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Quantum Gravity, or The Art of Building Spacetime[1]

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Abstract

The method of four-dimensional Causal Dynamical Triangulations provides a background-independent definition of the sum over geometries in quantum gravity, in the presence of a positive cosmological constant. We present the evidence accumulated to date that a macroscopic four-dimensional world can emerge from this theory dynamically. Using computer simulations we observe in the Euclidean sector a universe whose scale factor exhibits the same dynamics as that of the simplest mini-superspace models in quantum cosmology, with the distinction that in the case of causal dynamical triangulations the effective action for the scale factor is not put in by hand but obtained by integrating out *in the quantum theory* the full set of dynamical degrees of freedom except for the scale factor itself.

[1]Contribution to the book " Approaches to Quantum Gravity", edited by D. Oriti, to appear at Cambridge University Press.

1 Introduction

What is more natural than constructing space from elementary geometric building blocks? It is not as easy as one might think, based on our intuition of playing with Lego blocks in three-dimensional space. Imagine the building blocks are d-dimensional flat simplices all of whose side lengths are a, and let $d > 2$. The problem is that if we glue such blocks together carelessly we will *with probability one* create a space of no extension, in which it is possible to get from one vertex to any other in a few steps, moving along the one-dimensional edges of the simplicial manifold we have created. We can also say that the space has an extension which remains at the 'cut-off' scale a. Our intuition coming from playing with Lego blocks is misleading here because it presupposes that the building blocks are embedded geometrically faithfully in Euclidean $I\!R^3$, which is not the case for the intrinsic geometric construction of a simplicial space.

By contrast, let us now be more careful in our construction work by assigning to a simplicial space $\mathcal{T}$ – which we will interpret as a (Euclidean) spacetime – the weight $e^{-S(\mathcal{T})}$, where $S(\mathcal{T})$ denotes the Einstein action associated with the piecewise linear geometry uniquely defined by our construction[2]. As long as the (bare) gravitational coupling constant G_N is large, we have the same situation as before. However, upon lowering G_N we will eventually encounter a phase transition beyond which the geometry is no longer crumpled into a tiny ball, but maximally extended. Such a geometry is made out of effectively one-dimensional filaments[3] which can branch out, and are therefore called *branched polymers* or *trees* [1, 2]. The transition separating the two phases [3, 4] is of first order, which implies that there is no smooth change between the two pathological types of minimally or maximally extended "universes".

In order for the sum over geometries to produce a quantum theory of gravity in which classical geometry is reproduced in a suitable limit, we therefore need a different principle for selecting the geometries to be included in this sum. Below we will introduce such a principle: our prescription will be to sum over a class of (Euclidean) geometries which are in one-to-one correspondence with Lorentzian, *causal* geometries. At the discretized level, where we use a specific set of building blocks and gluing rules to constructively define the path integral, we call these geometries *causal dynamical triangulations* (CDT) [5, 6, 7, 8].

Before discussing CDT in more detail let us comment on the nature of the geometries contributing to the path integral. It is important to emphasize that in a *quantum* theory of gravity a given spacetime geometry as such has no immediate

[2]There exists a natural, coordinate-independent definition of the Einstein action for piecewise linear geometries called the Regge action.

[3]The d-dimensional building blocks are arranged such that $(d-1)$ "transverse" dimensions have a size of only a few lattice spacings.

physical meaning. The situation is really the same as in ordinary quantum field theory or even quantum mechanics, where individual field configurations $\phi(x,t)$ or particle paths $x(t)$ are *not* observable. Only certain expectation values related to the fields or paths can be observed in experiments. This does not mean there cannot exist limits in which it is appropriate to talk about a particular field configuration or the path of a particle in an approximate sense. In the case of our actual universe, down to the smallest distances that have been probed experimentally, it certainly does seem adequate to talk about a fixed classical spacetime geometry. Nevertheless, at sufficiently small distances it will no longer make sense to ask classical questions about spacetime, at least if we are to believe in the principles of conventional quantum theory.

By way of illustration let us discuss the situation for the ordinary harmonic oscillator (or the free particle) and consider the path integral from (x_1,t_1) to (x_2,t_2). Precisely for the harmonic oscillator (or the free particle) the decomposition

$$x(t) = x_{cl}(t) + y(t), \qquad y(t_1) = y(t_2) = 0, \tag{1}$$

leads to an exact factorization of the path integral, because the action satisfies

$$S(x) = S(x_{cl}) + S(y). \tag{2}$$

This implies that the classical path $x_{cl}(t)$ contributes to the path integral with the classical action, and $y(t)$ with quantum fluctuations independent of this classical part. Taking the classical trajectory to be macroscopic one obtains the picture of a macroscopic path dressed with small quantum fluctuations; small because they are independent of the classical motion. An explicit Euclidean calculation yields the result

$$\left\langle \int_0^T \mathrm{d}t\; y^2(t) \right\rangle = \frac{\hbar}{2m\omega^2}\,(\omega T \tanh^{-1}\omega T - 1) \tag{3}$$

as a function of the oscillator frequency ω and mass m. Let us now consider a situation where we have chosen the "system size", i.e. $x_{cl}(t)$, to be macroscopic. According to (3), the quantum fluctuations around this path can then be considered small since $\hbar$ is small.

This is more or less the picture we envisage for our present-day universe in quantum gravity: the universe is of macroscopic size, governed by the classical equations of motion (the analogue of choosing "by hand" (x_1,t_1) and (x_2,t_2) to be macroscopic in the example above), and the small quantum fluctuations are dictated by the gravitational coupling constant (times $\hbar/c^3$).

A given configuration $x(t)$ in the path integral for the quantum-mechanical particle is (with probability one) a continuous, nowhere differentiable path, which moreover is fractal with Hausdorff dimension two, as we know from the rigorous construction of the Wiener measure on the set of parametrized paths. In the

case of quantum gravity we do not have a similar mathematically rigorously defined measure on the space of geometries, but it is natural to expect that *if* it exists, a typical geometry in the path integral will be continuous, but nowhere differentiable. By analogy, the piecewise linear geometries seem a good choice if we want to approximate the gravitational path integral by a set of geometries and subsequently take a limit where the approximation (the cut-off) is removed. Moreover, such simplicial manifolds possess a natural, geometric and coordinate-independent implementation of the Einstein-Hilbert action. With all local curvature degrees of freedom present (albeit in a discretized fashion), we also expect them to be suitably "dense" in the set of all continuous geometries.

The spirit is very much that of the standard lattice formulation of quantum field theory where (flat) spacetime is approximated by a hypercubic lattice. The ultraviolet cut-off in such field theories is given by the lattice spacing, i.e. the length of all one-dimensional lattice edges. We can in a similar and simple manner introduce a *diffeomorphism-invariant* cut-off in the sum over the piecewise linear geometries by restricting it to the *building blocks* mentioned earlier. A natural building block for a d-dimensional spacetime is a d-dimensional equilateral simplex with side-length a, and the path integral is approximated by performing the sum over all geometries (of fixed topology[4]) which can be obtained by gluing such building blocks together, each geometry weighted appropriately (for example, by e^{-S}, where S is the Einstein-Hilbert action). Afterwards we take the limit $a \to 0$. For a particular choice of the bare, dimensionless coupling constants one may be able to obtain a continuum limit, and thus extract a continuum theory. For other values, if the sum exists at all (possibly after renormalization), one will merely obtain a sum which has no continuum interpretation. This situation is precisely the same as encountered in ordinary lattice field theory in flat spacetime.

As mentioned earlier it has up to now not been possible to define construc-

[4]In classical General Relativity there is no motivation to consider spacetimes whose spatial topology changes in time, since their Lorentzian structure is necessarily singular. There is an interesting and long-standing discussion about whether one should include topology changes in a *quantum* theory of gravity. However, even in the case of two-dimensional Euclidean quantum gravity, where the classification of topology changes is simple, the summation over topologies has never been defined non-perturbatively in a satisfactory way, despite many attempts, in particular, in so-called non-critical string theory. (However, see [9] for how one may improve the convergence of the sum in two-dimensional *Lorentzian* quantum gravity by invoking not just the topological, but the causal, geometric structure of spacetime.) The situation becomes worse in higher dimensions. For instance, four-dimensional topologies are not classifiable, so what does it mean to sum over them in the path integral? The problem – even in dimension two – is that there are many more geometries of complicated topology than there are of simple topology, with the consequence that any sum over geometries will be (i) completely dominated by these complicated topologies, and (ii) plainly divergent in a way which (until now) has made it impossible to define the theory non-perturbatively in an unambiguous and physically satisfactory manner. In higher dimensions these problems are totally out of control.

tively a Euclidean path integral for gravity in four dimensions by following the philosophy just outlined. One simply has not succeeded in identifying a continuum limit of the (unrestricted) sum over Euclidean building blocks. Among the reasons that have been advanced to explain this failure, it is clear that the *entropy* of the various geometries plays an important role. We have already pointed out that the crumpled geometries of no extension dominate the space of all continuous geometries whenever the dimension of spacetime is larger than two. There is nothing wrong with this a priori; the path integral of any quantum field theory is dominated completely by wild UV-field fluctuations. However, in the case of *renormalizable* quantum field theories there exists a well-defined limiting procedure which allows one to extract "continuum" physics by fine-tuning the bare coupling constants of the theory. An analogous procedure in Euclidean quantum gravity still has not been found, and adding (bosonic) matter does not improve the situation. Instead, note that the Einstein-Hilbert action has a unique feature, namely, it is unbounded from below. The transition between the crumpled and the branched-polymer geometries can be seen as a transition from a phase where the entropy of configurations dominates over the action to a phase where the unboundedness of the Euclidean action becomes dominant[5]. The impossibility of finding a continuum limit may be seen as the impossibility of balancing the entropy of configurations against the action. We need another guiding principle for selecting Euclidean geometries in the path integral in order to obtain a continuum limit, and it is such a principle we turn to next.

2 Defining CDT

It has been suggested that the signature of spacetime may be explained from a dynamical principle [11]. Being somewhat less ambitious, we will assume it has Lorentzian signature and accordingly change our perspective from the Euclidean formulation of the path integral discussed in the previous section to a Lorentzian formulation, motivated by the uncontroversial fact that our universe has three space and one time dimension. A specific rotation to Euclidean signature introduced below will be needed in our set-up as a merely technical tool to perform certain sums over geometries. Unlike in flat spacetime there are no general theo-

[5]Although the action is not unbounded below in the regularized theory, this feature of the continuum action nevertheless manifests itself in the limit as the (discretized) volume of spacetime is increased, eventually leading to the above-mentioned phase transition at a particular value of the bare gravitational coupling constant. Remarkably, a related phenomenon occurs in bosonic string theory. If the world-sheet theory is regularized non-perturbatively in terms of triangulations (with each two-dimensional world-sheet glued from fundamental simplicial building blocks), the tachyonic sickness of the theory manifests itself in the form of surfaces degenerating into branched polymers [10].

rems which would allow us to relate the Euclidean and Lorentzian quantum field theories when dealing with quantum gravity.

Consider now a connected space-like hypersurface in spacetime. Any classical evolution in general relativity will leave the topology of this hypersurface unchanged, since otherwise spacetime would contain regions where the metric is degenerate. However, as long as we do not have a consistent theory of quantum gravity we do not know whether such degenerate configurations should be included in the path integral. We have already argued that the inclusion of arbitrary spacetime topologies leads to a path integral that has little chance of making sense. One might still consider a situation where the overall topology of spacetime is fixed, but where one allows "baby universes" to branch off from the main universe, without permitting them to rejoin it and thus form "handles". Apart from being a rather artificial constraint on geometry, such a construction is unlikely to be compatible with unitarity. We will in the following take a conservative point of view and only sum over geometries (with Lorentzian signature) which permit a foliation in (proper) time and are causally well-behaved in the sense that no topology changes are allowed as a function of time. In the context of a formal continuum path integral for gravity, similar ideas have earlier been advanced in [12].

Of the diffeomorphism-invariant quantities one can consider in the quantum theory, we have chosen a particular *proper-time propagator*, which can be defined constructively in a transparent way. We are thus interested in defining the path integral

$$G(g(0), g(T); T) = \int_{g(0)}^{g(T)} \mathcal{D}g \; \mathrm{e}^{iS[g]} \tag{4}$$

over Lorentzian geometries on a manifold $\mathcal{M}$ with topology $\Sigma \times [0,1]$, where Σ is a compact, connected three-dimensional manifold. The geometries included in the path integral will be such that the induced boundary three-geometries $g(0)$ and $g(T)$ are space-like and separated by a time-like geodesic distance T, with T an external (diffeomorphism-invariant) parameter.

We now turn to the constructive definition of this object in terms of building blocks. The discretized analogue of an infinitesimal proper-time "sandwich" in the continuum will be a finite sandwich of thickness $\Delta t = 1$ (measured in "building block units" a) of topology $\Sigma \times [0,1]$ consisting of a single layer of four-simplices. This layer has two spacelike boundaries, corresponding to two slices of constant (integer) "proper time" t which are one unit apart. They form two three-dimensional piecewise flat manifolds of topology Σ and consist of purely spacelike tetrahedra. By construction, the sandwich interior contains no vertices, so that any one of the four-simplices shares k of its vertices with the initial spatial slice and $5-k$ of them with the final spatial slice, where $1 \leq k \leq 4$. To obtain extended spacetimes, one glues together sandwiches pairwise along their

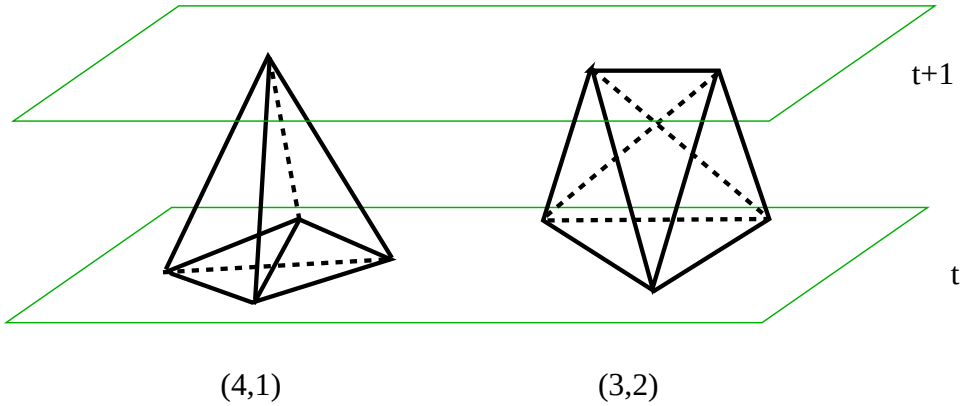

Figure 1: The two fundamental building blocks of causal dynamically triangulated gravity. The flat four-simplex of type (4,1) on the left has four of its vertices at time t and one at time $t+1$, and analogously for the (3,2)-simplex on the right. The "gap" between two consecutive spatial slices of constant integer time is filled by copies of these simplicial building blocks and their time-reversed counterparts, the (1,4)- and the (2,3)-simplices.

matching three-dimensional boundary geometries. We choose each four-simplex to have time-like links of length-squared a_t^2 and space-like links of length-squared a_s^2, with all of the latter located in spatial slices of constant integer-t.

Each spatial tetrahedron at time t is therefore shared by two four-simplices (said to be of type (1,4) and (4,1)) whose fifth vertex lies in the neighbouring slice of constant time $t-1$ and $t+1$ respectively. In addition we need four-simplices of type (2,3) and (3,2) which share one link and one triangle with two adjacent spatial slices, as illustrated in Fig. 1 (see [7] for details). The integer-valued proper time t can be extended in a natural way to the interiors of the four-simplices, leading to a global foliation of any causal dynamically triangulated spacetime into piecewise flat (generalized) triangulations for any constant real value of t [13]. Inside each building block this time coincides with the proper time of Minkowski space. Moreover, it can be seen that in the piecewise linear geometries the mid-points of all spatial tetrahedra at constant time t are separated a fixed time-like geodesic distance 1 (in units of a_t) from the neighbouring hypersurfaces at $t-1$ and $t+1$. It is in this sense that the "link distance" t, i.e. counting future-oriented time-like links between spatial slices is a discretized analogue of their proper-time distance.

Let us furthermore assume that the two possible link lengths are related by

$$a_t^2 = -\alpha a_s^2. \tag{5}$$

All choices $\alpha > 0$ correspond to Lorentzian and all choices $\alpha < -7/12$ to Euclidean signature, and a Euclideanization of geometry is obtained by a suitable

analytic continuation in α (see [7] for a detailed discussion of this "Wick rotation" where one finds $S_E(-\alpha) = iS_L(\alpha)$ for $\alpha > 7/12$).

Setting $\alpha = -1$ leads to a particularly simple expression for the (Euclidean) Einstein-Hilbert action of a given triangulation $\mathcal{T}$ (since all four-simplices are then identical geometrically), namely,

$$S_E(\mathcal{T}) = -k_0 N_0(\mathcal{T}) + k_4 N_4(\mathcal{T}), \tag{6}$$

with $N_i(\mathcal{T})$ denoting the number of i-dimensional simplices in T. In (6), k_0 is proportional to the inverse (bare) gravitational coupling constant, $k_0 \sim 1/G_N$, while k_4 is a linear combination of the cosmological and inverse gravitational coupling constants. The action (6) is calculated from Regge's prescription for piecewise linear geometries. If we take $\alpha \neq -1$ the Euclidean four-simplices of type (1,4) and type (2,3) will be different and appear with different weights in the Einstein-Hilbert action [7]. For our present purposes it is convenient to use the equivalent parametrization

$$S_E(\mathcal{T}) = -k_0 N_0(\mathcal{T}) + k_4 N_4(\mathcal{T}) + \Delta(2N_{14}(\mathcal{T}) + N_{23}(\mathcal{T})), \tag{7}$$

where $N_{14}(\mathcal{T})$ and $N_{23}(\mathcal{T})$ denote the combined numbers in $\mathcal{T}$ of four-simplices of types $(1,4)$ and $(4,1)$, and of types $(2,3)$ and $(3,2)$, respectively. The explicit map between the parameter Δ in eq. (7) and α can be readily worked out [14]. For the simulations reported here we have used Δ in the range 0.4–0.6.

The (Euclidean) discretized analogue of the continuum proper-time propagator (4) is defined by

$$G_{k_0,k_4,\Delta}(\mathcal{T}^{(3)}(0), \mathcal{T}^{(3)}(T), T) = \sum_{\mathcal{T}\in\mathcal{T}_T} \frac{1}{C_{\mathcal{T}}}\, e^{-S_E(\mathcal{T})}. \tag{8}$$

where the summation is over the set $\mathcal{T}_T$ of all four-dimensional triangulations of topology $\Sigma^3 \times [0,1]$ (which we in the following always choose to be S^3) and T proper-time steps, whose spatial boundary geometries at proper times 0 and T are $\mathcal{T}^{(3)}(0)$ and $\mathcal{T}^{(3)}(T)$. The order of the automorphism group of the graph $\mathcal{T}$ is denoted by $C_{\mathcal{T}}$. The propagator can be related to the quantum Hamiltonian conjugate to t, and in turn to the transfer matrix of the (Euclidean) statistical theory [7].

It is important to emphasize again that we rotate each configuration to a Euclidean "spacetime" simply in order to perform the summation in the path integral, and that this is made possible by the piecewise linear structure of our geometry and the existence of a proper-time foliation. Viewed from an inherently Euclidean perspective there would be no motivation to restrict the sum over geometries to "causal" geometries of the kind constructed above. We also want

to stress that the use of piecewise linear geometries has allowed us to write down a (regularized) version of (4) using only geometries, not metrics (which are of course not diffeomorphism-invariant), and finally that the use of building blocks has enabled the introduction of a diffeomorphism-invariant cut-off (the lattice link length a).

3 Numerical analysis of the model

While it may be difficult to find an explicit analytic expression for the full propagator (8) of the four-dimensional theory, Monte Carlo simulations are readily available for its analysis, employing standard techniques from Euclidean dynamically triangulated quantum gravity [15]. Ideally one would like to keep the *renormalized* [6] cosmological constant Λ fixed in the simulation, in which case the presence of the cosmological term $\Lambda \int \sqrt{g}$ in the action would imply that the four-volume V_4 fluctuated around $\langle V_4 \rangle \sim \Lambda^{-1}$. However, for simulation-technical reasons one fixes instead the number N_4 of four-simplices (or[7] the four-volume V_4) from the outset, working effectively with a cosmological constant $\Lambda \sim V_4^{-1}$.

3.1 The global dimension of spacetime

A "snapshot", by which we mean the distribution of three-volumes as a function of the proper time $0 \leq t \leq T$ for a spacetime configuration randomly picked from the Monte Carlo-generated geometric ensemble, is shown in Fig. 2. One observes a"stalk" of essentially no spatial extension (with spatial volumes close to the minimal triangulation of S^3 consisting of five tetrahedra) expanding into a universe of genuine "macroscopic" spatial volumes, which after a certain time $\tau \leq T$ contracts again to a state of minimal spatial extension. As we emphasized earlier, a single such configuration is unphysical, and therefore not observable. However, a more systematic analysis reveals that fluctuations around an overall "shape" similar to the one of Fig. 2 are relatively small, suggesting the existence of a *background geometry* with relatively small quantum fluctuations superimposed. This is precisely the scenario advocated in Sec. 1 and is rather remarkable, given that our formalism is background-independent. Our first major goal is to verify quantitatively that we are indeed dealing with an approximate four-dimensional background geometry [16, 17], and secondly to determine the effective action responsible for the observed large-scale features of this background geometry [18, 14] .

[6] For the relation between the bare (dimensionless) cosmological constant k_4 and the renormalized cosmological constant Λ see [1].

[7] For fixed α (or Δ) one has $\langle N_{14} \rangle \propto \langle N_{23} \rangle \propto \langle N_4 \rangle$. V_4 is given as (see [7] for details): $V_4 = a_s^4 (N_{14}\sqrt{8\alpha+3} + N_{23}\sqrt{12\alpha+7})$. We set $a_s = 1$.

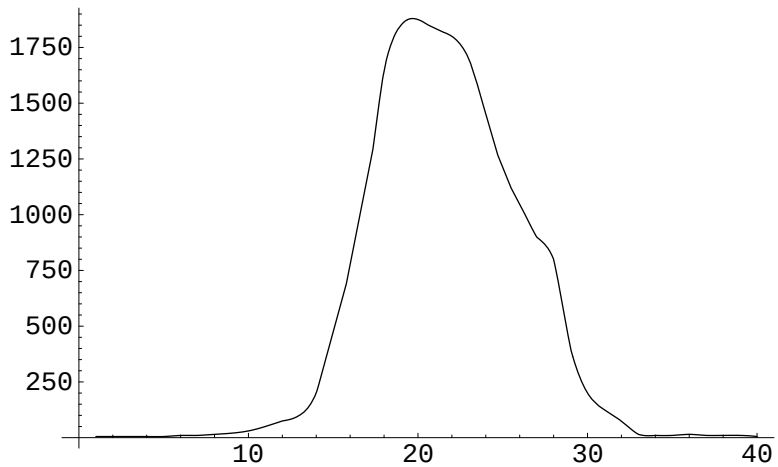

Figure 2: Snapshot of a "typical universe" consisting of approximately 91000 four-simplices as it appears in the Monte Carlo simulations at a given "computer time". We plot the three-volume at each integer step in proper time, for a total time extent of $T = 40$, in units where $a_s = 1$.

Important information is contained in how the expectation values of the volume V_3 of spatial slices and the total time extent τ (the proper-time interval during which the spatial volumes $V_3 \gg 1$) of the observed universe behave as the total spacetime volume V_4 is varied. We find that to good approximation the spatially extended parts of the spacetimes for various four-volumes V_4 can be mapped onto each other by rescaling the spatial volumes and the proper times according to

$$V_3 \to V_3/V_4^{3/4}, \qquad \tau \to \tau/V_4^{1/4}. \tag{9}$$

To quantify this we studied the so-called volume-volume correlator

$$\langle V_3(0)V_3(\delta)\rangle = \frac{1}{t^2}\sum_{j=1}^{t}\langle V_3(j)V_3(j+\delta)\rangle \tag{10}$$

for pairs of spatial slices an integer proper-time distance δ apart. Fig. 5 shows the volume-volume correlator for five different spacetime volumes V_4, using the rescaling (9)[8], and exhibiting that it is almost perfect. An error estimate yields $d = 4 \pm 0.2$ for the large-scale dimension of the universe [14].

[8]In (10) we use discrete units such that successive spatial slices are separated by 1. For convenience we periodically identify $\mathcal{T}^{(3)}(T) = \mathcal{T}^{(3)}(0)$ and sum over all possible three-geometries $\mathcal{T}^{(3)}(0)$, rather than working with fixed boundary conditions. In this way (10) becomes a convenient translation-invariant measure of the spatial and temporal extensions of the universe (see [8] for a detailed discussion).

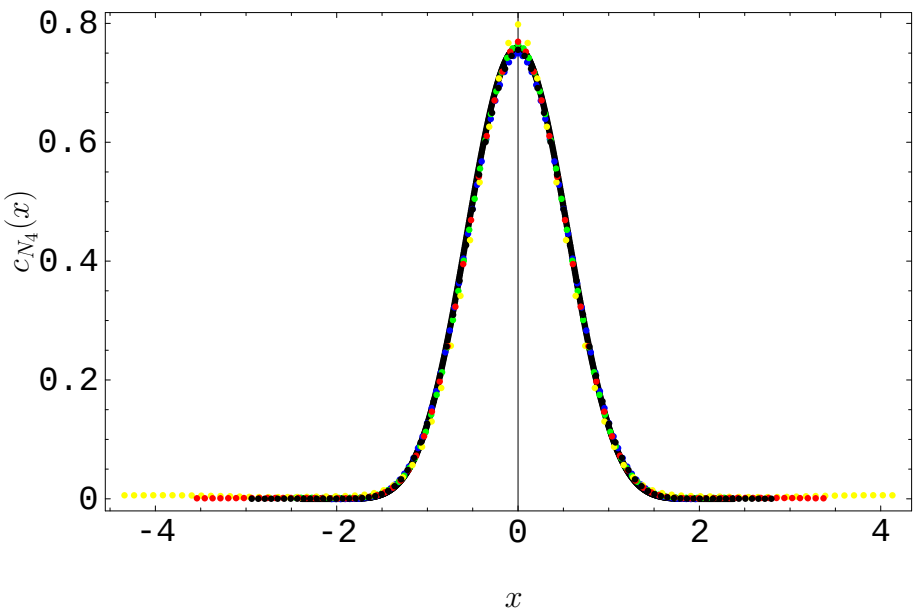

Figure 3: The scaling of the volume-volume correlator, as function of the rescaled time variable $x = \delta/(N_4)^{1/4}$. Data points come from system sizes $N_4 = 22500$, 45000, 91000, 181000 and 362000 at $\kappa_0 = 2.2$, $\Delta = 0.6$ and $T = 80$.

Another way of obtaining an effective dimension of the nonperturbative ground state, its so-called *spectral dimension* D_S, comes from studying a diffusion process on the underlying geometric ensemble. On a d-dimensional manifold with a fixed, smooth Riemannian metric $g_{ab}(\xi)$, the diffusion equation has the form

$$\frac{\partial}{\partial\sigma} K_g(\xi,\xi_0;\sigma) = \Delta_g K_g(\xi,\xi_0;\sigma), \tag{11}$$

where σ is a fictitious diffusion time, Δ_g the Laplace operator of the metric $g_{ab}(\xi)$ and $K_g(\xi,\xi_0;\sigma)$ the probability density of diffusion from point ξ_0 to point ξ in diffusion time σ. We will consider diffusion processes which initially are peaked at some point ξ_0, so that

$$K_g(\xi,\xi_0;\sigma=0) = \frac{1}{\sqrt{\det g(\xi)}}\,\delta^d(\xi-\xi_0). \tag{12}$$

For the special case of a flat Euclidean metric, we have

$$K_g(\xi,\xi_0;\sigma) = \frac{\mathrm{e}^{-d_g^2(\xi,\xi_0)/4\sigma}}{(4\pi\sigma)^{d/2}}, \qquad g_{ab}(\xi)=\delta_{ab}, \tag{13}$$

where d_g denotes the distance function associated with the metric g.

A quantity which is easier to measure in numerical simulations is the *average return probability* $P_g(\sigma)$, defined by

$$P_g(\sigma) := \frac{1}{V} \int d^d\xi \sqrt{\det g(\xi)}\; K_g(\xi,\xi;\sigma), \tag{14}$$

where V is the spacetime volume $V = \int d^d\xi \sqrt{\det g(\xi)}$. For an infinite flat space, we have $P_g(\sigma) = 1/(4\pi\sigma)^{d/2}$ and thus can extract the dimension d by taking the logarithmic derivative

$$-\,2\;\frac{d\log P_g(\sigma)}{d\log\sigma} = d, \tag{15}$$

independent of σ. For nonflat spaces and/or finite volume V, one can still use eq. (15) to extract the dimension, but there will be correction terms (see [14] for a detailed discussion).

In applying this set-up to four-dimensional quantum gravity in a path integral formulation, we are interested in measuring the expectation value of the average return probability $P_g(\sigma)$. Since $P_g(\sigma)$ defined according to (14) is invariant under reparametrizations, it makes sense to take its quantum average over all geometries of a given spacetime volume V_4,

$$P_{V_4}(\sigma) = \frac{1}{\tilde{Z}_E(V_4)} \int \mathcal{D}[g_{ab}]\; e^{-\tilde{S}_E(g_{ab})} \delta\Big(\int d^4x \sqrt{\det g} - V_4\Big)\, P_g(\sigma), \tag{16}$$

where $\tilde{Z}_E(V_4)$ is the quantum gravity partition function for spacetimes with constant four-volume V_4.

Our next task is to define a diffusion process on the class of metric spaces under consideration, the piecewise linear structures defined by the causal triangulations $\mathcal{T}$. We start from an initial probability distribution

$$K_{\mathcal{T}}(i, i_0; \sigma = 0) = \delta_{i,i_0}, \tag{17}$$

which vanishes everywhere except at a randomly chosen (4,1)-simplex i_0, and define the diffusion process by the evolution rule

$$K_{\mathcal{T}}(j, i_0; \sigma + 1) = \frac{1}{5} \sum_{k \to j} K_{\mathcal{T}}(k, i_0; \sigma), \tag{18}$$

where the diffusion time σ now advances in discrete integer steps. These equations are the simplicial analogues of (12) and (11), $k \to j$ denoting the five nearest neighbours of the four-simplex j. In this process, the total probability

$$\sum_j K_{\mathcal{T}}(j, i_0; \sigma) = 1 \tag{19}$$

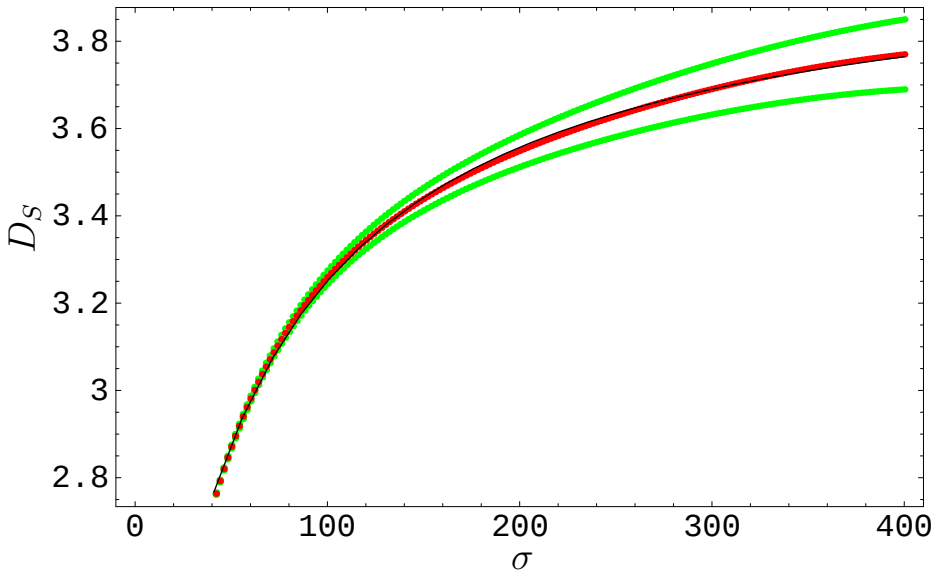

Figure 4: The spectral dimension D_S of the universe as function of the diffusion time σ, measured for $\kappa_0 = 2.2$, $\Delta = 0.6$ and $t = 80$, and a spacetime volume $N_4 = 181$k. The averaged measurements lie along the central curve, together with a superimposed best fit $D_S(\sigma) = 4.02 - 119/(54+\sigma)$ (thin black curve). The two outer curves represent error bars.

is conserved. The probability to return to the simplex i_0 is then defined as $P_{\mathcal{T}}(i_0;\sigma) = K_{\mathcal{T}}(i_0, i_0; \sigma)$ and its quantum average as

$$P_{N_4}(\sigma) = \frac{1}{\tilde{Z}_E(N_4)} \sum_{\mathcal{T}_{N_4}} e^{-\tilde{S}_E(\mathcal{T}_{N_4})} \, \frac{1}{N_4} \sum_{i_0 \in \mathcal{T}_{N_4}} K_{\mathcal{T}_{N_4}}(i_0, i_0; \sigma), \tag{20}$$

where $\mathcal{T}_{N_4}$ denotes a triangulation with N_4 four-simplices, and $\tilde{S}_E(\mathcal{T}_{N_4})$ and $\tilde{Z}_E(N_4)$ are the obvious simplicial analogues of the continuum quantities in eq. (16).

We can extract the value of the spectral dimension D_S by measuring the logarithmic derivative as in (15) above, that is,

$$D_S(\sigma) = -2 \, \frac{d \log P_{N_4}(\sigma)}{d \log \sigma}, \tag{21}$$

as long as the diffusion time is not much larger than N_4^{2/D_S}. The outcome of the measurements is presented in Fig. 4, with error bars included. (The two outer curves represent the envelopes to the tops and bottoms of the error bars.) The error grows linearly with σ, due to the presence of the $\log \sigma$ in (21).

The remarkable feature of the curve $D_S(\sigma)$ is its slow approach to the asymptotic value of $D_S(\sigma)$ for large σ. The new phenomenon we observe here is a *scale dependence of the spectral dimension*, which has emerged dynamically [17, 14].

As explained in [17], the best three-parameter fit which asymptotically approaches a constant is of the form

$$D_S(\sigma) = a - \frac{b}{\sigma + c} = 4.02 - \frac{119}{54 + \sigma}. \tag{22}$$

The constants a, b and c have been determined by using the data range $\sigma \in [40, 400]$ and the curve shape agrees well with the measurements, as can be seen from Fig. 4. Integrating (22) we obtain

$$P(\sigma) \sim \frac{1}{\sigma^{a/2}(1 + c/\sigma)^{b/2c}}, \tag{23}$$

from which we deduce the limiting cases

$$P(\sigma) \sim \begin{cases} \sigma^{-a/2} & \text{for large } \sigma, \\ \sigma^{-(a-b/c)/2} & \text{for small } \sigma. \end{cases} \tag{24}$$

Again we conclude that within measuring accuracy the large-scale dimension of spacetime in our model is four. We also note that the short-distance spectral dimension seems to be approximately $D_S = 2$, signalling a highly non-classical behaviour.

3.2 The effective action

Our next goal will be to understand the precise analytical form of the volume-volume correlator (10). To this end, let us consider the distribution of differences in the spatial volumes V_3 of successive spatial slices at proper times t and $t + \delta$, where δ is infinitesimal, i.e. $\delta = 1$ in lattice proper-time units. We have measured the probability distribution $P_{V_3}(z)$ of the variable

$$z = \frac{V_3(t + \delta) - V_3(t)}{V_3^{1/2}}, \qquad V_3 := V_3(t) + V_3(t + \delta). \tag{25}$$

for different values of V_3. As shown in Fig. 5 they fall on a common curve.[9] Furthermore, the distribution $P_{V_3}(z)$ is fitted very well by a Gaussian e^{-cz^2}, with a constant c independent of V_3. From estimating the entropy of spatial geome-

[9] Again we have applied finite-size scaling techniques, starting out with an arbitrary power V_3^α in the denominator in (25), and then determining $\alpha = 1/2$ from the principle of maximal overlap of the distributions for various V_3's.

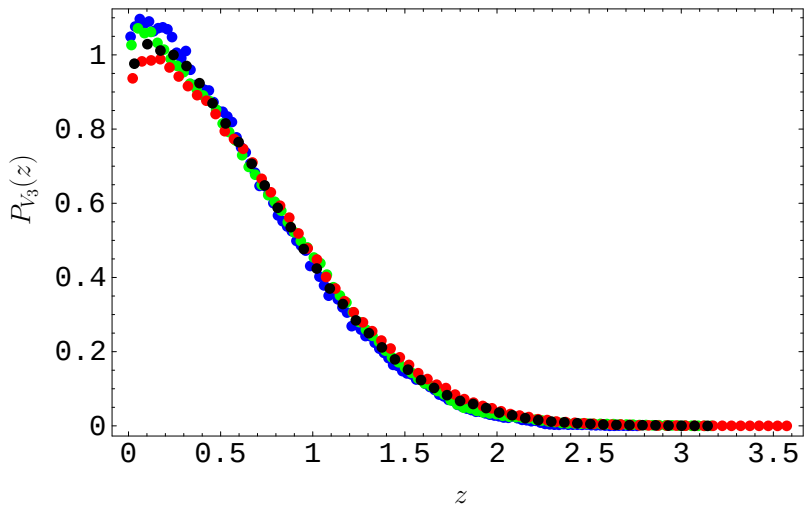

Figure 5: Distribution $P_{V_3}(z)$ of volume differences of adjacent spatial slices, for three-volumes $V_3 = 10.000$, 20.000, 40.000 and 80.000 tetrahedra.

tries, that is, the number of such configurations, one would expect corrections of the form V_3^α, with $0 \leq \alpha < 1$, to the exponent $c\,z^2$ in the distribution $P_{V_3}(z)$. Unfortunately it is impossible to measure these corrections directly in a reliable way. We therefore make a general ansatz for the probability distribution for *large* $V_3(t)$ as

$$\exp\left[-\frac{c_1}{V_3(t)}\left(\frac{\mathrm{d}V_3(t)}{\mathrm{d}t}\right)^2 - c_2 V_3^\alpha(t)\right], \tag{26}$$

where $0 \leq \alpha < 1$, and c_1 and c_2 are positive constants.

In this manner, we are led by "observation" to the effective action

$$S_{V_4}^{\mathit{eff}} = \int_0^T \mathrm{d}t\,\left(\frac{c_1}{V_3(t)}\left(\frac{\mathrm{d}V_3(t)}{\mathrm{d}t}\right)^2 + c_2 V_3^\alpha(t) - \lambda V_3(t)\right), \tag{27}$$

valid for large three-volume $V_3(t)$, where λ is a Lagrange multiplier to be determined such that

$$\int_0^T dt\, V_3(t) = V_4. \tag{28}$$

From general scaling of the above action it is clear that the only chance to obtain the observed scaling law, expressed in terms of the variable $t/V_4^{1/4}$, is by setting $\alpha = 1/3$. In addition, to reproduce the observed stalk for large times t the function $V_3^{1/3}$ has to be replaced by a function of V_3 whose derivative at 0 goes

like V_3^ν, $\nu \geq 0$, for reasons that will become clear below. A simple modification, which keeps the large-V_3 behaviour intact, is given by

$$V_3^{1/3} \to (1 + V_3)^{1/3} - 1, \tag{29}$$

but the detailed form is not important. If we now introduce the (non-negative) *scale factor* $a(t)$ by

$$V_3(t) = a^3(t), \tag{30}$$

we can (after suitable rescaling of t and $a(t)$) write the effective action as

$$S_{V_4}^{\mathit{eff}} = \frac{1}{G_N} \int_0^T \mathrm{d}t \, \left(a(t) \left(\frac{\mathrm{d}a(t)}{\mathrm{d}t} \right)^2 + a(t) - \lambda a^3(t) \right), \tag{31}$$

with the understanding that the linear term should be replaced using (30) and (29) for small $a(t)$. We emphasize again that we have been led to (31) entirely by "observation" and that one can view the small-$a(t)$ behaviour implied by (29) as a result of quantum fluctuations.

3.3 Minisuperspace

Let us now consider the simplest minisuperspace model for a closed universe in quantum cosmology, as for instance used by Hartle and Hawking in their semiclassical evaluation of the wave function of the universe [19]. In Euclidean signature and proper-time coordinates, the metrics are of the form

$$ds^2 = dt^2 + a^2(t) d\Omega_3^2, \tag{32}$$

where the scale factor $a(t)$ is the only dynamical variable and $d\Omega_3^2$ denotes the metric on the three-sphere. The corresponding Einstein-Hilbert action is

$$S^{\mathit{eff}} = \frac{1}{G_N} \int \mathrm{d}t \left(-a(t) \left(\frac{\mathrm{d}a(t)}{\mathrm{d}t} \right)^2 - a(t) + \lambda a^3(t) \right). \tag{33}$$

If no four-volume constraint is imposed, λ is the cosmological constant. If the four-volume is fixed to V_4, such that the discussion parallels the computer simulations reported above, λ should be viewed as a Lagrange multiplier enforcing a given size of the universe. In the latter case we obtain the same effective action as that extracted from the Monte Carlo simulations in (31), *up to an overall sign*, due to the infamous conformal divergence of the classical Einstein action evident in (33). From the point of view of the *classical* equations of motion this overall sign plays of course no role. Let us compare the two potentials relevant for the

calculation of semiclassical Euclidean solutions associated with the actions (33) and (31). The "potential"[10] is

$$V(a) = -a + \lambda a^3, \tag{34}$$

and is shown in Fig. 6, without and with small-a modification, for the standard minisuperspace model and our effective model, respectively. The quantum-

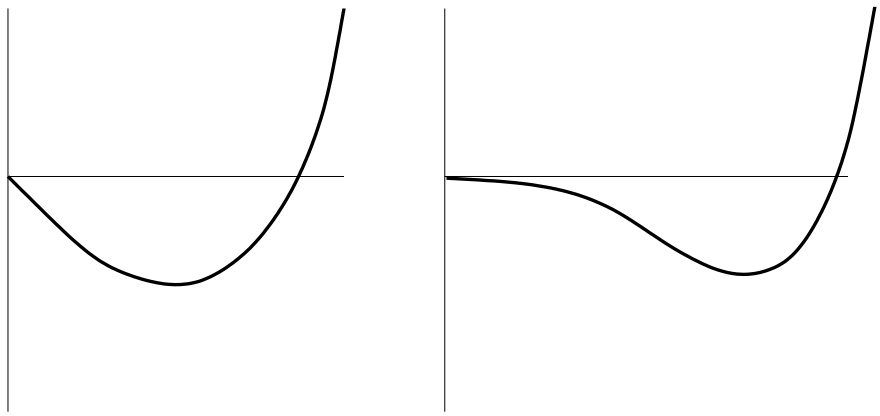

Figure 6: The potential $V(a)$ of (34) underlying the standard minisuperspace dynamics (left) and the analogous potential in the effective action obtained from the full quantum gravity model, with small-a modification due to quantum fluctuations (right).

induced difference for small a is important since the action (31) admits a classically stable solution $a(t) = 0$ which explains the "stalk" observed in the computer simulations (see Fig. 2). Moreover, it is appropriate to speak of a Euclidean "bounce" because $a = 0$ is a local maximum. If one therefore *naively* turns the potential upside down when rotating back to Lorentzian signature, the metastable state $a(t) = 0$ can tunnel to a state where $a(t) \sim V_4^{1/4}$, with a probability amplitude per unit time which is (the exponential of) the Euclidean action.

In order to understand how well the semiclassical action (31) can reproduce the Monte Carlo data, that is, the correlator (10) of Fig. 3, we have solved for the semiclassical bounce using (31), and presented the result as the black curve in Fig. 3. The agreement with the real data generated by the Monte Carlo simulations is clearly perfect.

The picture emerging from the above for the effective dynamics of the scale factor resembles that of a universe created by tunneling from nothing (see, for

[10]To obtain a standard potential – without changing "time" – one should first transform to a variable $x = a^{\frac{3}{2}}$ for which the kinetic term in the actions assumes the standard quadratic form. It is the resulting potential $\tilde{V}(x) = -x^{2/3} + \lambda x^2$ which in the case of (31) should be modified for small x such that $\tilde{V}'(0) = 0$.

example, [20, 21, 22]), although the presence of a preferred notion of time makes our situation closer to conventional quantum mechanics. In the set-up analyzed here, there is apparently a state of vanishing spatial extension which can "tunnel" to a universe of finite linear extension of order $a \sim V_4^{1/4}$. Adopting such a tunneling interpretation, the action of the bounce is

$$S_{V_4}^{eff} \sim \frac{V_4^{1/2}}{G_N}, \tag{35}$$

and the associated probability per unit proper time for the tunneling given by

$$P(V_4) \sim e^{-S_{V_4}^{eff}}. \tag{36}$$

4 Discussion

Causal dynamical triangulations (CDT) provide a regularized model of quantum gravity, which uses a class of piecewise linear geometries of Lorentzian signature (made from flat triangular building blocks) to define the regularized sum over geometries. The model is background-independent and has a diffeomorphism-invariant cut-off. For certain values of the bare gravitational and cosmological coupling constants we have found evidence that a continuum limit exists. The limit has been analyzed by rotating the sum over geometries to Euclidean signature, made possible by our use of piecewise linear geometries. The geometries included in the sum thus originate from Lorentzian-signature spacetimes, a class different from (and smaller than) the class of geometries one would naturally include in a "native" Euclidean path integral. We have concentrated on computing a particular diffeomorphism-invariant quantity, the proper-time propagator, representing the sum over all geometries whose space-like boundaries are separated by a geodesic distance T. The sum over such geometries allows a simple and transparent implementation in terms of the above-mentioned building blocks.

In the Euclidean sector of the model, which can be probed by computer simulations we observe a four-dimensional macroscopic universe that can be viewed as a "bounce". When we integrate out (*after* having constructed the full path integral) all geometric degrees of freedom except for the global scale factor, the large-scale structure of the universe (the bounce) is described by the classical general-relativistic solution for a homogenous, isotropic universe with a cosmological constant on which (small) quantum fluctuations are superimposed. We find this result remarkable in view of the difficulties – prior to the introduction of causal dynamical triangulations – to obtain dynamically any kind of "quantum geometry" resembling a four-dimensional universe. In our construction, the restrictions imposed by causality before rotating to Euclidean signature clearly have played a pivotal role.

A number of issues are being addressed currently to obtain a more complete understanding of the physical and geometric properties of the quantum gravity theory generated by CDT, and to verify in more detail that its classical limit is well defined. Among them are:
(i) A better understanding of the renormalization of the bare coupling constants in the continuum limit, with the currently favoured scenario being that of asymptotic safety [23]. There are very encouraging agreements between the results of CDT and those of a Euclidean renormalization group approach [24] (see [25] for older, related work). In particular, both approaches obtain a scale-dependent spectral dimension which varies between four on large and two on short scales.
(ii) An identification and measurement of the "transverse" gravitational degrees of freedom, to complement the information extracted so far for the scale factor only. For background-independent and coordinate-free formulations like CDT we still lack a simple and robust prescription for how to extract information about the transverse degrees of freedom, a quantity analogous to the Wilson loop in non-abelian gauge theories.
(iii) The inclusion of matter fields in the computer simulations. Of particular interest would be a scalar field, playing the role of an inflaton field. While it is straightforward to include a scalar field in the formalism, it is less obvious which observables one should measure, being confined to the Euclidean sector of the theory. Based on a well-defined CDT model for the nonperturbative quantum excitations of geometry and matter, moving the discussion of quantum cosmology and various types of inflation from handwaving arguments into the realm of quantitative analysis would be highly desirable and quite possibly already within reach.

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