Rokhaya Ba Tutoring Notes

MAT109: Calculus I
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Evaluating functions and expressions

Given that $f(x) = 3 - x^2$*, evaluate the expression* $\frac{f(2+h)-f(2)}{h}$.

It is very important to understand what is going on when we want to "evaluate" and *expression*. Here we are trying to evaluate this expression that can sort of be thought of as three parts:

1. The $f(2+h)$ in the numerator.
2. The $f(2)$ in the numerator.
3. The h in the denominator.

We can see that to determine the numerator of this expression, we have to evaluate the function $f(x)$ at two *specific* values. Remember, $f(x)$ is a *general* relationship that can be true for any value of x. It describes the transformation that the input x undergoes, and to emphasize this let's color-code it as

$$f(\mathtt{x}) = 3 - (\mathtt{x})^2,$$

and this would be true for *any* value of $\mathtt{x}$. So we will evaluate the expression in three steps:

1. Evaluate $f(2+h)$.
2. Evaluate $f(2)$.
3. Simplify.

For the first step, we are setting $\mathtt{x} = 2 + h$, so we get

$$f(2+h) = 3 - (2+h)^2 = 3 - (4 + 4h + h^2) = 3 - 4 - 4h - h^2 = -1 - 4h - h^2.$$

For the second step, we are setting $\mathtt{x} = 2$, so we get

$$f(2) = 3 - (2)^2 = 3 - 4 = -1.$$

Finally, we can combine our answers and simplify the expression:

$$\begin{aligned}
\frac{f(2+h)-f(2)}{h} &= \frac{(-1-4h-h^2)-(-1)}{h} \\
&= \frac{-1-4h-h^2+1}{h} \\
&= \frac{-4h-h^2}{h} \\
&= \frac{h(-4-h)}{h} \\
&= -4-h.
\end{aligned}$$

So for the final answer, we have the expression being equal to $-4 - h$.

Squaring and adding

We talked today about the fact that for any two numbers a and b, it is true that

$$(a+b)^2 = a^2 + 2ab + b^2,$$

and *not* $(a+b)^2 = a^2 + b^2$. To start, let's re-write the expression as

$$(a+b)^2 = (a+b)(a+b),$$

and treat the second $(a+b)$ as a single object for now. If we distribute, we get

$$(a+b)(a+b) = a(a+b) + b(a+b).$$

Now, if we no longer treat the $(a+b)$ as a single object, we can distribute again to get

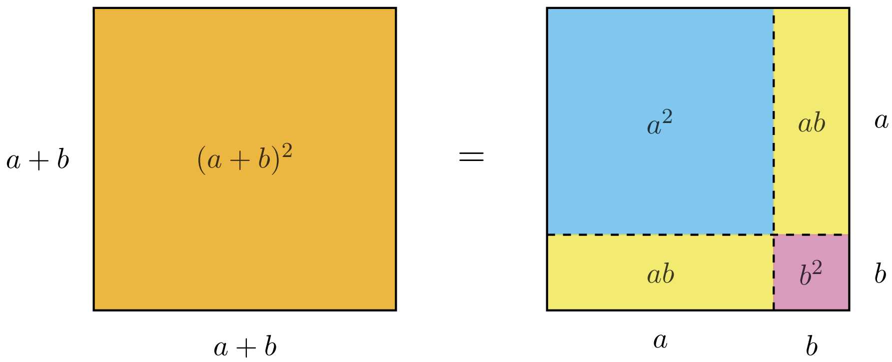
$$a(a+b) + b(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2,$$

since $ab = ba$. So we can see that $(a+b)^2 = a^2 + 2ab + b^2$.

For a more intuitive way to remember this, consider the diagram below. Think of the product $(a+b)^2$ as the area of a square that has a side length of $a+b$. The left square shows the original square, with its area given by $(a+b)^2$. The right square shows the original square broken up into four different components:

1. The first is the smaller blue square with a side length of a, and its area is given by a^2.
2. The second and third are the two yellow rectangles with a length of a and a width of b; the area for each is given by ab.
3. The fourth is the smaller pink square with a side length of b, and its area is given by b^2.

Adding up all four areas gives us $a^2 + 2ab + b^2$, and since we never changed the square at all, the two areas must be equal. Therefore, $(a+b)^2 = a^2 + 2ab + b^2$.



Domain and Range

We spent some time today talking about the *domain* and *range* of a function. We can describe them like this:

- A function describes an operation that is performed on a number. For example, the function $f(x) = \sqrt{x}$ says that for a given number, which is the *input*, we take its square root, which is the *output*.

- The domain is the list of possible numbers that is allowed to enter the function. So $f(-1)$ would *not* be allowed, since you cannot take the square root of a negative number. For this function, the domain includes all non-negative numbers, so $\mathcal{D} = [0, \infty)$.
- The range is the list of possible numbers that could potentially be outputs of the function, *given* the conditions of the range. For $f(x) = \sqrt{x}$, there is no value of x that could possibly give us a negative number; if we have $x = 0$, then $f(0) = \sqrt{0} = 0$, and plugging in larger values of x gives us larger outputs (for example, $f(4) = 2$). So for this function, $\mathcal{R} = [0, \infty)$.

There are two "problems" that we could come across when looking at functions (for now):

1. **Zero in the denominator**: if the function has any type of fraction, then this could happen, e.g. $f(x) = 1/x$, $g(x) = \frac{x^2-5}{2x}$, or $h(x) = \frac{3}{5x+2}$.
2. **Negative square roots**: if the function has a square root, then this could happen, e.g. $f(x) = \sqrt{x}$, $g(x) = 3\sqrt{x+1}$, or $h(x) = \sqrt{3-2x} + 5$.
3. Some combination of the two, e.g. $f(x) = \frac{1}{\sqrt{x}}$, $g(x) = \frac{\sqrt{3x-2}}{4x}$, or $h(x) = \frac{\sqrt{x-6}}{2\sqrt{5x+x^3}}$.

If neither of these problems are present, then the domain will always be $\mathcal{D} = (-\infty, \infty)$. The range can still be different, it depends on how the function is defined.

Composite functions

Toward the end we talked about composite functions. A composite function is when you have more than one function, and the *output* of one function is the *input* of the second function. As an example, lets say you have two generic functions $f(x)$ and $g(x)$, and you are asked to define the composite function $\big(g \circ f\big)(x)$. The steps to understanding the process are as follows:

1. Take your original input x, apply the function, and you will have your output $f(x)$.
2. Then, take the first output $f(x)$ and make this the *new* input that will be applied to $g(x)$.
3. Your final output will be $g\big(f(x)\big) = \big(g \circ f\big)(x)$.

The diagram below illustrates this process.

$$x \longrightarrow \boxed{\quad f \quad} \longrightarrow f(x) \longrightarrow \boxed{\quad g \quad} \longrightarrow \big(g \circ f\big)(x)$$

When we have $\big(g \circ f\big)(x)$, we apply $f(x)$ first and then $g(x)$; it seems kind of counter-intuitive, but to determine the order of how the functions are applied, we read from *right* to *left*. Similarly, if we have $\big(f \circ g\big)(x)$, it means that we apply $g(x)$ first and then $f(x)$.

For example, let's suppose $f(x) = \frac{1}{\sqrt{x}}$ and $g(x) = x^2 + 2$. Then

$$\big(f \circ g\big)(x) = f\big(g(x)\big) = \frac{1}{\sqrt{g(x)}} = \frac{1}{\sqrt{x^2+2}}$$

and

$$\big(g \circ f\big)(x) = g\big(f(x)\big) = \big(f(x)\big)^2 + 2 = \left(\frac{1}{\sqrt{x}}\right)^2 + 2 = \frac{1}{x} + 2.$$