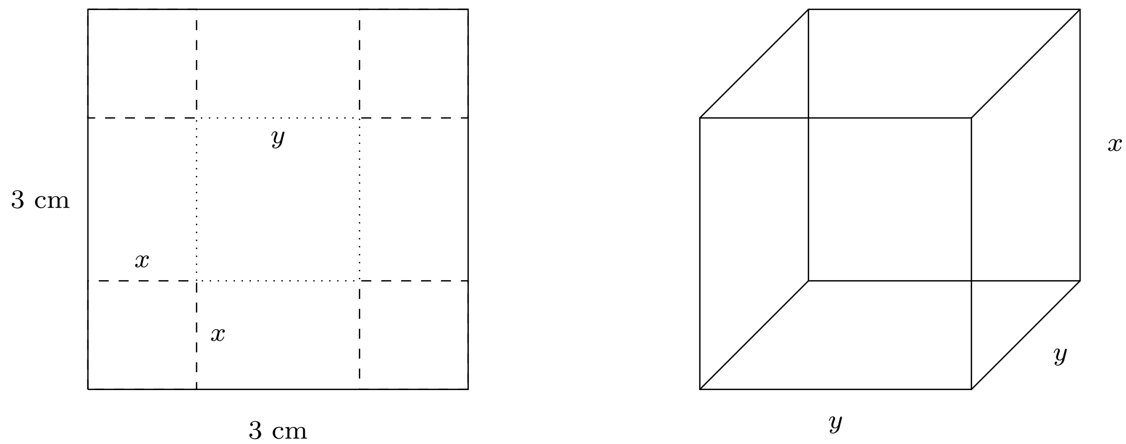
A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have. Let x denote the length of the side of the square being cut out, and let y denote the length of the base.

To start, its always important to visualize what the question is asking, so here is the diagram we drew today.



Now, we are trying to determine the *maximum* volume of the cube given the restrictions on the original piece of cardboard, and because we are able to vary the length, width, and height of the cube, we should write it in terms of the variables, namely,

$$V(x, y) = xy^2. \tag{1}$$

Now, we want to maximize this equation, and this is done by taking the derivative and setting it equal to 0. However, we need to write (1) in terms of either x or y, not both. This is done by looking at the original piece of cardboard; we see that $3 = 2x + y$, and we rewrite this in terms of y as $y = 3 - 2x$. Substituting this back into equation (1) and simplifying gives us

$$V(x) = 4x^3 - 12x^2 + 9x \tag{2}$$

Differentiating equation (2) gives us

$$V'(x) = 12x^2 - 24x + 9, \tag{3}$$

and now we must set this equal to 0 and determine the values of x (because this is a quadratic equation, there will be two solutions). When we do this, we can simplify enough to see that

$$\left(x - \frac{1}{2}\right)\left(x - \frac{3}{2}\right) = 0,$$

and so our two possible solutions are $x = \frac{1}{2}$ and $x = \frac{3}{2}$ (we can see this in hindsight, but if you cannot do this easily on another problem, remember to use the quadratic formula). We now have to determine which of these points is a max and which is a min, so to do this, we will check the concavity. We start by differentiating equation (3) to get

$$V''(x) = 24x - 24, \tag{4}$$

and plugging in our two possible values gives us

$$V''\left(\frac{1}{2}\right) = 24\left(\frac{1}{2}\right) - 24 = -12$$
$$V''\left(\frac{3}{2}\right) = 24\left(\frac{3}{2}\right) - 24 = 12.$$

When $x = \frac{1}{2}$, the concavity is negative, indicating that it is a max, so this is the value that will maximize the cube's volume. From this, we can see that $y = 3 - 2\left(\frac{1}{2}\right) = 2$, so to maximize the cube's volume, we must choose the dimensions $x = \frac{1}{2}$ and $y = 2$. Finally, to determine the cubes maximum volume, we plug these dimensions back into equation (1):

$$V_{MAX} = (2)^2\left(\frac{1}{2}\right) = 2,$$

so the maximum volume of the cube is 2 cm^3.