A fair 4-sided die, with numbers 1, 2, 3, and 4 is rolled twice. If the score on the second roll is strictly greater than the score on the first, you win the difference in dollars. If the score on the second roll is strictly less than the score on the first roll, you lose the difference in dollars. If the scores are equal, you neither win nor lose. If we let X denote your winnings, what is the probability mass function of X?

If X is the amount of money you win, then given that the die has four sides, the possible values for X are $x = -3, -2, -1, 0, 1, 2, 3$ (for example, if the first roll is 1 and the second is 3, $x = 2$).

		Second Roll			
	Die #	1	2	3	4
First Roll	1	0	1	2	3
	2	-1	0	1	2
	3	-2	-1	0	1
	4	-3	-2	-1	0

Table 1: The possible values of X and how they are obtained.

See Table 1 for how each possible value of X occurs. For each value in the table, the probability of its occurrence is $\frac{1}{16}$. This is because The chance of rolling any side of the die is $\frac{1}{4}$, and since rolling again is an independent even, the probability of rolling any combination of two sides of the die is $\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{16}$.

But as we know, we are not interested in the number on the side of the die. Rather, we want to know how much money was made or lost based off of the two rolls. We can find the probability that $X = x$ for each value of x by adding up how many times that x could occur, given that each occurrence has a $\frac{1}{16}$ chance of happening. In other words,

$$\begin{aligned}
P(X = -3) &= \frac{1}{16} \\
P(X = -2) &= \frac{1}{16} + \frac{1}{16} = \frac{2}{16} \\
P(X = -1) &= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{16} \\
P(X = 0) &= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{4}{16} \\
P(X = 1) &= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{16} \\
P(X = 2) &= \frac{1}{16} + \frac{1}{16} = \frac{2}{16} \\
P(X = 3) &= \frac{1}{16}
\end{aligned}$$

The results have been summarized in Table 2.

x	-3	-2	-1	0	1	2	3
$P(X=x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

Table 2: The possible values for X and their respective probabilities.

Now all that remains is to determine a valid pdf based on our results, which can be tricky to do. When trying to solve problems like these, the best step is to see if there are any patterns present.

Here, we can see that the denominator of each probability is 16, but the numerator is changing in a symmetric fashion, starting at 1, going to 4 at the midpoint, and returning to 1 by the end. When $x = \pm 3$, the numerator is 1, when $x = \pm 2$, the numerator is 2, when $x = \pm 1$, the numerator is 3, and when $x = 0$, the numerator is 4.

Again, figuring out the pattern on your own can be kind of tricky, but maybe you can see that if you take 4 and subtract the magnitude (i.e. absolute value) of x, we get the numerator. So one possible way of writing the pdf of X is

$$p(x) = P(X = x) = \begin{cases} \frac{4-|x|}{16}, & x = 0, \pm 1, \pm 2, \pm 3 \\ 0, & \text{elsewhere.} \end{cases}$$

If you do not feel comfortable with the absolute value, you could also express the pdf as

$$p(x) = P(X = x) = \begin{cases} \frac{4-x}{16}, & x = 0, 1, 2, 3 \\ \frac{4+x}{16}, & x = -3, -2, -1 \\ 0, & \text{elsewhere,} \end{cases}$$

but the resulting answer is not different. Alternatively, since $\binom{n}{1} = n$, we can re-write the pdf yet again as

$$p(x) = P(X = x) = \begin{cases} \frac{1}{16} \cdot \binom{4-|x|}{1}, & x = 0, \pm 1, \pm 2, \pm 3 \\ 0, & \text{elsewhere.} \end{cases}$$

All of these solutions are the same thing, and I just wanted to show that there are multiple ways to getting the same answer.