Exercise 1.1: Let P be a probability measure on $(\Omega, \mathcal{F})$

i) **Monotonicity:** If $A \subset B$ then $P(B) - P(A) = P(B - A) \geq 0$.

Let $C = B \cap A^c$. Then $A \cup C = B$. Because A and C are disjoint, we have that $P(B) = P(A \cup C) = P(A) + P(C)$. Then $P(B) - P(A) = P(C) = P(B \setminus A)$ which is greater or equal to zero since $P(C) \geq 0$.

ii) **Subadditivity:** For $\langle A_m \rangle \in \mathcal{F}$ and $A \subset \cup^\infty A_m$ it follows that $P(A) \leq \sum^\infty P(A_m)$

Since $\langle A_m \rangle \in \mathcal{F}$ and $\mathcal{F}$ is a sigma field, there exists $\langle B_m \rangle$ such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\cup^\infty A_m = \cup^\infty B_m$ [1, pp. 17–18]. From (i) we know that $A \subset \cup B_m \implies P(A) \leq P(\cup B_m)$. Then

$$P(A) \leq P(\cup^\infty B_m) = \sum^\infty P(B_m) \underset{*}{\leq} \sum^\infty P(A_m)$$

The $(*)$ inequality follows from how we define each element of $\langle B_m \rangle$:

$$\begin{aligned} B_i = A_i \setminus [A_1 \cup \cdots A_{i-1}] &\implies P(B_i) = P(A_i) - \sum^{i-1} P(A_j) \\ &\implies P(B_i) \leq P(A_i),\ \forall i \end{aligned}$$

iii) **Continuity from below:** If $A_i \uparrow A$ then $P(A_i) \uparrow P(A)$

By supposition, we know that for every i we have $A_i = \cup_{j=1}^i A_j$

$$\lim_{i\to\infty} P(A_i) = \lim_{i\to\infty} P(\cup^i A_j) = P(A)$$

(I'm not sure if this is rigorous enough but ...)

iv) **Continuity from above:** If $A_i \downarrow A$ then $P(A_i) \downarrow P(A)$

For every i we have that $A_i = \cap_{j=1}^i A_j$

$$\lim_{i\to\infty} P(A_j) = \lim_{i\to\infty} P(\cap^i A_j) = P(A)$$

Exercise 1.2:

i) If $\mathcal{F}_i,\ i \in I$ are σ-fields then $\cap_{i\in I}\mathcal{F}_i$ is.

> Suppose $\mathcal{F}_i$ are σ-fields.
>
> - Consider $A \in \cap_{i\in I}\mathcal{F}_i$. Then $\exists i$ such that $A \in \mathcal{F}_i$. So $A^c \in \mathcal{F}_i$ because $\mathcal{F}_i$ is a σ-field σ-field. Then $A^c \in \cap_{i\in I}\mathcal{F}_i$.
> - Consider $A_j \in \cap_{i\in I}\mathcal{F}_i$ a countable sequence of sets. Then $\forall j, \forall i,\ A_j \in \mathcal{F}_i$. Then $\cup A_j \in F_i,\ \forall i$ because $\mathcal{F}_i$ is a σ-field. Then $\cup A_j \in \cap_{i\in I}\mathcal{F}_i$

ii) Use the result in (i) to show if we are given a set Ω and a collection $\mathcal{A}$ of subsets of Ω then there is a smallest σ-field containing $\mathcal{A}$.

> Let $\mathcal{A}$ be a collection of subsets of Ω. let $\mathcal{F}_{\mathcal{A}}$ be the set of sigma fields that contain $\mathcal{A}$. Define $\mathcal{F} = \cap_{\mathcal{A}}\mathcal{F}_{\mathcal{A}}$. From (i) we know that $\mathcal{F}$ is a sigma field. By definition $\mathcal{F}$ is the smallest sigma field containing $\mathcal{A}$ since for sigma field $\mathcal{C}$ such that $\mathcal{A} \subset \mathcal{C}$, we have $\mathcal{F} = \cap_{\mathcal{A}}\mathcal{F}_{\mathcal{A}} \subset \mathcal{C}$

With $(\mathbb{R}, \mathcal{F}, P)$ and $\mathcal{B}$ the borel sets, define a random variable as a real valued function such that $X : \Omega \to \mathbb{R}$ is $\mathcal{F}$ measurable for every borel set

$$X^{-1}(B) \in \mathcal{F}, \quad B \in \mathcal{B}$$

Then X induces a probability measure on $\mathbb{R}$ called its distribution

$$\mu(A) = P(X \in A) = P(X^{-1}(A)), \quad A \in \mathcal{B}$$

The **distribution function** is defined as

$$F(x) = P(X \leq x)$$

When the distribution function $F(x) = P(X \leq x)$ has the form

$$F(x) = \int_{-\infty}^{x} f(y)dy$$

we say that X has **density function** f.

Exercise 1.5: A σ-field $\mathcal{F}$ is said to be **countably generated** if there is a countable collection $\mathcal{C} \subset \mathcal{F}$ so that $\sigma(\mathcal{C})$. Show that $\mathcal{R}^d$ is countably generated.

$\mathcal{R}^d$ are the Borel subsets of $\mathbb{R}^n$. First let's look at $\mathcal{R}$. We'll show that $\mathcal{G} = \{[q, \infty) : q \in \mathbb{Q}\}$ generates $\mathcal{R}$. Consider an arbitrary open interval $(a, b) \subset \mathbb{R},\ a < b$. See that

$$[b, \infty)^c \cap \bigcup_n^\infty [a + 1/n, \infty) = (-\infty, b) \cap (a, \infty) = (a, b) \tag{1}$$

Remembering that every open set of real numbers is the countable union of disjoint open intervals [1, p. 42], we observe that using 1 as a way to generate open intervals, we can also generate any open set. Therefore $\mathcal{R} \subset \sigma(\mathcal{G})$. To see that $\sigma(\mathcal{B}) \subset \mathcal{R}$, observe that any interval $[q, \infty)$ can be generated by unions, intersections, and complements of open sets (very easy to show). Therefore $\mathcal{R} = \sigma(\mathcal{G})$

Since $\mathcal{G}$ is countable, $\mathcal{G} \times \cdots \times \mathcal{G}$ is countable and $\sigma(\mathcal{G} \times \cdots \times \mathcal{G}) = \mathcal{R} \times \cdots \times \mathcal{R} = \mathcal{R}^d$

References

[1] H. L. Royden. *Real analysis / H.L. Royden.* eng. Third edition. New York: Macmillan, 1988 - 1988. ISBN: 0024041513.