# STA 601: Lab 6

Akbota Anuarbek

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \sim N(\mu, \Sigma), \text{ where } \mu = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 1 & 0.9 & 0.1 \\ 0.9 & 1 & 0.1 \\ 0.1 & 0.1 & 1 \end{bmatrix}$$

(1)

$X|(Y,Z) \sim N(\mu_x, \sigma_x^2)$, where

$$\mu_x = 0 + \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}^{-1} \left( \begin{bmatrix} y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \tfrac{1}{0.99} \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} y - 0.1z \\ z - 0.1y \end{bmatrix} = \tfrac{89y+z}{99}$$

$$\sigma_x^2 = 1 - \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix} = 1 - \tfrac{1}{0.99} \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 0.89 \\ 0.01 \end{bmatrix} = \tfrac{94}{495}$$

$Y|(X,Z) \sim N(\mu_y, \sigma_y^2)$, where

$$\mu_y = 0 + \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}^{-1} \left( \begin{bmatrix} x \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \tfrac{1}{0.99} \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} x - 0.1z \\ z - 0.1x \end{bmatrix} = \tfrac{89x+z}{99}$$

$$\sigma_y^2 = 1 - \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix} = 1 - \tfrac{1}{0.99} \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 0.89 \\ 0.01 \end{bmatrix} = \tfrac{94}{495}$$

$Z|(X,Y) \sim N(\mu_z, \sigma_z^2)$, where

$$\mu_z = 0 + \begin{bmatrix} 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}^{-1} \left( \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \tfrac{1}{0.19} \begin{bmatrix} 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} x - 0.9y \\ y - 0.9x \end{bmatrix} = \tfrac{x+y}{19}$$

$$\sigma_z^2 = 1 - \begin{bmatrix} 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} = 1 - \tfrac{1}{0.19} \begin{bmatrix} 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix} = \tfrac{94}{95}$$

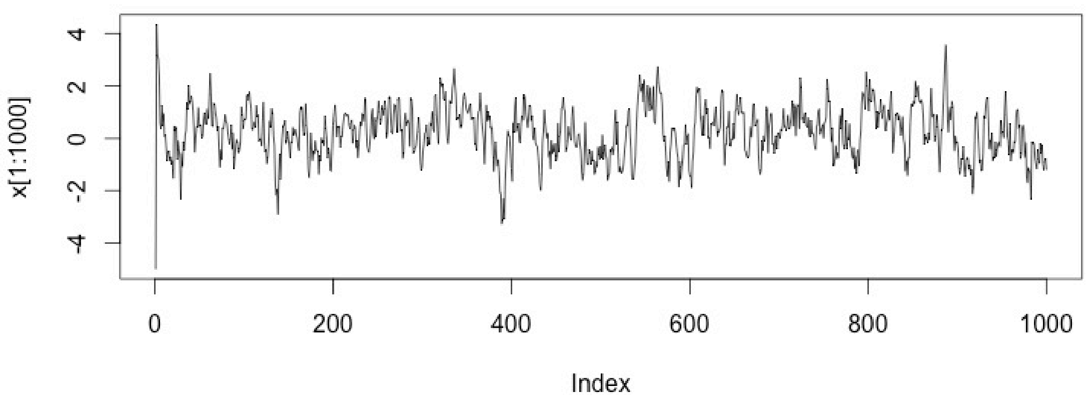
(2)

```
N<-10000

x<-c(-5)
y<-c(5)
z<-c(1)

for(i in 2:N){
  x[i]=rnorm(1, mean=(89*y[i-1]+z[i-1])/99, sd=sqrt(94/495))
  y[i]=rnorm(1, mean=(89*x[i]+z[i-1])/99, sd=sqrt(94/495))
  z[i]=rnorm(1, mean=(x[i]+y[i])/19, sd=sqrt(94/95))
}

plot(x[1:1000], type="lines")
```




From the trace plot of X above, we can see that it is not stationary, there is a high autocorrelation. As a result, simulated marginal distribuions may not converge to the joint distribution.

(3)

$(X, Y)|Z \sim N(\mu_{x,y}, \Sigma_{x,y})$, where

$$\mu_{x,y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} 1(z-0) = \begin{bmatrix} 0.1z \\ 0.1z \end{bmatrix}$$

$$\Sigma_{x,y} = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} 1 \begin{bmatrix} 0.1 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.99 & 0.89 \\ 0.89 & 0.99 \end{bmatrix}$$

Complete conditional of $Z|(X,Y)$ was found in part(1).

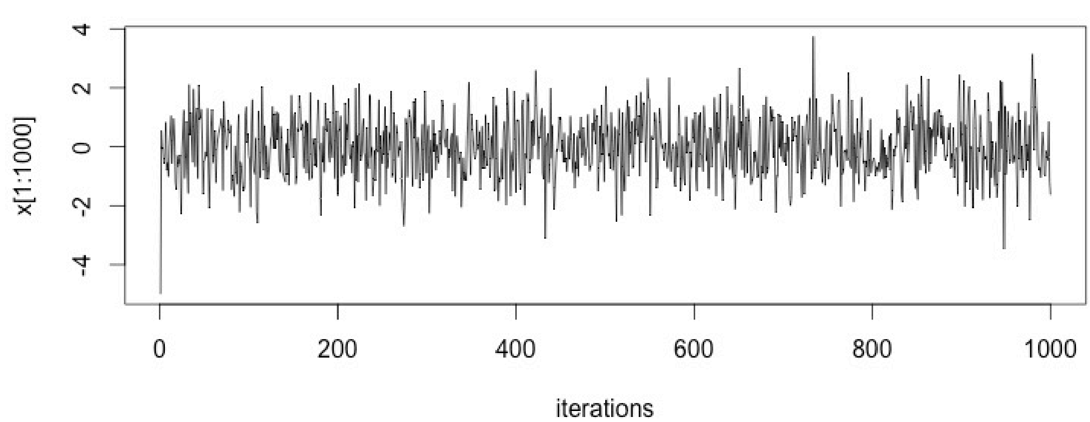
```
N<-10000

x<-c(-5)
y<-c(5)
z<-c(1)

Sigma_=matrix(c(0.99, 0.89, 0.89, 0.99), 2, 2)

for(i in 2:N){
  mu_=c(0.1*z[i-1], 0.1*z[i-1])
  xy=mvrnorm(n=1, mu=mu_, Sigma=Sigma_)
  x[i]=xy[1]
  y[i]=xy[2]
  z[i]=rnorm(1, mean=(x[i]+y[i])/19, sd=sqrt(94/95))
}

plot(x[1:1000], type="lines", xlab="iterations")
```



From the trace plot of X above, we can see that it quickly becomes stationary after the first several iterations. Thus, simulated marginal distributions converge to the joint distribution very fast.

(4)

Second Gibbs sampler is more efficient, because it converges faster and has lower autocorrelation.

In the first Gibbs sampler the simulated marginal distributions do not approximate the target distribution precisely because of very high correlation between X and Y. More precisely, each value of X is drawn from its full conditional distribution given previous value of Y, which is higly correlated with previous value of X. As a result, X is correlated with its previous values, same is true for values of Y.

In contrast, in the second Gibbs sampler, values of X and Y are drawn without conditioning on values of each other, so that high correlation between X and Y is not an issue.