Discrete Structures II

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Combinatorics and Probability

- Loosely speaking, combinatorics is the area of mathematics concerned with "counting". This has a natural relation to discrete mathematics. Recall countably infinite sets.
- Combinatorics deals with the processes of counting. That is, how to count. But it also deals with counts themselves. Finding the number of things that satisfy a certain property, for example.
- In this chapter we will explore some of the principles of counting. It's not as easy as it sounds! This includes formulas for counting, the inclusion-exclusion principle, the pigeonhole principle. We will explore applications in permutations, combinations, and discrete probability.

The Principle of Counting

- There are many algorithmic ways of counting.
  1. The Product Rule
  2. The Sum Rule
  3. The Sum and Product together

The Product Rule

Definition 1.1: Product Rule

Given two choices with n possibilities for the first and m possibilities for the second, then there are a total of $n \times m$ different combinations of choices.

- Consider the set $S = \{a, b, c\}$ How many different sequences of length 3 can be constructed using the elements of S?

$$\begin{array}{ccc} a,a,a & b,a,a & c,a,a \\ a,a,b & b,a,b & c,a,b \\ a,a,c & b,a,c & c,a,c \\ a,b,a & b,b,a & c,b,a \\ a,b,b & b,b,b & c,b,b \\ a,b,c & b,b,c & c,b,c \\ a,c,a & b,c,a & c,c,a \\ a,c,b & b,c,b & c,c,b \\ a,c,c & b,c,c & c,c,c \end{array}$$

- The first term has 3 possible choices a, b or c. The Second term has 3 possible choices a, b or c, Same as the third term i.e a, b or c.
- So the total possible combination of choices is $3 \times 3 \times 3 = 27$

The product rule for sets

- The product rule can be made formal by considering sets and their Cartesian product.
- Given two sets A and B, the number of elements in $A \times B$ is $|A| \cdot |B|$
- The Cartesian Product $A \times B$ is all tuples a, b where $a \in A$ and $b \in B$.
    1. For the first coordinate, we have $n = |A|$ possible choices.
    2. The second coordinate has $m = |B|$
    3. Therefore the total number of tuples a, b is $n \times m = |A| \cdot |B|$
- Generally, Let $A_1, A_2, A_3, \ldots, A_k$ be finite sets. Then

$$|A_1 \times A_2 \times A_3 \times \cdots \times A_k| = |A_1| \cdot |A_2| \cdot |A_3| \cdots |A_k|$$

The Product Rule

Example 1.1: Counting Licence Plates

- Consider a province where the license plate format is $XYZABC$. Where X, Y, Z are capital letters and A, B, C are decimal digits. How many possible license plates are there?
- We could proceed intuitively using the product rule. There are 26 possible choices of letters for each of X, Y, Z. There are 10 possible choices of digits for each of A, B, C
- Therefore, the total number of license plates is:

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$$

- Formally, Let $L = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \dots, \mathrm{Y}, \mathrm{Z}\}$ and $N = \{0, 1, 2, \cdots, 8, 9\}$
- Clearly $|L| = 26$ and $|N| = 10$
- A license plate of the form $XYZABC$ can be viewed as a tuple $(l_1, l_2, l_3, n_1, n_2, n_3)$ and thus as an element of $L \times L \times L \times N \times N \times N$
- From the product rule applied to the sets:

$$\begin{aligned} |L \times L \times L \times N \times N \times N| &= |L| \cdot |L| \cdot |L| \cdot |N| \cdot |N| \cdot |N| \\ &= 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \\ &= 17,576,000 \end{aligned}$$

Sum Rule

- The sum rule applies when you must make a single choice from multiple groups.

Definition 1.2: Sum Rule

Given a choice, where one can choose either from a group of n or a group of m possibilities, then a total of $n+m$ different choices can be made.

- Consider picking your outfit of the day. In your closet you have 10 pairs of jeans and 5 pairs of Dior.
- How many different choices of pants do you have? 15, of course.
- You can wear one of your 10 pairs of jeans or one of your 5 pairs of diors (but not both).

Sum Rule

Example 1.2: Counting Pets

You are adopting one pet from the local SPCA. Good job!
The shelter currently houses 17 cats and 12 dogs. How many choices do you have for your new pet?
You can either choose a cat or choose a dog. Therefore, there are $17 + 12 = 29$ choices

The Sum Rule for Sets

- For two sets A and B, the number of elements in $A \cup B$ is $|A| + |B|$ so long as A and B share no elements
- This follows naturally from the union operation of disjoint sets. If A and B have no common elements, i.e. $A \cap B = \emptyset$, then their union contains all the elements of A and all the elements of B with no dublicates.
- Formally then:
    - Let$A_1, A_2, A_3, \ldots, A_k$ be finite pairwise disjoint sets. Then

$$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k| = |A_1| + |A_2| + |A_3| + |A_k|$$

The Sum and Product Rule Together

- The Sum and Product rule can be combined together.
- Say you have to make one choice from group A and a second choice from group A or group B
- Assuming the first and second choice can be the same choice, both the product rule and the sum rule apply.
- For the first choice, there are $|A|$ choices. For the second choice, there are $|A| + |B|$ possible choices.
- The total number of choices is thus $|A| \cdot (|A| + |B|)$

The Sum and Product Rule together

Example 1.3: Counting Programming Language Variables

In most programming languages, the first character of a variable's name must be a letter. From then on, characters may be numbers of letters. Assuming variables are case-sensitive, how many possible 4-character variable names can you create?

There are 26 lower-case letters and 26 upper-case letters. There are also 10 individual decimal digit characters.

For a 4-character variable name we have 4 choices. The first choice must be a letter and thus there are $26 + 26 = 52$ possible choices. The second, third, and fourth choice can be a letter or a number, and thus there are $26 + 26 + 10 = 62$

In total we have:

$$52 \cdot 62 \cdot 62 \cdot 62 = 12,393,056$$

possible 4-character variable names.

The Sum and Product Rule together

Example 1.4: Counting Rules

In some computer system, accounts must have a case-insensitive password of 6 to 8 characters in length. Those characters must only be letters, numbers, or one of `!`, `$`, or, `?`
How many possible passwords are there to choose from?
Solution

The Sum and Product Rule together

Example 1.5: Counting Rules

In some computer system, accounts must have a case-insensitive password of 6 to 8 characters in length. Those characters must only be letters, numbers, or one of !, $, or, ?
How many possible passwords are there to choose from?

Solution

For each character we can choose a letter, a number, or a special character. This is $26 + 10 + 3 = 39$ choices
Passwords can be either 6, 7, or 8 characters long.
All length-6 passwords count: 39^6
All length-7 passwords count: 39^7
All length-8 passwords count: 39^8
Therefore, the total number of possible passwords is:

$$39^6 + 39^7 + 39^8 = 5,492,759,010,921$$

The Principle of Inclusion-Exclusion

- The sum rule assumes that the sets from which you are choosing are disjoint.
- This is a strong assumption. When the sets to choose from are not disjoint, special care is needed so that choices are not double counted.

Proposition 1.1: The Principle of Inclusion-Exclusion

or two sets A and B the cardinality of $A \cup B$ is given by:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- When A and B are disjoint sets, the sum rule tell us that $|A \cup B| = |A| + |B|$.
- When they are not disjoint, we must subtract the elements in $A \cap B$ to avoid double-counting those shared elements. This proposition can be called the subtraction rule.

The Principle of Inclusion-Exclusion

- More generally, the inclusion-exclusion principle applies to exclude items which are double counted, but then include items which have been doubly excluded.
- This is easiest to see with the union of three sets.

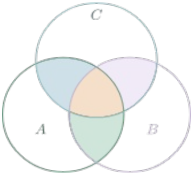

Figure: A Venn Diagram of three sets

- If we want to count the number of elements in the union of A, B and C, there are many possibilities for double counting. The full formula is:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

- This idea is, if you have 9 holes and 10 pegions, then one hole must have more than one pegion.
- When generalized, the idea is called the pigeon hole principle.

Theorem 1.1: Pegion Hole Principle

Let k be a positive integer. If there are more than k objects to be plasced into k boxes, then at least one box contains two or more objects.

Proof.

Proceed by contrapositive. We look to prove that if every box contains fewer than 2 objects, then there are k or fewer objects. There are k boxes. Each box contains at most one object. Thus, the total number of objects is at most k $\square$

Pigeon Holes as functions

- Recall from Injective, Surjective, Bijective an important fact about bijective functions. If a bijection exists between sets A and B then $|A| = |B|$
- What if $|A| > |B|$?
- Since functions are always total, For $f : A \rightarrow B$, every element a of A must have an associated output $f(a)$
- When $|A| > |B|$, notice what the total property implies: the pigeon hole principle!
- Let P be the set of pigeons and H be the set of pigeon holes with $|P| > |H|$, then figure 2 follows.

Pigeon Holes as functions

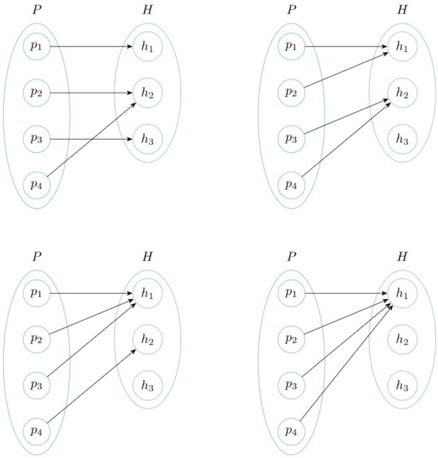

Figure: Pigeon hole as a function

Theorem 1.2: Generalized pigeon hole principle

If N objects are placed into k boxes, then there is at least one box containing $\lceil N/k \rceil$ or more objects

Proof.

Proceed by contrapositive. Suppose that none of the k boxes contain more than $\lceil N/k \rceil - 1$ objects. Then, the total number of objects must be $k \times (\lceil N/k \rceil - 1)$

However

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N$$

Thus there are strictly less than N objects, a conctradiction. Therefore, it must be that at least one box contains at least $\lceil N/k \rceil$ objects. □

Generalized Pigeon Hole

Example 1.6: Birthdays and generalized pigeons

In a group of100 people, there are at least 9 people which are born in the same month.

Why? Generalized pigeon hole principle. There are 12 months in the year. By the generalized pigeon hole principle, there must be at least one month with

$$\lceil 100/12 \rceil = 9$$

or more, people sharing that month as their birthday.

1. Recall that a byte consists of 8 binary digits. How many different bytes are there that begin with 1 or end with 0?
   Solution

1. Recall that a byte consists of 8 binary digits. How many different bytes are there that begin with 1 or end with 0?

   Solution

   - In a byte there are 8 bits, and each bit belongs to {0,1}
   - When a byte begins with 1, there are 7 remaining bits to choose from and so 2^7 possible bytes.
   - When a byte ends with 0, there are 7 remaining bits to choose from and so 2^7 possible bytes.
   - Now, how many bytes both start with 1 and end with 0? Those number of bytes were double counted. If we fix the first and last bit, there are 6 remaining bits to choose from and so 2^6 such bytes.
   - In total we have $2^7 + 2^7 - 2^6 = 192$ possible bites

2. How many cards must you draw from a standard 52-card deck to guarantee that you will have 4 cards of the same suit?

Exercises

3. On the internet, every computer must have an IP address. In Version 4 of the IP standard, there are 3 possible "classes" of IP addresses for computers connected to the internet. Class A is a 31-bit binary number. Class B is a 30-bit binary number. Class C is a 29-bit binary number. If no IP address can have every bit assigned to 1, nor every bit assigned to 0, how many possible IP addresses are there for computers connected to the internet?
4. 9 sports teams participate in a round robin, where every teams plays every other team, and ties are not allowed. Prove that if no team loses all of its games, then there must be at least two teams that end up with the same number of wins.
5. A class of 42 students is required to break into 12 groups of size no more than 6. Show that there are at least 5 groups of 3 or more students. (Hint: consider a proof by contradiction on the contrapositive)

- In this section we will extend the idea of counting to permutations and their closely related sibling, combinations.
- Both of these concepts extend the idea of choosing items from a set (product rule and sum rule) to consider additional replacement or, rather, lack thereof.

Definition 2.1

A permutation of a set of objects is an ordered arrangement of those objects. An ordered arrangement of r objects is a called an r-permutation.

- For a A of size n, an n-permutation of A is simply some ordering of the elements of A.

Example 2.1: Permutations of a Set

Let $A = \{a, b, c\}$.
A 3-permutation of A is a, b, c.
Another 3-permutation of A is c, a, b.
Another 3-permutation of A is b, a, c.
A 2 permutation of A is a, b another is b, a.

Permutations

- A fundamental consideration of permutations is the number of possible permutations.
- For a set of size n there is a precise number of possible r-permutations
- This number is denoted $P(n, r)$

Theorem 2.1

For positive integers n, r with $n \geq r$ the number of r-permutations on a set of n elements is

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

Proof.

Use the product rule. From a set of size n we have n possible choices After that choice, there are $n-1$ remaining elements of the set to choose from. After that choice, there are $n-2$ remaining elements to chose from. Continue in this way for r choices. By the product rule, this results in:

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

□

Permutations

Corollary 2.1

For positive integers n, r with $n \geq r$, we have

$$P(n,r) = \frac{n!}{(n-r)!}$$

Example 2.2: Counting Contest Winnder

Consider a contest with 56 participants. Assuming there are no ties, how many ways can first, second, and third place be awarded? To choose first place, there are 56 choices. To choose second, there are then 55 To choose third, there are 54 choices. By the product rule, there are $56 \cdot 55 \cdot 54 = 166320$ different ways to award first through third place.

Alternatively, we could use the formula for $P(56,3) = \frac{56!}{53!} = 166320$

Combinations are simply permutations where order does not matter.

Definition 2.2: Combination

An r combination is an unordered arrangement of r elements of a set. An r combination of a set is a subset with r elements

For a set A of size n and n-combinations of A is just A itself. Since both sets and combinations do not care about ordering, they are the same thing if they have the same number of elements.

Example 2.3: Combinations of a Set

Let $A = \{a, b, c\}$
The only 3-combination of A is a, b, c
The 2-combinations of A are a, b; a, c; and b, c

Combinations

The number of r-combinations of a set of size n is denoted by $C(n, r)$. Its value is given by the following theorem.

Theorem 2.2

For non-negative integers n, r with $n \geq r$, the number of r-combinations of a set of n elements is

$$C(n, r) = \frac{n!}{(n-r)! \cdot r!}$$

Proof.

Combinations are just permutations which do not care about order. Given an r-permutation, there are $P(r, r)$ possible orderings of that permutation. Therefore, for each r-combination there are $P(r, r)$ different possible permutations. Thus,

$$C(n, r) = P(n, r)/P(r, r) = \frac{n!}{(n-r)!} \,/\, r!$$

□

Combinations

Example 2.4: Counting soccer teams

In a grade 4 gym class, there are 28 students. How many ways can these students be separated into 2 soccer teams

The order in which students are assigned to each team does not matter, only the composition of the teams. Thus, this is a question about combinations.

$$C(28, 14) = \frac{28!}{14! \cdot 14!} = 40,116,600$$

Corollary 2.2

Let n, r be non-negative integers with $n \geq r$

$$C(n, r) = C(n, n - r)$$

Combinations

Proof of Corollary 2.2.

From theorem 2.2, we have:

$$C(n,r) = \frac{n!}{(n-r)! \cdot r!}$$

and

$$C(n,n-r) = \frac{n!}{(n-(n-r))! \cdot (n-r)!}$$

$$= \frac{n!}{(n-r)! \cdot r!}$$

□

- This corollary tells us that there are the same number of r-combinations as there are $(n-r)$-combinations
- $C(n,r)$ appears in many places throughout mathematics. Thus, we give it some special names and notations.
- $C(n,r) = \binom{n}{r}$ and is read "n choose r". $\binom{n}{r}$ is also called a binomial coefficient.

Binomial Coefficients

- A binomial is a polynomial with 2 terms. We all know that:

$$(a+b)(c+d) = ac + ad + bc + bd.$$

- When $a = c$ and $b = d$ this equation simplifies to

$$(a+b)^2 = a^2 + 2ab + b^2.$$

- This can be generalized to any power n. We can use the principle of counting to compute $(x+y)^n$ without doing any algebra.

Theorem 2.3: Binomial Theorem

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

Binomial Coefficients

- Expanding the binomial theorem for $n = 3$

$$(x+y)^3 = \binom{3}{0}x^3y^0 + \binom{3}{1}x^2y^1 + \binom{3}{2}x^1y^2 + \binom{3}{3}x^0y^3$$

$$= \binom{3}{3}x^3y^0 + \binom{3}{2}x^2y^1 + \binom{3}{1}x^1y^2 + \binom{3}{0}x^0y^3$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

Binomial Coefficients

Example 2.5

- What is the coefficient of $x^{13}y^{11}$ in $(x+y)^{24}$
- We have $x^{24-11}y^{11}$ By the binomial theorem, its coefficient is

$$\binom{24}{11} = \frac{24!}{(24-11)!(11!)} = 2496144.$$

Binomial Coefficients

Recall that combinations are really just subsets and $C(n, r) = \binom{n}{r}$. Thus, the following corollary follows naturally.

Corollary 2.3

For a non-negative integer n

$$\sum_{r=0}^{n} 2\binom{n}{r} = 2^n$$

Proof.

Apply the binomial theorem to $2^n = (1 + 1)^n$ □

Exercises

1. Suppose you are interviewing 25 people in order to fill 5 job vacancies. How many possible choices of 5 people are there to fill those vacancies?
2. Given a standard 52-deck of playing cards, how many different 5-hand flushes are there. That is how many different sets of 5 cards all have the same suit?
3. What is the coefficient of x^9y^{12} in $(4x - 9y)^{21}$

Discrete Probability

Now that we know how to count objects, permutations, and combinations, a natural extension is to ask what is the chance or probability of a particular permutation, combination, etc. actually occurring?

Finite Probability

- Finite probability derives from Pierre-Simon Laplace's classic theory of probability. In this context we have three key terms.

Definition 3.1: Experiment

An experiment is a procedure which yields a particular outcome from a set of possible outcomes.

Definition 3.2: Sample Space

The sample space of an experiment is the set of all possible outcomes.

Definition 3.3: Event

An event is a subset of a sample space.

Finite Probability

- The simplest method of determining the chance of a particular outcome occurring is counting the number of occurrence of that outcome and dividing by the total number of possible outcomes.
- If you roll a fair 6-sided die, there is $\frac{1}{6}$ chance of rolling a 4
- When we can do simple counting and division to compute probability, the outcomes are called equally likely.

Definition 3.4: Equally Likely Outcomes

In a finite sample space S if all outcomes are equally likely, then the probability of an event E occuring is $p(E) = |E|/|S|$

Finite Probability

- In a sample space of equally likely outcomes, we always have $0 \leq p(E) \leq 1$ for any event E
- Indeed, since $E \subset S$ and S is finite, then $0 \leq |E| \leq |S|$

Example 3.1: Rolling Dice

What is the probability of rolling two fair 6-sided dice and having their sum equal 9?

There are 36 possible outcomes when rolling two dice. Indeed, each outcome is $(d_1, d_2) \in \{1, 2, 3, 4, 5, 6\}^2$.

Of those outcomes there are $(3, 6), (4, 5), (5, 4)$ and (6,3) which add to 9. Therefore, the probability is $4/36 = 1/9$

With and without replacement

- When taking a sample (one step in an experiment), there are different ways for that sample to interact with the next.
  1. "With replacement" means that a sample does not affect the probability of the next.
  2. "Without replacement" means that a sample does affect the probability of the next.
- This terminology comes again from the classic example of choosing marbles from urns.

With and without replacement

Example 3.2: Urn Without Replacement

In an urn filled with 12 blue marbles and 6 green marbles, what is the probability that a marble chosen from the urn is blue?

There are 18 possible outcomes in the sample space because there are 18 total marbles.

The event E we are interested in is obtaining a blue marble. There are 12 of them. Thus:

$$p(E) = 12/18 = \frac{2}{3} = 0.\overline{6}$$

Example 3.3: Urn With and Without Replacement

Assume that an urn contains 20 marbles. 12 are blue and 8 are green. What is the probability of first drawing a blue marble and second drawing a green marble, when:

With and without replacement

Example: Urn With and Without Replacement Cont'd

1. The first marble is put back into the urn before choosing the second (with replacement), and
2. The first marble is not put back into the urn (without replacement).

Solution

1. With replacement, there are $20 \cdot 20 = 400$ possible outcomes by the product rule. On the first choice $12/20$ marbles are blue. On the second $8/20$ choice marbles are green. There is a $12/20$ chance of drawing blue on the first draw. There is an $8/20$ chance of drawing green on the second draw. In total, the number of outcomes which have blue first and green second is $12 \cdot 8 = 96.96/400 = 0.24$

Example: Urn With and Without Replacement Cont'd

2. Without replacement, there are $20 \cdot 19$ possible outcomes, since the first choice reduces the number of outcomes of the second choice. Therefore, 380 possible outcomes. There are still $12 \cdot 8$ outcomes which have blue first and green second. Thus, the total probability is $96/380 \approx 0.2526$

Probability of Complements and Unions

- If we know the probability of an event occurring, what is the probability of an event not occurring?

Theorem 3.1

Let E be an event in the sample space S The probability of the complementary event $\overline{E} = S \setminus E$

$$p(\overline{E}) = 1 - p(E)$$

Proof.

The sample space is finite. We know that $|\overline{E}| = |S| - |E|$. Thus

$$p(\overline{E}) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E).$$

$\square$

Probability of Complements and Unions

Example 3.4: Bits of probability

- The bits of a byte are chosen randomly. What is the probability that at least one bit is 1?
- Let E be the event that at least one of those bits is 1. Then, $\overline{E}$ is the vent that none of the bits are 1
- For each bit we have two choices:0 or 1. Thus, $p(\overline{E}) = \frac{1}{2^8}$. There is exactly 1 byte with all 0s. By the product rule, there are 2^8 possible choices for all the bits in a byte.
- Therefore, the probability of having at least one bit being 1 is $p(E) = 1 - \frac{1}{2^8} = \frac{255}{256}$

Probability of Complements and Unions

Theorem 3.2

Let E_1 and E_2 be two events in the sample space S. The probability that either event occurs is:

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Proof.

This follows directly from the inclusion-exclusion principle applied to the sets E_1 and E_2

$$\begin{aligned} p(E_1 \cup E_2) &= \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} \\ &= p(E_1) + p(E_2) - p(E_1 \cap E_2) \end{aligned}$$

□

Probability of divisibility

Example 3.5: Probability of divisibility

- Given a randomly chosen positive integer less than 1000, what is the probability that the chosen number is divisible by 3 or is divisible by 5?
- Every third number is divisible by 3, so there are $\lfloor 1000/3 \rfloor = 333$ numbers divisible by 3
- Every third number is divisible by 5, so there are $\lfloor 1000/5 \rfloor = 200$ numbers divisible by 5
- But, how many numbers are divisible by 3 and 5?. Since 3 and 5 are co-prime, the only way to be divisible by both is to be a multiple of 15. There are $\lfloor 1000/15 \rfloor = 66$ such numbers.
- Thus, the probability of the number being divisible by 3 or 5 is:

$$\frac{333}{1000} + \frac{200}{1000} - \frac{66}{1000} = \frac{467}{1000} = 0.467$$

Exercises

1. In a lottery, the grand prize occurs when a player picks the same four digits, in order, as those chosen by a random mechanical process. Assume repeated numbers are allowed. What is the probability of winning? What is the probability of getting 3 out of 4 numbers correct?
2. In a lottery, a winner is determined by choosing 6 different numbers from the set $\{1, 2, 3, \dots, 49\}$. What is the probability of choosing the same 6 numbers?
3. In a roll of two 6-sided fair dice, what is the probability that the sum of the numbers showing on the dice is less than 6?
4. In 5-card poker, 5 cards from a standard 52-card is dealt to each player.
   a. What is the probability of being dealt a flush (all 5 cards of the same suit)?
   b. What is the probability of being dealt a flush or a straight (5 cards, of any suit, in sequence; Ace is high or low)?