Lesson 6

Mapping by Elementary Functions

6.1 Introduction

Suppose $w = f(z) = u(x,y) + iv(x,y) = u + iv$ be a function of a complex variable z with

$$z = x + iy$$

To each pair of values (x, y) there correspond one value for u and another value for v, in

$$w = u + iv.$$

We utilize two separate complex planes for the representation of

$$z = (x, y) \text{ and } w = (u, v).$$

The two planes are called the z plane and the w plane respectively.

The relationship, $w = f(z)$ then establishes a connection between the points of a given region R in the z plane and the corresponding points of another region R' determined by

$$w = f(z) \text{ in the w plane}$$

In this lesson we shall study how various curves and regions in the z plane (with the x and the y axes) are mapped by elementary analytic functions on to the w plane (with the u and the v axes).

6.2 Objectives of the lesson

By the end of this lesson you will be able to:

i) define the mapping or transformation of points, curves and regions from z plane to w plane under a transformation function f(z).

ii) discuss the mapping by elementary functions such as polynomials, exponential, trigonometric and logarithmic functions.

6.3 Meaning of Mapping (or Transformation)

Consider

$$w = f(z) = 2z^2 + 3 \qquad (1)$$

where

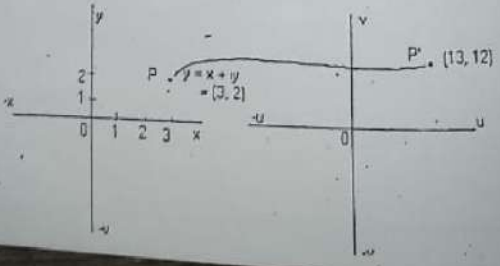
$$w = u + iv \text{ and } z = x + iy$$

Then

$$\begin{aligned} u + iv &= 2(x + iy)^2 + 3 \\ &= 2(x^2 - y^2 + 2xiy) + 3 \\ &= 2x^2 - 2y^2 + 3 + 2xiy \qquad (2) \end{aligned}$$

Equating the real and the imaginary parts in (2) we have

$$u = 2x^2 - 2y^2 + 3 \text{ and } v = 2xy$$



Consider any point say

$$z = 3 + 2i \text{ or } z = (3,2).$$

Then

$$u = 2x^2 - 2y^2 + 3 = 2(3)^2 - 2(2)^2 + 3 = 13$$

and

$$v = 2xy = 2(3)(2) = 12$$

Under the transformation,

$$w = 2z^2 + 3$$

where $z = (x, y) = (3, 2)$ and $w = (u, v) = (13, 12)$

We say that the point (3, 2) on the xy plane or z plane is mapped on to the point (13, 12) on the uv plane or w plane

The point (3, 2) is called the **object** on the z plane and the corresponding point P′ (13, 12) is called the **image** of P on the w plane.

$w = f(z)$ is called the **transformation function or mapping function.**

Example 1

Find the image of the point (4, 3) on z plane under the transformation $w = 2z^2 + 3$.

Solution

Since $u + iv = 2(x + iy)^2 + 3$

$$u = 2x^2 - 2y^2 + 3 = 2(16) - 2(9) + 3 = 17 \text{ and}$$

$$v = 2xy = 2(4)(3) = 24.$$

Hence the image of (4, 3) on z plane is (17, 24) on the w plane. We shall discuss some of the transformations in the following examples:

Example 2

Let $w = 3z + 4 - 5i = f(z)$

Find the values of w which corresponds to $z = -3 + i$ on the z plane.

Solution

Let $w = u + iv = 3z + 4 - 5i$ then,

Then

$$u + iv = 3(x + iy) + 4 - 5i$$

or

$$u + iv + (3x + 4) + i(3y - 5)$$

Then $u = 3x + 4$ and $v = 3y - 5$

When $x = -3$ and $y = 1$, $u = 3(-3) + 4 = -5$ and $v = 3(1) - 5 = -2$

thus the point $z = -3 + i$ is transformed (mapped) on to the point (-5, -2) on the w plane or $w = -5 - 2i$.

They are shown on the z plane and the w plane

Example 3

Explain the nature of the transformation $w = z^2$

Solution

Let $z = re^{i\theta}$ and $w = Re^{i\phi}$

Then $w = z^2$ becomes

$$Re^{i\phi} = \left(re^{i\theta}\right)^2$$

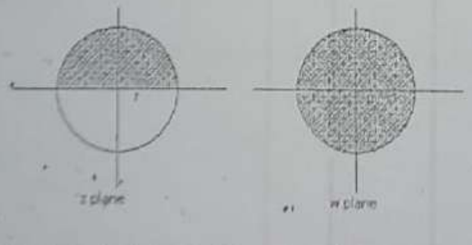
$$Re^{i\phi} = r^2 e^{i2\theta}$$

then $R = r^2$ and $\phi = 2\theta$

The range of variations of θ from $0 < \theta < \pi$ makes ϕ from $0 < \phi < 2\pi$

Let $|z| = r$ be the circle with radius r.

Points on the upper half of the circle on z plane map into the entire circle $|w| = r^2$



Thus the semi-circle of radius r in the z plane is mapped into the full circle of radius $R = r^2$ in the w plane.

6.4 The linear Transformation

The transformation

$$w = az + b$$

where a and b are real or complex constants is called a linear **Transformation**

6.5 The Bilinear (or Fractional) Transformation

The transformation

Substituting *(1)*, *(2)*, *(3)*, *(4)*, in $\frac{(w_4-w_1)(w_2-w_3)}{(w_2-w_1)(w_4-w_3)}$ we have

$$\frac{(w_4-w_1)(w_2-w_3)}{(w_2-w_1)(w_4-w_3)}=\frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)}$$

Hence the cross ratio of w_1, w_2, w_3, w_4 = cross ratio of z_1, z_2, z_3, z_4.

This is written as $(w_1, w_2, w_3, w_4)=(z_1, z_2, z_3, z_4)$

Example 4

Find a bilinear transformation which maps the point $z = 0, -i, -1$ on the z plane into $w = i, 1, 0$ respectively on the w plane.

Solution

Let the points $0, -i, -1, z$ on the z plane be transformed into the points $i, 1, 0, w$ respectively on the w plane.

The cross ratio of w_1, w_2, w_3, w_4 is the same as the cross ratio of z_1, z_2, z_3, z_4

Then $$\frac{(w_4-w_1)(w_2-w_3)}{(w_2-w_1)(w_4-w_3)}=\frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)} \quad , (1)$$

Substituting the values of z_1, z_2, z_3 and w_1, w_2, w_3 in the equation (1) and letting $w_4 =$ and $z_4 = z$ we will get the solution (Try this yourself!)

Example 5

Find a bilinear transformation which maps the points $z_1 = 2, z_2 = i, z_3 = -2$ into the poi $w_1 = 1, w_2 = i, w_3 = -1$ respectively.

Solution

Let a general point z be transformed into w under the same transformation.
Since the cross ratios of four points are preserved under a bilinear transformation
$(w, w_1, w_2, w_3)=(z, z_1, z_2, z_3)$

Then $$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{w-1}{w+1}=\frac{(z-2)(3+4i)}{5(z+2)i}=\frac{(z-2)(3i-4)}{(z+2)(-5)}$$

Substituting the given points we have,

$$w = \frac{az+b}{cz+d}, \quad \text{where} \qquad ad - bc \neq 0$$

is called a **bilinear Transformation.** It is also known as a **Fractional transformation** or **Mobias transformation.** Here, a, b, c, d are real or complex constants.

6.6 Cross Ratio of Four points z_1, z_2, z_3 and z_4

If z_1, z_2, z_3, z_4 are distinct then the ratio

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

is called cross ratio of z_1, z_2, z_3, z_4.

The cross ratios of z_1, z_2, z_3, z_4 can be written in six different ways

For example $\frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}$ is another way of writing the cross ratio of z_1, z_2, z_3, z_4. It is written as (z_1, z_2, z_3, z_4)

6.7 An important property of a Bilinear Transformation

If z_1, z_2, z_3, z_4 are four distinct points on the z plane and w_1, w_2, w_3, w_4 are the images of z_1, z_2, z_3, z_4 respectively under a bilinear transformation

$$w = \frac{az+b}{cz+d}$$

then the cross ratio of z_1, z_2, z_3, z_4 is equal to that of w_1, w_2, w_3, w_4

or
$$\frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)} = \frac{(w_4 - w_1)(w_2 - w_3)}{(w_2 - w_1)(w_4 - w_3)}$$

Proof

This property can be proved by direct substitution for w_1, w_2, w_3, w_4.

Consider $w_2 - w_3 = \frac{az_2 + b}{cz_2 + d} - \frac{az_3 + b}{cz_3 + d}$

$$= \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)} \qquad (1)$$

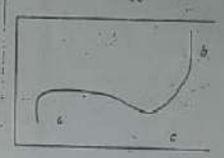
Similarly
$$w_4 - w_1 = \frac{(ad - bc)(z_4 - z_1)}{(cz_4 + d)(cz_1 + d)} \qquad (2)$$

$$w_2 - w_1 = \frac{(ad - bc)(z_2 - z_1)}{(cz_2 + d)(cz_1 + d)} \qquad (3)$$

$$w_4 - w_3 = \frac{(ad - bc)(z_4 - z_3)}{(cz_4 + d)(cz_3 + d)} \qquad (4)$$

I is called the line integral of $f(z)$ along curve c or the definite integral of $f(z)$ from a to b along the curve c.

7.4 Conditions for the limit I exists or $\int_c f(z)dz$ exists



If $f(z)$ is analytic at all points of a Region R and if the curve c is lying in R then the limit I exists and $f(z)$ is said to be integrable along c. The famous French Mathematician, Cauchy has discovered that $I = 0$ if the curve c is closed or a and b coincide.

7.5 All the formulae for integration of functions of real variables hold good for integration of functions of complex variables.

For example -

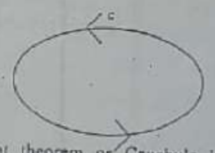
$$\int z^n dz = \frac{z^{n+1}}{n+1} + c, \quad \int \frac{dz}{z} = \ln z + c \text{ where } n \neq -1$$

$$\int \cos z dz = \sin z + c \text{ and } \int \sin z dz = -\cos z + c$$

Similarly all the other formulae for exponential, trigonometric and logarithmic functions hold good for the functions of complex variables.

7.6 Cauchy's Fundamental Theorem

If a function $f(z)$ is analytic inside and on a simple closed curve c then $\int_c f(z)dz = 0$



Cauchy's theorem is also called Cauchy - Goursat theorem or Cauchy's Integral Theorem.

7.7 Converse of Cauchy's Theorem or Morera's Theorem.

Let $f(z)$ be continuous in a simply connected region R and suppose that

$$\int_c f(z)dz = 0$$

around every simple closed curve c in R. Then $f(z)$ is analytic in R.

or $\frac{x^2}{c^2\cosh^2 v} + \frac{y^2}{c^2\sinh^2 v} = 1$ (3)

If $v = \pm\alpha$ a constant, then

$\frac{x^2}{c^2\cosh^2\alpha} + \frac{y^2}{c^2\sinh^2\alpha} = 1$ which represent an ellipse.

Also for $-\frac{\pi}{2} < u < \frac{\pi}{2}$ we have $\cos u$ is positive. For the line PQ, $v = \alpha$ and u varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, x varies from $-c\cos h\alpha$, to $+c\cos h\,\alpha$

Thus we conclude that the side PQ of the rectangle corresponds to upper half of the ellipse in z plane.

The line RS

In the same way we conclude that the side RS of the rectangle corresponds to the lower half of the ellipse (or y negative)

The line PS

$u = \frac{\pi}{2}$ and v varies from $-\alpha$ to α so that from (2), $v = 0$ $(\cos\frac{\pi}{2} = 0)$ and x varies from c $\cos h\alpha$ to c and then from c to $c\cos h\lambda$ according as v varies from $-\alpha$ to 0 and then from 0 to α Hence the rectangle enclosed by $u = \pm\frac{\pi}{2}$, $v = \pm\alpha$ in the w plane corresponds to the ellipse in the w plane corresponds to the ellipse in z plane with two slits.

Exercise 6

1. Define a mapping from z plane to w plane.
2. If the function is $f(z) = z + i$ find the image of the point p, $4 + 5i$ on the w plane.
3. A point $3 + bi$ on the z plane is mapped on to the point (11, c) on the w plane by the mapping function $f(z) = 2z^2 + 1$. find the values of b and c.
4. Define a bilinear Transformation.
5. Find a bilinear Transformation which maps $z = 1, i, -1$ respectively onto $w = i, 0, -1$.
6. Find a bilinear transformation which maps points $z = 0, -i, -1$ onto $w = i, 1, 0$ respectively
7. Find the invariant points of the transformation $w = \frac{2z-5}{z+4}$ (Hint: the fixed points are attained putting $z = \frac{2z-5}{z+4}$ and solving).
8. Define cross ratio of any four points z_1, z_2, z_3, z_4.

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)}=\frac{(z-2)(i+2)}{(z-i)(-2-z)}$$

$$\frac{(w-1)(1+i)^2}{(1-i)(1+i)(-1-w)}=\frac{(z-2)(2+i)^2}{(z-i)(z+i)(-2-z)}$$

$$\frac{(w-1)(2i)}{w+1}=\frac{(z-2)(3+4i)}{(z+2)(5)}$$

or $\dfrac{w-1}{w+1}=\dfrac{(z-2)(3+4i)}{5(z+2)i}=\dfrac{(z-2)(3i-4)}{(z+2)(-5)}$

or $\dfrac{w-1}{w+1}=\dfrac{3iz-4z-6i+8}{-5z-10}$ using componendo and dividendo

$\left(if\ \dfrac{a}{b}=\dfrac{c}{d}\text{ then }\dfrac{a+b}{b-a}=\dfrac{c+d}{d-c}\right)$

we have,

$$\frac{(w-1)+(w+1)}{(w+1)-(w-1)}=\frac{(3iz-4z-6i+\delta)+(-5z-10)}{(-5z-10)-(3iz-4z-6i+\delta)}$$

$$\frac{(w-1)+(w+1)}{(w+1)-(w-1)}=\frac{(3iz-4z-6i+\delta)+(-5z-10)}{(-5z-10)-(3iz-4z-6i+\delta)}$$

or $\dfrac{2w}{2}=\dfrac{-9z+3iz-6i-2}{-z-3iz+6i-18}$

or $w=\dfrac{3z(i-3)+2i(-3+i)}{iz(i-3)+6(i-3)}$

or $w=\dfrac{3z+2i}{(iz+6)}$ (2)

This is of the form $w=\dfrac{az+b}{cz+d}$. Hence the required transformation is given in 2.

6.8 The transformation $w=lnz$

Let $w=lnz$

Then $u+iv=ln(x+iy)$.

Raising both sides to the powers e we have

$$e^{u+iv}=x+iy$$

$$e^{u}e^{iv}=x+iy$$

$$e^{u}(\cos v+i\sin v)=x+iy$$

Then $e^{u}\cos v=x,\quad e^{u}\sin v=y$ (1)

Thus (x,y) becomes (e^u cos v, e^u sin v)

writing $w=lnz$ in another way we have

$u+iv=\ln\left(re^{i\theta}\right)$ since $z=re^{i\theta}$, we have $u+iv=lnr+i\,\theta$

Lesson 7

Complex Integration and Cauchy's Theorem

7.1 Introduction

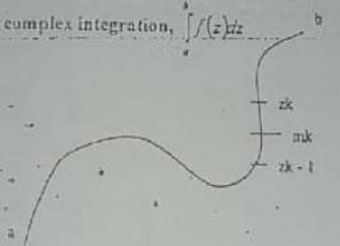
You have already studied the differentiation of the functions of a complex variables and Analytic function at a point on the z plane. In this lesson you will study the integration of functions of complex variables and Cauchy's fundamental Theorem in integration of $f(z)$ over a simple closed curve when $f(z)$ is analytic inside and on the closed curve.

7.2 Objectives of the lesson

By the end of this lesson you will be able to

i) state the meaning of complex integration along a curve or line,
ii) apply the meaning of complex integration along a line or curve.
iii) State Cauchy's fundamental theorem in complex analysis.
iv) Apply Cauchy's Theorem to evaluate integrals of functions of z.

7.3 Meaning of the complex integration, $\int_a^b f(z)dz$



let c be a curve of finite length and $f(z)$ be continuous at all points on the curve c. let the curve be subdivided into n parts by means of points $z_1\ z_2 \ldots\ldots z_{n-1}$ chosen arbitrarily.

Let us call the starting point a and the ending point b as z_0 and z_n respectively. Now the curve c is subdivided into n arcs from z_0 to z_n. Consider any one arc joining z_{k-1} to z_k. let m_k be a point on the arc $z_{k-1} z_k$ where k varies from 1 to n. let

$$\text{Let } s_n = f(m_1)(z_1 - a) + f(m_2)(z_2 - z_1) + \ldots f(m_n)(b - z_{n-1})$$

$$= \sum_{k=1}^{n} f(m_k)(z_k - z_{k-1})$$

$$= \sum_{k=1}^{n} f(m_k)\Delta z_k \text{ where } \Delta z_i = z_i - z_{i-1}$$

Let the number of subdivision, n increases and let the largest of the chord length $|\Delta z_i|$ tends to zero. Let the sum s_n approaches a limit I. We denote this limit I by

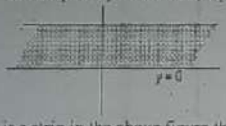
$$I = \int_a^b f(z)dz$$

iii) What will happen if $w = 0$?

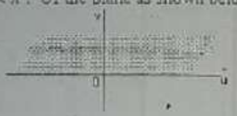
Since e^z is never zero $w = 0$ has no image on the z plane.

iv) What will be the image of $w = e^z$ if $0 < y < \pi$.

The strip $0 < y < \pi$ on the z plane



is a strip in the above figure then, taking logarithm on both sides of $e^z = w$ we have $z = \ln w + 2n\pi i$. Thus the infinite strip is mapped on the upper half $R > 0$ and $0 < \phi < \pi$. Of the plane as shown below:



Example 6

a) Find a bilinear transformation, which transforms the unit circle $|z| = 1$ into the real axis of the w plane in such a way that the points $z_1 = 1$, $z_2 = i$, $z_3 = -1$ are mapped onto $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$.

b) In what regions the interior and exterior of the circle are mapped.

Solution

Let a general point z be mapped onto w. Since bilinear transformations preserve cross ratio of four points we have

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

then

$$(w, 0, 1, \infty) = (z, 1, i, -1)$$

$$\frac{(w-0)(1-\infty)}{(0-1)(\infty-w)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}.$$

$$\frac{(w)(1-x)}{(-1)(x-w)} = \frac{(z-1)}{-1-z}\frac{(i+1)^2}{(1-i)(1+i)}, \qquad \text{where } x \to \infty$$

$$\frac{w\left(\frac{1}{x}-1\right)}{\left(\frac{x}{x}-\frac{w}{x}\right)} = \frac{(z-1)}{1+z}\frac{(2i)}{2}$$

$$\frac{w(-1)}{(1-0)} = \frac{-i(1-z)}{1+z}, \qquad \text{when } x \to \infty, \frac{1}{x} = \frac{w}{x} = 0$$

7.8 **Indefinite Integrals or antiderivative of $f(z)$**

If $f(z)$ and $F(z)$ are analytic in a region R and $F'(z) = f(z)$ then $F(z)$ is called an indefinite integral or $F(z)$ is called the anti-derivative of $f(z)$ denoted by

$$F(z) = \int f(z)dz \text{ or sometimes we write } F(z) = \int f(z)dz + c$$

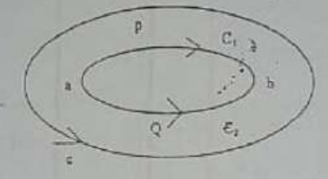
Example 1

Since we have $\frac{d}{dx}\left(5x^2 + e^{2x}\right) = 10x + 2e^{2x}$ we write $\int\left(10x + 2e^{2x}\right)dx = 5x + e^{2x} + c$

Here $5x^2 + e^{2x} + c$ is called the anti derivative of $10x + 2e^{2x}$.

Theorem

7.9 **If $f(z)$ is analytic inside and on a the boundary c of a simply connected region R, then $\int_a^b f(z)dz$ is independent of the path in R joining the points a and b in R.**



Proof

Let $\bar{C}$ be any simple closed curve enclosing the region R. Let a and b be two points in R. Let apb and aQb be two paths connecting a and b.

By Cauchy's Theorem

$$\int_{apbQa} f(z)dz = 0 \text{ since } f(z) \text{ is analytic inside and on } apbQa$$

$$\text{or } \int_{apb} f(z)dz + \int_{bQa} f(z)dz = 0$$

$$\text{or } \int_{apb} f(z)dz = -\int_{bQa} f(z)dz$$

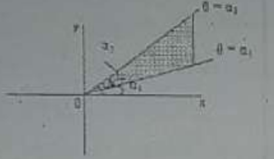
$$\text{or } \int_{apb} f(z)dz = -\int_{aQb} f(z)dz$$

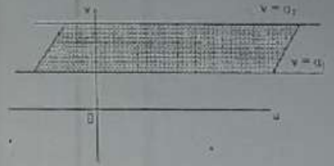
Hence the value of the integrals in the two paths apb and aQb are the same.

Theorem

Hence $u = lnr$ and $v = \theta$ (2)

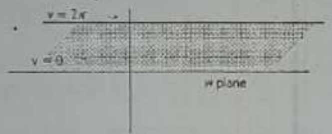
Consider the lines $\theta = \alpha_1$ and $\theta = \alpha_2$ on the z plane



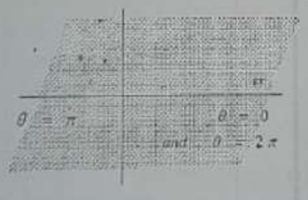


Hence we see that the area between $\theta = \alpha_1$ and $\theta = \alpha_2$ on the z plane is mapped on to the finite strip between $v = \alpha_1$ and $v = \alpha_2$ on the w plane.

The infinite strip $0 \le v \le 2\pi$



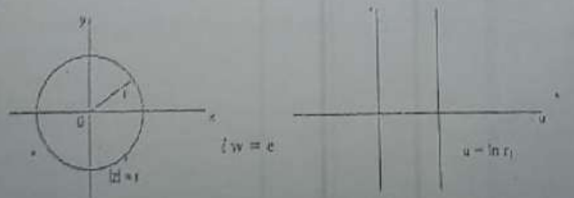
Let $v = 0$ on the w plane. Then $\theta = 0$ on the z plane. If $v = 2\pi$ on the w plane then $\theta = 2\pi$ on the z plane.



Hence the infinite strip $v = 0$ to $v = 2\pi$ is transformed into the whole of the z plane from $\theta = 0$ to $\theta = 2\pi$

The image of the circle with radius $r = r_1$

All the circles defined by $r = r_1$ in the z plane are mapped on to the straight lines $u = \ln r_1$



or $w = \frac{i(1-z)}{1+z}$

is the required transformation.

6.9 Fixed points of a bilinear transformation

If a point $z_1 = x_1 + iy$, on the z plane may have the image w = u + iv. The points which coincide with their images under a bilinear transformation are called **Fixed points** of the transformation.

If P is a fixed point of the bilinear transformation $w = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$ then w and z will be equal.

Hence $z = \frac{az+b}{cz+d}$

or $cz^2 + dz = az + b$

or $cz^2 + (d-a)z - b = 0$

then $z = \frac{(a-d) \pm \sqrt{(a-d)^2 - 4(c)(-b)}}{2c}$

or

$$z = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4(bc)}}{2c}$$

The two values of z are the **fixed points** of the bilinear transformation.

The nature of fixed points

i) If a = d, then the fixed points are $\pm \frac{2\sqrt{bc}}{2c} = \pm \frac{\sqrt{bc}}{c}$

ii) If c = 0 and $a \neq d$ we have one fixed point is finite and other is infinite.

iii) If $c \neq 0$ and $(a-d)^2 + 4bc$ is positive then there will be two finite fixed points.

Example 7

Find the fixed points of the bilinear transformation $w = \frac{3z-4}{3-1}$.

Solution

At the fixed points z = w.

Hence $z = \frac{3z-4}{3-1}$

or

$$z^2 - 4z + 4 = 0$$

or

$$(z-2)^2 = 0$$

then $z = 2$ is the only fixed point or we say that fixed points 2, 2 coincide.

Example 8

Find the fixed points of the bilinear transformation $w = \frac{z-1}{z+1}$

Solution
If the fixed points $z = w$
Hence

$$z = \frac{z-1}{z+1}$$

or

$$z^2 + 1 = 0 \quad \text{or} \quad z^2 = i^2$$

Hence $z = \pm i$ are the two distinct fixed points.

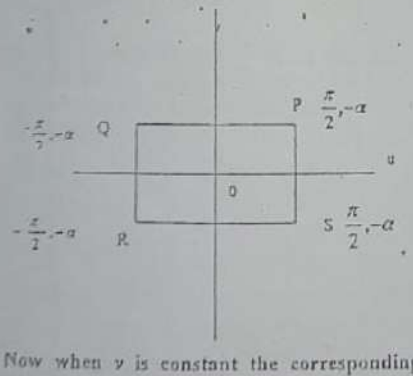
6.10 The transformation $z = c \sin w$, c being real:

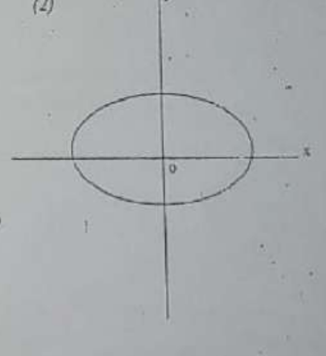
The image of the rectangle $u = \pm\frac{\pi}{2}$ and $v = \pm\alpha$ in w plane.

Let $z = x + iy$ and $w = u + iv$
$z = c \sin w$ becomes,
$x + iy = c \sin(u + iv)$

$$= c(\sin u \cos hv + i \cos u \sin v)$$

Hence $x = c \sin u \cos hv$ (1)
and $y = c \cos u \sin hv$ (2)



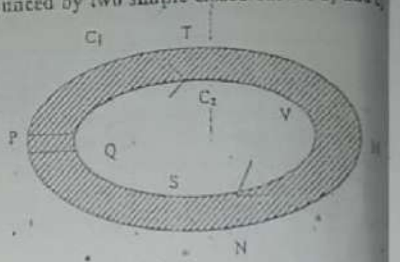


Now when v is constant the corresponding curves in z plane are obtained by eliminating u from (1) and (2). Then

$$\frac{x^2}{c^2\cosh^2 v} + \frac{y^2}{c^2\sinh^2 v} = \sin^2 u + \cos^2 u$$

7.10 If $f(z)$ is analytic in a region R bounced by two simple closed curves c_1 and c_2 and also on c_1 and c_2 then

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$$



where c_1 and c_2 are both

traversed in the anticlockwise direction relative to their interiors.

Proof

Construct a cross – cut PQ. Consider the curve $PQV\ SQPNMTP$ $f(z)$ is analytic inside the region R and on the curve $PQVS\ QPNMTP$.

By Cauchy's Theorem we have

$$\int_{PQVSQPNMTP} - f(z)dz = 0 \int_{c_1} f(z)dz$$

$$\text{or } \int_{PQ} f(z)dz + \int_{QVSQ} f(z)dz + \int_{QP} f(z)dz + \int_{PNMTP} f(z)dz = 0$$

$$\text{or } \int_{PVSQ} + \int_{PNMTP} = 0$$

since the integrals along PQ and QP cancel.

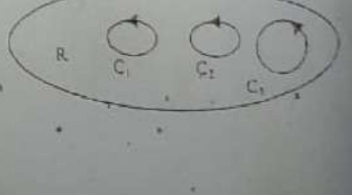
$$\text{Then } \int_{PVSQ} f(z)dz = - \int_{PNMTP} f(z)dz$$

$$\text{or} - \int_{QSVQ} f(z)dz = - \int_{PNMTP} f(z)dz$$

$$\text{or } \int_{QVSQ} f(z)dz = \int_{PNMTP}$$

$$\text{or } \int_{c_1} f(z)dz = \int_{c_2} f(z)dz$$

7.11 The above result can be extended when there are more than one region bounded by $c_1, c_2, \ldots c_n$



If $f(z)$ is analytic inside and on the curve c enclosing the region R and and $c_1, c_2, \ldots, c_n$ are closed curves in R. Then $\int_c f(z)dz = \int_{c_1} + \int_{c_1} + \ldots\ldots \int_{c_n}$

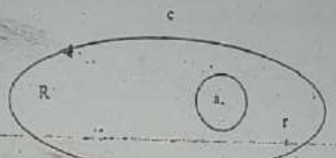
Theorem

7.12 Cauchy's Integral Formula.

If $f(z)$ is analytic inside and on the boundary c of a simply connected region R and a is any point inside the curve c then,

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$$

Proof



Let c be any simple curve enclosing the Region R and a be any point inside c. Draw a circle r of center a and radius c.

Now $\int_c \frac{f(z)}{z-a}dz = \int_r \frac{f(z)}{z-a}dz$ by Theorem.

Any point z on r is given by $z = a + \in e^{i\theta}$ where θ varies from 0 to 2π and $dz = \in e^{i\theta} i d\theta$

Hence $\int_r \frac{f(z)}{z-a}dz = \int_0^{2\pi} \frac{f(a+\in e^{i\theta})e^{i\theta} i d\theta}{\in e^{i\theta}}$

$$= i\int_0^{2\pi} f(a+\in e^{i\theta})d\theta$$

or $\int_r \frac{f(z)}{z-a}dz = i\int_0^{2\pi} f(a+\in e^{i\theta})d\theta$ (2)

Taking the limit $\in \to 0$, on both sides we have

$$\int_r \frac{f(z)}{z-a}dz = \lim_{\in\to 0} i\int_0^{2\pi} f(a+\in e^{i\theta})d\theta$$

$$= i\int_0^{2\pi} f(a)d\theta$$

$$= if(a)\int_0^{2\pi} d\theta$$

$$= 2\pi i f(a)$$

Hence $f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$

The above theorem can be extended by considering $\int_c \frac{f(z)}{(z-a)^2}$ within the curve c

Thus $f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)^2} dz$

Also $f^{(n)}(a) = \frac{n!}{2\pi i} \int \frac{f(z)}{(z-a)^{n+1}} dz$

The theorems are also applicable for multiply connected Regions. We can prove this by making a cut.

Example 2

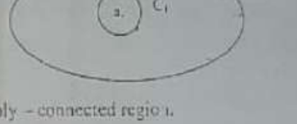
a) Evaluate $\int_c \frac{dz}{z-a}$ where c is any simple closed curve c and a is inside c.

b) What is the value of the integral if a is outside the closed curve?

Solution

Let c be any simple closed curve c and a be a point inside c

Let c_1 be a circle with center a and radius $\in$



Now $\int_c \frac{dz}{z-a} = \int_{c_1} \frac{dz}{z-a}$ By Cauchy's Theorem for multiply - connected region.

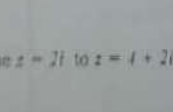
On c_1, $|z-a| = \in$ or $z - a = \in e^{i\theta}$

Hence $z = a + \in e^{i\theta}$ where $0 \leq \theta \leq 2\pi$ and $dz = i \in e^{i\theta} d\theta$

Then $\int_{c_1} \frac{dz}{z-a} = \int_{\theta=0}^{2\pi} \frac{i \in e^{i\theta} d\theta}{\in e^{i\theta}}$

$= i \int_0^{2\pi} d\theta = 2\pi i$

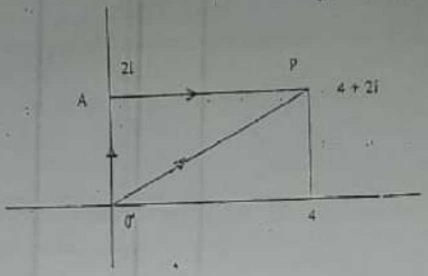
Hence $\int_c \frac{dz}{z-a} = 2\pi i$ if c is any simple closed curve and a is inside c.



b) If a is outside the curve then $f(z) = \frac{1}{z-a}$ is analytic inside and on the closed curve c and hence by Cauchy's fundamental Theorem $\int_c \frac{dz}{z-a} = 0$

Example 3

Evaluate $\int_c z dz$ from $z = 0$ to $z = 4 + 2i$ along the line $z = 2i$ to $z = 4 + 2i$



Solution

$$\int_c \bar{z}dz = \int_c (x-iy)d(x+iy) = \int_c (xdx + xidy - iydx + ydy)$$

$$= \int_c (xdx + ydy) + i\int (xdy - ydx)$$

The line from $z = 0$ to $z = 2i$ is OA. On OA $x = 0$, y is varying from 0 to 2.

Hence $\int \bar{z}dz = \int (0d(0) + ydy) + i\int 0dy - yd(0)$

$$= \int_0^2 ydy + i\int 0$$

$$= \int_0^2 ydy = \frac{y^2}{2}\Big|_0^2$$

$$= 2.$$

The line from $z = 2i$ to $z = 4 + 2i$ is the line AP on which $y = 2$ constant and x varies from 0 to 4

Hence $\int_{AP} \bar{z}dz = \int (x-iy)d(x+iy)$ becomes $\int_{AP} (x-iy)(dx+idy)$

$\int_{AP} (xdx + ixdy - iydx + ydy)$ on AP x is varying from 0 to 4 and $y = 2$ and $dy = 0$.

$$= \int (xdx + 0 - iydx + 0)$$

$$= \int_0^4 xdx - i\int_0^4 2dx$$

$$= \frac{x^2}{2}\Big|_0^4 - i2x\Big|_0^4$$

$$= 8 - i8$$

$$\int_c \bar{z}dz = \int_{OA} \bar{z}dz + \int_{AP} \bar{z}dz$$

$$= 2 + 8 - 8i$$
$$= 10 - 8i$$

Example 4

a) Integrate z^2 along the straight line OM (direct) and also along the path OLM consisting of two straight line segments OL and LM. O is the origin and M is the point $z = z + i$.

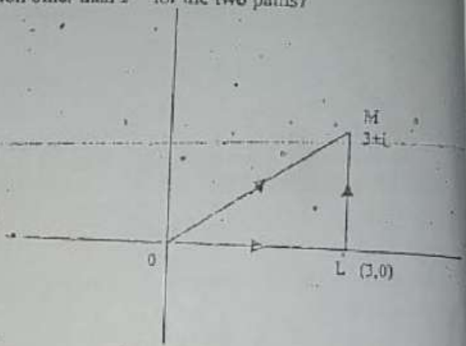
b) Show that the integral of z^2 along the two different paths are equal.

c) Is the result true for any function other than z^2 for the two paths?

Let $I_1 = \int_{OM} z^2 dz = \left[\frac{z^3}{3}\right]_0^{3+i}$

$$= \frac{(3+i)^3}{3} - 0$$

$$= \frac{18+26i}{3}$$



Let $I_2 = \int_{OL} z^2 dz + \int_{LM} z^2 dz$

$= \int_{OL} x^2 dx + \int_{LM} (3+iy)^2 i dy$ Note that $z = x$ and $y = 0$ at LM and $x = 3$ and $y - 3 + i$ and $dx = 0$ on LM

hence $I_2 = \left[\frac{x^3}{3}\right]_{x=0}^{3} + i\left[\frac{(3+iy)^3}{3}\right]_{y=1}^{3}$

$$= 9 + \frac{1}{3}(26i - 9)$$

$$= \frac{18+26i}{3}$$

Thus $I_2 = I_1$

d) The result will be true for any function $f(z)$ other than z^2 provided $f(z)$ is analytic in the region [illegible] OLM and on OLMO.

Exercise

1) State Cauchy's Integral Theorem.
2) State the converse of Cauchy's Theorem.
3) Give [illegible]xample of $f(z)$ and the anti derivative of $f(z)$.
4) If $f(z)$ is [illegible] and on a closed contour c of a simply connected region R, [illegible] that $\int f(z)dz$ is independent of the path in R joining the points a and b in R.

ii) Cauchy's fundamental Theorem: If $f(z)$ is analytic inside and on a simple closed curve C then $\int_c f(z)dz = 0$.

iii) Converse of Cauchy's Theorem (Morera's Theorem).

iv) Three important Theorems derived from Cauchy's Theorem.

v) Cauchy's Integral Formula.

If $f(z)$ is analytic inside and on the boundary of a simply connected region and a is any point inside c then $f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$ and its extensions.

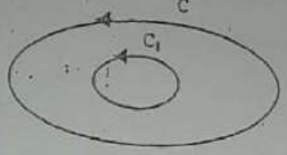
Further reading

1. Complex variables and Applications
   By R.V Churchill and others
   Mc Graw – Hill, Kogakusha Ltd
   Tokyo Singapore.

2. Complex variables
   By Murray R. Spiegel, Ph.D
   Schaum outline series.
   Mc Graw – Hill Book Company
   Singapore.

3. First Course in Complex Analysis
   By Dr. D. Sengottaiyan Ph. D
   Oxford Publications
   London Nairobi.

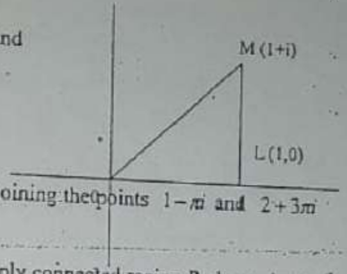
5) If $f(z)$ is analytic inside and on a closed curve C (as in the figure) show that $\int_{c} f(z)dz = \int_{c_1} f(z)dz$.



6) Find the value of the integral $I_1 = \int_{OL}(z^2 + 4iz)dz$ and

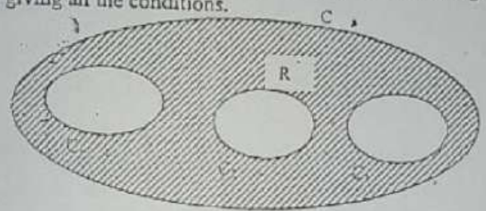
$I_2 = \int_{OL}(z^2 + 4iz)dz + \int_{LM}(z^2 + 4iz)^2 dz$

show that $I_1 = I_2$



7) Show that $\int_{c} e^{-2z} dz$ is independent of the path C joining the points $1 - \pi i$ and $2 + 3\pi i$ and determine its value.

8) Prove Cauchy – Goursat – Theorem for the multiply connected region R shown in the figure shaded giving all the conditions.



9) Show that $\int_{3+4i}^{4-3i}(6z^2 + 8iz)dz$ has the same value along the following paths C joining the points 3 + 4i and 4 – 3i along a straight line and also along the straight line 3 + 4i to 4 + 4i and then from 4 + 4i to 4 – 3i.

10) Evaluate $\int_{\pi i}^{2\pi i} e^{3z} dz$

11) Show that $\int_0^{\frac{\pi}{2}} \sin^2 z dz = \int_0^{\frac{\pi}{2}} \cos^2 z dz = \frac{\pi}{4}$

12) Show that $\int \frac{dz}{z^2 - a^2} = \frac{1}{2a} \ln\left(\frac{z-a}{z+a}\right) + c_1 \frac{1}{3} \ln(z^3 + 3z + 2) + c$

13) Evaluate $\int \frac{z^2 + 1}{z^3 + 3z + 2} dz$.

Summary of the lesson

You have learnt the following from this lesson

i) the meaning of complex Integration or the line integral of $f(z)$.

Lesson 8

Laurent series and Singularities of Functions

8.1 Introduction

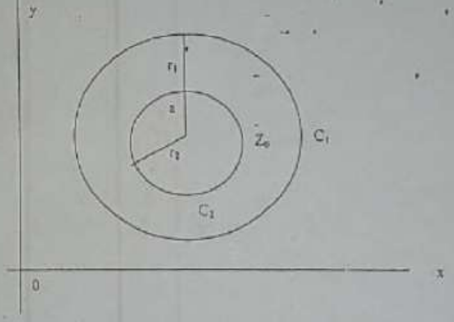
You have studied analytic functions in the previous lessons. In this lesson you will study the series representation of analytic functions. You shall study mainly Laurent series expansion in a ring shaped region between two concentric circles.

8.2 Objectives of the lesson.

By the end of this lesson you should be able to

i) state Laurent's series
ii) deduce Taylor's series from Laurent Series.
iii) define Singularities of functions
iv) classify the singularities of functions.
v) expand a function *f(z)* at its singularities.

8.3 Laurent series



Let c_1 and c_2 be two concentric circles of radii r_1 and r_2 respectively and cetre at a. suppose that a function *f(z)* is single valued and analytic on c_1 and c_2 and in the ring shaped region between c_1 and c_2 shown in the figure. If z_0 is any point in R, then we have

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{bn}{(z-z_0)^n} \qquad (1)$$

where $an=\frac{1}{2\pi i}\int_{c_1}\frac{f(z)}{(z-z_0)^{n+1}}$ n = 0, 1, 2 ...

$$bn=\frac{1}{2\pi}\int_{c_1}\frac{f(z)}{(z-z_0)^{-n+1}} \qquad n=1, 2 .....$$

The path of integration is taken counter clockwise.

The series (1) is called Laurent Series.

The part of Laurent Series $\sum_{n=0}^{\infty} an(z-z_0)^n$ is called the **analytic part** of Laurent Series and

$\sum_{n=1}^{\infty}\frac{bn}{(z-z_0)^n}$ is called the **principal part** of Laurent Series.

5.4 Taylor's Series from Laurent Series

If *f(z)* is analytic at all points inside and on c_1 the function $\frac{f(z)}{(z-z_0)^{-n+1}}$ is analytic inside and on c_2 since $-n+1\leq 0$. Hence by Cauchy's fundamental Theorem b_n becomes zero. In this case Laurent Series reduces to Taylors series. Then

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

where $a_n=\frac{1}{2\pi i}\int_{c_1}\frac{f(z)}{(z-z_0)^{n+1}}$ n=0, 1, 2, ….

5.5 Singular points or Singularities.

A point on the *z* plane at which *f(z)* fails to be analytic is called **a singular point** or singularity of *f(z)*. There are various types of singularities:

i) Singular points

A point $z=z_0$ is called an isolated singular point of *f(z)* if we can find $\delta>0$ such that the circle $|z-z_0|=\delta$ encloses no singular point other than z_0

In other words there exists a deleted neighborhood of z_0 containing no singularity. If no such δ can be found we call z_0 a non – isolated singularity.

ii) Poles

Consider a point z_0 on the *z* plane at which *f(z)* becomes infinite. If we can find a positive integer *n* such that $\lim_{z\to z_0}(z-z_0)^n f(z)=c\neq 0$. Then the point z_0 is called a pole of order *n*

Example 1

$f(z)=\frac{2z}{z-3}$ has a pole of order 1. Here $\frac{2z}{z-3}$ becomes ∞ at $z = 3$, but $\lim_{z\to 3}(z-3)\frac{2z}{z-3}=6$ which is not zero

Hence $f(z)=\frac{2z}{z-3}$ has a pole of order one.

Example 2

$\frac{2z+4}{(z-2)^3(z-1)^4(z+4)}$ has

a pole of order 3 at $z = 2$
a pole of order 4 at $z = 1$ and
a pole of order one at $z = -4$.
A pole of order one is called a simple pole.

For a multiple valued function all the Branch points are called singular points.

Examples: $f(z)=z^{\frac{1}{2}}$ has a branch point at $z = 0$

$f(z)=(z-5)^{\frac{1}{2}}$ has a branch point at $z = 5$.

$f(z)=\ln(z^2+3z-10)$ or $\ln(z-2)(z+5)$ has branch points at $z = 2$ and $z = -5$

iii) Removable singularities

If z_0 is a singular point of $f(z)$ but $\lim_{z\to z_0} f(z)$ exists, then the singular point z_0 is called a removable singularity of $f(z)$.

Examples

Consider

$f(z)=\frac{\sin z}{z}$. $z=0$ is not an ordinary point of $f(z)$ since it takes the form $\frac{0}{0}$. But $\lim_{z\to 0}\frac{\sin z}{z}=1$ by L Hospital Rule.

Hence $z=0$ is a removable singularity for $f(z)=\frac{\sin z}{z}$

iv) Essential Singularity

A function $f(z)$ may have a singular point $z = z_0$ but if this singular point is neither an isolated singularity, nor a pole or branch point or removable singularity then the singularity z_0 is called an essential singularity of $f(z)$.

Example 4

$f(z)=\frac{1}{e^{3-z}}$ has an essential singularity at $z = 3$.

Suppose $f(z)$ becomes infinity but we cannot find any positive integral n such that $\lim_{z\to z_0}(z-z_0)^n f(z)=R\neq 0$ then $z = z_0$ is an essential singularity.

v) Singularities at Infinity

A function $f(z)$ may not have any pole at $z = z_0$, but $f(\frac{1}{z})$ may have poles. Such poles are called singularities at infinity.

vi) Branch point

Example 5

Consider $f(z)=z^4$. $f(z)$ has actually no poles but $f(\frac{1}{z})=z^{\frac{1}{4}}$ has a pole at $z = 0$. Hence $f(z)$ has a pole of order 4 at $z = \infty$

8.6 Laurent series about the singularities.

If z_0 is any kind of a singularity for $f(z)$, we can expand the function $f(z)$ in an infinite series about the Singularity such series are called Laurent Series.

We shall consider some functions having singularities at $z = z_0$ and expand the functions at z_0 in the following examples.

Example 6

Consider the function $f(z)=\frac{e^{3z}}{(z-2)^3}$

a) state the singularity of $f(z)$.

b) what is the kind of the singularity of $f(z)$.

c) expand $f(z)$ in a Laurent series.

d) state the region of convergence of the series.

Solution

a) when $z=2$. $f(z)$ becomes infinite and it is not analytic Hence $z = 2$ is a singularity of $f(z$.

b) $\lim_{z\to 2}\frac{(z-2)^3 e^{3z}}{(z-3)^3}=e^{6z}$ which is not equal to zero Then $z = 2$ is a pole of order 3.

Let $f(z)=\frac{e^{3z}}{(z-2)^3}$ and let $z-2=u$ so that $z = u + 2$.

Then $\frac{e^{3z}}{(z-2)^3}=\frac{e^{3(u+2)}}{u^3}$

$$=\frac{e^{6+3u}}{u^3}$$

$$=\frac{e^6 . e^{3u}}{u^3}$$

$$=\frac{e^6}{u^3}\left[1+\frac{3u}{1!}+\frac{(3u)^2}{2!}+\frac{(3u)^3}{3!}+\frac{(3u)^4}{4!}+....\right]$$

$$=\frac{e^6}{u^3}+\frac{3e^6}{u^2}+\frac{9e^6}{u.2!}+\frac{27e^6}{3!}+\frac{81u}{4!}+....$$

$$=\frac{e^6}{(z-2)^3}+\frac{3e^6}{(z-2)^2}+\frac{9e^6}{2!(z-2)}+\frac{27e^6}{3!}+\frac{81u}{4!}+......$$

The series converges at all points except at $z = 2$.

Example 7

Consider the function $f(z)=\frac{z-\sin z}{z^3}$

a) state the singularity of $f(z)$.
b) what is the kind of singularity of $f(z)$.
c) find the Laurent series of $f(z)$.
~~d)~~ what is the region of convergence of $f(z)$?. → Ignore

Solution

Let $f(z)=\frac{z-\sin z}{z^2}$

a) $z = 0$ is a singularity since at $z = 0$ the function is not defined and hence not analytic.

b) $\lim\limits_{z\to 0}\frac{z-\sin z}{z^3}=\lim\limits_{z\to 0}\frac{1-\cos z}{2z}=\lim\limits_{z\to 0}\frac{1+\sin z}{2}=\frac{1}{2}$ (By L' Hosp tal Rule the limit exists). Hence $z = 0$ is a removable singularity.

c) Let $z = 0$

$$\frac{z-\sin z}{z^3}=\frac{1}{z^3}\left[z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+.....\right)\right]$$

$$=\frac{1}{z^3}\left[\frac{z^3}{3!}-\frac{z^5}{5!}+\frac{z^7}{7!}-.....\right]$$

$$=\frac{z}{3!}-\frac{z^3}{5!}+\frac{z^5}{7!}+........$$

d) The series converges for all values of z i.e. the series converges on the whole of z plane.

Example 8

Consider the function $f(z) = (z-4)\sin\frac{1}{z+2}$

a) state the singularity of $f(z)$
b) what is the kind of singularity of $f(z)$.
c) expand $f(z)$ in a Laurent series.
d) state the region of convergence of the series.

Solution

a) Let $f(z) = (z-4)\sin\frac{1}{z+2}$ when $z = -2$ $f(z)$ is not defined and hence it is not analytic. Hence $z = -2$ is a singularity of $f(z)$.

b) $z = -2$ is neither a pole nor Branch point. It is an essential singularity.

c) Let $z + 2 = u$ so that $z = u - 2$

Then $(z-4)\sin\frac{1}{z+2} = (u-6)\sin\frac{1}{u}$

$$= (u-6)\left[\frac{1}{u} - \frac{1}{3!u^3} + \frac{1}{5!u^5} + \dots\dots\right]$$

$$= 1 - \frac{1}{3!u^2} + \frac{1}{5!u^4} + \dots\dots - \frac{6}{u} + \frac{6}{3!u^3} - \frac{6}{5!u^5} + \dots\dots$$

$$= 1 - \frac{6}{u} - \frac{1}{3!u^2} + \frac{6}{3!u^3} + \frac{1}{5!u^4} - \frac{6}{5!u^5} + \dots$$

$$= 1 - \frac{6}{z+2} - \frac{1}{3!(z+2)^2} + \frac{6}{3!(z+2)^3} + \frac{1}{5!(z+2)^4} - \frac{6}{5!(z+2)^5}$$

The series converges at all points except at $z = -2$.

Example 9

Expand $f(z) = \frac{3}{z^2(z-3)^2}$ in a Laurent Series at $z = 3$.

Solution

We can expand $f(z)$ using Binomial theorem.

$$f(z) = \frac{3}{z^2(z-3)^2}$$

$z = 0$ and $z = 3$ are poles of $f(z)$.
Let $z - 3 = u$ or $z = u + 3$.

$$f(z)=\frac{3}{z^2(z-3)^2}=\frac{3}{(u+3)^2(u^2)}$$

$$=\frac{3}{\left[3\left(1+\frac{u}{3}\right)\right]^2 u^2}$$

$$=\frac{3\left(1+\frac{u}{3}\right)^{-2}}{9u^2}$$

$$=\frac{1}{3u^2}\left[1+(-2)\frac{u}{3}+\frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3+\ldots\ldots\right]$$

$$=\frac{1}{3u^2}-\frac{2}{9u}+\frac{1}{9}-\frac{4}{81}u+\ldots.$$

$$f(z)=\frac{1}{3(z-3)^2}-\frac{2}{9(z-3)}+\frac{1}{9}-\frac{4}{81}(z-3)-\ldots..$$

This series converges for all values of z such that $0<|z-3|<3$.

Example 10

Expand $f(z)=\frac{1}{(z+3)(z+1)}$ is a Laurent series valid for $0<|z+1|<2$.

Solution

Let $f(z)=\frac{1}{(z+3)(z+1)}$ (1)

Let $(z+1)=u$ then

$$f(z)=\frac{1}{u(u+2)}=\frac{1}{2u\left(1+\frac{u}{2}\right)}$$

$$=\frac{1}{2u}\left(1-\frac{u}{2}+\frac{u^2}{4}-\frac{u^3}{8}+\ldots\right) \quad (2)$$

$$=\frac{1}{2(z+1)}-\frac{1}{4}+\frac{1}{8}(z+1)-\frac{1}{16}(z+1)^2+\ldots.$$

The expansion of Binomial theorem is valid when $\frac{u}{2}<1$ or $\frac{z+1}{2}<1$ or $z+1<2$ or $|z+1|<2$.

Example 11

Expand $f(z)=\frac{1}{(z+1)(z+3)}$

Solution

Let $f(z) = \frac{1}{(z+1)(z+3)}$

$$= \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\frac{1}{(z+3)} \text{ Resolving into partial fraction}$$

$$\frac{1}{2(z+1)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3}\right) = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \dots \qquad (2)$$

$$\frac{1}{2(z+3)} = \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \dots\right) \qquad (3)$$

The two expansions are valid if $\left|\frac{1}{z}\right| < 1$ and $\left|1+\frac{z}{3}\right| < 1$ or if $|z| > 1$ and $|z| < 3$ or $1 < |z| < 3$. The required Laurent series is the sum of the series in (2) and (3). Hence

$$f(z) = \frac{1}{(z+1)(z+3)} = \dots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} \dots\dots$$

Exercise 8

1) State Laurent Series for a function *f(z)*.
2) Derive Taylor's series from Laurent series.
3) Define singularity of a function *f(z)*.
4) Give one example for each of the following
    - i) Isolated singularity
    - ii) Poles
    - iii) Branch point
    - iv) Removable singularity
    - v) Essential singularity
    - vi) Singularity at infinity.
5) Expand $f(z) = \frac{1}{z-3}$ in a Laurent series valid for i) $|z| < 3$ and ii) $|z| > 3$.
6) Expand $f(z) = \frac{1}{z(z-2)}$ in a Laurent series valid for a) $0 < |z| < 2$ b) $|z| > 2$.
7) Find the singularities of the functions $\frac{z}{e^z - 1}$ and classify the singularity.
8) a) Expand $f(z) = \frac{z}{e^{z-2}}$ in a Laurent series about $z = 2$ and determine the region of convergence of this series.

b) Classify the singularities of $f(z)$.

Summary of the lesson

You have learnt the following from this lesson.

i) Laurent series expansion of an analytic function in a ring shaped region.
ii) Taylor's series derived from Laurent's series.
iii) Singularities of $f(z)$ and their classification.
iv) Expansion of functions at the singularities.

Further Reading

1. Complex Variables and Applications
   By R.V Churchill and others
   Mc Graw – Hill, KOGAKUSHA Ltd
   Tokyo Singapore.

2. Complex Variables
   By Murray R. Spiegel, Ph.D
   Schaum outline series.
   Mc Graw – Hill Book Company
   Singapore.

3. First Course in Complex Variables
   By Dr. D. Sengottaiyan Ph. D
   Oxford Publications
   London Nairobi.

Lesson 9

Poles and Residues of a Function

9.1 Introduction

You have studied Cauchy's fundamental theorem which states that if a function is analytic every where inside and on a simple closed contour (curve) c, then the integral of a function around that contour is zero. If however the functions fails to be analytic at a finite number of points inside C those points may be called poles of the function. Each of these points contributes to the value of the integral. These contributions are called the Residues of the function. You will learn, in this lesson to determine the poles and the residues at the poles of a function of complex variables.

9.2 Objectives of the lesson

By the end of this lesson you will be able to: -

i). define the pole of a function $f(z)$

ii). determine the pole of $f(z)$

iii). define the Residues of a function at its poles.

iv). determine the residue of a function $f(z)$ at its poles of order 1, 2...n.

9.3 Definition of poles of a function $f(z)$

Let $f(x)$ be any function of z. Generally $\lim_{z \to a}(z-a)f(z) = 0$

Suppose $\lim_{z \to a}(z-a)f(z) = A$ which is not zero. Then z = a is called a pole of order one of $f(z)$.

Similarly if $\lim_{z \to a}(z-a)^m f(z) = A, \neq 0$ then z = a is called a pole of order m of $f(z)$.

9.4 Determination of the Poles of f(z) at its poles.

At the pole the function $f(z)$ becomes infinite. Hence to find the poles of $f(z)$ we put $f(z) = \infty$ and find z which are poles. If $f(z) = \dfrac{\phi(z)}{g(z)}$ we solve $g(z) = 0$ and the roots are poles of $f(z)$.

Example 1

Find the poles of $\dfrac{z^3}{(z-1)(z+3)^2(z-8)^5}$ and state the order of each pole.

Solution

At the pole $f(z)$ becomes infinite. If $(z-1)(z+3)^2(z-8)^5 = 0$, $f(z)$ becomes infinite.

Hence z = 1 is pole of order 1

z = -3 is pole of order 2

z = 8 is pole of order 5

Generally to find the pole of $f(z)$ put the denominator of $f(z)$ to zero and solve for z.

Example 2

Determine the poles of $\dfrac{e^{iz}}{z^2(z^2+2z+2)}$

Solution

Let $f(z)\dfrac{e^{iz}}{z^2(z^2+2z+2)}$

The poles of *f(z)* are obtained by solving $z^2(z^2+2z+2)=0$
One pole is z = 0 of order 2.

Solving $z^2+2z+2=0$ we get $z=\dfrac{-2+\sqrt{4-8}}{2}$ or z = -1 + i and – 1 – i both of them are simple poles.

Thus z = 0, 0, -1 + i, -1 – i are the four poles.

Example 3

Find the poles of $\dfrac{2z^2+5}{z^4+16}$

Solution
The poles of *f(z)* are obtained by solving the equation $z^4 + 16 = 0$

or $z^4 = -16$
or $z^4 = 16\ (-1)$

Hence $z=2(-1)^{\frac{1}{4}}$

$$=2[\cos(2n+1)\pi+i\sin(2n+1)\pi]^{\frac{1}{4}}$$

$$=2\left[\cos\frac{(2n+1)\pi}{4}+i\sin\frac{(2n+1)\pi}{4}\right]$$

n = 0, 1, 2, 3 (by Demoivre's Theorem).

If n = 0, $z=2\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)=2\left(\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}\right)$

If n = 1, $z=2\left(\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)=2\left(-\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}\right)$

If n = 2, $z = 2\left(\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 2\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

If n = 3, $z = 2\left(\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

Thus the poles of $\frac{2z^2+5}{z^4+16}$ are given by $z = \sqrt{2} \pm i, \ \sqrt{2} \pm i$

9.5 Residue of $f(z)$ at its pole

Let $f(z)$ be single valued and analytic inside and on a circle c whose center is a $f(z)$ is not analytic at the point z = a (center of the circle)

Then $f(z)$ has a Laurent series about z = a given by:-

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n \qquad (1)$$

$$\text{where } a_n = \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}} dz \quad n = 0, \pm 1, \pm 2, \ldots\ldots\ldots \qquad (2)$$

suppose n = -1 we have from (2) $\int f(z)dz = 2\pi i \ \ a_{-1}$ (3)

(3) involves only the coefficient a_{-1} in (1). We call a_{-1} **the residue** of $f(z)$ at z = a. It is denoted by R.

Useful formula for the residue of $f(z)$ at the pole $z = a$

9.6 Determination of Residues of $f(z)$ at its poles

i). If z = a is a simple pole for $f(z)$ then the residue of $f(z)$ at a is given by

$$R = \lim_{z \to a} (z-a) f(z)$$

ii). If z = a is a pole of order two then

$$R = \lim_{z \to a} \frac{1}{1!} \frac{d}{dz}\left[(z-a)^2 f(z)\right]$$

iii). If z = a is a pole of order n then

$$R = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]$$

The following examples illustrate the method of finding the Residues of *f(z)* at its poles.

Example 4

Let $f(x) = \frac{z^3 + 5z + 1}{z - 2}$

a) Determine the pole of *f(z)*
b) Calculate the residue of *f(z)* at its pole.

Solution
a) The pole *f(z)* is obtained by solving the denominator $z - 2 = 0$
Hence $z = 2$ is a simple pole of *f(z)*

b) The Residue of *f(z)* at $z = a$ is given by

$$\begin{aligned} R &= \lim_{z \to 2} (z-2) f(z) \\ &= \lim_{z \to} (z-2) \frac{(z^3 + 5z + 1)}{z - 2} \\ &= \lim_{z \to 2} (z^3 + 5z + 1) \\ &= 8 + 10 + 1 \\ &= 19 \end{aligned}$$

Then the Residue of *f(x)* at $z = 2$ is 19

Example 5

Let $f(z) = \frac{z^2 - 7z + 10}{(z-3)^2}$

Determine the pole of *f(z)*. Calculate the pole of *f(z)* at its poles.

Solution
a) The pole of *f(z)* is obtained by equating *f(z)* = ∞ or by solving the denominator
$(z-3)^2 = 0$
$(z-3)^2 = 0$ **gives** $z = 3, 3$ **Hence** $z = 3$ **is a pole of order 2.**

If $z = a$ is a pole of order n, then $R = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]$

Hence if $z = 3$ is a pole of order 2, we have

$$R = \lim_{z \to 3} \frac{1}{(2-1)!} \frac{d}{dz} \left[(z-3)^2 \frac{(z^2 - 7z + 10)}{(z-3)^2} \right]$$

$$= \lim_{z \to 3} (z^2 - 7z + 10)$$

$$= 9 - 7(3) + 10 = -2$$

Exercise 9

1. Define the pole of a function *f(z)*
2. State the formula for finding the pole of $f(z) = \frac{\phi(z)}{g(z)}$
3. Let z = a be a simple pole for *f(z)* state the formula for finding the Residue of *f(z)* at the pole z = a.
4. Let z = a be a pole of order 4 for the function *f(z)*. Write down the formula for finding the Residue of *f(z)* at *z = a*.

For each of the following functions determine the pole and the Residues at the poles (5 to 9).--

5. $\frac{-2z+1}{z^2-z-2}$
6. $\frac{(z+1)^2}{(z-1)^2}$
7. $\frac{1}{z^4+1}$
8. $\frac{3z^3+2}{(z-1)(z^2+9)}$
9. $\frac{1}{z^2(z+4)}$
10. $\frac{1}{z^4+81}$

11. a). Find the 6 poles of the function $\frac{1}{z^6+1}$

    b). Determine the residues at each pole of *f(z)*

12. a). Find the three poles of $f(z) = \frac{1}{z^3-1}$

    b). Determine the residues of *f(z)* at its 3 poles.

Summary of the lesson

You have learnt the following from this lesson.

1. Definition of the poles of a function *f(z)*:

LESSON 10

Residue Theorem and its Applications to Integration

10.1 Introduction

You have already studied Cauchy's fundamental Theorem when a function is analytic inside and on a simple closed contour C then the integral of the function along the curve is zero. What will be the value of the integral if the function has some finite number of poles inside the closed curve? In this chapter we shall find the answer which is called th Residue Theorem.

10.2 Objectives of the lesson

By the end of this lesson you will be able to: -

i). state Residue Theorem

ii). apply Residue Theorem for the evaluation of three types of integrals:

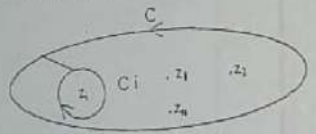
a) improper integrals of the type $\int_{-\infty}^{\infty} f(x)dx$

b) definite integrals of the Trigonometric type $\int f(\sin\theta,\ \cos\theta)d\theta$ and

c) integration round a Branch point

iii). state some theorems useful for integration.

10.3 Residue Theorem



If $f(z)$ is analytic inside and on a closed curve C except at a finite number of poles a_1, a_2 ...... a_n inside C at which the residues are R_1, R_2...R_n respectively, then

$$\int_c f(z)dz = 2\pi i(R_1 + R_2 + ...R_n)$$

$$= 2\pi i \sum \text{Residues}$$

Proof

Let z_i be one pole inside C. Draw a small circle C_i such that c_i is inside C and no two poles inside the circle C_i.

According to extension of Cauchy's fundamental theorem for the multiply connecte region

$$\int f(z)dz - \int_{c_i} f(z)dz = 0$$

$$\int_{c} f(z)dz = \int_{c_1} f(z)dz = 2\pi i R_1$$

In the same way taking $i = 1, 2, \ldots n$ we have

$$\int_{c} f(z)dz = 2\pi i(R_1 + R_2 + \ldots R_n)$$

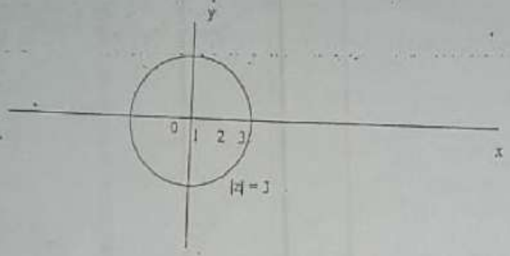
$$= 2\pi i \text{ (sum of Residues)}$$

Example 1

Evaluate $\int_{c} \frac{2z+3}{z-1}dz$ around the circle $|z| = 3$

Solution

The integrand $\frac{2z+3}{z-1}$ has a pole at z = 1. (put the denominator = 0 anc solve for z).



This pole z = 1 + 0i is inside the circle with center origin and radius 3 units. The Residue at z = 1 is given by

$$\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{(z-)(2z+3)}{z-1}$$

$$= \lim_{z\to 1}(2z+3) = 5$$

There is only one pole and one Residue = 5. Hence by Residue Theorem,

$$\int_{c}\frac{(2z+3)}{z-1}dz = 2\pi i \text{ (sum of residues) where C is} |z|=3$$

$$= 2\pi i(5)$$

$$= 10\pi i$$

Example 2

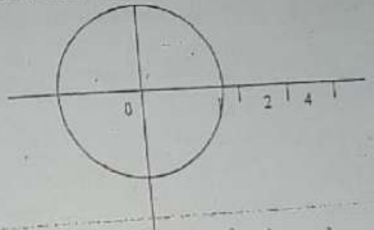
Evaluate $\int_{c}\frac{e^z}{(z-2)(z-4)}dz$

when

there is only one pole $z = 2$ inside $|z| = 3$ and the corresponding Residue at $z = 2$ is $\frac{e^2}{-2}$.

Hence $\int_c \frac{e^z\ dz}{(z-2)(z-4)} = \frac{2\pi i e^2}{-2} = -\pi i e^2$ if c is $|z| = 3$

iii). If $|z| = 1$ or c is the circle with center origin and radius 1 unit,



there is no pole inside the circle $|z| = 1$. Since both the poles $z = 2$ and $z = 4$ are outside the circle.

Hence by Cauchy's fundamental Theorem

$\int_c \frac{e^z\ dz}{(z-2)(z-4)} = 0$; (since no pole no Residue)

10.4 Application of Residue theorem for various Types of Integrals

Using Cauchy's Residue Theorem we can evaluate the following types of Integrals.

i). Improper Real integrals of the type $\int_{-\infty}^{\infty} f(x)dx$, provided the integral is convergent.

ii). Definite integrals of the Trigonometric functions of the type $\int f(\sin\theta, \cos\theta)d\theta$

iii). Integration round a branch point.

We shall consider some examples in each of the three types of Integrals.

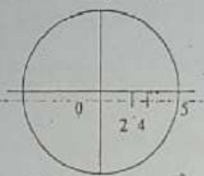
i). c is the circle $|z| = 5$

ii). c is the circle $|z| = 3$

iii). c is the circle $|z| = 1$

Solution

i). The integrand $\frac{e^z}{(z-2)(z-4)}$ has two simple poles at $z = 2$ and $z = 4$ inside $|z|$ (since the denominator $(z-2)(z-4) = 0$ gives $z = 2$ and $z = 4$].



Residue at $z = 2$ is

$$\lim_{z\to 2}\frac{(z-2)e^z}{(z-2)(z-4)} = \lim_{z\to 2}\frac{e^z}{(z-4)}$$

$$= \frac{e^2}{2-4} = \frac{e^2}{-2}$$

Residue at $z = 4$ is

$$\lim_{z\to 4}\frac{(z-4)e^z}{(z-2)(z-4)} = \lim_{z\to 4}\frac{e^z}{z-2} = \frac{e^4}{2}$$

$$\text{Sum of Residues} = \frac{e^2}{2} + \frac{e^4}{2} = \frac{e^4 - e^2}{2}$$

Hence, $\int_c \frac{e^z}{(z-2)(z-4)}dz = 2\pi i$ (sum of residues)

$$= 2\pi i\frac{(e^4 - e^2)}{2}$$

(if c is $|z| = 5$)

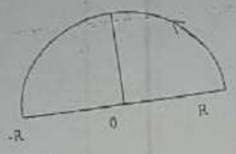
$$= \pi i\,(e^4 - e^2)$$

ii). If $|z| = 3$, i.e. c is the circle with center origin and radius 3 units

10.5 Improper integrals of the type

$$\int_{-\infty}^{\infty} f(x)dx \quad \text{and} \quad \int_{0}^{\infty} g(x)dx$$

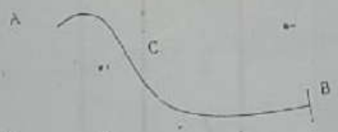
Some of the integrals of the above type can be evaluated without the help of complex integration, but the harder type of such integrals can be evaluated using Cauchy's Residue Theorem. We consider the curve C, in this case, as a semi-circle with center origin and radius R units and finally we take the limit R → ∞



10.6 Some important Theorems

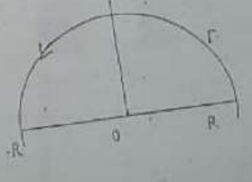
Theorem 1
If $f(z)$ is analytic then it is bounded. Hence $|f(z)| \leq M$ where M is an upper bound of $|f(z)|$

Theorem 2
If $f(z)$ is analytic



then $\left|\int_c f(z)dz\right| \leq ML$

where M is the upper bound of $f(z)$ on the curve c and L is the length of the curve c from A to B.



Theorem 3

If $\left|\int f(z)dz\right| \leq \frac{M}{R^k}$ then

$$\lim_{R\to\infty}\int_{\Gamma} f(z) = 0 \text{ if } k > 1$$

where Γ is a semi circular arc whose center is origin and radius R units.

Proof

From Theorem (2) $\left|\int_c f(z)dz\right| \le ML$

For a semicircle

The length of the arc $= \dfrac{2\pi R}{2} = \pi R$

Hence $\left|\int_{\Gamma} f(z)dz\right| \le \dfrac{M}{R^k}\pi R$

$$\le \frac{M\pi}{R^{k-1}}$$

if k − 1 > 0 or k > 1 then Mπ is finite and $R^{k-1} \to \infty$

Hence $\dfrac{M\pi}{R^{k-1}} \to 0$ and it cannot be negative.

Then $\left|\int_{\Gamma} f(z)dz\right| = 0$

The following examples illustrate the evaluation of the improper integrals of real variables.

Example 3

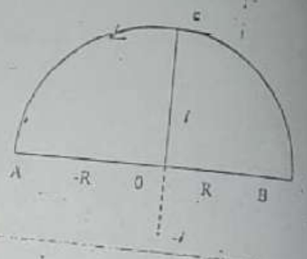
Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{x^2+1}$

Solution

(i) We consider the integral

$\int_c \frac{dz}{z^2+1}$ and

ii). The curve c is a semicircle with center at origin and Radius R (where $R \to \infty$)



iii). The poles of $f(z) = \frac{1}{z^2+1}$ are obtained by solving $z^2+1=0$ or $z^2 = -1$. Hence $z = \sqrt{-1} = \pm i$ are the two simple poles inside the circle $|z| = R$ within the semicircle there is only one pole at z = i, the other being outside the semi-circle.

The residue at z = i for $f(z) = \frac{1}{z^2+1}$ is given by $\lim_{z\to i} \frac{(z-i)}{z^2+1} = \lim_{z\to i} \frac{(z-i)}{(z+i)(z-i)} = \frac{1}{2i}$

Now $\int_c f(z)dz = \int_{AB} f(z)dz + \int_\Gamma f(z)dz = 2\pi i R_1$ where AB is on the x axis and Γ is the semicircular arc.

On the line AB, z = x+ iy becomes z = x since y = 0 on the x axis.
On the semicircle $z = Re^{i\theta}$ $dz = Re^{i\theta} i d\theta$

$$\text{Hence } \int_c f(z)dz = \int_{-R}^{R} f(x)dx + \int_c \frac{1}{z^2+1}dz$$

$$= \int_{-R}^{R} \frac{1}{x^2+1}dx + \int_{\theta=0}^{\pi} \frac{Re^{i\theta} i d\theta}{\left(Re^{i\theta}\right)^2+1}$$

$$= \int_{-\infty}^{\infty} \frac{1}{x^2+1}dx + 0 \text{ when } R \to \infty$$

Since $\int \frac{R e^{i\theta} i d\theta}{R^2 e^{2i\theta} + 1} = \int \frac{e^{i\theta} i d\theta}{R e^{2i\theta} + 0}$ when R → ∞

$= 0$ when R → ∞.

$$\int_c f(z)dz = \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = 2\pi i \text{ (sum of Residues)}$$

$$\text{or } \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = 2\pi i \left(\frac{1}{2i}\right) = \pi$$

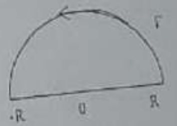
Since the integrand $\frac{1}{x^2+1}$ is even function

$$2\int_0^{\infty} \frac{dx}{x^2+1} = \pi$$

$$\text{or } \int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}.$$

Example 4

Prove that $\int_0^{-\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$



Solution

Consider $\int_c \frac{dz}{z^4+1}$, where c is the closed contour, consisting of the x-axis from – R to R and the semi circle Γ traversed in the anticlockwise direction.

The poles of the integrand is obtained by solving

$z^4 + 1 = 0$ or $z^4 = -1$

or $z^4 = \cos(2r+1)\pi + i \sin(2r+1)\pi$ r = 0, 1, 2, 3

or $z^4 = e^{(2r+1)\pi i}$

Hence $z = e^{\frac{(2r+1)\pi i}{4}}$ r = 0, 1, 2, 3

or $z = e^{\frac{\pi}{4}}$, $e^{\frac{3\pi}{4}}$, $e^{\frac{5\pi i}{4}}$, $e^{\frac{7\pi i}{4}}$ only the poles $e^{\frac{\pi}{4}}$ and $e^{\frac{3\pi}{4}}$, lie within c, other two are below the x axis.

Residue at $z = e^{\frac{\pi}{4}} = \lim_{z \to e^{\frac{\pi}{4}}} \left[\frac{\left(z - e^{\frac{\pi}{4}}\right)}{z^4 + 1} \right]$

$$= \lim_{z \to e^{\frac{\pi}{4}}} \left[\frac{1}{4z^3} \right] \text{ using L' Hospital Rule}$$

$$= \frac{1}{4} e^{\frac{-3\pi i}{4}}$$

Residue at $z = e^{\frac{3\pi}{4}} = \lim_{z \to e^{\frac{3\pi}{4}}} \left[\frac{z - e^{\frac{3\pi}{4}}}{z^4 + 1} \right]$

$$= \lim_{z \to e^{\frac{3\pi}{4}}} \left[\frac{1}{4z^3} \right] \text{ using L' Hospital Rule}$$

$$= \frac{1}{4} e^{\frac{-9\pi i}{4}}$$

Thus $\int_C \frac{dz}{z^4 + 1} = 2\pi i \left[\frac{1}{4} e^{\frac{-3\pi i}{4}} + \frac{1}{4} e^{\frac{-9\pi i}{4}} \right]$

$$= \frac{\pi i}{2} \left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4} \right]$$

$$= \frac{\pi i}{2} [\cos 135 - i\sin 135 + \cos 405 + i\sin 405]$$

$$= \frac{\pi i}{2} \left[-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right]$$

$$= \frac{\pi i}{2} \left(\frac{-2i}{\sqrt{2}} \right)$$

$$= \frac{\pi}{\sqrt{2}}$$

ter two are

10.7 Definite Integrals of the Trigonometric functions of the type $\int f(\sin\theta, \cos\theta)d\theta$

Consider a real integral $\int_0^{2\pi} f(\sin\theta, \cos\theta)d\theta$ (1)

The evaluation of such integrals as (1) can be reduced to the calculation of a rational function of z along the **Unit Circle** $|z| = 1$.

Since rational functions have no singularities other than poles, the Residue theorem provides a simple means for evaluating integrals of the form (1).

We set $z = e^{i\theta}$ so that $dz = e^{i\theta} i d\theta$
If $z = e^{i\theta} = \cos\theta + i\sin\theta$
then $\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

Hence $z + \frac{1}{z} = 2\cos\theta$

and $z - \frac{1}{z} = 2i\sin\theta$

Thus we substitute

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and

$dz = e^{i\theta} i d\theta$.

The following examples illustrate the method of evaluating integrals of the type $\int_c f(\sin\theta, \cos\theta)d\theta$ where c is the unit circle $|z| = 1$

Example 5

Evaluate $\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$

Solution

Substituting $z = e^{i\theta}$

$$dz = e^{i\theta} i d\theta \text{ or } d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = -\frac{1}{2i}\left(z-\frac{1}{z}\right)$$

we have

$$\boxed{\begin{aligned}\cos\theta &= \frac{z+\frac{1}{z}}{2}.\\ \sin\theta &= \frac{z-\frac{1}{z}}{2i}\\ d\theta &= \frac{dz}{iz}\end{aligned}}$$

in (1) we have

$$\int_0^{2\pi} f(\sin\theta,\cos\theta)d\theta = \int_c \phi(z)dz = 2\pi i\sum R$$

where c is the unit circle.

The method of evaluating the integrals of the type (1) is illustrate in the following examples.

Example 6

Use residue theorem to how that

$$\int \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}$$

Solution

We put

$$z = \cos\theta + i\sin\theta = e^{i\theta}$$

$$z^{-1} = \cos\theta - i\sin\theta = e^{-i\theta}$$ and take the contour $|z| = 1$

$$dz = ie^{i\theta}d\theta \quad \text{or} \quad d\theta = \frac{dz}{iz}$$

$$\int_0^{2\pi}\frac{d\theta}{5+4\sin\theta} = \int_c \frac{dz}{iz\left[5+4(z-z^{-1}).\frac{1}{2i}\right]}$$

The integrand $\frac{z^{p-1}}{1+z}dz$ has a simple pole at $z=-1=\cos\pi i+i\sin\pi i=e^{\pi i}$

Residue R at z = -1 is given by

$$R=\lim_{z\to e^{\pi i}}\frac{(z+1)z^{p-1}}{z+1}=\left(e^{\pi i}\right)^{p-1}$$

Then $\int_c \frac{z^{p-1}}{1+z}=2\pi i$ (sum of Residues)

$$=2\pi i\left(e^{\pi i}\right)^{p-1}$$

$$\text{or } \int_c \frac{z^{p-1}}{1+z}dz=\int_{PQ}+\int_{QMNTR}+\int_{RS}+\int_{SVP} \qquad (3)$$

Any point z, on PQ is given by z = x + iy = x

Any point z, on the circle QMNTR is given by $z=Re^{i\theta}$ where R is the radius of the outer circle.

Any point on RS is given by z = x + iy = x and the points on the circle SVP is given by $z=\in e^{i\theta}$ where $\in$ is the radius of the inner circle.

Hence (3) becomes

$$\int_c \frac{z^{p-1}}{1+z}dz=\int_\in^R \frac{x^{p-1}}{1+x}dx+\int_0^{2\pi}\frac{\left(Re^{i\theta}\right)^{p-1}Re^{i\theta}id\theta}{1+Re^{i\theta}}+\int_R^\in \frac{\left(xe^{2\pi i}\right)^{p-1}}{1+xe^{2\pi i}}dx+\int_{2\pi}^0 \frac{\left(\in e^{i\theta}\right)^{p-1}\in e^{i\theta}id\theta}{1+\in e^{i\theta}}d\theta$$

Taking the limit $\in\to 0$ and $R\to\infty$ we have the second and the fourth in egral tend to zero.

Hence using Residue Theorem;

$$\int_c \frac{z^{p-1}}{1+zdz}=\int_0^\infty \frac{x^{p-1}}{1+x}dx+\int_\infty^0 \frac{e^{2\pi i(p-1)}}{1+x}dx=2\pi i e^{(p-1)\pi i}$$

$$\text{or } \int_0^\infty \frac{x^{p-1}}{1+x}dx-\int_0^\infty \frac{e^{2\pi i(p-1)}x^{p-1}}{1+x}dx=2\pi i e^{(p-1)\pi i}$$

$$\text{or } \int_0^\infty \frac{x^{p-1}}{1+x}dx-e^{2\pi i(p-1)}\int_0^\infty \frac{x^{p-1}}{1+x}dx=2\pi i e^{(p-1)\pi i}$$

$$\text{or } \int_0^\infty \frac{x^{p-1}}{1+x}dx-\left[1-e^{2\pi i(p-1)}\right]=2\pi i e^{(p-1)\pi i}$$

or $\int_0^\infty \frac{x^{p-1}}{1+x}dx = \frac{2\pi i\, e^{(p-1)\pi i}}{1 - e^{2\pi i(p-1)}}$

Dividing the Numerator and denominator of the Right Hand side by $e^{(p-1)\pi i}$ we have

$$\int_0^\infty \frac{x^{p-1}}{1+xdx} = \frac{2\pi i}{e^{-(p-1)\pi i} - e^{(p-1)\pi i}}$$

$$= \frac{2\pi i}{e^{-p\pi i}.e^{\pi i} - e^{p\pi i}e^{-\pi i}}$$

$$= \frac{2\pi i}{e^{-p\pi i}(-1) - e^{p\pi i}(-1)}$$

$$= \frac{2\pi i}{e^{p\pi i} - e^{-p\pi i}}$$

$$= \frac{2\pi i}{2i\sin p\pi}$$

$$= \frac{\pi}{\sin p\pi}$$

Summary

You have learnt the following from this lesson: -

i) Statement of Cauchy's Residue Theorem namely:
If f(z) is analytic inside and on a closed curve C except at a finite number of poles at $a_1, a_2, \ldots a_n$ inside C at which the Residues are $R_1, R_2, \ldots R_n$ respectively, then,
$\int_C f(z)dz = 2\pi i(R_1 + R_2 + \ldots R_n)$

ii) To apply Residue Theorem for the evaluation of

a). Improper integrals of the type
$\int_{-\infty}^{\infty} f(x)dx$

b). Definite integrals of the Trigonometric type $\int f(\sin\theta, \cos\theta)d\theta$

c). Integration round a branch point of the type $\int_0^\infty \frac{x^{p-1}}{1+x}dx$ when $0 < p < 1$

Exercise 10

1. State the Residue Theorem

$$= \frac{2}{8\left(-\frac{i}{2}\right)+10i} = \frac{2}{-4i+10i} = \frac{1}{3i}$$

Thus $$\int_c^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_c \frac{2dz}{4z^2+10iz-4} = 2\pi i \sum R$$

$$= 2\pi i\left(\frac{1}{3i}\right)$$

$$= \frac{2\pi}{3}$$

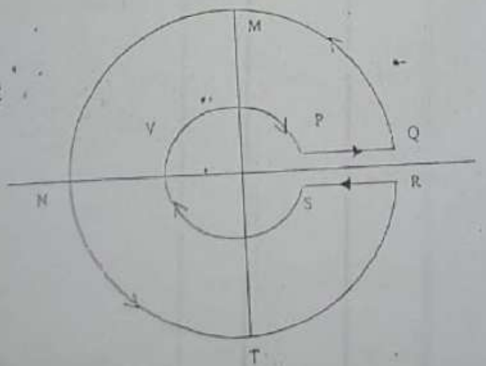
10.8 Integration round a Branch point

We have already studied the definition of the branch points and we now consider an example involving Branch points and Branch cuts. We shall evaluate the improper real integral,

$$\int_0^{\infty} \frac{x^{p-1}}{1+x} dx \qquad (1)$$

where p is a positive proper fraction or $0 < p < 1$.

To evaluate the integral (1) we consider $\int \frac{z^{p-1}}{1+z} dz$ since $z = 0$ is a branch point, we can choose c as the contour where the positive x axis is the branch line PQ and RS coincident with x axis but shown separated



$$= \int_c \frac{2idz}{iz(10i+4z-4z^{-1})}$$

$$= \int_c \frac{2dz}{4z^2+10iz-4}$$

where c is a circle with center origin and radius 1 unit.

The poles of $\Gamma = \int_c \frac{2}{4z^2+10iz-4}$ are obtained by solving $z^2+10iz-4=0$ and are given by

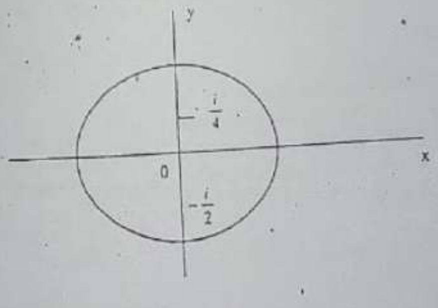
$$z = \frac{-10i \pm \sqrt{-100+64}}{8}$$

$$= \frac{-10i \pm \sqrt{-36}}{8}$$

$$= \frac{-10i \pm 6i}{8} = \frac{-5i \pm 3i}{4}$$

$$= -\frac{i}{2} \text{ and } -2i$$

only $-\frac{i}{2}$ lies inside the unit circle $|z| = 1$



Residue at $z = -\frac{i}{2}$ is $\lim_{z \to \frac{-i}{2}} \frac{\left(z+\frac{i}{2}\right).2}{4z^2+10iz-4}$

$$= \lim_{z \to -\frac{i}{2}} \frac{2}{(8z+10i)} \text{ Using L' Hospital Rule}$$