I wanted to make sure you understood exactly the motivations behind some of the things about power series today, so here are some of the things we went over.

Taylor Series

Most functions $f(x)$ can be represented as an infinite series known as a **power series**, which is written in the form

$$f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \dots, \tag{1}$$

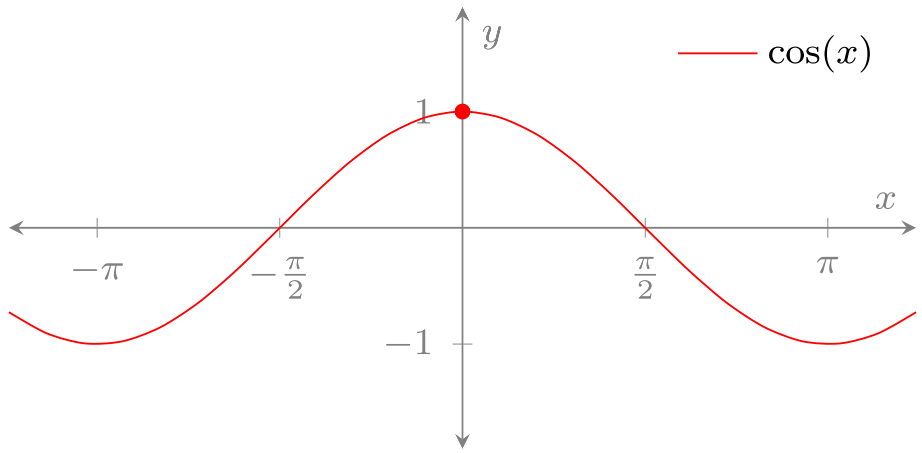
and we say that the power series is **centered at** a. This is done because we want to express the function $f(x)$, which can sometimes be more complicated, and write it as a **polynomial**. Polynomials are generally considered to be "easier functions," in the sense that they are easier to evaluate, differentiate, integrate, and apply other operations to.

For **any** function $f(x)$ that can be represented as a power series, the power series can be written in the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \dots . \end{aligned} \tag{2}$$

In other words, the coefficients C_n from equation (1) can be written as $C_n = \frac{f^{(n)}(a)}{n!}$.

Why are we doing this? It may sort of seem like we pulled this out of thin air, but there is actually a really intuitive way of thinking about it. To start, lets consider the function $f(x) = \cos(x)$, and say that we want to estimate the value of this function at points near $x = 0$.



I'm doing this for the sake of example; we could just use a calculator or something to figure out the values of $\cos(x)$, but if you had a more complicated function (something like $\frac{5x^3}{(x+7)^4}$, for example), it would be better to approximate values of the function by using its Taylor Series.

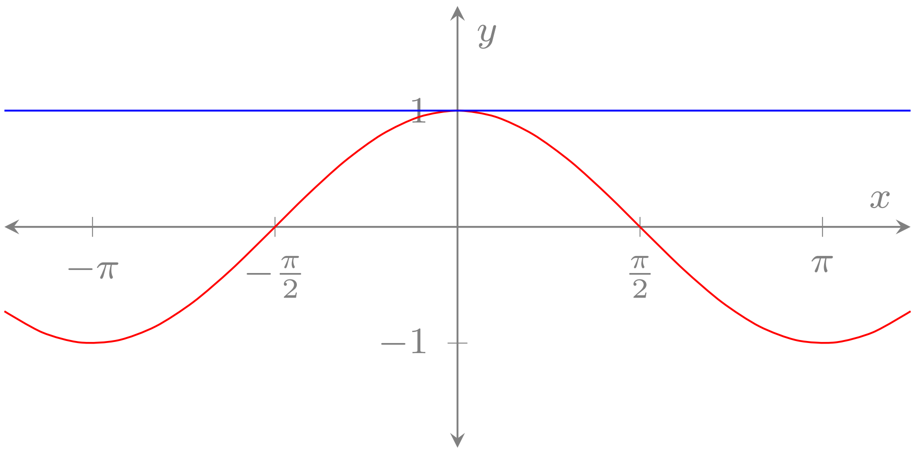
Now, pretend that we do not know the equation for a Taylor series that we wrote in (2), and instead we only know that we want to represent $\cos(x)$ as an infinite polynomial. That is, we want to choose the coefficients C_n such that

$$\cos(x) = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots .$$

To start, what would be a good choice of C_0? It would make sense that at the point $x = 0$, our approximation polynomial equals $\cos(0)$, so when we plug in 0 to the polynomial, we see that

$$\cos(0) = C_0,$$

and so we set $C_0 = 1$.



As we can see, as we move away from the point $x = 0$, our approximate polynomial is no longer a good estimate for $\cos(x)$. This is because we are not accounting for the rate of change (i.e. the derivative) of $\cos(x)$. So to account for this, we want the derivative of our approximate polynomial to equal the derivative of $\cos(x)$ at the point $x = 0$.

When we differentiate both sides, we see that we want

$$-\sin(x) = C_1 + 2C_2 x + 3C_3 x^2 + \dots ,$$

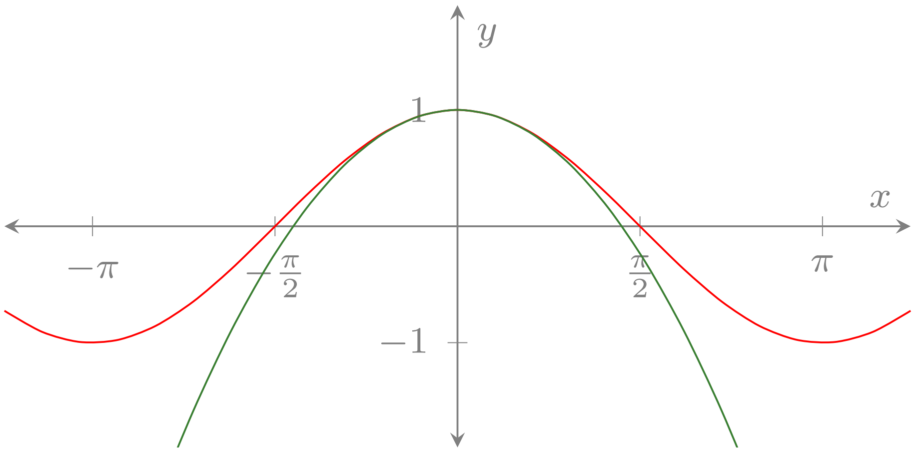
and so at 0, $C_1 = -\sin(0) = 0$. Observe that when we evaluate this at $x = 0$, all of the other terms that have an x will disappear.

We repeat this step now as many times as we need to; at the point $x = 0$, we want every derivative of $\cos(x)$ to be the same at the derivative of the approximate polynomial.

If we differentiate again and evaluate at the point $x = 0$, we see that we want

$$-\cos(0) = 2C_2,$$

and so $C_2 = -\frac{1}{2}$.



Things are looking promising now, as our polynomial is becoming a better approximation of $\cos(x)$ around the point $x = 0$. All we have to do is keep iterating this step, and by matching up the derivatives of the polynomial with the derivative of $\cos(x)$, our approximation will get better and better.

Let's keep going! If we differentiate again and evaluate at 0, we see that we want $\sin(0) = 3 \cdot 2 \cdot 1 \cdot C_3$, so $C_3 = 0$. The reason I wrote the terms like this is because this is where the $n!$ in equation (2) comes from; every time you differentiate the polynomial, the power comes down to the front, and so we have $f'''(x) = 3!C_3$, and so $C_3 = \frac{f'''(x)}{3!}$.

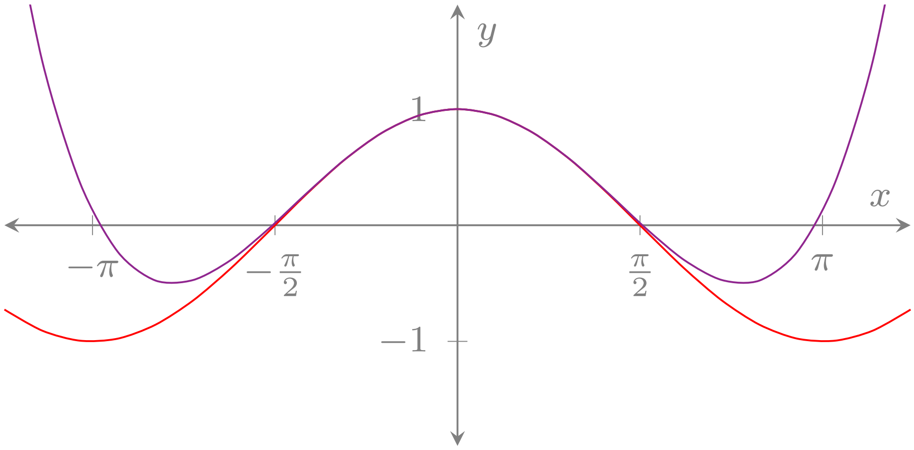
Let's look at one more step before we wrap up this example an generalize it. Differentiating yet again and evaluating at the point $x = 0$ shows that we want

$$\cos(0) = 4 \cdot 3 \cdot 2 \cdot 1 \cdot C_4,$$

and so $C_4 = \frac{1}{24}$.

Now, let's recap where we are so far. We have been determining the coefficients of an infinite polynomial that give us the best approximation for $\cos(x)$ at the point $x = 0$. So far, we have determined the following coefficients: $C_0 = 1, C_1 = 0, C_2 = -\frac{1}{2}, C_3 = 0$, and $C_4 = \frac{1}{24}$. So if we wrote out our approximation polynomial with the coefficients we determined so far, we would be able to approximate $\cos(x)$ around $x = 0$ by using

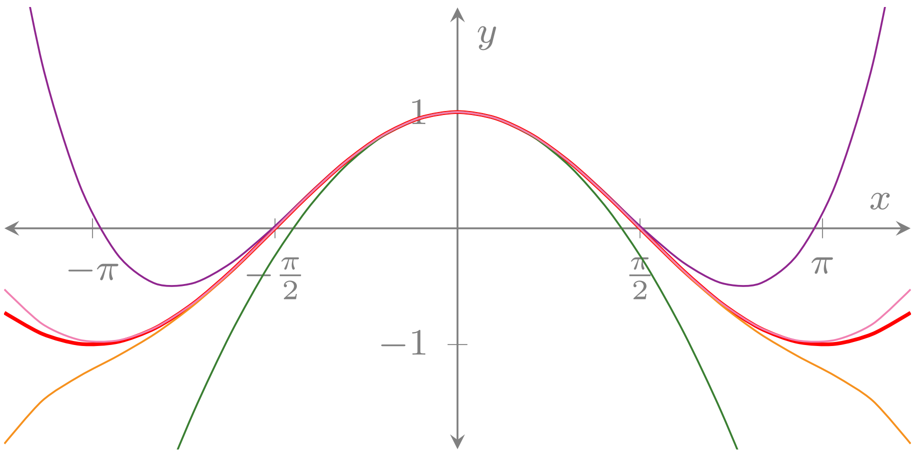
$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$



We saw before that $C_3 = \frac{f'''(x)}{3!}$, and this is what happens when we try and do this step for any C_n. after differentiating the polynomial n times, and evaluating the result at $x = 0$ (at least in this case), we have $n! \cdot C_n = f^{(n)}(0)$, and so

$$C_n = \frac{f^{(n)}(0)}{n!},$$

and this is exactly where the coefficients for a Taylor series come from.



You have to admit that this looks kind of cool, but also notice that the pink line (the approximation polynomial to the eighth degree) is almost equal to $\cos(x)$. And this is exactly the point of all this; as we add more terms of the polynomial, we get a better and better approximation of $\cos(x)$.

What is the actual formula for this polynomial? The key is to determining the pattern that arises when you differentiate $\cos(x)$; we know that the derivatives of $\cos(x)$ are going to "cycle" through each other, because

$$\begin{aligned}
f(x) &= \cos(x),\\
f'(x) &= -\sin(x),\\
f''(x) &= -\cos(x),\\
f'''(x) &= \sin(x),\\
f^{(4)}(x) &= \cos(x),
\end{aligned}$$

and so on. Because $\sin(0) = 0$, every odd derivative is going to disappear, and because $\cos(0) = 1$, every even derivative is going to oscillate between -1 and 1. So we need to account for the oscillation and some of the disappearing in our sum when determining the C_n's.

To make the pattern clearer, when we write out the polynomial's first five nonzero terms, we see that

$$\cos(x) + 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} + \dots.$$

If we leave it like this, the pattern really isn't clear, so let's re-write this as

$$\cos(x) = \frac{x^0}{0!} + \frac{(-1)x^2}{2!} + \frac{x^4}{4!} + \frac{(-1)x^6}{6!} + \frac{x^8}{8!} + \dots.$$

It looks like that every even term of the polynomial is negative, the factorials in the denominators are each for every even number, and the powers of x are only even as well. So, we see that the Taylor series expansion for $\cos(x)$, centered at 0, is

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

Now, what happens when the power series is not centered at 0? For example, what if we wanted to represent $\cos(x)$ as a power series centered at $\frac{\pi}{2}$?

I am not going to go through the steps again, but the main idea is that instead of matching up the polynomial's derivatives at 0, we are now matching them up at the point $x = \frac{\pi}{2}$, and instead of having x^n, we have $(x - \frac{\pi}{2})^n$. This is because we want the cancellation at the point $x = \frac{\pi}{2}$ (see the red text from before), and we cannot have that unless we make this adjustment.

As it turns out, the power series of $\cos(x)$ will converge for all values of x. In other words, it has an infinite radius of convergence. We can see this using the ratio test:

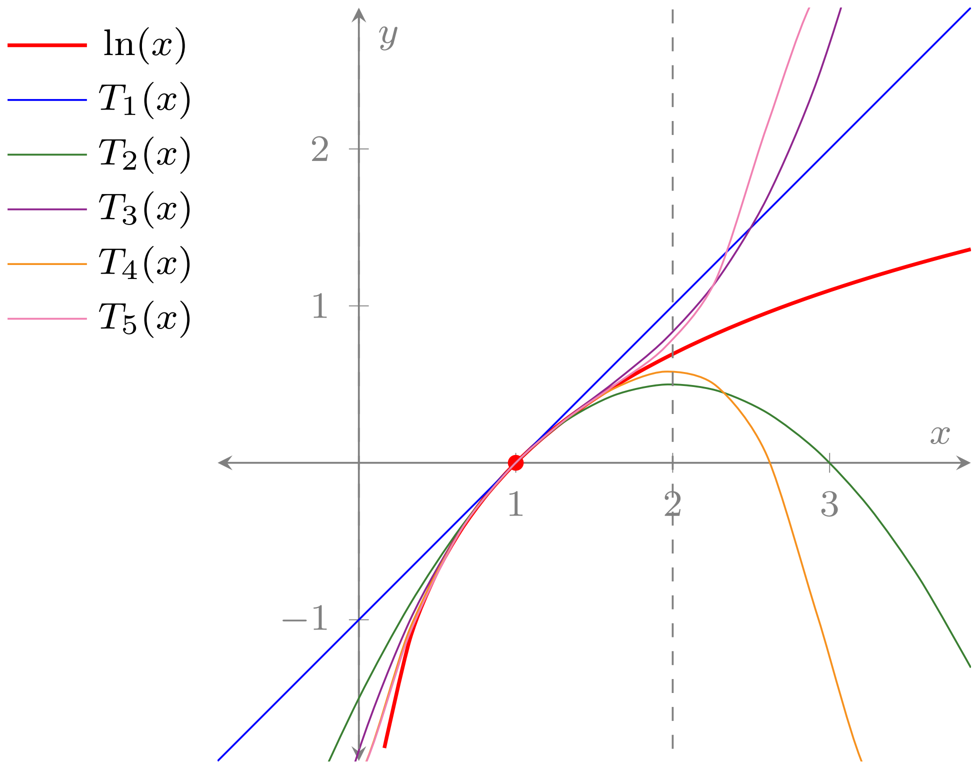
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!}}{\frac{(-1)^n x^{2n}}{(2n)!}} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+2}(2n)!}{x^{2n}(2n+2)!} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right|$$

$$= \lim_{n\to\infty} |x^2| \cdot \left| \frac{1}{(2n+2)(2n+1)} \right| = \lim_{n\to\infty} |x^2| \cdot \lim_{n\to\infty} \left| \frac{1}{(2n+2)(2n+1)} \right| = |x^2| \cdot 0 = 0$$

Because $0 < 1$ for any value of x, this means that the Taylor Series for $\cos(x)$ will converge for any x. In other words, the Taylor Series of $\cos(x)$ can be used to approximate $\cos(x)$ for any value of x.

Here is another example: let's try and determine the Taylor series for the function $f(x) = \ln(x)$ centered at the point $x = 1$. You can try figuring this out yourself by determining the pattern that arises when you keep differentiating $\ln(x)$ and evaluating at 1; the Taylor series is given by

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots.$$



Now, for notation purposes, I wrote $T_1(x)$ to represent the Taylor series when you only have the first term added to it, $T_2(x)$ when you have the first and second terms, and so on.

The point of this is to show how the Taylor series evolves as you add more of the terms to it. We saw that the Taylor series of $\cos(x)$ became a better and better approximation of $\cos(x)$ *for any value of* x as more terms of the polynomial were added, and we said that this was because the Taylor series has an infinite radius of convergence. It turns out that this is not the case for $\ln(x)$; using the ratio test, you can show that the Taylor series for $\ln(x)$ (when centered at the point $x = 1$) has an interval of convergence given by $(0, 2]$, since its radius of convergence is 1 (you would also have to check for endpoints).

Trying to visualize this is a bit tricky, as a Taylor series is an infinite sum, and we are looking at only a finite number of terms of the polynomial. But by looking at the point $x = 2$, we can start to see a pattern; for any x that is less that 2, the Taylor series gets closer and closer to the actual value of $\ln(x)$ as we include more and more terms, and for any x that is greater than 2, the Taylor series never really approaches the actual value of $\ln(x)$ (or any value at all, for that matter). And as it turns out, if you were to include an infinite amount of terms, the Taylor series would actually diverge for values of x greater than 2.

If you want to see where I learned this from (and some beautiful animations), check out *this video.*

Example

Let's say that we want to determine the power series representation of

$$f(x) = \frac{5x^2}{(x-6)^2}.$$

As you now know, you could take the derivative of this function and determine the power series using the Taylor polynomials. But there is another way: to modify a known power series. To start, lets consider the function

$$f(x) = \frac{1}{1-\frac{x}{6}} = \sum_{n=0}^{\infty} \left(\frac{x}{6}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{6^n}. \tag{3}$$

This is a geometric series with $a = 1$ and $r = \frac{x}{6}$, and this is nice because we can easily determine the radius of convergence. Since we need $\left|\frac{x}{6}\right| < 1$, we can see that $|x| < 6$. This is good because applying an operation to the power series will not change the radius of convergence.

Now, let's multiply both sides of (3) by $\frac{1}{6}$, which gives us

$$\frac{1}{6}\left(\frac{1}{1-\frac{x}{6}}\right) = \frac{1}{6}\left(\sum_{n=0}^{\infty} \frac{x^n}{6^n}\right)$$

$$\frac{1}{6-x} = \sum_{n=0}^{\infty} \frac{x^n}{6^{n+1}} \tag{4}$$

Now, if we differentiate both sides of (4), we get

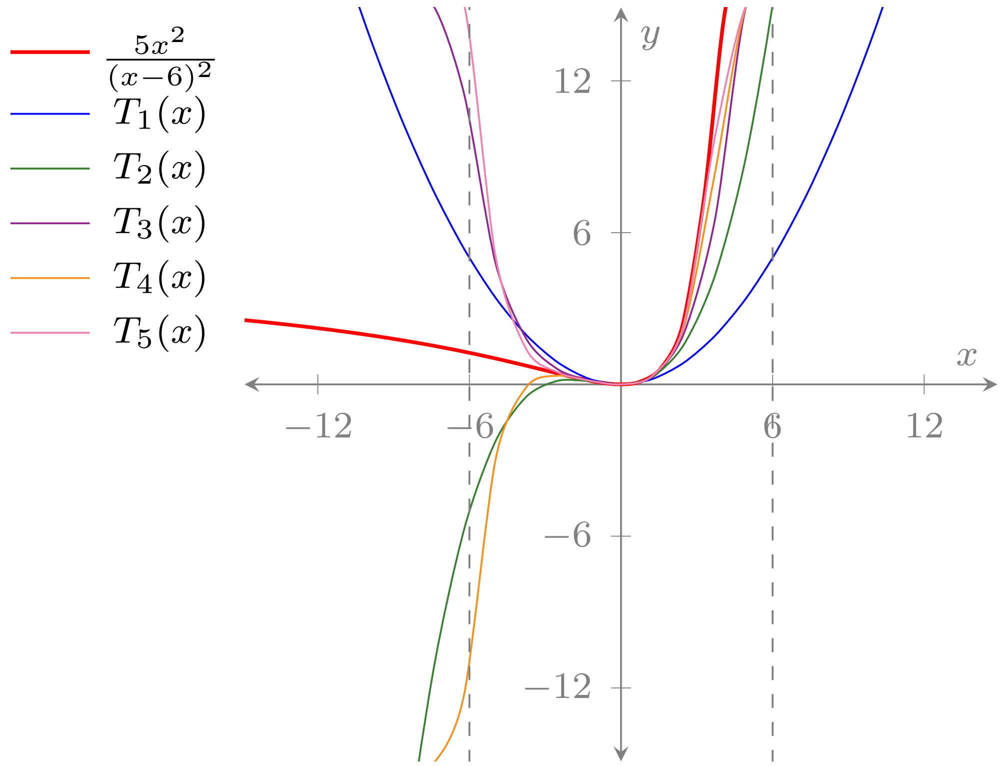
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{6-x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty} \frac{x^n}{6^{n+1}}\right)$$

$$\frac{1}{(6-x)^2} = \frac{1}{(x-6)^2} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{6^{n+1}} \tag{5}$$

A subtle note, $(6-x)^2 = (x-6)^2$, and this is true for any difference squared (i.e. $(a-b)^2 = (b-a)^2$ for all a and b). Also note that we are starting the polynomial at $n = 1$, not $n = 0$; this is because if we start with $n = 0$, the first term of the polynomial is 0, so it is redundant to include it. Finally, if we multiply both sides of (5) by $5x^2$, we get

$$\begin{aligned} 5x^2\left(\frac{1}{(x-6)^2}\right) &= 5x^2\left(\sum_{n=1}^{\infty}\frac{nx^{n-1}}{6^{n+1}}\right) \\ \frac{5x^2}{(x-6)^2} &= \sum_{n=1}^{\infty}\frac{5nx^{n+1}}{6^{n+1}}. \end{aligned} \tag{6}$$

Now, remember that the radius of convergence is 6, and we see that this power series is centered at 0. To find the interval of convergence, we have to plug in -6 and 6; neither of these converge, and so the interval of convergence of the power series is $(-6, 6)$.



You would have to add more terms to the polynomial to see that the power series converges between -6 and 6, but hopefully you get the idea by now!