Poisson Point Processes

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October 11, 2018

Recall a realization of the basic noise for Gaussian processes looked like that in Figure 1. Now, arrows are either muted or (rarely) point up. See Figure 2.

1 Motivation

Suppose in some space X we lay down a large number of LED lights, each with their own battery, with density given by a σ-finite measure μ. We do this in a way so that, for each region $\mathrm{A} \subset \mathrm{X}$, we put down about $\mathrm{M}\mu(\mathrm{A})$ lights in that region, where M is some large number. Independently we turn on each light with probability M^{-1}, and leave off otherwise.

We would like to answer the following question: how many lights in A are on? To that end, let $\mathrm{N}(\mathrm{A})$ denote the number of lights on in A and compute

$$\mathbb{E}\left[\mathrm{N}(\mathrm{A})\right] = \mathbb{E}\left[\sum_{\text{lights in A}} \mathbb{1}_{\{\text{lights on}\}}\right] = \sum_{\text{lights in A}} \mathbb{P}\left\{\text{light is on}\right\} = \mathrm{M}\mu(\mathrm{A})\left(\frac{1}{\mathrm{M}}\right) = \mu(\mathrm{A}).$$

Thus μ gives the expected density for the set of lights that are on in A. By construction, we know $\mathrm{N}(\mathrm{A}) \sim \mathrm{Binom}\left(\mathrm{M}\mu(\mathrm{A}), \mathrm{M}^{-1}\right)$, and hence $\mathrm{N}(\mathrm{A}) \sim \mathrm{Pois}(\mu(\mathrm{A}))$. To see this, put $\mathrm{L} = \mathrm{M}\mu(\mathrm{A})$ and observe,

$$\begin{aligned}
\mathbb{P}\left\{\mathrm{N}(\mathrm{A}) = n\right\} &= \binom{\mathrm{L}}{n}\left(\frac{1}{\mathrm{M}}\right)^{n}\left(1 - \frac{1}{\mathrm{M}}\right)^{\mathrm{L}-n} \\
&= \frac{\mathrm{L}(\mathrm{L}-1)\cdots(\mathrm{L}-n+1)}{n!\mathrm{M}^{n}}\left(1 - \frac{1}{\mathrm{M}}\right)^{\mathrm{L}-n} \\
&\simeq \frac{1}{n!}\left(\frac{\mathrm{L}}{\mathrm{M}}\right)^{n}\exp\left(-\frac{\mathrm{L}}{\mathrm{M}}\right) + \mathcal{O}\left(\frac{1}{\mathrm{M}}\right) \\
&\simeq \frac{\mu(\mathrm{A})^{n}}{n!}e^{-\mu(\mathrm{A})}
\end{aligned}$$

This motivates the following definition.

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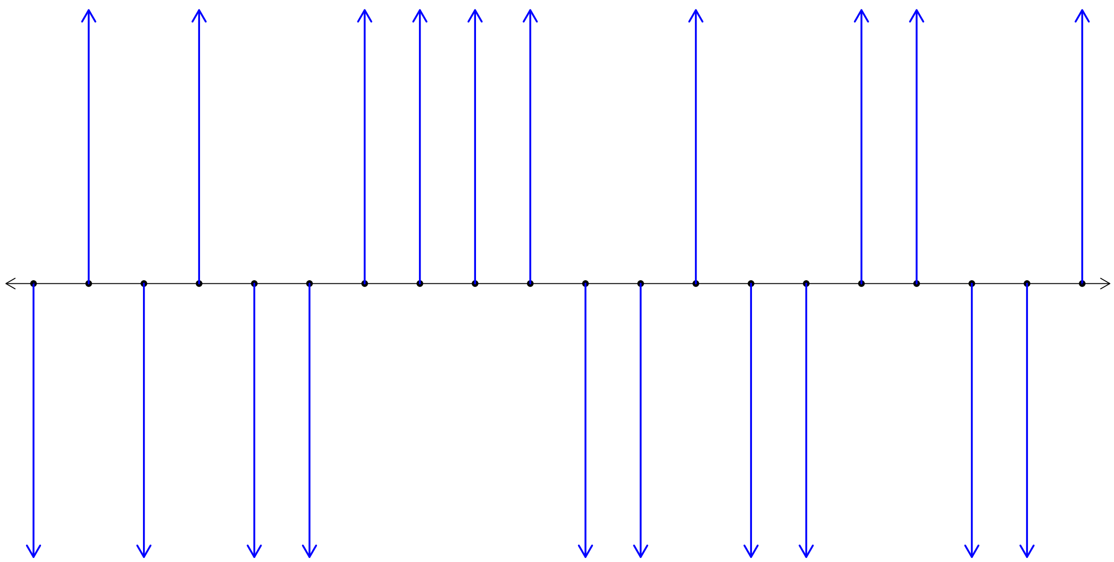

Figure 1: A realization of the basic noise used to construct a Gaussian process.

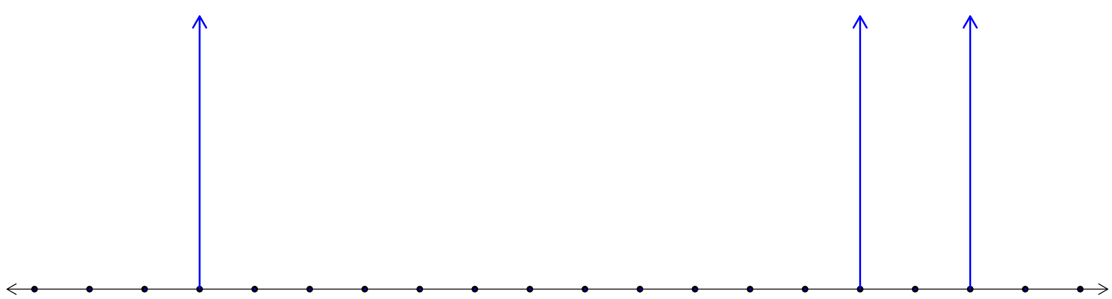

Figure 2: A realization of the basic noise used to construct a Poisson process.

DEFINITION 1.1. Let μ be a σ-finite measure on some space X. A *Poisson Point Process* (PPP) on X with *mean measure* (or, *intensity*) μ is a random point measure N such that:

(a) For any Borel set $\mathrm{A} \subset \mathrm{X}$, we have $\mathrm{N}(\mathrm{A}) \in \mathbb{Z}_{\geq 0}$ and $\mathrm{N}(\mathrm{A}) \sim \mathrm{Pois}(\mu(\mathrm{A}))$, i.e.

$$\mathbb{P}\{\mathrm{N}(\mathrm{A}) = n\} = \frac{\mu(\mathrm{A})^n}{n!} e^{-\mu(\mathrm{A})}. \tag{1.1}$$

(b) If A and B are disjoint Borel subsets of X, then N(A) and N(B) are independent random variables.

Recall a point measure is just a measure whose mass is atomic. That is, if $\{x_i\} \subset \mathrm{X}$ then a point measure is of the form

$$\mu = \sum_i a_i \delta_{x_i}$$

where δ_x is the unit point mass at x.

2 PPP Properties

It is sometimes useful to think of a PPP as a random collection of points. With this in mind, we list some important properties of $\mathrm{N} \sim \mathrm{PPP}(\mu)$ on some space X:

- *Enumeration:* It is always possible to enumerate the points of N, i.e. there is a random collection of points $\{x_i\} \subset \mathrm{X}$ such that

$$\mathrm{N} = \sum_i \delta_{x_i}.$$

- *Mean measure:* If $f : \mathrm{X} \to \mathbb{R}$ then

$$\mathbb{E}\left[\int f(x)\, d\mathrm{N}(x)\right] = \int f(x) d\mu(x). \tag{2.1}$$

  Note: This is a more general property of point processes, as any point process has a mean measure. To see (2.1) holds without needing N to be a *Poisson* point process, let f be a simple function, i.e.

$$f(x) = \sum_{i=1}^{n} f_i \mathbb{1}_{\mathrm{A}_i}(x), \qquad \text{where} \qquad \mathrm{X} = \bigsqcup_i \mathrm{A}_i.$$

  Then we compute

$$\mathbb{E}\left[\int_{\mathrm{X}} f(x)\, d\mathrm{N}(x)\right] = \mathbb{E}\left[\sum_i f_i \mathrm{N}(\mathrm{A}_i)\right] = \sum_i f_i \mathbb{E}\left[\mathrm{N}(\mathrm{A}_i)\right] = \sum_i f_i \mu(\mathrm{A}_i) = \int_{\mathrm{X}} f(x)\, d\mu(x).$$

- *Thinning:* Independently discard each point of N with probability $1-p(x)$ for a point at $x \in \mathrm{X}$. The result is a $\mathrm{PPP}(\nu)$, where

$$\nu(\mathrm{A}) = \int_{\mathrm{A}} p(x)\, d\mu(x). \tag{2.2}$$

In other words, if $\mathrm{N} = \sum_i \delta_{x_i}$ and $a_i = 1$ with probability $p(x_i)$ and $a_i = 0$ otherwise, then

$$\widetilde{\mathrm{N}} = \sum_i a_i \delta_{x_i} \sim \mathrm{PPP}(\nu).$$

- *Additivity:* If $\mathrm{N}_1 \sim \mathrm{PPP}(\mu_1)$ and $\mathrm{N}_2 \sim \mathrm{PPP}(\mu_2)$ are independent on X, then $\mathrm{N}_1 + \mathrm{N}_2 \sim \mathrm{PPP}(\mu_1 + \mu_2)$. That is to say, if $\mathbb{P}\{\mathrm{X} = n\} = \frac{\lambda^n}{n!} e^{-\lambda}$ and $\mathbb{P}\{\mathrm{Y} = n\} = \frac{\nu^n}{n!} e^{-\nu}$ are independent, then

$$\mathbb{P}\{\mathrm{X} + \mathrm{Y} = n\} = \frac{(\lambda+\nu)^n}{n!} e^{-(\lambda+\nu)}.$$

- *Labeling:* For each point in a PPP, associate an independent label from a space Y according to some distribution ν. Let $\mathrm{N} = \sum_i \delta_{x_i}$ for $\{x_i\} \subset \mathrm{X}$ and let $\mathrm{G}_1, \mathrm{G}_2, \ldots \in \mathrm{Y}$ be iid with density ν. Then

$$\overline{\mathrm{N}} := \sum_i \delta_{(x_i, \mathrm{G}_i)} \sim \mathrm{PPP}(\mu \times \nu) \tag{2.3}$$

on $\mathrm{X} \times \mathrm{Y}$.

3 Examples

Henceforth, let λ denote Lebesgue measure.

Example 3.1. Let $\mathrm{N} \sim \mathrm{PPP}(\lambda)$ on $\mathbb{R}_{\geq 0}$, where λ is Lebesgue measure.

(a) How far until the first light?

(b) Suppose each light is independently either red or green with probability $\frac{1}{2}$. How far until the first red light?

Solution. Let $\mathrm{N} = \sum_i \delta_{x_i}$ and put $\mathrm{T} = \min\{x_i\}$. Using (1.1) we compute

$$\mathbb{P}\{\mathrm{T} > t\} = \mathbb{P}\{\mathrm{N}([0, t]) = 0\} = e^{-t}.$$

This solves part (a). For the colorblind readers, this also solves part (b).

Now let $\{\tilde{x}_i\} \subset \{x_i\}$ be the (random) set of red lights and define $\widetilde{\mathrm{N}} = \sum_i \delta_{\tilde{x}_i}$, the point process for the red lights from N. By the thinning property (2.2), $\widetilde{\mathrm{N}} \sim \mathrm{PPP}\left(\frac{1}{2}\lambda\right)$. Similarly define $\widetilde{\mathrm{T}} = \min\{\tilde{x}_i\}$ and observe

$$\mathbb{P}\left\{\widetilde{\mathrm{T}} > t\right\} = \mathbb{P}\left\{\widetilde{\mathrm{N}}([0,t]) = 0\right\} = e^{-t/2},$$

thus (b) is solved. ♦

<u>Example</u> 3.2. Rain falls for 10 minutes on a large patio at a rate of $\nu = 12$ drops per minute per square meter. Each drop splatters to a radius $\mathrm{R} \sim \mathrm{Exp}(100)$, so that $\mathbb{E}[\mathrm{R}] = 1$ cm, with a thickness of 0.001 meters. Assume the set of locations of the raindrops is a PPP.

(a) What is the mean and variance of the total amount of water falling on a square with area 1 m^2?

(b) A very small ant is running around the patio. See Figure 3.1. What is the chance the ant gets hit?

<u>Solution</u>. Let $\mathrm{N} = \sum_i \delta_{(x_i, y_i)}$ where (x_i, y_i) is the center of the ith drop. Let us restrict to $[0,1]^2$ and take $\mathrm{N} \sim \mathrm{PPP}(\nu\lambda)$. Let M denote the number of drops in $[0,1]^2$, so that $\mathrm{M} = \mathrm{N}([0,1]^2) \sim \mathrm{Pois}(\nu)$. Then the total volume V is

$$\mathrm{V} = \sum_{i=1}^{\mathrm{M}} \frac{\pi}{10^3} \mathrm{R}_i^2$$

where R_i is the radius of the ith drop. Note this is a sum of random variables where the number of terms is also a random variable. Thus we use Wald's equation (3.2) to obtain

$$\mathbb{E}[\mathrm{V}] = \frac{\pi}{10^3}\mathbb{E}[\mathrm{M}]\mathbb{E}[\mathrm{R}_1^2] = \frac{\pi}{10^3} \cdot \nu \cdot \frac{2}{100^2} = \frac{2\pi}{10^7}\nu \tag{3.1}$$

The second step in (3.1) was obtained from the fact that an exponentially distributed random variable X with mean β^{-1} has higher moments given by

$$\mathbb{E}[\mathrm{X}^n] = \frac{n!}{\beta^n}.$$

This is proved by an iterated application of integration by parts, and the result gives rise to

$$\mathrm{var}[\mathrm{X}^n] = \mathbb{E}\left[\mathrm{X}^{2n}\right] - \mathbb{E}[\mathrm{X}^n]^2 = \frac{(2n)! - (n!)^2}{\beta^{2n}}.$$

The $n = 2$ case will turn out to be useful when computing the variance of V.

Indeed, to compute the variance we utilize the variance decomposition formula. Observe,

$$\begin{aligned}
\operatorname{var}[V] &= \mathbb{E}\left[\operatorname{var}[V \mid M]\right] + \operatorname{var}\left[\mathbb{E}[V \mid M]\right] \\
&= \mathbb{E}\left[M\left(\frac{\pi}{10^3}\right)^2 \operatorname{var}(R^2)\right] + \operatorname{var}\left[M\left(\frac{\pi}{10^3}\right)\mathbb{E}[R^2]\right] \\
&= \nu\left(\frac{\pi}{10^3}\right)^2\left(\frac{20}{100^4}\right) + \nu\left(\frac{\pi}{10^3}\right)^2\left(\frac{2}{100^2}\right)^2 \\
&= \left(\frac{\pi}{10^3}\right)^2\left(\frac{24}{100^4}\right)\nu.
\end{aligned}$$

This solves part (a).

Now, part (b) can be solved by way of the labeling property. Here, we use the radius R_i of the ith drop to label the point (x_i, y_i). Define a measure μ on $X := \mathbb{R}^2 \times [0, \infty)$ by

$$\mu(A) = \int_A 100\nu \exp(-100r)\, dxdydr.$$

So X is the (closed) upper half plane in $\mathbb{R}^3$ where the third coordinate is a realization of R. By the labeling property (2.3), $\overline{N} := \sum_i \delta_{(x_i, y_i, R_i)} \sim \mathrm{PPP}(\mu)$ on X. For the ant to remain dry, any drop with radius r must land outside the circle of radius r centered at the ant. Viewed from the space X, we want to integrate over the cone with its tip at the ant, whose horizontal cross-section at height r is a circle of radius r. From this we compute

$$\mathbb{P}\{\text{ant is dry}\} = \mathbb{P}\left\{\overline{N}(A) = 0\right\} = \exp(-\mu(A)) = \exp\left(-100\pi\nu\int_0^\infty r^2 e^{-100r}\, dr\right) = \exp\left(-\frac{\pi\nu}{5000}\right).$$

Plugging in $\nu = 12$ yields $\mathbb{P}\{\text{ant is dry}\} \approx 0.9925$, so either the ant has an umbrella or the rain is very light. ♦

3.1 Wald's Equation

The following is the statement of Wald's equation, taken from Wikipedia[1].

<u>Theorem</u> 3.3 (Wald's Equation). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued, independent and identically distributed random variables and let N be a nonnegative integer-valued random variable that is independent of the sequence $(X_n)_{n\in\mathbb{N}}$. Suppose that N and the X_n have finite expectations. Then*

$$\mathbb{E}\left[\sum_{i=1}^{N} X_i\right] = \mathbb{E}[N]\,\mathbb{E}[X_1]. \tag{3.2}$$

[1] The proof is also on Wikipedia.

Figure 3: A realization of the ant from Example 3.2. Looks like he had an umbrella after all.