Notes on Probability Mass Functions

Probability Mass Functions

Note: I'm not going to refer to the sample space Ω in the main body of these notes.

In class we talked about random variables X, which we think of formally as numerical summaries of a random process. For example:

- You roll a die several times and want to summarize this process with X, the number of unique faces that we saw.
- You enter a raffle many times, and want to summarize this process with X, the amount of money that you won overall.
- You record the license plates of cars that drive by your apartment and summarize this process with X, the proportion of license plates that come from other states (not California).

Just as a random process can give you a different outcome each time you repeat it, the random variable X that summarizes that process can take on different values. We call the set of values that X can potentially take the *range* or *support* of X, and write it as $\mathcal{X}$. The *distribution* of X defines how likely X is to take any one of the values in its range $\mathcal{X}$. For the examples we consider in this class, where the range $\mathcal{X}$ is finite (that is, X can only take finitely many values), we can represent this distribution using a *probability mass function*, or pmf for short[1].

A probability mass function (pmf) takes any value $k \in \mathcal{X}$ (that is, any value, which we will write as k, from the range $\mathcal{X}$), and returns the probability that X will take that value. We write this function using the notation $P(X = k)$.

Note that the symbol k here is just a placeholder variable for the *argument* of the pmf. We could equivalently use x or $\heartsuit$ here, and write the same pmf as $P(X = x)$ or $P(X = \heartsuit)$. The important thing is that the pmf is a *function* of the placeholder variable.

Example: Binomial Random Variable

In class, we talked a binomial random variable X that we define as follows: Flip a coin n times. Each flip of the coin is independent, and has probability p of coming up heads. Define X as the number of heads that we see after flipping the coin n times.

In this case, the range of X is $\mathcal{X} = \{0, \cdots, n\}$, since we can see as few as 0 heads or as many as n heads. The probability mass function for X is

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{, or equivalently,} \quad P(X = \heartsuit) = \binom{n}{k} p^{\heartsuit} (1-p)^{n-\heartsuit}$$

[1] The range of $\mathcal{X}$ of a random variable X is *not* the same as the sample space Ω. Ω is the *domain* of X; X takes outcomes in the sample space Ω and summarizes them with a number contained in its range $\mathcal{X}$. In general, Ω is a set of full descriptions of ways that the random process could have turned out (e.g., sequences of heads and tails), but $\mathcal{X}$ is a set of numbers.

for any $k \in \mathcal{X}$ (or any $\heartsuit \in \mathcal{X}$).

Using this function, you can answer any question of the form "What is the probability that we see m heads?" by plugging in m for k and evaluating the expression. For example, if n is greater than or equal to 10 and we ask, "What is the probability that we see 10 heads?", we can plug in 10 for k and get our answer:

$$P(X = 10) = \binom{n}{10} p^{10}(1-p)^{n-10}.$$

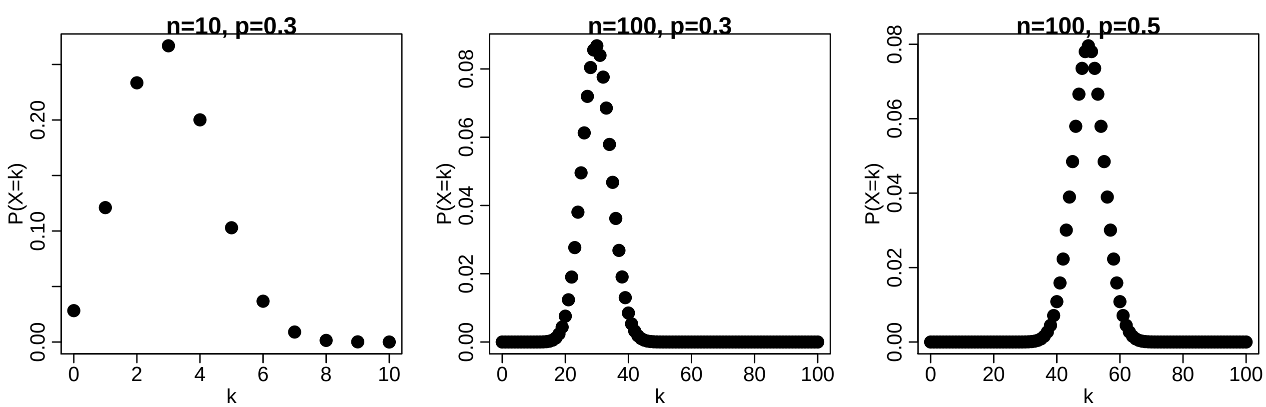
Visualization

Probability mass functions are useful for visualizing the distributions of random variables. For each value $k \in \mathcal{X}$, we can plot the probability that X takes this value. Thus, the probability mass function tells you exactly how to draw a distribution plot. It can be intuitive to think that whenever we define a probability mass function, we are actually defining one of these plots.

Example: Binomial Distribution Plots

Using the pmf defined in the last example, if we supply a number of coin flips n and a probability of flipping heads p, we can plot the distribution of X, the number of heads that we would see if we flipped this coin n times.

For example, here are the distribution plots for three different settings of n and p:



These visualizations help us make observations like the following. When we run a coin flipping procedure like the one described:

- In all three cases, the most likely value that X (the number of heads) will take is np.
- It is very unlikely that we'll see X take a value in certain parts of the range $\mathcal{X}$. For example, in both cases where $n = 100$, it's very rare to see a number of heads that differs from the most likely value np by more than 10 heads.