We are asked to write the sum

$$-1 + 0 + 3 + 8 + 15 + 24 + 35 + 48 + 63$$

in summation form.

I figured two ways to go about doing this. The first, and what I believe is the intention, is much easier. You have to see that we can re-write each of the terms as follows:

$$\underbrace{(-1)(1)}_{-1} + \underbrace{(0)(2)}_{0} + \underbrace{(1)(3)}_{3} + \underbrace{(2)(4)}_{8} + \underbrace{(3)(5)}_{15} + \underbrace{(4)(6)}_{24} + \underbrace{(5)(7)}_{35} + \underbrace{(6)(8)}_{48} + \underbrace{(7)(9)}_{63}.$$

From here the pattern should be clear, and so you can express the sum as

$$-1 + 0 + 3 + 8 + 15 + 24 + 35 + 48 + 63 = \sum_{m=1}^{9} m(m-2) = 195. \tag{1}$$

The second way is much more difficult, but it has to do with adding up the odd numbers like we saw before, so I figured we should include it. We identified the pattern before as adding each odd number to the current term to get the next term, starting with -1. This can be expressed as

$$\underbrace{(-1)}_{-1} + \underbrace{(-1+1)}_{0} + \underbrace{(0+3)}_{3} + \underbrace{(3+5)}_{8} + \underbrace{(8+7)}_{15} + \underbrace{(15+9)}_{24} + \underbrace{(24+11)}_{35} + \underbrace{(35+13)}_{48} + \underbrace{(48+15)}_{63}.$$

Now, to get each term, we are essentially adding up however many odd numbers and subtracting 1; the number of odd numbers that we add increase with each term. So for the first term, we are adding no odd numbers (i.e. 0) and then subtracting 1, the second term we are adding one odd number (i.e. 1) and then subtracting 1, the third term we are adding two odd numbers (i.e. $1+3$) and then subtracting 1, and repeating until we get to the 8^{th} odd number. Hopefully you see the pattern here: for the n^{th} term of this sum, we are adding $n-1$ odd numbers to -1.

It may be good to remind yourself of adding odd numbers, which can be expressed as

$$\sum_{n=1}^{m} 2n - 1 = m^2.$$

So we can re-write our sum as

$$\underbrace{[-1]}_{-1} + \underbrace{\left[-1 + \sum_{n=1}^{1} 2n-1\right]}_{0} + \underbrace{\left[-1 + \sum_{n=1}^{2} 2n-1\right]}_{3} + \ldots + \underbrace{\left[-1 + \sum_{n=1}^{8} 2n-1\right]}_{63}.$$

We can rearrange this to get

$$\left[\sum_{m=1}^{8}\left(\sum_{n=1}^{m} 2n-1\right)\right] - \underbrace{[1+1+\ldots+1]}_{9\text{ times}},$$

and simplifying gives us

$$\left[\sum_{m=1}^{8} m^2\right] - 9.$$

Now, one of the useful sums that you should know is

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

and so our sum becomes

$$\left[\sum_{m=1}^{8} m^2\right] - 9 = \frac{8(8+1)(16+1)}{6} - 9 = 204 - 9 = 195,$$

which is what we got before.

Now, the question asks for an explicit sum notation, so we should generalize this result. We understand that for the n^{th} term in this sum, we add $n-1$ odd integers to -1. Writing out our sum gives us

$$[-1] + \left[-1 + \sum_{k=1}^{1} 2k-1\right] + \left[-1 + \sum_{k=1}^{2} 2k-1\right] + \ldots + \left[-1 + \sum_{n=1}^{n} 2k-1\right],$$

and this can be arranged to

$$\left[\sum_{m=1}^{n}\left(\sum_{k=1}^{m} 2k-1\right)\right] - \underbrace{[1+1+\ldots+1]}_{n+1 \text{ times}} = \left[\sum_{m=1}^{n} m^2\right] - [n+1]. \tag{2}$$

Reducing further gives us

$$\begin{aligned}\left[\sum_{m=1}^{n} m^2\right] - [n+1] &= \left[\frac{n(n+1)(2n+1)}{6}\right] - [n+1] \\ &= \frac{n(n+1)(2n+1)}{6} - \frac{6(n+1)}{6} \\ &= \frac{2n^3+3n^2-5n-6}{6}. \end{aligned} \tag{3}$$

Plugging in $n = 8$ gives us 195, which is the result we wanted.

Up until this point, we have two possible formulas for the sum, (1) and (2), and 1 is much easier to work with. However, we know that they are the same, and (2) also has a closed form (3), which would be nice to have, so we now wish to apply this closed form to (1). This can get kind of confusing, as the end term for each sum is slightly different.

To see this, let's compare the two sums:

$$195 = \underbrace{\sum_{m=1}^{9} m(m-2)}_{(1)} = \underbrace{\left[\sum_{m=1}^{8} m^2\right]}_{(2)} - 9 = \underbrace{\frac{2(8)^3+3(8)^2-5(8)-6}{6}}_{(3)}$$

For (1), we see that the last number included in the sum is 9, which we will call n (i.e. $n = 9$), but for the second sum, the last number included is 8, which we will call k (i.e. $k = 8$). That is, the last term for (2) is one less than (1), i.e. $k = n - 1$, so all we have to do is write (3) in terms of n, not k. We go about that as follows:

$$\begin{aligned}\frac{2k^3 + 3k^2 - 5k - 6}{6} &= \frac{2(n-1)^3 + 3(n-1)^2 - 5(n-1) - 6}{6} \\ &= \frac{2n^3 - 3n^2 - 5n}{6}\end{aligned}$$

We finally have not only a sum representation for this sum, but also a *closed form*:

$$\sum_{m=1}^{n} m(m-2) = \frac{2n^3 - 3n^2 - 5n}{6} \tag{4}$$

The *closed form* give us the value of the sum depending on how many terms in the sum you want to include using this pattern. So if we were to add 11 numbers that continued to increase in this pattern, the sum would be

$$\sum_{m=1}^{11} m(m-2) = \frac{2(11)^3 - 3(11)^2 - 5(11)}{6} = 374.$$

I included all of this stuff about closed form, but for the purposes of what you were asked to do, this is way more than necessary. All you needed to know was the sum notation, which is given by

$$\sum_{m=1}^{9} m(m-2).$$