx} = \frac{1}{2}\left(cx - \frac{1}{cx}\right).$$

$\therefore$ integrating again, we have $y = \dfrac{1}{2}\left(c\dfrac{x^2}{2} - \dfrac{1}{c}\log x\right) + c'$ as the required solution.

Example 15.5. *Solve* $\dfrac{d^4y}{dx^4}\cdot\dfrac{d^3y}{dx^3} = 1.$

Solution. Putting $d^3y/dx^3 = p$ and $d^4y/dx^4 = dp/dx$, the given equation becomes $\dfrac{dp}{dx}\,p = 1.$

Integrating w.r.t. x, $\int p\,dp = x + c_1$, *i.e.* $p^2/2 = x + c_1$ or $(p =)\ d^3y/dx^3 = \sqrt{2}(x+c_1)^{1/2}$.

Integrating thrice successively, we get

$$\frac{d^2y}{dx^2} = \sqrt{2}\frac{(x+c_1)^{3/2}}{3/2} + c_2,\ \frac{dy}{dx} = \frac{2\sqrt{2}}{3}\cdot\frac{(x+c_1)^{5/2}}{5/2} + c_2x + c_3$$

$$y = \frac{4\sqrt{2}}{15}\,\frac{(x+c_1)^{7/2}}{7/2} + c_2\frac{x^2}{2} + c_3x + c_4$$

Hence $y = \dfrac{8\sqrt{2}}{105}(x+c_1)^{7/2} + \dfrac{1}{2}c_2x^2 + c_3x + c_4$ is the desired solution.

PROBLEMS 15.3

Solve the following equations :

1. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 6x = 0.$
2. $(1+x^2)\dfrac{d^2y}{dx^2} + 1 + \left(\dfrac{dy}{dx}\right)^2 = 0.$
3. $2x\dfrac{d^3y}{dx^3}\cdot\dfrac{d^2y}{dx^2} = \left(\dfrac{d^2y}{dx^2}\right)^2 - a^2.$
4. $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2} = a\dfrac{d^2y}{dx^2}.$
5. A particle of mass m grammes is constrained to move in a horizontal circular path of radius a cm and is subjected to a resistance proportional to the square of the speed at any instant. Show that the differential equation of motion is of the form $m\dfrac{d^2\theta}{dt^2} + \mu a\left(\dfrac{d\theta}{dt}\right)^2 = 0$. If the particle starts with an angular velocity ω, find its angular displacement θ at time t sec.
6. When the inner of two concentric spheres of radii r_1 and $r_2 (r_1 < r_2)$ carries an electric charge, the differential equation for the potential v at any point between two spheres at a distance r from their common centre is $\dfrac{d^2v}{dt^2} + \dfrac{2}{r}\dfrac{dv}{dr} = 0$. Solve for v given $v = v_1$ when $r = r_1$ and $v = v_2$ when $r = r_2$.

15.5 EQUATIONS WHICH DO NOT CONTAIN x

A second order equation of this form is

$$f(d^2y/dx^2, dy/dx, y) = 0.$$

On putting $\dfrac{dy}{dx} = p$ and $\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx} = \dfrac{dp}{dy}\cdot\dfrac{dy}{dx} = p\dfrac{dp}{dy}$, it becomes

$$f(p\,dp/dy, p, y) = 0.$$

This is an equation of the first order in y and p and can, therefore, be solved easily.

Example 15.6. *Solve* $y\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\left(\dfrac{dy}{dx} - 2y\right) = 0.$

Solution. On putting $dy/dx = p$ and $d^2y/dx^2 = p\,dp/dy$, the given equation becomes

$$yp\frac{dp}{dy} + p(p - 2y) = 0.$$

This gives *either* $p = 0$, of which the solution is $y = c$;

or $$\left(y\frac{dp}{dy} + p\right) - 2y = 0 \quad i.e., \quad (ydp + pdy) = 2ydy \text{ } i.e., d(py) = 2ydy.$$

Integrating, $\quad py = 2\int ydy + c_1 = y^2 + c_1.$

Separating the variables and integrating, we get

$$\int \frac{ydy}{y^2 + c_1} = \int dx + c_2 \quad \text{or} \quad \frac{1}{2}\log(y^2 + c_1) = x + c_2 \text{ whence } y^2 + c_1 = c_3e^{2x}$$

Hence the required solutions are $y = c$ and $y^2 + c_1 = c_3\,e^{2x}$.

Example 15.7. *Find the curve in which the radius of curvature is twice the normal and in the opposite direction.*

Solution. At any point $P(x, y)$ of a curve, the radius of curvature

$$\rho = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Bigg/ \frac{d^2y}{dx^2}$$

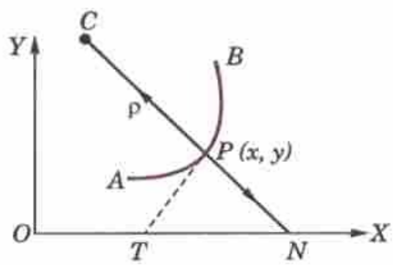

Fig. 15.2

and the length of the normal (*PN*)

$$= y\sqrt{[1 + (dy/dx)^2]}.$$

Also we know that ρ is measured inwards and the normal is measured outwards, *i.e.*, both of them are positive when measured in opposite directions. So the sign will be positive (or negative) according as ρ and the normal run in the opposite (or same) directions.

Thus for the given curve $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2} \div \dfrac{d^2y}{dx^2} = 2y\sqrt{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}$

or $$1 + \left(\frac{dy}{dx}\right)^2 = 2y\frac{d^2y}{dx^2}.$$

On putting $dy/dx = p$ and $d^2y/dx^2 = p\,dp/dy$, the given equation becomes

$$1 + p^2 = 2y\,.\,p\,dp/dy.$$

$\therefore$ separating variables and integrating, we have

$$\int \frac{2pdp}{1 + p^2} = \int \frac{dy}{y} + \text{constant}$$

or $$\log(1 + p^2) = \log y + \log a = \log ay$$

$\therefore$ $$1 + p^2 = ay \text{ or } (p =)\, dy/dx = \sqrt{(ay - 1)}$$

$\therefore$ separating the variables and integrating, we get

$$\int dx + b = \int (ay-1)^{-1/2} dy$$

or $$x + b = \frac{2}{a}(ay-1)^{1/2} \quad \text{or} \quad a^2(x+b)^2 = 4(ay-1)$$

which is required equation of the curve and represents a system of parabolas having axes parallel to y-axis.

PROBLEMS 15.4

Solve the following equations :

1. $2\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + 4 = 0.$
2. $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1.$
3. $y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = y^2 \log y.$
4. $y(1-\log y)\frac{d^2y}{dx^2} + (1+\log y)\left(\frac{dy}{dx}\right)^2 = 0.$
5. Find the curve in which the radius of curvature is equal to the normal and is in the same direction.

15.6 EQUATIONS WHOSE ONE SOLUTION IS KNOWN

Consider the equation $d^2y/dx^2 + P\,dy/dx + Q = R$, where P, Q and R are functions of x only. If $y = u(x)$ is a known solution of this equation, then put $y = uv$ in it. It *reduces the differential equation to one of first order in dv/dx which can be completely solved.*

One integral belonging to the C.F. can be found by inspection as follows ;

(*i*) If $1 + P + Q = 0$, then $y = e^x$ is a solution,

(*ii*) If $I - P + Q = 0$, then $y = e^{-x}$ is a solution,

(*iii*) If $P + Qx = 0$, then $y = x$ is a solution.

Example 15.8. *Solve* $x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = 0.$ *(Bhopal, 2008 S)*

Solution. The given equation is $\frac{d^2y}{dx^2} - \left(2 - \frac{1}{x}\right)\frac{dy}{dx} + \left(1 - \frac{1}{x}\right)y = 0$ ...(*i*)

Here $1 + P + Q = 1 - (2 - 1/x) + (1 - 1/x) = 0$

$\therefore$ $y = e^x$ is a part of C.F. of (*i*)

Now let $y = e^x v$ ...(*ii*)

so that $$\frac{dy}{dx} = e^x v + e^x \frac{dv}{dx} \quad ...(iii) \qquad \text{and} \qquad \frac{d^2y}{dx^2} = e^x v + 2e^x \frac{dv}{dx} + e^x \frac{d^2v}{dx^2} \quad ...(iv)$$

Substituting (*iv*), (*iii*) and (*ii*) in (*i*), we get

$$x\left(e^x v + 2e^x \frac{dv}{dx} + e^x \frac{d^2v}{dx^2}\right) - (2x-1)\left(e^x v + e^x \frac{dv}{dx}\right) + (x-1)e^x v = 0$$

or cancelling e^x, it becomes $x\frac{d^2v}{dx^2} + \frac{dv}{dx} = 0$ or $x\frac{dp}{dx} + p = 0$, where $p = \frac{dv}{dx}$.

Integrating, we get $\int \frac{dp}{p} = -\int \frac{dx}{x} + c$ or $\log p = -\log x + \log c_1$

i.e., $$p = \frac{c_1}{x} \quad \text{or} \quad \frac{dv}{dx} = \frac{c_1}{x}.$$

Again integrating, we obtain $v = c_1 \log x + c_2$

Hence the complete solution of (*i*) is $y = e^x(c_1 \log x + c_2)$.

Example 15.9. *Solve $(1 - x^2) y'' - 2xy' + 2y = 0$ given that $y = x$ is a solution.* (B.P.T.U., 2005 S)

Solution. Let $y = xv$ so that $y' = v + x \dfrac{dv}{dx}$

and $$y'' = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Substituting these in the given equation, we get

$$(1 - x^2)\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - 2x\left(v + x\frac{dv}{dx}\right) + 2xv = 0$$

or $$(x - x^3)\frac{d^2v}{dx^2} + (2 - 4x^2)\frac{dv}{dx} = 0$$

or $$(x - x^3)\frac{dp}{dx} + (2 - 4x^2)\,p = 0 \text{ where } p = \frac{dv}{dx}$$

Integrating, we get $\displaystyle\int \frac{dp}{p} + \int \frac{2 - 4x^2}{x - x^3}\,dx = c$

or $$\log p + \int \frac{2}{x}dx - \int \frac{dx}{1 - x} - \int \frac{dx}{1 + x} = c$$

or $$\log p + 2\log x + \log(1 - x) - \log(1 + x) = \log c_1$$

$$px^2(1 - x)/(1 + x) = c_1 \text{ or } \frac{dv}{dx} = \frac{c_1(1 + x)}{x^2(1 - x)}$$

Again integrating, $v = c_1 \displaystyle\int \left(\frac{2}{x} + \frac{1}{x^2} + \frac{2}{1 - x}\right)dx + c_2$

or $$v = c_1\{2\log(x/1 - x) - 1/x\} + c_2$$

Hence the required complete solution is $y = x\,[c_1\{\log(x/1 - x)^2 - 1/x\} + c_2]$

Obs. Here $P + Qx = 0$. That is why $y = x$ is a solution of the given equation.

PROBLEMS 15.5

1. If $y = e^{x^2}$ is a solution of $y'' - 4xy' + (4x^2 - 2)y = 0$, find a second independent solution. (U.P.T.U., 2004)
2. Solve $x^2y'' - (x^2 + 2x)y' + (x + 2)y = x^3e^x$.
3. Solve $x\dfrac{d^2y}{dx^2} - (2x - 1)\dfrac{dy}{dx} + (x - 1)y = e^x$ given that $y = e^x$ is one integral. (Bhopal, 2007 S)
4. Solve $\sin^2 x\dfrac{d^2y}{dx^2} = 2y$, given that $y = \cot x$ is a solution. (Bhopal, 2007)
5. Solve $\dfrac{d^2y}{dx^2} - \cot x\dfrac{dy}{dx} - (1 - \cot x)y = e^x \sin x$.

15.7 EQUATIONS WHICH CAN BE SOLVED BY CHANGING THE INDEPENDENT VARIABLE

Consider the equation $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$ ...(1)

To change the independent variable x to z, let $z = f(x)$

Then $$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} \quad \text{...(2)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right) = \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dx^2} \quad \text{...(3)}$$

Substituting (2) and (3) in (1), we get $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1$ ...(4)

where $$P_1 = \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\Big/\left(\frac{dz}{dx}\right)^2, Q_1 = Q\Big/\left(\frac{dz}{dx}\right)^2, R_1 = R\Big/\left(\frac{dz}{dx}\right)^2$$

Now equation (4) can be solved by taking Q_1 = a constant.

Example 15.10. *Solve, by changing the independent variable,* $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = x^5$ *(U.P.T.U., 2003)*

Solution. Given equation is $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4$ ...(*i*)

Here $P = -1/x$, $Q = 4x^2$ and $R = x^4$.

Choose z so that $Q/(dz/dx)^2$ = const. or $(dz/dx)^2 = 4x^2$ (say)

or $$\frac{dz}{dx} = 2x \quad \text{or} \quad z = x^2$$

Changing the independent variable x to z by $z = x^2$, we get

$$\frac{d^2y}{dz^2} + P\cdot\frac{dy}{dz} + Q_1y = R_1 \quad ...(ii)$$

where $P_1 = \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\Big/\left(\frac{dz}{dx}\right)^2 = [2 + (-x^{-1})\,2x]/4x^2 = 0$

$$Q_1 = \frac{Q}{(dz/dx)^2} = \frac{4x^2}{4x^2} = 1, R_1 = \frac{R}{(dz/dx)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4} = \frac{z}{4}$$

$\therefore$ (*ii*) takes the form $\frac{d^2y}{dz^2} + y = \frac{z}{4}$ or $(D^2 + 1)\,y = \frac{z}{4}$

Its A.E. is $D^2 + 1 = 0$, *i.e.*, $D = \pm i$

$$\text{C.F.} = c_1 \cos z + c_2 \sin z$$

$$\text{P.I.} = \frac{1}{D^2+1}\frac{z}{4} = \frac{1}{4}(1 + D^2)^{-1}z = \frac{1}{4}(1 - D^2 \text{.........})\,z = \frac{z}{4}.$$

Hence the complete solution of (*i*) is

$$y = c_1 \cos z + c_2 \sin z + \frac{z}{4} \quad \text{or} \quad y = c_1 \cos x^2 + c_2 \sin x^2 + \frac{x^2}{4}.$$

Example 15.11. *Solve* $\frac{d^2y}{dx^2} + \cot x\frac{dy}{dx} + 4y\operatorname{cosec}^2 x = 0.$ ...(*i*)

Solution. Here $P = \cot x$, $Q = 4\operatorname{cosec}^2 x$

Choosing z so that $Q\Big/\left(\frac{dz}{dx}\right)^2$ = const. or $\left(\frac{dz}{dx}\right)^2 = \operatorname{cosec}^2 x$ (say)

$$dz/dx = \operatorname{cosec} x \text{ or } z = \int \operatorname{cosec} x\, dx = \log\tan x/2$$

Changing the independent variable x to z, we get

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1 \quad ...(ii)$$

where $P_1 = \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\Big/\left(\frac{dz}{dx}\right)^2 = (-\operatorname{cosec} x \cot x + \cot x \operatorname{cosec} x)/\operatorname{cosec}^2 x = 0$

$$Q_1 = Q\Big/\left(\frac{dz}{dx}\right)^2 = \frac{4\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x} = 4, R_1 = 0$$

$\therefore$ (*ii*) takes the form $\dfrac{d^2y}{dz^2} + 4y = 0$

Its solution is $y = c_1 \cos(2z) + c_2 \sin(2z)$

i.e., $$y = c_1 \cos(2 \log \tan x/2) + c_2 \sin(2 \log \tan x/2)$$

This is the required complete solution of (*i*).

PROBLEMS 15.6

Solve the following equations (by changing the independent variable) :

1. $\dfrac{d^2y}{dx^2} + \tan x \dfrac{dy}{dx} + y \cos^2 x = 0.$ (*Bhopal, 2005*)
2. $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} + \dfrac{y}{x^4} = 0.$
3. $\dfrac{d^2y}{dx^2} - \cot x \dfrac{dy}{dx} - \sin^2 xy = 0.$
4. $x\dfrac{d^2y}{dx^2} + (4x^2 - 1)\dfrac{dy}{dx} + 4x^3y = 2x^3.$ (*U.P.T.U., 2006*)
5. $\cos x \dfrac{d^2y}{dx^2} + \sin x \dfrac{dy}{dx} - 2y \cos^3 x = 2\cos^5 x.$ (*Bhopal, 2006 S*)

15.8 TOTAL DIFFERENTIAL EQUATIONS

(1) An ordinary differential equation of the first order and first degree involving three variables is of the form

$$P + Q\frac{dy}{dx} + R\frac{dz}{dx} = 0 \qquad ...(1)$$

where P, Q, R are functions of x, y, z and x is the independent variable.

In terms of differentials, (1) can be written as

$$Pdx + Qdy + Rdz = 0 \qquad ...(2)$$

which is integrable only if

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \qquad ...(3)$$

(2) Rule *to solve* $Pdx + Qdy + Rdz = 0$

If the condition of integrability is satisfied, consider one of the variables say : z, as constant so that dz = 0. Then integrate the equation Pdx + Qdy = 0. Replace the arbitrary constant appearing in its integral by φ(z). Now differentiate the integral just obtained with respect to x, y, z. Finally, compare this result with the given differential equation to determine φ(z).

Example 15.12. *Solve* $(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0.$

Solution. Here $P = y^2 + yz,\ Q = z^2 + zx,\ R = y^2 - xy.$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (y^2 + yz)\,[2z + x - (2y - x)] + (z^2 + zx)\,[-y - y] + (y^2 - xy)\,[(2y + z) - z] = 0$$

Hence the condition of integrability is satisfied.

Considering z as constant, the given equation becomes

$$(y^2 + yz)dx + (z^2 + zx)dy = 0, \quad \text{or} \quad \frac{dx}{z(z + x)} + \frac{dy}{y(y + z)} = 0$$

Integrating and noting that z is a constant, we get

$$\frac{1}{z}\int \frac{dx}{z + x} + \frac{1}{z}\int \left(\frac{1}{y} - \frac{1}{y + z}\right) dy = \text{constant}$$

i.e., $$\log(z + x) + \log y - \log(y + z) = \text{constant.}$$

i.e., $$\frac{y(z + x)}{y + z} = \text{constant} = \phi(z), \text{ say} \qquad ...(i)$$

or $$y(z + x) - (y + z)\,\phi(z) = 0$$

Differentiating w.r.t. x, y, z, we obtain

$$y(dz + dx) + (z + x)dy - [(y + z)\phi'(z)dz + (dy + dz)\phi(z)] = 0$$

or $$ydx + [z + x - \phi(z)]dy + [y - (y + z)\,\phi'(z) - \phi(z)]\,dz = 0 \qquad ...(ii)$$

Comparing (*ii*) with the given differential equation, we get

$$\frac{y^2 + yz}{y} = \frac{z^2 + zx}{z + x - \phi(z)} = \frac{y^2 - xy}{y - (y + z)\phi'(z) - \phi(z)}.$$

The relation $\dfrac{y^2 + yz}{y} = \dfrac{z^2 + zx}{z + x - \phi(z)}$ reduces to (*i*). $\therefore$ it gives no information about $\phi(z)$.

Taking $\dfrac{y^2 + yz}{y} = \dfrac{y^2 - xy}{y - (y + z)\phi'(z) - \phi(z)}$, we get

$$y^2 - xy = (y + z)\,[y - (y + z)\phi'(z) - \phi(z)] = y^2 + yz - (y + z)^2\,\phi'(z) - (y + z)\,\phi(z)$$

$$= y^2 + yz - (y + z)^2\,\phi'(z) - y(z + x) \qquad \text{[From } (i)]$$

$$= y^2 - xy - (y + z)^2\,\phi'(z)$$

i.e., $(y + z)^2\,\phi'(z) = 0$, *i.e.*, $\phi'(z) = 0$ so that $\phi(z) = c$

Hence the required solution is $y(z + x) = (y + z)\,c$. [From (*i*)]

Obs. *Sometimes the integral is readily obtained by simply regrouping the terms in the given equation* as is illustrated below.

Example 15.13. *Solve* $xdx + zdy + (y + 2z)\,dz = 0.$

Solution. Regrouping the terms, we can write the given equation as

$$xdx + (ydz + zdy) + 2z\,dz = 0$$

of which the integral is $\dfrac{x^2}{2} + yz + z^2 = c.$

PROBLEMS 15.7

Solve :

1. $(mz - ny)dx + (nx - lz)dy + (ly - mx)dz = 0.$
2. $(y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0.$
3. $yzdx - 2zxdy - 3xydz = 0.$
4. $(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0.$
5. $(x + z)^2dy + y^2(dx + dz) = 0.$
6. $(yz + xyz)dx + (zx + xyz)\,dy + (xy + xyz)\,dz = 0.$

15.9 SIMULTANEOUS TOTAL DIFFERENTIAL EQUATIONS

These equations in three variables are given by

$$\left.\begin{aligned} Pdx + Qdy + Rdz &= 0 \\ P'dx + Q'dy + R'dz &= 0 \end{aligned}\right\} \qquad ...(1)$$

where P, Q, R and P', Q', R' are any functions of x, y, z.

(*a*) If each of these equations is integrable and have solutions $f(x, y, z) = c$ and $Y(x, y, z) = c¢$ respectively, then these taken together constitute the solution of the simultaneous equations (1).

(*b*) If one or both the equations (1) is not integrable, then we write these as follows :

$$\frac{dx}{QR' - Q'R} = \frac{dy}{RP' - R'P} = \frac{dz}{PQ' - P'Q}$$

and solve these by the methods explained below.

15.10 EQUATIONS OF THE FORM dx/P = dy/Q = dz/R

(1) Method of grouping

See if it is possible to take two fractions $dx/P = dz/R$ from which y can be cancelled or is absent, leaving equations in x and z only.

If so, integrate it by giving $\phi(x, z) = c$. ...(1)

Again see if one variable say : x is absent or can be removed may be with the help of (1), from the equation $dy/Q = dz/R$.

Then integrate it by giving $\psi(y, z) = c'$ ...(2)

These two independent solutions (1) and (2) taken together constitute the complete solution required.

Example 15.14. *Solve* $\dfrac{dx}{z^2y} = \dfrac{dy}{z^2x} = \dfrac{dz}{y^2x}$.

Solution. Taking the first two fractions and cancelling z^2, we get

$$\frac{dx}{y} = \frac{dy}{x} \quad \text{or} \quad xdx - ydy = 0$$

which on integration gives $x^2 - y^2 = c$. ...(*i*)

Again taking the second and third fractions and cancelling x, we have

$$\frac{dy}{z^2} = \frac{dz}{y^2}, \text{ i.e., } y^2dy - z^2dz = 0.$$

Its integral is $y^3 - z^3 = c'$. ...(*ii*)

Thus (*i*) and (*ii*) taken together constitute the required solution of the given equations.

(2) Method of multipliers

By a proper choice of the multipliers l, m, n which are not necessarily constants, we write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR} \text{ such that } lP + mQ + nR = 0.$$

Then $ldx + mdy + ndz = 0$ can be solved giving the integral $\phi(x, y, z) = c$ ...(1)

Again search for another set of multipliers λ, μ, γ

so that $\lambda P + \mu Q + \gamma R = 0$

giving $\lambda dx + \mu dy + \gamma dz = 0$,

which on integration gives the solution $\psi(x, y, z) = c'$ ...(2)

These two solutions (1) and (2) taken together constitute the required solution.

Example 15.15. *Solve* $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{-y(z^2 + x^2)} = \dfrac{dz}{z(x^2 + y^2)}$.

Solution. Using the multipliers x, y, z

each fraction $= \dfrac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = \dfrac{xdx + ydy + zdz}{0}$

$\therefore$ $xdx + ydy + zdz = 0$, which on integration gives the solution $x^2 + y^2 + z^2 = c$ ...(*i*)

Again using the multipliers $1/x, -1/y, -1/z$

each fraction $= \dfrac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)} = \dfrac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{0}$ so that $\dfrac{dx}{x} - \dfrac{dy}{y} - \dfrac{dz}{z} = 0$

which on integration gives $\log x - \log y - \log z = $ constant or $yz = c'x$. ...(*ii*)

Hence the solution of the given equation is $x^2 + y^2 + z^2 = c$; $yz = c'x$.

PROBLEMS 15.8

Solve :

1. $\dfrac{xdx}{y^2z} = \dfrac{dy}{xz} = \dfrac{dz}{y^2}$.
2. $\dfrac{dx}{mz - ny} = \dfrac{dy}{nx - lz} = \dfrac{dx}{ly - mx}$.
3. $\dfrac{dx}{x^2 - yz} = \dfrac{dy}{y^2 - zx} = \dfrac{dz}{z^2 - xy}$.
4. $\dfrac{dx}{y - zx} = \dfrac{dy}{yz + x} = \dfrac{dz}{x^2 + y^2}$.
5. $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{y(z^2 - x^2)} = \dfrac{dz}{z(x^2 - y^2)}$.
6. $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$.