STA601: Homework 5

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$y_i \sim N(\mu, \sigma^2)$

(1) Prior: $\pi(\mu, \sigma^{-2}) \propto N(\mu; \mu_0, k_0\sigma^2)Gamma(\sigma^{-2}; a, b)$

Posterior: $\pi(\mu, \sigma^{-2}|y_1, ..., y_n) \propto \pi(\mu, \sigma^{-2}) \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} exp[-\frac{1}{2\sigma^2}(y_i - \mu)^2]$

Let $\phi = \sigma^{-2}$, then

$\pi(\mu, \phi|y_1, ..., y_n) \propto \sqrt{\phi} e^{-\phi\frac{1}{2k_0}(\mu-\mu_0)^2} \phi^{a-1} e^{-b\phi} \prod_{i=1}^{n} \sqrt{\phi} e^{-\frac{1}{2}\phi(y_i-\mu)^2} \propto$

$\sqrt{\phi} e^{-\frac{\phi}{2}\left(\frac{\mu^2}{k_0} - 2\frac{\mu_0}{k_0}\mu + \frac{{\mu_0}^2}{k_0} + \sum_{i=1}^{n} {y_i}^2 - 2\mu\sum_{i=1}^{n} y_i + n\mu^2\right)} \phi^{a+\frac{n}{2}-1} e^{-b\phi} \propto$

$\sqrt{\phi} e^{-\frac{\phi}{2}\frac{nk_0+1}{k_0}\left(\mu^2 - 2\mu\left(\frac{\mu_0+k_0\sum y_i}{nk_0+1}\right) + \left(\frac{\mu_0+k_0\sum y_i}{nk_0+1}\right)^2 - \left(\frac{\mu_0+k_0\sum y_i}{nk_0+1}\right)^2 + \frac{{\mu_0}^2+k_0\sum {y_i}^2}{nk_0+1}\right)} \phi^{a+\frac{n}{2}-1} e^{-b\phi}$

Let $\mu_1 = \frac{\mu_0+k_0\sum y_i}{nk_0+1}$, $k_1 = \frac{k_0}{nk_0+1}$, then

$\propto \sqrt{\phi} e^{-\phi\frac{1}{2k_1}(\mu-\mu_1)^2} e^{-\frac{\phi}{2}\frac{nk_0+1}{k_0}\left(\frac{{\mu_0}^2+k_0\sum {y_i}^2}{nk_0+1} - \left(\frac{\mu_0+k_0\sum y_i}{nk_0+1}\right)^2\right)} \phi^{a+n/2-1} e^{-b\phi}$

$\propto \sqrt{\phi} e^{-\phi\frac{1}{2k_1}(\mu-\mu_1)^2} \phi^{a+n/2-1} e^{-\phi\left(b + \frac{\sum_{i=1}^{n} {y_i}^2}{2} + \frac{n{\mu_0}^2 - k_0(\sum_{i=1}^{n} y_i)^2 - 2\mu_0\sum_{i=1}^{n} y_i}{2(nk_0+1)}\right)}$

Let $a_1 = a + \frac{n}{2}$

$b_1 = b + \frac{\sum_{i=1}^{n} {y_i}^2}{2} + \frac{n{\mu_0}^2 - k_0(\sum_{i=1}^{n} y_i)^2 - 2\mu_0\sum_{i=1}^{n} y_i}{2(nk_0+1)}$, then

$\propto \sqrt{\phi} e^{-\phi\frac{1}{2k_1}(\mu-\mu_1)^2} \phi^{a_1-1} e^{-\phi b_1} \propto N(\mu; \mu_1, k_1\phi^{-1})Gamma(\phi; a_1, b_1)$

Thus, we obtain posterior:

$\pi(\mu, \sigma^{-2}|y_1, ..., y_n) \propto N(\mu; \mu_1, k_1\sigma^2)Gamma(\sigma^{-2}; a_1, b_1)$, where

$\mu_1 = \frac{\mu_0 + k_0 \sum y_i}{nk_0+1}$

$k_1 = \frac{k_0}{nk_0+1}$

$a_1 = a + \frac{n}{2}$

$b_1 = b + \frac{\sum_{i=1}^n {y_i}^2}{2} + \frac{n{\mu_0}^2 - k_0(\sum_{i=1}^n y_i)^2 - 2\mu_0 \sum_{i=1}^n y_i}{2(nk_0+1)}$

So we can see that the posterior is of the same form as the prior, so prior is conjugate.

(2) Marginal posterior for μ:

$\pi(\mu|y_1, ...y_n) = \int_0^\infty \pi(\mu, \sigma^{-2}|y_1, ..., y_n)d\sigma^{-2} = \int_0^\infty N(\mu; \mu_1, k_1\sigma^2)Gamma(\sigma^{-2}; a_1, b_1)d\sigma^{-2}$

Let $\sigma^{-2} = \phi$, then

$\pi(\mu|y_1, ...y_n) = \int_0^\infty N(\mu; \mu_1, k_1\phi)Gamma(\phi; a_1, b_1)d\phi$
$= \int_0^\infty \frac{\sqrt{\phi}}{\sqrt{2\pi k_1}} e^{-\phi \frac{1}{2k_1}(\mu-\mu_1)^2} \frac{{b_1}^{a_1}}{\Gamma(a_1)} \phi^{a_1-1} e^{-b_1\phi} d\phi$
$= \frac{1}{\sqrt{2\pi k_1}} \frac{{b_1}^{a_1}}{\Gamma(a_1)} \int_0^\infty \phi^{a_1+\frac{1}{2}-1} e^{-\phi\left(b_1+\frac{1}{2k_1}(\mu-\mu_1)^2\right)} d\phi$
$= \frac{1}{\sqrt{2\pi k_1}} \frac{{b_1}^{a_1}}{\Gamma(a_1)} \frac{\Gamma(a_1+\frac{1}{2})}{\left(b_1+\frac{1}{2k_1}(\mu-\mu_1)^2\right)^{(a_1+\frac{1}{2})}} \int_0^\infty Gamma\Big(\phi; b_1 + \frac{1}{2k_1}(\mu-\mu_1)^2, a_1+\frac{1}{2}\Big)d\phi$
$= \frac{1}{\sqrt{2\pi k_1}} \frac{{b_1}^{a_1}}{\Gamma(a_1)} \frac{\Gamma(a_1+\frac{1}{2})}{\left(b_1+\frac{1}{2k_1}(\mu-\mu_1)^2\right)^{(a_1+\frac{1}{2})}}$
$= \frac{1}{\sqrt{2\pi}} \frac{\Gamma\left(a_1+\frac{1}{2}\right)}{\Gamma(a_1)} (b_1k_1)^{-\frac{1}{2}} \left(1 + \frac{1}{2a_1}\frac{a_1}{b_1k_1}(\mu-\mu_1)^2\right)^{-(a_1+1/2)}$
$= \frac{1}{\sqrt{2a_1\pi}} \frac{\Gamma\left(a_1+\frac{1}{2}\right)}{\Gamma(a_1)} \sqrt{\frac{a_1}{b_1k_1}} \left(1 + \frac{1}{2a_1}\frac{a_1}{b_1k_1}(\mu-\mu_1)^2\right)^{-(a_1+1/2)}$
$= T_{2a_1}(\mu; \mu_1, \frac{b_1k_1}{a_1})$

Here, $\mu_1 = \frac{\mu_0 + k_0 \sum y_i}{nk_0+1}$

$k_1 = \frac{k_0}{nk_0+1}$

$a_1 = a + \frac{n}{2}$

$b_1 = b + \frac{\sum_{i=1}^{n} {y_i}^2}{2} + \frac{n{\mu_0}^2 - k_0(\sum_{i=1}^{n} y_i)^2 - 2\mu_0 \sum_{i=1}^{n} y_i}{2(nk_0+1)}$

(3) Prior: $\pi(\mu, \sigma^{-2}) = \pi(\mu)\pi(\sigma^{-2}) = N(\mu; \mu_0, \sigma_0) Gamma(\sigma^{-2}; c, d)$

Posterior: $\pi(\mu, \sigma^{-2}|y_1, ..., y_n) \propto \pi(\mu)\pi(\sigma^{-2}) \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} exp[-\frac{1}{2\sigma^2}(y_i - \mu)^2]$

$\propto e^{-\frac{(\mu-\mu_0)^2}{2{\sigma_0}^2}} (\sigma^{-2})^{c-1} e^{-d\sigma^{-2}} \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2}$

$\pi(\mu|\sigma, y) \propto e^{-\frac{(\mu-\mu_0)^2}{2{\sigma_0}^2}} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2}$

$\propto e^{-\frac{1}{2}\left(\frac{1}{{\sigma_0}^2}\mu^2 - 2\mu\frac{1}{{\sigma_0}^2}\mu_0 + \frac{n}{\sigma^2}\mu^2 - 2\mu\frac{\sum_{i=1}^{n} y_i}{\sigma^2}\right)}$

$\propto e^{-\frac{1}{2}\frac{\sigma^2 + n{\sigma_0}^2}{\sigma^2{\sigma_0}^2}\left(\mu - \frac{\mu_0\sigma^2 + {\sigma_0}^2 \sum_{i=1}^{n} y_i}{\sigma^2 + n{\sigma_0}^2}\right)^2}$

$\propto N(\mu; M, S^2)$, where

$M = \frac{\mu_0\sigma^2 + {\sigma_0}^2 \sum_{i=1}^{n} y_i}{\sigma^2 + n{\sigma_0}^2}$

$S = \left(\frac{\sigma\sigma_0}{\sqrt{\sigma^2 + n{\sigma_0}^2}}\right.$

$\pi(\sigma|\mu, y) \propto (\sigma^{-2})^{c-1}\sigma^{-n} e^{-d\sigma^{-2} - \sigma^{-2}\frac{\sum_{i=1}^{n}(y_i-\mu)^2}{2}}$

$\propto (\sigma^{-2})^{c+n/2-1} e^{-\sigma^{-2}\left(d + \frac{\sum_{i=1}^{n}(y_i-\mu)^2}{2}\right)}$

$\propto Gamma(c + \frac{n}{2}, d + \frac{\sum_{i=1}^{n}(y_i-\mu)^2}{2})$

(4) Simulate independent indentically distributed data from $y_i = N(\mu, \sigma^2)$, where

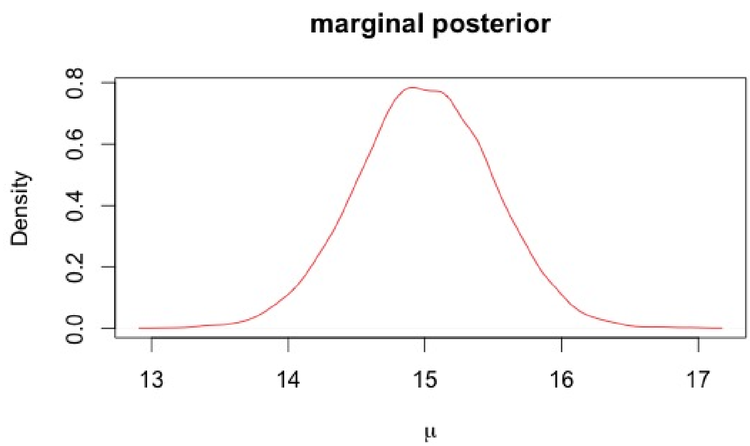
$\mu = 15, \sigma^2 = 25$

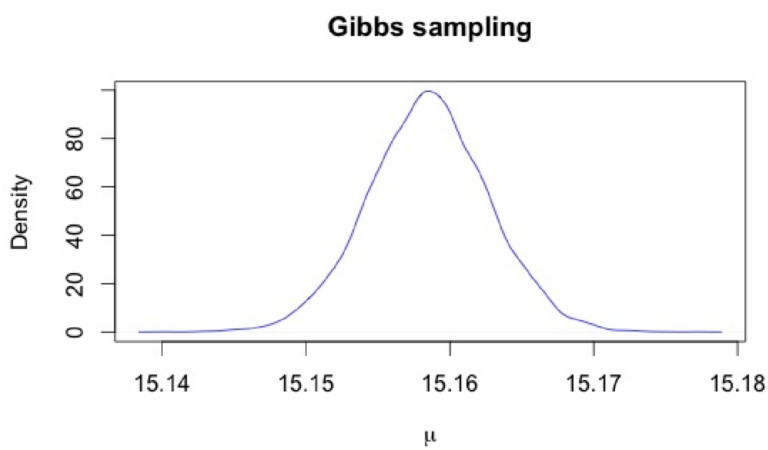
1. randomly generate data from marginal posterior distribution of μ

$\mu_0 = 11, k_0 = 4, a = 1, b = 0.5$

2. run Gibbs sampling with $\mu_0 = 11, \sigma_0 = 6, c = 1, d = 0.5$. Burn-in first 200 observations.

Below are posterior distribution of μ using two different methods. Overall, two methods give us very similar results, both distributions are centered very close to 15, as is illustrated in figures below. However, sample derived using Gibbs sampling has smaller variance.





	mean	variance
marginal posterior	14.99996	0.2479583
gibbs sampling	15.15852	1.765673e-05

R-code

```
n<-100
N<-10000
m=15
s=5
y<-rnorm(n,mean=m,sd=s)



m_0=11
k_0=4
a=1
b=0.5

m_1=(m_0+k_0*sum(y))/(n*k_0+1)
k_1=k_0/(n*k_0+1)
a_1=a+n/2
b_1=b+sum(y^2)/2+(n*m_0^2-k_0*(sum(y))^2-2*m_0*sum(y))/(2*(n*k_0+1))

s=sqrt(b_1*k_1/a_1)
v=2*a_1
```

```
mu_1<-rt(N, v)*s+m


m_0=11
sigm_0=6
c=1
d=0.5

sigm_2<-c(1)
mu_2<-c(1)



for(i in 2:(N+200)){
  M=(m_0*sigm_2[i-1]^2+sigm_0^2*sum(y))/(sigm_2[i-1]^2+n*sigm_0^2)
  S=(sigm_2[i-1]*sigm_0)/(sqrt(sigm_2[i-1]^2+n*sigm_0^2))

  mu_2[i]=rnorm(1,M,S)
  sigm_2[i]=rgamma(1,shape=c+n/2,rate=d+sum(((y-mu_2[i])^2)/2))

}

mu_2_post_burnin<-mu_2[201:(N+200)]

plot(density(mu_1), col="red", xlab=expression(mu), main="marginal posterior")
```

```
plot(density(mu_2_post_burnin), col="blue", xlab=expression(mu), main="Gibbs_sar

mean(mu_1)
mean(mu_2_post_burnin)

var(mu_1)
var(mu_2_post_burnin)
```