Series Rearrangement

Real Analysis-MA4.101

A Kishore Kumar (2020909016)
Nitin Rajesakar (2020909008)
Rohan Gupta (2020909094)

November 2020

Contents

1 Acknowledgement

We are grateful to the director Dr. P. J. Narayanan, the dean of academics Dr. Jayanthi Sivaswamy, our professor in Real Analysis, Dr. Lakshmi Burra, our teaching assistants Mr. Vivek Chandela, Mr. Soham Choudhuri and Mr. Aman Goel for providing us the opportunity to present this project on Series Rearrangement. In particular we appreciate the efforts made by professor Lakshmi Burra and the teaching assistants of this course in guiding, teaching and encouraging us not only in the subject but our academic careers as a whole. We are thankful for the opportunity to apply what we have learnt from the course in the form of this project.

Abstract

Infinite series have been a very common ingredient of proofs that look like they break mathematics. A lot of them stem from incorrect usage of the sum of a conditionally converging series. In this paper, we will visit a famous conditionally converging sequence: the alternating harmonic series and discuss a couple theorems on rearrangements from Riemanns series rearrangement to Pringsheim and even run algorithms that help visualize series rearrangement and it's effect on their sum first hand. The goal of this paper is to leave behind our intuition about sums of finite sets of numbers and explore theorems and operations over the infinite.

2 Introduction

Over the course of this paper, we will be discussing the Rearrangement of Series. We attempt to address the following questions: What is a series? What does it mean to find the sum of an infinite series? And most importantly, **when** does it make sense to talk about the sum of an infinite series? We will find the sum of the alternating harmonic series without any rearrangement.

We then provide an account of the Riemann rearrangement theorem a generalizations of it and Pringsheim's theorem, all accompanied with their respective proofs. Over the course of this paper, we hope to provide mind bending results related to infinite series.

2.1 Georg Friedrich Bernhard Riemann

Riemann, a German mathematician, is one of the most famous names in Mathematics. To many, he is a hero in the field of Mathematics. He has made numerous contributions to the field of Mathematical analysis. He was one of the first to formulate a rigorous definition for the integral of a function over an interval, the *Riemann integral.* He introduced the world of Complex Analysis to concepts such as Riemann surfaces and the Riemann mapping theorem. An entire field of geometry is named after this man, *Riemann geometry.* His contribution to the field of analytic number theory includes the Riemann zeta function. His conjecture on this function, more famously known as the *Riemann Hypothesis* is one of the Clay Mathematics Institute's Millennium Prize Problems. It is considered by many to be one of the most important unsolved problems in pure math. Then there is the Riemann mapping theorem, the Cauchy-Riemann equations, the Riemann tensor, and so many more works of mathematical art that this man has contributed to. This paper is dedicated to exploring one of his numerous contributions in detail: The Riemann rearrangement theorem.

3 Finite and Infinite Sums

To understand the rearrangement theorem and it's many implications, we must first fully understand what a series is and what it means to find the sum of an **infinite** series. Let us begin by addressing the questions we raised in the introduction segment.

3.1 Series

A series can be described as the sum of the terms of a given numerical sequence. Recall that a numerical sequence is simply an ordered list or collection of numbers where repetition is allowed. Consider the finite sequence of natural numbers from 1 to 10: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
The series corresponding to the terms of this sequence is:

$$\sum_{k=1}^{10} k = 1 + 2 + 3 + \ldots + 10 = 55$$

This was an example of a finite sequence. Similarly, just like we can have finite and infinite sequences, we can have an infinite series. An infinite series is simply the sum of an infinite number of terms of a corresponding sequence. For example, if we extend the previous finite series to include the sum of all the natural numbers, we get:

$$\sum_{k=1}^{\infty} k = 1 + 2 + 3 + \ldots$$

But what is the sum of an infinite series equal to? To answer this, let us begin by defining 2 different types of series. **Convergent** series and **divergent** series.

3.1.1 Convergent series

We define a convergent series as a series who's sequence of partial sums tends to a limit.

Consider the following series,

$$S = \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

We now define a new term, **partial sums.** The k^{th} partial sum of a series is defined as the sum of the first k terms of that series. The set of all the partial sums of an infinite series form an infinite sequence where the n^{th} term of the sequence is equal to the n^{th} partial sum of the series. Let S_k denote the k^{th} partial sum of our above defined series S. Then, the first few partial sums are as follows:

$$S_1 = \sum_{k=1}^{1} \frac{1}{2^k} = \frac{1}{2} = 1 - \frac{1}{2}$$

$$S_2 = \sum_{k=1}^{2} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4}$$

$$S_3 = \sum_{k=1}^{3} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = 1 - \frac{1}{8}$$

$$S_4 = \sum_{k=1}^{4} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} = 1 - \frac{1}{16}$$

We can write down these partial sums as the beginning few terms of an infinite sequence,

$$\left\{ \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \dots \right\}$$

Recall that the n^{th} term of this sequence is the n^{th} partial sum of the series. The n^{th} partial sum of the series will also be the sum of our infinite series. We can now define the sum of an infinite series as the limit of the sequence of partial sums as n tends to infinity. If such a limit does not exist, we say that the series does not have a sum. Let us attempt to find the general term of this sequence,

$$2s_k = \frac{2}{2} + \frac{2}{4} + \frac{2}{8} + \cdots + \frac{2}{2^k} = 1 + \left[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}}\right] = 1 + \left[s_k - \frac{1}{2^k}\right].$$

$$s_k = 1 - \frac{1}{2^k}$$

We can notice this occurrence in our listing of the first 4 partial sums of this series. As k tends to infinity, we have:

$$\lim_{k \to \infty} 1 - \frac{1}{2^k} = 1$$

Since the sequence of its partial sums tends to a limit, this is a convergent series who's sum is equal to the n^{th} term of the sequence as n tends to infinity. Here,

$$S = \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1$$

3.1.2 Divergent Series

A divergent series can be simply defined as a series which does not converge, i.e., a series is divergent if the infinite sequence of its partial sums does not have a finite limit. This is true if either the limit does not exist or it is equal to $\pm\infty$.

Consider the simple harmonic series,

$$S = \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \ldots$$

Let S_k represent the sum of the terms between the terms indexed 2^k (excluded) and 2^{k-1} (included) We make the following observations:

$$S_1 = 1 > \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{3} > \frac{1}{2}$$

$$S_3 = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} > \frac{4}{8} = \frac{1}{2}$$

$$S_4 = \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} > \frac{8}{16} = \frac{1}{2}$$

$$\vdots$$

We can write the series S as the sum of these segments S_1, S_2, S_3... etc. Now, since each segment individually sums to a number greater than $\frac{1}{2}$, we can make the following relation:

$$S = S_1 + S_2 + S_3 + S_4 + \ldots > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$$

The sequence of partial sums of the series on the right $\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\right)$ tends to ∞. Hence, we can say that, our simple harmonic series S diverges to ∞.

3.2 Commutativity of absolute convergence

If $\sum_{i=n}^{m} a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum_{i=n}^{m} a_n$ converges to the same sum.

Now we know that since a_n is absolutely convergent,

$$\sum_{i=n}^{m} |a_i| \leq \epsilon$$

if $m \geq n \geq N$ for an integer N , given an $\epsilon > 0$.

We choose p such that the integers 1,2,3..N are all present i the set $k_1, k_2, k_3 ... k_p$, which is the rearranged sequence . So if n>p, the numbers, $a_1, a_2, a_3 ... a_n$ cancel out in the difference $s_n - S_n$, where S_n is the sum of the rearranged sequence . Thus we get that, $|s_n - S_n \leq \epsilon$ .Hence S_n converges to the same value as s_n .

Commutative law for addition: a+b= b+a

That is, the commutative law says rearranging the summands in a finite sum does not change the total.

But, that's not true for conditionally convergent series

Ex:(Alternating harmonic series)

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + ... = ln2$$

Consider:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + ...$$

$$= (1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + (\frac{1}{5} - \frac{1}{10}) - \frac{1}{12} + ...$$

$$\frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - ...) = \frac{1}{2} ln2$$

We explained earlier that sum of a series is a limit of partial sums as $n \to \infty$

SO, Rearranging terms of a series changes the partial sums .

and a result this changes limit of the partial sums

3.2.1 Conditionally Convergent Series

A series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if and only if the sequence of its partial sums, i.e., $\lim_{m\to\infty} \sum_{n=1}^{m} a_n$ exists and evaluates to some finite number while $\lim_{m\to\infty} \sum_{n=1}^{m} |a_n|$ diverges to ∞.

3.3 Rearrangement:

Definition: Let $k_n, n = 1, 2, 3, ...,$ be an integer valued positive sequence in which every positive integer appears once and only once (that is, $k_n = k_{n'}$ and only if $n = n'$). Given a series $\sum a_n$, Put

$$a'_n = a_{k_n} \qquad\qquad (n = 1, 2, 3...)$$

we say that $\sum a'_n$ is a rearrangement of $\sum a_n$.

4 Sum of Alternating Harmonic Series

A power series (centered at 0) is a function of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

If the series converges for non-zero x, there is an $R > 0$ so that the series converges in the open interval $-R < x < R$. In this interval, the series can be differentiated and integrated term by term and the resulting series also converge in this open interval.

Abel's Theorem.

If $\sum a_n$ converges , and if $f(x) = \sum a_n x^n$, then

$$\sum a_n = \lim_{x \to 1^-} f(x)$$

Abel's theorem and results on integration and differentiation of series allow us to find sums of series like the AHS.

To sum

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + ...$$

let,

$$f(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 - ...$$

This power series converges in the open interval $-1 < x < 1$.

let $F(x) = f'(x)$ so that

$$F(x) = f'(x) = 1 - \frac{1}{2}2x + \frac{1}{3}3x^2 - \frac{1}{4}4x^3 + \frac{1}{5}5x^4 - ...$$

$$= 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 + ...$$

$$= \frac{1}{1+x}$$

Since $f'(x) = \frac{1}{1+x}$ we can see $f(x) = ln(1+x)$

Now Abel's theorem says

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - ... = \lim_{x \to 1} ln(1+x) = ln2$$

5 Riemann Series Rearrangement

Take an arbitrary infinite sequence of real numbers $(a_1, a_2, a_3, \ldots)$ such that $\sum\limits_{n=1}^{\infty} a_n$ is conditionally convergent. Let K be any number belonging to the set of the extended real numbers. Then there exists a permutation

$$g : \mathbb{N} \to \mathbb{N} \text{ such that } \sum_{n=1}^{\infty} a_{g(n)} = K$$

5.1 Proof

5.1.1 Existence of a rearrangement that converges to a finite real number

Consider K to be any real positive number. Let the series be denoted by $S = \sum\limits_{i=0}^{\infty} a_i$. It is conditionally convergent. This means that it has an infinite number of positive and an infinite number of negative terms each. Let us denote them as follows:

Let $(p_1, p_2, p_3, \ldots)$ denote the sub-sequence of all positive terms in S and $(n_1, n_2, n_3, \ldots)$ denote the sub-sequence of all negative terms in S. Since the series is conditionally convergent, both the positive and negative series (p_i) & (n_i) will diverge to $\pm\infty$. Hence, we have:

$$\sum p = +\infty$$

$$\sum n = -\infty$$

Since $\sum p$ tends to ∞, it implies that there exist a minimum natural number N_1 such that for all $N \geq N_1$ the following holds true: If S_k denotes the partial sum of the first k terms of this rearranged series,

$$S_N = \sum_{i=1}^{N} p_i > K$$

Since N_1 is the minimum such number, it implies that:

$$\sum_{i=1}^{N_1 - 1} p_i \leq K < \sum_{i=1}^{N_1} p_i \tag{1}$$

We can begin to develop a mapping $\sigma : \mathbb{N} \to \mathbb{N}$ such that,

$$\sum_{i=1}^{N_1} p_i = \big(a_{\sigma(1)} + a_{\sigma(2)} + a_{\sigma(3)} + \cdots + a_{\sigma(N_1)}\big)$$

Now, since $\sum n$ also diverges to ∞, it is possible to add just enough terms from (n_i) so that the resulting sum

$$S_{N_1+M} = \sum_{i=1}^{N_1} p_i + \sum_{i=1}^{M} n_i \leq K$$

Let M_1 be the minimum number of terms required from (n_i) for the above statement to hold true. This implies that,

$$\sum_{i=1}^{N_1} p_i + \sum_{i=1}^{M_1-1} n_i > K \geq S_{N_1+M_1} \tag{2}$$

Consider equation (1), if we subtract S_{N_1} from the inequality and flip the signs, we get:

$$0 \leq S_{N_1} - K < p_{N_1}$$

In equation (2), if we subtract $S_{N_1+M_1}$ from the inequality, we get:

$$0 \leq K - S_{N_1+M_1} < -n_{M_1}$$

Now, we can write $S_{N_1+M_1}$ as

$$S_{N_1+M_1} = a_{\sigma(1)}+a_{\sigma(2)}+a_{\sigma(3)}+\cdots+a_{\sigma(N_1)}+a_{\sigma(N_1+1)}+a_{\sigma(N_1+2)}+a_{\sigma(N_1+3)}+\cdots+a_{\sigma(N_1+M_1)}$$

Notice, that this mapping of σ is injective. Now, we can repeat the process we performed above. Add just enough positive terms from $\sum p$ till the partial sum of this new rearranged series is just greater than K, then add enough negative terms from $\sum n$ till the partial sum is lesser than or equal to K. Because both $\sum n$ & $\sum p$ diverge to infinity, this process can be carried out infinitely many times.

In general, our rearranged series would look like

$$p_1+p_2+\cdots+p_{N_1}+n_1+n_2+\cdots+n_{M_1}+p_{N_1+1}+p_{N_1+2}+\cdots+p_{N_2}+n_{M_1+1}+n_{M_1+2}\cdots+n_{M_2}+\ldots$$

Note that for every partial sum who's last summation step was adding terms from the positive series,

$$S_{p_i} - K < p_{N_i}$$

and for every partial sum who's last summation step was adding terms from the negative series,

$$K - S_{n_i} < n_{M_i}$$

More generally, we can say that at every "change in direction" or magnitude, the partial sum of the rearranged series at that point differs from our real number K by at most $|p_{N_i}|$ or $|n_{M_i}|$. But we know that $\sum_{i=n}^{\infty} a_n$ converges. Therefore, as n tends to ∞, a_n also tends to 0. Consequentially, $|p_{N_i}|$ & $|n_{M_i}|$ must also tend to 0. From the above two observations, we can say that the following is true:

As n tends to ∞, the partial sums of our rearranged series $\sum a_{\sigma(n)}$ tends to K.

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = K$$

This same method can be used to show convergence to any negative real number K or K = 0.

5.1.2 Existence of a rearrangement that diverges to infinity

Let the series be denoted by $S = \sum_{i=0}^{\infty} a_i$. It is conditionally convergent. This means that it has an infinite number of positive and an infinite number of negative terms each. Let us denote them as follows:

Let $(p_1, p_2, p_3, \ldots)$ denote the sub-sequence of all positive terms in S and $(n_1, n_2, n_3, \ldots)$ denote the sub-sequence of all negative terms in S. Since the series is conditionally convergent, both the positive and negative series (p_i) & (n_i) will diverge to $\pm\infty$. Hence, we have:

$$\sum p = +\infty$$

$$\sum n = -\infty$$

Since $\sum p$ tends to ∞, it implies that there exist a minimum natural number N_1 such that for all $N \geq N_1$ the following holds true:

$$\sum_{i=1}^{N_1} p_i > |n_1| + c$$

Where c is some constant positive real number. Similarly we can find a N_2 such that it is the smallest natural number for which the following holds true:

$$\sum_{i=N_1+1}^{N_2} p_i > |n_2| + c$$

We can do this repeatedly an infinite number of times because the subsequence of positive terms diverges.

This gives us our rearranged series:

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = p_1 + p_2 + \cdots + p_{N_1} + n_1 + p_{N_1+1} + p_{N_1+2} + \cdots p_{N_2} + n_2 + p_{N_2+1} + \ldots$$

Owing to the way we chose N_1, the first $N_1 + 1$ terms of the series have a partial sum that is at least c and no partial sum in this group is negative. Similarly, the partial sum of the first $N_2 + 1$ terms of this series are at least greater than $2c$ and no partial sum in this group is negative. In general, for any $N_i + 1$ terms of this series, we can say that the partial sum is at least $N_i * c$ and no partial sum in that group is negative. Hence, we can say that as n tends to ∞, N_i tends to ∞ and therefore, the sequence of partial sums of the series tends to ∞.

5.2 Algorithm to analyze series rearrangement!

All programs related to this paper can be found here:
https://github.com/akcube/notes/tree/main/series-rearrangement

From the proof, we can observe the algorithm one can use to rearrange a conditionally convergent series to sum up to any such real number K. Here, we will attempt to do two things.

We will observe how the rearranged series looks like for it to converge to some real number M. You can use the programs in the above repo to print the series up to a certain number of terms for any real number M. Here, we will attach the output for what the beginning of the series looks like when we attempt to rearrange it to sum to 0.534. We ran the program like so

```
./print_rearrangement 0.535 100 100
```

The program will print what the series looks like for the first 100 groups of positive and negative terms.

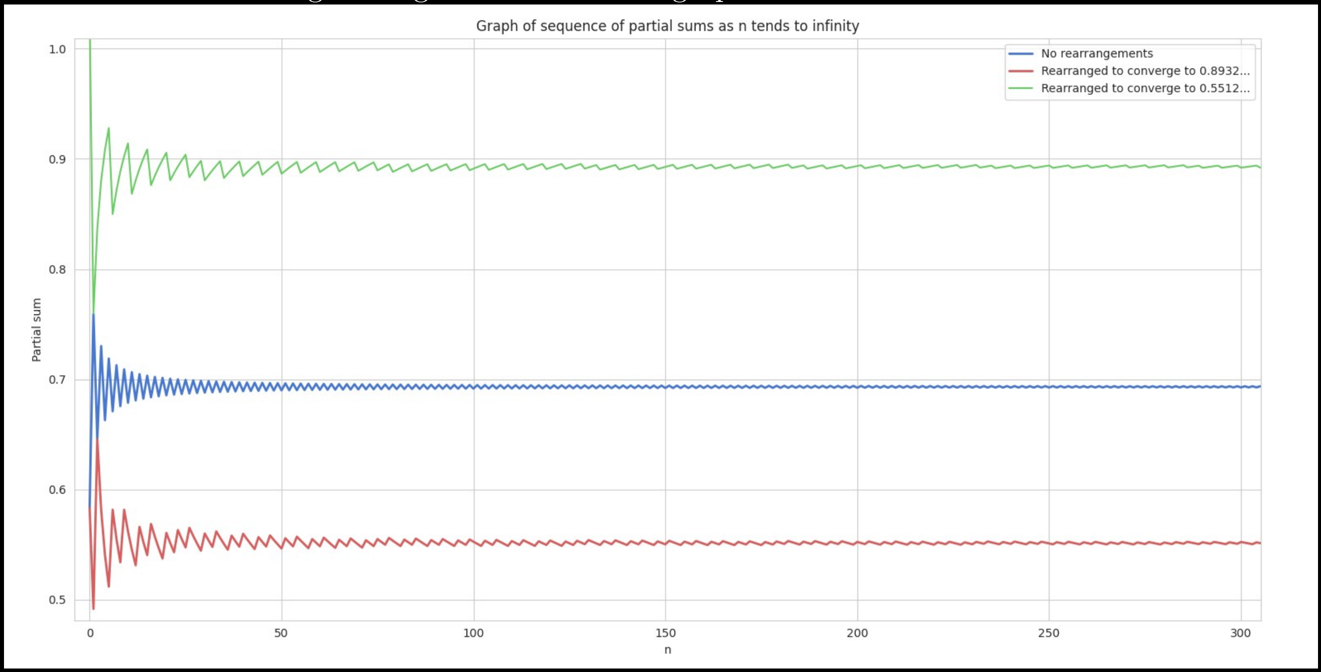
```
+ 1/1    - 1/2    + 1/3    - 1/4    - 1/6    + 1/5    - 1/8    + 1/7    - 1/10   - 1/12   + 1/9    - 1/14   + 1/11   - 1/16   + 1/13   - 1/18
- 1/20   + 1/15   - 1/22   + 1/17   - 1/24   + 1/19   - 1/26   - 1/28   + 1/21   - 1/30   + 1/23   - 1/32   + 1/25   - 1/34   - 1/36   + 1/27
- 1/38   + 1/29   - 1/40   - 1/42   + 1/31   - 1/44   + 1/33   - 1/46   + 1/35   - 1/48   - 1/50   + 1/37   - 1/52   + 1/39   - 1/54   + 1/41
- 1/56   - 1/58   + 1/43   - 1/60   + 1/45   - 1/62   - 1/64   + 1/47   - 1/66   + 1/49   - 1/68   + 1/51   - 1/70   - 1/72   + 1/53   - 1/74
+ 1/55   - 1/76   + 1/57   - 1/78   - 1/80   + 1/59   - 1/82   + 1/61   - 1/84   - 1/86   + 1/63   - 1/88   + 1/65   - 1/90   + 1/67   - 1/92
- 1/94   + 1/69   - 1/96   + 1/71   - 1/98   + 1/73   - 1/100  - 1/102  + 1/75   - 1/104  + 1/77   - 1/106  - 1/108  + 1/79   - 1/110  + 1/81
- 1/112  + 1/83   - 1/114  - 1/116  + 1/85   - 1/118  + 1/87   - 1/120  + 1/89   - 1/122  - 1/124  + 1/91   - 1/126  + 1/93   - 1/128  - 1/130
+ 1/95   - 1/132  + 1/97   - 1/134  + 1/99   - 1/136  ...

Sum: 0.535745703610323
```

Further, we can use prog1.cpp and prog2.cpp to generate data points for plotting. prog1.c will generate data points of the partial sums of the alternating harmonic series up to a given number of terms. We can use prog2.c to generate data points of the partial sums for a rearranged alternating harmonic series that converges to some given real number M.

For the sake of illustration, we have chosen to plot the partial sums as n keeps increasing for the following rearrangements.

- The normal alternating harmonic series
- A rearrangement of the alternating harmonic series that sums to 0.5512
- A rearrangement of the alternating harmonic series that sums to 0.8932

Plotting them gives us the below graph.



Hopefully, this graph is able to paint more intuition as to why rearranging the terms of an infinite conditionally convergent series changes its sum. By rearranging the terms such that the sequence of its partial sums oscillates around a limit of our choice, we're able to effectively choose the limit we wish the sequence of partial sums to approach. This is due to the fact that the sum of the positive and negative terms individually diverge to infinity. But the series itself converges to some limit, hence the n^{th} term of the series approaches 0.

Both these properties are true for conditionally convergent series and it is due to this very reason that we're able to rearrange the infinite sum to converge to whatsoever real sum of our choice.

6 A more general version of Riemann's rearrangement

6.1 Theorem:

Let $\sum a_n$ be a series of real numbers which converges but not absolutely. Suppose

$$-\infty \leq \alpha \leq \beta \leq \infty$$

Then there exists a rearrangement $\sum a'_n$ with partial sums s'$_n such that$

$$\lim_{n\to\infty} inf\ s'_n = \alpha \qquad \lim_{n\to\infty} sup\ s'_n = \beta$$

PROOF:

$$p_n = \frac{|a_n| + a_n}{2}, \qquad q_n = \frac{|a_n| - a_n}{2} \qquad (n = 1, 2, ...)$$

Then $p_n - q_n = a_n$, $p_n + q_n = |a_n|,\ p_n \geq 0,\ q_n \geq 0$. The series $\sum p_n, \sum q_n$ must both diverge.

For if both were convergent, then

$$\sum (p_n + q_n) = \sum |a_n|$$

would converge, contrary to hypothesis. Since

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} (p_n - q_n) = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n$$

divergence of $\sum p_n$ and convergence of $\sum q_n$ (or vice versa) implies divergence of $\sum a_n$, again contrary to hypothesis.

Now let $P_1, P_2, P_3...$ denote the nonnegative terms of $\sum a_n$, in the order in which they occur and let Q_1, Q_2, Q_3....be the absolute values of negative terms of $\sum a_n$, also in their original order.

The series $\sum P_n, \sum Q_n$ differ from $\sum p_n, \sum q_n$ only by zero terms , and are therefore divergent.

We shall construct sequence m_n, k_n, such that the series

$$(1) P_1 + ... + P_{m1} - Q_1 - .... - Q_{k1} + P_{m1+1} + ... + P_{m2}$$
$$-Qk1 + 1 - ... - Q_{k2} + ...$$

which clearly is a rearrangement of $\sum a_n$,

Choose real valued sequence α_n, β_n such that $\alpha_n \to \alpha, \beta_n \to \beta, \alpha_n < \beta_n, \beta_1 > 0$.

Let m_1, k_1 be the smallest integers such that

$$P_1 + ... + P_{m1} > \beta_1,$$
$$P_1 + ... + P_{m1} - Q_1 - .... - Q_{k1} < \alpha_1;$$

let m_2, k_2 be the smallest integers such that

$$P_1 + ... + P_{m1} - Q_1 - .... - Q_{k1} + P_{m1+1} + ... + P_{m2} > \beta_2,$$

$$P_1 + ... + P_{m1} - Q_1 - .... + P_{m1+1} + ... + P_{m2} - Q_{k1+1} - .... - Q_{k2} < \alpha_2;$$

and continue in this way. This is possible since $\sum P_n$ and $\sum Q_n$ diverge

If x_n, y_n denote the partial sums of (1) whose last term are $P_{m_n}, -Q_{k_n}$, then.

$$|x_n - \beta_n| \leq P_{m_n}, \qquad |y_n - \alpha_n| \leq Q_{k_n}$$

Since $P_n \to 0$ and $\mathbb{Q}_n \to 0$ as $n \to \infty$, we see that $x_n \to \beta, y_n \to \alpha$.

Finally, it is clear that no number less than α or greater than β can be subsequential limit of

the partial sums of (1).

7 Pringsheim's theorem

If $\sum_{n=1}^{\infty} f^n$ is a rearrangement of the alternating harmonic series, then it converges to an extended real number if and only if the asymptotic density of positive terms (α) exists, in which case

$$\sum_{n=1}^{\infty} f^n = ln(2) + \frac{1}{2}\ln(\frac{\alpha}{1-\alpha}).$$

Let $\sum_{n=1}^{\infty} a_n$ be a simple rearrangement of the alternating harmonic series. Let p_k be the number of positive terms in $\{a_1, a_2, a_3 ... a_k\}$, and q_k be the number of negative terms .

As the given series is a simple rearrangement of the Alternating Harmonic Series,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{p_k} \frac{1}{2j-1} - \sum_{n=1}^{q_k} \frac{1}{2j}$$

For every positive integer m, let $E_m = \sum_{n=1}^{m} \frac{1}{n}$ - $ln(m)$

The sequence is a decreasing sequence of positive numbers whose limit is γ .

7.1 Proof that E_m does converge to a constant

By integral estimation,

$$1 + \frac{1}{2} + \frac{1}{3} ... + \frac{1}{n} > \int_1^{n+1} \frac{dx}{x} = ln(n+1)$$

and

$$\frac{1}{2} + \frac{1}{3} ... + \frac{1}{n} < \int_1^{n} \frac{dx}{x} = ln(n)$$

Now consider δ_n to be $H_n - ln(n)$

So,

$$ln(1+n) < H_n < 1 + ln(n)$$

Thus

$$ln(1+\frac{1}{n}) < H_n - ln(n) < 1$$

where

$$n > 1$$

Thus, every δ_n is a positive number not greater than 1 .

Now,

$$\delta_n - \delta_{n+1} = (H_n - ln(n)) - (H_{n+1} - ln(n+1))$$

$$= ln(n+1) - ln(n) - \frac{1}{n+1}$$

$$= \int_n^{n+1} \frac{dx}{x} - \frac{1}{n+1} > 0$$

Now as δ_n is monotone decreasing and is always > 0 , the series must converge .

$\lim_{n\to\infty}(H_n - ln(n))$ is found to be a constant $\gamma \approx 0.5772$

Now,

$$\sum_{n=1}^{m} \frac{1}{n} = ln(m) + E_m$$

, so we can see that

$$\sum_{j=1}^{q_k} \frac{1}{2j} = \frac{1}{2}\sum_{j=1}^{q_k} \frac{1}{j} = \frac{1}{2}ln(q_k) + \frac{1}{2}E_{q_k}$$

And

$$\sum_{j=1}^{p_k} \frac{1}{2j-1} = \sum_{l=1}^{2p_k} \frac{1}{l} - \sum_{l=1}^{p_k} \frac{1}{2l} = (ln(2p_k) + E_{2p_k}) - (\frac{1}{2}ln(p_k) + \frac{1}{2}E_{p_k})$$

Thus,

$$\sum_{n=1}^{k} a_n = \sum_{j=1}^{p_k} \frac{1}{2j-1} - \sum_{j=1}^{q_k} \frac{1}{2j} = ln(2_{p_k}) + E_{2p_k} - \frac{1}{2}ln(p_k) - E_{p_k} - \frac{1}{2}ln(q_k) - E_{q_k}$$

$$= ln(2) + ln(p_k) - \frac{1}{2}ln(p_k) - \frac{1}{2}ln(q_k) + E_{2p_k} - \frac{1}{2}ln(E_{p_k}) - \frac{1}{2}ln(E_{q_k})$$

$$= ln(2) + \frac{1}{2}ln(\frac{p_k}{q_k}) + E_{2p_k} - \frac{1}{2}E_{p_k} - \frac{1}{2}E_{q_k}$$

Now as k approaches infinity,

$$\frac{p_k}{q_k} = \frac{\frac{p_k}{k}}{1 - \frac{p_k}{k}} = \frac{\alpha}{1-\alpha}$$

Now, $E_{2p_k} = E_{p_k} = E_{q_k} = \gamma$
Thus,

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} (ln(2) + \frac{1}{2}ln(\frac{p_k}{q_k}) + E_{2p_k} - \frac{1}{2}E_{p_k} - \frac{1}{2}E_{q_k})$$

$$= ln(2) + \frac{1}{2}ln(\frac{\alpha}{1-\alpha}) + \gamma - \frac{1}{2}\gamma - \frac{1}{2}\gamma = ln(2) + \frac{1}{2}ln(\frac{\alpha}{1-\alpha})$$

which is what we set out to prove .

Theorem:

$$\lim_{N\to\infty}(F(2p_N)-F(2q_N))=2A-2T$$

Here,

F(x) is defined as $\displaystyle\int_1^x f(t)dt$

T is defined as

$$T=\lim_{n\to\infty}T_N=lim_{n\to\infty}\sum_{k=1}^{N}a_k(-1)^{k-1}$$

A is the real number to which the Nth partial sum of a rearrangement of the series converges.

Now let us assume that $p_n\geq q_n$ for large enough values of N.

We may prove the opposite case in a similar manner .

Now,

$$S_N(A)=\sum_{k=1}^{p_N}a_{2k-1}-\sum_{k=1}^{q_N}a_{2k}$$

$$=\sum_{k=1}^{2q_N}a_k(-1)^{k-1}+\sum_{k=q_N+1}^{p_N}a_{2k-1}$$

$$=T_2q_n+\sum_{k=q_N+1}^{p_N}a_{2k-1}$$

Now by integral estimation,

$$2\sum_{k=q_n+2}^{p_N}a_{2k-1}\leq\int_{2q_N}^{2p_N}f(t)dt$$

$$2\sum_{k=q_n+1}^{p_N}a_{2k-1}-2a_{2_{q_N}+1}\leq\int_{2q_N}^{2p_N}f(t)dt$$

Also,

$$2\sum_{k=q_n}^{p_N}a_{2k-1}\geq\int_{2q_N}^{2p_N}f(t)dt$$

$$2\sum_{k=q_n+1}^{p_N} a_{2k-1} + 2a_{2_{q_N}-1} \geq \int_{2q_N}^{2p_N} f(t)dt$$

Combining the above equations,

$$\int_{2q_N}^{2p_N} f(t)dt - 2a_{2q_{N-1}} \leq 2\sum_{k=q_n+1}^{p_N} a_{2k-1} \leq \int_{2q_N}^{2p_N} f(t)dt + 2a_{2q_{N+1}}$$

$$\int_{2q_N}^{2p_N} f(t)dt - 2a_{2q_{N-1}} \leq 2S_N(A) - 2T_{2q_N} \leq \int_{2q_N}^{2p_N} f(t)dt + 2a_{2q_{N+1}}$$

Now,

$$\lim_{N\to\infty} 2a_{2q_{N-1}} = lim_{N\to\infty} 2a_{2q_{N+1}} = 0$$

Thus taking the limit as N→0,

$$\lim_{N\to\infty} (F(2p_N) - F(2q_N)) = 2A - 2T$$

8 Conclusion

Infinite series were always always a favorite tool for creating crazy paradoxes. Playing with infinity was a sure fire way to find ridiculous proofs that seemed right on a cursory glance. Rearranging conditionally convergent series and equating their sums was one of those methods. The rearrangement theorem and other such theorems about the same laid a more solid understanding of the sum of such infinite series and a way to meaningfully use them in equations and told us **when** we could equate rearranged series and when we couldn't. This may seem trivial for simple sums over $\mathbb{N}$, but it is quite useful when dealing with sums of $\mathbb{N}^k$. For instance it makes it much easier to prove the following important result:

$$exp(x)exp(y) = \sum_{m,n\geq 0} \frac{x^m y^n}{m!n!} = \sum_{l\geq 0} \frac{1}{l!} \left(\sum_{m+n=l} \frac{l!}{m!n!} x^m y^n \right) = \sum_{l\geq 0} \frac{(x+y)^l}{l!} = exp(x+y)$$

And we can extend this to even integrals, which gives us the famous Fubini's theorem.

In short, series rearrangement made a lot of weird math make sense and made it very clear that our usual intuition while working with finite sums does not extend to infinite sums. We learned that it is possible to rearrange the sum of an infinite conditionally converging series to make it converge to any real number we wanted or diverge to infinity if we so chose. It helped define some rules that made sense of the addition of infinite sums and was definitely a very fun topic to research about, to say the least!

9 Results

1) If $\sum_{k=1}^{\infty} f^n$ is a rearrangement of the alternating harmonic series, then it converges to an extended real number if and only if the asymptotic density of positive terms (α) exists, in which case

$$\sum_{n=1}^{\infty} f^n = ln(2) + \frac{1}{2}\ln(\frac{\alpha}{1-\alpha}).$$

2)

$$\lim_{N\to\infty}(F(2p_N) - F(2q_N)) = 2A - 2T$$

3) If $\sum_{i=n}^{m} a_n$ is an absolutely convergent sequence,every rearrangement of $\sum_{i=n}^{m} a_n$ converges to the same sum.

4) Take an arbitrary infinite sequence of real numbers $(a_1, a_2, a_3, \ldots)$ such that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. Let K be any number belonging to the set of the extended real numbers. Then there exists a permutation

$$g : \mathbb{N} \to \mathbb{N} \text{ such that } \sum_{n=1}^{\infty} a_{g(n)} = K$$

5)

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - ... = \lim_{x\to1} ln(1+x) = ln2$$

6) Let $\sum a_n$ be a series of real numbers which converges but not absolutely. Suppose

$$-\infty \leq \alpha \leq \beta \leq \infty$$

Then there exists a rearrangement $\sum a'_n$ with partial sums s'$_n such that$

$$\lim_{n\to\infty} inf\ s'_n = \alpha \qquad \lim_{n\to\infty} sup\ s'_n = \beta$$

10 Bibliography

- The classical theory of rearrangements, Agana, M. J.,

  $https : //scholarworks.boisestate.edu/cgi/viewcontent.cgi?article = 2052amp$

- Riemann series theorem, Wikipedia,

  $https : //en.wikipedia.org/wiki/Riemann_s eries_t heorem$

- Riemann's paradox: pi = infinity minus infinity, Mathologer

  $https : //www.youtube.com/watch?v = -EtHF5ND3_s$

- Infinite Series - Numberphile

  $https : //www.youtube.com/watch?v = Jwtn5_d 2YCs$

- Riemann - Wikipedia

  $https : //en.wikipedia.org/wiki/Bernhard_R iemann$

- Principles of Mathematical Analysis, Walter Rudin

- On Conditionally Convergent Series, Werner Horn and Madjiguene Ndiaye

  $http : //www.csun.edu/\ hcmth017/riemann1/riemann1.pdf$

- Calculus Notes, Grinshpan

  $https : //www.math.drexel.edu/\ tolya/123_h armonic.pdf$

- Rearranging the Alternating Harmonic Series, Carl C. Cowen

  $https : //www.math.iupui.edu/\ ccowen/ButlerAHslides.pdf$