Since we were having a bit of trouble with these question during tutoring, I want to make sure that you understand them.

Modeling with Differential Equations

Q: Set up and solve the equation to determine the amount of salt in a tank that holds 400 liters of solution, if it starts with 3 kg of salt and water containing 0.05 kg of salt per liter is poured in at a rate of 8 liters per minute, while the thoroughly mixed solution drains out at the same rate.

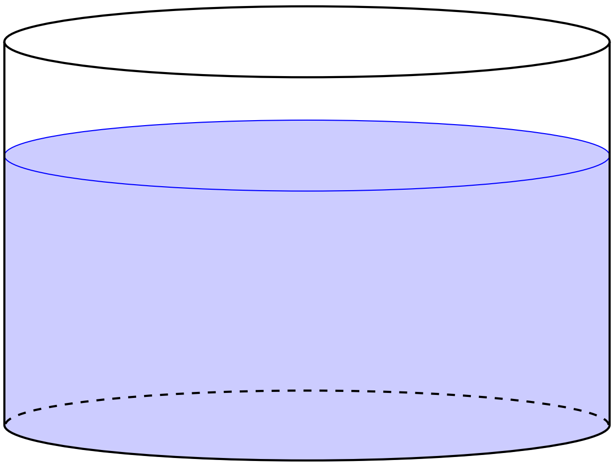

Figure 1: The tank with 400 L of solution in it.

To start, it is important to be able to pick out all of the information given to us in the problem:

- The tank holds 400 L of *solution*.
- The initial amount of salt in the tank is 3 kg.
- The in rate of the water into the tank is 8 L/min, and this water has a salt concentration of 0.05 kg/L.
- The out rate of the solution in the tank is the same as the in rate (8 L/min), and because of this, the total amount of solution in the tank will not change.

It is important to recognize the subtle but very distinct difference between *solution* and *salt concentration*; they are **not** the same!

Now, we are trying to determine a function that represents the *amount* of salt in the tank at a given time; we will call this $S(t)$. We will start by analyzing the *rate* at which salt enters and exits the tank, which we can see generally as

$$S'(t) = S'(t)_{in} - S'(t)_{out}.$$

Let's first look at the in rate; we see that the water (**not** the salt) flows in at a rate of 8 L/min, and the salt concentration of that water is a constant 0.05 kg/L, so we can find the rate at which *salt* enters the tank by

$$S'(t)_{in} = \left(\frac{8\text{ L}}{\text{min}}\right)\left(\frac{0.05\text{ kg}}{\text{L}}\right) = 0.4\text{ kg/min}.$$

The next step is to find the rate at which salt leaves the tank. This is a little trickier; we just saw now that to find the rate that *salt* leaves the tank is to take the rate at which the *total solution* is leaving the tank, and multiply it by the concentration of salt at that given time. This gives us the rate that *salt* leaves the tank.

The difference now is that at any given time, we are not (yet) sure of the salt concentration in the tank. This is because the amount of salt is **changing** as time passes (this is what we are solving for in the first place). At any given time, the concentration of salt in the tank is given by $\frac{S(t)}{400\text{ L}}$, and so

$$S'(t)_{out} = \left(\frac{8\text{ L}}{\text{min}}\right)\left(\frac{S(t)}{400\text{ L}}\right) = \frac{S(t)}{50\text{ min}},$$

since the in rate and out rate of solution are the same. Don't forget that $S(t)$ is measured in kg.

Finally, we can see that the differential equation is given by

$$S'(t) = 0.4 - \frac{S(t)}{50}. \tag{1}$$

Solving for $S(t)$ from (1) is beyond the scope of MAT110, but I have included the solution because it is good to know. Equation (1) is known as a **linear first order differential equation**, and is the subject of MAT229.

To start, re-write (1) as

$$S'(t) + \frac{S(t)}{50} = 0.4.$$

The **integrating factor** for (1) is $e^{\int 1/50\text{d}t} = e^{t/50}$, so we multiply both sides of (1) by this to get

$$S'(t)\cdot e^{t/50} + \frac{S(t)\cdot e^{t/50}}{50} = 0.4e^{t/50}.$$

We can see that $S'(t)\cdot e^{t/50} + \frac{S(t)\cdot e^{t/50}}{50} = \frac{\text{d}}{\text{d}t}\left(S(t)\cdot e^{t/50}\right)$ (think of integration by parts), and so we make the substitution

$$\frac{\text{d}}{\text{d}t}\left(S(t)\cdot e^{t/50}\right) = 0.4e^{t/50}.$$

We will now integrate both sides with respect to t, and we can see that

$$\begin{aligned}\int \frac{\text{d}}{\text{d}t}\left(S(t)\cdot e^{t/50}\right) &= \int 0.4e^{t/50} \\ S(t)\cdot e^{t/50} &= 20e^{t/50} + C,\end{aligned}$$

where C is some arbitrary constant. Finally, dividing both sides by $e^{t/50}$ gives us

$$S(t) = 20 + \frac{C}{e^{t/50}}. \tag{2}$$

To determine the constant C in (2), we must use the initial condition $S(0) = 3$, so

$$S(0) = 3 = 20 + \frac{C}{e^0} \quad\Rightarrow\quad C = -17.$$

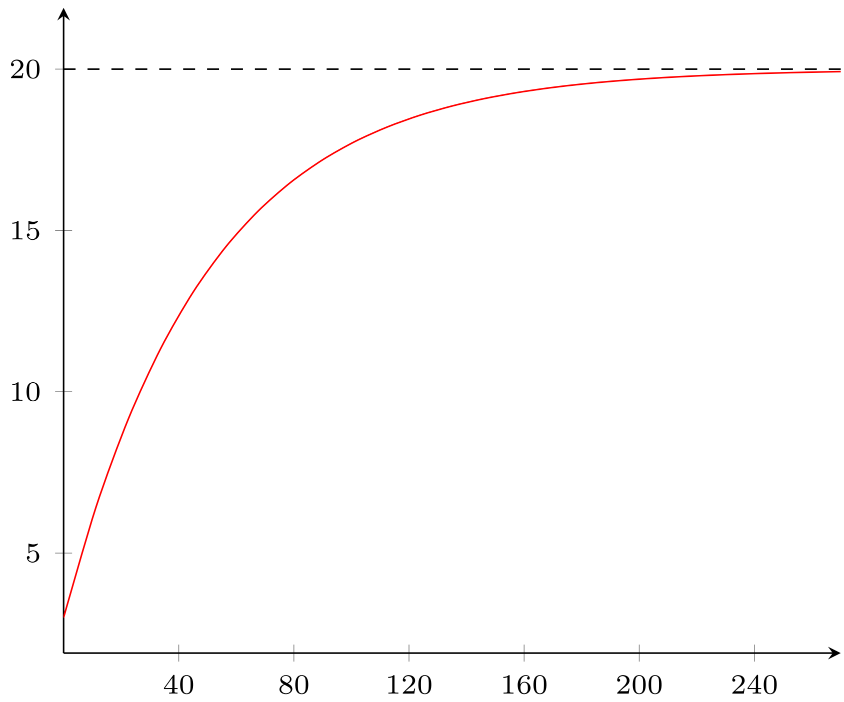

Figure 2: $S(t)$ for the first four hours.

And that's that! The amount of salt in the tank at any given minute of time can be expressed at

$$S(t) = 20 - \frac{17}{e^{t/50}}. \tag{3}$$

From the graph of the equation in Figure 2, we can see that as time passes by, the amount of salt in the tank approached 20 kg. Intuitively, this should make sense, since as the concentration of salt in the tank increases over time, the in rate and out rate of salt gets more and more similar, so the rate of change of salt approaches 0. Mathematically, we can see that

$$\lim_{t\to\infty} S(t) = \lim_{t\to\infty}\left(20 - \frac{17}{e^{t/50}}\right) = 20,$$

and since $S'(t) = 0.4 - \frac{S(t)}{50}$,

$$\lim_{t\to\infty} S'(t) = \lim_{t\to\infty}\left(0.4 - \frac{S(t)}{50}\right) = 0.4 - \frac{20}{50} = 0.$$

Like I said, anything that has to do with solving this would not be expected of you now, so if it doesn't make sense, don't worry about it too much.

Solving a Differential Equation

Q: Find the function y such that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^3}{2x^2}$$

and $y(1) = -1$.

We can see that this is a **separable differential equation**, and so

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y^3}{2x^2} \\
\frac{\mathrm{d}y}{y^3} &= \frac{\mathrm{d}x}{2x^2} \\
\int y^{-3}\mathrm{d}y &= \frac{1}{2}\int x^{-2}\mathrm{d}x \\
\frac{1}{2y^2} &= \frac{1}{2x} + C \\
\frac{1}{y^2} &= \frac{1}{x} + C \\
y^2 &= \frac{1}{C + \frac{1}{x}} \\
y &= \pm\sqrt{\frac{1}{C + \frac{1}{x}}}. \qquad (4)
\end{aligned}$$

To solve for C in (4), we simply use the initial condition. There are two possibilities, so we must test both.

$$\begin{aligned}
&-1 = \sqrt{\frac{1}{C+1}} \\
&(-1)^2 = \frac{1}{C+1} \\
&1 = \frac{1}{C+1} \\
&C + 1 = 1 \\
&C = 0
\end{aligned}
\qquad\qquad
\begin{aligned}
&-1 = -\sqrt{\frac{1}{C+1}} \\
&(1)^2 = \frac{1}{C+1} \\
&1 = \frac{1}{C+1} \\
&C + 1 = 1 \\
&C = 0
\end{aligned}$$

We can see that regardless of the possible solution, $C = 0$, and so we conclude that

$$y = \sqrt{x}. \qquad (5)$$

We can check to see that this is in fact a solution to the differential equation. To start, consider

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}y &= \frac{\mathrm{d}}{\mathrm{d}x}x^{1/2} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2x^{1/2}}.
\end{aligned}$$

Now all we have to do is plug back in to the differential equation:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y^3}{2x^2} \\
\frac{1}{2x^{1/2}} &= \frac{(x^{1/2})^3}{2x^2} \\
\frac{1}{2x^{1/2}} &= \frac{x^{3/2}}{2x^{4/2}} \\
\frac{1}{2x^{1/2}} &= \frac{1}{2x^{1/2}}.
\end{aligned}$$

So (5) is a solution to the differential equation.