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MAT110: Nonlinear Dynamics
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We were trying to determine the value of the definite integral

$$\int_{1/3\sqrt{3}}^{3\sqrt{3}} \frac{\mathrm{d}x}{x^{4/3} + x^{2/3}}.$$

By using u-substitution, we can let $u = x^{1/3}$. Note that this also means $u^3 = x$. Solving for $\mathrm{d}u$ gives us

$$\mathrm{d}u = \frac{1}{3}x^{-2/3}\,\mathrm{d}x = \frac{1}{3}(u^3)^{-2/3}\,\mathrm{d}x = \frac{1}{3}u^{-2}\,\mathrm{d}x \quad \Rightarrow \quad 3u^2\,\mathrm{d}u = \mathrm{d}x.$$

When transforming this from an integral with respect to x to an integral with respect to u, we must also account for the changing upper and lower bounds. We have

$$\begin{aligned} x &= \frac{1}{3\sqrt{3}} \quad \Rightarrow \quad u = \left(\frac{1}{3\sqrt{3}}\right)^{1/3} = \frac{1}{\sqrt{3}} \\ x &= 3\sqrt{3} \quad \Rightarrow \quad u = \left(3\sqrt{3}\right)^{1/3} = \sqrt{3} \end{aligned}$$

Substituting all of this into the integral gives us

$$\begin{aligned} \int_{1/3\sqrt{3}}^{3\sqrt{3}} \frac{\mathrm{d}x}{x^{4/3} + x^{2/3}} &= \int_{1/3\sqrt{3}}^{3\sqrt{3}} \frac{\mathrm{d}x}{\left(x^{1/3}\right)^4 + \left(x^{1/3}\right)^2} = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{3u^2}{u^4 + u^2}\,\mathrm{d}u \\ &= 3\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u^2}{u^2(u^2+1)}\,\mathrm{d}u = 3\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{\mathrm{d}u}{u^2+1} = 3\arctan(u)\Big|_{1/\sqrt{3}}^{\sqrt{3}} \\ &= 3\left(\arctan(\sqrt{3}) - \arctan(1/\sqrt{3})\right) = 3\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 3\left(\frac{\pi}{6}\right) = \frac{\pi}{2}. \end{aligned}$$