MLTT — CT — Proof Theory

$x : A$ — x is a object of type A
$x = y : A$ — x and y are definitionaly equal objects of type A

Nat : U n — constant functor
List (A : U n) : U n — functor
List Nat : U n — constant functor

$$\text{List } A = 1 \longrightarrow \text{List } A \longrightarrow \text{List } A \longleftarrow A : U$$
$$\text{nil} : 1 \longrightarrow \text{List } A$$
$$\text{cons} : A \longrightarrow \text{List } A \longrightarrow \text{List } A$$

$$\text{Nat} = 1 \longrightarrow N \longrightarrow N : U$$
$$\text{zero} : 1 \longrightarrow N$$
$$\text{succ} : N \longrightarrow N$$

Pure Type System

Infinity Topoi

Grothendieck Universum

$U_0 : U_1 : U_2 : U_3 : \ldots \infty$

U_0 — propositions
U_1 — sets
U_2 — types
U_3 — sorts

$S\ (n : nat) = U\ n$
$A_1\ (n\ m : nat) = U\ n : U\ m$ where $m > n$ — cumulative
$R_1\ (m\ n : nat) = U\ m \longrightarrow U\ n : U\ (max\ m\ n)$ — predicative

$A_2\ (n : nat) = U\ n : U\ (n + 1)$ — non-cumulative
$R_2\ (m\ n : nat) = U\ m \longrightarrow U\ n : U\ n$ — impredicative

$Prop = Large\ \Omega_0 = U_0$

$\mathbf{\Sigma} = Large\ \Omega_1 = U_1$

Pi Type

Functional Completness

```
data O₁ := U : nat → O₁
         | Var: name → O₁
         | App: O₁ → O₁ → O₁
         | Lambda: name → O₁ → O₁ → O₁
         | Arrow: O₁ → O₁ → O₁
         | Pi: name → O₁ → O₁ → O₁.
```

∀ x: A, B x : U — formation rule
λ x: A, b : B x — introduction
app f a : B x — elimination
app (**λ** o:A, b) a = b [a/o] : B x — equation

```
record Pi (A: Type) :=
    intro: (A → Type) → Type
    fun: (B: A → Type) → ∀ (a: A) → B a → intro B
    app: (B: A → Type) → intro B → ∀ (a: A) → B a
    app-lam (B: A → Type) (f: ∀ (a: A) → B a): ∀ (a: A) → app (fun f) a ==> f a
    lam-app (B: A → Type) (p: intro B): fun (λ (a: A) → app p a) ==> p.
```

Sigma Types

Contextual Completness

```
data O₂ := O₁
         | Sigma: name → O₂ → O₂ → O₂
         | Pair: O₂ → O₂ → O₂
         | Fst: O₂ → O₂
         | Snd: O₂ → O₂.
```

$\Sigma\, x : A, B\, x : U$ — formation rule
$\mathrm{pair}\ (x : A)\ (y : B\, x)$ — introduction
$\mathrm{pr}_1\ s : A$ — elimination
$\mathrm{pr}_2\ s : B\, x$ — elimination

Identity Type a la Martin-Löf

```
record Id (A: Type): Type :=
       Id: A → A → Type
       refl (a: A): Id a a
       Predicate: ∀ (x,y: A) → Id x y → Type
       Forall (C: Predicate): ∀ (x,y: A) → ∀ (p: Id x y) → C x y p
       Δ (C: Predicate): ∀ (x: A) → C x x (refl x)
       axiom-J (C: Predicate): Δ C → Forall C
       computation (C: Predicate) (t: Δ C): ∀ (x: A) → (J C t x x (refl x)) ==> (t x) )

record Subst (A: Type): Type :=
       intro (P (a: A): Type) (a1,a2: A) : Id a1 a2 → P a1 → P a2 :=
              Id.axiom-J (λ a1 a2 p12 → P a1 → P a2))
```

K UIP Congruence

```
record UIP (A: Type): Type :=
       intro (A: Type) (a,b: A) (x,y: Id a b) : Id (Id A a b) x y)

record K (A: Type): Type :=
       PredicateK: ∀ (a: A) → Id a a → Type
       ForallK (C:PredicateK): ∀ (a: A) → ∀ (p: Id a a) → C a p
       ΔK (C: PredicateK) : ∀ (a: A) → C a (Id.refl a)
       axiom-K (C: Predicate): ΔK C → ForallK C)

 define Respect (A,B: Type) (C: A → Type) (D: B → Type) (R: A → B → Prop)
       (Ro: ∀ (x: A) (y: B) → C x → D y → Prop) : (∀ (x: A) → C x) → (∀ (x: B) → D x) → Prop
   := λ (f,g: Type → Type) → (∀ (x,y: Type) → R x y) → Ro x y (f x) (g y)
```

Category

```
record Cat: U :=
    Ob: U
    Hom: (dom,cod: Ob) → Setoid
    Id:     (x: Ob) → Hom x x
    Comp:  (x,y,z: Ob) → Hom x y → Hom y z → Hom x z
    Dom1∘:  (x,y: Ob) (f: Hom x y) → (Hom.Equ x y (Comp x x y id f) f)
    Cod1∘:  (x,y: Ob) (f: Hom x y) → (Hom.Equ x y (Comp x y y f id) f)
    Subst1: (x,y,z: Ob) → Proper (Respect Equ (Respect Equ Equ)) (Comp x y z)
    Subst∘: (x,y,z: Ob) (f1,f2: Hom x y) (g1,g2: Hom y z)
            → (Hom.Equ x y f1 f2) → (Hom.Equ y z g1 g2)
            → (Hom.Equ x z (Comp x y z f1 g1) (Comp x y z f2 g2))
    Assoc∘: (x,y,z,w: Ob) (f: Hom x y) (g: Hom y z) (h: Hom z w)
            → (Hom.Equ x w (Comp x y w f (Comp y z w g h))
                      (Comp x z w (Comp x y z f g) h))
```

Setoid

```
record Setoid: Type :=
    Carrier: Type
    Equ: Carrier → Carrier → Prop
    Refl: (x: Carrier) → Equ x x
    Trans: (x₁,x₂,x₃: Carrier) → Equ x₁ x₂ → Equ x₂ x₃ → Equ x₁ x₃
    Sym: (x₁,x₂: Carrier) → Equ x₁ x₂ → Equ x₂ x₁
```

Application 1. Logic

```
data Proper (A: Type) (R: A → A → Prop) (m: A): Prop := intro: R m m
data Inhabited (A: Type): Prop := intro: A → Inhabited A
data True: Prop := intro: () → True
data False: Prop := ()
data Eq (A : Type): A → A → Type := refl: ∀ (x: A) → Eq A x x
data Exists (A: Type): A → Type → Type := intro: ∀ (P: A → Type) →
                                                ∀ (x: A) → P x → Exists A P

record SKI := S : (p → q → r) → (p → q) → p → r
              K :  p → (q → p)
              I : p -> p
```

Application 2. Control

```
record pure (P: Type → Type) (A: Type): Type := return: P A
record functor (F: Type → Type) (A,B: Type): Type := map: (A → B) → F A → F B
record applicative (F: Type → Type) (A,B: Type): Type :=
        pure: pure F A
        functor: functor F A B
        ap: F (A → B) → F A → F B

record monad (F: Type → Type) (A,B: Type): Type :=
        pure: pure F A
        functor: functor F A B
        join: F (F A) → F B
```

Infinity Language

```
data Infinity := O₂
              | Where:   MLTT → Decls → MLTT
              | Con:     Label → list MLTT → MLTT
              | Split:   Loc → list Branch → MLTT
              | Sum:     Binder → NamedSum → MLTT
              | HIT:     HomotopyCalculus → MLTT
              | PI:      PiCalculus → MLTT
              | EFF:     EffectCalculus → MLTT
              | STREAM:  StreamCalculus → MLTT.
```

Homotopy Calculus

```
Require Import core.

Inductive HomotopyCalculus :=
 | Id             | Refl          | Inh              | Inc
 | Squash         | InhRec        | TransU           | TransInvU
 | TransURef      | Singl         | MapOnPath        | AppOnPath
 | HExt           | EquivEq       | EquivEqRef       | TransUEquivEq
 | IdP            | MapOnPathD    | IdS              | MapOnPathS
 | AppOnPathD     | Circle        | Base             | HLoop
 | CircleRec      | I             | I0               | I1
 | Line           | IntRec        | Undef: Loc  ->  HomotopyCalculus.
```

Equiv Squash
Id Inh Inc

```
Id : (A : U) (a b : A) → U
IdP : (A B : U) → Id U A B → A → B → U
refl : (A : U) (a : A) → Id A a a
inh : U → U
inc : (A : U) → A → inh A
squash : (A : U) → prop (inh A)
inhrec : (A : U) (B : U) (p : prop B) (f : A → B) (a : inh A) → B
contrSingl : (A : U) (a b : A) (p : Id A a b) → Id (singl A a) (a, refl A a) (b, p)
equivEq : (A B : U) (f : A → B) (s : (y : B) → fiber A B f y)
        (t : (y : B) → (v : fiber A B f y) →
        Id (fiber A B f y) (s y) v) → Id U A B
equivEqRef : (A : U) → (s : (y : A) → pathTo A y) →
          (t : (y : A) → (v : pathTo A y) →
          Id (pathTo A y) (s y) v) →
          Id (Id U A A) (refl U A) (equivEq A A (id A) s t)
```

Proposition Fibration
Path Singleton

```
id : (A : U) → A → A
id A a = a
sId : (A : U) (a : A) → pathTo A a
sId A a = (a, refl A a)
singl : (A : U) → A → U
singl A a = Sigma A (Id A a)
pathTo : (A:U) → A → U
pathTo A = fiber A A (id A)
```

```
prop : U → U
prop A = (a b : A) → Id A a b
Sigma : (A : U) (B : A → U) → U
Sigma A B = (x : A) * B x
fiber : (A B : U) (f : A → B) (y : B) → U
fiber A B f y = Sigma A ( λ x → Id B (f x) y)
```

```
IdS : (A : U) (F : A → U) (a0 a1 : A) (p : Id A a0 a1) → F a0 → F a1 → U
IdS A F a0 a1 p = IdP (F a0) (F a1) (mapOnPath A U F a0 a1 p)
```

Transport

```
transport : (A B : U) → Id U A B → A → B
transpInv : (A B : U) → Id U A B → B → A
transportRef : (A : U) (a : A) → Id A a (transport A A (refl U A) a)
transpEquivEq : (A B : U) → (f : A → B) (s : (y : B) → fiber A B f y) →
          (t : (y : B) → (v : fiber A B f y) → Id (fiber A B f y) (s y) v) →
          (a : A) → Id B (f a) (transport A B (equivEq A B f s t) a)
```

App

```
appOnPath : (A B : U) (f g : A → B) (a b : A) (q : Id (A → B) f g) (p : Id A a b) → Id B (f a) (g b)
appOnPathD :  (A : U) (F : A → U) (f g : (x : A) → F x) → Id ((x : A) → F x) f g →
         (a0 a1 : A) (p : Id A a0 a1) → IdS A F a0 a1 p  (f a0) (g a1)
```

Map

```
mapOnPath : (A B : U) (f : A → B) (a b : A) (p : Id A a b) → Id B (f a) (f b)
mapOnPathD : (A:U) (F: A → U) (f: (x:A) -> F x) (a0 a1: A) (p: Id A a0 a1) → IdS A F a0 a1 p  (f a0) (f a1)
mapOnPathS : (A:U) (F: A → U) (C: U) (f: (x:A) → F x → C)
        (a0 a1 : A) (p : Id A a0 a1) (b0 : F a0) (b1 : F a1)
        (q : IdS A F a0 a1 p b0 b1) → Id C (f a0 b0) (f a1 b1)
```

Extensionality

```
funHExt : (A : U) (B : A → U) (f g : (a : A) → B a) →
        ((x y : A) → (p : Id A x y) → IdS A B x y p (f x) (g y)) → Id ((y : A) → B y) f g
```

Interval

```
record I : U :=
    I0 : I
    I1 : I
    line : Id I I0 I1
    rec : (F : I → U) (s : F I0) (e : F I1) (l : IdS I F I0 I1 line s e) (x : I) → F x
```

Circle

```
record S¹ : U :=
    base : S¹
    loop : Id S¹ base base
    rec : (F : S¹ → U) (b : F base) (l : IdS S¹ F base base loop b b) (x : S¹) → F x
```

Ladder to computable HITs

1. Barendregt. The Lambda Calculus with Types
2. Martin-Löf. Intuitionistic Type Theory
3. Awodey. Category Theory
4. Hermida, Jacobs. Fibrations with indeterminates
5. Jacobs. Categorical Logic
6. Voevodsky et all. Homotopy Type Theory
7. Huber, Coquand. Cubical Type Theory