The Two-Phase Simplex Method

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Preview

- When a basic feasible solution is not readily available, the two-phase simplex method may be used as an alternative to the big M method.
- In the two-phase simplex method, we add artificial variables to the same constraints as we did in big M method. Then we find a bfs to the original LP by solving the Phase I LP.
- In the Phase I LP, the objective function is to minimize the sum of all artificial variables.
- At the completion of Phase I, we reintroduce the original LP’s objective function and determine the optimal solution to the original LP.

The two-phase simplex method

- Step 1

  Modify the constraints so that the right-hand side of each constraint is nonnegative. This requires that each constraint with a negative right-hand side be multiplied through by -1.
- Step 1'

  Identify each constraint that is now an = or ≥ constraint. In step 3, we will add an the artificial variable to each of these constraints.
- Step 2

  Convert each inequality constraint to standard form.

  For ≤ constraint i, we add a slack variable s_i;

  For ≥ constraint i, we add an excess variable e_i;

The two-phase simplex method

- Step 4

For now, ignore the original LP's objective function. Instead solve an LP whose objective function is min w'=(sum of all the artificial variables). This is called the **Phase I LP**. The act of solving the phase I LP will force the artificial variables to be zero.

The two-phase simplex method

Since each $a_i \geq 0$, solving the Phase I LP will result in one of the following three cases:

- Case 1

The optimal value of w' is greater than zero. In this case, the original LP has no feasible solution.

- Case 2

The optimal value of w' is equal to zero, and no artificial variables are in the optimal Phase I basis. In this case, we drop all columns in the optimal Phase I tableau that correspond to the artificial variables. We now combine the original objective function with the constraints from the optimal Phase I tableau. This yields the Phase II LP. The optimal solution to the Phase II LP is the optimal solution to the original LP.

The two-phase simplex method

- Case 3

  The optimal value of w' is equal to zero and at least one artificial variable is in the optimal Phase I basis. In this case, we can find the optimal solution to the original LP if at the end of Phase I we drop from the optimal Phase I tableau all nonbasic artificial variables and any variable from the original problem that has a negative coefficient in row 0 of the optimal Phase I tableau.

Phase I and II feasible solutions

- Suppose the original LP is infeasible. Then the only way to obtain a feasible solution to the Phase I LP is to let at least one artificial variable be positive. In this situation, $w' > 0$ will result.
- On the other hand, if the original LP has a feasible solution, this feasible solution is feasible in the Phase I LP and yields $w' = 0$. This means that if the original LP has a feasible solution, the optimal Phase I solution will have $w' = 0$.

Example 5

- The Bevco problem

$$\begin{aligned} \min\ z &= 2x_1+3x_2 \\ \text{s.t.}\quad \tfrac{1}{2}x_1+\tfrac{1}{4}x_2 &\le 4 \quad \text{(sugar constraint)} \\ x_1+3x_2 &\ge 20 \quad \text{(vitamin C constraint)} \\ x_1+x_2 &= 10 \quad \text{(10 oz in bottle of Oranj)} \\ x_1,x_2 &\ge 0 \end{aligned}$$

Example 5

- **Solution**

As in the Big M method, step 1-3 transform the constraints into

$$
\begin{aligned}
1/2\, x_1 + 1/4 x_2 + s_1 \quad &= 4 \\
x_1 + 3x_2 - e_2 + a_2 \quad &= 20 \\
x_1 + x_2 + a_3 &= 10
\end{aligned}
$$

Example 5

Step 4 yields the following Phase I LP:

$$\begin{array}{llllll}
\min w' = & & & & a_2+a_3 & \\
\text{s.t. } \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 & & & & & =4 \\
x_1 + & 3x_2 & & -e_2+a_2 & & =20 \\
x_1 + & x_2 & & & +a_3 & =10
\end{array}$$

This set of equations yields a starting bfs for Phase I($s_1=4, a_2=20, a_3=10$).

Example 5

- Note:

The row 0 for this tableau ($w'-a_2-a_3=0$) contains the basic variables a_2 and a_3.as in the Big M method, a_2 and a_3 must be eliminated from row 0 before we can solve Phase I.

To eliminate a_2 and a_3 from row 0,

Row 0:	w'	$-a_2-a_3$	$=0$
+Row 2:		$x_1+3x_2-e_2+a_2$	$=20$
+Row 3:		$x_1+x_2+a_3$	$=10$
=New row 0:	w'	$+2x_1+4x_2-e_2$	$=30$

Example 5

Combining the new row 0 with the Phase I constraints yields the initial Phase I tableau:

Initial Phase I tableau for Bevco

W'	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic variable	ratio
1	2	4	0	-1	0	0	30	W'=30	
0	½	¼	1	0	0	0	4	s_1=4	16
0	1	3	0	-1	1	0	20	a_2=20	20/3*
0	1	1	0	0	0	1	10	a_3=10	10

Note: Phase I problem is always a min problem, even if the original LP is a max problem.

Example 5

After performing the necessary ero's, we obtain the second tableau.

Phase I tableau for Bevco after one iteration

W'	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic variable	ratio
1	2/3	0	0	1/3	-4/3	0	10/3	W'=10/3	
0	5/12	0	1	1/12	-1/12	0	7/3	s_1=7/3	28/5
0	1/3	1	0	-1/3	1/3	0	20/3	x_2=20/3	20
0	2/3	0	0	1/3	-1/3	1	10/3	a_3=10/3	5*

Example 5

After performing the necessary ero’s, we obtain the third tableau.

Phase I tableau for Bevco after two iteration

W’	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic variable
1	0	0	0	0	-1	-1	0	W’=0
0	0	0	1	-1/8	1/8	-5/8	¼	s_1=1/4
0	0	1	0	-1/2	½	-1/2	5	x_2=5
0	1	0	0	1/2	-1/2	3/2	5	x_1=5

Example 5

- Since w'=0, Phase I has been concluded. The basic feasible solution $s_1=1/4, x_2=5, x_1=5$ has been found.
- Since no artificial variables are in the optimal Phase I basis, the problem is an example of Case 2.
- We now drop the columns for the artificial variables a_2 and a_3 and reintroduce the original objective function.

  min $z=2x_1+3x_2$ or $z-2x_1-3x_2=0$

Example 5

Since x_1 and x_2 are both in the optimal Phase I basis, they must be eliminated from the Phase II row 0.

Phase II row 0:	$z-2x_1-3x_2$	$=0$
+3(row 2):	$3x_2-3/2e_2$	$=15$
+2(row 3):	$2x_1 \quad +e_2$	$=10$
= New Phase II row 0: z	$-1/2e_2$	$=25$

We now begin Phase II with the following set of equations:

$$
\begin{array}{ll}
\min z & -1/2e_2=25 \\
\text{s.t} & s_1-1/8e_2=1/4 \\
 & x_2 \quad -1/2e_2=5 \\
 & x_1 \quad +1/2e_2=5
\end{array}
$$

This is optimal.

The optimal solution is $z=25$ $x_1=5$ $x_2=5$ $s_1=1/4$ $e_2=0$.

Example 6

- We modify Bevco's problem that 36 mg of vitamin C are required. Then, we know that this problem is infeasible. We begin with the original problem:

  $\min z = 2x_1 + 3x_2$

  s.t. $\frac{1}{2}x_1 + \frac{1}{4}x_2 \leq 4$ (sugar constraint)

  $x_1 + 3x_2 \geq 36$ (vitamin C constraint)

  $x_1 + x_2 = 10$ (10 oz in bottle of Oranj)

  $x_1, x_2 \geq 0$

Example 6

- **Solution**

After completing steps 1-4 of the two-phase simplex, we obtain the following Phase I problem:

$$\begin{aligned} \min w' = &\quad a_2 + a_3 \\ \text{s.t. } &\tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 = 4 \\ &x_1 + 3x_2 - e_2 + a_2 = 36 \\ &x_1 + x_2 + a_3 = 10 \end{aligned}$$

From this set of equations, we see that the initial Phase I bfs is s1=4,a2=36 and a3=10.

Example 6

Since the basic variables a2 and a3 occur in the Phase I objective function, they must be eliminated from the Phase I row 0.

Row 0:	w'	$-a_2-a_3$	$=0$
+Row 2:	$x_1+3x_2-e_2+a_2$		$=36$
+Row 3:	$x_1+\ x_2$	$+a_3$	$=10$
=New row 0:	$w'\ +2x_1+4x_2-e_2$		$=46$

Example 6

With the new row 0, the initial Phase I tableau is:

W'	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic variable	ratio
1	2	4	0	-1	0	0	46	W'=46	
0	½	¼	1	0	0	0	4	$s_1=4$	16
0	1	3	0	-1	1	0	36	$a_2=36$	12
0	1	1	0	0	0	1	10	$a_3=10$	10*

x_2 to enter, a_3 to leave the basis, the new tableau is:

W'	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic variable
1	-2	0	0	-1	0	-4	6	W'=6
0	¼	0	1	0	0	-1/4	3/2	$s_1=3/2$
0	-2	0	0	-1	1	-3	6	$a_2=6$
0	1	1	0	0	0	1	10	$x_2=10$

Example 6

- Since no variable in row 0 has a positive coefficient, this is an optimal Phase I tableau.
- Since the optimal value of w' is 6>0, the original LP must have no feasible solution.

Remarks

- As with the Big M method, the column for any artificial variable may be dropped from future tableaus as soon as the artificial variable leaves the basis.
- The Big M method and Phase I of the twp-phase method make the same sequence of pivots. The two-phase method does not cause roundoff errors and other computational difficulties.