Algorithm for the semi-analytic method for steady-state heat transfer

October 28, 2016

The governing equation is

$$\nabla \cdot (k\nabla T) = 0.$$

In this case, the geometry involves an infinitely long hollow cylinder with inner radius r_i and outer radius r_o. The thermal conductivity is given.

A linear system of equations results from discretizing using the finite element method:

$$\mathbf{K}\mathbf{u} = \mathbf{0} \tag{1}$$

Differentiating (2) with respect to the design variables, $\mathbf{x}$, yields

$$\mathbf{K}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = -\frac{\partial \mathbf{K}}{\partial \mathbf{x}}\mathbf{u} \tag{2}$$

In the semi-analytic method, the sensitivity of the stiffness matrix with respect to the design variables is computed using finite differences. My understanding of how this is done is below:

1. Find forward solution, $\mathbf{u}$.
2. Set step size for the i-th variable: $\Delta \mathbf{x}_i$.
3. Compute sensitivity of the stiffness matrix to $\Delta \mathbf{x}_i$:

$$\frac{\Delta \mathbf{K}}{\Delta x_i} \approx \frac{\mathbf{K}(\mathbf{x} + \Delta \mathbf{x}_i) - \mathbf{K}(\mathbf{x} - \Delta \mathbf{x}_i)}{2\,\Delta x_i}$$

4. Compute the sensitivity of the solution to the design variables:

$$\frac{\Delta \mathbf{u}}{\Delta x_i} = \mathbf{K}^{-1}\frac{\Delta \mathbf{K}}{\Delta x_i}\mathbf{u}$$

5. Compute the sensitivity of the objective function with respect to the design variables:

$$\frac{\partial f}{\partial x_i} \approx \frac{f(\mathbf{u} + \Delta \mathbf{u}) - f(\mathbf{u} - \Delta \mathbf{u})}{2\Delta \mathbf{u}}\frac{\Delta \mathbf{u}}{\Delta x_i} = \frac{f(\mathbf{u} + \Delta \mathbf{u}) - f(\mathbf{u} - \Delta \mathbf{u})}{2\Delta x_i}$$