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DISCRETE QUANTUM GRAVITY: THE REGGE CALCULUS APPROACH [1]

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Abstract

After a brief introduction to Regge calculus, some examples of its application in quantum gravity are described. In particular, the earliest such application, by Ponzano and Regge, is discussed in some detail and it is shown how this leads naturally to current work on invariants of three-manifolds.

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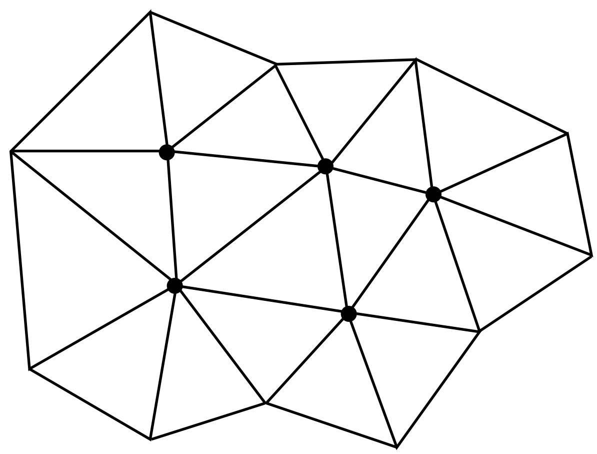
1 REGGE CALCULUS

The essential idea of the discrete approach to general relativity known as Regge calculus [Regge 1961] is the use of piecewise linear spaces, in which the curvature is restricted to subspaces of codimension two, the "hinges". For example, a two-dimensional curved surface can be approximated arbitrarily closely by a network of flat triangles and the curvature resides at the vertices. The deficit angle at a vertex, defined by

$$\varepsilon = 2\pi - \sum_{\substack{\text{triangles}\\ \text{meeting}\\ \text{at vertex}}} \text{vertex angles} \ , \tag{1}$$

gives a measure of the curvature there. In three dimensions, the lattice consists of flat tetrahedra, with curvature restricted to the one-simplices or edges. Similarly in four dimensions, the curvature is distributed over the triangles.



Regge showed that the discrete form of the Einstein action

$$I = \frac{1}{2} \int R\sqrt{g}\ d^n x \tag{2}$$

is given by

$$I_R = \sum_{\text{hinges i}} |\sigma^i| \varepsilon_i \ , \tag{3}$$

where $|\sigma^i|$ is the volume of a hinge σ^i and ε_i is the deficit angle there. The edge lengths, which completely specify a simplicial space, are the variables of the theory, and so the requirements for such a space to be a classical solution of Einstein's equations may be obtained by varying the Regge action with respect to these lengths. Regge [1961] showed that the term involving the variation of the deficit angle vanishes, and so the Regge calculus analogue of Einstein's equations is

$$\sum_i \frac{\partial |\sigma^i|}{\partial l_j} \varepsilon_i = 0 \ . \tag{4}$$

Simplicial spaces are of great interest in their own right, especially in the light of theories that the underlying structure of space–time may be discrete. If, on the other hand, they are regarded as approximations to spaces where the curvature is distributed continuously, it is important to study their convergence properties in the continuum limit. Cheeger, Müller and Schrader [1984] have shown that the Regge action converges to the continuum action in the sense of measures, provided that certain conditions on the fatness of the simplices are satisfied. Barrett, in a series of papers (see, e.g., Barrett 1988 and references therein) has studied the relationship between solutions of the Regge equations and continuum solutions of Einstein's equation, and has formulated a convergence criterion in the linearized case.

There are many mathematical results on piecewise linear spaces which have potential use in a theory of quantum gravity based on Regge calculus. Let me mention some work with Martin Roček, in which we have obtained some new formulae for the Euler characteristic of a simplicial manifold of dimension $2n$, firstly in terms of the numbers of even-dimensional subsimplices and secondly in terms of even-dimensional deficit angles [Roček and Williams 1991]. The starting point is the well-known formula for the Euler characteristic

$$\chi^{(2n)} = \sum_{i=0}^{2n} (-1)^i N_i^{(2n)} \tag{5}$$

where $N_i^{(2n)}$ is the number of subsimplices of dimension i. For a given simplicial complex, the numbers $N_i^{(2n)}$ are not independent but satisfy a set of linear equations called the Dehn–Sommerville equations. These can be used to eliminate the $N_i^{(2n)}$'s for odd i from (5) to give

$$\chi^{(2n)} = \sum_{i=0}^{n} a_{i+1} N_{2i}^{(2n)} \ , \tag{6}$$

with

$$a_i = \frac{2}{i}(2^{2i}-1)B_{2i} \ , \tag{7}$$

where $\{B_{2i}\}$ are the even Bernoulli numbers. Now a set of equations exactly equivalent to the Dehn–Sommerville ones, is also satisfied by the angle sums $S_i^{(2n)}$ defined by

$$S_i^{(2n)} = \sum_{\sigma^i \subset \sigma^{2n}} (i, 2n) \ , \tag{8}$$

where (i, j) is the normalized interior dihedral angle at the subsimplex σ^j in the simplex σ^i, and we have defined

$$S_{-1}^{(2n)} \equiv 0 \ , \quad N_{-1}^{(2n)} \equiv \chi \ . \tag{9}$$

Therefore the result for the angle sums analogous to (6) is

$$0 = \sum_{i=0}^{n} \ a_{i+1} \ S_{2i}^{(2n)} \ . \tag{10}$$

We can then use (7) and (10) to show that

$$\chi^{(2n)} = \sum_{i=0}^{n-1} a_{i+1} \sum_{\sigma^{2i}} \varepsilon_{2_i}^{(2n)} \ , \tag{11}$$

where the normalized k-dimensional deficit angle at a simplex σ^j is defined by

$$\varepsilon_j^{(k)} = 1 - \sum_{\sigma^k \supset \sigma^j} (j, k) \ . \tag{12}$$

For example, in four dimensions, (12) gives

$$\chi = \sum_{\sigma^0} \varepsilon_0^{(4)} - \frac{1}{2} \sum_{\sigma^2} \varepsilon_2^{(4)} ,$$

where $\varepsilon_0^{(4)}$ is the "solid" deficit angle at a vertex σ^0, and $\varepsilon_2^{(4)}$ is the deficit angle at a triangle (the usual deficit angle appearing in four-dimensional Regge calculus). The appearance of the Bernoulli numbers in these formulae is intriguing; characteristically they appear in expressions that are in some sense approximations (for example in finite difference approximations to derivatives) and their presence here may reflect the idea that a piecewise linear space is in a sense an approximation to a space with continuously distributed curvature.

2 QUANTUM APPLICATIONS

Rather than producing an exhaustive list of work on quantum gravity using Regge calculus (which can be found in Williams and Tuckey, 1991), I shall just mention a few typical applications in different categories.

2.1 Perturbative Calculations

Analytic calculations in four dimensions using Regge calculus are very complicated, so most work of this type has involved the weak field limit. Roček and Williams [1984] calculated the graviton propagator by considering small fluctuations about a flat-space background and showed that it agreed with the continuum propagator in the long wavelength limit.

2.2 Numerical Simulations

Recent progress in the understanding of functional integral methods for simplicial quantum gravity, and the need for a non-perturbative approach, have led to some large-scale numerical studies, using ideas developed in lattice gauge theories. The basic idea is to start with, say, a flat lattice and then to allow the system to evolve, using a Monte Carlo algorithm, to an equilibrium configuration, about which the edge lengths make quantum fluctuations. The expectation values of certain operators (e.g. volume, curvature) can then be measured. The inclusion of a curvature-squared term to ensure positivity of the action and hence convergence of the functional integral, was considered by Hamber and Williams [1984, 1986a, 1986b] in their calculations in two and four dimensions. For a lattice with the topology of a four-torus, it was found that at strong coupling the system developed an average negative curvature. Hamber [1990, 1991] has carried out further investigation of the phase diagram and critical exponents for pure gravity in four dimensions, using lattices of size up to 16^4. Berg [1986] has also performed numerical simulations of gravity in four dimensions, keeping the total volume constant.

The two-dimensional Regge calculus numerical simulations I have just mentioned are not the same in method as the large number of recent two-dimensional simulations using equilateral triangles and summing over triangulations. These calculations use the Regge action but there is no integration over edge lengths. However it seems that the results from the two approaches are compatible (see, e.g., Gross and Hamber 1991). Work is currently in progress on three-dimensional simulations using both the standard Regge calculus approach with fixed triangulation and integration over edge lengths, and the dynamical triangulation method with equilateral tetrahedra, and it will be very interesting to see how the results compare.

2.3 Quantum Cosmology

In quantum cosmology, it has been proposed that the wave function of the Universe, $\psi(\mathcal{G}^{(3)})$, for a given three-geometry $\mathcal{G}^{(3)}$, can be calculated from the functional integral,

$$\psi(\mathcal{G}^{(3)}) = \int D[\mathcal{G}^{(4)}] \exp[-I(\mathcal{G}^{(4)}] \tag{13}$$

where the integral is over all four-geometries $\mathcal{G}^{(4)}$ having the required three-geometry as boundary, $I[\mathcal{G}^{(4)}]$ is the Einstein action with cosmological constant and $D[\mathcal{G}^{(4)}]$ is some measure on the space of four-geometries. The integral can be approximated by a sum over discrete geometries described by a finite number of parameters, and Regge calculus provides the natural way of specifying such geometries. In a series of papers on simplicial minisuperspace, Hartle [1985a, 1986, 1989] has described how this idea may be implemented and has performed a sample calculation using a five-simplex model. Hartle [1985b] has also discussed the need to sum over topologies as well as over geometries, and this means that triangulations are needed for many different four-manifolds. Hartle, Sorkin and Williams [1991] have studied triangulations on product manifolds.

2.4 The work of Ponzano and Regge

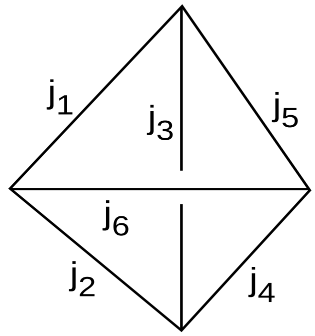
Long before any of the applications described so far, Ponzano and Regge [1968] had pointed out an extremely interesting connection between a sum involving the asymptotic values of $6j$-symbols associated with a triangulated three-manifold, and the Feynman path integral for three-dimensional simplicial quantum gravity with the Regge action. This was discussed further by Hasslacher and Perry [1981] and Lewis [1983]. Let us look at the Ponzano and Regge result in detail to see how it anticipates current work on invariants of three-manifolds, which will be described in the next section.

It is well known that $6j$-symbols, which arise in the recoupling of angular momenta, have tetrahedral symmetry: to the tetrahedron with edge lengths $j_1, j_2, ..., j_6$, we associate the $6j$-symbol

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

Ponzano and Regge showed that in the semi-classical limit, when the j's all become large (the edge lengths of the tetrahedron are really $j_i\hbar$, so we are taking the $\hbar \to 0$ limit while keeping the edge lengths finite), the $6j$-symbol has the asymptotic value

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \simeq \frac{1}{(12\pi V)^{1/2}} \cos\left(\sum_{\substack{\text{edges} \\ \text{i}}} j_i\theta_i + \frac{\pi}{4} \right) \tag{14}$$



where V is the volume of the associated tetrahedron and θ_i is the angle between the outer normals to the faces meeting at edge i (thus $\pi - \theta_i$ is the interior dihedral angle at edge i).

Now consider a triangulated closed two-surface D and dissect its interior into p tetrahedra. Label the external edges (i.e. those on D) by $l_j, j = 1, 2, ..., s$, and the internal edges by $x_i, i = 1, 2, ..., t$. Define

$$A(x_1, ..., x_t) \equiv \prod_{k=1}^{p} [T_k](-1)^x \prod_{i=1}^{t} (2x_i + 1) \ . \qquad (15)$$

where $[T_k]$ is the $6j$-symbol associated with tetrahedron k, and x gives a phase factor. We now sum over internal edges to obtain

$$S \equiv \sum_{x_1, ..., x_t} A(x_1, ..., x_t) \ . \qquad (16)$$

The triangulated two-surface D may be interpreted as a diagrammatic representation of a $3nj$-symbol with value denoted by $[D]$. In evaluating S to find its relation to $[D]$, it is of interest to consider two cases. When there are no internal vertices, the sum is finite, with $S = [D]$. In the sum over internal edge lengths, the large values dominate, so we may replace the sum over x by an integral and also use the asymptotic formula (**??**) for the values of the $6j$-symbols in (**??**). The most important contributions to the integral are from the points of stationary phase, corresponding to the conditions

$$\sum_{\substack{\text{tetrahedra} \\ \text{k meeting} \\ \text{on edge i}}} (\pi - \theta_i^k) = 2\pi \ . \qquad (17)$$

This just states that the sum of the dihedral angles in the tetrahedra meeting on each internal edge x_i is 2π; in other words, the space is flat, as it should be in three dimensions in the absence of matter, according to Einstein's equations.

Since it is always possible to triangulate the interior of D without introducing internal vertices, it is not strictly necessary to consider the second case, that with internal vertices, when evaluating S. In fact, in this case, the sum is infinite but is is possible to introduce a cut-off and obtain a finite value for S. The reason for considering this case is that when the number of vertices and edges on D and inside D is very large, the triangulation looks like a better approximation to the continuum and the analogy with a Feynman path integral is more plausible. When the summation over x is replaced by an integral and the asymptotic value for the $6j$-symbols is used, S takes the form

$$S \simeq \frac{1}{(12\pi)^{1/2}} \int \prod_i dx_i (2x_i+1)(-1)^x \prod_{\text{tetrahedra k}} \frac{1}{V_k^{1/2}} \cos\left(\sum_{\substack{\text{edges } l \\ \text{in tetrahedron k}}} j_l \theta_l^k + \frac{\pi}{4} \right) \tag{18}$$

By writing the cosine as a sum of exponentials and interchanging the orders of summation over tetrahedra and summation over edges within a given tetrahedron, we can show that S contains a term of the form

$$\int \prod_i dx_i (2x_i+1) \exp i \sum_{\text{edges } l} j_l \left[2\pi - \sum_{\substack{\text{tetrahedra} \\ \text{k containing} \\ \text{edge } l}} (\pi - \theta_l^k) \right] = \int \prod_i dx_i (2x_i+1) \exp\left(i \sum_l j_l \varepsilon_l \right) . \tag{19}$$

This looks like a Feynman sum over histories with the Regge calculus action for three-dimensional gravity:

$$\int \prod_i d\mu(x_i) \exp(iI_R) \tag{20}$$

with

$$I_R = \sum_l j_l \varepsilon_l \ ,$$

where $d\mu$ is some measure on the space of edge lengths. Ponzano and Regge speculated that other terms in S correspond to different orientations of the tetrahedra.

3 INVARIANTS OF THREE-MANIFOLDS

At present there is a great deal of work being done on defining new invariants of three-manifolds. These invariants can be calculated in various ways and we are interested here in a method which involves triangulating the manifold, as formulated by Turaev and Viro [1990]. I will now describe their construction in a way which emphasizes its relationship to the work of Ponzano and Regge.

Fix a positive integer r and consider the set $I_r = (0, 1/2, 1, ..., (r-2)/2)$. The edges of a triangulated three-manifold M are "coloured" by assigning a value from I_r to each edge of M, subject to certain admissibility conditions, that the colourings of any three edges which form a triangle should satisfy the triangle inequalities and that their sum should be an integer less than or equal to $r-2$. For simplicity we take M to be closed. (Strictly speaking, we should really consider a manifold with boundary for the comparison with the Ponzano–Regge result, but the formula for a closed manifold is simpler.) Suppose that M has v vertices, t edges and p tetrahedra. We define

$$|M|_\phi = \omega^{-2v} \prod_{k=1}^{p} |T_k^\phi| \prod_{i=1}^{t} \omega_{\phi(i)}^2 \ , \tag{21}$$

where ϕ is an admissible colouring of the edges, with $\phi(i)$ being the colouring of edge i. The other symbols appearing in (??) are defined as follows:

$$\omega_j \equiv (-1)^j [2j+1]^{1/2} \ , \tag{22}$$

where

$$[n] \equiv \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \tag{23}$$

and

$$q \equiv e^{2\pi i/r} \ ; \tag{24}$$

$$\omega^2 \equiv \sum_{j \in I_r} \omega_j^4 \ . \tag{25}$$

$|T_k^\phi|$ is the quantum $6j$-symbol corresponding to tetrahedron k with colouring ϕ. It is related to the q-deformed Racah–Wigner $6j$-symbol

$$\left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\}_q$$

by

$$\begin{vmatrix} i & j & k \\ l & m & m \end{vmatrix} = (-1)^{i+j+k+l+m+n} \left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\}_q \ . \tag{26}$$

The quantum Racah–Wigner $6j$-symbol is defined by the same formula as the classical Racah–Wigner $6j$-symbol used by Ponzano and Regge, with the vital difference that each factor n is replaced by its q-deformed analogue $[n]$.

Summing (**??**) over admissible colourings, adm(M), we obtain

$$|M| = \sum_{\phi \in \mathrm{adm}(M)} |M|_\phi \ . \tag{27}$$

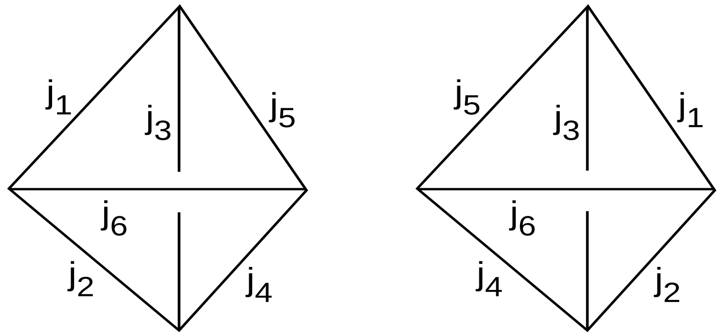
We need only to compare (**??**) and (**??**) with (**??**) and (**??**) to see the close similarity between the two formulations. In fact the Ponzano–Regge expression is simply the classical limit of the corresponding Turaev–Viro expression, obtained in the $q \to 1$ or $r \to \infty$ limit. Turaev and Viro proved that their expression is a manifold invariant, which means that it is independent of the choice of triangulation, by showing that it is invariant under the Alexander moves [Alexander 1930] which can be used to relate any two combinatorially equivalent triangulations. (It is interesting that the Alexander moves have also been rediscovered in the context of numerical simulations of three-dimensional gravity using dynamical triangulations, as the generalizations of the flip moves in two-dimensional simulations.) Ponzano and Regge clearly understood that their expression was independent of triangulation, although they did not prove it explicitly.

The obvious question now is whether there is a connection between the Turaev–Viro invariant and simplicial quantum gravity, analogous to the Ponzano–Regge result in the semi-classical limit. Witten has conjectured that the Turaev–Viro invariant is equivalent to the Feynman path integral with the Chern–Simons action for $SU(2)_k \otimes SU(2)_k$:

$$\mathcal{L} = \frac{k}{4\pi} \int_m \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) - \frac{k}{4\pi} \int_M \mathrm{Tr} \left(B \wedge dB + \frac{2}{3} B \wedge B \wedge B \right) \ , \tag{28}$$

where $A = A_i^a T_a dx^i$, $B = B_i^a T_a dx^i$, with T_a $(a = 1, 2, 3)$ being a basis of an $SU(2)$ Lie algebra, satisfying $[T_a, T_b] = \epsilon_{abc} T^b$. With the change of variables

$$A_i^a = \omega_i^a + \frac{1}{k} e_i^a \text{ and } B_i^a = \omega_i^a - \frac{1}{k} e_i^a \ , \tag{29}$$



where e_i^a is the dreibein and ω_i^a is defined in terms of the connection 2-form ω_i^{ab} by $\omega_i^a \equiv \frac{1}{2}\epsilon^{abc}\omega_{ibc}$, the Lagrangian above becomes the Einstein–Hilbert one with cosmological constant proportional to $1/k^2$. The precise relation conjectured is then

$$|M| = Z(M; SU(2)_k \otimes SU(2)_{-k}) = |Z(M; SU(2)_k)|^2 \ , \tag{30}$$

where $|M|$ is the Turaev–Viro invariant for a manifold M and $Z(M;G)$ is the partition function for M with gauge group G. The conjecture (**??**) has now been proved for closed three-manifolds by Walker [1991] and by Turaev [1990]. (More precisely, Turaev has proved that the Turaev–Viro invariant for a closed three-manifold is equal to the modulus-squared of the corresponding Reshetikhin–Turaev [1991] invariant, which is essentially the mathematically rigorous formulation of the Feynman path integral.)

Before commenting further on the relationship summarized in (**??**), let me illustrate it with a simple calculation (which should also demonstrate that it is not always so hard to work with $6j$-symbols!). Let us calculate the Turaev–Viro invariant for S^3 and compare it with the corresponding partition function. We use the simplest possible triangulation of S^3 which consists of a tetrahedron and its mirror image, with faces identified pairwise to form a closed manifold. We assign colourings $j_1, j_2, ..., j_6$ to the edges where each j_i is an integer or half-integer from the set $(0, 1/2, 1, ..., (r-2)/2)$, with r a fixed integer ≥ 3. According to (**??**) and (**??**), the Turaev–Viro invariant for this triangulation with four vertices, six edges and two tetrahedra is

$$|M(S^3)| = \omega^{-8} \sum_{j_i,\ i=1,\dots,6} \omega_{j_1}^2 \dots \omega_{j_6}^2 \begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix}^2 \ , \tag{31}$$

where the sum is over all admissible values of the j_i's. We use the orthogonality relationship satisfied by the $6j$-symbols

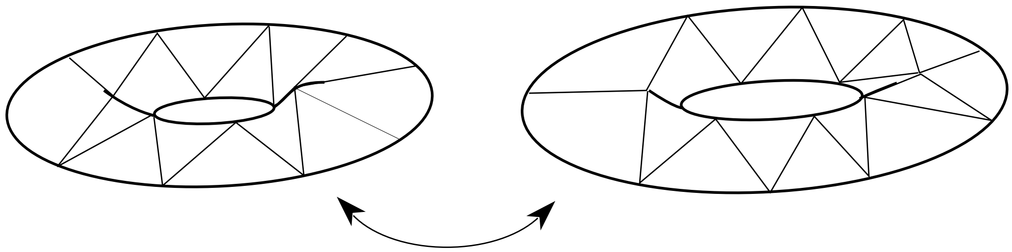
$$\sum_{j_3} \omega_{j_3}^2 \omega_{j_6}^2 \begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix} \begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6' \end{vmatrix} = \delta_{j_6 j_6'} \tag{32}$$

to perform the summation over j_3 and reduce (**??**) to

$$|M(S^3)| = \omega^{-8} \sum_{\substack{\mathrm{j_1 j_2 j_4 j_5 j_6} \\ \text{such that}(\mathrm{j_1 j_5 j_6}), \\ (\mathrm{j_2 j_4 j_6})\ \text{adm.}}} \omega_{j_1}^2 \omega_{j_2}^2 \omega_{j_4}^2 \omega_{j_5}^2 \ . \tag{33}$$

The ω_j's satisfy a further useful identity

$$\sum_{\substack{\mathrm{j,k} \\ \text{such that} \\ (\mathrm{i,j,k})\text{adm.}}} \omega_j^2 \omega_k^2 = \omega^2 \omega_i^2 \tag{34}$$



and then the sums over j_1 and j_5, and over j_2 and j_4 lead to

$$|M(S^3)| = \omega^{-4} \sum_{j_6} \omega_{j_6}^4 \ . \tag{35}$$

By (**??**) this gives simply

$$|M(S^3)| = \omega^{-2} \ . \tag{36}$$

Combining (**??**)–(**??**), we can show that the final value for $|M|$ is

$$|M(S^3)| = \frac{2}{r} \sin^2\left(\frac{\pi}{r}\right) \tag{37}$$

We may compare this with the partition function for S^3 with gauge group $SU(2)_k$ [Witten 1989]:

$$Z(S^3) = \sqrt{\frac{2}{k+2}} \ \sin\left(\frac{\pi}{k+2}\right) \tag{38}$$

and, making the required identification of r with $k+2$, we obtain

$$|M(S^3)| = |Z(S^3)|^2 \ , \tag{39}$$

in agreement with (**??**).

I shall now mention briefly some joint work with Francis Archer in which we have been looking at the relationship between the Turaev–Viro invariants, and the partition function for three-dimensional gravity with a cosmological constant. The basic line of investigation goes as follows. It is known that any three-manifold can be constructed by taking a Heegaard decomposition into two handlebodies of appropriate genus and identifying their surfaces under homeomorphism. The interiors of the handlebodies may be taken to be Euclidean space, which means that all the information about the topology of the manifold being constructed can be carried by the Heegaard diagram and the homeomorphism. Thus there is a relation between the Turaev–Viro invariant for a triangulated three-manifold and the triangulations induced on the surfaces of the handlebodies, which is precisely one of the ideas which Ponzano and Regge [1968] were using. However, this works in the most naive way only for handlebodies with trivial topology; for surfaces of non-zero genus, one also needs a system of meridional discs (Turaev 1991). The duals of the surface triangulations may then be treated as generalized spin networks on the surfaces and the values of the spin networks have been related precisely to the corresponding Turaev–Viro invariants. (This is in parallel with the work of Kauffman and Lins [1991].) Now it is well known [Kauffman 1990] that the evaluation of generalized spin networks leads to products of bracket polynomials, which in turn are related to the Jones polynomials. The final link in the chain is the work of Witten [1989] relating the Jones polynomial to expectation values of Wilson loops for quantum field theories with a Chern–Simons action, of which, as we have already noted, three-dimensional gravity with cosmological constant is one particular example.

Clearly, there are many open questions here, at all levels. Quite apart from technical points in proving the equivalences between the theories discussed, there is also the question of precisely how the Turaev–Viro type calculations are related to conformal field theory. One may try to understand why $6j$-symbols from the theory of angular momentum should play a role in describing the quantum structure of space–time. One may also ask whether there is anything to be learned from the remarkable new results on three-manifolds and three-dimensional gravity, which would be at all useful in studying gravity in four dimensions.

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