Implicit Differentiation

Use implicit differentiation to determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ if

$$x^2 = \frac{2x+5y}{2x-5y}. \tag{1}$$

By using implicit differentiation, we must first differentiate both sides of the equation with respect to x. Doing so gives us

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}x^2 &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x+5y}{2x-5y}\right)\\ 2x &= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}x}(2x+5y)\right)(2x-5y)-\left(\frac{\mathrm{d}}{\mathrm{d}x}(2x-5y)\right)(2x+5y)}{(2x-5y)^2}\\ 2x &= \frac{\left(2+5\frac{\mathrm{d}y}{\mathrm{d}x}\right)(2x-5y)-\left(2-5\frac{\mathrm{d}y}{\mathrm{d}x}\right)(2x+5y)}{(2x-5y)^2}.\end{aligned}$$

Remember why we are doing this: we want to determine $\frac{\mathrm{d}y}{\mathrm{d}x}$, and now that it has appeared in our equation, we can begin to solve for it. First, let's multiply both sides by $(2x-5y)^2$, which gives us

$$\left(2+5\frac{\mathrm{d}y}{\mathrm{d}x}\right)(2x-5y)-\left(2-5\frac{\mathrm{d}y}{\mathrm{d}x}\right)(2x+5y) = 2x(2x-5y)^2.$$

Now we should multiply out all of the terms on the left so we can pick out $\frac{\mathrm{d}y}{\mathrm{d}x}$, and we get

$$\begin{aligned}\left(4x-10y+10x\frac{\mathrm{d}y}{\mathrm{d}x}-25y\frac{\mathrm{d}y}{\mathrm{d}x}\right)-\left(4x+10y-10x\frac{\mathrm{d}y}{\mathrm{d}x}-25y\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 2x(2x-5y)^2\\ 4x-10y+10x\frac{\mathrm{d}y}{\mathrm{d}x}-25y\frac{\mathrm{d}y}{\mathrm{d}x}-4x-10y+10x\frac{\mathrm{d}y}{\mathrm{d}x}+25y\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x(2x-5y)^2\\ 20x\frac{\mathrm{d}y}{\mathrm{d}x}-20y &= 2x(2x-5y)^2.\end{aligned}$$

Adding $20y$ to both sides gives us

$$20x\frac{\mathrm{d}y}{\mathrm{d}x} = 2x(2x-5y)^2+20y,$$

and by dividing both sides by 20x, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x(2x-5y)^2+20y}{20x} = \frac{(2x-5y)^2}{10}+\frac{y}{x}. \tag{2}$$

Logarithmic Differentiation

We first took the natural logarithm of both sides, i.e.

$$\ln\left(x^2\right) = \ln\left(\frac{2x+5y}{2x-5y}\right),$$

and by using the properties of logarithms, we simplified the equation to

$$2\ln(x) = \ln(2x+5y)-\ln(2x-5y).$$

Now, differentiating both sides with respect to x gives us

$$\frac{2}{x} = \frac{2+5\frac{\mathrm{d}y}{\mathrm{d}x}}{2x+5y}-\frac{2-5\frac{\mathrm{d}y}{\mathrm{d}x}}{2x+5y}.$$

Doing some more algebra, we were able to show that

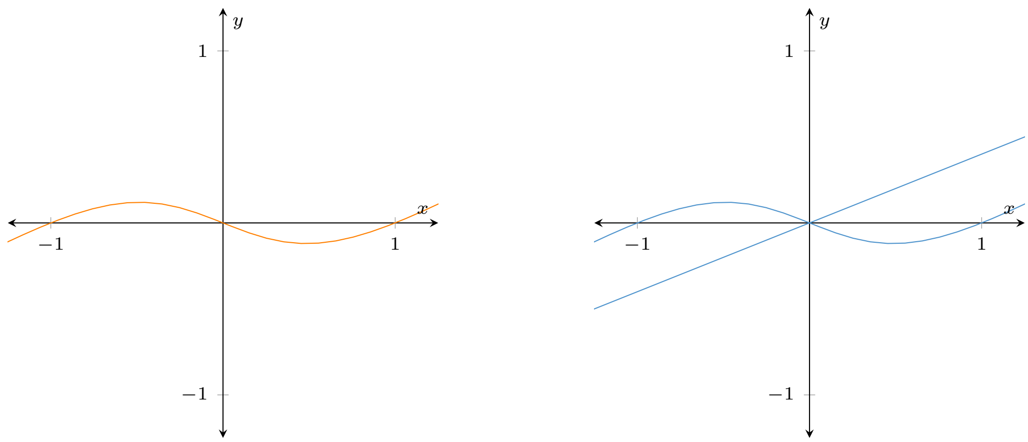
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dfrac{2}{x} + \dfrac{2}{2x-5y} - \dfrac{2}{2x+5y}}{\dfrac{5}{2x+5y} + \dfrac{5}{2x-5y}}. \tag{3}$$

Despite the fact these two answers look completely different, they are in fact equal. However, actually re-writing (3) in the form of (2) is incredibly challenging, and since there is no need for you to do so, I won't put it here.

Instead, we can look at more intuitive line of reasoning to show that they are equal. We will do the following:

1. Graph the initial relation given by (1); this can be seen in the left plot of Figure 1.
2. Set (2) and (3) equal to each other and plot them against each other; this can be seen in the right plot of Figure 1.

The purpose of this is to see *where* (2) and (3) are equal to each other. Any point where the two are equal will be plotted on the graph. That being said, if the two are equal at all the points given by (1), then we can say that the two derivatives are equal.



(A) *Plotting (1).* (B) *Plotting (2) against (3).*

FIGURE 1: *Plots to show that the two derivatives are equal.*

From the figure, we can clearly see that the two derivatives are equal at all of the points where (1) occurs, and so we can conclude that the derivatives are equal to each other. Notice that the two derivatives are equal at other places at well (here, where $y = 2x/5$), but that is not important.