NTP Server Evaluation: A Mathematical Model

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1 Estimating a System's Drift Constant

Time laboratories, such as NIST's Time and Frequency Division, distribute their own realization of standard time scales such as UTC. One method of the distribution of their time is by the Network Time Protocol (NTP) method, in which packets are sent over the internet so that other systems may synchronize their time with that of NIST. However, there is uncertainty associated with this distribution method, on the order of hundreds of microseconds to milliseconds, according to NIST. As such, we model the output of NIST's time distribution via NTP as a non-deterministic process with some uncertainty.

Let $N(t)$ denote the offset between a realization of UTC laboratory time—for example UTC(NIST)—and the distributed time via NTP, at some time t. That is, $N(t) = t - \text{NTP}(t)$, where t is NIST's realization of UTC time and NTP(t) is the time distributed by NIST's NTP server at time t. Thus, $N(t)$ gives the error associated with the distribution of the time via NTP at time t. Because the $N(t)$ are non-deterministic, we will model them as random variables. In particular, we assume that the $N(t)$ are distributed i.i.d. according to a Normal distribution with mean zero and variance σ^2, for all t.

Now, let $S(t)$ be the system time at time t. By "time t," we mean the time as determined by a time laboratory such as NIST. Because computer clocks are very inaccurate and unstable compared to laboratory clocks, they will exhibit the phenomenon of "clock drift," whereby they drift apart from a standardized timescale such as UTC and its realizations by laboratory clocks. We will assume that the system clock exhibits a linear frequency drift, i.e. $S(t) = Dt + b$ for a drift constant D and constant b.

When we query a NTP server, we are given NTP$(t) = t - N(t)$ in response. We can also easily calculate the offset between the NTP time and the system clock's time, with NTP$(t) - S(t) = t - N(t) - S(t)$. For a time t_i, let us denote this offset quantity by $\Delta(t_i) \equiv \text{NTP}(t_i) - S(t_i) = t_i - N(t_i) - S(t_i)$, and note that $\Delta(t_i)$ is an observable quanity, like NTP(t) and $S(t)$. Also note that $\Delta(t_i)$ is distributed Normally with mean $t_i - S(t_i) = (1-D)t_i - b$ and variance σ^2.

Now, we wish to estimate the constants D and b. In order to obtain estimates for these constants, we will have to observe many instances of $\Delta(t)$ at different times. We will then use the maximum likelihood method to estimate D and b.

Consider n observations of our observed quantity, $\Delta(t_1), ..., \Delta(t_n)$. Since the $\Delta(t_i)$ are distributed Normally with mean $(1-D)t_i - b$ and variance σ^2, The likelihood function for our data is.

$$\mathcal{L}(D, b, \sigma^2; \Delta(t_1), ..., \Delta(t_n)) = \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \exp[-\frac{1}{2\sigma^2}(\Delta(t_i) - (1-D)t_i + b)^2] \tag{1}$$

$$= (2\pi\sigma^2)^{-n/2} \exp[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\Delta(t_i) - (1-D)t_i + b)^2] \tag{2}$$

And note that the right-hand expression is maximized when $\sum_{i=1}^{n}(\Delta(t_i) - (1-D)t_i + b)^2$ is minimized. Further note that this is a sum of squares of differences between $\Delta(t_i)$ and $(1-D)t_i - b$. Therefore, if we plot pairs $(t_i, \Delta(t_i))$ and perform a linear least squares fit on this data, we will obtain a line with slope $1 - D$ and with an intercept of $-b$, allowing us to estimate both of these quantities. It might be noted that t_i is not an observable quantity, which is indeed true. However, recall that $t = N(t) + \text{NTP}(t)$, and that $N(t)$ has mean zero. Therefore, $\text{NTP}(t)$ is an unbiased estimator for t since $\mathbb{E}(\text{NTP}(t)) = \mathbb{E}(t - N(t)) = t$. Thus, we may instead plot observable pairs $(\text{NTP}(t_i), \Delta(t_i))$ and perform our linear least squares fit procedure on this data.

To summarize, we have shown that estimates of the drift constant D and the constant b can be obtained by plotting observable pairs $(\text{NTP}(t_i), \Delta(t_i))$ and then performing a linear least squares fit on the data. This will yield a line with slope $1 - D$ and intercept $-b$, allowing us to estimate D and b with $\hat{D} = 1 -$ slope and $\hat{b} = -$intercept.

2 Evaluating NTP Servers

Once we have estimates of D and b, we can obtain an estimate of the true time t, which is unobservable, with $\hat{t}_i = (S(t_i) - \hat{b})/\hat{D}$. In contrast to $\text{NTP}(t)$, $\hat{t}_i$ does not require us to query a NTP server for time, and provides an estimate of t based purely on the system clock's time.

3 References