## Notes on Probability Mass Functions

### Probability Mass Functions

**Note:** I'm not going to refer to the sample space $\Omega$ in the main body of these notes.

In class we talked about random variables $X$, which we think of formally as numerical summaries of a random process. For example:

- You roll a die several times and want to summarize this process with $X$, the number of unique faces that we saw.
- You enter a raffle many times, and want to summarize this process with $X$, the amount of money that you won overall.
- You record the license plates of cars that drive by your apartment and summarize this process with $X$, the proportion of license plates that come from other states (not California).

Just as a random process can give you a different outcome each time you repeat it, the random variable $X$ that summarizes that process can take on different values. We call the set of values that $X$ can potentially take the *range* or *support* of $X$, and write it as $\mathcal{X}$. The *distribution* of $X$ defines how likely $X$ is to take any one of the values in its range $\mathcal{X}$. For the examples we consider in this class, where the range $\mathcal{X}$ is finite (that is, $X$ can only take finitely many values), we can represent this distribution using a *probability mass function*, or pmf for short[1].

A probability mass function (pmf) takes a particular value $k \in \mathcal{X}$ (that is, a particular value from the range $\mathcal{X}$, which we will write as $k$), and returns the probability that $X$ could take that value. We write this function using the notation $P(X = k)$.

Note that $k$ here is just a placeholder value, and we could just as easily have used $x$ or $a$ to represent the argument to the pmf. In those two cases, we would write the same pmf as $P(X = x)$ or $P(X = a)$. The important thing here is that the pmf is a *function* of the placeholder value.

### Example: Binomial Random Variable

In class, we talked a Binomial random variable $X$ that we define as follows: Flip a coin $n$ times. Each flip of the coin is independent, and has probability $p$ of coming up heads. Define $X$ as the number of heads that we see after flipping the coin $n$ times. A random variable defined in this way is called a *Binomial* random variable.

In this case, the range of $X$ is $\mathcal{X} = \{0, \cdots, n\}$, since we can see as few as 0 heads or as many as $n$ heads. The probability mass function for $X$ is

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k},$$

[1] The range of $\mathcal{X}$ of a random variable $X$ is *not* the same as the sample space $\Omega$. $\Omega$ is the *domain* of $X$; $X$ takes outcomes in the sample space $\Omega$ and summarizes them with a number contained in its range $\mathcal{X}$. In general, $\Omega$ is a set of full descriptions of ways that the random process could have turned out (e.g., sequences of heads and tails), but $\mathcal{X}$ is a set of numbers.

for any $k \in \mathcal{X}$.

Using this function, you can answer any question of the form "What is the probability that we see $m$ heads?" by plugging in $m$ for $k$ and evaluating the expression. For example, if $n$ is greater than or equal to 10 and we ask, "What is the probability that we see 10 heads?", we can plug in 10 for $k$ and get our answer:

$$P(X = 10) = \binom{n}{10} p^{10}(1-p)^{n-10}.$$

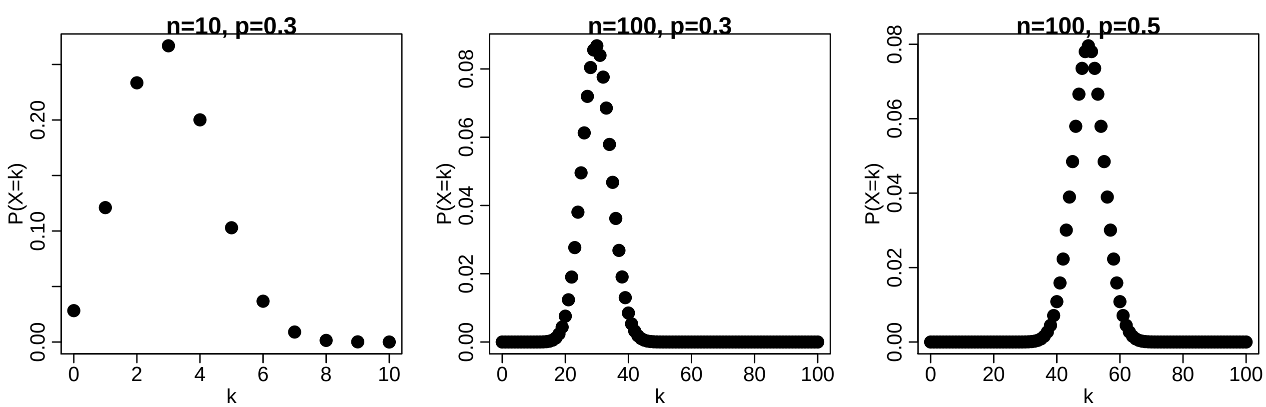
**Visualization**

Probability mass functions are useful for visualizing the distributions of random variables. For each value $k \in \mathcal{X}$, we can plot the probability that $X$ takes this value. Thus, the probability mass function tells you exactly how to draw a distribution plot. It can be intuitive to think that whenever we define a probability mass function, we are actually defining one of these plots.

**Example: Binomial Distribution Plots**

Using the pmf defined in the last example, if we supply a number of coin flips $n$ and a probability of flipping heads $p$, we can plot the distribution of $X$, the number of heads that we would if we flipped this coin $n$ times.

For example, here are the distribution plots for three different settings of $n$ and $p$:




These visualizations help us make observations like the following. When we run a coin flipping procedure like the one described:

- The most likely value that $X$ (the number of heads) will take is $np$.
- It is very unlikely that we'll see $X$ take a value in certain parts of the range $\mathcal{X}$. For example, when $n = 100$ and $p$ is moderate, it's very rare to see a number of heads that differs from the maximum value $np$ by more than 10 heads.