Daisy Grossman Tutoring Notes

MAT109: Calculus I
Professor Wendell Ressler
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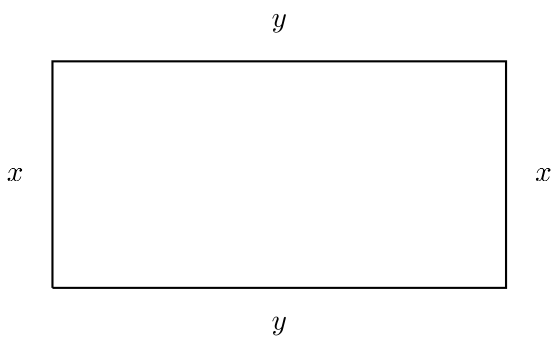
Question: *A rancher wants to fence in an area of 3,000,000 square feet in a rectangular field and then divide it in half with a fence down the middle, parallel to one side. What is the shortest length of fence that the rancher can use?*

Before we start, it is always good to have a visual understanding of the situation. We know that the field is a rectangle, but we are not exactly sure about the lengths of each side (this is what we are trying to figure out). So let's let x denote the shorter of the two sides, and y denote the longer of the two sides. We can now say two things:

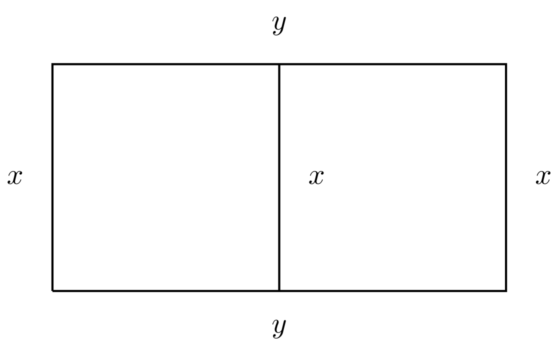
1. The *perimeter* of this rectangle is given by $P = 2x + 2y$.

2. The *area* of this rectangle is given by $A = xy$.

The diagram below shows a picture of the situation.



Now, in addition to the fencing that makes up the perimeter of the field, the rancher also wants to add another fence dividing the field in half. We have two options: the fence divides the shorter side and has length y, or the fence divides the longer side and has length x. Regardless of your choice, you will get the same answer, so let's make the fence divide the longer side and have length x. The diagram below shows a picture of this situation.



We can see that the total amount of fencing used in this situation is $F = 3x + 2y$. Our goal is to find the smallest possible value of F such that the field has an area of $3,000,000$.

We can see that F is a function of x and y, and we want to find the values of x and y that minimize F. However, because we know that $A = 3000000$, we can say that

$$A = 3000000 = xy \quad \rightarrow \quad y = \frac{3000000}{x},$$

and plugging this into F gives us

$$F = 3x + 2 \cdot \frac{3000000}{x} = 3x + \frac{6000000}{x}.$$

We now have F as a function that only depends on x, so we can use what we know about Calculus I to optimize F. Recall that the minimum of F is a *critical point*, and those are points where $\mathrm{d}F/\mathrm{d}x = 0$. So to find the minimum value of F, we have to find the critical points, check to see that the critical point is a minimum, and then plug the critical point back into F.

Differentiating F with respect to x and setting equal to 0 gives us

$$\frac{\mathrm{d}F}{\mathrm{d}x} = 3 - \frac{3000000}{x^2} \stackrel{\text{set}}{=} 0.$$

Doing some more algebra to solve for x, we have

$$\begin{aligned} 3 - \frac{6000000}{x^2} &= 0 \\ \Rightarrow\ 3 &= \frac{6000000}{x^2} \\ \Rightarrow\ 3x^2 &= 6000000 \\ \Rightarrow\ x^2 &= 2000000 \\ \Rightarrow\ x &= \pm 1000\sqrt{2}. \end{aligned}$$

So our two critical points are $x = -1000\sqrt{2}$ and $x = 1000\sqrt{2}$. Thinking practically, in this situation it is impossible for x to be negative, as we cannot have a fence of negative length. So $x = 1000\sqrt{2}$ is the only critical point we will consider.

We next have to check that $x = 1000\sqrt{2}$ really is a minimum. To do this, we check the concavity of F at $x = 1000\sqrt{2}$, and if the concavity is positive, we have a minimum. Differentiating F with respect to x again gives us

$$\frac{\mathrm{d}^2F}{\mathrm{d}x^2} = \frac{6000000}{x^3},$$

and when $x = 1000\sqrt{2}$, we can see that $\mathrm{d}^2F/\mathrm{d}x^2 > 0$, and so $x = 1000\sqrt{2}$ is indeed a minimum. Note that due to the physical limitations of this situation, we were able to disregard one of the critical points, but there are optimization problems where we are not able to do this; in those problems, checking the concavity is much more important!

Now, it remains to evaluate F at $x = 1000\sqrt{2}$ to give us the minimum amount of fencing required. We have

$$F\big(1000\sqrt{2}\big) = 3 \cdot 1000\sqrt{2} + \frac{6000000}{1000\sqrt{2}} = 3000\sqrt{2} + \frac{6000}{\sqrt{2}} = 6000\sqrt{2},$$

so $F = 6000\sqrt{2}$ is the smallest amount of fencing that the rancher can use in order to fence in 3000000 square feet.