Exercise 1.1: Let P be a probability measure on $(\Omega, \mathcal{F})$

i) **Monotonicity:** If $A \subset B$ then $P(B) - P(A) = P(B - A) \geq 0$.

Let $C = B \cap A^c$. Then $A \cup C = B$. Because A and C are disjoint, we have that $P(B) = P(A \cup C) = P(A) + P(C)$. Then $P(B) - P(A) = P(C) = P(B \setminus A)$ which is greater or equal to zero since $P(C) \geq 0$. □

ii) **Subadditivity:** For $\langle A_m \rangle \in \mathcal{F}$ and $A \subset \cup^\infty A_m$ it follows that $P(A) \leq \sum^\infty P(A_m)$

Since $\langle A_m \rangle \in \mathcal{F}$ and $\mathcal{F}$ is a sigma field, there exists $\langle B_m \rangle$ such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\cup^\infty A_m = \cup^\infty B_m$ [2, pp. 17–18]. From (i) we know that $A \subset \cup B_m \implies P(A) \leq P(\cup B_m)$. Then

$$P(A) \leq P(\cup^\infty B_m) = \sum^\infty P(B_m) \underset{*}{\leq} \sum^\infty P(A_m)$$

The $(*)$ inequality follows from how we define each element of $\langle B_m \rangle$:

$$\begin{aligned} B_i = A_i \setminus [A_1 \cup \cdots A_{i-1}] &\implies P(B_i) = P(A_i) - \sum^{i-1} P(A_j) \\ &\implies P(B_i) \leq P(A_i),\ \forall i \end{aligned}$$

□

iii) **Continuity from below:** If $A_i \uparrow A$ then $P(A_i) \uparrow P(A)$

By supposition, we know that for every i we have $A_i = \cup_{j=1}^i A_j$

$$\lim_{i\to\infty} P(A_i) = \lim_{i\to\infty} P(\cup^i A_j) = P(A)$$

(I'm not sure if this is rigorous enough but ...) □

iv) **Continuity from above:** If $A_i \downarrow A$ then $P(A_i) \downarrow P(A)$

For every i we have that $A_i = \cap_{j=1}^{i} A_j$

$$\lim_{i\to\infty} P(A_j) = \lim_{i\to\infty} P(\cap^i A_j) = P(A)$$

$\square$

Exercise 1.2:

i) If $\mathcal{F}_i,\ i \in I$ are σ-fields then $\cap_{i\in I}\mathcal{F}_i$ is.

Suppose $\mathcal{F}_i$ are σ-fields.

- Consider $A \in \cap_{i\in I}\mathcal{F}_i$. Then $\exists i$ such that $A \in \mathcal{F}_i$. So $A^c \in \mathcal{F}_i$ because $\mathcal{F}_i$ is a σ-field σ-field. Then $A^c \in \cap_{i\in I}\mathcal{F}_i$.
- Consider $A_j \in \cap_{i\in I}\mathcal{F}_i$ a countable sequence of sets. Then $\forall j, \forall i,\ A_j \in \mathcal{F}_i$. Then $\cup A_j \in F_i,\ \forall i$ because $\mathcal{F}_i$ is a σ-field. Then $\cup A_j \in \cap_{i\in I}\mathcal{F}_i$

$\square$

ii) Use the result in (i) to show if we are given a set Ω and a collection $\mathcal{A}$ of subsets of Ω then there is a smallest σ-field containing $\mathcal{A}$.

Let $\mathcal{A}$ be a collection of subsets of Ω. let $\mathcal{F}_\mathcal{A}$ be the set of sigma fields that contain $\mathcal{A}$. Define $\mathcal{F} = \cap_\mathcal{A}\mathcal{F}_\mathcal{A}$. From (i) we know that $\mathcal{F}$ is a sigma field. By definition $\mathcal{F}$ is the smallest sigma field containing $\mathcal{A}$ since for sigma field $\mathcal{C}$ such that $\mathcal{A} \subset \mathcal{C}$, we have $\mathcal{F} = \cap_\mathcal{A}\mathcal{F}_\mathcal{A} \subset \mathcal{C}$ $\square$

With $(\mathbb{R}, \mathcal{F}, P)$ and $\mathcal{B}$ the borel sets, define a random variable as a real valued function such that $X : \Omega \to \mathbb{R}$ is $\mathcal{F}$ measurable for every borel set

$$X^{-1}(B) \in \mathcal{F}, \quad B \in \mathcal{B}$$

Then X induces a probability measure on $\mathbb{R}$ called its distribution

$$\mu(A) = P(X \in A) = P(X^{-1}(A)), \quad A \in \mathcal{B}$$

The **distribution function** is defined as

$$F(x) = P(X \leq x)$$

When the distribution function $F(x) = P(X \leq x)$ has the form

$$F(x) = \int_{-\infty}^{x} f(y)dy$$

we say that X has **density function** f.

Exercise 1.5: A σ-field $\mathcal{F}$ is said to be **countably generated** if there is a countable collection $\mathcal{C} \subset \mathcal{F}$ so that $\sigma(\mathcal{C})$. Show that $\mathcal{R}^d$ is countably generated.

$\mathcal{R}^d$ are the Borel subsets of $\mathbb{R}^n$. First let's look at $\mathcal{R}$. We'll show that $\mathcal{G} = \{[q, \infty) : q \in \mathbb{Q}\}$ generates $\mathcal{R}$. Consider an arbitrary open interval $(a, b) \subset \mathbb{R},\ a < b$. See that

$$[b, \infty)^c \cap \bigcup_{n}^{\infty} [a + 1/n, \infty) = (-\infty, b) \cap (a, \infty) = (a, b) \tag{1}$$

Remembering that every open set of real numbers is the countable union of disjoint open intervals [2, p. 42], we observe that using 1 as a way to generate open intervals, we can also generate any open set. Therefore $\mathcal{R} \subset \sigma(\mathcal{G})$. To see that $\sigma(\mathcal{B}) \subset \mathcal{R}$, observe that any interval $[q, \infty)$ can be generated by unions, intersections, and complements of open sets (very easy to show). Therefore $\mathcal{R} = \sigma(\mathcal{G})$
Since $\mathcal{G}$ is countable, $\mathcal{G} \times \cdots \times \mathcal{G}$ is countable and $\sigma(\mathcal{G} \times \cdots \times \mathcal{G}) = \mathcal{R} \times \cdots \times \mathcal{R} = \mathcal{R}^d$. $\square$

Exercise 1.6: Suppose X and Y are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.

We want to show that $Z^{-1}(B) \in \mathcal{F}$ given $B \in \mathcal{R}$ (borel set). By supposition we have $Z^{-1}(B) \cap A = X^{-1}(B) \cap A$ and $Z^{-1}(B) \cap A^c = Y^{-1}(B) \cap A^c$. Therefore,

$$\begin{aligned} Z^{-1}(B) &= (Z^{-1}(B) \cap A) \cup (Z^{-1}(B) \cap A^c) \\ &= (X^{-1}(B) \cap A) \cup (Y^{-1}(B) \cap A^c) \overset{*}{\in} \mathcal{F} \end{aligned}$$

To show $(*)$, observe that since X and Y are random variables, we know that $X^{-1}(B), Y^{-1}(B) \in \mathcal{F}$. Then since $A, A^c \in \mathcal{F}$ it's clear that both $X^{-1}(B) \cap A$ and $Y^{-1}(B) \cap A^c \in \mathcal{F}$. So their intersection must also be an element of $\mathcal{F}$. $\square$

Exercise 1.8: Show that a distribution function has at most countably many discontinuities.

Let $D = \{x \in \mathbb{R} : F(x-) \neq F(x+)\}$. Since F is increasing, $x \in D \implies F(x-) < F(x+) \implies F(x+) - F(x-) > 0$. However, it's also clear from the fact that F is increasing and $F(-\infty) = 0, F(\infty) = 1$ that $\sum_{x \in D} F(x+) - F(x-) \leq 1 < \infty$. Therefore D must be countable infinite since the sum over the terms is finite [1, p. 11]. $\square$

Alternate solution partially inspired from a clever answer on stack exchange

Let D be the set of points of discontinuity. Since $F(x-) < F(x+)$ for all $x \in D$, choose a rational r_x such that $F(x-) < r_x < F(x+)$. Because F is increasing, the intervals $(F(x-), F(x+))$ are mutually disjoint. So $x \to r_x$ is an injective function from $D \to \mathbb{Q}$. Therefore D must be countably infinite. $\square$

Exercise 1.9: Show that if $F(x) = P(X \leq x)$ is continuous then $Y = F_X(X)$ has a uniform distribution on $(0, 1)$. That is, if $y \in [0, 1]$, $P(Y \leq y) = y$

Suppose $F_X(x) = P(X \leq x)$ is continuous. Let $Y = F_X(X)$. Then for $y \in [0, 1]$

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(F_X(X) \leq y) \\
&= P(X \leq F_X^{-1}(y)) && \text{(using } F^{-1} \text{ on page 6)} \\
&= F_X(F_X^{-1}(y)) = y
\end{aligned}$$

$\square$

Exercise 1.10: Suppose that X has density f, $P(\alpha \leq X \leq \beta) = 1$ and g is a function that is increasing and differentiable on (α, β). Then $g(X)$ has density $f(g^{-1}(x))/g'(g^{-1}(x))$ for $x \in (g(\alpha), g(\beta))$ and 0 otherwise. When $g(x) = ax + b$ with $a \geq 0$ the answer is $f((y - b)/a)/a$.

Let $F_{g(X)}(x)$ be the distribution function for $g(x)$. Then the density function is $h = \frac{d}{dx} F_{g(X)}(x)$. Then

$$\begin{aligned}
F_{g(X)}(x) &= P(g(X) \leq x) \\
&= P(X \leq g^{-1}(x)) && \text{(this exists because } g \text{ is increasing)} \\
&= F_X(g^{-1}(x))
\end{aligned}$$

So by definition

$$\begin{aligned}
h = F_{g(X)}(x) &= \frac{d}{dx}F_X(g^{-1}(x)) \\
&= f(g^{-1}(x))\left(\frac{d}{dx}g^{-1}(x)\right) \\
&= f(g^{-1}(x))\frac{1}{g'(g^{-1}(x))} = \frac{f(g^{-1}(x))}{g'(g^{-1}(x))}
\end{aligned}$$

Also notice that

$$P(\alpha \leq X \leq \beta) = 1 \implies P(g(\alpha) \leq g(X) \leq g(\beta)) = 1 \implies x \in (g(\alpha), g(\beta))$$

Exercise 1.11: Suppose X has a normal distribution. Use the previous exercise to compute the density of $\exp(X)$. This is called the **lognormal distribution**.

I'm assuming X has a standard normal distribution ($\mu = 0$, $\sigma = 1$). Then using the previous problem we have that $g(X) = \exp(X)$ has density

$$\begin{aligned}
\frac{f(g^{-1}(x))}{g'(g^{-1}(x))} &= \frac{f(\ln(x))}{\exp(\ln(x))} \\
&= \frac{f(\ln(x))}{x} \\
&= \frac{1}{x\sqrt{2\pi}}\exp\left(\frac{-\ln(x)^2}{2}\right), \quad \text{for } x \in (\exp(-\infty), \exp(\infty)) = (0, \infty)
\end{aligned}$$

Exercise 1.12:

i) Suppose X has density function f. Compute the distribution function of X^2 and then differentiate to find its density function.

ii) Work out the answer when X has a standard normal distribution to find the density of the **chi-square distribution**.

References

[1] G.B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2013. ISBN: 9781118626399.

[2] H. L. Royden. *Real analysis / H.L. Royden.* eng. Third edition. New York: Macmillan, 1988 - 1988. ISBN: 0024041513.