Malachi Longmore Tutoring Notes

MAT109: Calculus I
Professor Annalisa Crannell
September 1, 2019

Even and odd functions

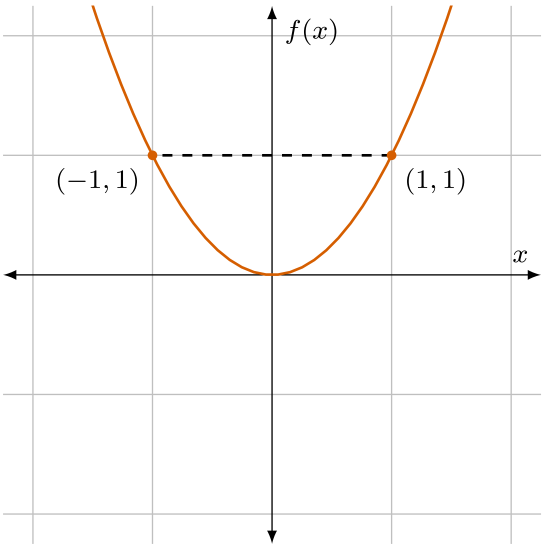
Today we talked about "even" and "odd" functions, two special classes of functions that you will encounter. A function $f : \mathbb{R} \to \mathbb{R}$ is *even* if

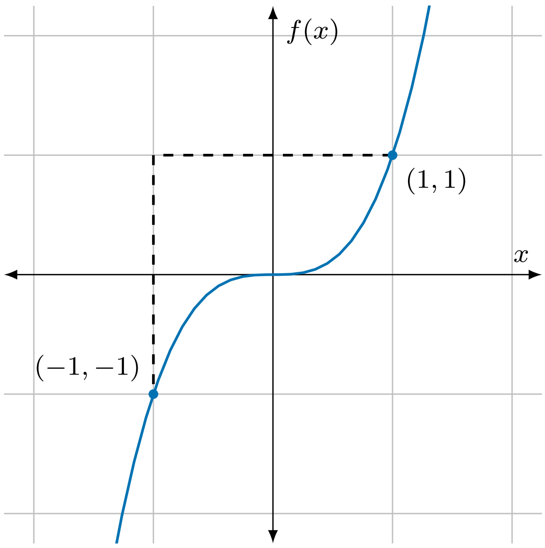
$$f(-x) = f(x)$$

for all $x \in \mathbb{R}$, and a function is *odd* if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$. The left graph below shows an example of an even function, and the right graph below shows an example of an odd function. An even function has a symmetry over the y-axis, while an odd function has two symmetries: first, a symmetry over the y-axis, and then another symmetry over the x-axis.





Domain and Range

We spent some time today talking about the *domain* and *range* of a function. We can describe them like this:

- A function describes an operation that is performed on a number. For example, the function $f(x) = \sqrt{x}$ says that for a given number, which is the *input*, we take its square root, which is the *output*.

- The domain is the list of possible numbers that is allowed to enter the function. So $f(-1)$ would *not* be allowed, since you cannot take the square root of a negative number. For this function, the domain includes all non-negative numbers, so $\mathcal{D} = [0, \infty)$.

- The range is the list of possible numbers that could potentially be outputs of the function, *given* the conditions of the range. For $f(x) = \sqrt{x}$, there is no value of x that could possibly give us a negative number; if we have $x = 0$, then $f(0) = \sqrt{0} = 0$, and plugging in larger values of x gives us larger outputs (for example, $f(4) = 2$). So for this function, $\mathcal{R} = [0, \infty)$.

There are two "problems" that we could come across when looking at functions (for now):

1. **Zero in the denominator**: if the function has any type of fraction, then this could happen, e.g. $f(x) = 1/x$, $g(x) = \frac{x^2-5}{2x}$, or $h(x) = \frac{3}{5x+2}$.

2. **Negative square roots**: if the function has a square root, then this could happen, e.g. $f(x) = \sqrt{x}$, $g(x) = 3\sqrt{x+1}$, or $h(x) = \sqrt{3-2x} + 5$.

3. Some combination of the two, e.g. $f(x) = \frac{1}{\sqrt{x}}$, $g(x) = \frac{\sqrt{3x-2}}{4x}$, or $h(x) = \frac{\sqrt{x-6}}{2\sqrt{5x+x^3}}$.

If neither of these problems are present, then the domain will always be $\mathcal{D} = (-\infty, \infty)$. The range can still be different, it depends on how the function is defined.

Composite functions

Toward the end we talked about composite functions. A composite function is when you have more than one function, and the *output* of one function is the *input* of the second function. As an example, lets say you have two generic functions $f(x)$ and $g(x)$, and you are asked to define the composite function $(g \circ f)(x)$. The steps to understanding the process are as follows:

1. Take your original input x, apply the function, and you will have your output $f(x)$.

2. Then, take the first output $f(x)$ and make this the *new* input that will be applied to $g(x)$.

3. Your final output will be $g\big(f(x)\big) = \big(g \circ f\big)(x)$.

The diagram below illustrates this process.

$$x \longrightarrow \boxed{\quad f \quad} \longrightarrow f(x) \longrightarrow \boxed{\quad g \quad} \longrightarrow \big(g \circ f\big)(x)$$

When we have $\big(g \circ f\big)(x)$, we apply $f(x)$ first and then $g(x)$; it seems kind of counter-intuitive, but to determine the order of how the functions are applied, we read from *right* to *left*. Similarly, if we have $\big(f \circ g\big)(x)$, it means that we apply $g(x)$ first and then $f(x)$.

For example, let's suppose $f(x) = \frac{1}{\sqrt{x}}$ and $g(x) = x^2 + 2$. Then

$$\big(f \circ g\big)(x) = f\big(g(x)\big) = \frac{1}{\sqrt{g(x)}} = \frac{1}{\sqrt{x^2+2}}$$

and

$$\big(g \circ f\big)(x) = g\big(f(x)\big) = \big(f(x)\big)^2 + 2 = \left(\frac{1}{\sqrt{x}}\right)^2 + 2 = \frac{1}{x} + 2.$$

Combining functions

In addition to composite functions, we talked about combining functions in other ways. If we have two functions f and g, we would like to find something like $f+g$, $f-g$, fg, or f/g. For example, if we have $f(x) = x - 8$ and $g(x) = 5x^2$, then $\big(f+g\big)(x) = x - 8 + 5x^2$.

While it is easy enough to add together functions, it is more important to intuitively understand what is going on. The diagram below shows what is happening for $\big(f+g\big)(x)$. We take our input x and then apply it to both f and g to get $f(x)$ and $g(x)$. Then we take these two outputs and *combine them*, in this case by adding them, to get out final output, $\big(f+g\big)(x)$.