WEAK ASCENT SEQUENCES AND RELATED COMBINATORIAL STRUCTURES

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ABSTRACT. In this paper we introduce weak ascent sequences, a class of number sequences that properly contains ascent sequences. We show how these sequences uniquely encode each of the following objects: permutations avoiding a particular length-4 bivincular pattern; upper-triangular binary matrices that satisfy a column-adjacency rule; factorial posets that are weakly (3+1)-free. We also show how weak ascent sequences are related to a class of pattern avoiding inversion sequences that has been a topic of recent research by Auli and Elizalde. Finally, we consider the problem of enumerating these new sequences and give a closed form expression for the number of weak ascent sequences having a prescribed length and number of weak ascents.

1. INTRODUCTION

Ascent sequences [5] are rich number sequences in that they uniquely encode four different combinatorial objects and thereby induce bijections between these objects. These objects are (2+2)-free posets; Fishburn permutations; upper-triangular matrices of non-negative integers having neither columns nor rows of only zeros; and Stoimenow matchings. Statistics on those objects have been shown to be related to natural considerations on the ascent sequences to which they correspond.

In this paper we will define a new sequence that we term a *weak ascent* sequence and study the rich connections these sequences have to other combinatorial objects that are similar in spirit to those mentioned above. Given a sequence of integers $x = (x_1, \ldots, x_n)$, we say there is a *weak ascent* at position i if $x_i \leq x_{i+1}$. We denote by $\mathrm{wasc}(x)$ the number of weak ascents in the sequence x. Throughout this paper we will use the notation $[a, b]$ for the set $\{a, a+1, a+2, \ldots, b\}$.

Definition 1. We call a sequence of integers $a = (a_1, \ldots, a_n)$ a *weak ascent sequence* if $a_1 = 0$ and $a_{i+1} \in [0, 1 + \mathrm{wasc}(a_1, \ldots, a_i)]$ for all $i \in [0, n-1]$. Let WAsc_n be the set of weak ascent sequences of length n.

In Table 1 we list all weak ascent sequences of length at most four.

To contrast this with the original ascent sequences, recall that a sequence of integers $x = (x_1, \ldots, x_n)$ has an *ascent* at position i if $x_i < x_{i+1}$. An *ascent sequence* is a sequence of integers $a = (a_1, \ldots, a_n)$ with $a_1 = 0$ and $a_{i+1} \in [0, 1+\mathrm{asc}(a_1, \ldots, a_i)]$ for all $i \in [0, n-1]$, where asc denotes the number of ascents in the sequence. Clearly, ascent sequences are weak ascent sequences.

In this paper we will show how these weak ascent sequences uniquely encode each of the following objects: permutations avoiding a particular length-4 bivincular pattern (in

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n	WAsc_n
1	(0)
2	$(0,0), (0,1)$
3	$(0,0,0), (0,0,1), (0,0,2), (0,1,0), (0,1,1), (0,1,2)$
4	$(0,0,0,0), (0,0,0,1), (0,0,0,2), (0,0,0,3), (0,0,1,0), (0,0,1,1), (0,0,1,2),$ $(0,0,1,3), (0,0,2,0), (0,0,2,1), (0,0,2,2), (0,0,2,3), (0,1,0,0), (0,1,0,1),$ $(0,1,0,2), (0,1,1,0), (0,1,1,1), (0,1,1,2), (0,1,1,3), (0,1,2,0), (0,1,2,1),$ $(0,1,2,2), (0,1,2,3)$

TABLE 1. All weak ascent sequences of length at most 4.

Section 2); upper-triangular binary matrices that satisfy a column-adjacency rule (in Section 3); factorial posets that contain no *weak* (3+1) subposet (in Section 4). We show in Section 5 how weak ascent sequences are related to a class of pattern avoiding inversion sequences that has been a topic of recent research by Auli and Elizalde [2, 3, 4]. In Section 5 we also consider the problem of enumerating these new sequences and give a closed form expression for the number of weak ascent sequences having a prescribed length and number of weak ascents.

The objects that we study in this paper are summarised in Figure 1, along with the names of the bijections that we construct and prove between these objects. In that diagram we also include the section numbers where each of the bijections may be found.
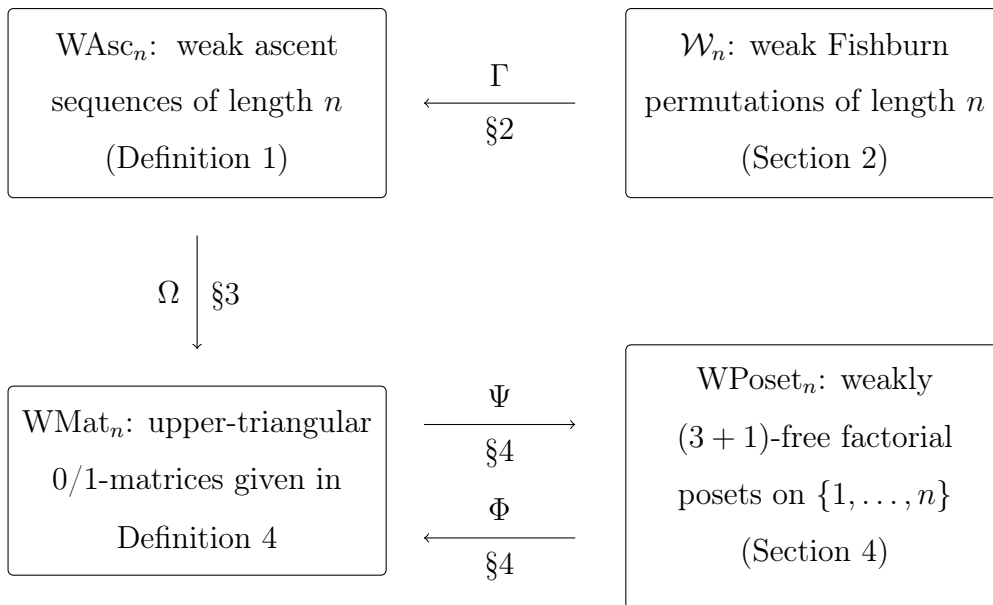

FIGURE 1. Diagrammatic summary of the sets and bijections of interest.

2. WEAK FISHBURN PERMUTATIONS

Let S_n be the set of permutations of the set $\{1, \ldots, n\}$. Given a pattern P, in the pattern-avoidance literature the convention is to denote by $S_n(P)$ the set of permutations in S_n

that do not contain the pattern P. The set of Fishburn permutations [5, 15], $\mathcal{F}_n = S_n(F)$, are those that avoid the bivincular pattern

$$F = \big(231, \, [0,3] \times \{1\} \cup \{1\} \times [0,3] \big) = \;\;\raisebox{-0.5\height}{}\;\; ,$$

here defined and depicted as a mesh pattern [6]. The inclusion of shaded rows and columns indicates that in an occurrence of such a pattern in a permutation, there should be no other permutation dots in the shaded zones when this pattern is placed over a permutation. Bousquet-Mélou et al. [5] gave a bijection between ascent sequences and Fishburn permutations. More precisely, ascent sequences encode the so called active sites of the Fishburn permutations.

We define the bivincular pattern

$$W = \big(3412, \, [0,4] \times \{2\} \cup \{1\} \times [0,4] \big) = \;\;\raisebox{-0.5\height}{}$$

and call $\mathcal{W}_n = S_n(W)$ the set of *weak Fishburn permutations*. For the benefit of readers not familiar with bivincular or mesh patterns we also give an elementary definition of the *weak Fishburn pattern* W: A permutation $\pi \in S_n$ contains W if there are four indices $1 \le i < j < k < \ell \le n$ such that $j = i + 1$, $\pi_i = \pi_\ell + 1$ and $\pi_k < \pi_\ell < \pi_i < \pi_j$. In this case we also say that $\pi_i \pi_j \pi_k \pi_\ell$ is an *occurrence* of W in π. If there are no occurrences of W in π, then we say that π *avoids* W.

If $\pi_i \pi_j \pi_k \pi_\ell$ is an occurrence of W then $\pi_i \pi_j \pi_\ell$ is an occurrence of F. In other words, every Fishburn permutation is a weak Fishburn permutation and we have $\mathcal{F}_n \subseteq \mathcal{W}_n$.

We can construct permutations in $\mathcal{W}_n$ inductively: Let π be a permutation in $\mathcal{W}_n$ with $n > 0$. Suppose that τ is obtained from π by deleting the entry n. Then $\tau \in \mathcal{W}_{n-1}$. To see why this must be the case, let $\tau_i \tau_{i+1} \tau_k \tau_\ell$ be an occurrence of W in τ but not in π. This can only happen if $\pi_{i+1} = n$. However, this implies that $\pi_i \pi_{i+1} \pi_{k+1} \pi_{\ell+1}$ is an occurrence of a W in π.

Given $\tau \in \mathcal{W}_{n-1}$, let us call the sites where the new maximal value n can be inserted in τ so as to produce an element of $\mathcal{W}_n$ *active sites*. The site before τ_1 and the site after τ_{n-1} are always active. Determining whether the site between entries τ_i and τ_{i+1} is active depends on whether $\tau_i \le 2$ or if there does not exist $(\tau_i, t, \tau_i - 1)$ in τ with $t < \tau_i - 1$. This latter (non-existence) condition is somewhat hard to absorb, so let us disentangle it as follows.

The site between entries τ_i and τ_{i+1} is active if

- $\tau_i \le 2$, or
- $\tau_i - 1$ is to the left of τ_i, or
- $\tau_i - 1$ is to the right of τ_i and there is no value $t < \tau_i - 1$ between τ_i and $\tau_i - 1$.

With this notion of active sites let us label the active sites, from left to right, with $\{0, 1, 2, \dots\}$.

We will now introduce a map Γ from $\mathcal{W}_n$ to WAsc_n, the set of weak ascent sequences of length n, that we then show (in Theorem 3) to be a bijection. This mapping is defined recursively. For $n = 1$, we define $\Gamma(1) = (0)$. Next let $n \ge 2$ and suppose that $\pi \in \mathcal{W}_n$ is

obtained by inserting n into active site labeled i of $\tau \in \mathcal{W}_{n-1}$. The sequence associated with π is then $\Gamma(\pi) = (x_1, \ldots, x_{n-1}, i)$, where $(x_1, \ldots, x_{n-1}) = \Gamma(\tau)$.

Example 2. The permutation $\pi = 62754138$ corresponds to the sequence $x = (0, 0, 2, 1, 1, 0, 1, 5)$. It is obtained through the following recursive insertion of new maximal values into active sites. The subscripts indicate the labels of the active sites.

$$_01_1 \xrightarrow{x_2=0} {}_02_11_2$$
$$\xrightarrow{x_3=2} {}_02_11_23_3$$
$$\xrightarrow{x_4=1} {}_02_14\ 1_23_3$$
$$\xrightarrow{x_5=1} {}_02_15_24\ 1_33_4$$
$$\xrightarrow{x_6=0} {}_06\ 2_15_24\ 1_33_4$$
$$\xrightarrow{x_7=1} {}_06\ 2_17_25_34\ 1_43_5$$
$$\xrightarrow{x_8=5} 6\ 2\ 7\ 5\ 4\ 1\ 3\ 8.$$

In our terminology, we thus have $\Gamma(6, 2, 7, 5, 4, 1, 3, 8) = (0, 0, 2, 1, 1, 0, 1, 5)$.

Theorem 3. *The map Γ is a bijection from $\mathcal{W}_n$ to WAsc_n. Furthermore, given $\pi \in \mathcal{W}_n$ with $x = (x_1, \ldots, x_n) = \Gamma(\pi)$, $\mathrm{wasc}(x) = \mathrm{numact}(\pi)$ and $x_n = \mathrm{lastact}(\pi)$ where numact is the number of active sites in π and $\mathrm{lastact}$ is the label of the site located just before the largest entry of π.*

Proof. The integer sequence $\Gamma(\pi)$ encodes the construction of the W-avoiding permutation π so the map Γ is injective. In order to prove that Γ is bijective we must show that the image $\Gamma(\mathcal{W}_n)$ is the set WAsc_n. Let $\mathrm{numact}(\pi)$ be the number of active sites in the permutation π. The recursive description of the map Γ tells us that $x = (x_1, \ldots, x_n) \in \Gamma(\mathcal{W}_n)$ if and only if

$$x' = (x_1, \ldots, x_{n-1}) \in \Gamma(\mathcal{W}_{n-1}) \quad \text{and} \quad 0 \leq x_n \leq \mathrm{numact}\left(\Gamma^{-1}(x')\right) - 1 \qquad (1)$$

Recall that the leftmost active site is labeled 0 and the rightmost active site is labeled $\mathrm{numact}(\pi) - 1$. We will now prove by induction (on n) that for all $\pi \in \mathcal{W}_n$, with associated sequence $\Gamma(\pi) = x = (x_1, \ldots, x_n)$, one has

$$\mathrm{numact}(\pi) = 2 + \mathrm{wasc}(x) \quad \text{and} \quad \mathrm{lastact}(\pi) = x_n, \qquad (2)$$

where $\mathrm{lastact}(\pi)$ is the label of the site located just before the largest entry of π. This will convert the above description (1) of $\Gamma(\mathcal{W}_n)$ into the definition of weak ascent sequences, thereby concluding the proof.

Let us focus on the properties (2). It is straightforward to see that they hold for $n = 1$. Next let us assume they hold for some $n - 1$ where $n \geq 2$. Let $\pi \in \mathcal{W}_n$ be obtained by inserting n into the active site labeled i of $\tau \in \mathcal{W}_{n-1}$. Then $\Gamma(\pi) = x = (x_1, \ldots, x_{n-1}, i)$ where $\Gamma(\tau) = x' = (x_1, \ldots, x_{n-1})$.

Every entry of the permutation π that is smaller than n is followed in π by an active site if and only if it was followed in τ by an active site. The leftmost site in π also remains active. The label of the active site preceding n in π is $i = x_n$, and this proves the second property. In order to determine $\mathrm{numact}(\pi)$, we must see whether the site following n is active in π. There are three cases to consider. Before doing this recall that, by the induction hypothesis, we have $\mathrm{numact}(\tau) = 2 + \mathrm{wasc}(x')$ and $\mathrm{lastact}(\tau) = x_{n-1}$.

Case 1. If $0 \leq i < \mathrm{lastact}(\tau) = x_{n-1}$ then $\mathrm{wasc}(x) = \mathrm{wasc}(x')$ and the entry n in π is to the left of $n-1$ and there is at least one element in-between these. This 'in-between' element must be $< n-1$, so the site after n in π cannot be active since it would lead to the creation of a W-pattern. The number of active sites remains unchanged and $\mathrm{numact}(\pi) = \mathrm{numact}(\tau) = 2 + \mathrm{wasc}(x') = 2 + \mathrm{wasc}(x)$.

Case 2. If $i = \mathrm{lastact}(\tau) = x_{n-1}$ then $\mathrm{wasc}(x) = 1 + \mathrm{wasc}(x')$ and the entry n in π is immediately to the left of $n-1$ in π. Furthermore, there are no elements in-between n and $n-1$. The site that follows n is therefore active, and $\mathrm{numact}(\pi) = 1 + \mathrm{numact}(\tau) = 3 + \mathrm{wasc}(x') = 2 + \mathrm{wasc}(x)$.

Case 3. If $i > \mathrm{lastact}(\tau) = x_{n-1}$ then $\mathrm{wasc}(x) = 1 + \mathrm{wasc}(x')$ and the entry n in π is to the right of $n-1$. The site that follows n is therefore active, and $\mathrm{numact}(\pi) = 1 + \mathrm{numact}(\tau) = 3 + \mathrm{wasc}(x') = 2 + \mathrm{wasc}(x)$. $\qquad\square$

3. A CLASS OF UPPER-TRIANGULAR BINARY MATRICES

Dukes and Parviainen [13] showed how the set of upper triangular integer matrices whose entries sum to n and which contain no zero rows or columns are in one-to-one correspondence with ascent sequences. A property of that correspondence is that the number of ascents in an ascent sequence equals the dimension of the corresponding matrix less one, while the depth of the first non-zero entry in the rightmost column corresponds to one plus the final entry of the ascent sequence.

In this section we will present a similar construction for weak ascent sequences. This correspondence is different to [13] in that the matrix entries are binary and rows of zeros will be allowed. The reason for this is that we would like the dimension of a matrix to be one more than the number of weak ascents in the corresponding weak ascent sequence. Further to this, and in keeping with the spirit of [13], we also wish to preserve the second property "the depth of the first non-zero entry in the rightmost column corresponds to the final entry of the weak-ascent sequence".

Let us first define the class of matrices we will be interested in and then present the correspondence between these matrices and weak ascent sequences. The notation $\dim(A)$ refers to the dimension of the matrix A.

Definition 4. Let WMat_n be the set of upper triangular square $0/1$-matrices A that satisfy the following properties:

(a) There are n 1s in A.
(b) There is at least one 1 in every column of A.
(c) For every pair of adjacent columns, the topmost 1 in the left column is weakly above the bottommost 1 in the right column.

All of the matrices in $\mathrm{WMat}_1, \ldots, \mathrm{WMat}_4$ are shown in Table 2.

Our goal is to present and prove a bijection between the set of weak ascent sequences and the set of the matrices we have just introduced. First, we give a recursive description of the bijection that allows us to prove some local properties of the correspondence in a simple way. Following this we present a more direct definition of the bijection that involves a decomposition of the weak ascent sequence into decreasing subsequences.

Given a matrix $A \in \mathrm{WMat}$, let $\mathrm{topone}(A)$ be the index such that $A_{\mathrm{topone}(A),\dim(A)} = 1$ is the topmost 1 in the rightmost column of A. Such a value always exists since, by

n	WMat_n
1	$\begin{bmatrix} 1 \end{bmatrix}$
2	$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
3	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
4	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

TABLE 2. Matrices in our class of interest.

Definition 4(b), there is at least one 1 in every column of A. Let us define

$$\mathrm{reduce}(A) = \big(B, \mathrm{topone}(A) - 1\big)$$

where B is a copy of A except that $B_{\mathrm{topone}(A),\dim(A)} = 0$ (this entry was 1 in A), and if this results in B having a final column of all zeros then we delete that column and the bottommost row so that B has dimension 1 less than A.

Lemma 5. *If $A \in \mathrm{WMat}_n$, with $n \geq 2$, and $\mathrm{reduce}(A) = \big(B, \mathrm{topone}(A) - 1\big)$, then $B \in \mathrm{WMat}_{n-1}$.*

Proof. Suppose n and A are as stated and $\mathrm{reduce}(A) = (B, \mathrm{topone}(A) - 1)$. Let us first observe that the number of 1s in B is one less than the number of 1s in A, and is $n - 1$. This shows property (a) of Definition 4 is satisfied. Since $A \in \mathrm{WMat}_n$ there is at least one 1 in every column of A, let us consider what happens in the reduction from A to B. If there was a single 1 in the rightmost column of A, then it is removed along with that column and bottom row so that there is at least one 1 in every column of B. Alternatively, if there was more than one 1 in the rightmost column of A, then changing the 1 at position $(\mathrm{topone}(A), \dim(A))$ to 0 will still ensure there is at least one more 1 in that column. This shows property (b) of Definition 4 is satisfied.

Showing that property (c) in Definition 4 is preserved is a little bit more delicate. Notice that in terms of our reduction we need only consider property (c) and how things change in terms of the rightmost two columns. If there was only one 1 in the rightmost column of A, then that column will not appear in B so property (c) certainly holds true in this case. If there is more than one 1 in the rightmost column of A, then removing it does not change the depth of the bottommost one in that column, so property (c) will still hold true. In both cases, property (c) still holds. Therefore $B \in \mathrm{WMat}_{n-1}$. $\qquad\square$

Next we will define a matrix insertion operation that is complementary to the removal operation reduce.

Definition 6. Given a matrix $A \in \mathrm{WMat}_n$ and an integer $i \in [0, \dim(A)]$, let us define $\mathrm{expand}(A, i)$ as follows.

(a) If $i < \mathrm{topone}(A)-1$ then let $\mathrm{expand}(A,i)$ be the matrix A with $A_{i+1,\dim(A)}$ changed from 0 to 1.

(b) If $i \geq \mathrm{topone}(A) - 1$ then let $\mathrm{expand}(A,i)$ be the matrix A with a new column of 0s added to the right and a new row of 0s appended to the bottom. Then change the 0 at position $(i+1, \dim(A)+1)$ in $\mathrm{expand}(A,i)$ to 1.

To illustrate Definition 6 let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\mathrm{expand}(A,2) = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathrm{expand}(A,4) = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & 1 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Lemma 7. *Let $n \geq 2$ and $B \in \mathrm{WMat}_{n-1}$. Let $i \in [0, \dim(A)]$ and define $A = \mathrm{expand}(B,i)$. Then $A \in \mathrm{WMat}_n$ and $\mathrm{topone}(A) = i+1$.*

Proof. Let n, i, and B be as stated in the lemma. In Definition 6, the two operations (a) and (b) increase the number of 1s in the matrix by 1, so the number of 1s in the matrix $A = \mathrm{expand}(B,i)$ will be n. This means matrix A satisfies Definition 4(a). Similarly, if addition rule (a) is used then the number of 1s in a column of A is at least as many as in B, so A satisfies Definition 4(b). If addition rule (b) is used then the new column that appears has precisely one 1, so again A satisfies Definition 4(b). Let us now consider the cases $i < \mathrm{topone}(B) - 1$ and $i \geq \mathrm{topone}(B) - 1$ separately.

If $i < \mathrm{topone}(B)-1$ then matrix A is created by inserting a 1 into position $(i+1, \dim(B))$ of B which is above the topmost 1 in that column. Consequently in the new matrix A one has $\mathrm{topone}(A) = i+1$. Furthermore, the positions of the topmost 1 in the second to last column and the bottommost 1 in the final column remain unchanged, so A satisfies Definition 4(c).

If $i \geq \mathrm{topone}(B) - 1$ then A is created from B by adding a new column and row, and inserting a 1 at position $(i+1, \dim(B)+1)$. Notice that the topmost 1 in column $\dim(B)$ is at position $(\mathrm{topone}(B), \dim(B))$. The bottommost 1 in the final column of A is now at position $(i+1, \dim(B)+1)$. Since $\mathrm{topone}(B) - 1 \leq i$ we have $\mathrm{topone}(B) \leq i+1$ and again A satisfies Definition 4(c). $\qquad\square$

Lemma 8. *Let $B \in \mathrm{WMat}_n$ and let $i \in [0, \dim(B)]$. Then*

$$\mathrm{reduce}\big(\mathrm{expand}(B,i)\big) = (B,i)$$

and, if $n \geq 2$,

$$\mathrm{expand}\big(\mathrm{reduce}(B)\big) = B.$$

Proof. Let $A = \mathrm{expand}(B,i)$. From Lemma 7 we have $\mathrm{topone}(A) = i+1$ and the removal operation when applied to A will yield (C,i) for some matrix C. We need to show $B = C$ for the two different cases of Definition 6. Suppose that $i < \mathrm{topone}(B) - 1$. Then A is a copy of B with a new 1 at position $(i+1, \dim(B))$, which becomes the topmost one in that column. The reduction operation applied to A removes that topmost one in the rightmost column and the resulting matrix is $C = B$. A similar argument holds for the case $i \geq \mathrm{topone}(B) - 1$. This establishes the first part of our lemma.

Suppose $B \in \mathrm{WMat}_n$ is a matrix that has only one 1-entry in the last column. Then reduce$(B) = (C, i)$, where C is the matrix that we obtain by deleting the rightmost column and bottommost row of B and $i = \mathrm{topone}(B) - 1$. By the property (c) of Definition 4 $\mathrm{topone}(C) \leq \mathrm{topone}(B)$. So, $i \geq \mathrm{topone}(C) - 1$ and by Definition 6(b) we have that the matrix $A = \mathrm{expand}(C, i)$ is the matrix that we obtain by appending a column with a single 1 in row $i + 1$ to the right and an all-zeros row to the bottom of C. Hence $A = B$. Suppose now that $B \in \mathrm{WMat}_n$ is a matrix with more than one 1 in the rightmost column. Then reduce$(B) = (C, i)$, where C is the matrix that we obtain by exchanging the topmost 1 in the rightmost column to 0 and $i = \mathrm{topone}(B) - 1$. Note that due to this we have $\mathrm{topone}(C) > \mathrm{topone}(B)$ and $i < \mathrm{topone}(C) - 1$. By Definition 6(a) $A = \mathrm{expand}(C, i)$ is the matrix that we obtain by changing the 0 in the $(i + 1)$th row in the rightmost column of C to 1, hence we have $A = B$. $\qquad \square$

Let us now define a mapping Ω from WMat_n to integer sequences of length n.

Definition 9. For $n = 1$ let $\Omega([\mathsf{1}]) = (0)$. Now let $n \geq 2$ and suppose that the removal operation, when applied to $A \in \mathrm{WMat}_n$ gives reduce$(A) = (B, i)$. Then the sequence associated with A is $\Omega(A) = (x_1, \ldots, x_{n-1}, i)$ where $(x_1, \ldots, x_{n-1}) = \Omega(B)$.

Theorem 10. *The mapping* $\Omega : \mathrm{WMat}_n \to \mathrm{WAsc}_n$ *is a bijection.*

Proof. Since the sequence $\Omega(A)$ encodes the construction of the matrix A, the mapping Ω is injective. We have to prove that the image of WMat_n is the set WAsc_n. By definition, $x = (x_1, \ldots, x_n) \in \Omega(\mathrm{WMat}_n)$ if and only if

$$x' = (x_1, \ldots, x_{n-1}) \in \Omega(\mathrm{WMat}_{n-1}) \quad \text{and} \quad x_n \in [0, \dim(\Omega^{-1}(x'))]. \qquad (3)$$

We will prove by induction on n that for all $A \in \mathrm{WMat}_n$, with associated sequence $\Omega(A) = x = (x_1, \ldots, x_n)$, one has

$$\dim(A) = \mathrm{wasc}(x) + 1 \quad \text{and} \quad \mathrm{topone}(A) = x_n + 1. \qquad (4)$$

This will convert the description (3) above into the definition of weak ascent sequences, thus concluding the proof.

Let us examine the two statements of (4) more closely. They hold for $n = 1$. Assume they hold for $n - 1$ with $n \geq 2$ and let $A = \mathrm{expand}(B, i)$ for $B \in \mathrm{WMat}_{n-1}$. If $\Omega(B) = (x_1, \ldots, x_{n-1})$ then $\Omega(A) = (x_1, \ldots, x_{n-1}, i)$. Lemma 7 gives us that $\mathrm{topone}(A) = i + 1$ and it follows that

$$\dim(A) = \begin{cases} \dim(B) = \mathrm{wasc}(x') + 1 = \mathrm{wasc}(x) + 1 & \text{if } i < x_{n-1} \\ \dim(B) + 1 = \mathrm{wasc}(x') + 1 + 1 = \mathrm{wasc}(x) + 1 & \text{if } i \geq x_{n-1}. \end{cases}$$

The result follows from this. $\qquad \square$

Example 11. Let us construct the matrix A that corresponds to the weak ascent sequence $x = (0, 0, 2, 1, 1, 0, 1, 5) \in \mathrm{WAsc}_8$; that is, $\Omega(A) = x$. To begin, we have $\Omega([\mathsf{1}]) = (0)$.

From this,

$$[\,1\,] \xrightarrow{x_2=0} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{x_3=2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{x_4=1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{x_5=1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{x_6=0} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{x_7=1} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{x_8=5} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = A$$

We can offer a more direct description of the bijection in terms of the decreasing subsequence decomposition of a weak ascent sequence. A weak ascent sequence can be seen as a sequence of decreasing subsequences or runs. For instance, $w = 00211015$ can be decomposed into decreasing runs as $w = 0|0|21|10|1|5$. Construct the matrix M from w as follows: column i of M is formed from the ith decreasing run of w whereby there is a 1-entry at position (i, j) if the ith decreasing run contains the value $j - 1$.

Let us also mention that the inverse mapping to this consists of labelling all the 1-entries of a matrix $M \in \mathrm{WMat}_n$ with labels 1 to n (in the manner specified in Definition 17). Then the ith element of the weak ascent sequence is equal to j if and only if the entry labelled i is in the $(j + 1)$th row of the matrix M.

It is straightforward to see how several simple statistics get translated through the bijection Ω.

Proposition 12. *Let $w = (w_1, \ldots, w_n) \in \mathrm{WAsc}_n$ and suppose that $M \in \mathrm{WMat}_n$ is such that $\Omega(M) = w$. Then*

- *the number of occurrences of j in w equals the sum of the entries of the $(j + 1)$th row in M,*
- *the number of weak ascents in w equals the dimension of M reduced by 1,*
- *the length of the final decreasing run is the sum of the entries in the rightmost column of M,*
- *w_n, the last entry of w, is equal to $\mathrm{topone}(M) - 1$.*

Next, we will characterize those matrices that correspond, via our bijection Ω, to ascent sequences.

Given a word $w = w_1 \ldots w_n$ let an entry w_i with $w_{i-1} = w_i$ be called a *plateau*. Each weak ascent "creates" a new column. An ascent corresponds to a bottommost 1-entry in a new column which is strictly south-east of the topmost 1 in the previous column. On the other hand a plateau corresponds to a bottommost 1-entry in a new column which is in the same row as the topmost 1 in the previous column.

We introduce the terminology *weak* and *strong* entries for these two special kind of 1s in the matrix.

Definition 13. Let a 1-entry in a 0/1-matrix be called *weak* entry if (a) there are only 0s below the 1-entry in its column, and (b) there is a 1 to the left of it in the same row such that there are only 0s above this 1. On the other hand, let a 1-entry in a 0/1-matrix be called *strong* entry if (a) there are only 0s below the 1-entry in its column, and (b) the top most 1 to the column to the left is in a smaller indexed row.

Example 14. In matrix C the entry at position $(2,4)$ is the only weak entry, $C_{2,2}$, $C_{3,3}$, $C_{2,4}$ and $C_{5,5}$ are strong 1-entries. In the matrix D the entries at positions $(1,2)$ and $(2,5)$ are the weak entries and $D_{2,3}$, $D_{3,4}$ and $D_{3,6}$ are the strong 1-entries.

$$C = \begin{bmatrix} 1 & 1 & \mathbf{0} & 1 & 0 & 0 \\ 0 & 1 & \mathbf{1} & 1 & 1 & 0 \\ 0 & 0 & 1 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} \mathbf{1} & \mathbf{1} & 0 & \mathbf{0} & 1 & 1 \\ 0 & \mathbf{0} & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 0 & 0 \end{bmatrix}$$

The main observation is that in an ascent sequence the appearance of a plateau w_i restricts the possible values of w_j for $j \geq i+1$, though it does not influence the values in a weak ascent sequence. Hence, if w is an ascent sequence then in the corresponding matrix a 1 can only appear in row i in the jth column if there are in the previous $j-1$ columns at least $i-2$ strong entries (column 1 cannot by definition contain a strong entry).

Thus, we have the following corollary of our bijection.

Corollary 15. *A matrix $A \in \mathrm{WMat}$ corresponds to an ascent sequence via the bijection Ω if for every 1-entry $A_{i,j}$ there are at least $i-2$ strong entries in columns $1, 2, \ldots, j-1$.*

4. A CLASS OF FACTORIAL POSETS

In this section we will define a mapping from the set of matrices WMat to a set of labeled posets and prove that this mapping is a bijection. First let us recall the definition of a factorial poset from [8]. To begin, a poset P on the elements $\{1, \ldots, n\}$ is *naturally labeled* if $i <_P j$ implies $i < j$. The poset whose Hasse diagram is depicted in Figure 2 is a naturally labeled poset.

Definition 16 ([8]). A naturally labeled poset P on $[1, n]$ such that, for all $i, j, k \in [1, n]$, we have

$$i < j \text{ and } j <_P k \implies i <_P k$$

is called a *factorial poset*.

Factorial posets are (2+2)-free. This fact, and further properties of factorial posets can be found in [8].

Definition 17 (The mapping Ψ). Let $A \in \mathrm{WMat}_n$. Form a matrix B as follows. Make a copy of A. Beginning with the leftmost column, and within each column one goes from bottom to top, replace every 1 that appears with the elements $1, 2, \ldots, n$. Further, define a partial order $(P, <)$ on $[1, n]$ as follows: $i <_P j$ if the index of the column that contains i is strictly less than the index of the row that contains j. Let

$$P = \Psi(A)$$

be the resulting poset.

Diagrammatically the relation in Definition 17 is equivalent to i being north-west of j in the matrix and the "lower hook" of i and j being strictly beneath the diagonal:
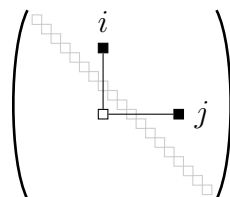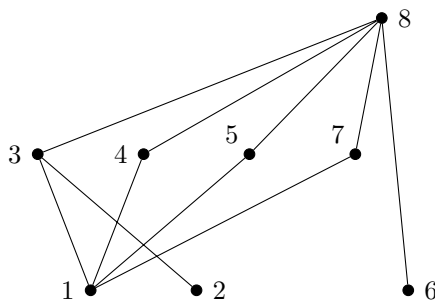
FIGURE 2. A weakly $(3+1)$-free factorial poset.

Note that the set of entries contained in the first s columns for an s is the complete set $\{1, 2, \ldots, s_k\}$ for some s_k.

Example 18. Consider the matrix A from Example 11. Form matrix B by relabeling the 1s in the matrix according to the rule.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mapsto B = \begin{bmatrix} 1 & 2 & 0 & 6 & 0 & 0 \\ 0 & 0 & 4 & 5 & 7 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

This gives the poset $(P, <) = \Psi(A)$ with the following relations:

- $1 <_P 3, 4, 5, 7, 8$
- $2 <_P 3, 8$
- $3, 4, 5, 6, 7 <_P 8$

The Hasse diagram of this poset is illustrated in Figure 2.

The mapping Ψ is a mapping from WMat_n to a set of labeled $(2+2)$-free posets on the set $[1, n]$, which we will now define.

Let P be a factorial poset on $[1, n]$. We say that P contains a *special 3+1* if there exist four distinct elements $i < j < j + 1 < k$ such that the poset P restricted to $\{i, j, j+1, k\}$ induces the 3+1 poset with $i <_P j <_P k$:
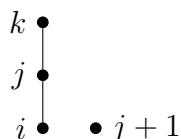
If P does not contain a special 3+1 we say that P is *weakly $(3+1)$-free*. Let WPoset_n be the set of weakly $(3+1)$-free factorial posets on $[1, n]$.

Theorem 19. *Let Ψ be as in Definition 17. If $A \in \mathrm{WMat}_n$, then the poset $P = \Psi(A)$ is factorial and weakly $(3+1)$-free. That is $P \in \mathrm{WPoset}_n$ so that*

$$\Psi : \mathrm{WMat}_n \to \mathrm{WPoset}_n.$$

Proof. Let $A \in \mathrm{WMat}_n$ and $P = \Psi(A)$. Given $i \in [1, n]$, we define the strict downset $D(i) = \{j \in [1, n] : j <_P i\}$ of i. A defining characteristic of a $(2+2)$-free poset is that

the collection $\{D(i) : i \in [1, n]\}$ of strict downsets can be linearly ordered by inclusion. Similarly, a defining characteristic of a factorial poset is that each strict downset is of the form $[1, k]$ for some $k < n$. From Definition 17, it is clear why this must be the case for P: In constructing P from the matrix A, the intermediate matrix B contains all entries in $[1, n]$ exactly once. If j appears in row t of B, then the strict downset $D(j)$ will consist of all i's that appear in columns 1 through to $t - 1$ (inclusive). All elements that appear in row t of B have the same strict downset. Furthermore, since the entries 1, 2, ..., n in B are such that their indices appear from left to right in increasing order, the strict downset of every element must be $[1, k]$ for some $k < n$. Thus P is factorial.

Let us now suppose that P contains an induced subposet on the four elements $i < j < j+1 < k$ that forms a special 3+1. In particular, $i <_P j <_P k$. Consider the matrix entries in B that correspond to i, j, $j+1$ and k. Suppose that i is at position (i_1, i_2) in B, and that j and k are at positions (j_1, j_2) and (k_1, k_2), respectively. The hooks formed from (i, j) and (j, k) must be beneath the diagonal, so we must have $i_1 \leq i_2 < j_1 \leq j_2 < k_1 \leq k_2$. Consider now $\ell = j + 1$ that is at position (ℓ_1, ℓ_2) in B. This element must appear either (a) in the same column as j and strictly above it, or (b) in the next column and in a row weakly below j. For case (a) this means $\ell_1 \leq \ell_2 = j_2$, from which we find that $\ell <_P k$, but this cannot happen since ℓ and k are incomparable. For case (b) this means $j_2 \leq \ell_1 \leq \ell_2$, from which we find that $i <_P \ell$, but this cannot happen since ℓ and i are incomparable. Therefore P cannot contain a special 3+1. In other words, P is weakly (3+1)-free and $\Psi : \mathrm{WMat}_n \to \mathrm{WPoset}_n$.   □

We next define a function Φ that maps posets in WPoset_n to matrices. We shall show that Φ is the inverse of Ψ.

Definition 20. Given $P \in \mathrm{WPoset}_n$, suppose there are k different strict downsets of the elements of P, and that these are $D_0 = \emptyset$, D_1, ..., D_{k-1}. By convention we also let $D_k = [1, n]$. Suppose that L_i is the set of elements $p \in P$ such that $D(p) = D_i$; these are called *level sets*. Let C be the matrix with $C_{i,j} = L_{i-1} \cap (D_j \setminus D_{j-1})$ for all $i, j \in [1, k]$, see [11] for details. Start with B as a copy of C and then repeat the following steps until there is no i that satisfies the condition of A1:

A1. Choose the first column, i say, of B that contains either a set of size > 1 or two elements such that the element in the higher row is less is value to the element in the lower row.

A2. With respect to the usual order on $\mathbb{N}$, let ℓ be the smallest entry in column i.

A3. Introduce a new empty column between columns $i-1$ and i so that the old column i is now column $i+1$.

A4. Move ℓ one column to the left (to the current column i) and set $j = 1$.

A5. If $\ell + j$ is strictly above $\ell + j - 1$, then move it one column to the left, increase j by 1, and repeat A5. Otherwise, go to A6. (The outcome of step A5 will be that the non-empty singleton sets in column i, from bottom to top, are $\ell, \ell + 1, \ldots, \ell + t$ for some $t \geq 0$.)

A6. Introduce a new row of empty sets between rows i and $i + 1$ of B. Matrix B will have increased in dimension by 1.

Finally, let $\Phi(P)$ be the result of replacing the singletons in B with ones and the empty sets in B with zeros.

Example 21. Let P be the poset in Figure 2. The strict downsets of P are

$$D_0 = \emptyset, D_1 = \{1\}, D_2 = \{1, 2\} \text{ and } D_3 = \{1, 2, 3, 4, 5, 6, 7\}.$$

The level sets of P are

$$L_0 = \{1, 2, 6\}, L_1 = \{4, 5, 7\}, L_2 = \{3\} \text{ and } L_3 = \{8\}.$$

This gives the matrix

$$C = \begin{bmatrix} \{1\} & \{2\} & \{6\} & \emptyset \\ & \emptyset & \{4, 5, 7\} & \emptyset \\ & & \{3\} & \emptyset \\ & & & \{8\} \end{bmatrix}.$$

Before continuing we would like to point out that the matrix obtained from C by replacing each set $C_{i,j}$ by its cardinality $|C_{i,j}|$ corresponds, via [11, §3.1], to the unlabeled version of the poset which is (2+2)-free. Continuing, by applying A1 we find that $i = 3$ is the first column satisfying A1. The smallest number in this column is 3. Furthermore, 4 is above 3 so we move 4 to the 3rd column, but 5 is in the same row as 4. So the largest contiguous sequence starting from 3 as we go up from it in that column (and skip rows if we wish) is $\{3, 4\}$. We next go to step A6 and insert a new row with empty sets below the row of $\{3\}$. Hence, the outcome of A6 will be

$$B = \begin{bmatrix} \{1\} & \{2\} & \emptyset & \{6\} & \emptyset \\ & \emptyset & \{4\} & \{5, 7\} & \emptyset \\ & & \{3\} & \emptyset & \emptyset \\ & & & \emptyset & \emptyset \\ & & & & \{8\} \end{bmatrix}.$$

As column $i = 4$ satisfies the statement of A1, we go through steps A2-A6 and find the outcome of A6 to be

$$B = \begin{bmatrix} \{1\} & \{2\} & \emptyset & \{6\} & \emptyset & \emptyset \\ & \emptyset & \{4\} & \{5\} & \{7\} & \emptyset \\ & & \{3\} & \emptyset & \emptyset & \emptyset \\ & & & \emptyset & \emptyset & \emptyset \\ & & & & \emptyset & \emptyset \\ & & & & & \{8\} \end{bmatrix}.$$

Since there are now no columns that satisfy A1, we replace singletons with 1s and emptysets with 0s to find

$$\Phi(P) = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Theorem 22. *Let Φ be the mapping from Definition 20. If P is a poset in WPoset_n, then the matrix $\Phi(P)$ is in WMat_n so that*

$$\Phi : \mathrm{WPoset}_n \to \mathrm{WMat}_n.$$

Proof. Let $P \in \mathrm{WPoset}_n$. Since P is a factorial poset, we know that the strict downsets of elements all have the same contiguous form $[1, \ell]$ for some $\ell \in [0, n-1]$. Let us suppose that there are k different strict downsets of the elements of P, and that these are $D_0 = \emptyset$, D_1, ..., D_{k-1}. Let us further suppose that L_i is the set of elements $p \in P$ such that $D(p) = D_i$, the elements at *level* i of the poset. Next let C be the matrix with

$$C_{i,j} = L_{i-1} \cap (D_j \backslash D_{j-1}) \text{ for all } i, j \in [1, k].$$

The matrix C is upper triangular and is in fact a partition matrix. A partition matrix is an upper triangular matrix whose entries form a set partition of an underlying set with the additional property: for all $1 \leq a < b \leq n$, the column containing b cannot be left of that containing a. See [7] for further details on partition matrices.

Moreover, since P is weakly (3+1)-free, the structure of the matrix C is further restricted in the following sense:

Property 1: The matrix C does not contain four entries $i, j, j+1$ and k such that the hooks for the pairs (i, j) and (j, k) are both below the main diagonal, whereas the hooks, if defined between $j + 1$ and each of i, j, k are on or above the main diagonal.

(Note that the hook for a pair (i, j) is only defined when the entries are in strict north-west relative position. When the hook for two entries is not defined, then the corresponding entries in the poset are incomparable.)

Next, let us consider B that is constructed from C in Definition 20. When a column of B is split into two (as per A3), a new empty row is added in step A6 which preserves the upper-triangular property. Also, there can be no empty columns in B since the dissection step A4 ensures a set of size at least 2 is split into a singleton set (that will appear in the new left column) and the set difference (that will appear one place to its right). Since C is upper triangular, the construction of B ensures it is upper triangular.

Furthermore, by construction, it can never be the case that on completion of all A1–A6, the entry $a + 1$ appears above a in the column to its right. If it were, then rule A5 would not have been executed properly. So the matrix B is such that the highest indexed entry in every column is the highest entry in that column, a say, is weakly above the smallest entry (with respect to the order on $\mathbb{N}$) in the subsequent column $a + 1$ (which is also the lowest in that column). This condition absorbs Property 1 when one considers the final pair of columns and the entries j and $j + 1$.

The replacement of all singleton sets with ones and empty sets with zeros results in a matrix $\Phi(P)$ with the following properties:

- $\Phi(P)$ contains n ones and is upper triangular.
- There are no columns consisting of all-zeros, but there can be rows of all-zeros.
- The topmost one in every column is weakly above the bottommost 1 in the column to its right.

Therefore, we have $\Phi(P) \in \mathrm{WMat}_n$. $\qquad \square$

Theorem 23. *The mapping* $\Psi : \mathrm{WMat}_n \to \mathrm{WPoset}_n$ *is a bijection.*

Proof. We start by showing that Ψ is injective. Suppose that A and A' are two different matrices in WMat_n. As there are n 1s in each of the matrices A and A', and they are different, there must be at least two positions in which they differ. Consider the intermediate matrices B and B' in the construction and let a be the smallest label that appears in a different position in B and in B'. Fix i, i', j, j' so that $B_{i,j} = a = B'_{i',j'}$. If $i \neq i'$ then the strict downset of a in $\Psi(A)$ is different to its strict downset in $\Psi(A')$. If $i = i'$ then $j \neq j'$. Let us suppose that $j < j'$. We necessarily have $j' = j + 1$ as otherwise the column $j + 1$ would be empty in B' and A'. Consequently the label $a - 1$ is in column j in both B and B' (otherwise column j would be empty in B' and A'), and therefore $a - 1$ is in a row below row i. However this implies that, in B', the topmost entry in

column j (equal to $a - 1$) is below the bottommost entry in column $j + 1$ (equal to a), contradicting $A' \in \mathrm{WMat}_n$. Therefore $\Psi(A) \neq \Psi(A')$.

To prove that Ψ is surjective, we will show that $\Psi(\Phi(P)) = P$, thereby establishing Φ as the inverse of Ψ. Let $P \in \mathrm{WPoset}_n$ that has k different levels $L_0, \ldots, L_{k-1}$ and down sets $D_0, \ldots, D_{k-1}$. Let $M = \Phi(P)$ be the matrix that satisfies Definition 4. Suppose that $Q = \Psi(\Phi(P)) = \Psi(M)$.

As $M \in \mathrm{WMat}_n$ we know, by Theorem 22, that $Q \in \mathrm{WPoset}_n$. The poset $Q = \Psi(M)$ that we construct using Definition 17 is such that the level j of the poset Q corresponds to the set of elements in the $(j + 1)$th non-zero row of M once the labelleing of the 1s in M that is described in Definition 17 has taken place. Given that the $(j + 1)$th non-zero row of M corresponded to the jth level of P, we have that $L_{j+1}(Q) = L_{j+1}(P)$ for all j. Let $i_1, \ldots, i_k$ denote the indices of the k non-empty rows of M. The elements of $D_j(Q)$ are all of those matrix entries (with the labelling of Definition 17) weakly to the right of column i_j. As M was constructed from P using a process of separating out columns while creating empty rows, the jth downset of $D_j(P)$ will coincide with that of $D_j(Q)$. $\qquad\square$

Just as we did in the previous section, we can see how statistics between these two sets are translated:

Proposition 24. *Suppose that $M \in \mathrm{WMat}_n$ and $P = \Psi(M)$. Then*

- *the sum of the top row of M is the number of minimal elements in P,*
- *the number of non-zero rows in M equals the number of levels in the poset P.*

5. PATTERN-AVOIDING INVERSION SEQUENCES AND ENUMERATION

The study of patterns in inversion sequences was recently considered by Corteel et al. [10] and continued throughout several papers [1, 2, 3, 4, 18, 19]. In a recent paper Auli and Elizalde [4] focused on vincular patterns in inversion sequences. We recall some important definitions.

It is well known that a permutation of $[1, n]$ can be encoded by an integer sequence $e_1 e_2 \ldots e_n$, where e_i is the number of larger elements to the left of the entry π_i. It is easy to see that every sequence $e_1 e_2 \ldots e_n$ with the property that $e_i \in [0, i - 1]$ for all i corresponds uniquely to a permutation. Such a sequence is called an *inversion sequence*.

A *vincular pattern* is a sequence $p = p_1 p_2 \ldots p_r$, where some disjoint subsequences of two or more adjacent entries may be underlined, satisfying $p_i \in [0, r - 1]$ for each i, where any value $j > 0$ can only appear in p if $j - 1$ appears as well. The *standardization* of a word $w = w_1 w_2 \ldots w_k$ is the word obtained by replacing all instances of the ith smallest entry of w with i.

An inversion sequence e *avoids* the vincular pattern p if there is no subsequence $e_{i_1} e_{i_2} \ldots e_{i_r}$ of e whose standardization is p, and such that $i_{s+1} = i_s + 1$ whenever p_s and p_{s+1} are part of the same underlined subsequence. $\mathcal{I}_n(p)$ denotes the set of inversion sequences of size n that avoid the vincular pattern p and $I_n(p)$ denotes the size of this set.

Auli and Elizalde [4] showed that $I_n(\underline{10}0) = I_n(\underline{10}1)$. The known numbers of this sequence [20, A336070] are

$$1, 1, 2, 6, 23, 106, 567, 3440, 23286, 173704, 1414102.$$

Given a sequence of integers $x = x_1 \ldots x_n$, we say that there is a *descent* at position i if $x_i > x_{i+1}$; we denote by

$$D(x) = \{i : x_i > x_{i+1}\} \quad \text{and} \quad \text{desbot}(x) = \{x_{i+1} : i \in D(x)\}$$

the set of descents and *descent bottoms*, respectively. The main result of this section is the following theorem.

Theorem 25. *The number of length-n weak ascent sequences is the same as the number of length-n inversion sequences that avoid the vincular pattern $\underline{10}0$. Further, there is a descent preserving bijection between the two sets.*

We prove Theorem 25 by establishing a bijection ϕ between the sets WAsc_n and $\mathcal{I}_n(\underline{10}0)$. First, we recall a crucial property of the elements of $\mathcal{I}_n(\underline{10}0)$ from Auli and Elizalde [4]: For each $e = e_1 \ldots e_n \in \mathcal{I}_n(\underline{10}0)$,

$$e_n \in [0, n-1] \setminus \text{desbot}(\bar{e}),$$

where $\bar{e} = e_1 \ldots e_{n-1}$ denotes the inversion sequence that we obtain by deleting the last entry of e. Another important observation is that the descent bottoms of e are distinct elements. Indeed, if e_i and e_j, with $i < j$, are descent bottoms and $e_i = e_j$, then the subword $e_{i-1}e_ie_j$ would form a $\underline{10}0$ pattern.

On the other hand, for any weak ascent sequence $w = w_1 \ldots w_n$ we have, by definition, $w_n \in [0, \text{wasc}(\bar{w}) + 1]$, and $\text{wasc}(\bar{w}) = |\bar{w}| - 1 - |D(\bar{w})|$. That is,

$$w_n \in [0, n - 1 - |D(\bar{w})|].$$

Proof of Theorem 25. Given $w = w_1 \ldots w_n$ in WAsc_n and $e = e_1 \ldots e_n$ in $\mathcal{I}_n(\underline{10}0)$, let $\bar{w} = w_1 \ldots w_{n-1}$ and $\bar{e} = e_1 \ldots e_{n-1}$ (as above) so that we can write $w = \bar{w}w_n$ and $e = \bar{e}e_n$. We shall define $\phi : \text{WAsc}_n \to \mathcal{I}_n(\underline{10}0)$ using recursion. The base case, $n = 0$, is that ϕ maps the empty word to the empty word. Assume $n \geq 1$. Let $\phi(\bar{w}w_n) = \bar{e}e_n$, where $\bar{e} = \phi(\bar{w})$ and e_n is the w_nth element of $[0, n-1] \setminus \text{desbot}(\bar{e})$ when listed in increasing order. In other words, if $[0, n-1] \setminus \text{desbot}(\bar{e}) = \{\ell_0, \ldots, \ell_k\}$ and $\ell_0 < \cdots < \ell_k$, then $e_n = \ell_i$ with $i = w_n$. We claim that

    (1) ϕ is a well defined mapping,
    (2) ϕ is bijective, and
    (3) ϕ preserves descent; i.e. if $\phi(w) = e$, then $D(w) = D(e)$.

Assume that $w \in \text{WAsc}_n$ and $\phi(w) = e$. By definition we have $\phi(\bar{w}) = \bar{e}$, and by induction we may further assume that $D(\bar{w}) = D(\bar{e})$. We now prove each of the three claims separately.

Proof of (1)*.* The only thing that can go wrong in the definition of ϕ is that the w_nth element of $[0, n-1] \setminus \text{desbot}(\bar{e})$ may, a priori, not exist. Recall that, for $e \in \mathcal{I}(\underline{10}0)$, the descent bottoms of e are distinct. Thus, $|\text{desbot}(\bar{e})| = |D(\bar{e})|$. Further, $D(\bar{e}) = D(\bar{w})$ (by induction) and thus there are as many elements in $[0, n-1] \setminus \text{desbot}(\bar{e})$ as there are in $[0, n-1-|D(\bar{w})|]$, which is the set that w_n belongs to.

Proof of (2)*.* Assume that $\phi(u) = \phi(v) = e$. Then $\phi(\bar{u}) = \phi(\bar{v})$ and, by induction, $\bar{u} = \bar{v}$. Also, if $[0, n-1] \setminus \text{desbot}(\bar{e}) = \{\ell_0, \ldots, \ell_k\}$ and $e_n = \ell_i$, then $u_n = i = v_n$. Thus ϕ is injective. To see that ϕ is surjective, let $e \in \mathcal{I}_n(\underline{10}0)$ be given. By induction there is a $\bar{w}$ in WAsc_{n-1} such that $\phi(\bar{w}) = \bar{e}$. Further, with $e_n = \ell_i$, we let $w_n = i$. Then $\phi(w) = e$.

Proof of (3)*.* By the induction hypothesis we have $D(\bar{w}) = D(\bar{e})$. What remains to show is that $n - 1 \in D(w)$ if and only if $n - 1 \in D(e)$. Note that e_{n-1} is a member of

$[0, n-1] \setminus \mathrm{desbot}(\bar{e})$; let this set be $\{\ell_0, \ldots, \ell_r\}$. Further, let $w_{n-1} = i$, so that ϕ maps w_{n-1} to $e_{n-1} = \ell_i$. There are two possibilities: If e_{n-1} is a descent bottom, then e_n belongs to the set $[0, n] \setminus \{\mathrm{desbot}(\bar{e}) \cup \{e_{n-1}\}\} = \{\ell_0, \ell_1, \ldots, \ell_{i-1}, \ell_{i+1}, \ldots, \ell_r, \ell_{r+1}\}$, where $\ell_{r+1} = n$. Let $j = w_n$. If $j < i$ then $n-1$ is both in $D(w)$ and $D(e)$. If $j \geq i$, then $w_{n-1} \leq w_n$ and $n-1 \notin D(w)$, but then also $e_n \geq \ell_{i+1} > \ell_i$, and hence $n-1 \notin D(e)$. If e_{n-1} is not a descent bottom, then e_n is from the set $[0, n] \setminus \mathrm{desbot}(\bar{e}) = \{\ell_0, \ell_1, \ldots, \ell_r, \ell_{r+1}\}$, where $\ell_{r+1} = n$. In this case the statement is clear. $\qquad\square$

Example 26. With $w = 010101$ we find that $e = \phi(w) = 010213$ as detailed in the following table:

n	w	e	$[0, n] \setminus \mathrm{desbot}(e)$
0	ϵ	ϵ	0
1	0	0	0, 1
2	01	01	0, 1, 2
3	010	010	1, 2, 3
4	0101	0102	1, 2, 3, 4
5	01010	01021	2, 3, 4, 5
6	010101	010213	2, 3, 4, 5, 6

E.g. when constructing the row for $n = 4$ we are given $w = 0101$ and $\bar{e} = 010$ (from the preceding row). The value of e_4 is $\ell_{w_4} = \ell_1$, where $(\ell_0, \ell_1, \ell_2) = (1, 2, 3)$ are the values in the last column of the preceding row. That is, $e_4 = 2$.

We give another set of inversion sequences that are also in one-to-one correspondence with weak ascent sequences. Let $\mathcal{I}_n(D)$ denote the sequences of integers $w = w_1 \ldots w_n$ with

$$w_i \in [0, i-1] \setminus D(w_1 \ldots w_{i-1}),$$

i.e., the set of length-n inversion sequences where the positions of descents are forbidden as entries. Another way to describe this is as the set of inversion sequences that contain only entries at which positions a weak ascent occur. The sequence 0102 is not in $\mathcal{I}_4(D)$, because w_2 is a descent top, hence the value 2 is forbidden. All other length-4 inversion sequences are in $\mathcal{I}_4(D)$.

While we do not give a formal proof of the following result, let us note that it follows by an argument similar to that given for Theorem 25 in conjunction with the bijection Λ from partition matrices to inversion sequences that was given in [7].

Proposition 27. *The set $\mathcal{I}_n(D)$ is equinumerous with the set* WAsc_n.

Note that the inductive construction is very similar in each case, weak-ascent sequences, inversion sequences avoiding the vincular pattern, and the weak Fishburn permutations. In each case there is a set of possible values for the jth entry that is determined in the prefix of length $j-1$. Auli and Elizalde [4] use the method of generating trees to derive an expression for the generating function

$$A(z) = \sum_{n \geq 0} I_n(\underline{100})z^n = \sum_{n \geq 0} I_n(\underline{101})z^n.$$

Proposition 28 ([4] Proposition 3.12)**.** *We have that $A(z) = G(1, z)$, where $G(u, z)$ is defined recursively by*

$$G(u, z) = u(1 - u) + uG(u(1 + z - uz), z).$$

This expression and the bijection from the proof of Theorem 25 imply that if we denote by A_n the enumeration sequence that counts the number of weak ascent sequences of length n, we have $A_n = \sum_{k=0}^{n} a_{n,k}$, where $a_{n,k}$ is given by the following formula. The initial values $a_{0,0} = 1$, $a_{n,0} = a_{0,k} = 0$ and

$$a_{n,k} = \sum_{i=0}^{n} \sum_{j=0}^{k-1} (-1)^j \binom{k-j}{i} \binom{i}{j} a_{n-i,k-j-1}. \tag{5}$$

Proposition 29. *The number of weak ascent sequences of length n having k weak ascents is $a_{n,k+1}$.*

Proof. Let $W_{n,k}$ denote the set of weak ascent sequences with k weak ascents. After the last weak ascent in a weak ascent sequence the entries are in decreasing order. Let us consider the objects where some of these descents are marked. More precisely, let $W_{n,k}^{(a)}$ be the set of weak ascent sequences of length n with k weak ascents, where there are a marked descents in the last maximal decreasing subsequence. For instance, 0012012115320 is an element of $W_{13,7}^{(2)}$, because the last decreasing subsequence is $(5, 3, 2, 0)$, from which 3 and 0 is marked. Let further $S_0 = W_{n,k-1} \cup W_{n,k-2} \cup W_{n,k-2}^{(1)}$ and for $1 \leq j \leq k - 1$

$$S_j = W_{n,k-j-1}^{(j-1)} \cup W_{n,k-j-1}^{(j)} \cup W_{n,k-j-2}^{(j)} \cup W_{n,k-j-2}^{(j+1)}.$$

It is clear then that

$$|W_{n,k-1}| = |S_0| - |S_1| + |S_2| - \cdots + (-1)^{k-1}|S_{k-1}|. \tag{6}$$

In order to enumerate the sets S_j, we describe the way of its elements are constructed. For a given i, $0 \leq i \leq n$, consider the set of weak ascent sequences of length $n - i$ with $k - 2$ weak ascents $W_{n-i,k-2}$. We want to augment a sequence from the set $W_{n-i,k-2}$ by a decreasing sequence of i elements in order to obtain a valid weak ascent sequence of length n. Because of the definition of a weak ascent sequence, these elements are from the set $\{0, 1, \ldots, k-1\}$, so we can choose from the k possible values i and attach to the underlying sequence in decreasing order. However, two cases can happen. If the greatest chosen value is greater than or equal to the $(n-i)$th entry, then we obtain a weak ascent sequence with $k - 1$ weak ascents, hence an object from $W_{n,k-1}$ such that the last weak ascent is at the $(n-i+1)$th position. So, letting i go from 1 to n we generated all the objects in $W_{n,k-1}$ exactly once.

On the other hand, if the greatest chosen value is smaller than the $(n-i)$th entry, then we obtain a weak ascent sequence with $k - 2$ weak ascents. In this case we mark the $(n-i+1)$th entry, which is a descent in the last decreasing subsequence. Letting i go from 1 to n, we obtain this way every object exactly once from the set $W_{n,k-2}^{(1)}$.

Consider now the general set S_j. For $i \geq j$, take a weak ascent sequence of length $n - i$ with $k - j - 2$ weak ascents. Choose from the $k - j$ available values i and attach them in decreasing order to obtain a valid weak ascent sequence. After, choose from the i values j out, mark those that are descents. (It can happen that the first value is chosen and it is a weak ascent, in this case do not mark this entry.)

If $i = j$, we do not have any free choice in this last case (j elements out of i), but there are two possibilities:

    I.0 The greatest chosen value is greater than or equal to the $(n-i)$th entry. Then we obtained a weak ascent sequence with $k - j - 1$ weak ascents with $j - 1$ marks on all the descents of the last decreasing subsequence.

II.0 The greatest chosen value is smaller than the $(n-i)$th entry. Then we obtained a weak ascent sequence with $k-j-2$ weak ascents and j marks on the last decreasing subsequence.

For $i > j$ there are four cases:

I. The greatest chosen value is greater than or equal to the $(n-i)$th entry.
  1. If this greatest value was chosen among the j (out of the i) then we obtain a weak ascent sequence with $k-j-1$ weak ascents and $j-1$ marked descents in the last decreasing subsequence.
  2. If this greatest value was not chosen among the j (out of the i) then we obtain a weak ascent sequence with $k-j-1$ weak ascents and j marked descents in the last decreasing subsequence.
II. The greatest chosen value is smaller than the $(n-i)$th entry.
  1. If the greatest value was chosen among the j then we obtain a weak ascent sequence with $k-j-2$ weak ascents and j marked descents in the last decreasing subsequence.
  2. If the greatest value was not chosen among the j then mark this descent also. This way we obtain a weak ascent sequence with $k-j-2$ weak ascents and $j+1$ marked descents in the last decreasing subsequence.

Letting i go from $i = j$ to n we have that in the cases I.0 and I.1 together all the objects in $W_{n,k-j-1}^{(j-1)}$ are constructed exactly once. Similarly, during the above construction we get in the case I.2 the objects in $W_{n,k-j-1}^{(j)}$, in the cases II.0 and II.1 the objects in $W_{n,k-j-2}^{(j)}$ and in the case II.2 the objects $W_{n,k-j-2}^{j+1}$.

The counting formula for the size of the sets S_j is straightforward from the construction:

$$|S_j| = \sum_{i=j}^{n} \binom{k-j}{i}\binom{i}{j}|W_{n-i,k-j-2}|. \tag{7}$$

and hence, by Equation (6) we have

$$|W_{n,k-1}| = \sum_{j=0}^{k-1}(-1)^j \sum_{i=j}^{n} \binom{k-j}{i}\binom{i}{j}|W_{n-i,k-j-2}|.$$

We see that $|W_{n,k-1}|$ satisfies the same recurrence relation as $a_{n,k}$ in Equation (5). □

Note that $a_{n,n}$ are the Catalan numbers, which is clear, since weak ascent sequences that have only ascents are in a trivial bijection for instance with Dyck paths.

Remark 30. Since the sequence A_n has a rapid growth, greater than $n^{n/2}$, the series $\sum_{n=0}^{\infty} A_n z^n$ converges only for $z = 0$. On the other hand, since $A_n \leq n!$, the exponential generating function $\sum_{n=0}^{\infty} A_n \frac{t^n}{n!}$ determines an analytic function on a certain domain. However, it could be difficult to represent it as a function by using classical functions. We did not manage to derive a nice closed formula for it.

6. Concluding remarks

Experimentation with restricted classes of weak ascent sequences has shown that there are relationships to other known number sequences. As an example, we offer the following simple Catalan result:

Proposition 31. *The number of weak ascent sequences $w = (w_1, \ldots, w_n)$ that are weakly-increasing, i.e. $w_i \leq w_{i+1}$ for all i, is given by the Catalan numbers.*

Proof. If a weak-ascent sequence is weakly increasing then there is no restriction on the entries, so the set of weakly-increasing weak ascent sequences is the same as the set of nondecreasing sequences of integers a_i with $0 \leq a_i \leq i$ which are known to be enumerated by the Catalan numbers. □

A slightly different restriction gives rise to the following conjecture:

Conjecture 32. *The number of weak ascent sequences $w = (w_1, \ldots, w_n)$ that satisfy $w_{i+1} \geq w_i - 1$ for all i equals OEIS [20, A279567] "Number of length n inversion sequences avoiding the patterns 100, 110, 120, and 210."*

The paper [12] probed restrictions on ascent sequences and how such restrictions played out in the bijective correspondences. The above proposition and conjecture represent a first step in that direction for weak ascent sequences.

Research into pattern avoidance in ascent sequences (see Duncan and Steingrímsson [14]) proved to be a fruitful avenue of research that produced a wealth of enumerative identities and conjectures, some of which are still open. The asymptotics of generating functions for these has recently been investigated by Conway et al. [9]. We posit that a similarly rich collection of results are to be discovered by exploring pattern avoidance for weak ascent sequences and that equidistribution results hold for multivarite statistics on weak ascent sequences in the same way they were shown to hold for ascent sequences [16].

Acknowledgments

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