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Constructive Proofs of Heterogeneous Equalities in Cubical Type Theory

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Abstract

Cubical Base Library

We demonstrate the usage of part of base library by showing how to build a constructive proof of heterogeneous equality, the simple and elegant formulation of the equality problem, that was impossible to achieve in pure Martin-Löf Type Theory (MLTT). The machinery used in this article unveils the internal aspect of path equalities and isomorphism, used e.g. for proving univalence axiom, that became possible only in CTT.

Structure of Presentation

From proving to Extraction, Linking and Running

1) Type Theory

— Pi Type
— Sigma Type
— Equ Type

2) Equality Problem

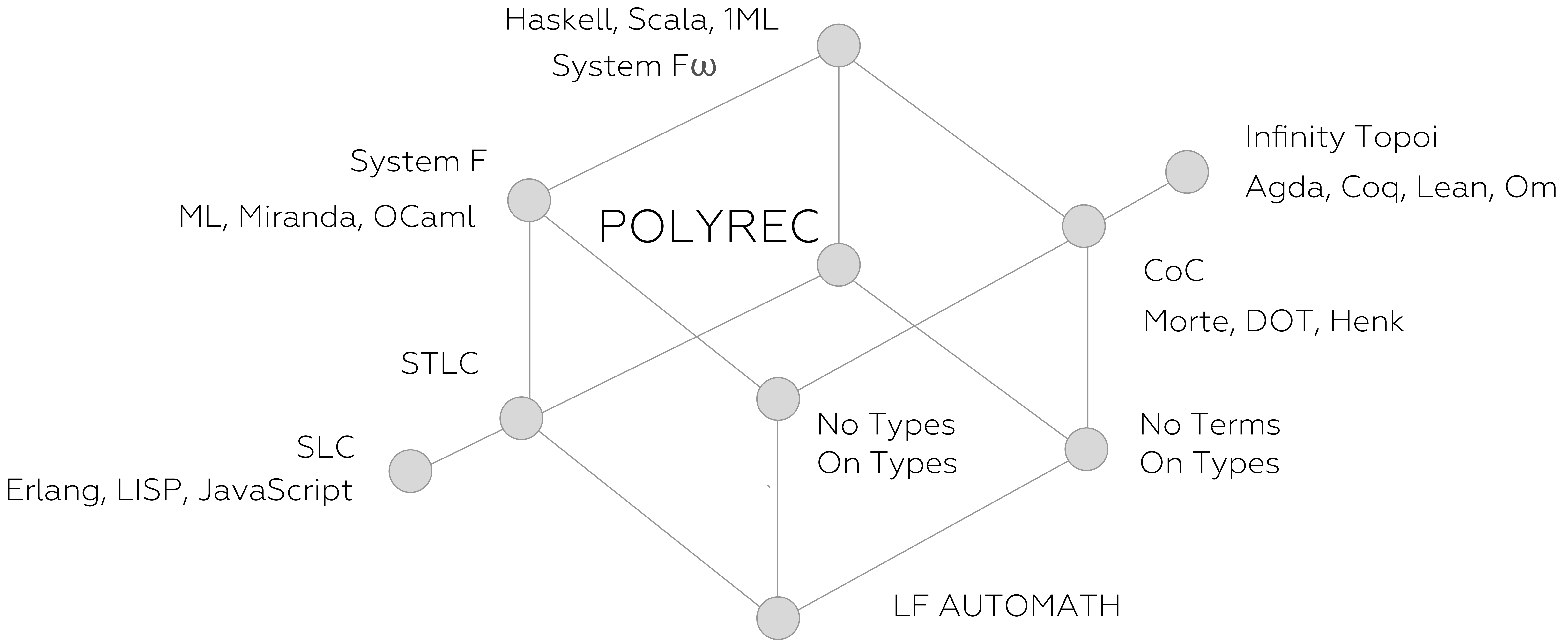
— Definitional Equality
— Propositional Equality
— Heterogeneous and Globular Equality

3) Computational Semantics — Cubical Type Theory 2017

— CCHM Fibrations
— Homotopy Path Type as Equality with de Morgan Algebra
— Composition, Gluening, Filling
— Cubical Base Library and further work

Programs as Proofs

in Extended Lambda Cube



Model Verification Process

From proving to Extraction, Linking and Running

1) Models

— Real Analysis
— Abstract Algebra
— Homotopy Theory

2) Core — Infinity Language

— Model Verification
— Normalization
— Bidirectional Checking

— Pure Type System (Om)
— Identity
— Induction
— Homotopy Interval [0,1]

3) Extraction

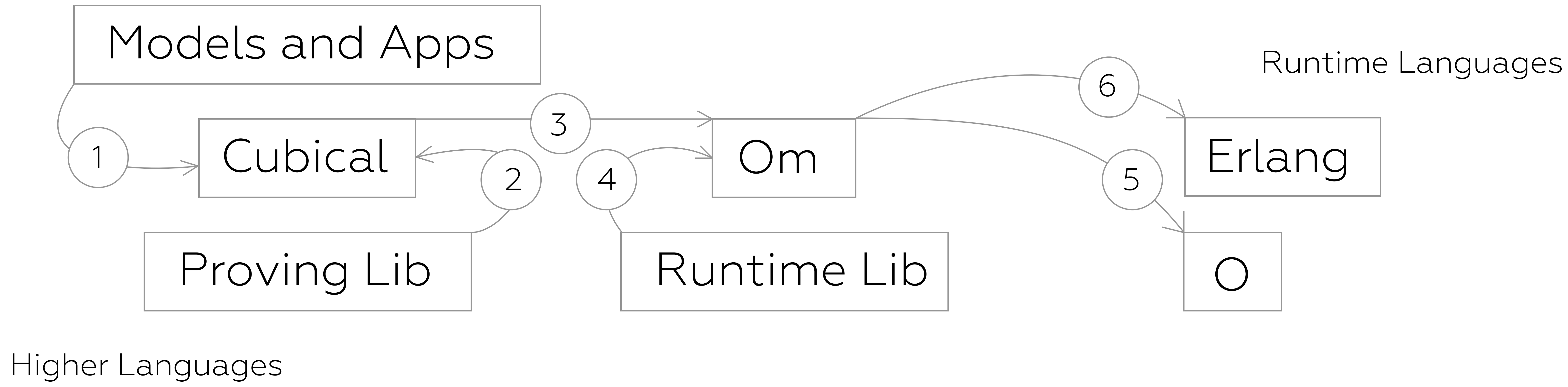
— LLVM
— Interpreters
— Detyping
— Optimization
— Linking

4) Runtimes

— O
— Erlang
— V8
— JVM

Prover Structure

Models, Languages, Libraries, Applications



[1,2] cover the presented work, [3,4,5,6] cover by other articles

MLTT 1984

Type Theory as new Foundations of Mathematics

$U_0 : U_1 : U_2 : U_3 : \dots \infty$ — universes
x : A — x is a point in space A
y = [x : A] — x and y are definitionaly equal objects of type A

	$\prod$	Σ	=
1. Formation Rules	x:A -> B(x)	x:A * B(x)	x:A = y:A
2. Introduction Rules	\ (x: A) -> B(x)	(x,B(x))	refl A x
3. Elimination Rules	f a = B(a)	pr1, pr2	J
4. Computational Rules	Equalities	Equalities	Equalities

Pi Type

Inductive Type, AST, Logical Framework

```
data pts
   = star (n: nat)
   | var (x: name) (l: nat)
   | pi (x: name) (l: nat) (d c: lang)
   | lambda (x: name) (l: nat) (d c: lang)
   | app (f a: lang)
```

```
<> ::= #option
T ::= #identifier
U ::= ∗ < #number >
O ::= U | T | ( O ) | O O | O -> O
     | \ (I: O) -> O | (I: O) -> O
```

```
Pi (A:U)(P:A->U):U=(x:A)->P(x)
lambda (A : U) (B: A -> U) (a : A) (b: B a): A -> B a = \(x: A) -> b
app (A : U) (B: A -> U) (a : A) (f: A -> B a): B a = f a
```

Sigma Type

Inductive Type, AST, Logical Framework

```
data exists
    = sigma (n: name) (a b: lang)
    | pair (a b: lang)
    | fst (p: lang)
    | snd (p: lang)
```

O := Σ (x: O), O | (O,O) | O.1 | O.2

```
Sigma (A : U) (B : A -> U) : U = (x : A) * B x
pair (A : U) (B: A -> U) (a : A) (b: B a): Sigma A B = (a,b)
pr1 (A: U) (B: A -> U) (x: Sigma A B): A = x.1
pr2 (A: U) (B: A -> U) (x: Sigma A B): B (pr1 A B x) = x.2
```

Sigma Type in Pi

Typing and Introduction Rules

```
-- Sigma/@
\ (A: *)
-> \ (P: A -> *)
-> \ (n: A)
-> \/ (Exists: *)
-> \/ (Intro: A -> P n -> Exists)
-> Exists
```

```
-- Sigma/Intro
\ (A: *)
-> \ (P: A -> *)
-> \ (x: A)
-> \ (y: P x)
-> \ (Exists: *)
-> \ (Intro: \/ (x:A) -> P x -> Exists)
-> Intro x y
```

Sigma Type in Pi

Eliminators

```
-- Sigma/fst
\ (A: *)
-> \ (B: A -> *)
-> \ (n: A)
->\ (S: #Sigma/@ A B n)
-> S A ( \(x: A) -> \(y: B n) -> x)
```

```
-- Sigma/snd
\ (A: *)
-> \ (B: A -> *)
-> \ (n: A)
->\ (S: #Sigma/@ A B n)
-> S (B n) ( \( : A) -> \(y: B n) -> y )
```

Equ Type in Any ?

```
-- Equ/@
\ (A: *)
-> \ (x: A)
-> \ (y: A)
-> \/ (Equ: A -> A -> *)
-> \/ (Refl: \/ (z: A) -> Equ z z) -> Equ x y

-- Equ/Refl \ (A: *)
-> \ (x: A)
-> \ (Equ: A -> A -> *)
-> \ (Refl: \/ (z: A) -> Equ z z)
-> Refl x
```

```
-- Equ/J
???
```

Definitional Equality

Definitional, built into Type Checker

```
eq (:star ,N)        (:star ,N)         → true
   (:var,N,I)        (:var,(N,I))       → true
   (:remote ,N)      (:remote ,N)       → true
   (:pi,N1,0,I1,O1)  (:pi,N2,0,I2,O2)   → let :true = eq I1 I2
                                            in eq O1 (subst (shift O2 N1 0) N2 (:var,N1,0) 0)
   (:fn,N1,0,I1,O1)  (:fn,N2,0,I2,O2)   → let :true = eq I1 I2
                                            in eq O1 (subst (shift O2 N1 0) N2 (:var,N1,0) 0)
   (:app,F1,A1)      (:app,F2,A2)       → let :true = eq F1 F2 in eq A1 A2
   (A,               B)                 → (:error ,(:eq,A,B))
```

Propositional Logic

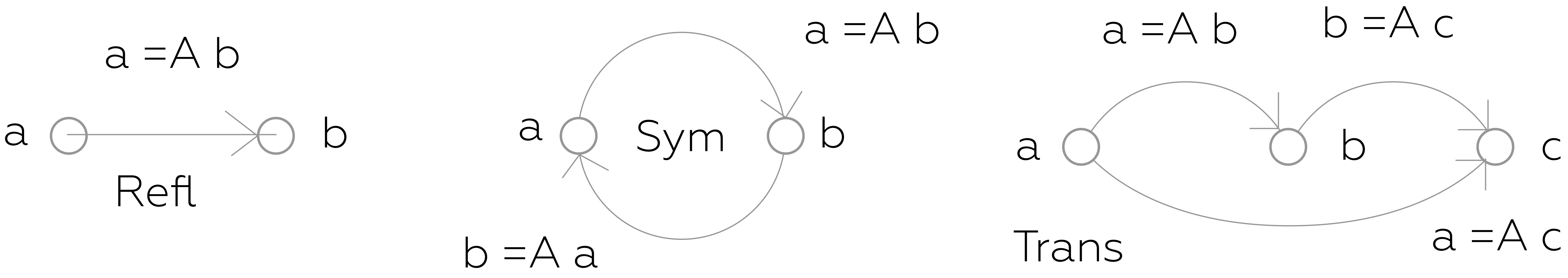
According to Brouwer–Heyting–Kolmogorov interpretation

∀,∏	∃,Σ	=	0	1	+
x:A -> B(x)	x:A * B(x)	x:A = y:A	data empty	data unit	data either (A B:: U)
\ (x: A) -> B(x)	(x,B(x))	refl A x		= tt	= inl (a: A) | inr (b: B)
f a = B(a)	pr1, pr2	J	elim0	elim1	elimEither

Bishop’s Constructive Analysis

Reflexivity, Transitivity, Symmetry

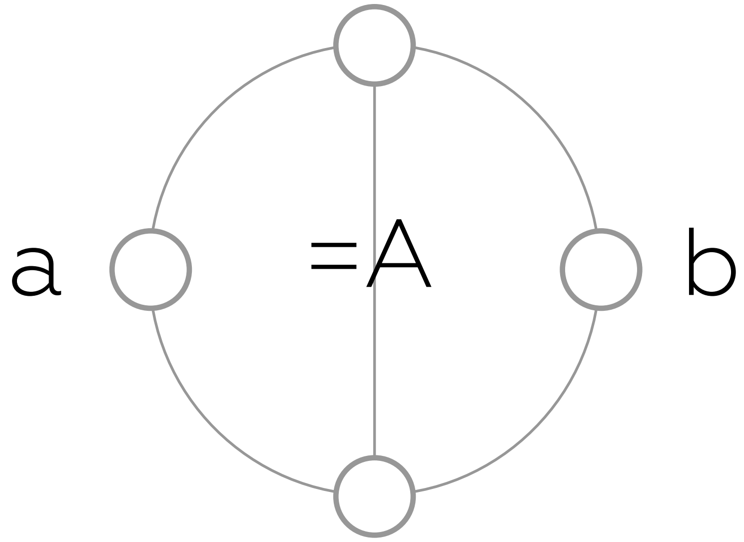
Setoid (A: U): U
  = (Carrier: A)
  * (Equ: (a b: A) -> Path A a b)
  * (Refl: (x: A) -> Equ x x)
  * (Trans: (x_1,x_2,x_3: A) -> Equ x_1 x_2 -> Equ x_2 x_3 -> Equ x_1 x_3)
  * (Sym: (x_1,x_2: A) -> Equ x_1 x_2 -> Equ x_2 x_1)



Globular Theory

Multidimentional Equality

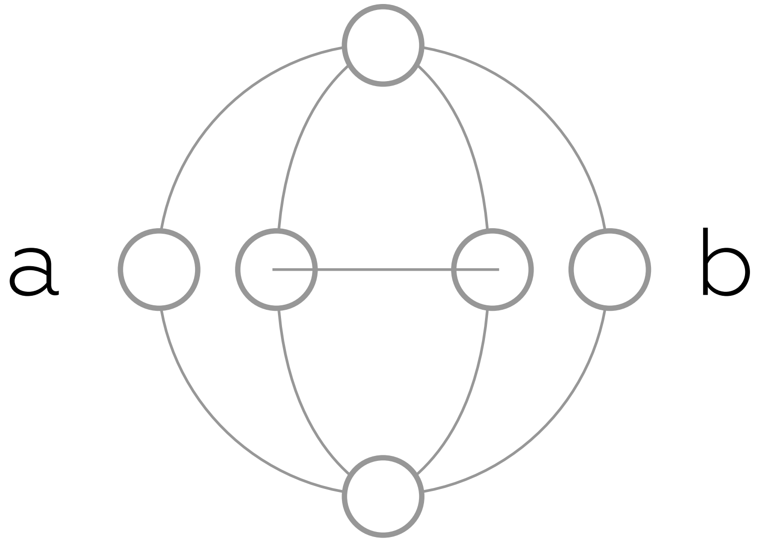
a =A b



((a =A b) =(=A) (a =A b))

a =A b

((a =A b) =(=A) (a =A b)) =(=A) ((a =A b) =(=A) (a =A b))



a =A b

Equ Type a la Martin-Löf

```
Path (A: U) (a b: A): U = axiom — PathP (<i>A) a b
HeteroEqu (A B: U) (a: A) (b: B) (P: Path U A B) : U = axiom — PathP P a b

Equ      (A: U) (x y: A): U = HeteroEqu A A x y (<i>A)
reflect  (A: U) (a: A): Equ A a a = <i> a
D        (A: U) : U = (x y: A) -> Equ A x y -> U
singl    (A: U) (a: A): U = (x : A) * Equ A a x
eta      (A: U) (a: A): singl A a = (a,reflect A a)
cong     (A B: U) (f: A->B) (a b: A) (p: Equ A a b): Equ B (f a) (f b)
subst    (A: U) (P: A->U) (a b: A) (p: Equ A a b) (e: P a): P b

J  (A: U) (a: A)  (C: (x : A) -> Path A a x -> U)
   (d: C a (refl A a)) (x: A) (p: Path A a x): C x p
```

Path Types as Cubes

Syntax and Model

```
data hts
    = path (a b: lang)
    | path_lam (n: name) (a b: lang)
    | path_app (f: name) (a b: lang)
    | comp_ (a b: lang)
    | fill_ (a b c: lang)
    | glue_ (a b c: lang)
    | glue_elem (a b: lang)
    | unglue_elem (a b: lang)
```

x : [PathP p a b, p = (i: I) -> A]

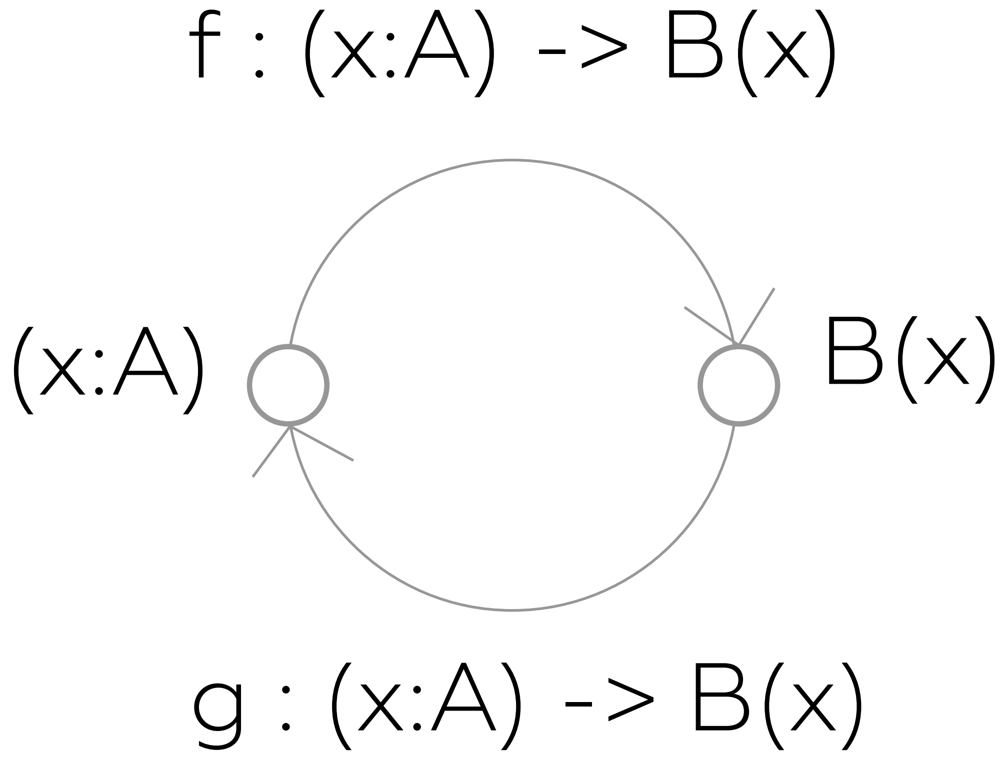
a : A  b : A

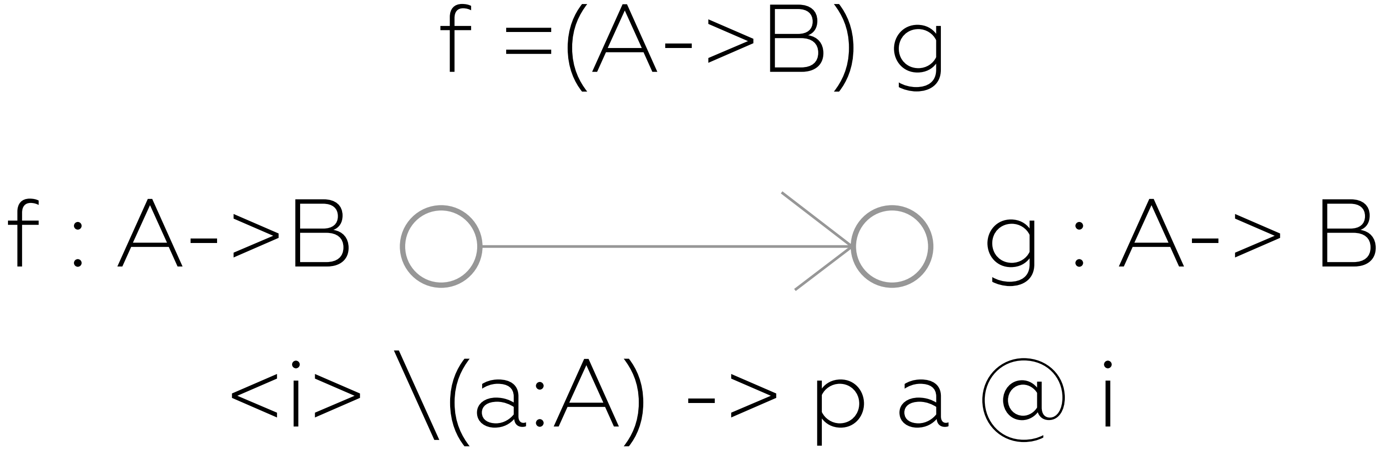
de Morgan: 1-i | i | i $\wedge$ j | i $\vee$ j

FunExt

Syntax and Model

```
piExt  (A: U) (B: A -> U) (f g: (x:A) -> B x)
       (p: (x:A) -> Path (B x) (f x) (g x))
     : Path ((y:A) -> B y) f g
     = <i> \(a: A) -> (p a) @ i
```



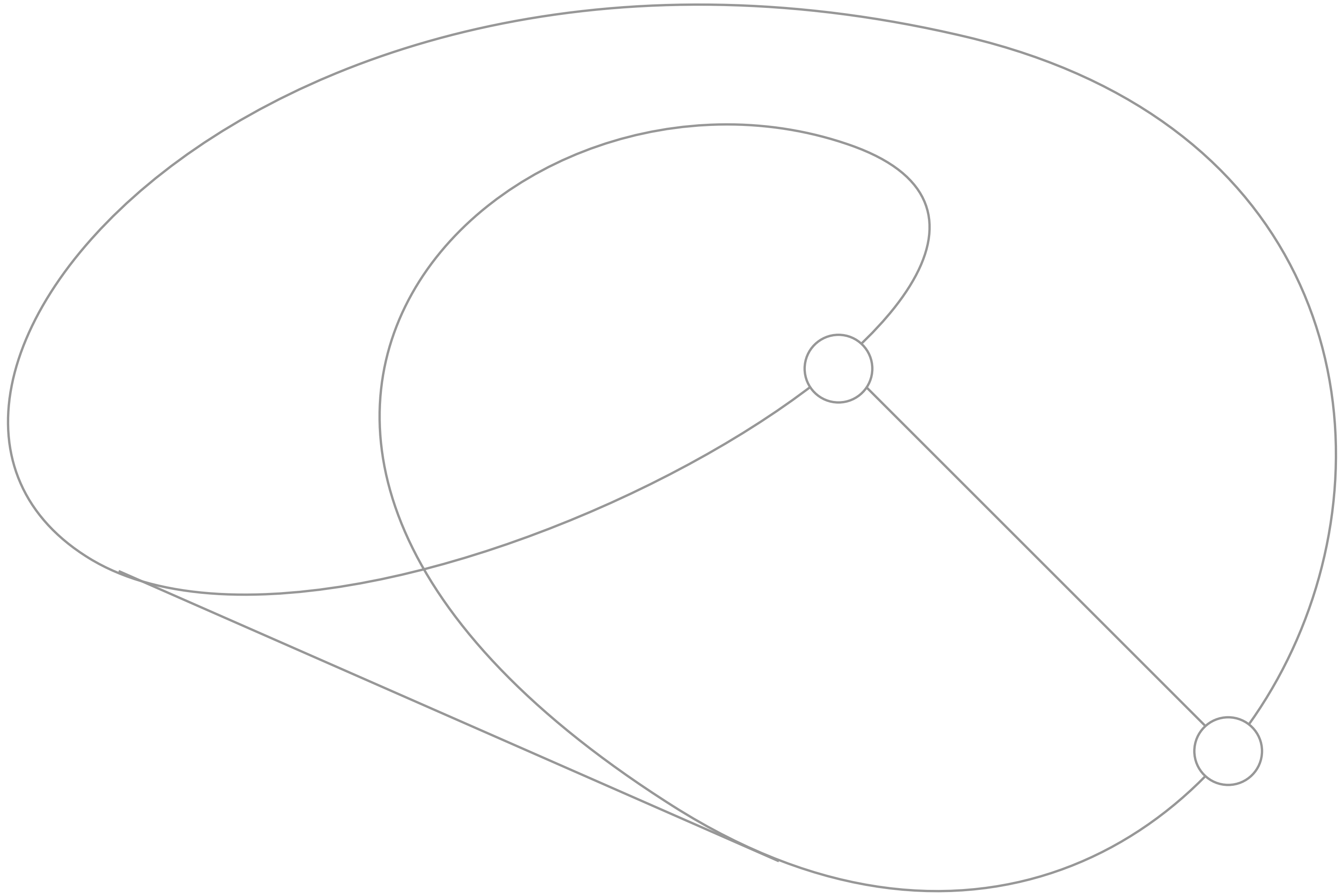


h-Types

```
data N = Z  | S (n: N)
n_grpd (A: U) (n: N): U = (a b: A) -> rec A a b n where
   rec (A: U) (a b: A) : (k: N) -> U
     = split { Z -> Path A a b ; S n -> n_grpd (Path A a b) n }

inf_grpd (A: U): U = (carrier: A) * (eq: (a b: A) -> Path A a b)
                                   * ((a b: A) -> inf_grpd (Path A a b))

isContr (A: U): U = (x: A) * ((y: A) -> Path A x y)
isProp (A: U): U = n_grpd A Z
isSet (A: U): U = n_grpd A (S Z)
isGroupoid  (A: U): U = n_grpd A (S (S Z))

PROP : U = (X:U) * isProp X
SET : U = (X:U) * isSet X
GROUPOID : U = (X:U) * isGroupoid X
INF_GROUPOID : U = (X:U) * isInfinityGroupoid X
```

Weak Equivalence

```
fiber (A B: U) (f: A -> B) (y: B): U = (x: A) * Path B y (f x)
isEquiv (A B: U) (f: A -> B): U = (y: B) -> isContr (fiber A B f y)
equiv (A B: U): U = (f: A -> B) * isEquiv A B f
```



Fiber Bundle: F -> E -> B
Moebius E = S^1 ‘twisted *’ [0,1]
Trivial: E = B * F
p : total -> B
F = fiber : B -> total
total = (y: B) * fiber(y)

```
Fiber=Pi (B: U) (F: B -> U) (y: B)
  : Path U (fiber (total B F) B (trivial B F) y) (F y)
```

Isomorphism

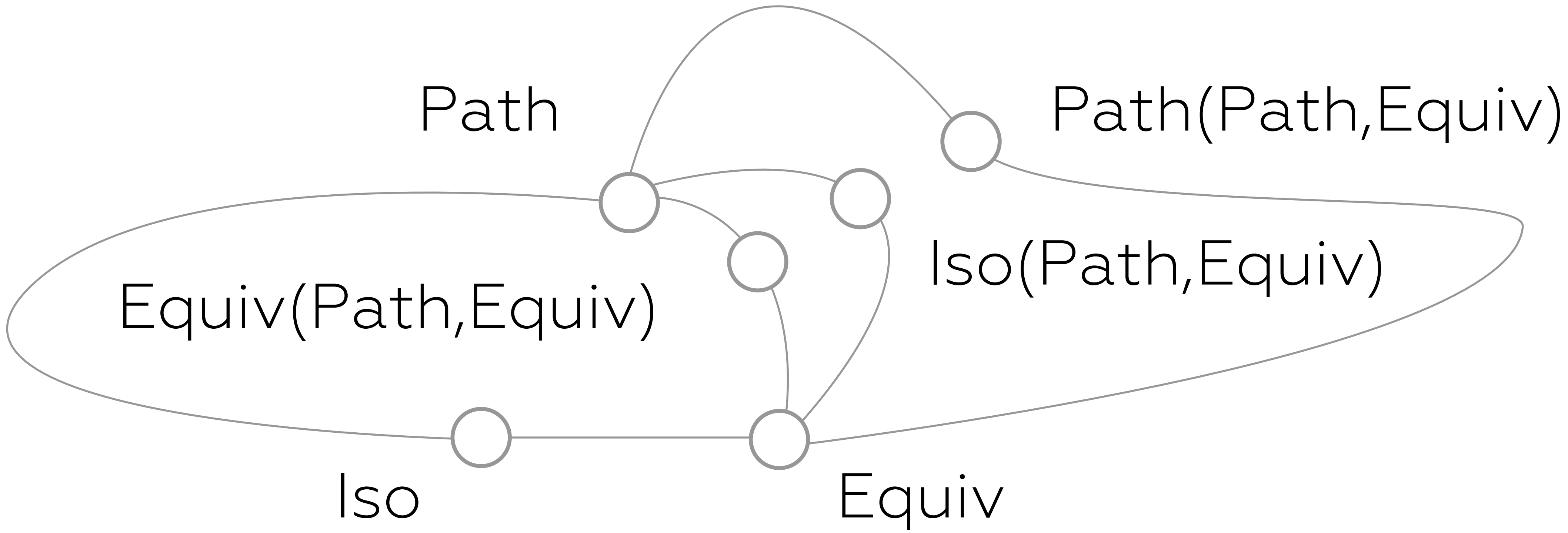
```
isIso (A B: U): U
  = (f: A -> B)
  * (g: B -> A)
  * (s: section A B f g)
  * (t: retract A B f g)
  * unit

iso: U
  = (A: U)
  * (B: U)
  * isIso A B
```

```
isoPath (A B: U) (f: A -> B) (g: B -> A)
        (s: section A B f g) (t: retract A B f g): Path U A B
  = <i> Glue B [ (i = 0) -> (A,f,isoToEquiv A B f g s t),
                 (i = 1) -> (B,idfun B,idIsEquiv B) ]

isoToPath (i: iso): Path U i.1 i.2.1
  = isoPath i.1 i.2.1 i.2.2.1 i.2.2.2.1 i.2.2.2.2.1 i.2.2.2.2.2.1
```

```
section (A B: U) (f: A -> B) (g: B -> A): U = (b: B) -> Path B (f (g b)) b
retract (A B: U) (f: A -> B) (g: B -> A): U = (a: A) -> Path A (g (f a)) a
```

Univalence Axiom

All Equalities Should Be Equal



Iso = Equiv = (=)

```
lemIso (A B: U) (f: A -> B) (g: B -> A) (s: section A B f g) (t: retract A B f g)
      (y: B) (x0 x1: A) (p0: Path B y (f x0)) (p1: Path B y (f x1))
    : Path (fiber A B f y) (x0,p0) (x1,p1) = <i> (p @ i,sq1 @ i) where
  rem0: Path A (g y) x0 = <i> comp (<k> A) (g (p0 @ i)) [ (i = 1) -> t x0, (i = 0) -> <k> g y ]
  rem1: Path A (g y) x1 = <i> comp (<k> A) (g (p1 @ i)) [ (i = 1) -> t x1, (i = 0) -> <k> g y ]
  p:    Path A x0 x1    = <i> comp (<k> A) (g y)        [ (i = 0) -> rem0, (i = 1) -> rem1 ]
  fill0: Square A (g y)(g (f x0)) (g y) x0 (<i> g (p0 @ i)) rem0 (<i> g y) (t x0) =
        <i j> comp (<k> A) (g (p0 @ i)) [ (i = 1) -> <k> t x0 @ j /\ k,
                                          (i = 0) -> <k> g y,
                                          (j = 0) -> <k> g (p0 @ i) ]
  fill1: Square A(g y)(g(f x1))(g y) x1 (<i>g (p1@i)) rem1 (<i> g y) (t x1) =
        <i j> comp (<k> A) (g (p1 @ i)) [ (i = 1) -> <k> t x1 @ j /\ k,
                                          (i = 0) -> <k> g y,
                                          (j = 0) -> <k> g (p1 @ i) ]
  fill2: Square A (g y) (g y) x0 x1 (<k> g y) p rem0 rem1 =
              <i j> comp (<k> A) (g y) [ (i = 0) -> <k> rem0 @ j /\ k,
                                          (i = 1) -> <k> rem1 @ j /\ k,
                                          (j = 0) -> <k> g y ]
  sq: Square A(g y)(g y)(g(f x0))(g(f x1))(<i>g y) (<i>g (f(p@i)))(<j>g(p0@j))(<j>g(p1@j)) =
      <i j> comp (<k> A) (fill2 @ i @ j) [ (i = 0) -> <k> fill0 @ j @ -k,
                                          (i = 1) -> <k> fill1 @ j @ -k,
                                          (j = 0) -> <k> g y,
                                          (j = 1) -> <k> t (p @ i) @ -k ]
  sq1: Square B y y (f x0) (f x1) (<k>y) (<i> f (p @ i)) p0 p1 =
      <i j> comp (<k> B) (f (sq @ i @j)) [ (i = 0) -> s (p0 @ j),
                                          (i = 1) -> s (p1 @ j),
                                          (j = 1) -> s (f (p @ i)),
                                          (j = 0) -> s y ]
```

Trivial Fiber = Pi

```
lem2 (B: U) (F: B -> U) (y: B) (x: F y)
   : Path (F y) (comp (<i>F (refl B y @ i)) x []) x
   = <j> comp (<i>F ((refl B y) @ j/\i)) x [(j=1) -> <k>x]

FiberPi (B: U) (F: B -> U) (y: B)
  : Path U (fiber (total B F) B (trivial B F) y) (F y)
  = isoPath T A f g s t where
    T: U = fiber (total B F) B (trivial B F) y
    A: U = F y
    f: T -> A = encode B F y
    g: A -> T = decode B F y
    s (x: A): Path A (f (g x)) x = lem2 B F y x
    t (x: T): Path T (g (f x)) x = lem3 B F y x
```

Fix Maybe = Nat

```
data maybe  (A: U) = nothing | just (a: A)
data fix (F:U->U) = Fix (point: F (fix F))

data nat = zero | succ (n: nat)
```

```
natToMaybe: nat -> fix maybe = split
  zero   -> Fix nothing
  succ n -> Fix (just (natToMaybe n))

maybeToNat: fix maybe -> nat = split
  Fix m  -> go m where go: maybe (fix maybe) -> nat = split
          nothing -> zero
          just f -> succ (maybeToNat f)

natMaybeIso: (a: nat) -> Path nat (maybeToNat (natToMaybe a)) a = split
  zero   -> <i> zero
  succ n -> <i> succ (natMaybeIso n @ i)

maybeNatIso : (a : fix maybe) -> Path (fix maybe) (natToMaybe (maybeToNat a)) a = split
  Fix m  -> go m where go: (a: maybe (fix maybe)) -> Path (fix maybe) (natToMaybe (maybeToNat (Fix a))) (Fix a) = split
          nothing -> <i> Fix nothing
          just f -> <i> Fix (just (maybeNatIso f @ i))

maybenat: Path U (fix maybe) nat = isoPath (fix maybe) nat maybeToNat natToMaybe natMaybeIso maybeNatIso
HeteroEqu (A B:U)(a:A)(b:B)(P:Path U A B):U = PathP P a b

> HeteroEqu (fix maybe) nat (Fix nothing) zero maybenat

> transNeg (fix maybe) (nat) maybenat (succ (succ zero))
EVAL: Fix (just (Fix (just (Fix nothing))))
> trans (fix maybe) (nat) maybenat (Fix nothing)
EVAL: zero
```

Thank You!

https://groupoid.space