Primal and Dual LP Problems

Economic theory indicates that scarce (limited) resources have value. In LP models, limited resources are allocated, so they should be, valued.

Whenever we solve an LP problem, we implicitly solve two problems: the primal resource allocation problem, and the dual resource valuation problem.

Here we cover the resource valuation, or as it is commonly called, the Dual LP

Primal

$$\begin{array}{llccl} Max & \sum_j c_j X_j & & & \\ s.t. & \sum_j a_{ij} X_j & \leq & b_i & \text{for all } i \\ & X_j & \geq & 0 & \text{for all } j \end{array}$$

Dual

$$\begin{array}{llccl} \text{Min} & \sum_i U_i b_i & & & \\ \text{s.t.} & \sum_i U_i a_{ij} & \geq & c_j & \text{for all } j \\ & U_i & \geq & 0 & \text{for all } i \end{array}$$

Primal Dual Pair and Their Units

Primal

$$\begin{aligned} &Max \quad \sum_j c_j X_j \\ &s.t. \quad \sum_j a_{ij} X_j \leq b_i \quad \text{for all } i \\ &\qquad\qquad X_j \geq 0 \quad \text{for all } j \end{aligned}$$

where x is the variable and equals units sold

max sum (per unit profits) * (units sold)
s.t. sum (per unit res. use)*(units sold) $\leq$ res on hand

Dual

$$\begin{aligned} &\text{Min} \quad \sum_i U_i b_i \\ &\text{s.t.} \quad \sum_i U_i a_{ij} \geq c_j \quad \text{for all } j \\ &\qquad\qquad U_i \geq 0 \quad \text{for all } i \end{aligned}$$

U is the variable and equals per unit resource value

min sum (per unit res value) * (res on hand)
s.t. sum (per unit res value) * (per unit res use)
$\geq$ per unit profits

So resource values – shadow prices are set up so per unit profits are exhausted.

Primal Dual Pair and Example

$$\begin{array}{llllllll}
\max & 2000X_{fancy} & + & 1700X_{fine} & + & 1200X_{new} & & \\
\text{s.t.} & X_{fancy} & + & X_{fine} & + & X_{new} & \leq & 12 \\
 & 25X_{fancy} & + & 20X_{fine} & + & 19X_{new} & \leq & 280 \\
 & X_{fancy} & , & X_{fine} & , & X_{new} & \geq & 0
\end{array}$$

$$\begin{array}{lllllll}
 & \text{(van cap)} & & \text{(labor)} & & & \text{(profits)} \\
\min & 12U_1 & + & 280U_2 & & & \text{(Resource Payments)} \\
\text{s.t.} & U_1 & + & 25U_2 & \geq & 2000 & (X_{fancy}) \\
 & U_1 & + & 20U_2 & \geq & 1700 & (X_{fine}) \\
 & U_1 & + & 19U_2 & \geq & 1200 & (X_{new}) \\
 & U_1 & , & U_2 & \geq & & \text{(nonegativity)}
\end{array}$$

Primal rows become dual columns

Primal Dual Objective Correspondence

Given Feasible X* and U*

$$CX^{*} \leq U^{*\prime} AX^{*} \leq U^{*\prime} b$$

$$CX^{*} \leq U^{*\prime} b$$

Dual Variable
Relation
to Partial
Derivative

$$u^{*} = C_{B} B^{-1} = \frac{\partial Z}{\partial b}$$

Complementary Slackness
At Optimal X*, u*

$$U^{*\prime} (b - AX^{*}) = 0$$

$$(U^{*\prime} A - C) X^{*} = 0.$$

Zero Profits
Given Optimal

$$u' b = c x$$

Primal Dual Interrelations

Constructing Dual Solutions

Note we can construct a dual solution from optimal primal solution without resolving the problem.

Given optimal primal $X_B^* = B^{-1}b$ and $X_{NB}^* = 0$. This solution must be feasible;

$$X_B^* = B^{-1}b \geq 0 \text{ and } X \geq 0$$

and must satisfy nonnegative reduced cost

$$C_B B^{-1} A_{NB} - C_{NB} \geq 0.$$

Given this, suppose try $U^{*\prime} = C_B B^{-1}$ as a dual solution.

First, is this feasible in the dual constraints.
To be feasible, we must have $U^{*\prime} A \geq C$ and $U^* \geq 0$.

We know $U^{*\prime} A_{NB} - C_{NB} \geq 0$ because at optimality this is equivalent to the reduced cost criteria,

$$C_B B^{-1} A_{NB} - C_{NB} \geq 0.$$

Further, for basic variables reduced cost is

$$C_B B^{-1} A_B - C_B = C_B B^{-1} B - C_B = C_B - C_B = 0,$$

So unifying these statements, we know when $U^* A \geq C$

Primal Dual Interrelations

Constructing Dual Solutions

Now are dual nonnegativity conditions $U \geq 0$ satisfied.

We can look at this by looking the implication of the nonnegative reduced costs over the slacks ($UA \geq C$).

For the slacks, A contains an identity matrix and the associated entries in C are all 0's.

Thus, $UA = UI = U \geq C = 0$ or $U \geq 0$.

So the U's are non-negative. Thus, $U = C_B B^{-1}$ is a feasible dual solution.

Primal Dual Interrelations

Constructing Dual Solutions

Now the question becomes, is this choice optimal?

In this case the primal objective function Z equals $CX^* = C_B X_B^* + C_{NB} X_{NB}^* = C_B B^{-1} b + C_{NB} 0 = C_B B^{-1} b$.

Simultaneously, the dual objective equals $Ub = C_B B^{-1} b$ which equals the primal objective. Therefore, the primal and dual objectives are equal at optimality.

Furthermore, since for a feasible primal we have $AX \leq b$ and for the dual $UA \geq C$ then multiplying both so term on left is UAX we see $Ub \geq Cx$ or all feasible primal objectives must be less than or equal to all feasible dual objectives.

Thus $C_B B^{-1}$ is an optimal dual solution.

This demonstration shows that given the solution from the primal the dual solution can simply be computed without need to solve the dual problem.

Primal Dual Interrelations

Interpreting Dual Solutions

In addition given the derivation in the last chapter we can establish the interpretation of the dual variables. In particular, since the optimal dual variables equal $C_B B^{-1}$ (which are called the primal shadow prices) then the dual variables are interpretable as the marginal value product of the resources since we showed

$$\frac{\partial Z}{\partial b} = C_B B^{1} = U^{*\prime}.$$

Also

Complementary Slackness
At Optimal X*, U*

$$U^{*\prime}(b - AX^{*}) \quad = \quad 0$$

$$(U^{*\prime}A - C)X^{*} \quad = \quad 0.$$

Zero Profits
Given Optimal

$$U^{*\prime}b = cx^{*}$$

Primal Dual Interrelations

Primal Dual Objective Correspondence

Primal Solution Item Primal Solution Information	Dual Solution Item Corresponding Dual Solution Information
Objective function	Objective function
Shadow prices	Variable values
Slacks	Reduced costs
Variable values	Shadow prices
Reduced costs	Slacks

Degeneracy and Duality

The above interpretations for the dual variables depend upon whether the basis still exists after the change occurs.

When a basic primal variable equals zero, dual has alternative optimal solutions. The cause of this situation is generally primal constraints are redundant at the solution point and the range of right hand sides is zero.

$$
\begin{array}{llcccl}
\text{Max} & 3X_1 & + & 2X_2 & & \\
 & X_1 & + & X_2 & \leq & 100 \\
 & X_1 & & & \leq & 50 \\
 & & & X_2 & \leq & 50 \\
 & X_1 & , & X_2 & \geq & 0
\end{array}
$$

At the optimal solution, $X_1 = 50$, $X_2 = 50$, constraints are redundant.

If first slack variable is basic then $X_1 = 50$, $X_2 = 50$, $S_1 = 0$ while S_1 is basic. Shadow prices are 0, 3, and 2.

If S_3 basic $X_1 = 50$, $X_2 = 50$ $S_3 = 0$ with shadow prices 2, 1, 0. Same objective value -- multiple solutions.

Degeneracy and Duality

$$
\begin{array}{lrcrcr}
\text{Max} & 3X_1 & + & 2X_2 & & \\
 & X_1 & + & X_2 & \leq & 100 \\
 & X_1 & & & \leq & 50 \\
 & & & X_2 & \leq & 50 \\
 & X_1 & , & X_2 & \geq & 0
\end{array}
$$

u1 u2 u3 = [0 3 2] or [2 1 0]

The main difficulty with degeneracy is in interpreting the shadow prices as they take on a direction.

If one were to increase the first right hand side from 100 to 101 this would lead to a zero change in the objective function and X_1 and X_2 would remain at 50.

Decrease first constrain rhs from 100 to 99 then objective function which is two units smaller because X_2 would need to be reduced from 50 to 49.

This shows that the two alternative shadow prices for the first constraint (i.e., 0 and 2) each hold in a direction.

Similarly if bound on X_1 51, obj increases by 1, whereas, if moved downward to 49, it would cost 3.
Meanwhile, reducing X_2 bound costs 2 and increasing by

0. this explains all shadow prices

Primal Columns are Dual Constraints

Columns in the primal, form constraints on the dual shadow price information.

Thus, for example, when a column is entered into a model indicating as much of a resource can be purchased at a fixed price as one wants, then this column forms an upper bound on the shadow price of that resource.

Note that it would not be sensible to have a shadow price of that resource above the purchase price since one could purchase more of that resource.

Similarly, allowing goods to be sold at a particular price without restriction provides a lower bound on the shadow price.

In general, the structure of the columns in a primal linear programming model should be examined to see what constraints they place upon the dual information.

The linear programming modeling chapter extends this discussion.