

We are told that this playlist has 100 songs, where 10 are by the Beatles. If the songs are chosen randomly, what is the probability that the first Beatles song played is *the fifth song*? There are two ways to go about doing this, so we will go through both.

The fact that it has to be the fifth song is a subtle but extremely important point. For a moment, let's not worry about that, and instead focus on a slightly different question: if 10 songs are selected, what is the probability that *any one* of them is by the Beatles (as opposed to having to have the fifth song be the Beatles)?

Let's define A_i as the event that *the i^{th}* song is by the Beatles, and B as the event that *any one* of the five songs is by the Beatles. Because these events are independent and the chances of each A_i occurring is the same, we have

$$\begin{aligned} P(B) &= P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) \\ &= P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5) = 5 \cdot P(A_5). \end{aligned}$$

Since there are 100 songs and we are choosing 10 of them, there are $\binom{100}{5}$ different possible choices of 10 songs. Of the 100 songs, 10 of them are by the Beatles, so there are $\binom{10}{1}$ ways to choose one song by the Beatles. On the other hand, 90 of the songs are *not* by the Beatles, so there are $\binom{90}{4}$ ways to choose 4 songs that are not by the Beatles. As a result, there are $\binom{10}{1} \binom{90}{4}$ ways to choose 5 songs such that only one of them are by the Beatles. Therefore, the probability of choosing 5 songs and having one of them be by the Beatles is

$$P(B) = \frac{\binom{10}{1} \binom{90}{4}}{\binom{100}{5}}.$$

We said before that B is the event that any one of the five songs are by the Beatles, but we are interested in the probability that only the fifth song is by the Beatles. Therefore,

$$P(A_5) = \frac{P(B)}{5} = \frac{1}{5} \cdot \left[\frac{\binom{10}{1} \binom{90}{4}}{\binom{100}{5}} \right] \approx 0.0679.$$

Another way to solve this is to assume that songs *cannot* be repeated. Then the chance of the fifth song being by the Beatles is

$$P(A_5) = \left(\frac{90}{100} \right) \cdot \left(\frac{89}{99} \right) \cdot \left(\frac{88}{98} \right) \cdot \left(\frac{87}{97} \right) \cdot \left(\frac{10}{96} \right) \approx 0.0679.$$

I am not so sure about the second solution; this was how Professor Draguljić solved it, but I do not like the fact that the songs cannot be repeated, when they can in the first.