

**Exercise 1.1:** Let  $P$  be a probability measure on  $(\Omega, \mathcal{F})$

i) **Monotonicity:** If  $A \subset B$  then  $P(B) - P(A) = P(B - A) \geq 0$ .

Let  $C = B \cap A^c$ . Then  $A \cup C = B$ . Because  $A$  and  $C$  are disjoint, we have that  $P(B) = P(A \cup C) = P(A) + P(C)$ . Then  $P(B) - P(A) = P(C) = P(B \setminus A)$  which is greater or equal to zero since  $P(C) \geq 0$ .

ii) **Subadditivity:** For  $\langle A_m \rangle \in \mathcal{F}$  and  $A \subset \cup^\infty A_m$  it follows that  $P(A) \leq \sum^\infty P(A_m)$

Since  $\langle A_m \rangle \in \mathcal{F}$  and  $\mathcal{F}$  is a sigma field, there exists  $\langle B_m \rangle$  such that  $B_i \cap B_j = \emptyset$  for  $i \neq j$  and  $\cup^\infty A_m = \cup^\infty B_m$  [1, pp. 17–18]. From (i) we know that  $A \subset \cup B_m \implies P(A) \leq P(\cup B_m)$ . Then

$$P(A) \leq P(\cup^\infty B_m) = \sum^\infty P(B_m) \underset{*}{\leq} \sum^\infty P(A_m)$$

The (\*) inequality follows from how we define each element of  $\langle B_m \rangle$ :

$$\begin{aligned} B_i &= A_i \setminus [A_1 \cup \dots \cup A_{i-1}] \implies P(B_i) = P(A_i) - \sum_{j=1}^{i-1} P(A_j) \\ &\implies P(B_i) \leq P(A_i), \forall i \end{aligned}$$

iii) **Continuity from below:** If  $A_i \uparrow A$  then  $P(A_i) \uparrow P(A)$

By supposition, we know that for every  $i$  we have  $A_i = \cup_{j=1}^i A_j$

$$\lim_{i \rightarrow \infty} P(A_i) = \lim_{i \rightarrow \infty} P(\cup^i A_j) = P(A)$$

(I'm not sure if this is rigorous enough but ...)

iv) **Continuity from above:** If  $A_i \downarrow A$  then  $P(A_i) \downarrow P(A)$

For every  $i$  we have that  $A_i = \cap_{j=1}^i A_j$

$$\lim_{i \rightarrow \infty} P(A_i) = \lim_{i \rightarrow \infty} P(\cap^i A_j) = P(A)$$

**Exercise 1.2:**

i) If  $\mathcal{F}_i$ ,  $i \in I$  are  $\sigma$ -fields then  $\cap_{i \in I} \mathcal{F}_i$  is.

Suppose  $\mathcal{F}_i$  are  $\sigma$ -fields.

- Consider  $A \in \cap_{i \in I} \mathcal{F}_i$ . Then  $\exists i$  such that  $A \in \mathcal{F}_i$ . So  $A^c \in \mathcal{F}_i$  because  $\mathcal{F}_i$  is a  $\sigma$ -field. Then  $A^c \in \cap_{i \in I} \mathcal{F}_i$ .
- Consider  $A_j \in \cap_{i \in I} \mathcal{F}_i$  a countable sequence of sets. Then  $\forall j, \forall i, A_j \in \mathcal{F}_i$ . Then  $\cup A_j \in \mathcal{F}_i, \forall i$  because  $\mathcal{F}_i$  is a  $\sigma$ -field. Then  $\cup A_j \in \cap_{i \in I} \mathcal{F}_i$ .

ii) Use the result in (i) to show if we are given a set  $\Omega$  and a collection  $\mathcal{A}$  of subsets of  $\Omega$  then there is a smallest  $\sigma$ -field containing  $\mathcal{A}$ .

Let  $\mathcal{A}$  be a collection of subsets of  $\Omega$ . let  $\mathcal{F}_{\mathcal{A}}$  be the set of sigma fields that contain  $\mathcal{A}$ . Define  $\mathcal{F} = \cap_{\mathcal{A}} \mathcal{F}_{\mathcal{A}}$ . From (i) we know that  $\mathcal{F}$  is a sigma field. By definition  $\mathcal{F}$  is the smallest sigma field containing  $\mathcal{A}$  since for sigma field  $\mathcal{C}$  such that  $\mathcal{A} \subset \mathcal{C}$ , we have  $\mathcal{F} = \cap_{\mathcal{A}} \mathcal{F}_{\mathcal{A}} \subset \mathcal{C}$

With  $(\mathbb{R}, \mathcal{F}, P)$  and  $\mathcal{B}$  the borel sets, define a random variable as a real valued function such that  $X : \Omega \rightarrow \mathbb{R}$  is  $\mathcal{F}$  measurable for every borel set

$$X^{-1}(B) \in \mathcal{F}, \quad B \in \mathcal{B}$$

Then  $X$  induces a probability measure on  $\mathbb{R}$  called its distribution

$$\mu(A) = P(X \in A) = P(X^{-1}(A)), \quad A \in \mathcal{B}$$

The **distribution function** is defined as

$$F(x) = P(X \leq x)$$

When the distribution function  $F(x) = P(X \leq x)$  has the form

$$F(x) = \int_{-\infty}^x f(y) dy$$

we say that  $X$  has **density function**  $f$ .

**Exercise 1.5:** A  $\sigma$ -field  $\mathcal{F}$  is said to be **countably generated** if there is a countable collection  $\mathcal{C} \subset \mathcal{F}$  so that  $\sigma(\mathcal{C})$ . Show that  $\mathcal{R}^d$  is countably generated.

$\mathcal{R}^d$  are the Borel subsets of  $\mathbb{R}^n$ . First let's look at  $\mathcal{R}$ . We'll show that  $\mathcal{G} = \{[q, \infty) : q \in \mathbb{Q}\}$  generates  $\mathcal{R}$ . Consider an arbitrary open interval  $(a, b) \subset \mathbb{R}$ ,  $a < b$ . See that

$$[b, \infty)^c \cap \bigcup_n [a + 1/n, \infty) = (-\infty, b) \cap (a, \infty) = (a, b) \quad (1)$$

Remembering that every open set of real numbers is the countable union of disjoint open intervals [1, p. 42], we observe that using 1 as a way to generate open intervals, we can also generate any open set. Therefore  $\mathcal{R} \subset \sigma(\mathcal{G})$ . To see that  $\sigma(\mathcal{B}) \subset \mathcal{R}$ , observe that any interval  $[q, \infty)$  can be generated by unions, intersections, and complements of open sets (very easy to show). Therefore  $\mathcal{R} = \sigma(\mathcal{G})$ . Since  $\mathcal{G}$  is countable,  $\mathcal{G} \times \cdots \times \mathcal{G}$  is countable and  $\sigma(\mathcal{G} \times \cdots \times \mathcal{G}) = \mathcal{R} \times \cdots \times \mathcal{R} = \mathcal{R}^d$

## References

- [1] H. L. Royden. *Real analysis / H.L. Royden*. eng. Third edition. New York: Macmillan, 1988 - 1988. ISBN: 0024041513.