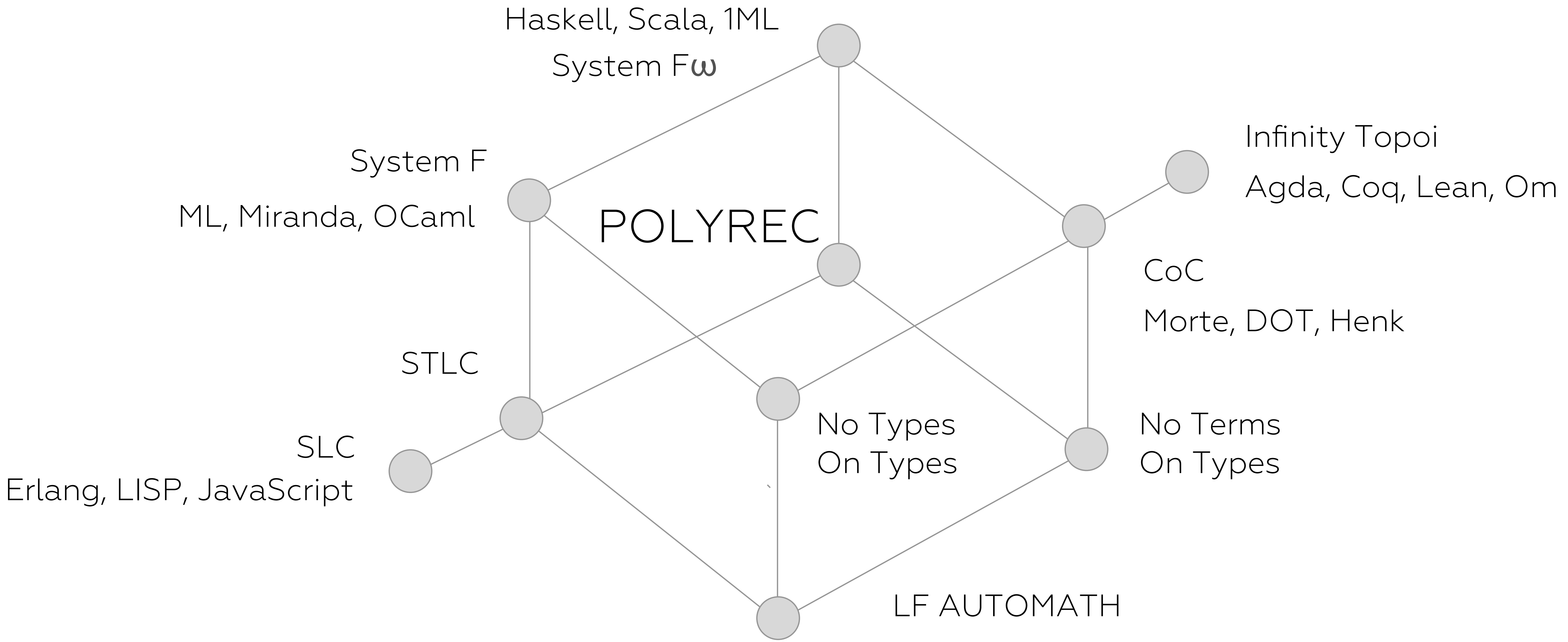
The Systems Engineering of
Consistent Pure Language with Effect Type System
for Certified Applications and Higher Languages

PTS-infinity

MMCTSE 2018 144 Maksym Sokhatskyi

Programming Languages

in Barendregt Cube



Model Verification Process

From proving to and Extraction, Linking

1) Models

— IR/II
— Bohm
— HoTT

2) Core — Infinity Language

— Model Verification
— Normalization
— Bidirectional Checking

— Pure Type System (Om)
— Identity
— Induction
— Homotopy Interval [0,1]

3) Extraction

— LLVM
— Interpreters
— Detyping
— Optimization
— Linking

4) Runtimes

— O
— Erlang
— V8
— JVM

Runtime Languages

Through a Prism of Engineering

JIT	Interpreters	LLVM	Non-LLVM
LuaJIT	K	Rust	OCaml
V8	BEAM/Erlang	Julia	GHC
SpiderMonkey	O	C/C++	Spiral
EDGE			
JVM/HotSpot			
CLR			

Higher Languages

for Proving and Model Checking

Target	Class	Higher Language	Type Theory
CPU	Non-LLVM	Spiral	System F
JVM	JIT	Scala	System F-omega
GHC	Non-LLVM	Morte	CoC
Erlang	Interpreter	Om	PTS-infinity
O	Interpreter	Om	PTS-infinity

MLTT — CT — Proof Theory

Syntax — Semantics — Logic

$x : A$ — x is a object of type A
$y = [\, x : A \,]$ — x and y are definitionaly equal objects of type A

Nat : U n — constant functor
List (A : U n) : U n — functor
List Nat : U n — constant functor

$$\text{List } A = 1 \longrightarrow \text{List } A \longrightarrow \text{List } A \longleftarrow A : U$$
$$\text{nil} : 1 \longrightarrow \text{List } A$$
$$\text{cons} : A \longrightarrow \text{List } A \longrightarrow \text{List } A$$

$$\text{Nat} = 1 \longrightarrow N \longrightarrow N : U$$
$$\text{zero} : 1 \longrightarrow N$$
$$\text{succ} : N \longrightarrow N$$

Pure Type System

Infinity Topoi

$U_0 : U_1 : U_2 : U_3 : \ldots \infty$

U_0 — propositions
U_1 — sets
U_2 — types
U_3 — sorts

$S\ (n : nat) = U\ n$
$A_1\ (n\ m : nat) = U\ n : U\ m$ where $m > n$ — cumulative
$R_1\ (m\ n : nat) = U\ m \longrightarrow U\ n : U\ (\max\ m\ n)$ — predicative

$A_2\ (n : nat) = U\ n : U\ (n + 1)$ — non-cumulative
$R_2\ (m\ n : nat) = U\ m \longrightarrow U\ n : U\ n$ — impredicative

$Prop = Large\ \Omega_0 = U_0$

$\boldsymbol{\Sigma} = Large\ \Omega_2 = U_2$

Pi Type

Functional Completness

```
data O₁ := U : nat → O₁
         | Var: Ident → O₁
         | App: O₁ → O₁ → O₁
         | Lambda: Binder → O₁ → O₁ → O₁
         | Arrow: O₁ → O₁ → O₁
         | Pi: name → O₁ → O₁ → O₁.
```

∀ x: A, B x : U — formation rule
λ x: A, b : B x — introduction
app f a : B x — elimination
app (λ o:A, b) a = b [a/o] : B x — computation

```
record Pi (A: Type) :=
    intro: (A → Type) → Type :=
    fun: (B: A → Type) → ∀ (a: A) → B a → intro B
    app: (B: A → Type) → intro B → ∀ (a: A) → B a
    app-fun (B: A → Type) (f: ∀ (a: A) → B a): ∀ (a: A) → app (fun f) a ==> f a
    fun-app (B: A → Type) (p: intro B): fun (λ (a: A) → app p a) ==> p
```

Shifting

Modified version of De Bruin indeces

```
sh (:star, X)          N P → (:star, X)
   (:var, N, I)        N P → (:var, N, I+1) when I >= P
                           → (:var, N, I)
   (:remote, X)        N P → (:remote, X)
   (:pi, N, 0, I, O)   N P → (:pi, N, 0, sh I N P, sh O N (P+1))
   (:fn, N, 0, I, O)   N P → (:fn, N, 0, sh I N P, sh O N (P+1))
   (:app, L, R)        N P → (:app, L, R)
```

Substitution

Replacing variable occurance in terms

```
sub (:star, X)         N V L → (:star, X)
    (:var, N, L)       N V L → V
    (:var, N, I)       N V L → (:var, N, I−1) when I > L
    (:remote, X)       N V L → (:remote, X)
    (:pi, N, 0, I, O)  N V L → (:pi, N, 0, sub I N V L, sub O N (sh V N 0) L+1)
    (:pi, F, X, I, O)  N V L → (:pi, F, X, sub I N V L, sub O N (sh V F 0) L)
    (:fn, N, 0, I, O)  N V L → (:fn, N, 0, sub I N V L, sub O N (sh V N 0) L+1)
    (:fn, F, X, I, O)  N V L → (:fn, F, X, sub I N V L, sub O N (sh V F 0) L)
    (:app, F, A)       N V L → (:app, sub F N V L, sub A N V L)
```

Normalization

Replacing variable occurance in terms

```
type (:star, N)          D → (:star, N+1)
     (:var, N, I)        D → :true = proplists:defined N B, om:keyget N D I
     (:remote, N)        D → om:cache(typeND)
     (:pi, N, 0, I, O)   D → (:star ,h (star (type I D)), star (type O [(N,norm I)|D]))
     (:fn, N, 0, I, O)   D → let star (typeID), NI=norm I
                                in (:pi,N,0,NI,type(O,[(N,NI)|D]))
     (:app, F, A)        D → let T = type(F,D), (:pi,N,0,I,O) = T, :true = eq I (type AD)
                                 in norm (subst O N A)
```

Type Inferenece

Type Checker

```
type (:star, N)          D → (:star,N+1)
     (:var, N, I)        D → :true = proplists:is defined N B, om:keyget N D I
     (:remote, N)        D → om:cache(typeND)
     (:pi, N, 0, I, O)   D → (:star ,h(star(type I D)),star(type O [(N,norm I)|D]))
     (:fn, N, 0, I, O)   D → let star (typeID), NI = norm I
                                 in (:pi,N,0,NI,type(O,[(N,NI)|D]))
     (:app, F, A)        D → let T = type(F,D), (:pi,N,0,I,O) = T, :true = eq I (type AD)
                                 in norm (subst O N A)
```

Equality

Definitional, Built Into Type Checker

```
eq (:star ,N)         (:star ,N)          → true
     (:var,N,I)         (:var,(N,I))        → true
     (:remote ,N)     (:remote ,N)       → true
     (:pi,N1,0,I1,O1) (:pi,N2,0,I2,O2)  → let :true = eq I1 I2
                                              in eq O1 (subst (shift O2 N1 0) N2 (:var,N1,0) 0)
     (:fn,N1,0,I1,O1) (:fn,N2,0,I2,O2) → let :true = eq I1 I2
                                              in eq O1 (subst (shift O2 N1 0) N2 (:var,N1,0) 0)
     (:app,F1,A1)      (:app,F2,A2)       → let :true = eq F1 F2 in eq A1 A2
     (A,                  B)                   → (:error ,(:eq,A,B))
```

Language Usage

Erlang or UNIX shell

```
$ ./om help me
[{a,[expr],"to parse. Returns { , } or {error , }."} ,
{type,[term],"typechecks and returns type."},
{erase ,[ term ] ," to untyped term . Returns {   ,   }."} ,
{norm,[term],"normalize term. Returns term's normal form."},
{file ,[name],"load file as binary."},
{str ,[binary],"lexical tokenizer."},
{parse ,[ tokens ] ," parse given tokens into {   ,   } term ."} ,
{fst ,[{x,y}],"returns first element of a pair."},
{snd ,[{x,y}] ," returns second element of a pair ."} ,
{debug ,[ bool ] ," enable / disable debug output ."} , {mode,[name],"select metaverse folder."},
{modes ,[] ," list all metaverses ."}]
```

Language Usage

Erlang or UNIX shell

```
$ ./om print fst erase norm a "#List/Cons"

    \ Head
 -> \ Tail
 -> \ Cons
 -> \ Nil
 -> Cons Head (Tail Cons Nil)

ok
```

Applications: Sigma Types

Syntax and Model

data $O_2 := O_1$
| Sigma: name → O_2 → O_2 → O_2
| Pair: O_2 → O_2 → O_2
| Fst: O_2 → O_2
| Snd: O_2 → O_2.

Σ x: A, B x : U — formation rule
pair (x : A) (y : B x) — introduction
pr_1 s : A — elimination
pr_2 s : B x — elimination

```
data   Sigma (A: Type) (P: A -> Type) (x: A): Type =
        intro: P x -> Sigma A P
```

Sigma Types

Typing and Introduction Rules

```
-- Sigma/@ \ (A: *)
-> \ (P: A -> *)
-> \ (n: A)
-> \/ (Exists: *)
-> \/ (Intro: A -> P n -> Exists)
-> Exists
```

```
-- Sigma/Intro \ (A: *)
-> \ (P: A -> *)
-> \ (x: A)
-> \ (y: P x)
-> \ (Exists: *)
-> \ (Intro: \/ (x:A) -> P x -> Exists)
-> Intro x y
```

Sigma Types

Dependent Eliminators

```
-- Sigma/fst \ (A: *)
-> \ (B: A -> *)
-> \ (n: A)
->\ (S: #Sigma/@ABn)
-> S A ( \(x: A) -> \(y: B n) -> x)
```

```
-- Sigma/snd \ (A: *)
-> \ (B: A -> *)
-> \ (n: A)
->\ (S: #Sigma/@ABn)
-> S (B n) ( \( : A) -> \(y: B n) -> y )
```

Equ Type a la Martin-Löf

```
record Id (A: Type): Type :=
    Id: A → A → Type
    refl (a: A): Id a a
    Predicate: ∀ (x,y: A) → Id x y → Type
    Forall (C: Predicate): ∀ (x,y: A) → ∀ (p: Id x y) → C x y p
    Δ (C: Predicate): ∀ (x: A) → C x x (refl x)
    axiom-J (C: Predicate): Δ C → Forall C
    computation (C: Predicate) (t: Δ C): ∀ (x: A) → (J C t x x (refl x)) ==> (t x) )

record Subst (A: Type): Type :=
    intro (P (a: A): Type) (a1,a2: A) : Id a1 a2 → P a1 → P a2 :=
        Id.axiom-J (λ a1 a2 p12 → P a1 → P a2))
```

Equ Type in Om

```
-- Equ/Refl \ (A: *)
-> \ (x: A)
-> \ (Equ: A -> A -> *)
-> \ (Refl: \/ (z: A) -> Equ z z)
-> Refl x
```

```
-- Equ/@
\ (A: *)
-> \ (x: A)
-> \ (y: A)
-> \/ (Equ: A -> A -> *)
-> \/ (Refl: \/ (z: A) -> Equ z z) -> Equ x y
```

Effects Protocol

Type Spec

```
String: Type = List Nat

data IO: Type =
        getLine: (String -> IO) -> IO
        putLine: String -> IO
        pure: () -> IO
```

```
-- IO/@
      \ (a : *)
-> \/ (IO : *)
-> \/ (GetLine : (#IO/data -> IO) -> IO)
-> \/ (PutLine : #IO/data -> IO -> IO)
-> \/ (Pure : a -> IO) -> IO
```

Replication

Church Encoded Identity Folding

```
-- IO/replicateM
    \ (n: #Nat/@)
-> \ (io: #IO/@ #Unit/@)
-> #Nat/fold n (#IO/@ #Unit/@)
               (#IO[>>] io)
               (#IO/pure #Unit/@ #Unit /Make)

— Nat/fold
#id #Nat/@
```

Runtime Recursion Sample

Recursion Elimination

```
-- Recursion
((#IO/replicateM #Nat/Five)
  ((((#IO/[>>=] #IO/data) #Unit/@) #IO/getLine)
    #IO/putLine))
```

Infinity I/O

Corecursion Fixpoint

```
-- IOI/@
     \   (r : *)
->  \/ (x : *)
-> (\/ (s : *)
-> s
-> (s -> #IOI/F r s) -> x)
-> x
```

```
-- IOI/F
   \ (a : *)
-> \ (State : *)
-> \/ (IOF : *)
-> \/ (PutLine : #IOI/data -> State -> IOF)
-> \/ (GetLine : (#IOI/data -> State) -> IOF)
-> \/(Pure :a->IOF)
-> IOF
```

Infinity I/O Construction

Corecursion Introduction

```
-- IOI/MkIO
    \ (r : *)
-> \ (s : *)
-> \ (seed: s)
-> \ (step: s -> #IOI/F r s)
-> \ (x : *)
-> \ (k : forall (s : *) -> s -> (s -> #IOI/F r s) -> x)
-> k s seed step
```

Infinity I/O Process

Corecursion Elimination

```
-- Corecursion
( \ (r: *1) -> ( (((#IOI/MkIO r) (#Maybe/@ #IOI/data))
        (#Maybe/Nothing #IOI/data))
    ( \ (m: (#Maybe/@ #IOI/data))
      -> (((((#Maybe/maybe #IOI/data) m)
           ((#IOI/F r) (#Maybe/@ #IOI/data)))
           ( \ (str: #IOI/data)
             -> ((((#IOI/putLine r ) (#Maybe/@ #IOI/data)) str)
                  (#Maybe/Nothing #IOI/data))))
                (((#IOI/getLine r ) (#Maybe/@ #IOI / data ))
                  (#Maybe/Just #IOI/data))))))
```

Application 1. Logic

```
data Proper (A: Type) (R: A → A → Prop) (m: A): Prop := intro: R m m
data Inhabited (A: Type): Prop := intro: A → Inhabited A
data True: Prop := intro: () → True
data False: Prop := ()
data Eq (A : Type): A → A → Type := refl: ∀ (x: A) → Eq A x x
data Exists (A: Type): A → Type → Type := intro: ∀ (P: A → Type) →
                                                   ∀ (x: A) → P x → Exists A P

        record SKI := S : (p → q → r) → (p → q) → p → r
                      K :  p → (q → p)
                      I : p -> p
```

Application 2. Control

```
record pure (P: Type → Type) (A: Type): Type := return: P A
record functor (F: Type → Type) (A,B: Type): Type := map: (A → B) → F A → F B
record applicative (F: Type → Type) (A,B: Type): Type :=
          pure: pure F A
          functor: functor F A B
          ap: F (A → B) → F A → F B

record monad (F: Type → Type) (A,B: Type): Type :=
          pure: pure F A
          functor: functor F A B
          join: F (F A) → F B
```

Application 3. Setoid

```
record Setoid: Type :=
    Carrier: Type
    Equ: Carrier → Carrier → Prop
    Refl: (x: Carrier) → Equ x x
    Trans: (x₁,x₂,x₃: Carrier) → Equ x₁ x₂ → Equ x₂ x₃ → Equ x₁ x₃
    Sym: (x₁,x₂: Carrier) → Equ x₁ x₂ → Equ x₂ x₁
```

Application 4. Category

```
record Cat: U :=
    Ob: U
    Hom: (dom,cod: Ob) → Setoid
    Id:     (x: Ob) → Hom x x
    Comp:  (x,y,z: Ob) → Hom x y → Hom y z → Hom x z
    Dom1∘:  (x,y: Ob) (f: Hom x y) → (Hom.Equ x y (Comp x x y id f) f)
    Cod1∘:  (x,y: Ob) (f: Hom x y) → (Hom.Equ x y (Comp x y y f id) f)
    Subst1: (x,y,z: Ob) → Proper (Respect Equ (Respect Equ Equ)) (Comp x y z)
    Subst∘: (x,y,z: Ob) (f1,f2: Hom x y) (g1,g2: Hom y z)
          → (Hom.Equ x y f1 f2) → (Hom.Equ y z g1 g2)
          → (Hom.Equ x z (Comp x y z f1 g1) (Comp x y z f2 g2))
    Assoc∘: (x,y,z,w: Ob) (f: Hom x y) (g: Hom y z) (h: Hom z w)
          → (Hom.Equ x w (Comp x y w f (Comp y z w g h))
                  (Comp x z w (Comp x y z f g) h))
```

Infinity Language

```
data Infinity := O₂
          | Where:   MLTT → Decls → MLTT
          | Con:     Label → list MLTT → MLTT
          | Split:   Loc → list Branch → MLTT
          | Sum:     Binder → NamedSum → MLTT
          | HIT:     HomotopyCalculus → MLTT
          | PI:      PiCalculus → MLTT
          | EFF:     EffectCalculus → MLTT
          | STREAM: StreamCalculus → MLTT.
```

Ladder to computable HITs

1. Barendregt. The Lambda Calculus with Types http://5ht.co/pts.pdf
2. Martin-Löf. Intuitionistic Type Theory http://5ht.co/mltt.pdf
3. Awodey. Category Theory http://5ht.co/cat.pdf
4. Hermida, Jacobs. Fibrations with indeterminates http://5ht.co/completeness.pdf
5. Jacobs. Categorical Logic http://5ht.co/fibrations.pdf
6. Streicher. The groupoid interpretation of type theory http://5ht.co/groupoid.pdf
7. Voevodsky et all. Homotopy Type Theory http://5ht.co/hott.pdf
8. Huber, Coquand. Cubical Type Theory http://5ht.co/cubicaltt.pdf