Test Question

An oil company will drill a succession of holes in a given area to find a productive well. The probability of finding a productive well on any drilled hole is p, where $0 < p < 1$. Let Y be the number of holes drilled until a productive well is found.

(a) What is the p.m.f. of Y, i.e. identify the possible values for Y and identify the probability of each one of those values.

(b) If the company can afford to drill at most 3 wells, what is the probability that it will find a productive well?

(c) Let c_1 and c_2 be two positive integers such that $c_1 < c_2$. Show that $P(Y > c_2 \mid Y > c_1) = P(Y > c_2 - c_1)$.

You had no trouble with (a) and (b); the important part going forward is that Y has a geometric distribution, and so its p.m.f. is given by

$$f(y) = \begin{cases} p(1-p)^{y-1}, & y = 1, 2, \ldots, c_1, \ldots, c_2, \ldots \\ 0, & \text{elsewhere.} \end{cases}$$

Part (c) is difficult, we want to show that

$$P(Y > c_2 \mid Y > c_1) = P(Y > c_2 - c_1).$$

For problems like these, it is usually best to simplify each one of these quantities before doing any algebra. Using conditional probability, the left side becomes

$$P(Y > c_2 \mid Y > c_1) = \frac{P(Y > c_2 \cap Y > c_1)}{P(Y > c_1)} = \frac{P(Y > c_2)}{P(Y > c_1)} = \frac{1 - P(Y \leq c_2)}{1 - P(Y \leq c_1)},$$

since if Y is greater than c_2 it will automatically be greater than c_1. Also, it is always useful to convert from $P(X > x)$ to $1 - P(x \leq x)$. Now, the right side becomes

$$\begin{aligned} P(Y > c_2 - c_1) &= 1 - P(Y \leq c_2 - c_1) = 1 - \sum_{y=1}^{c_2 - c_1} p(1-p)^{y-1} \\ &= 1 - p\left[1 + (1-p) + (1-p)^2 + \ldots + (1 - p^{c_2 - c_1 - 1})\right]. \end{aligned}$$

This is not too useful now, but is often good to try and expand what you have as much as possible. That way, if it comes up while you are working on the problem, you know you are getting close.

Back to the left side of the equation; we can simplify to

$$\begin{aligned}
P(Y > c_2 \mid Y > c_1) &= \frac{1 - P(Y \leq c_2)}{1 - P(Y \leq c_1)} = \frac{1 - \sum_{y=1}^{c_2} p(1-p)^{y-1}}{1 - \sum_{y=1}^{c_1} p(1-p)^{y-1}} \\
&= \frac{1 - p\left[1 + (1-p) + (1-p)^2 + \ldots + (1-p)^{c_1-1} + \ldots + (1-p)^{c_2-1}\right]}{1 - p\left[1 + (1-p) + (1-p)^2 + \ldots + (1-p)^{c_1-1}\right]} \\
&= 1 - \frac{p\left[(1-p)^{c_1} + (1-p)^{c_1+1} + (1-p)^{c_2+2} + \ldots + (1-p)^{c_2-1}\right]}{1 - p\left[1 + (1-p) + (1-p)^2 + \ldots + (1-p)^{c_1-1}\right]}.
\end{aligned}$$

This results from us splitting the numerator into two parts: the first part is the sum of all the terms that match the denominator (so it becomes 1), and the second is what remains.

Let's look at the numerator of the fraction, which is

$$p\left[(1-p)^{c_1} + (1-p)^{c_1+1} + (1-p)^{c_2+1} + \ldots + (1-p)^{c_2-1}\right].$$

This kind of looks like part of the reduced sum that we calculated for $P(Y > c_2 - c_1)$, which was given by

$$p\left[1 + (1-p) + (1-p)^2 + \ldots + (1 - p^{c_2-c_1-1})\right],$$

the only difference is that what we have now is multiplied by $(1-p)^{c_1}$. This is good, since we seem to be getting closer to what we want, so we should factor out $(1-p)^{c_1}$ and keep reducing. We then get

$$\begin{aligned}
&P(Y > c_2 \mid Y > c_1) \\
&\quad = 1 - \frac{p\left[(1-p)^{c_1} + (1-p)^{c_1+1} + (1-p)^{c_2+2} + \ldots + (1-p)^{c_2-1}\right]}{1 - p\left[1 + (1-p) + (1-p)^2 + \ldots + (1-p)^{c_1-1}\right]} \\
&\quad = 1 - \frac{p(1-p)^{c_1}\left[1 + (1-p) + (1-p)^2 + \ldots + (1-p)^{c_2-c_1-1}\right]}{1 - p\left[1 + (1-p) + (1-p)^2 + \ldots + (1-p)^{c_1-1}\right]} \\
&\quad = 1 - \frac{(1-p)^{c_1}\sum_{y=1}^{c_2-c_1} p(1-p)^{y-1}}{1 - \sum_{y=1}^{c_1} p(1-p)^{y-1}}.
\end{aligned}$$

It looks like we are getting close, but we still have the $(1-p)^{c_1}$ in the numerator and the entire denominator, so let's try and simplify them. As a reminder, to evaluate a geometric series up to a finite number of terms, we use the formula

$$\sum_{i=1}^{n} cr^i = \frac{c(1-r^n)}{1-r},$$

where $r \neq 1$. So for that geometric series in the denominator, we have $n = c_1$, $c = p$, and $r = (1-p) \neq 1$, so we should be good to go! The series then sums to

$$\sum_{y=1}^{c_1} p(1-p)^{y-1} = \frac{p(1-(1-p)^{c_1})}{1-(1-p)} = \frac{p(1-(1-p)^{c_1})}{p} = 1 - (1-p)^{c_1}.$$

So we can simply $P(Y > c_2 \mid Y > c_1)$ to

$$\begin{aligned}
P(Y > c_2 \mid Y > c_1) &= 1 - \frac{(1-p)^{c_1} \sum_{y=1}^{c_2-c_1} p(1-p)^{y-1}}{1 - \sum_{y=1}^{c_1} p(1-p)^{y-1}} \\
&= 1 - \frac{(1-p)^{c_1} \sum_{y=1}^{c_2-c_1} p(1-p)^{y-1}}{1 - [1-(1-p)^{c_1}]} \\
&= 1 - \frac{(1-p)^{c_1} \sum_{y=1}^{c_2-c_1} p(1-p)^{y-1}}{(1-p)^{c_1}} \\
&= 1 - \sum_{y=1}^{c_2-c_1} p(1-p)^{y-1} = 1 - P(Y \le c_2 - c_1) = P(Y > c_2 - c_1),
\end{aligned}$$

and so we have shown that

$$P(Y > c_2 \mid Y > c_1) = P(Y > c_2 - c_1).$$

Moment-Generating Functions

We also touched on moment-generating functions, useful tools that will make some calculations easier, namely expected values and variances of random variables. For any random variable X with all possible values in the set $\mathcal{D}$, its m.g.f. is defined as

$$M_X(t) = E[e^{xt}] = \sum_{x \in \mathcal{D}} e^{xt} \cdot f(x),$$

and we can use $M_X(t)$ to calculate the r^{th} moment of X, given by

$$E[X^r] = \frac{\mathrm{d}^r}{\mathrm{d}t^r} M_X(0).$$

Going with the example we did today, a random variable X with a geometric distribution has a general p.d.f. given by

$$f(x) = \begin{cases} p(1-p)^{y-1}, & x = 1, 2, 3, \ldots \\ 0, & \text{elsewhere.} \end{cases}$$

It is difficult to find $E[X]$ by our normal procedures, as we saw today; it is given by

$$\begin{aligned}
E[X] &= \sum_{x=1}^{\infty} x \cdot p(1-p)^{x-1} = p \sum_{x=1}^{\infty} x(1-p)^{x-1} = -p \sum_{x=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}p}(1-p)^x \\
&= -p \cdot \frac{\mathrm{d}}{\mathrm{d}p} \sum_{x=1}^{\infty} (1-p)^x = -p \cdot \frac{\mathrm{d}}{\mathrm{d}p}\left[-1 + \sum_{x=0}^{\infty}(1-p)^x\right] = -p \cdot \frac{\mathrm{d}}{\mathrm{d}p}\left[-1 + \frac{1}{p}\right] \\
&= -p\left[-\frac{1}{p^2}\right] = \frac{p}{p^2} = \frac{1}{p}.
\end{aligned}$$

You would never think to evaluate this sum the way we did, but this is an example of where the m.g.f. is incredibly useful. We can find $M_X(t)$ by

$$\begin{aligned} M_X(t) &= \sum_{x=1}^{\infty} e^{xt} \cdot p(1-p)^{x-1} = pe^t \sum_{x=1}^{\infty} (e^t)^{x-1} \cdot (1-p)^{x-1} \\ &= pe^t \sum_{x=1}^{\infty} [e^t(1-p)]^{x-1} = pe^t \left(\frac{1}{1-(1-p)e^t} \right) = \frac{pe^t}{1-(1-p)e^t}. \end{aligned}$$

Differentiating $M_X(t)$ and evaluating at 0 will give us $E[X]$, so

$$\begin{aligned} E[X] &= \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{pe^t}{1-(1-p)e^t} \right]_{t=0} = \frac{(pe^0)(1-(1-p)e^0) - (-(1-p)e^0)(pe^0)}{(1-(1-p)e^0)^2} \\ &= \frac{(p)(p)-(p-1)(p)}{p^2} = \frac{p^2-p^2+p}{p^2} = \frac{p}{p^2} = \frac{1}{p}, \end{aligned}$$

and so we got the same answer is a much easier way.