Enumerative aspects of Caylerian polynomials

Giulio Cerbai*[1] and Anders Claesson[1]

[1]Department of Mathematics, University of Iceland, Reykjavik, Iceland,
akc@hi.is, giulio@hi.is.

30 July 2025

Abstract

Eulerian polynomials record the distribution of descents over permutations. Caylerian polynomials likewise record the distribution of descents over Cayley permutations, where a Cayley permutation is a word of positive integers such that if a number appears in the word then all positive integers less than that number also appear in the word. Using combinatorial species and sign-reversing involutions we derive counting formulas and generating functions for the Caylerian polynomials as well as for related refined polynomials.

1 Introduction

A *Cayley permutation* is a word of positive integers such that if a number appears in the word then all positive integers less than that number also appear in the word. We [9] recently introduced the *Caylerian polynomials*, which record the distribution of descents over Cayley permutations, and provided a framework in which Burge matrices and Burge words underpin these polynomials, allowing for generalizations of several results and combinatorial constructions originally defined for the Eulerian polynomials. In this sequel to the aforementioned paper [9], we apply our framework to obtain counting formulas and generating functions for the Caylerian polynomials, as well as for some related polynomials and combinatorial sets.

Our main results are as follows. In Theorem 4.9 we provide an explicit formula for the nth Caylerian polynomial, $C_n(t)$, in terms of Fubini numbers and Stirling numbers of the first and second kind:

$$C_n(t) = \frac{1}{n!}\sum_{k,i} \mathrm{fub}(k) \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ i \end{Bmatrix} i!(t-1)^{n-i}.$$

Carlitz's [6] identity for the Eulerian polynomials is $tA_n(t)/(1-t)^{n+1} = \sum_{m\geq 1} m^n t^m$ and in Theorem 4.12 we give an analogous identity for the Caylerian polynomials:

$$\sum_{n\geq 0} \frac{tC_n(t)}{(1-t)^{n+1}} x^n = \sum_{m\geq 1} \frac{(1-x)^m}{2(1-x)^m - 1} t^m.$$

*G.C. is a member of the Gruppo Nazionale Calcolo Scientifico–Istituto Nazionale di Alta Matematica (GNCS-INdAM).

There is a natural two-sided generalization, $\hat{B}_n(s,t)$, of the Caylerian polynomials and in Theorem 4.8 we give a formula for those polynomials:

$$\hat{B}_n(s,t) = \frac{1}{n!}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \big|\mathrm{Bal}^s[k]\big|\big|\mathrm{Bal}^t[k]\big|$$

Here

$$|\mathrm{Bal}^t[n]| = \sum_{w\in \mathrm{Bal}[n]} t^{\mathrm{blocks}(w)}$$

is a polynomial recording the distribution of the number of blocks over ballots (ordered set partitions); exact definitions will be given below.

In Section 2 we recall the definitions of (weak and strict) Caylerian polynomials and associated combinatorial sets. To simultaneously handle the structural and enumerative aspects of these objects we shall use combinatorial species, which we briefly introduce in Section 3. A particular species of matrices of linear orders is defined in Section 4 and through species equations we derive counting formulas and generating functions. In Sections 5 and 6 we provide two sign-reversing involutions to enumerate Cayley permutations with prescribed ascent set and binary Burge matrices, respectively. In particular, we derive the following formula (see Proposition 5.1) for the number of Cayley permutations of $[n]$ whose ascent set is a subset of $S = \{s_1, \ldots, s_r\}$:

$$\kappa_n(S) = \sum_{k,i} (-1)^i \binom{k}{i} \prod_{j=0}^{r} \binom{k-i}{s_{j+1}-s_j}.$$

In Section 7 we make some closing remarks and summarize the formulas proved in this paper; see tables 1, 2 and 3.

2 Caylerian polynomials

A *Cayley permutation* of $[n]$ is a map $w : [n] \to [n]$ with $\mathrm{Im}(w) = [k]$ for some $k \le n$, where $[0] = \emptyset$ and $[n] = \{1, 2, \ldots, n\}$ if $n \ge 1$, and we identify the function w with the word $w = w(1)\ldots w(n)$ whenever it is convenient. Denote by $\mathrm{Cay}[n]$ the set of Cayley permutations on $[n]$. For instance, $\mathrm{Cay}[1] = \{1\}$, $\mathrm{Cay}[2] = \{11, 12, 21\}$ and

$$\mathrm{Cay}[3] = \{111, 112, 121, 122, 123, 132, 211, 212, 213, 221, 231, 312, 321\}.$$

The term Cayley permutation was first used by Mor and Fraenkel [17] in 1983 and as the name suggests their history traces back to Cayley. See the recent paper by Cerbai, Claesson, Ernst, and Golab [10], for a short account of the history and the plethora of guises that Cayley permutations have appeared under.

A *ballot*, or *ordered set partition*, of $[n]$ is a list of disjoint blocks (nonempty sets) $B_1B_2\ldots B_k$ whose union is $[n]$. Let $\mathrm{Bal}[n]$ be the set of ballots of $[n]$. It is well known that Cayley permutations encode ballots: $B_1B_2\ldots B_k \in \mathrm{Bal}[n]$ is encoded by $w \in \mathrm{Cay}[n]$ where $i \in B_{w(i)}$. For instance, $\{2,3,5\}\{6\}\{1,7\}\{4\}$ in $\mathrm{Bal}[7]$ is encoded by 3114123 in $\mathrm{Cay}[7]$. As a consequence, $|\mathrm{Cay}[n]|$ is equal to the nth Fubini number; see sequence A000670 in the OEIS [21].

The *descent set* and *strict descent set* of $w \in \mathrm{Cay}[n]$ are defined, respectively, as

$$D(w) = \{i \in [n-1] : w(i) \geq w(i+1)\};$$
$$D^\circ(w) = \{i \in [n-1] : w(i) > w(i+1)\}.$$

We also let $\mathrm{des}(w) = |D(w)|$ and $\mathrm{des}^\circ(w) = |D^\circ(w)|$ denote their cardinalities. The sets $A(w)$ and $A^\circ(w)$ of *ascents* and *strict ascents*, as well as $\mathrm{asc}(w)$ and $\mathrm{asc}^\circ(w)$, are defined analogously. We will often add the word *weak* to ascents and descents to distinguish them from their strict counterparts. The *nth (weak) Caylerian polynomial* and the *nth strict Caylerian polynomial* are defined by

$$C_n(t) = \sum_{w \in \mathrm{Cay}[n]} t^{\mathrm{des}(w)} \quad \text{and} \quad C_n^\circ(t) = \sum_{w \in \mathrm{Cay}[n]} t^{\mathrm{des}^\circ(w)}.$$

Let $w \in \mathrm{Cay}[n]$. Define the *reverse* of w by $w^r(i) = w(n+1-i)$, for each $i \in [n]$. Also, define the *complement* of w by $w^c(i) = \max(w) + 1 - w(i)$, for each $i \in [n]$, where $\max(w) = \max\{w(i) : i \in [n]\}$. The symmetry group generated by reverse and complement acts on Cayley permutations, and for this reason we can replace (weak) descents with (weak) ascents in the definition of the Caylerian polynomials. Furthermore, since $\mathrm{des}^\circ(w) = n - 1 - \mathrm{asc}(w)$, the coefficients of the strict Caylerian polynomials $C_n^\circ(t)$ are simply the reverse of the coefficients of $C_n(t)$:

$$C_n^\circ(t) = t^{n-1} C_n\big(1/t\big).$$

The resulting triangle of coefficients is A366173 [21].

The set $\mathrm{Bur}[n]$ of *Burge words* [1, 4, 8, 9] of size n is defined as

$$\mathrm{Bur}[n] = \{(u, v) \in \mathrm{I}[n] \times \mathrm{Cay}[n] : D(u) \subseteq D(v)\},$$

where $\mathrm{I}[n]$ denotes the set of weakly increasing Cayley permutations of size n. A *Burge matrix* [8, 9] is a matrix with nonnegative integer entries whose every row and column has at least one nonzero entry. The size of a Burge matrix is the sum of its entries and we let $\mathrm{Mat}[n]$ denote the set of Burge matrices of size n. A bijection between $\mathrm{Bur}[n]$ and $\mathrm{Mat}[n]$ is obtained by mapping the Burge word (u, v) to the matrix $A = (a_{ij})$ where a_{ij} is equal to the number of pairs $(u(\ell), v(\ell)) = (i, j)$ in (u, v). For instance, the Burge word below corresponds to the Burge matrix to its right:

$$\begin{pmatrix} 1\,1\,1\,2\,3\,3\,3\,3\,3 \\ 3\,3\,1\,3\,4\,2\,2\,2\,2 \end{pmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix}.$$

The inverse map is obtained by associating each matrix $A = (a_{ij})$ in $\mathrm{Mat}[n]$ with the biword $\binom{u}{v}$ of size n where any column $\binom{i}{j}$ appears a_{ij} times, and the columns are sorted in ascending order with respect to the top entry, breaking ties by sorting in descending order with respect to the bottom entry.

Under the bijection described above, the set $\mathrm{Mat}^{01}[n]$ of *binary Burge matrices* corresponds to

$$\mathrm{Bur}^{01}[n] = \big\{(u, v) \in \mathrm{Bur}[n] : D(u) \subseteq D^\circ(v)\big\}.$$

The sequence of cardinalities $|\mathrm{Mat}[n]| = |\mathrm{Bur}[n]|$ is recorded as A120733 [21], while its binary counterpart $|\mathrm{Mat}^{01}[n]| = |\mathrm{Bur}^{01}[n]|$ gives A101370.

There is a significant interplay between Cayley permutations and Burge structures. Indeed, we [9, Theorem 5.1] showed that:

$$\begin{aligned} C_n(2) &= \left|\mathrm{Bur}[n]\right| = \left|\mathrm{Mat}[n]\right|; \\ C_n^{\circ}(2) &= \left|\mathrm{Bur}^{01}[n]\right| = \left|\mathrm{Mat}^{01}[n]\right|, \end{aligned}$$

a generalization of the well-known fact that the nth Eulerian polynomial evaluated at 2 is equal to $|\mathrm{Bal}[n]|$, the nth Fubini number. Pushing the interplay between Caylerian polynomials and Burge structures further, we [9] defined the *weak* and *strict two-sided Caylerian polynomials* by

$$\begin{aligned} \hat{B}_n(s,t) &= \sum_{A \in \mathrm{Mat}[n]} s^{\mathrm{row}(A)} t^{\mathrm{col}(A)}; \\ \hat{B}_n^{\circ}(s,t) &= \sum_{A \in \mathrm{Mat}^{01}[n]} s^{\mathrm{row}(A)} t^{\mathrm{col}(A)}, \end{aligned}$$

where $\mathrm{row}(A)$ and $\mathrm{col}(A)$ denote the number of rows and columns of A, respectively, and showed that [9, Corollary 6.5]:

$$\begin{aligned} C_n(t) &= (t-1)^n \hat{B}_n\left(1, \frac{1}{t-1}\right); \\ C_n^{\circ}(t) &= (t-1)^n \hat{B}_n^{\circ}\left(1, \frac{1}{t-1}\right). \end{aligned}$$

3 $\mathbb{L}$-species

To delve deeper into the enumerative aspects of Caylerian polynomials and Burge matrices we will use combinatorial species. The main references for the theory of combinatorial species are Joyal's seminal paper [12] and the book by Bergeron, Labelle and Leroux [3]. Shorter introductions can be found in the papers by Claesson [11] and—in the context of (pattern-avoiding) Cayley permutations—Cerbai et al. [10].

The prototypical combinatorial species are the $\mathbb{B}$-*species*, and they are endofunctors on the category $\mathbb{B}$ whose objects are finite sets and whose morphisms are bijections. We will, however, be using so called $\mathbb{L}$-species, where $\mathbb{L}$ denotes the category of finite totally ordered sets with order-preserving bijections as morphisms.

An $\mathbb{L}$-*species* is a functor $F : \mathbb{L} \to \mathbb{B}$ or, unwinding the definition of a functor, an $\mathbb{L}$-species is a rule F that associates

- to each finite totally ordered set ℓ, a finite set $F[\ell]$;
- to each order preserving bijection $\sigma : \ell_1 \to \ell_2$, a bijection $F[\sigma] : F[\ell_1] \to F[\ell_2]$ such that $F[\sigma \circ \tau] = F[\sigma] \circ F[\tau]$ for all order preserving bijections $\sigma : \ell_1 \to \ell_2$, $\tau : \ell_2 \to \ell_3$, and $F[\mathrm{id}_\ell] = \mathrm{id}_{F[\ell]}$.

We will need the following species whose careful definitions can be found in the aforementioned references [3, 10, 11, 12]:

(a) 1: characteristic of empty set;

(b) X: singletons;

(c) E: sets;

(d) E_+: nonempty sets;

(e) L: linear orders;

(f) $\mathcal{S}$: permutations;

(g) Par: set partitions;

(h) Bal: ballots.

An element $s \in F[\ell]$ is called an *F-structure* on ℓ, and the function $F[\sigma]$ is called the *transport of F-structures along σ*. We shall often use $\ell = [n]$ with the natural order as our underlying totally ordered set, and for ease of notation we write $F[n] = F[[n]]$. The *(exponential) generating function* of the species F is the formal power series

$$F(x) = \sum_{n\geq 0} |F[n]|\frac{x^n}{n!}.$$

For instance, $1(x) = 1$, $X(x) = x$, $E(x) = e^x$, and $E_+(x) = e^x - 1$. In general, if F is a species, then F_+ denotes the species of nonempty F-structures, and hence $F = 1 + F_+$.

Two species are considered (combinatorially) equal if there is a natural isomorphism between them. Clearly, if $F = G$, then $F(x) = G(x)$. For $\mathbb{B}$-species, the converse is false, the typical example being $L \neq \mathcal{S}$, but $L(x) = \mathcal{S}(x) = 1/(1-x)$. For $\mathbb{L}$-species it is, however, the case that $F = G \Leftrightarrow F(x) = G(x)$. This is a consequence of there being a unique order preserving bijection between any given pair of finite totally ordered sets of the same cardinality. So while transport of structure plays a crucial role in the theory of $\mathbb{B}$-species it is less important in the context of $\mathbb{L}$-species, and will usually be left implicit. Henceforth, all species are assumed to be $\mathbb{L}$-species and we drop the prefix $\mathbb{L}$.

We can construct new species from existing species using operations such as addition, multiplication, cartesian multiplication, and composition.

The *sum* of two species F and G is simple to define: for any finite totally ordered set ℓ, let $(F+G)[\ell] = F[\ell] \sqcup G[\ell]$, where $\sqcup$ denotes disjoint union. Clearly, $(F+G)(x) = F(x) + G(x)$.

For the *product*, an $(F \cdot G)$-structure on ℓ is a pair (s,t), where s is an F-structure on a subset ℓ_1 of ℓ and t is a G-structure on the remaining elements $\ell_2 = \ell \setminus \ell_1$. That is, $(F \cdot G)[\ell] = \bigsqcup (F[\ell_1] \times G[\ell_2])$ in which the union is over all pairs (ℓ_1, ℓ_2) such that $\ell = \ell_1 \cup \ell_2$ and $\ell_1 \cap \ell_2 = \emptyset$. It is easy to see that $(FG)(x) = F(x)G(x)$.

The *cartesian product* is defined by $(F \times G)[\ell] = F[\ell] \times G[\ell]$. In other words, an $(F \times G)$-structure on ℓ is the superposition of an F-structure and a G-structure on ℓ. Clearly, $|(F \times G)[n]| = |F[n]| \cdot |G[n]|$ and in terms of exponential generating functions this corresponds to the (exponential) Hadamard product of $F(x)$ and $G(x)$.

Assume $G[\emptyset] = \emptyset$, that is, there are no G-structures on the empty set. The *(partitional) composition* of F and G, denoted $F \circ G$ or $F(G)$, is defined by

$$(F \circ G)[\ell] \;=\; \bigcup_{\beta \in \mathrm{Par}[\ell]} F[\beta] \times \prod_{B \in \beta} G[B],$$

in which $\prod$ denotes the set-theoretical cartesian product. In terms of cardinalities,

$$|(F \circ G)[n]| = \sum_{\beta \in \mathrm{Par}[n]} |F[\beta]| \prod_{B \in \beta} |G[B]| \tag{1}$$

and one can show that $(F \circ G)(x) = F(G(x))$. Intuitively, an $(F \circ G)$-structure is a generalized partition in which each block of the partition carries a G-structure, and the blocks are structured by F. For instance, a set partition is a set of nonempty sets, while a ballot (ordered set partition) is a linear order of nonempty sets:

$$\mathrm{Par} = E \circ E_+ \quad \text{and} \quad \mathrm{Bal} = L \circ E_+,$$

from which $\mathrm{Par}(x) = \exp(e^x - 1)$ and $\mathrm{Bal}(x) = 1/(2 - e^x)$ follow.

Cayley permutations can also be defined as a species [10]:

$$\mathrm{Cay}[\ell] = \{w \in [n]^\ell \colon \mathrm{Im}(w) = [k] \text{ for some } k \le n\},$$

where $n = |\ell|$ (and with transport of structure defined by $\mathrm{Cay}[\sigma](w) = w \circ \sigma^{-1}$). As previously mentioned there is a natural bijection between Cayley permutations and ballots, and hence $\mathrm{Cay} = \mathrm{Bal}$.

Let us now return to the species L of linear orders. An L-structure on ℓ is a bijection $w : [n] \to \ell$, which we may represent using one-line notation as $w = w(1) \dots w(n)$. More generally, an L^m-structure on ℓ is an m-tuple, or vector, of linear orders on disjoint underlying sets $\ell_1 \cup \dots \cup \ell_m = \ell$. And, an $(L^m)_+$-structure is such a vector with at least one nonempty entry. The generating function of L^m is

$$L^m(x) = \frac{1}{(1-x)^m} = (1 + x + x^2 + \cdots)^m = \sum_{n \ge 0} \left(\!\!\binom{m}{n}\!\!\right) x^n,$$

where the multichoose coefficient $\left(\!\binom{m}{n}\!\right) = \binom{m+n-1}{n}$ is the number of multisets of cardinality n over $[m]$. Thus,

$$|L^m[n]| = n! \left(\!\!\binom{m}{n}\!\!\right). \tag{2}$$

4 Matrices of linear orders

An $(L \circ (L^m)_+)$-structure is obtained by placing an $(L^m)_+$-structure on every block of a set partition of ℓ, and then a linear order on the blocks. Viewing an $(L^m)_+$-structure as a column vector with m components, it is natural to view an $(L \circ (L^m)_+)$-structure as a matrix with m rows. We shall give an alternative description of those matrices, but first we make a couple of definitions. For $(\alpha_1, \alpha_2) \in L^2[\ell]$ we use juxtaposition, $\alpha_1\alpha_2 \in L[\ell]$, to denote the concatenation of α_1 and α_2. For instance, if $(\alpha_1, \alpha_2) = (451, 32) \in L^2[5]$, then $\alpha_1\alpha_2 = 45132 \in L[5]$. Similarly, for a vector $\mathbf{a} = (\alpha_1, \dots, \alpha_m) \in L^m[\ell]$ of linear orders, let $\prod \mathbf{a} = \alpha_1 \dots \alpha_m$. We now have the following characterization of the matrices in question.

Lemma 4.1. *Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k \end{bmatrix}$ be a matrix whose ith column is the vector $\mathbf{a}_i$ and whose entries are linear orders of disjoint sets. Then, A is an $(L \circ (L^m)_+)$-structure on ℓ if and only if A has m rows and it satisfies the following two conditions:*

1. $\prod A := \prod \mathbf{a}_1 \prod \mathbf{a}_2 \cdots \prod \mathbf{a}_k \in L[\ell]$*;*

2. $\prod \mathbf{a}_j \neq \epsilon$ *for each* $j \in [k]$*.*

In other words, a matrix A with m rows whose entries are linear orders belongs to $(L \circ (L^m)_+)\,[\ell]$ if different entries (linear orders) have disjoint underlying sets, the union of the underlying sets is ℓ, and each column contains at least one nonempty linear order. For instance, there are fourteen $(L \circ (L^2)_+)$-structures on $\{1, 2\}$:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 12 \\ \epsilon \end{bmatrix} \begin{bmatrix} 21 \\ \epsilon \end{bmatrix} \begin{bmatrix} \epsilon \\ 12 \end{bmatrix} \begin{bmatrix} \epsilon \\ 21 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \epsilon \\ \epsilon & 2 \end{bmatrix} \begin{bmatrix} 2 & \epsilon \\ \epsilon & 1 \end{bmatrix} \begin{bmatrix} \epsilon & 1 \\ 2 & \epsilon \end{bmatrix} \begin{bmatrix} \epsilon & 2 \\ 1 & \epsilon \end{bmatrix} \begin{bmatrix} 1 & 2 \\ \epsilon & \epsilon \end{bmatrix} \begin{bmatrix} 2 & 1 \\ \epsilon & \epsilon \end{bmatrix} \begin{bmatrix} \epsilon & \epsilon \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \epsilon & \epsilon \\ 2 & 1 \end{bmatrix}.$$

Of these fourteen matrices seven are such that $\prod A = 12$. In general, the species of matrices in $(L \circ (L^m)_+)\,[\ell]$ such that $\prod A$ is the increasing linear order of ℓ can be defined as follows.

Definition 4.2. For a totally ordered set $\ell = \{u_1, \dots, u_n\}$ with $u_1 < \cdots < u_n$ we define that an m-row matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k \end{bmatrix}$ is an $\mathfrak{Mat}_m$-structure on ℓ if its entries are linear orders of disjoint sets and it satisfies the following two conditions:

1. $\prod A = u_1 u_2 \dots u_n$;
2. $\prod \mathbf{a}_j \neq \epsilon$ for each $j \in [k]$.

In the prequel [9] the same notation, $\mathfrak{Mat}_m[n]$, was used to denote the set of m-row matrices with nonnegative integer entries whose total sum is n and have at least one positive entry in each column. The difference is mostly superficial and such matrices are obtained from the matrices defined here by taking the length of each entry. The slightly more complex definition given here better fits the species framework we would like to work within.

We will now define a natural action of permutations on $\mathfrak{Mat}_m$ that will directly lead to our next simple lemma. Recall that a permutation $w \in \mathcal{S}[\ell]$ is a bijection $w : \ell \to \ell$. Permutations of a totally ordered set can also be represented in one-line notation: $w = w(u_1) \dots w(u_n)$, where $\ell = \{u_1, \dots, u_n\}$ and $u_1 < \cdots < u_n$. Now, a permutation $w \in \mathcal{S}[\ell]$ acts on a matrix $A = (a_{ij}) \in \mathfrak{Mat}_m[\ell]$ by replacing each entry $a_{ij} = c_1 c_2 \dots c_r$ with $w \cdot a_{ij} := w(c_1) w(c_2) \dots w(c_r)$, and we denote the resulting matrix by $w \cdot A = (w \cdot a_{ij})$.

Lemma 4.3. *For each* $m \geq 0$*,*

$$\mathcal{S} \times \mathfrak{Mat}_m = L \circ (L^m)_+.$$

Proof. To prove this species identity, it suffices to give a size-preserving bijection from $(\mathcal{S} \times \mathfrak{Mat}_m)$-structures to $(L \circ (L^m)_+)$-structures, and we claim that

$$(w, A) \mapsto w \cdot A =: M$$

is such a map. Indeed, it is easy to see that since A satisfies the two conditions of Definition 4.2, the matrix M satisfies the corresponding two conditions of Lemma 4.1. Also, the inverse map is $M \mapsto (w,\, w^{-1} \cdot M)$ where $w = \prod M$. □

Example 4.4. Let $(w, A) \in (\mathcal{S} \times \mathfrak{Mat}_4)\,[9]$ be given by

$$w = 784652391 \quad \text{and} \quad A = \begin{bmatrix} \cdot & 5 & 67 \\ 123 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 4 & \cdot & 89 \end{bmatrix}.$$

The bijection described in the proof of Lemma 4.3 maps (w, A) to the matrix

$$M = \begin{bmatrix} \cdot & 5 & 23 \\ 784 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 6 & \cdot & 91 \end{bmatrix} \in \big(L \circ (L^4)_+\big)[9].$$

Note also that $w = \prod M$, $w^{-1} = 967354128$, and $w^{-1} \cdot M = A$.

Next we shall derive a counting formula for $\mathfrak{Mat}_m$ from the species equation given in Lemma 4.3, but first we need to introduce some auxiliary notation. Let us write $(a_1, \ldots, a_k) \models n$ to indicate that $(a_1, \ldots, a_k)$ is a *composition* of n; that is, each a_i is a positive integer and $\sum_{i=1}^k a_i = n$. Moreover, for $\alpha = (a_1, \ldots, a_k) \models n$, let

$$\binom{n}{\alpha} = \binom{n}{a_1, \ldots, a_k} = n! \,/ \prod_{a \in \alpha} a!$$

denote the associated multinomial coefficient. Now, for each $m, n \geq 0$,

$$\begin{aligned} |\mathfrak{Mat}_m[n]| &= \frac{1}{n!}\big|(L \circ (L^m)_+)\,[n]\big| \\ &= \frac{1}{n!} \sum_{\beta \in \mathrm{Par}[n]} |L[\beta]| \prod_{B \in \beta} |L^m[B]| \\ &= \frac{1}{n!} \sum_{\beta \in \mathrm{Par}[n]} k! \prod_{B \in \beta} \left(\!\!\binom{m}{|B|}\!\!\right) |B|! \\ &= \frac{1}{n!} \sum_{\alpha \models n} \binom{n}{\alpha} \prod_{a \in \alpha} \left(\!\!\binom{m}{a}\!\!\right) a! = \sum_{\alpha \models n} \prod_{a \in \alpha} \left(\!\!\binom{m}{a}\!\!\right). \end{aligned} \tag{3}$$

While this formula has a certain aesthetic appeal, it also has 2^{n-1} terms and is thus an ineffective way of counting these matrices. To arrive at an effective formula we will need to derive another species identity, but let us first give a simple general lemma concerning the composition of species.

Lemma 4.5. *For any species F and G, with $G[\emptyset] = \emptyset$,*

$$|(F \circ G)[n]| = \sum_{k=0}^{n} |F[k]| \cdot |E_k(G)[n]|,$$

where E_k denotes the species characteristic of sets of cardinality k, defined by $E_k[\ell] = \{\ell\}$ if $|\ell| = k$ and $E_k[\ell] = \emptyset$ otherwise.

Proof. We have

$$
\begin{aligned}
|(F \circ G)[n]| &= \sum_{\beta \in \mathrm{Par}[n]} |F[\beta]| \prod_{B \in \beta} |G[B]| \\
&= \sum_{k=0}^{n} |F[k]| \sum_{\beta \in \mathrm{Par}[n]} |E_k[\beta]| \prod_{B \in \beta} |G[B]| = \sum_{k=0}^{n} |F[k]| \cdot |E_k(G)[n]|. \qquad \square
\end{aligned}
$$

In further preparation for the next result, let us introduce the species of connected linear orders. Suppose that the species F and G are related by $F = E(G)$, so that an F-structure is a set of G-structures. We may then call G the species of *connected F-structures*, denoted by $G = F^c$. A classic example is $\mathcal{S} = E(\mathcal{C})$, where $\mathcal{C}$ is the species of cycles. That is, a connected permutation is a cycle, and any permutation decomposes uniquely as a set of cycles. The (n, k)th Stirling number of the first kind counts permutations of $[n]$ with k cycles:

$$
\begin{bmatrix} n \\ k \end{bmatrix} = |E_k(\mathcal{C})[n]|.
$$

A similar decomposition is obtained by splitting linear orders by their left-to-right minima. Here, a *left-to-right minimum* of a linear order w is an entry $w(i)$ such that $w(i) < w(j)$ for each $j < i$. For instance, the linear order $w = 784652391$ of Example 4.4 decomposes as

$$
w \;=\; 78\,|\,465\,|\,239\,|\,1.
$$

In this sense, a connected linear order is a linear order that begins with the minimum of the underlying set, and any linear order decomposes uniquely as a set of connected linear orders. To reconstruct the linear order we juxtapose the connected components with their minima in decreasing order, so that the minima of the connected components are the left-to-right minima of the resulting linear order. In particular, from $L(x) = 1/(1-x)$, $E(x) = e^x$, and $L = E(L^c)$, it follows that

$$
L^c(x) = \log\left(\frac{1}{1-x}\right).
$$

Proposition 4.6. *For each $n, m \geq 0$,*

$$
\mathcal{S} \times \mathfrak{Mat}_m = \mathrm{Bal}(mL^c) \quad \textit{and} \quad |\mathfrak{Mat}_m[n]| = \frac{1}{n!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} m^k \mathrm{fub}(k),
$$

where $\mathrm{fub}(k)$ *denotes the kth Fubini number.*

Proof. By Lemma 4.3 it suffices to prove that $L \circ (L^m)_+ = \mathrm{Bal}(mL^c)$. We have

$$
\begin{aligned}
L \circ (L^m)_+ &= L \circ \big(L^m - 1\big) \\
&= L \circ \big(E(L^c)^m - 1\big) \\
&= L \circ \big(E(mL^c) - 1\big) \\
&= L \circ E_+ \circ (mL^c) \\
&= \mathrm{Bal}(mL^c).
\end{aligned}
$$

Furthermore,

$$\begin{aligned} n!\big|\mathfrak{Mat}_m[n]\big| &= \big|(\mathrm{Bal}(mX)\circ L^c)[n]\big| \\ &= \sum_{k=0}^{n} |\mathrm{Bal}(mX)[k]|\cdot|E_k(L^c)[n]| \\ &= \sum_{k=0}^{n} |(\mathrm{Bal}\times E(mX))[k]|\cdot|E_k(L^c)[n]| \\ &= \sum_{k=0}^{n} m^k \mathrm{fub}(k)\begin{bmatrix} n \\ k \end{bmatrix}, \end{aligned}$$

where the second equality follows from Lemma 4.5, and the third equality uses the general identity $F(mX) = F \times E(mX)$ for any species F. □

In the proof of Proposition 4.6 we derived the identity $L \circ (L^m)_+ = \mathrm{Bal}(mL^c)$ by a sequence of species equations. We shall now give an alternative bijective proof of this identity. To an $(L\circ(L^m)_+)$-structure M we associate a $\mathrm{Bal}(mL^c)$-structure B as follows. Each entry of the matrix M is a linear order and by splitting each entry according to its left-to-right minima we obtain a set of connected linear orders. For instance, consider the matrix

$$M = \begin{bmatrix} \cdot & 5 & 23 \\ 784 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 6 & \cdot & 91 \end{bmatrix} \in (L\circ(L^m)_+)\,[9]$$

of Example 4.4. Here, 784 splits into $78|4$ and 91 splits into $9|1$, and the resulting set of connected linear orders is

$$\{78, 4, 6, 5, 23, 9, 1\}.$$

Let us call a connected linear order arising this way an *atom*. To determine B, we need to place a $(\mathrm{Bal}\times E(mX))$-structure on the set of atoms. The ballot is defined by collecting in the jth block the atoms contained in the jth column of M. To place an $E(mX)$-structure means to assign a color in $[m]$, and we let the color of each atom be equal to its row index in M. Here, we obtain

$$B = \{78_2, 4_2, 6_4\}\{5_1\}\{23_1, 9_4, 1_4\} \in \mathrm{Bal}(mL^c)[9],$$

where colors are indicated by the subscripts. To prove that the map $M \mapsto B$ is a bijection, we describe its inverse. Given B, the (i, j)th entry of M is the linear order obtained by sorting the i-colored atoms in the jth block of B decreasingly with respect to their first letter and then concatenating them.

By composing the map of Lemma 4.3 with the map described above, we obtained a bijection between $\mathcal{S}\times\mathfrak{Mat}_m$ and $\mathrm{Bal}(mL^c)$, which leads to a formula for $|\mathfrak{Mat}_m[n]|$. It is easy to adjust this approach to prove an analogous equation for Burge matrices and obtain a formula for $|\mathrm{Mat}[n]|$.

Proposition 4.7. *For each $n \geq 0$,*

$$\mathcal{S}\times\mathrm{Mat} = (\mathrm{Bal}\times\mathrm{Bal})\circ L^c \quad \textit{and} \quad |\mathrm{Mat}[n]| = \frac{1}{n!}\sum_{k=0}^{n}\mathrm{fub}(k)^2\begin{bmatrix} n \\ k \end{bmatrix}.$$

Proof. In Lemma 4.3 and in the previous paragraph we gave two bijections

$$(w, A) \longleftrightarrow M \quad \text{and} \quad M \longleftrightarrow B,$$

where (w, A) is an $(\mathcal{S} \times \mathfrak{Mat}_m)$-structure, M is an $(L \circ (L^m)_+)$-structure, and B is a $\mathrm{Bal}(mL^c)$-structure. Recall that a matrix A of $\mathfrak{Mat}_m$ is a Burge matrix if and only if it has no empty rows, which in turn is equivalent to each row of M containing at least one nonempty linear order. In terms of $\mathrm{Bal}(mL^c)$, the structure determining the row indices of the atoms in each column is more restrictive than $E(mX)$, since empty rows are not allowed. In fact, it is the same structure as the one used for the (nonempty) columns, that is, the ballot defined by collecting in the ith block the atoms contained in the ith row of A. Conversely, the (i, j)th entry of A contains the linear order obtained by juxtaposing the atoms that are simultaneously in the ith block of the ballot determining the row indices and in the jth block of the ballot determining the column indices. Consequently, a species identity for $\mathcal{S} \times \mathrm{Mat}$ is obtained by simply replacing $E(mX)$ with Bal:

$$\mathcal{S} \times \mathrm{Mat} = (\mathrm{Bal} \times \mathrm{Bal}) \circ L^c.$$

In particular,

$$\begin{aligned} n!|\mathrm{Mat}[n]| &= \left|\big((\mathrm{Bal} \times \mathrm{Bal}) \circ L^c\big)[n]\right| \\ &= \sum_{k=0}^{n} \left|(\mathrm{Bal} \times \mathrm{Bal})[k]\right| \cdot \left|E_k(L^c)[n]\right| \qquad \text{(by Lemma 4.5)} \\ &= \sum_{k=0}^{n} \mathrm{fub}(k)^2 \begin{bmatrix} n \\ k \end{bmatrix}. \end{aligned}$$

□

We wish to refine the previous result to obtain a formula for the (weak) two-sided Caylerian polynomials

$$\hat{B}_n(s, t) = \sum_{A \in \mathrm{Mat}[n]} s^{\mathrm{row}(A)} t^{\mathrm{col}(A)}$$

defined in Section 2. Let $\mathrm{Mat}^{s,t}$ be the ($\mathbb{Z}[s, t]$-weighted) species of Burge matrices where every row is marked by s and every column is marked by t. More formally, we assign to a matrix $A \in \mathrm{Mat}[n]$ the weight $s^{\mathrm{row}(A)} t^{\mathrm{col}(A)}$, so that

$$\hat{B}_n(s, t) = \sum_{A \in \mathrm{Mat}[n]} s^{\mathrm{row}(A)} t^{\mathrm{col}(A)} = |\mathrm{Mat}^{s,t}[n]| \tag{4}$$

is the total weight of all the matrices in $\mathrm{Mat}[n]$. From the proof of Proposition 4.7 it follows that the joint distribution of rows and columns on Burge matrices can be obtained by tracking the number of blocks in $(\mathrm{Bal} \times \mathrm{Bal})$-structures. Let Bal^t be the species of ballots where each block is marked by t. Then

$$|\mathrm{Bal}^t[n]| = \sum_{w \in \mathrm{Bal}[n]} t^{\mathrm{blocks}(w)} \quad \text{and} \quad \mathrm{Bal}^t(x) = L(tE_+)(x) = \frac{1}{1 - t(e^x - 1)}.$$

Expressed differently, $\mathrm{Bal}^t = L \circ tX \circ E_+$, where tX denotes the species of singletons where each singleton is marked with a t. In particular, the equation

$$\mathcal{S} \times \mathrm{Mat} = (\mathrm{Bal} \times \mathrm{Bal}) \circ L^c$$

is refined by

$$\mathcal{S} \times \mathrm{Mat}^{s,t} = \big(\mathrm{Bal}^s \times \mathrm{Bal}^t\big) \circ L^c. \tag{5}$$

Using Equation (4), we immediately obtain the following identity for the weak two-sided Caylerian polynomials:

Theorem 4.8. *We have*

$$\hat{B}_n(s,t) = \frac{1}{n!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \big|\mathrm{Bal}^s[k]\big|\big|\mathrm{Bal}^t[k]\big|.$$

This identity, in turn, leads to the explicit formula for $C_n(t)$:

Theorem 4.9. *The nth Caylerian polynomial satisfies*

$$C_n(t) = \frac{1}{n!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \mathrm{fub}(k) \sum_{i=0}^{k} \begin{Bmatrix} k \\ i \end{Bmatrix} i!(t-1)^{n-i},$$

where $\left\{{n \atop k}\right\} = |E_k(E_+)[n]|$ *denotes the* (n,k)*th Stirling number of the second kind, counting set partitions of* $[n]$ *with* k *blocks.*

Proof. We have

$$\begin{aligned} C_n(t) &= (t-1)^n \hat{B}_n\left(1, \frac{1}{t-1}\right) && \text{[9, Corollary 6.5]} \\ &= (t-1)^n \frac{1}{n!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \big|\mathrm{Bal}^1[k]\big|\big|\mathrm{Bal}^{\frac{1}{t-1}}[k]\big| \\ &= (t-1)^n \frac{1}{n!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \mathrm{fub}(k) \left(\sum_{i=0}^{k} \begin{Bmatrix} k \\ i \end{Bmatrix} i! \frac{1}{(t-1)^i} \right) \end{aligned}$$ □

The results proven thus far can be extended to binary matrices, as we sketch below.

Proposition 4.10. *For each* $n, m \geq 0$*,*

$$\mathcal{S} \times \mathfrak{Mat}_m^{01} = L\big((1+X)^m - 1\big) \quad \textit{and} \quad |\mathfrak{Mat}_m^{01}[n]| = \sum_{\alpha \models n} \prod_{a \in \alpha} \binom{m}{a}.$$

Proof. By restricting the size-preserving bijection from $(\mathcal{S} \times \mathfrak{Mat}_m)$-structures to $(L \circ (L^m)_+)$-structures described in Lemma 4.3 to pairs (w, A) where A is binary, one obtains matrices $M = w \cdot A$ whose entries are linear orders of size at most one. In other words, each column is a nonempty $(1+X)^m$-structure and we get

$$\mathcal{S} \times \mathfrak{Mat}_m^{01} = L\big((1+X)^m - 1\big).$$

The proof of the equation for $|\mathfrak{Mat}_m^{01}[n]|$ is similar to the proof of Equation (3) for $|\mathfrak{Mat}_m[n]|$. The multichoose coefficient $\left(\!\binom{m}{n}\!\right)$ is, however, replaced by the binomial coefficient $\binom{m}{n}$ since the generating function $1/(1-x)^m = \sum_{n\geq 0} \left(\!\binom{m}{n}\!\right) x^n$ is replaced by the generating function $(1+x)^m = \sum_{n \geq 0} \binom{m}{n} x^n$. □

Next, we wish to compute the *ordinary* generating functions of the two sequences $|\mathfrak{Mat}_m[n]|$ and $|\mathfrak{Mat}^{01}_m[n]|$. In general, if F is a species, then the ordinary generating function $\sum_{n\geq 0}|F[n]|x^n$ is equal to the (exponential) generating function of $\mathcal{S}\times F$:

$$\sum_{n\geq 0}|F[n]|x^n = \sum_{n\geq 0} n!|F[n]|\frac{x^n}{n!} = (\mathcal{S}\times F)(x).$$

Proposition 4.11. *We have*

$$\sum_{n\geq 0}|\mathfrak{Mat}_m[n]|x^n = \frac{(1-x)^m}{2(1-x)^m-1} \quad \text{and} \quad \sum_{n\geq 0}|\mathfrak{Mat}^{01}_m[n]|x^n = \frac{1}{2-(1+x)^m}.$$

Proof. Using Lemma 4.3 and Proposition 4.10, respectively, we get

$$\begin{aligned}\sum_{n\geq 0}|\mathfrak{Mat}_m[n]|x^n &= \big(\mathcal{S}\times\mathfrak{Mat}_m\big)(x)\\ &= \left(L\circ(L^m)_+\right)(x)\\ &= \frac{1}{1-\big((1-x)^{-m}-1\big)} = \frac{(1-x)^m}{2(1-x)^m-1}\end{aligned}$$

and

$$\begin{aligned}\sum_{n\geq 0}|\mathfrak{Mat}^{01}_m[n]|x^n &= \big(\mathcal{S}\times\mathfrak{Mat}^{01}_m\big)(x)\\ &= L\big((1+x)^m-1\big)\\ &= \frac{1}{1-\big((1+x)^m-1\big)} = \frac{1}{2-(1+x)^m}.\end{aligned}$$

□

To prove the next result, we will need two equations involving the weak and strict Caylerian polynomials [9, Theorem 7.6]:

$$\frac{tC_n(t)}{(1-t)^{n+1}} = \sum_{m\geq 1}|\mathfrak{Mat}_m[n]|t^m; \tag{6}$$

$$\frac{tC^{\circ}_n(t)}{(1-t)^{n+1}} = \sum_{m\geq 1}|\mathfrak{Mat}^{01}_m[n]|t^m. \tag{7}$$

These equations were originally stated in terms of (generalized) Burge words. For convenience, we have here reformulated them in terms of the equinumerous structures $\mathfrak{Mat}_m$ and $\mathfrak{Mat}^{01}_m$. We note that identities (6) and (7) have the same flavor as the following identity due to Carlitz [6], often used as an alternative definition of the Eulerian polynomials $A_n(t)$:

$$\frac{tA_n(t)}{(1-t)^{n+1}} = \sum_{m\geq 1} m^n t^m.$$

Theorem 4.12. *We have*

$$\sum_{n\geq 0}\frac{tC_n(t)}{(1-t)^{n+1}}x^n = \sum_{m\geq 1}\frac{(1-x)^m}{2(1-x)^m-1}t^m$$

and

$$\sum_{n\geq 0}\frac{tC^{\circ}_n(t)}{(1-t)^{n+1}}x^n = \sum_{m\geq 1}\frac{1}{2-(1+x)^m}t^m.$$

Proof. Using Proposition 4.11 and Equation (6),

$$\sum_{n\geq 0}\frac{tC_n(t)}{(1-t)^{n+1}}x^n = \sum_{n\geq 0}\sum_{m\geq 1}|\mathfrak{Mat}_m[n]|t^m x^n$$
$$= \sum_{m\geq 1}\left(\sum_{n\geq 0}|\mathfrak{Mat}_m[n]|x^n\right)t^m = \sum_{m\geq 1}\frac{(1-x)^m}{2(1-x)^m-1}t^m.$$

The second identity is similarly obtained by Proposition 4.11 and Equation (7). □

5 Cayley permutations with a prescribed ascent set

In Section 2, we defined the weak ascent set $A(w)$ and the strict ascent set $A^\circ(w)$ of a Cayley permutation w, and we shall use the same definitions here in the context of linear orders. Let n be a positive integer. Consider a subset $S=\{s_1,\dots,s_r\}$ of $[n-1]$, with $s_1<s_2<\cdots<s_r$, and let

$$\alpha_n(S) = |\{w\in L[n] : A(v)\subseteq S\}|$$

be the number of linear orders on $[n]$ whose ascent set is contained in S. Then (see MacMahon [14, vol. 1, p. 190] or Stanley [22, Proposition 1.4.1]), $\alpha_n(S)$ is a polynomial in n given by the multinomial coefficient

$$\alpha_n(S) = \binom{n}{s_1, s_2-s_1, s_3-s_2,\dots,n-s_r} = \frac{n!}{s_1!(s_2-s_1)!(s_3-s_2)!\cdots(n-s_r)!}.$$

Indeed, to obtain any $w\in L[n]$ with $A(w)\subseteq S$, first pick s_1 elements $w(1)>w(2)>\cdots>w(s_1)$ from $[n]$, then s_2-s_1 elements $w(s_1+1)>w(s_1+2)>\cdots>w(s_2)$ from the remaining elements, and so on. Equivalently, $\alpha_n(S)$ counts ballots on $[n]$ whose block sizes are given by the vector

$$\Delta(S) := (s_1, s_2-s_1,\dots,n-s_r).$$

We [9] have extended $\alpha_n(S)$ to Cayley permutations by letting

$$\kappa_n(S) = |\{w\in \mathrm{Cay}[n] :\ A(w)\subseteq S\}|;$$
$$\kappa_n^\circ(S) = |\{w\in \mathrm{Cay}[n] :\ A^\circ(w)\subseteq S\}|.$$

We [9, Theorem 8.2] then showed that

$$\kappa_n^\circ(S) = |\mathrm{Mat}(S)[n]| \quad\text{and}\quad \kappa_n(S) = |\mathrm{Mat}^{01}(S)[n]|, \tag{8}$$

where $\mathrm{Mat}(S)$ denotes the set of Burge matrices whose vector of row sums is equal to $\Delta(S)$, and $\mathrm{Mat}^{01}(S)$ denotes such matrices that are binary.

The main goal of this section is to establish the following result.

Proposition 5.1. *We have*

$$|\mathfrak{Mat}_m[n]| = \sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\left(\!\!\binom{m(k-i)}{n}\!\!\right); \tag{9}$$

$$|\mathfrak{Mat}_m^{01}[n]| = \sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{m(k-i)}{n}. \tag{10}$$

Furthermore, for $S = \{s_1, \dots, s_r\} \subseteq [n-1]$ *with* $s_1 < s_2 < \cdots < s_r$,

$$\kappa_n^\circ(S) = \sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\prod_{j=0}^{r}\left(\!\!\binom{k-i}{s_{j+1}-s_j}\!\!\right); \tag{11}$$

$$\kappa_n(S) = \sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\prod_{j=0}^{r}\binom{k-i}{s_{j+1}-s_j}, \tag{12}$$

where $s_0 = 0$ *and* $s_{r+1} = n$.

Munarini, Poneti and Rinaldi [18] used the inclusion–exclusion principle to obtain identities (9) and (11) in the context of composition matrices (see equation (12) and equation (13) in their paper, respectively). Equation (11) also appears in a book by Andrews [2, equation (4.3.3)] as a counting formula for so-called *vector compositions*, structures previously studied by MacMahon [15] under the name of *compositions of multipartite numbers*. Here, we provide sign-reversing involution proofs of these equations that allow us to generalize them to binary matrices $\mathfrak{Mat}_m^{01}$ and to $\kappa_n(S)$. Let us start with the first two equations, (9) and (10).

For $m \geq 0$, define $\mathcal{G}_m$ as the species obtained from $\mathfrak{Mat}_m$ by allowing empty columns, but no more than n columns in total, and then giving each column a sign such that nonempty columns are positive, but empty columns can be negative or positive. More precisely, for a totally ordered set $\ell = \{u_1, \dots, u_n\}$ with $u_1 < \cdots < u_n$, we define a $\mathcal{G}_m$-structure on ℓ as a pair (A, ξ), where $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k \end{bmatrix}$ is a matrix with m rows and $k \leq n$ columns, $\xi : [k] \to \{-1, 1\}$, and the following two conditions are satisfied:

1. $\prod A = u_1 u_2 \dots u_n$;
2. $\prod \mathbf{a}_i \neq \epsilon \implies \xi(i) = 1$.

For simplicity, we will refer to the pair (A, ξ) simply as a (signed) matrix and use A as its identifier. By way of illustration, we may write $A \in \mathcal{G}_4[9]$ where the matrix A together with the values of ξ (on top of each column) are displayed below:

$$A = \begin{array}{c} \begin{matrix} - & + & + & - & - & + & + \end{matrix} \\ \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 5 & 67 \\ \cdot & 123 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 4 & \cdot & \cdot & \cdot & \cdot & 89 \end{bmatrix} \end{array}$$

As an additional illustration, the five matrices in $\mathcal{G}_1[2]$ are displayed below:

$$\begin{array}{c} + \;\; + \\ \begin{bmatrix} 1 & 2 \end{bmatrix} \end{array} \qquad \begin{array}{c} + \;\; + \\ \begin{bmatrix} \cdot & 12 \end{bmatrix} \end{array} \qquad \begin{array}{c} - \;\; + \\ \begin{bmatrix} \cdot & 12 \end{bmatrix} \end{array} \qquad \begin{array}{c} + \;\; + \\ \begin{bmatrix} 12 & \cdot \end{bmatrix} \end{array} \qquad \begin{array}{c} + \;\; - \\ \begin{bmatrix} 12 & \cdot \end{bmatrix} \end{array}$$

To construct a matrix A in $\mathcal{G}_m[n]$ with k columns, first choose $i \leq k$ negative columns in $\binom{k}{i}$ ways. Next, fill in the remaining $mk - mi = m(k-i)$ entries of the matrix with linear orders so that $\prod A = 12 \dots n$. It is enough to pick the sizes of the linear orders and they form a weak composition of n into $m(k-i)$ parts; that is, an integer composition with nonnegative parts. There are $\left(\!\binom{m(k-i)}{n}\!\right)$ such compositions and we have arrived at the formula

$$|\mathcal{G}_m[n]| = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{k}{i} \left(\!\!\binom{m(k-i)}{n}\!\!\right).$$

By abuse of notation, we define the sign of $A = \begin{bmatrix} \mathbf{a}_1 \, \mathbf{a}_2 \, \cdots \, \mathbf{a}_k \end{bmatrix} \in \mathcal{G}_m[n]$ by

$$\xi(A) = \xi(1)\xi(2)\cdots\xi(k)$$

In other words, $\xi(A) = (-1)^i$ where i is the number of negative columns. Now, given $A \in \mathcal{G}_m[n]$, let $\lambda(A) \in \{0, 1, \ldots, k\}$ be the index of the leftmost empty column of A; if there is no such column, let $\lambda(A) = 0$. Note that $A \in \mathfrak{Mat}_m[n]$ if and only if $\lambda(A) = 0$. Finally, define the map $\gamma : \mathcal{G}_m[n] \to \mathcal{G}_m[n]$ by letting $\gamma(A)$ be the matrix obtained from A by reversing the sign of column $\lambda(A)$ if $\lambda(A) \neq 0$, and letting $\gamma(A) = A$ otherwise. It is clear that γ is an involution on $\mathcal{G}_m[n]$:

$$\gamma\big(\gamma(A)\big) = A.$$

Furthermore, let $\mathrm{Fix}(\gamma) = \{A : \gamma(A) = A\}$ be the set of *fixed points* of γ. Then

$$\mathrm{Fix}(\gamma) = \{A : \lambda(A) = 0\} = \mathfrak{Mat}_m[n].$$

On the other hand, if A is not fixed, then $\xi(\gamma(A)) = -\xi(A)$. Consequently, γ is a sign-reversing involution on $\mathcal{G}_m[n]$ whose set of fixed points is $\mathfrak{Mat}_m[n]$. Since fixed points have positive sign, Equation (9) follows:

$$\begin{aligned} |\mathfrak{Mat}_m[n]| &= \sum_{A \in \mathrm{Fix}(\gamma)} \xi(A) \\ &= \sum_{A \in \mathcal{G}_m[n]} \xi(A) = \sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \left(\!\!\binom{m(k-i)}{n}\!\!\right). \end{aligned}$$

The same technique applies to binary matrices in $\mathfrak{Mat}_m^{01}[n]$. Here, the nonempty linear orders have size one and hence the weak composition in the above argument has parts of size at most one. Clearly, there are $\binom{m(k-i)}{n}$ such compositions of n into $m(k-i)$ parts. Following what has by now become a familiar pattern in this article, the formula for $\mathfrak{Mat}_m^{01}$ is simply obtained by replacing the multichoose coefficient with a binomial coefficient in Equation (9):

$$|\mathfrak{Mat}_m^{01}[n]| = \sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{m(k-i)}{n}.$$

A simple adjustment of the same approach allows us to prove the equations for $\kappa_n^\circ(S)$ and $\kappa_n(S)$. By Equation (8), it suffices to count (binary) Burge matrices whose vector of row sums is equal to $\Delta(S)$. Given a matrix $A = (a_{ij})$ in $\mathcal{G}_m[n]$, let $\hat{A} = (|a_{ij}|)$ denote the matrix obtained from A by replacing each linear order with its size. Also, let $\mathbf{1}$ denote the all ones vector. Then $\hat{A} \cdot \mathbf{1}$ is the row sum vector of $\hat{A}$: its ith component equals the total size of the linear orders on row i. Letting

$$\mathcal{G}_m(S)[n] = \{A \in \mathcal{G}_m[n] : \hat{A} \cdot \mathbf{1} = \Delta(S)\}$$

we have

$$|\mathcal{G}_m(S)[n]| = \sum_{k=0}^{n}\sum_{i=0}^{k}\binom{k}{i}\left(\!\!\binom{k-i}{s_1}\!\!\right)\left(\!\!\binom{k-i}{s_2-s_1}\!\!\right)\cdots\left(\!\!\binom{k-i}{n-s_r}\!\!\right).$$

Indeed, the same argument used to count $\mathcal{G}_m[n]$ holds, except that here we have to pick the right sizes of the nonempty linear orders in each row; that is, we pick a weak composition of s_1 among the $k-i$ nonempty columns for the first row, a weak composition of $s_2 - s_1$ for the second row, and so on. Now, the sign ξ and the sign-reversing involution γ defined previously on $\mathcal{G}_m[n]$ work analogously on $\mathcal{G}_m(S)[n]$, and the fixed points are the matrices in $\mathcal{G}_m(S)[n]$ where every column contains at least one nonempty linear order. Further, due to the equality $\hat{A}\cdot\mathbf{1} = S$, each row contains at least one nonempty linear order. To summarize, the fixed points are the Burge matrices whose row sums are given by $\Delta(S)$, and identity (11) now immediately follows from identity (8):

$$\begin{aligned}\kappa_n^{\circ}(S) &= |\mathrm{Mat}(S)[n]|\\ &= \sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\left(\!\!\binom{k-i}{s_1}\!\!\right)\left(\!\!\binom{k-i}{s_2-s_1}\!\!\right)\cdots\left(\!\!\binom{k-i}{n-s_r}\!\!\right).\end{aligned}$$

Finally, the same reasoning applies to binary matrices, yielding

$$\begin{aligned}\kappa_n(S) &= |\mathrm{Mat}^{01}(S)[n]|\\ &= \sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{k-i}{s_1}\binom{k-i}{s_2-s_1}\cdots\binom{k-i}{n-s_r}.\end{aligned}$$

This completes the proof of Proposition 5.1.

We end this section with a remark about Cayley permutations whose ascent set *equals* the set $S = \{s_1, s_2, \dots, s_k\}$. Using the principle of inclusion-exclusion, the number $\lambda_n(S) := |\{w \in \mathrm{Cay}[n] : A(w) = S\}|$ can be expressed in terms of the formula for $\kappa_n(S)$, but the result would be a rather unpleasing triple sum. There is a nice determinant formula for the number $\beta_n(S) = |\{v \in \mathcal{S}[n] : A(v) = S\}|$ of permutations of $[n]$ whose ascent set is equal to S, namely $\beta_n(S) = n!\det\big[1/(s_j - s_{i-1})\big]$. See Stanley [22, p. 229] for further details. We have not been able to find a corresponding formula for $\lambda_n(S)$.

6 Binary Burge matrices

In Section 4, we proved the species identities

$$\mathcal{S} \times \mathfrak{Mat}_m = \big(\mathrm{Bal} \times E(mX)\big) \circ L^c; \tag{13}$$

$$\mathcal{S} \times \mathrm{Mat} = (\mathrm{Bal} \times \mathrm{Bal}) \circ L^c. \tag{14}$$

Here, we aim to define two sign-reversing involutions on the structures of these species whose fixed points are $(\mathcal{S}\times\mathfrak{Mat}_m^{01})$-structures and $(\mathcal{S}\times\mathrm{Mat}^{01})$-structures, respectively. This way we shall obtain new counting formulas for $\mathfrak{Mat}_m^{01}$ and Mat^{01}.

Consider Equation (13) and let (w, A) be an $(\mathcal{S}\times\mathfrak{Mat}_m)$-structure on ℓ. By Lemma 4.3, we can identify (w, A) with the matrix $M = w \cdot A$ obtained by letting w act on A. Following Example 4.4,

$$\text{if } w = 784652391 \text{ and } A = \begin{bmatrix} \cdot & 5 & 67 \\ 123 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 4 & \cdot & 89 \end{bmatrix}, \text{ then } M = \begin{bmatrix} \cdot & 5 & 23 \\ 784 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 6 & \cdot & 91 \end{bmatrix}.$$

Alternatively, in terms of the species on the right-hand side of Equation (13), M is obtained by arranging atoms (i.e., connected linear orders) in a matrix where the column index of each atom is determined by the first component of the cartesian product $\mathrm{Bal} \times E(mX)$, and the row index is determined by the second component. In the matrix M of the previous example, the atoms are $\{78, 4, 6, 5, 23, 9, 1\}$. The same reasoning applies to the species of Equation (14). In a pair $(w, A) \in \mathcal{S}\times\mathrm{Mat}$ the rows of A are, however, required to be nonempty. Hence the row index is determined by a Bal-structure, and we arrive at $(\mathrm{Bal} \times \mathrm{Bal}) \circ L^c$.

Now, let $T = \mathcal{S} \times \mathfrak{Mat}_m = L \circ (L^m)_+ = \big(\mathrm{Bal} \times E(mX)\big) \circ L^c$ be the species of Equation (13). We shall identify a T-structure on ℓ with a matrix in $(L \circ (L^m)_+)\,[\ell]$, but we will keep track of its atoms by separating the atoms in each entry with vertical bars. The usual example matrix M is encoded as

$$M = \begin{bmatrix} \cdot & 5 & 23 \\ 78|4 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 6 & \cdot & 9|1 \end{bmatrix}.$$

Define the sign of $M \in T[n]$ as $\xi(M) = (-1)^{n-k}$, where k is the number of atoms of M. Furthermore, define a mapping $\tau : T[n] \to T[n]$ by letting $\tau(M)$ be the matrix obtained from M in the following manner. Read the entries of M in some canonical order; e.g., read them in the same order as they appear in the concatenation $\prod M$. Relative to this order, find the first entry m_{ij} of M such that $|m_{ij}| \geq 2$ and transpose the first two letters of m_{ij}. If there is no such entry m_{ij}, then let $\tau(M) = M$. The matrix M illustrated previously has seven atoms and sign $\xi(M) = (-1)^{9-7} = 1$. Further,

$$\tau(M) = \begin{bmatrix} \cdot & 5 & 23 \\ 8|7|4 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 6 & \cdot & 9|1 \end{bmatrix}$$

is obtained from M by the transposition $78 \mapsto 8|7$. Since $\tau(M)$ has one more atom than M, the sign is reversed: $\xi(\tau(M)) = -\xi(M)$.

It is clear that τ is an involution on $T[n]$. In particular, M is a fixed point of τ if and only if each entry of M is a linear order of size at most one. If (w, A) is the pair in $(\mathcal{S} \times \mathfrak{Mat}_m)\,[n]$ corresponding to M, this is the same as saying that A is binary. In other words, the set of fixed points of τ corresponds to the pairs in $\big(\mathcal{S} \times \mathfrak{Mat}^{01}_m\big)\,[n]$. On the other hand, if M is not a fixed point, then $\tau(M)$ has the opposite sign of M. Indeed, suppose that τ acts on M by transposing the two letters yz. If $y < z$, then y and z belong to the same atom in M, but to different atoms in $\tau(M)$, and thus $\tau(M)$

has one more atom than M. Conversely, if $y > z$, then $\tau(M)$ has one less atom than M. Therefore, τ is a sign-reversing involution on T and

$$\begin{aligned}|\mathfrak{Mat}_m^{01}[n]| &= \frac{1}{n!}\sum_{M\in\mathrm{Fix}(\tau)}\xi(M)\\ &= \frac{1}{n!}\sum_{M\in T[n]}\xi(M) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\mathrm{fub}(k)m^k.\end{aligned}$$

The same reasoning applies to matrices in T whose every row contains at least one nonempty linear order, that is, to the species $(\mathrm{Bal}\times\mathrm{Bal})\circ L^c$. The signed sum of such matrices of size n is $\sum_{k=0}^{n}(-1)^{n-k}\left[{n \atop k}\right]\mathrm{fub}(k)^2$ and an analogous equation for $|\mathrm{Mat}^{01}[n]|$ follows. We collect the formulas derived in this section in the next proposition.

Proposition 6.1. *For each $n, m \geq 0$,*

$$\begin{aligned}|\mathfrak{Mat}_m^{01}[n]| &= \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\mathrm{fub}(k)m^k;\\ |\mathrm{Mat}^{01}[n]| &= \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\mathrm{fub}(k)^2.\end{aligned}$$

Let $L^c_{(-1)}$ denote the species of *signed connected linear orders*, i.e., connected linear orders $w \in L^c[n]$ with sign $(-1)^{n-1}$. In particular,

$$L^c_{(-1)}(x) = \sum_{n\geq 0}(-1)^{n-1}(n-1)!\frac{x^n}{n!} = \log\left(1+x\right).$$

The sign-reversing involutions that lead to Proposition 6.1 are embodied by the equations in the following proposition (whose straightforward proof we omit).

Proposition 6.2. *For $m \geq 0$,*

$$\begin{aligned}\mathcal{S}\times\mathfrak{Mat}_m^{01} &= \big(\mathrm{Bal}\times E(mX)\big)\circ L^c_{(-1)};\\ \mathcal{S}\times\mathrm{Mat}^{01} &= (\mathrm{Bal}\times\mathrm{Bal})\circ L^c_{(-1)}.\end{aligned}$$

We end this section by noting that the sign-reversing involution τ can be applied to the species $\mathcal{S}\times\mathrm{Mat}^{s,t}$ and that the result is a formula for the strict two-sided Caylerian polynomials that is analogous to Theorem 4.8 for the weak counterpart:

$$\hat{B}^{\circ}_n(s,t) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\big|\mathrm{Bal}^s[k]\big|\big|\mathrm{Bal}^t[k]\big|. \tag{15}$$

A formula for the strict Caylerian polynomials can be obtained in a similar fashion, this time in analogy with Theorem 4.9:

$$C^{\circ}_n(t) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\mathrm{fub}(k)\sum_{i=0}^{k}\begin{Bmatrix}k\\i\end{Bmatrix}i!(t-1)^{n-i}. \tag{16}$$

7 Final remarks

In our recent paper [9], we developed a framework relating the Caylerian polynomials and their variants with Burge matrices and Burge words. Here, using combinatorial species and sign-reversing involutions, we built on the interplay between these structures to obtain several counting formulas and species equations, most of which are collected in tables 1, 2 and 3. For a list of open problems and suggestions for future work, we refer to the same paper [9]. We end with a couple of further remarks and questions.

The OEIS [21] entries for the counting sequences of Burge matrices (A120733) and binary Burge matrices (A101370) contain the generating functions

$$\begin{aligned}\sum_{n\geq 0}|\mathrm{Mat}[n]|x^n &= \sum_{m\geq 0}\frac{1}{2^{m+1}}(\mathcal{S}\times\mathfrak{Mat}_m)(x);\\ \sum_{n\geq 0}|\mathrm{Mat}^{01}[n]|x^n &= \sum_{m\geq 0}\frac{1}{2^{m+1}}(\mathcal{S}\times\mathfrak{Mat}^{01}_m)(x).\end{aligned}$$

Can these two identities be proved with the tools developed here?

Maia and Mendez [16, equation (92)] defined the *modified arithmetic product* of two species F and G as

$$(F\boxast G)\,[U]=\sum_{(\pi,\tau)}F[\pi]\times G[\tau],$$

where the sum ranges over all the partial rectangles (π,τ) over U. Further, they [16, equation (116)] showed that

$$(F\boxast G)\,(x)=\big(F(E_+)\times G(E_+)\big)(x)\circ\log(1+x).$$

The notion of arithmetic product allows us to write $\mathcal{S}\times\mathrm{Mat}^{01}=L\boxast L$, and thus

$$\big(\mathcal{S}\times\mathrm{Mat}^{01}\big)\,(x)=(L\boxast L)\,(x)=(\mathrm{Bal}\times\mathrm{Bal})\,(x)\circ\log(1+x),$$

which gives an alternative path to the second equation of Proposition 6.1. Using another identity of Maia and Mendez [16, Equation (143)], we also get

$$|\mathrm{Mat}^{01}[n]|=\sum_{r,s\geq 0}\frac{1}{2^{r+s+2}}\binom{rs}{n}.$$

We could not find a proof of the corresponding equation for $\mathrm{Mat}[n]$ (A120733 [21]):

$$|\mathrm{Mat}[n]|=\sum_{r,s\geq 0}\frac{1}{2^{r+s+2}}\left(\!\!\binom{rs}{n}\!\!\right).$$

Munarini, Poneti and Rinaldi [18] considered *matrix compositions without zero rows.* In our setting, such matrix compositions of n correspond to Burge matrices in $\mathrm{Mat}[n]$. The interested reader is invited to consult Section 7 of their paper in which they present more enumerative results concerning these matrices.

Burge matrices	**Reference**
$\lvert\mathfrak{Mat}_m[n]\rvert = \sum_{\alpha \models n} \prod_{a \in \alpha} \left(\!\binom{m}{a}\!\right)$	Equation (3)
$\lvert\mathfrak{Mat}_m[n]\rvert = \frac{1}{n!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} m^k \mathrm{fub}(k)$	Proposition 4.6
$\lvert\mathfrak{Mat}_m[n]\rvert = \sum_{k,i} (-1)^i \binom{k}{i} \left(\!\binom{m(k-i)}{n}\!\right)$	Equation (9)
$\lvert\mathfrak{Mat}^{01}_m[n]\rvert = \sum_{\alpha \models n} \prod_{a \in \alpha} \binom{m}{a}$	Proposition 4.10
$\lvert\mathfrak{Mat}^{01}_m[n]\rvert = \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \mathrm{fub}(k) m^k$	Proposition 6.1
$\lvert\mathfrak{Mat}^{01}_m[n]\rvert = \sum_{k,i} (-1)^i \binom{k}{i} \binom{m(k-i)}{n}$	Equation (10)
$\lvert\mathrm{Mat}[n]\rvert = \frac{1}{n!} \sum_{k=0}^{n} \mathrm{fub}(k)^2 \begin{bmatrix} n \\ k \end{bmatrix}$	Proposition 4.7
$\lvert\mathrm{Mat}^{01}[n]\rvert = \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \mathrm{fub}(k)^2$	Proposition 6.1
$\sum_{n \geq 0} \lvert\mathfrak{Mat}_m[n]\rvert x^n = \frac{(1-x)^m}{2(1-x)^m - 1}$	Proposition 4.11
$\sum_{n \geq 0} \lvert\mathfrak{Mat}^{01}_m[n]\rvert x^n = \frac{1}{2-(1+x)^m}$	Proposition 4.11

Table 1: Formulas for (variants of) Burge matrices

Caylerian polynomials	**Reference**
$C_n(t) = \dfrac{1}{n!}\sum_{k,i} \begin{bmatrix} n \\ k \end{bmatrix} \mathrm{fub}(k) \begin{Bmatrix} k \\ i \end{Bmatrix} i!(t-1)^{n-i}$	Theorem 4.9
$C_n^\circ(t) = \dfrac{1}{n!}\sum_{k,i} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \mathrm{fub}(k) \begin{Bmatrix} k \\ i \end{Bmatrix} i!(t-1)^{n-i}$	Equation (16)
$\sum_{n\geq 0} \dfrac{tC_n(t)}{(1-t)^{n+1}} x^n = \sum_{m\geq 1} \dfrac{(1-x)^m}{2(1-x)^m - 1} t^m$	Theorem 4.12
$\sum_{n\geq 0} \dfrac{tC_n^\circ(t)}{(1-t)^{n+1}} x^n = \sum_{m\geq 1} \dfrac{1}{2-(1+x)^m} t^m$	Theorem 4.12
$\hat{B}_n(s,t) = \dfrac{1}{n!}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \lvert\mathrm{Bal}^s[k]\rvert \lvert\mathrm{Bal}^t[k]\rvert$	Theorem 4.8
$\hat{B}_n^\circ(s,t) = \dfrac{1}{n!}\sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \lvert\mathrm{Bal}^s[k]\rvert \lvert\mathrm{Bal}^t[k]\rvert$	Equation (15)
$\kappa_n(S) = \sum_{k,i} (-1)^i \binom{k}{i} \prod_{j=0}^{r} \binom{k-i}{s_{j+1}-s_j}$	Equation (12)
$\kappa_n^\circ(S) = \sum_{k,i} (-1)^i \binom{k}{i} \prod_{j=0}^{r} \left(\!\!\binom{k-i}{s_{j+1}-s_j}\!\!\right)$	Equation (11)

Table 2: Formulas for Caylerian polynomials

Species equation	Reference
$\mathcal{S} \times \mathfrak{Mat}_m = L \circ (L^m - 1)$	Lemma 4.3
$\mathcal{S} \times \mathfrak{Mat}_m^{01} = L \circ \big((1+X)^m - 1\big)$	Proposition 4.10
$\mathcal{S} \times \mathfrak{Mat}_m = \big(\mathrm{Bal} \times E(mX)\big) \circ L^c$	Proposition 4.6
$\mathcal{S} \times \mathfrak{Mat}_m^{01} = \big(\mathrm{Bal} \times E(mX)\big) \circ L^c_{(-1)}$	Proposition 6.2
$\mathcal{S} \times \mathrm{Mat} = (\mathrm{Bal} \times \mathrm{Bal}) \circ L^c$	Proposition 4.7
$\mathcal{S} \times \mathrm{Mat}^{01} = (\mathrm{Bal} \times \mathrm{Bal}) \circ L^c_{(-1)}$	Proposition 6.2
$\mathcal{S} \times \mathrm{Mat}^{s,t} = \big(\mathrm{Bal}^s \times \mathrm{Bal}^t\big) \circ L^c$	Equation (5)

Table 3: Species equations

References

[1] P. Alexandersson and J. Uhlin, *Cyclic sieving, skew Macdonald polynomials and Schur positivity*, Algebraic Combinatorics, Vol. 3(4), pp. 913–939, 2020.

[2] G. E. Andrews, *The theory of partitions*, Cambridge University Press, 1984.

[3] F. Bergeron, G. Labelle, P. Leroux, *Combinatorial species and tree-like structures*, Volume 67 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1998.

[4] W. H. Burge, *Four correspondences between graphs and generalized Young tableaux*, Journal of Combinatorial Theory, Series A, Vol. 17, pp. 12–30, 1974.

[5] L. Carlitz, D. P. Roselle, R. A. Scoville, *Permutations and sequences with repetitions by number of increases*, Journal of Combinatorial Theory, Vol. 1(3), pp. 350–374, 1966.

[6] L. Carlitz, *A Combinatorial property of q-Eulerian numbers*, The American Mathematical Monthly, Vol. 82(1), pp. 51–54, 1975.

[7] A. Cayley, *On the analytical forms called trees, second part*, Philosophical Magazine, Vol. 18, pp. 374–378, 1859.

[8] G. Cerbai and A. Claesson, *Transport of patterns by Burge transpose*, European Journal of Combinatorics, Vol. 108, 2023.

[9] G. Cerbai and A. Claesson, *Caylerian polynomials*, Discrete Mathematics, Vol. 347(12), 2024.

[10] G. Cerbai, A. Claesson, D. Ernst, H. Golab, *Pattern-avoiding Cayley permutations via combinatorial species*, arXiv:2407.19583, 2024.

[11] A. Claesson, *A species approach to Rota's twelvefold way*, Expositiones Mathematicae, 2019.

[12] A. Joyal, *Une théorie combinatoire des séries formelles*, Advances in Mathematics, 42(1):1–82, 1981.

[13] Z. Lin, *Proof of Gessel's γ-positivity conjecture*, The Electronic Journal of Combinatorics, Vol. 23 #P3.15, 2016.

[14] P. A. MacMahon, *Combinatory analysis*, Chelsea, New York, 1960.

[15] P. A. MacMahon, *Yoke-Chains and Multipartite Compositions in connexion with the Analytical forms called "Trees"*, Proceedings of the London Mathematical Society, Vol. s1-22(1), pp. 330–346, 1890.

[16] M. Maia, M. Méndez, *On the arithmetic product of combinatorial species*, Discrete Mathematics, Vol. 308(23), pp. 5407–5427, 2008.

[17] M. Mor and A. S. Fraenkel, *Cayley permutations*, Discrete mathematics, Vol. 48(1), pp. 101–112, 1984.

[18] E. Munarini, M. Poneti, S. Rinaldi, *Matrix Compositions*, Journal of Integer Sequences, Vol. 12(#0948), 2009.

[19] T. K. Petersen, *Two-sided Eulerian numbers via balls in boxes*, Mathematics Magazine, Vol. 86(3), pp. 159–176, 2013.

[20] J. Riordan, P. R. Stein, *Arrangements on chessboards*, Journal of Combinatorial Theory, Series A, Vol. 12(1), pp. 72–80, 1972.

[21] N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, at oeis.org.

[22] R. P. Stanley, *Enumerative combinatorics*, Vol. I, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks, 1986.