

# THE INTERVAL POSETS OF PERMUTATIONS SEEN FROM THE DECOMPOSITION TREE PERSPECTIVE

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**ABSTRACT.** The interval poset of a permutation is the set of intervals of a permutation, ordered with respect to inclusion. It has been introduced and studied recently in [Ten21]. We study this poset from the perspective of the decomposition trees of permutations, describing a procedure to obtain the former from the latter. We then give alternative proofs of some of the results in [Ten21], and we solve the open problems that it posed (and some other enumerative problems) using techniques from symbolic and analytic combinatorics. Finally, we compute the Möbius function on such posets.

## 1. INTRODUCTION

Recently, Bridget Tenner defined the *interval posets* associated with permutations, and described some properties of these posets in [Ten21]. We noted that the so-called *decomposition trees* of permutations, which encode their *substitution decomposition*, provide an alternative way of describing the set of intervals of any permutation. We therefore started a project to investigate the relations between decomposition trees and interval posets. The present paper reports on the results obtained in this project.

While [Ten21] makes use of the substitution decomposition of permutations, it does not mention the associated decomposition trees. Our first contribution is then to complement to the original work [Ten21] by introducing decomposition trees in the picture, and demonstrating the strong relation between decomposition trees and interval posets. Once this has been established, we can use it to solve completely the open questions posed in the final section of [Ten21].

The rest of this paper is organized as follow. In Section 2, we review the necessary definitions regarding the interval posets of permutations, the substitution decomposition, and the resulting decomposition trees. In Section 3, we present a procedure to compute the interval poset of a permutation from its decomposition tree. The procedure allows to provide an answer to Question 7.3 of [Ten21], describing the common properties of all permutations having the same interval poset. We then show in Section 4 how the decomposition trees can be used to give alternative proofs of some of the results of [Ten21] regarding structural properties of interval posets. Next, Section 5 focuses on enumerative properties of interval posets. In addition to solving the two enumerative questions left open in [Ten21] (Questions 7.1 and 7.2), we compute (exactly and asymptotically) the number of interval posets of any size. These results are achieved using the decomposition trees and classical tools from combinatorics of trees. Finally, in Section 6, we compute the Möbius function of any interval of an interval poset.

## 2. BACKGROUND: INTERVAL POSETS AND DECOMPOSITION TREES

**2.1. The interval poset of a permutation.** In the context of this work, a permutation  $\sigma = \sigma_1\sigma_2\dots\sigma_n$  of size  $n$  is a word on alphabet  $\{1, 2, \dots, n\}$  containing exactly once each letter. For example,  $\sigma = 456793128$  is a permutation of size 9. The size of a permutation  $\sigma$  is denoted  $|\sigma|$ .

Intervals in permutations can be defined in several ways (focusing on the indices – a.k.a. positions – in  $\sigma = \sigma_1\sigma_2\dots\sigma_n$  or on the values). Here, we follow the definition of [Ten21], and define an *interval* of a permutation  $\sigma = \sigma_1\sigma_2\dots\sigma_n$  as an interval  $[j, j+h]$  of values (for some  $1 \leq j \leq n-1$  and some  $0 \leq h \leq n-j$ ) which is the image by  $\sigma$  of an interval  $[i, i+h]$  of positions (for some  $1 \leq i \leq n-h$ ). Namely,  $[j, j+h]$  is an interval of  $\sigma$  when there exists an  $i$  satisfying  $\{\sigma_i, \dots, \sigma_{i+h}\} = [j, j+h]$ . The singletons  $\{1\}, \{2\}, \dots, \{n\}$  and the complete interval  $[1, n]$  are always intervals of  $\sigma$ , and are called *trivial*. The empty set is also an interval of  $\sigma$ , although most often we do not include it in the set of intervals of  $\sigma$ . Non-trivial and non-empty intervals are called *proper*. Continuing our example  $\sigma = 456793128$ , its proper intervals are  $[4, 5], [4, 6], [4, 7], [5, 6], [5, 7], [6, 7], [1, 2]$  and  $[1, 3]$ .

The inclusion relation naturally equips the set of intervals (proper ones and trivial ones) with a poset structure: the elements of this poset are the intervals, and the relation is the set inclusion. We can consider two versions of this poset: a first one in which the empty interval is an element (hence, the only minimal element in the poset), and a second one which excludes the empty interval (the minimal elements being then the singletons  $\{1\}$  through  $\{n\}$ ).

While posets are essentially “unordered” objects, we follow [Ten21] and consider particular plane embeddings of these posets.

**Definition 1.** Let  $\sigma$  be a permutation. The (original) interval poset of  $\sigma$ , denoted  $P(\sigma)$ , is the plane embedding of the poset of the non-empty intervals of  $\sigma$  where the minimal elements appear in the order  $\{1\}, \{2\}, \dots, \{n\}$  from left to right.

We denote  $P_\bullet(\sigma)$  the poset of the possibly empty intervals of  $\sigma$  with the same plane embedding:  $P_\bullet(\sigma)$  is just  $P(\sigma)$  with a new minimum smaller than all minimal elements of  $P(\sigma)$ .

The left part of Fig. 1 shows the interval poset of our running example. Clearly, in this figure as well as in general, every element of the poset represents the interval which consists of the set of values below it in the poset.

Deviating from [Ten21], we also wish to consider a second embedding of the interval poset of a permutation, illustrated on the right part of Fig. 1. We believe that this second embedding is also very natural, maybe even more natural than the first one once compared with the decomposition trees of permutations, as we shall see later.

**Definition 2.** Let  $\sigma$  be a permutation. The modified interval poset of  $\sigma$ , denoted  $\tilde{P}(\sigma)$ , is the plane embedding of the poset of the non-empty intervals of  $\sigma$  where the minimal elements appear in the order  $\{\sigma_1\}, \{\sigma_2\}, \dots, \{\sigma_n\}$  from left to right.

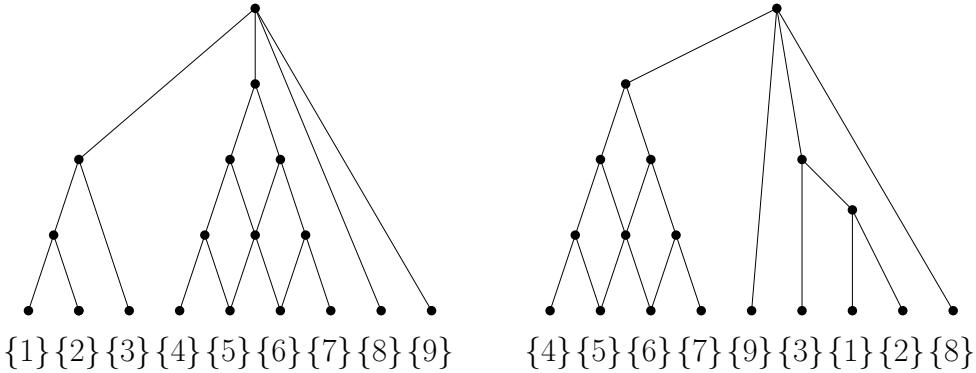


FIGURE 1. From left to right: The original interval poset  $P(\sigma)$ , and the modified interval poset  $\tilde{P}(\sigma)$ , for  $\sigma = 456793128$ .

We also define  $\tilde{P}_\bullet(\sigma)$  adding a new minimum, as in Definition 1.

Although these two embeddings are different, they can be connected by the following remark. It is illustrated comparing the left parts of Figs. 1 and 2.

**Proposition 3.** *For any permutation  $\sigma$ ,  $P(\sigma) = \tilde{P}(\sigma^{-1})$  (and consequently, we also have  $P_\bullet(\sigma) = \tilde{P}_\bullet(\sigma^{-1})$ ).*

*Proof.* First, observe that there is a bijective correspondence between the intervals of  $\sigma$  and those of  $\sigma^{-1}$ . Namely, every interval  $[j, j+h]$  of  $\sigma$  such that  $\{\sigma_i, \dots, \sigma_{i+h}\} = [j, j+h]$  corresponds to the interval  $[i, i+h]$  of  $\sigma^{-1}$  – indeed,  $[i, i+h] = \{\sigma_j^{-1}, \dots, \sigma_{j+h}^{-1}\}$ . Next observe that the inclusion relation is preserved by this correspondence: for  $I$  and  $J$  two intervals of  $\sigma$ , and  $I'$  and  $J'$  the corresponding intervals of  $\sigma^{-1}$ , it holds that  $I \subseteq J$  if and only if  $I' \subseteq J'$ . Therefore, the nonplane poset structures of  $P(\sigma)$  and  $\tilde{P}(\sigma^{-1})$  are identical.

We are left with proving that the plane embeddings of  $P(\sigma)$  and  $\tilde{P}(\sigma^{-1})$  are the same. For this, we identify the minimal elements of  $P(\sigma)$  and  $\tilde{P}(\sigma^{-1})$  by a pair  $(i, j)$  where  $i$  is the position of this element and  $j$  its value. Therefore, in  $P(\sigma)$ , the  $i$ -th minimal element in the left-to-right order has value  $i$  hence corresponds to the pair  $(\sigma_i^{-1}, i)$ . On the other hand, in  $\tilde{P}(\sigma^{-1})$ , the  $i$ -th minimal element in the left-to-right order is at position  $i$  in  $\sigma^{-1}$ , hence also corresponds to the pair  $(\sigma_i^{-1}, i)$ , concluding the proof.  $\square$

In the present paper, we focus on statements regarding the posets  $\tilde{P}(\sigma)$ , but Proposition 3 then of course allows to interpret them on the original posets  $P(\sigma)$  by considering the inverse permutation.

**2.2. Substitution decomposition and decomposition trees.** While interval posets of permutations have been defined and studied only recently, the inclusion relations among the intervals of permutations have been the subject of many studies, in the algorithmic and in the combinatorial literature. In both cases, the set of intervals is

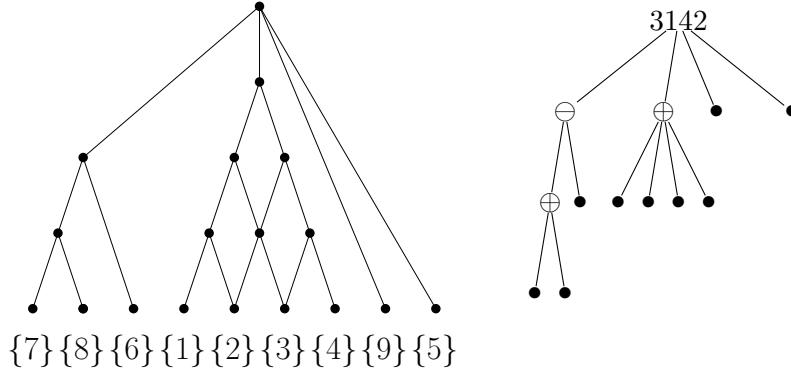


FIGURE 2. From left to right: The interval poset  $\tilde{P}(\sigma^{-1})$ , and the decomposition tree  $T(\sigma^{-1})$ , for  $\sigma = 456793128$ , i.e.,  $\sigma^{-1} = 786123495$ . The substitution decomposition of  $\sigma^{-1}$  is indeed  $\sigma^{-1} = 3142[\ominus[\oplus[1, 1], 1], \oplus[1, 1, 1, 1], 1, 1]$ .

represented by means of a tree, which is called *strong interval tree* or *(substitution) decomposition tree* depending on the context. For historical references, we refer to the introduction of [Vat15, Section 3.2], and to our own work [BMN20, Section 3] which in addition explains the equivalence between the two approaches (we are not aware of many papers which make this equivalence explicit).

Below, we review the definition of these trees, following essentially the combinatorial approach introduced in [AA05]. This is also the approach presented in the survey [Vat15, Section 3.2]. We need to recall some terminology.

A permutation is *simple* if it is of size at least 4 and its only intervals are the trivial ones. For example, there are two simple permutations of size 4 (namely, 2413 and 3142) and a simple permutation of size 7 is 5247316.

Given  $\pi$  a permutation of size  $k$  and  $k$  permutations  $\alpha_1, \dots, \alpha_k$ , the *inflation* of  $\pi$  by  $\alpha_1, \dots, \alpha_k$  (a.k.a. *substitution* of  $\alpha_1, \dots, \alpha_k$  in  $\pi$ ), denoted  $\pi[\alpha_1, \dots, \alpha_k]$ , is the permutation obtained from  $\pi$  by replacing each element  $\pi_i$  by an interval  $I_i$ , such that all elements in  $I_i$  are larger than all elements in  $I_j$  whenever  $\pi_i > \pi_j$ , and such that the elements of  $I_i$  form a subsequence order-isomorphic to  $\alpha_i$ . For instance  $312[12, 231, 4321] = 892317654$ . For any  $k \geq 2$ , we write inflations in  $\pi = 12\dots k$  (resp.  $\pi = k\dots 21$ ) as inflations in  $\oplus$  (resp.  $\ominus$ ), the value of  $k$  being simply determined by the number of components in the inflation. For instance,  $\oplus[1, 3412, 21, 12]$  means  $1234[1, 3412, 21, 12] = 145237689$ . Finally, we say that a permutation  $\sigma$  is  $\oplus$ - (resp.  $\ominus$ )-indecomposable when there does not exist any  $k$  and  $\alpha_i$  such that  $\sigma = \oplus[\alpha_1, \dots, \alpha_k]$  (resp.  $\sigma = \ominus[\alpha_1, \dots, \alpha_k]$ ).

**Theorem 4.** [AA05] *Every permutation  $\sigma$  of size at least 2 can be uniquely decomposed as  $\pi[\alpha_1, \dots, \alpha_k]$  with one of the following satisfied:*

- $\pi$  is simple;
- $\pi = 12\dots k$  for some  $k \geq 2$  and all  $\alpha_i$  are  $\oplus$ -indecomposable;

- $\pi = k \dots 21$  for some  $k \geq 2$  and all  $\alpha_i$  are  $\ominus$ -indecomposable.

The description of  $\sigma$  as  $\pi[\alpha_1, \dots, \alpha_k]$  satisfying the above is called the substitution decomposition of  $\sigma$ .

**Remark 5.** In the first item, we recall our (somewhat unusual) convention that simple permutations are of size at least 4. In the second item, the statement of [AA05] is rather  $\pi = \oplus[\alpha, \beta]$  with only  $\alpha$   $\oplus$ -indecomposable. It is easily seen to be equivalent to our second item above, decomposing recursively inside  $\beta$  until reaching a second component in  $\oplus$  which is itself  $\oplus$ -indecomposable. A similar remark obviously applies to the third statement.

Applying the substitution decomposition recursively inside the  $\alpha_i$  of Theorem 4 until we reach permutations of size 1, we can represent every permutation by a tree, called its *decomposition tree*, which we denote  $T(\sigma)$ . See an example on the right side of Fig. 2.

**Definition 6.** A decomposition tree of size  $n$  is a rooted tree with  $n$  leaves and in which every internal vertex  $v$  satisfies the following:

- either  $v$  is labeled by a simple permutation, whose size is equal to the number of children of  $v$ ,
- or  $v$  is labeled by  $\oplus$  or  $\ominus$  and has at least 2 children.

Decomposition trees are plane, meaning that the children of every internal vertex are ordered from left to right.

In addition, decomposition trees are required to not contain any  $\oplus - \oplus$  nor any  $\ominus - \ominus$  edge.

The following theorem follows immediately from Theorem 4 (the absence of  $\oplus - \oplus$  and  $\ominus - \ominus$  edges echoing the conditions in the second and third items of Theorem 4).

**Theorem 7.** The correspondence between permutations and decomposition trees is a size-preserving bijection. Under this correspondence,  $\sigma_i$  corresponds to the  $i$ -th leaf of  $T(\sigma)$  in the left-to-right order.

We do not discuss here in details the relation between the intervals of a permutation and its decomposition tree. We point out to the interested readers that the nodes of  $T(\sigma)$  are the so-called *strong intervals* of  $\sigma$ . Those are defined as the intervals of  $\sigma$  which do not overlap any other interval of  $\sigma$ , two intervals  $I$  and  $J$  being overlapping when  $I \cap J$  is neither empty nor equal to  $I$  or  $J$ . Details can be found in [BMN20, Section 3] where references are also given.

However, the relation between interval posets and decomposition trees – which we present in the next section – will hopefully clarify the link between decomposition trees and intervals of permutations, even without going back to these references.

### 3. COMPUTING THE POSET FROM THE DECOMPOSITION TREE

We start by a few definitions (illustrated by Fig. 3), and some easy facts from [Ten21].

The *dual claw poset of size k* is the poset with  $k + 1$  elements, one being larger than all others, which are incomparable among them. Note that it has  $k$  minimal elements. It was observed in [Ten21, Proposition 4.3] that it is the interval poset of all simple permutations of size  $k$  (and only those).

The *argyle poset of size k* is the interval poset of the permutation  $12 \dots k$ . It has  $k$  minimal elements. It was observed in [Ten21, Proposition 4.4] that it is the interval poset of exactly two permutations:  $12 \dots k$  and  $k \dots 21$ .

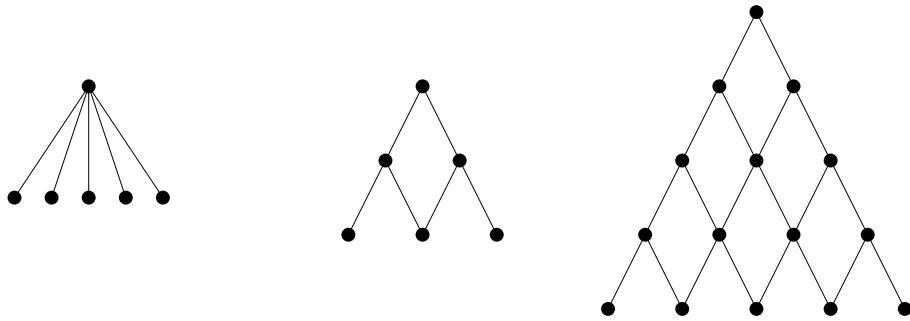


FIGURE 3. From left to right: the dual claw poset of size 5, and the argyle posets of size 3 and 5.

Now, consider the following procedure, which takes as input a permutation  $\sigma$  (or equivalently, its decomposition tree  $T(\sigma)$ ), and returns a plane embedding of a poset (which we denote  $Q(\sigma)$  for the moment).

- (1) If  $\sigma = 1$  (*i.e.*  $T(\sigma) = \bullet$ ), we set  $Q(\sigma)$  to be the poset containing only one element.
- (2) Otherwise, we consider the substitution decomposition  $\sigma = \pi[\alpha_1, \dots, \alpha_k]$  of  $\sigma$ .
- (3) If  $\pi$  is simple, we let  $R$  be the dual claw of size  $k$ . Otherwise (*i.e.* if  $\pi$  is  $\oplus$  or  $\ominus$ ) we denote by  $R$  the argyle poset of size  $k$ .
- (4) We let the minimal elements of  $R$  be  $p_1, \dots, p_k$ , in this order from left to right.
- (5) The poset  $Q(\sigma)$  is obtained by taking  $R$  and replacing, for each  $i \in [1, k]$ ,  $p_i$  by the recursively-obtained poset  $Q(\alpha_i)$ .

We can observe that the auxiliary poset  $R$  in this procedure actually is  $R = \tilde{P}(\pi)$ .

**Proposition 8.** *For every permutation  $\sigma$ ,  $Q(\sigma) = \tilde{P}(\sigma)$ .*

In other words, the interval poset  $\tilde{P}(\sigma)$  of any permutation  $\sigma$  can be obtained from  $T(\sigma)$  by replacing any internal node labeled by a simple permutation by a single element, and any internal node labeled by  $\oplus$  or  $\ominus$  having  $k$  children by several elements arranged in

an argyle poset structure with  $k$  minimal elements. This is illustrated comparing the two pictures of Fig. 2.

We note that it was observed in [Ten21, Section 2] that the substitution decomposition lays the ground work for interval posets. Specifically, Proposition 8 can actually be seen as a rephrasing of [Ten21, Theorem 4.8]. Nevertheless, for completeness, we provide a proof of Proposition 8 below.

Similarly, it was observed in [Ten21, Section 6] that the “separation trees” of separable permutations bear some resemblance with interval posets, although they are not the same. Actually, these separation trees are a restricted version of the decomposition trees in the special case of separable permutations, so our procedure explains precisely this resemblance and allows to go from one representation to the other.

*Proof of Proposition 8.* The proof is by induction on the depth (counted in number of edges) of  $T(\sigma)$ . The statement is clear when  $T(\sigma)$  is of depth 0, *i.e.*,  $T(\sigma) = \bullet$ . So, let us assume that  $T(\sigma)$  is of depth at least 1, and let  $\sigma = \pi[\alpha_1, \dots, \alpha_k]$  be the substitution decomposition of  $\sigma$ . By induction hypothesis, for each  $i$ ,  $Q(\alpha_i) = \tilde{P}(\alpha_i)$ .

If  $\pi$  is simple, then every interval of  $\sigma$  is either  $[1, |\sigma|]$  or included in some  $\alpha_i$ , implying that  $Q(\sigma) = \tilde{P}(\sigma)$ .

If  $\pi$  is  $\oplus$  or  $\ominus$ , the intervals of  $\sigma$  are either included in some  $\alpha_i$  or consist of a union of  $\alpha_i$ 's for a set of consecutive indices  $i$ . Such intervals being represented exactly by the elements of an argyle poset, this implies that  $Q(\sigma) = \tilde{P}(\sigma)$  also in this case.  $\square$

**Remark 9.** With the above procedure and Proposition 8, it is natural to refer to the elements of  $\tilde{P}_\bullet(\sigma)$  as elements of smaller interval posets. More precisely, denoting  $\pi[\alpha_1, \dots, \alpha_k]$  the substitution decomposition of  $\sigma$ , we can see  $\tilde{P}_\bullet(\sigma)$  as the poset obtained by identifying the minimal elements of  $\tilde{P}(\pi)$  with the maxima of the  $\tilde{P}(\alpha_i)$ 's, and then adding a minimum  $\emptyset$ . Therefore we will refer to the elements of  $\tilde{P}_\bullet(\sigma)$  different from  $\emptyset$  using the corresponding elements of  $\tilde{P}(\pi)$  or  $\tilde{P}(\alpha_i)$ .

While the material presented in this section is rather simple and not new, it allows to answer one of the open questions of [Ten21], namely Question 7.3. This question is interested in a description of the shared properties of the *interval generators* of an interval poset. We prefer to call *realizers* these interval generators of [Ten21], in line with the usual terminology in the algorithmic literature.

Given an interval poset  $P$ , a *realizer* of  $P$  is a permutation  $\sigma$  such that  $\tilde{P}(\sigma) = P$ .

Given a decomposition tree  $T$ , we define its *skeleton*  $sk(T)$  as the plane tree with the same set of nodes and the same genealogy, but where internal nodes have different labels. An internal node of  $sk(T)$  is labeled *prime* (*resp. linear*) when its label in  $T$  is a simple permutation (*resp.  $\oplus$  or  $\ominus$* ).

**Proposition 10.** *Let  $P$  be an interval poset, and  $\sigma$  be any permutation such that  $\tilde{P}(\sigma) = P$ . Then a permutation  $\tau$  is a realizer of  $P$  if and only if  $sk(T(\tau)) = sk(T(\sigma))$ .*

*Proof.* It follows immediately from Proposition 8 and the description of the procedure computing  $Q(\sigma)$ .  $\square$

For example, this allows to compute the set of realizers of the poset (without labels on the minimal elements) displayed in Fig. 2, left. This set is obtained by considering the skeleton of the tree shown in Fig. 2, right, and by considering all possible labels for its prime root (here 2413 or 3142), all possible labels for the second child of the root (here  $\oplus$  or  $\ominus$ ), and all possible labels on the left branch starting at the first child of the root (here  $\oplus - \ominus$  or  $\ominus - \oplus$ , keeping in mind that decomposition trees do not contain any  $\oplus - \oplus$  nor  $\ominus - \ominus$  edge). This yields the following set of eight realizers:  $\{342678915, 324678915, 342987615, 324987615, 786123495, 768123495, 786123495, 768123495\}$ .

We note that this idea was already present in [Ten21, Theorem 5.1], but only with the purpose of computing the *number* of realizers of an interval poset. Of course, the point of view of decomposition trees allows to provide an alternative proof of [Ten21, Theorem 5.1], although we do not provide details here, since they are very close to the proof of Theorem 4.1 in [Ten21].

Some of the results of [Ten21] can be seen as consequences (or special cases) of Proposition 10.

**Corollary 11.** *The following claims hold.*

- For every  $k \geq 4$ , the dual claw poset of size  $k$  is the poset of all simple permutations of size  $k$ , and only those (see [Ten21, Proposition 4.3]).
- For every  $k \geq 2$ , the argyle poset of size  $k$  is the poset of exactly two permutations:  $12\dots k$  and  $k\dots 21$ . (see [Ten21, Proposition 4.4]).
- For every permutation  $\sigma = \sigma_1\sigma_2\dots\sigma_n$ , denoting  $\sigma^R$  its reverse  $\sigma_n\dots\sigma_2\sigma_1$ , we have  $P(\sigma^R) = P(\sigma)$  (see [Ten21, Lemma 2.5]).

*Proof.* For the first item, we simply observe that the simple permutations  $\sigma$  of size  $k$  are exactly those such that  $sk(T(\sigma))$  consists of a prime node at the root with only  $k$  leaves pending under it.

Similarly, for the second item, we observe that  $sk(T(\sigma))$  consists of a linear node at the root with only  $k$  leaves pending under it if and only if  $\sigma = 12\dots k$  or  $k\dots 21$ .

For the third item, we first recall that for all  $\sigma$ ,  $P(\sigma^R) = \tilde{P}((\sigma^R)^{-1})$  by Proposition 3. Second, we observe that  $(\sigma^R)^{-1} = (\sigma^{-1})^C$  where for any permutation  $\pi = \pi_1\pi_2\dots\pi_n$ ,  $\pi^C$  denotes the complement  $(n+1-\pi_1)(n+1-\pi_2)\dots(n+1-\pi_n)$  of  $\pi$ . So, our claim is equivalent to showing that  $\tilde{P}(\pi) = \tilde{P}(\pi^C)$  for all  $\pi$ . And this holds since  $T(\pi^C)$  is obtained from  $T(\pi)$  by complementing each simple permutation at a node, and by changing each  $\oplus$  (resp.  $\ominus$ ) into a  $\ominus$  (resp.  $\oplus$ ), noting that these operations do not affect the prime or linear labels of the nodes of  $sk(T(\pi))$ .  $\square$

In addition, [Ten21, Theorem 4.8] states that interval posets are the posets which can be constructed starting from the 1-element poset, and recursively replacing minimal elements with dual claw posets, argyle posets or binary tree posets (defined in [Ten21, Definition 4.2]). Our Proposition 8 states that interval posets are those that can be constructed starting from the 1-element poset, and recursively replacing minimal elements with dual claw posets or argyle posets. Since binary tree posets can straightforwardly be obtained from the 1-element poset recursively replacing minimal elements with argyle posets with two minimal elements, Proposition 8 therefore implies [Ten21, Theorem 4.8]. It also allows to identify more clearly which “building blocks” are needed, namely dual claw posets and argyle posets, without explicitly needing binary tree posets.

#### 4. ALTERNATIVE PROOFS OF KNOWN STRUCTURAL RESULTS

In this section, we review several structural properties of the interval posets, which were already proved in [Ten21]. We believe that the approach through decomposition trees allows to provide more straightforward proofs of these statements.

We briefly recall some classical terminology regarding properties of posets. More details can be found in [Sta11]. Let  $P$  be a generic poset, whose partial order is denoted by  $\leq$ .

For  $a$  and  $b$  be two elements of  $P$ , we say that  $a$  *covers*  $b$  when  $b < a$  and there is no element  $c$  of  $P$  such that  $b < c < a$ . The *Hasse diagram* of  $P$  is a drawing of the graph whose vertices are the elements of  $P$ , and whose edges are the covering relations in  $P$ , with  $a$  being drawn higher than  $b$  whenever  $b < a$ . We say that  $P$  is *planar* when its Hasse diagram can be drawn in such a way that no two edges cross.

For any two elements  $a$  and  $b$  in  $P$ , their *meet* (resp. *join*) denoted  $a \vee b$  (resp.  $a \wedge b$ ) is the smallest element  $c$  such that  $a \leq c$  and  $b \leq c$ . (resp. the largest element  $c$  such that  $c \leq a$  and  $c \leq b$ ), if such an element exists. We say that  $P$  is a *lattice* when, for any two elements  $a$  and  $b$  of  $P$ , both  $a \vee b$  and  $a \wedge b$  exist. We also say that  $P$  is *modular* when, for any two elements  $a$  and  $b$  of  $P$ , they both cover  $a \wedge b$  if and only if  $a \vee b$  covers them both.

**Theorem 12.** ([Ten21, Theorem 3.2]) *For every permutation  $\sigma$ , the posets  $\tilde{P}(\sigma)$  and  $\tilde{P}_\bullet(\sigma)$  are planar.*

*Proof.* The proof heavily relies on Proposition 8 and the computation of  $\tilde{P}(\sigma)$  from the decomposition tree  $T(\sigma)$  of  $\sigma$  which is the core of Section 3. Clearly, for any permutation  $\sigma$ , the tree  $T(\sigma)$  is planar, and the transformations performed to obtain  $\tilde{P}(\sigma)$  from it maintain the planar property. This shows that  $\tilde{P}(\sigma)$  is planar (in addition with a planar drawing where all minimal elements can be placed on a horizontal line at the bottom of the picture). Since  $\tilde{P}_\bullet(\sigma)$  is obtained by adding to  $\tilde{P}(\sigma)$  a new minimum smaller than all minimal elements of  $P(\sigma)$ , the above ensures that  $\tilde{P}_\bullet(\sigma)$  is also planar.  $\square$

**Theorem 13.** ([Ten21, Theorem 3.3]) *For every permutation  $\sigma$ , the poset  $\tilde{P}_\bullet(\sigma)$  is a lattice.*

*Proof.* First, we note the following fact: if  $I$  and  $J$  are two elements of some poset  $\tilde{P}_\bullet(\sigma)$  such that  $I \subseteq J$  we have  $I \wedge J = I$  and  $I \vee J = J$ . We shall use this fact repeatedly, particularly when one of  $I$  or  $J$  is  $\emptyset$ .

Now, we prove the statement by structural induction on the substitution decomposition of  $\sigma$ .

If  $\sigma = 1$ , our claim follows immediately from the above fact (since the only two elements of  $\tilde{P}_\bullet(\sigma)$  are  $\{1\}$  and  $\emptyset$ ).

If  $\sigma$  is a simple permutation, then  $\tilde{P}_\bullet(\sigma)$  is a dual claw poset with an added minimum  $\emptyset$ . Clearly, any pair of elements have a meet and a join in this poset.

If  $\sigma$  is increasing or decreasing then  $\tilde{P}_\bullet(\sigma)$  is an argyle poset with an added minimum. The elements of this poset correspond to the intervals  $[a, b]$  for  $1 \leq a \leq b \leq |\sigma|$ , with the addition of  $\emptyset$ . Obviously, for all such intervals,  $[a, b] \vee [a', b'] = [\min(a, a'), \max(b, b')]$  and  $[a, b] \wedge [a', b'] = [\max(a, a'), \min(b, b')]$  with the convention that  $[x, y] = \emptyset$  whenever  $x > y$ . Using also the fact observed at the beginning of the proof in the case that one of the considered element is the emptyset, it follows that for increasing or decreasing permutations  $\sigma$ ,  $\tilde{P}_\bullet(\sigma)$  is a lattice.

Otherwise, we consider the substitution decomposition  $\pi[\alpha_1, \dots, \alpha_k]$  of  $\sigma$ , for which it holds that  $\pi \neq \sigma$ . Note that in this case we can apply the induction hypothesis to  $\pi$  as well as to each  $\alpha_i$ .

Let  $I$  and  $J$  be two elements of  $\tilde{P}_\bullet(\sigma)$ . If  $I$  or  $J$  is  $\emptyset$ , then  $I \vee J$  and  $I \wedge J$  exist from the fact noted earlier. Therefore, let us assume that  $I \neq \emptyset$  and  $J \neq \emptyset$ .

If  $I$  and  $J$  are elements of  $\tilde{P}(\pi)$ , we have  $I \wedge J$  and  $I \vee J$  in  $\tilde{P}_\bullet(\pi)$  through the induction hypothesis. Then,  $I \vee J$  is unchanged in  $\tilde{P}_\bullet(\sigma)$  and  $I \wedge J$  also stays unchanged, unless it is the minimal element ( $\emptyset$ ) of  $\tilde{P}_\bullet(\pi)$ . In this case, since there is no relation between the  $\tilde{P}(\alpha_i)$ , we have  $I \wedge J$  is the minimal element ( $\emptyset$ ) of  $\tilde{P}_\bullet(\sigma)$ .

If  $I$  and  $J$  are elements of the same subposet  $\tilde{P}(\alpha_i)$ , we have  $I \wedge J$  and  $I \vee J$  in  $\tilde{P}_\bullet(\sigma)$  through the induction hypothesis similarly to the previous case.

Otherwise,  $I$  and  $J$  belong to different subposets  $\tilde{P}(\alpha_i)$ , and we define  $I_\pi$  and  $J_\pi$  the smallest elements of  $\tilde{P}(\pi)$  that contain respectively  $I$  and  $J$ . By transitivity, we have  $I \vee J = I_\pi \vee J_\pi$  which we know exists due to the cases considered earlier. As for  $I \wedge J$ ,  $I \cap J$  is empty, and thus  $I \wedge J$  is the minimal element ( $\emptyset$ ) of  $\tilde{P}_\bullet(\sigma)$ .

This concludes our inductive proof that for any permutation  $\sigma$ ,  $\tilde{P}_\bullet(\sigma)$  is a lattice.  $\square$

**Theorem 14.** ([Ten21, Theorem 3.5]) *For every permutation  $\sigma$ , the poset  $\tilde{P}_\bullet(\sigma)$  is modular if and only if  $\sigma$  is a simple permutation or 1 or 12 or 21.*

While the proof of this theorem in [Ten21] is based on a characterization of modularity by sublattice avoidance, our proof relies only on the definition of a modular lattice.

*Proof.* First, let  $\sigma$  be a permutation whose decomposition tree  $T$  has depth at least 2. Denote by  $\pi[\alpha_1, \dots, \alpha_k]$  the substitution decomposition of  $\sigma$ . Let  $I$  be an interval (of size 1) of  $\sigma$  corresponding to a leaf of  $T$  at maximal depth. Let  $i$  be the index such that  $I$  lies in  $\tilde{P}(\alpha_i)$ . Of course,  $I$  is a minimal element of  $\tilde{P}(\sigma)$ . Let  $J$  be another minimal element of  $\tilde{P}(\sigma)$  which lies in  $\tilde{P}(\alpha_j)$  for some  $j \neq i$ .

Then, in  $\tilde{P}_\bullet(\sigma)$ ,  $I \wedge J = \emptyset$ , which they cover. Defining  $p_i$  and  $p_j$  as the maximal elements of  $\tilde{P}(\alpha_i)$  and  $\tilde{P}(\alpha_j)$  respectively, we have  $I \vee J = p_i \vee p_j$  (as in the proof of Theorem 13). It is possible that  $p_i \vee p_j$  covers  $p_i$  and  $p_j$ , and it is possible that  $p_j = J$ . However, because  $I$  corresponds to a leaf of depth at least 2 in  $T$ , it holds that  $p_i \neq I$ . Therefore  $I \vee J$  does not cover  $I$ , and  $\tilde{P}_\bullet(\sigma)$  cannot be modular in this case.

Now, assume that  $\sigma = 12\dots k$  or  $k\dots 21$  for some  $k \geq 3$ . Consequently,  $\tilde{P}(\sigma)$  is an argyle poset with  $k \geq 3$  minimal elements. Taking  $I = \{1\}$  and  $J = \{k\}$ , we see that they both cover their joint  $\emptyset$  in  $\tilde{P}_\bullet(\sigma)$ . However, their meet is  $[1, k]$  which does not cover them due to the argyle structure itself.

We are left with the cases  $\sigma = 1, 12, 21$  or  $\sigma$  is simple. In such cases, denoting  $n = |\sigma|$ , observe that all elements in  $\tilde{P}_\bullet(\sigma)$  are either  $\emptyset$ ,  $[1, n]$ , or cover the former while being covered by the latter. Therefore,  $\tilde{P}_\bullet(\sigma)$  is modular, concluding the proof.  $\square$

Finally, in [Ten21, Theorem 3.11] it is shown that  $\tilde{P}_\bullet(\sigma)$  is distributive if and only if  $\sigma$  is 1 or 12 or 21. For this particular statement, decomposition trees do not allow for a proof substantially different from [Ten21], so we leave this property outside of the present work.

## 5. ENUMERATIVE PROPERTIES OF INTERVAL POSETS

Proposition 10 allows us to identify interval posets with trees in a certain family. Specifically, the following holds.

**Corollary 15.** *Interval posets with  $n$  minimal elements are in bijection with trees of the form  $sk(T)$  for  $T$  a decomposition tree with  $n$  leaves.*

This perspective on interval posets is very useful to derive enumeration results on interval posets, using classical tools from tree enumeration.

**5.1. The number of realizers of a given interval poset.** Section 5 of [Ten21] is interested in computing the number of realizers of a given interval poset. The statement proved in [Ten21] can be rephrased in terms of decomposition trees, and we state it here for completeness. The proof is straightforward from the results of our Section 3, and this is essentially how the statement is proved in [Ten21] (although the language is a little bit different). Therefore, the statement is given without proof here.

**Theorem 16.** ([Ten21, Theorem 5.1]) *Let  $P$  be an interval poset. Denote by  $rl(P)$  the number of realizers of  $P$ , that is to say the number of permutations  $\sigma$  such that  $P = \tilde{P}(\sigma)$ .*

Let  $\sigma$  be one permutation such that  $P = \tilde{P}(\sigma)$ , and let  $T$  be the skeleton of the decomposition tree of  $\sigma$ . Then

$$rl(P) = \prod_{v \text{ non-leaf vertex of } T} rl(v)^{\varepsilon_v},$$

where the  $rl(v)$  are given by

$$rl(v) = \begin{cases} 2 & \text{if } v \text{ is linear,} \\ \text{number of simple permutations of size } k & \text{if } v \text{ is prime with } k \text{ children,} \end{cases}$$

and the exponents  $\varepsilon_v$  are given by

$$\varepsilon_v = \begin{cases} 1 & \text{if } v \text{ is the root of } T, \\ 1 & \text{if } v \text{ is prime,} \\ 1 & \text{if } v \text{ is linear with a prime parent,} \\ 0 & \text{if } v \text{ is linear with a linear parent.} \end{cases}$$

Equivalently,  $\varepsilon_v$  is 0 if  $v$  is linear with a linear parent, 1 otherwise.

Note that the last case in the definition of  $\varepsilon_v$  echoes the fact that there are no edges between two linear nodes with the same  $\oplus$  or  $\ominus$  labels in decomposition trees, leaving therefore no choice for the label of a linear node whose parent is also linear.

**5.2. Interval posets with exactly two realizers.** In Question 7.2 of [Ten21], B. Tenner asked the following: how many interval posets have exactly two realizers? We solve this question, and give a precise description of these interval posets. To state our results, we need some terminology.

A permutation is *separable* if it avoids the patterns 2413 and 3142. The definition of pattern-avoidance is omitted here (the reader can for example refer to [Vat15]), because separable permutations enjoy several other characterizations, including some which are more adapted for our purpose. Specifically, a permutation is separable if and only if it can be obtained from permutations of size 1 by repeated applications of the operations  $\oplus$  and  $\ominus$ . This is essentially how separable permutations were first considered in [BBL98], by means of their “separating trees”. With the point of view of substitution decomposition used throughout this paper, the characterization of [BBL98] is then equivalent to saying that a permutation is separable if and only if its decomposition tree contains only  $\oplus$  and  $\ominus$  nodes.

Separable permutations made an appearance in [Ten21], where Theorem 6.2 states that an interval poset  $P(\sigma)$  is binary if and only if  $\sigma$  is separable (an interval poset being by definition binary when it does not contain any dual claw with more than two minimal elements). Since separable permutations are stable by taking the inverse, it follows from Proposition 3 that also  $\tilde{P}(\sigma)$  is binary if and only if  $\sigma$  is separable.

The answer to Question 7.2 of [Ten21] actually also involves separable permutations.

**Theorem 17.** *An interval poset  $P = \tilde{P}(\sigma)$  is such that  $P$  has exactly two realizers if and only if  $\sigma$  is a separable permutation of size at least 2, or  $\sigma = 2413$  or  $3142$ .*

*As a consequence, the number  $a_n$  of interval posets with exactly two realizers is given by the sequence  $a_1 = 0$ ,  $a_2 = s_2 = 1$ ,  $a_3 = s_3 = 3$ ,  $a_4 = s_4 + 1 = 12$  and  $a_n = s_n$  for all  $n \geq 5$ , with  $s_n$  the  $n$ -th little Schröder number (see [SI20, sequence A001003]).*

*Proof.* Consider an interval poset  $P = \tilde{P}(\sigma)$ . We also denote by  $T$  the decomposition tree of  $\sigma$ .

Of course, if  $\sigma = 1$ , then 1 is the only realizer of  $P$ .

Since there are more than two simple permutations of any size at least 5, if  $T$  contains a prime node with at least 5 children, then  $P$  has more than two realizers.

If  $T$  contains a prime node  $v$  with 4 children, then we have two choices for the simple permutation labeling  $v$ , namely 2413 and 3142. This shows that  $P$  has two realizers when  $\sigma = 2413$  or  $3142$ . However, if  $T$  contains some other internal node  $u \neq v$ , then we can also choose (among at least two possibilities) the label of  $u$  (may  $u$  be prime or linear). Therefore, in such cases,  $P$  has more than two realizers.

We are left with the case where  $\sigma \neq 1$  is separable, corresponding to  $T \neq \bullet$  containing only linear nodes. The skeleton of  $T$  together with the  $\oplus$  or  $\ominus$  label of the root determines  $T$  (and hence  $\sigma$ ) completely, because  $\oplus$  and  $\ominus$  labels are required to alternate in decomposition trees. Therefore  $P$  has exactly two realizers:  $\sigma$  and the permutation whose decomposition tree is  $T$  with all  $\oplus$  changed into  $\ominus$  and conversely (which is none other than the complement of  $\pi$  – see the proof of Corollary 11, third item).

We now turn to the proof of the second part of our statement. The fact that  $a_1 = 0$  follows from the particular case of  $\sigma = 1$  above. Otherwise, except when  $n = 4$ ,  $a_n$  is half the number of separable permutations of size  $n$ . The number of separable permutations of size  $n$  is the  $n$ -th large Schröder number, whose half is called little Schröder number. In the particular case  $n = 4$ , we need to add 1 to  $s_n$ , to account for the dual claw poset with 4 minimal elements, whose two realizers are 2413 and 3142.  $\square$

**5.3. Counting interval posets.** Although the question of counting interval posets with  $n$  minimal elements was neither studied nor posed in [Ten21], we believe it is natural. We answer this question completely in this subsection.

Let  $\mathcal{P}$  be the family of rooted plane trees, where internal nodes carry a type which is either prime or linear, in which the size is defined as the number of leaves, and in which the number of children of any linear (resp. prime) node is at least 2 (resp. at least 4). Let  $\mathcal{P}_n$  be the set of trees of size  $n$  in  $\mathcal{P}$ . Clearly,  $\mathcal{P}_n$  is the set of skeletons of decomposition trees of permutations of size  $n$ , and from Corollary 15 we can identify  $\mathcal{P}_n$  with the set of interval posets with  $n$  minimal elements.

We now use the approach of symbolic combinatorics (see [FS08, Part A] for example) to obtain the enumeration of trees in  $\mathcal{P}$ , or equivalently of interval posets.

By definition of  $\mathcal{P}$ , it follows that  $\mathcal{P}$  satisfies the following combinatorial specification, where  $\bullet$  denotes a leaf,  $\uplus$  is the disjoint union, and  $Seq_{\geq k}$  is the sequence operator restricted to sequences of at least  $k$  components:

$$\mathcal{P} = \bullet \uplus Seq_{\geq 2}(\mathcal{P}) \uplus Seq_{\geq 4}(\mathcal{P}).$$

Indeed, the first term corresponds to the tree consisting of a single leaf, the second to trees with a linear root, and the third to trees with a prime root.

Denoting  $p_n$  the cardinality of  $\mathcal{P}_n$ , we let  $P(z) = \sum_{n \geq 0} p_n z^n$  be the ordinary generating function of  $\mathcal{P}$ . The combinatorial specification above indicates that  $P(z)$  satisfies the following equation:

$$(1) \quad P(z) = z + \frac{P(z)^2}{1 - P(z)} + \frac{P(z)^4}{1 - P(z)}.$$

Equivalently, this can be rewritten as

$$(2) \quad P(z) = z\phi(P(z)) \text{ with } \phi(u) = \frac{1}{1 - u \left( \frac{1+u^2}{1-u} \right)}.$$

From there, the Lagrange inversion formula (see, e.g. [FS08, Theorem A.2]) can be applied to obtain an explicit formula for  $p_n$ .

**Theorem 18.** *The number  $p_n$  of interval posets with  $n$  minimal elements is*

$$p_n = \begin{cases} 1 & \text{if } n = 1, \\ \frac{1}{n} \sum_{i=1}^{n-1} \sum_{k=0}^{\min\{i, \frac{n-i-1}{2}\}} \binom{n+i-1}{i} \binom{i}{k} \binom{n-2k-2}{i-1} & \text{if } n > 1. \end{cases}$$

The first terms of this sequence (starting from  $p_1$ ) are 1, 1, 3, 12, 52, 240, 1160, 5795, 29681. We contributed this sequence to the OEIS [SI20], where it is now sequence A348479.

*Proof.* Applying the Lagrange inversion theorem to Eq. (2), we have  $p_n = \frac{1}{n}[u^{n-1}]\phi(u)^n$ . In our computation of  $\phi(u)^n$ , we make use of the following identity, valid for any  $n \geq 1$ :

$$(3) \quad \left( \frac{1}{1-z} \right)^n = \sum_{i \geq 0} \binom{n+i-1}{i} z^i.$$

We derive

$$\begin{aligned}
\phi(u)^n &= \left( \frac{1}{1-u \left( \frac{1+u^2}{1-u} \right)} \right)^n = \sum_{i \geq 0} \binom{n+i-1}{i} u^i \left( \frac{1+u^2}{1-u} \right)^i \\
&= \sum_{i \geq 0} \binom{n+i-1}{i} u^i (1+u^2)^i \left( \frac{1}{1-u} \right)^i \\
&= \sum_{i \geq 0} \binom{n+i-1}{i} u^i \sum_{k=0}^i \binom{i}{k} u^{2k} \left( \frac{1}{1-u} \right)^i \\
(4) \quad &= 1 + \sum_{i>0} \binom{n+i-1}{i} u^i \sum_{k=0}^i \binom{i}{k} u^{2k} \sum_{j \geq 0} \binom{i+j-1}{j} u^j \\
(5) \quad &= 1 + \sum_{i>0} \sum_{k=0}^i \sum_{j \geq 0} \binom{n+i-1}{i} \binom{i}{k} \binom{i+j-1}{j} u^{i+2k+j}.
\end{aligned}$$

The reason why we isolate the term for  $i = 0$  in Eq. (4) is in order to apply Eq. (3) with a positive power of  $\frac{1}{1-u}$ .

We now want to compute  $[u^{n-1}] \phi(u)^n$ . Since  $p_1 = 1$  is obvious, we can assume  $n > 1$ . The exponent of  $u$  in Eq. (5) is  $i+2k+j$ , so we want  $n-1 = i+2k+j$ , i.e.,  $j = n-i-2k-1$ . Since  $j \geq 0$ ,  $i$  cannot be greater than  $n-1$ , while  $k$  cannot be greater than  $\frac{n-i-1}{2}$ . Since  $k$  is also at most  $i$ , we have  $k \leq \min\{i, \frac{n-i-1}{2}\}$ . Therefore

$$p_n = \frac{1}{n} [u^{n-1}] \phi(u)^n = \frac{1}{n} \sum_{i=1}^{n-1} \sum_{k=0}^{\min\{i, \frac{n-i-1}{2}\}} \binom{n+i-1}{i} \binom{i}{k} \binom{n-2k-2}{n-i-2k-1}.$$

To conclude the proof we just note that  $\binom{n-2k-2}{n-i-2k-1} = \binom{n-2k-2}{i-1}$ .  $\square$

From Eq. (1), applying the methods of analytic combinatorics (see [FS08, Part B]), we can also derive the asymptotic behavior of  $p_n$ .

**Theorem 19.** *Let  $\Lambda$  be the function defined by  $\Lambda(u) = \frac{u^2+u^4}{1-u}$ .*

*The radius of convergence  $\rho$  of the generating function  $P(z)$  of interval posets is given by  $\rho = \tau - \Lambda(\tau)$ , where  $\tau$  is the unique solution of  $\Lambda'(u) = 1$  such that  $\tau \in (0, 1)$ .*

*The behavior of  $P(z)$  near  $\rho$  is given by*

$$P(z) = \tau - \sqrt{\frac{2\rho}{\Lambda''(\tau)}} \sqrt{1 - \frac{z}{\rho}} + \mathcal{O}\left(1 - \frac{z}{\rho}\right).$$

*Numerically, we have  $\tau \approx 0.2708$ ,  $\rho \approx 0.1629$ ,  $\sqrt{\frac{2\rho}{\Lambda''(\tau)}} \approx 0.2206$ .*

As a consequence, the number  $p_n$  of interval posets with  $n$  minimal elements satisfies, as  $n \rightarrow \infty$ ,

$$p_n \sim \sqrt{\frac{\rho}{2\pi\Lambda''(\tau)}} \frac{\rho^{-n}}{n^{3/2}}.$$

Numerically, we have  $\sqrt{\frac{\rho}{2\pi\Lambda''(\tau)}} \approx 0.0622$ ,  $\rho^{-1} \approx 6.1403$ .

*Proof.* To prove this theorem we just need to prove that  $\Lambda$  satisfies the hypothesis of [BMN20, Theorem 1], which is an adaptation of [FS08, Proposition IV.5 and Theorem VI.6] to the setting where trees are counted by the number of *leaves* (as opposed to the more classical counting by the number of *nodes*). Specifically, we can immediately see from their definitions that  $\Lambda$  is analytic at 0, has non-negative Taylor coefficients, and has radius of convergence 1, and that  $P(z)$  is aperiodic. Finally, since  $\lim_{u \rightarrow 1} \Lambda'(u) = +\infty > 1$ , the result follows immediately from [BMN20, Theorem 1].  $\square$

We note that the classical theorems [FS08, Proposition IV.5 and Theorem VI.6] could also have been applied to obtain Theorem 19, starting from Eq. (2) instead of Eq. (1). (And we can check that both approaches indeed yield the same results.)

**Remark 20.** Remember from Definition 1 that interval posets are defined as *plane embeddings* of some posets. It would also make sense to consider *non-plane* versions of these posets, defining a non-plane interval poset with  $n$  minimal elements simply as the poset of the non-empty intervals of a permutation of size  $n$ . Following the same approach to put in correspondence interval posets with trees, it follows that the family  $\mathcal{Q}$  of non-plane interval posets satisfies the following combinatorial specification, where  $MSet_{\geq k}$  is the multi-set operator restricted to multi-sets of at least  $k$  components:

$$\mathcal{Q} = \bullet \uplus MSet_{\geq 2}(\mathcal{Q}) \uplus MSet_{\geq 4}(\mathcal{Q}).$$

While this specification can be translated on generating functions, the resulting equation involves Pólya operators, making its resolution much harder, even if just numerically. Nevertheless, from this equation, it can be proved that the sequence  $(q_n)$  enumerating non-plane interval posets behaves asymptotically like  $\alpha n^{-3/2} \beta^n$ , following the approach of [FS08, Section VII.5] or [HRS75]. Iterating the equation for the generating function, we computed the first 400 terms of the sequence  $(q_n)$ , which allowed to find loose numerical estimates for  $\alpha$  and  $\beta$  as  $\tilde{\alpha} = 0.1964$  and  $\tilde{\beta} = 3.7545$ .

**5.4. Counting tree interval posets.** A *tree interval poset* is an interval poset which is a tree. Of course, this definition applies to interval posets  $P(\sigma)$  (or  $\tilde{P}(\sigma)$ ) since intervals posets  $P_\bullet(\sigma)$  (or  $\tilde{P}_\bullet(\sigma)$ ) are never trees.

Put in our language, [Ten21, Theorem 6.1] characterizes the tree interval posets as the  $\tilde{P}(\sigma)$  such that the substitution decomposition of  $\sigma$  does not involve any  $\oplus$  or  $\ominus$  with at least three components. This result can be recovered from the procedure described in Section 3, where we can see that an interval poset  $\tilde{P}(\sigma)$  is a tree if and only if the decomposition tree of  $\sigma$  has no linear node with more than two children. Indeed, every internal node of the tree is substituted with a dual claw or an argyle poset, and the

resulting poset is a tree if and only if the substituted posets are themselves trees. Now, among the posets which can be substituted, the only ones that are not trees are the argyle posets with more than two minimal elements. Consequently, for  $\tilde{P}(\sigma)$  to be a tree, the decomposition tree of  $\sigma$  must be free of linear nodes with more than two children.

In [Ten21, Question 7.1], B. Tenner also asks how many tree interval posets have  $n$  minimal elements. We solve this question using the same techniques as the previous subsection, giving a closed formula for the number of tree interval posets with  $n$  minimal elements and its asymptotic behavior.

Let  $\mathcal{T}$  be the family of rooted plane trees, where internal nodes carry a type which is either prime or linear, in which the size is defined as the number of leaves, and in which the number of children of any linear node is exactly 2, while the number of children of any prime node is at least 4. Let  $\mathcal{T}_n$  be the set of trees of size  $n$  in  $\mathcal{T}$ . Clearly,  $\mathcal{T}_n$  is the set of skeletons of decomposition trees of permutations of size  $n$  whose interval poset is a tree, and we can identify  $\mathcal{T}_n$  with the set of tree interval posets with  $n$  minimal elements.

We now enumerate trees in  $\mathcal{T}$  using the approach of symbolic combinatorics in the same fashion as we did to enumerate trees in  $\mathcal{P}$ .

Like in the previous subsection, we derive that  $\mathcal{T}$  satisfies the following combinatorial specification, where  $\bullet$  denotes a leaf,  $\uplus$  is the disjoint union,  $\times$  is the Cartesian product, and  $Seq_{\geq k}$  is the sequence operator restricted to sequences of at least  $k$  components:

$$\mathcal{T} = \bullet \uplus (\mathcal{T} \times \mathcal{T}) \uplus Seq_{\geq 4}(\mathcal{T}).$$

Denoting  $t_n$  the cardinality of  $\mathcal{T}_n$ , we let  $T(z) = \sum_{n \geq 0} t_n z^n$  be the ordinary generating function of  $\mathcal{T}$ . The combinatorial specification above indicates that  $T(z)$  satisfies the following equation:

$$(6) \quad T(z) = z + T(z)^2 + \frac{T(z)^4}{1 - T(z)}.$$

Equivalently, this can be rewritten as

$$(7) \quad T(z) = z\psi(T(z)) \text{ with } \psi(u) = \frac{1}{1 - u \left(1 + \frac{u^2}{1-u}\right)}.$$

From there, we apply again the Lagrange inversion formula to obtain an explicit formula for  $t_n$ .

**Theorem 21.** *The number  $t_n$  of tree interval posets with  $n$  minimal elements is*

$$t_n = \begin{cases} 1 & \text{if } n = 1, \\ \frac{1}{n} \left[ \sum_{i=1}^{n-3} \sum_{k=1}^{\min\{i, \frac{n-i-1}{2}\}} \binom{n+i-1}{i} \binom{i}{k} \binom{n-i-k-2}{k-1} + \binom{2n-2}{n-1} \right] & \text{if } n > 1. \end{cases}$$

The first terms of this sequence (starting from  $t_1$ ) are 1, 1, 2, 6, 21, 78, 301, 1198, 4888. This is sequence A054515 in the OEIS [SI20].

*Proof.* The proof follows the same steps as that of Theorem 18. Applying the Lagrange inversion theorem to Eq. (7), we have  $t_n = \frac{1}{n}[u^{n-1}]\psi(u)^n$ . In our computation of  $\psi(u)^n$ , we make use again of the identity expressed in Eq. (3), valid for any  $n \geq 1$ .

We derive

$$\begin{aligned}
\psi(u)^n &= \left( \frac{1}{1-u\left(1+\frac{u^2}{1-u}\right)} \right)^n = \sum_{i \geq 0} \binom{n+i-1}{i} u^i \left(1 + \frac{u^2}{1-u}\right)^i \\
(8) \quad &= \sum_{i \geq 0} \binom{n+i-1}{i} u^i \left(1 + \sum_{k=1}^i \binom{i}{k} \left(\frac{u^2}{1-u}\right)^k\right) \\
&= \sum_{i \geq 0} \binom{n+i-1}{i} u^i \left(1 + \sum_{k=1}^i \binom{i}{k} u^{2k} \sum_{j \geq 0} \binom{k+j-1}{j} u^j\right) \\
(9) \quad &= \sum_{i \geq 1} \sum_{k=1}^i \sum_{j \geq 0} \binom{n+i-1}{i} \binom{i}{k} \binom{k+j-1}{j} u^{i+2k+j} + \sum_{i \geq 0} \binom{n+i-1}{i} u^i.
\end{aligned}$$

Note that we isolated the term for  $k = 0$  in Eq. (8), in order to apply Eq. (3) with a positive power of  $\frac{1}{1-u}$ .

We now want to compute  $[u^{n-1}]\psi(u)^n$ . Since  $t_1 = 1$  is obvious, we can assume  $n > 1$ . The exponent of  $u$  in the first term of Eq. (9) is  $i+2k+j$ , so we want  $n-1 = i+2k+j$ , *i.e.*,  $j = n-i-2k-1$ . Since  $j \geq 0$  and  $k \geq 1$ ,  $i$  cannot be greater than  $n-3$ , while  $k$  cannot be greater than  $\frac{n-i-1}{2}$ . Since  $k$  is also at most  $i$ , we have  $k \leq \min\{i, \frac{n-i-1}{2}\}$ . On the other hand, the coefficient of  $u^{n-1}$  is the second term of Eq. (9) is  $\binom{2n-2}{n-1}$ . Therefore  $t_n = \frac{1}{n}[u^{n-1}]\psi(u)^n$  gives

$$t_n = \frac{1}{n} \left[ \sum_{i=1}^{n-3} \sum_{k=1}^{\min\{i, \frac{n-i-1}{2}\}} \binom{n+i-1}{i} \binom{i}{k} \binom{n-i-k-2}{n-i-2k-1} + \binom{2n-2}{n-1} \right].$$

To conclude the proof we just note that  $\binom{n-i-k-2}{n-i-2k-1} = \binom{n-i-k-2}{k-1}$ .  $\square$

As in the previous subsection, we can obtain the asymptotic behavior of  $t_n$  using analytic combinatorics, either from Eq. (6) (which is the version we present) or from Eq. (7). The following can be proved exactly like Theorem 19.

**Theorem 22.** *Let  $\Lambda$  be the function defined by  $\Lambda(u) = u^2 + \frac{u^4}{1-u}$ .*

*The radius of convergence  $\rho$  of the generating function  $T(z)$  of tree interval posets is given by  $\rho = \tau - \Lambda(\tau)$ , where  $\tau$  is the unique solution of  $\Lambda'(u) = 1$  such that  $\tau \in (0, 1)$ .*

The behavior of  $T(z)$  near  $\rho$  is given by

$$T(z) = \tau - \sqrt{\frac{2\rho}{\Lambda''(\tau)}} \sqrt{1 - \frac{z}{\rho}} + \mathcal{O}\left(1 - \frac{z}{\rho}\right).$$

Numerically, we have  $\tau \approx 0.3501$ ,  $\rho \approx 0.2044$ ,  $\sqrt{\frac{2\rho}{\Lambda''(\tau)}} \approx 0.2808$ .

As a consequence, the number  $t_n$  of tree interval posets with  $n$  minimal elements satisfies, as  $n \rightarrow \infty$ ,

$$t_n \sim \sqrt{\frac{\rho}{2\pi\Lambda''(\tau)}} \frac{\rho^{-n}}{n^{3/2}}.$$

Numerically, we have  $\sqrt{\frac{\rho}{2\pi\Lambda''(\tau)}} \approx 0.0792$ ,  $\rho^{-1} \approx 4.8920$ .

**Remark 23.** Like in Remark 20, we can find an equation for the generating function of non-plane interval posets which are trees. And similarly, we can deduce from this equation that the asymptotic behavior of the sequence enumerating these objects is of the form  $\alpha n^{-3/2} \beta^n$ . Here, the loose numerical estimates for  $\alpha$  and  $\beta$  which we obtain in the same fashion as in Remark 20 are  $\tilde{\alpha} = 0.2597$  and  $\tilde{\beta} = 2.9784$ .

## 6. THE MÖBIUS FUNCTION ON INTERVAL POSETS

In this section we will calculate the Möbius function on interval posets  $\tilde{P}_\bullet(\sigma)$ . We first recall some basic concepts. We refer the reader to [Sta11] or [God18] for details.

**Definition 24.** Let  $(P, \leq)$  be a partially ordered set. If  $P$  has a maximum element  $M$ , then the elements covered by  $M$  are called coatoms.

**Definition 25.** Let  $(P, \leq)$  be a partially ordered set and let  $a, b \in P$ . The interval  $[a, b]$  of  $P$  is the set  $[a, b] = \{x \in P \mid a \leq x \leq b\}$ . If every interval of  $P$  is finite, then  $P$  is said to be locally finite.

In this paper we use the term interval to denote both the intervals of a permutation and the intervals of a poset. To avoid ambiguity, we will specify every time that we refer to the interval of a poset  $P$  by writing *interval of  $P$* .

**Definition 26.** Let  $(P, \leq)$  be a partially ordered set which is locally finite, and let  $a, b \in P$ . The Möbius function between  $a$  and  $b$  is recursively defined as

$$\mu_P(a, b) = \begin{cases} 1 & \text{if } a = b, \\ - \sum_{x: a < x \leq b} \mu_P(x, b) & \text{if } a < b, \\ 0 & \text{otherwise.} \end{cases}$$

Whenever  $P$  is clear from the context, we write just  $\mu$  instead of  $\mu_P$ .

Here we used the “top-down” definition of the Möbius function, but we point out that the classical definition is the (obviously equivalent) “bottom-up” definition, given by  $\mu(a, b) = -\sum_{x:a \leq x < b} \mu(a, x)$ , for  $a < b$ . For our purpose, the top-down definition is more convenient, because in  $\tilde{P}_\bullet(\sigma)$  it is simpler to start from the top and recursively compute the Möbius function with the elements below.

The following lemma is an immediate consequence of [God18, Lemma 10.4], and describes a simple case where the Möbius function is 0. This special case arises often in  $\tilde{P}_\bullet(\sigma)$ . We also present a brief, self-contained proof of the lemma.

**Lemma 27** ([God18]). *Let  $[a, b]$  be an interval of  $P$ . If there exists an element  $x \in [a, b]$ ,  $x \neq a, b$ , which is comparable with every element in the interval  $[a, b]$  of  $P$ , then  $\mu(a, b) = 0$ .*

*Proof.* Let  $x \in [a, b]$  be an element satisfying the properties of the statement. By definition of the Möbius function, we have  $\sum_{y:x \leq y \leq b} \mu(y, b) = 0$ . Consider an element  $c$  covered by  $x$ . Since  $x$  is comparable with every element of  $[a, b]$ , there is no other element of  $[a, b]$  which covers  $c$ . This implies that  $\mu(c, b) = -\sum_{y:c < y \leq b} \mu(y, b) = -\sum_{y:x \leq y \leq b} \mu(y, b) = 0$ . Reasoning by induction, we can see that for every element  $z \neq x$  in the interval  $[a, x]$  of  $P$  it holds that  $\mu(z, b) = 0$ . Indeed,

$$\mu(z, b) = \sum_{y:z < y < x} \mu(y, b) + \sum_{y:x \leq y \leq b} \mu(y, b) = \sum_{y:z < y < x} 0 + 0 = 0.$$

In particular,  $\mu(a, b) = 0$ . □

**Theorem 28.** *Let  $\sigma$  be a permutation of size  $n$  whose substitution decomposition is  $\pi[\alpha_1, \dots, \alpha_k]$ . For any  $I \in \tilde{P}_\bullet(\sigma)$ , it holds that*

$$\mu(I, [1, n]) = \begin{cases} 1 & \text{if } I = [1, n], \\ -1 & \text{if } I \text{ is covered by } [1, n] \text{ (i.e., } I \text{ is a coatom),} \\ k-1 & \text{if } I = \emptyset \text{ and } \pi \text{ is either simple or 12 or 21,} \\ 1 & \text{if } \pi \text{ is 12\dots k or k\dots 21 for some } k \geq 3 \\ & \quad \text{and } I \text{ is covered by the two coatoms of } \tilde{P}_\bullet(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* If  $I = [1, n]$  then  $\mu(I, [1, n]) = 1$ , while if  $I$  is a coatom then  $\mu(I, [1, n]) = -1$ , by definition. Suppose now that  $I$  is neither  $[1, n]$  nor a coatom.

We now distinguish cases according to the substitution decomposition  $\pi[\alpha_1, \dots, \alpha_k]$  of  $\sigma$ .

If  $\pi$  is simple, then  $\tilde{P}_\bullet(\sigma)$  is obtained by identifying the  $k$  minimal elements of the dual claw poset  $\tilde{P}(\pi)$  with the maxima of  $\tilde{P}(\alpha_i)$ , for  $1 \leq i \leq k$ , and adding the minimum  $\emptyset$ . This is also true when  $\pi = 12$  or  $21$ , because the argyle poset with two minimal elements is equal to the dual claw poset with two minimal elements. Consider an element  $I$  of  $\tilde{P}_\bullet(\sigma)$  such that  $I$  is neither  $\emptyset$ , nor a coatom, nor  $[1, n]$ . Let  $i$  be the index such that  $\tilde{P}(\alpha_i)$  contains  $I$ . Note that  $I$  is not the maximum of  $\tilde{P}(\alpha_i)$ , since it is not a coatom.

Then,  $\mu(I, [1, n]) = 0$  by Lemma 27, where the element  $x$  of the lemma is the maximum of  $\tilde{P}(\alpha_i)$ . Finally, we consider the case  $I = \emptyset$ . From the above results and the definition of  $\mu$ , we have

$$\begin{aligned}\mu(\emptyset, [1, n]) &= -\sum_{J \in \tilde{P}(\sigma)} \mu(J, [1, n]) = -\mu([1, n], [1, n]) - \sum_{J \text{ coatom}} \mu(J, [1, n]) \\ &= -1 - \sum_{J \text{ coatom}} (-1) = -1 + k.\end{aligned}$$

We are left with the case where  $\pi$  is  $12\dots k$  or  $k\dots 21$  for some  $k \geq 3$ . In this case,  $\tilde{P}_\bullet(\sigma)$  is obtained by identifying the  $k$  minimal elements of the argyle poset  $\tilde{P}(\pi)$  with the maxima of  $\tilde{P}(\alpha_i)$ , for  $1 \leq i \leq k$ , and adding the minimum  $\emptyset$ . We can easily compute the Möbius function for every element  $I \in \tilde{P}(\pi)$ . We have seen that  $\mu(I, [1, n]) = 1$  (resp.  $-1$ ) if  $I$  is  $[1, n]$  (resp. a coatom). It follows that  $\mu(I, [1, n]) = 1$  if  $I$  is the element covered by both coatoms, and that  $\mu(I, [1, n]) = 0$  for all the others  $I \in \tilde{P}(\pi)$  – see Fig. 4.

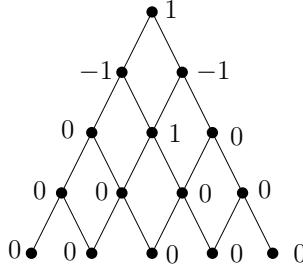


FIGURE 4. The Möbius function between the maximum and any element in an argyle poset.

We now consider an element  $I \in \tilde{P}(\alpha_i)$  for some  $i$ . If  $I$  is the maximum of  $\tilde{P}(\alpha_i)$ , then it is also a minimal element of  $\tilde{P}(\pi)$ , and hence  $\mu(I, [1, n]) = 0$  as we have seen. Otherwise, we apply again Lemma 27 (with  $x$  the maximum of  $\tilde{P}(\alpha_i)$ ), and we obtain  $\mu(I, [1, n]) = 0$ . Finally, if  $I = \emptyset$ , then we have

$$\begin{aligned}\mu(\emptyset, [1, n]) &= -\sum_{J \in \tilde{P}(\sigma)} \mu(J, [1, n]) = -\mu([1, n], [1, n]) - \sum_{J \text{ coatom}} \mu(J, [1, n]) - \mu(\bar{I}, [1, n]) = \\ &= -1 + 2 - 1 = 0,\end{aligned}$$

concluding the proof of the theorem.  $\square$

It may seem that Theorem 28 only allows to compute the Möbius function on intervals of  $\tilde{P}_\bullet(\sigma)$  whose largest element is the maximum of  $\tilde{P}_\bullet(\sigma)$ . The following remark shows that Theorem 28 is actually easily extended to all intervals of  $\tilde{P}_\bullet(\sigma)$ .

**Remark 29.** Let  $\sigma$  be a permutation and  $J$  be an element of  $\tilde{P}_\bullet(\sigma)$ . Define  $j = |J|$ . Let  $\mathcal{J}$  be the subposet of  $\tilde{P}_\bullet(\sigma)$  consisting of the elements in the interval  $[\emptyset, J]$  of  $\tilde{P}_\bullet(\sigma)$ . There exists a permutation  $\tau$  (of size  $j$ ) such that  $\tilde{P}_\bullet(\tau)$  is isomorphic to  $\mathcal{J}$ .

*Proof.* Let  $\hat{\tau}$  be the subsequence of  $\sigma$  composed by the elements of  $J$ . Note that the values occurring in  $\hat{\tau}$  form an interval of integers ( $J$  being an interval of  $\sigma$ ). We then define  $\tau$  as the permutation obtained by rescaling  $\hat{\tau}$  to the set  $\{1, \dots, j\}$ . Since the relative order among the elements remains unchanged, the intervals of  $\tau$  correspond to the subsets of  $J$  that are intervals of  $\sigma$ . Therefore the poset  $\tilde{P}_\bullet(\tau)$  is isomorphic to the poset  $\mathcal{J}$ .  $\square$

As a consequence, for any  $I, J \in \tilde{P}_\bullet(\sigma)$ , we can compute  $\mu_{\tilde{P}_\bullet(\sigma)}(I, J)$  using the Möbius function on  $\tilde{P}_\bullet(\tau)$ . More precisely, letting  $I'$  be the interval obtained rescaling  $I$  by the same value that we used to rescale  $J$  into  $[1, j]$ , we have  $\mu_{\tilde{P}_\bullet(\sigma)}(I, J) = \mu_{\tilde{P}_\bullet(\tau)}(I', [1, j])$ .

For example, let  $\sigma = 456793128$ , whose interval poset  $\tilde{P}_\bullet(\sigma)$  is represented in Fig. 1, right. If we want to calculate  $\mu_{\tilde{P}_\bullet(\sigma)}(\{5\}, [4, 7])$ , we consider  $\tau = 1234$  (corresponding to  $\tau' = 4567$  rescaled by 3) and compute  $\mu_{\tilde{P}_\bullet(\sigma)}(\{5\}, [4, 7]) = \mu_{\tilde{P}_\bullet(\tau)}(\{2\}, [1, 4]) = 0$ .

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