The Fundamental Theorem of Calculus

1. The *Fresnel function* is given by

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)\,\mathrm{d}t,$$

and is used to model the diffraction of light waves, as well as in the design of highways and roller coasters. Determine $S'(1)$.

2. In probability and statistics, the *normal distribution cdf* is given by

$$\Phi(x) = \int_{-\infty}^x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t,$$

where μ and σ are just arbitrary constants. Determine the *normal distribution pdf*, given by $\phi(x) = \Phi'(x)$.

3. Find $F'(x)$ if

(a) $\displaystyle\int_{100}^{3x+9} \frac{t}{3}\,\mathrm{d}t$ (b) $\displaystyle F(x) = \int_{\cos(x)}^{5} t^3\,\mathrm{d}t$ (c) $\displaystyle F(x) = \int_{3x}^{2x^3} \sin(\pi t)\,\mathrm{d}t$

4. Question 8 of the 2018 Stanford Math Tournament asked participants to evaluate

$$\lim_{n\to\infty} n^2 \int_0^{1/n} x^{2018x+1}\,\mathrm{d}x.$$

To get you started, observe that you can re-write this as

$$\lim_{n\to\infty} \frac{1}{1/n^2} \int_0^{1/n} x^{2018x+1}\,\mathrm{d}x.$$

You should see that this limit takes the form of 0/0, so you should use l'Hospital's Rule to evaluate it.

5. Evaluate the following definite integrals using the Fundamental Theorem of Calculus:

(a) $\displaystyle\int_1^2 \frac{x^3+3x^6}{x^4}\,\mathrm{d}x$ (b) $\displaystyle\int_0^1 x^{4/5}\,\mathrm{d}x$

(c) $\displaystyle\int_0^1 10^x\,\mathrm{d}x$ (d) $\displaystyle\int_0^{\pi/4} \sec(x)\tan(x)\,\mathrm{d}x$

(e) $\displaystyle\int_1^9 \frac{1}{2x}\,\mathrm{d}x$ (f) $\displaystyle\int_0^{3\pi/2} |\sin(x)|\,\mathrm{d}x$

6. A few weeks ago, myself and Yoni, the head mathematics tutor, were in the process of interviewing new candidates for math tutors next year. On the mock tutor portion of the interview, we asked them to differentiate

$$F(x) = \int_0^{x^2} \sin^2(t)\,\mathrm{d}t,$$

and many of them actually could not solve it. Find the derivative of this function. Would you have qualified to be a math tutor? ☺

Integration Techniques

1. Use substitution to evaluate the following indefinite integrals:

   (a) $\int e^{\cos(x)} \sin(x) \, \mathrm{d}x$

   (b) $\int \frac{3x^2}{1+x^6} \, \mathrm{d}x$

   (c) $\int \frac{1}{x\sqrt{\ln(3x)}} \, \mathrm{d}x$

   (d) $\int \frac{1}{\cos^2(x)\sqrt{1+\tan(x)}} \, \mathrm{d}x$

2. Use integration by parts to evaluate the following indefinite integrals:

   (a) $\int x^2 \ln(x) \, \mathrm{d}x$

   (b) $\int \frac{\ln(x)}{\sqrt{x}} \, \mathrm{d}x$

   (c) $\int x \sin(\pi x) \, \mathrm{d}x$

   (d) $\int x^3 \sqrt{1+x^2} \, \mathrm{d}x$

3. Using whatever method(s) you deem appropriate (e.g. substitution, by parts, integral table), evaluate the following indefinite integrals:

   (a) $\int \frac{14}{\sqrt{5-x^2}} \, \mathrm{d}x$

   (b) $\int x^2 e^{7x} \, \mathrm{d}x$

   (c) $\int x^2 \cos(\pi x) \, \mathrm{d}x$

   (d) $\int x^3 e^{-3x^2} \, \mathrm{d}x$

   (e) $\int \sqrt{\frac{x}{5-x}} \, \mathrm{d}x$

   (f) $\int x(x+3)^4 \, \mathrm{d}x$

   (g) $\int e^{-9x} \sin(2\pi x) \, \mathrm{d}x$

   (h) $\int \cos^9(6x) \, \mathrm{d}x$

   (i) $\int \frac{\ln(x)}{x} \, \mathrm{d}x$

   (j) $\int \sec(x) \, \mathrm{d}x$

Geometric Applications

1. Suppose that $f(x) = x^x$ from $0 \leq x \leq 3$. Set up, but do not try and solve (you actually will not be able to), the appropriate integrals for the following quantities:

   (a) The area under the curve from 0 to 1.

   (b) The volume of the solid of revolution from rotating this curve around the y-axis.

   (c) The arclength of the curve.

2. Let $\mathcal{R}$ be the region enclosed by $y = x$ and $y = x^2$. If this region is rotated about the x-axis, find the volume of the resulting solid of revolution.

3. In high school, you probably learned that the volume of a sphere is given by $4\pi r^3/3$. Confirm that this is true using solids of revolution. More formally, let $y = \sqrt{r^2 - x^2}$ from $-r \leq x \leq r$. Rotate this around the x-axis and show that its volume is $4\pi r^3/3$.

Differential Equations

1. Certain broods of cicadas, a type of insect, are known as *periodic cicadas*, since they spend most of their lives underground only to emerge periodically ever few years. The *Great Eastern Brood* is a brood that spends 17 years underground before emerging, lay eggs, and die off after several weeks. Thousands can emerge at a single time, and given that they are clumsy and easy to catch by predators, they die off rather quickly. The population dies off exponentially and is governed by the differential equation

$$p' = -\mu p,$$

where $p = p(t)$ is the population equation and μ is the mortality rate.

   (a) Solve for the general solution of $p(t)$.

   (b) If $p(0) = 100000$ and $p(2) = 5000$, what is the mortality rate μ?

2. Solve the following differential equations:

   (a) $y' = xy^2$ (b) $y' = \dfrac{x}{e^y}$ (c) $y' = \dfrac{\ln(x)}{xy}$

   (d) $y' = \sqrt{xy}$ (e) $y' = \dfrac{xy\sin(x)}{y+1}$ (f) $y' = \dfrac{xe^x}{y\sqrt{1+y^2}}$

3. Research in biology and medicine are always looking to understand how cancer cells multiply. One model of tumor growth is the *Gompertz model*, given by

$$R' = -aR\ln\left(\frac{R}{k}\right).$$

Here, $R = R(t)$ is the radius of the tumor at time t and a and k are positive constants. Solve for $R(t)$. It is okay to leave this solution in implicit form.

4. In psychology, it is sometimes of interest to model an individual's performance over time. To do this, they use *learning curves*. One plausible learning curve is governed by the differential equation

$$P' = k(M - P),$$

where $P = P(t)$ is the individual's performance over time, M is the maximum level of performance, and k is an arbitrary constant.

   (a) Determine the general solution of $P(t)$.

   (b) If $P(0) = 0$ and $P(100) = 5$, what is the arbitrary constant k?

5. Verify that $y = -x\cos(x) - x$ is a solution of the initial value problem

$$xy' = y + x^2\sin(x), \quad y(\pi) = 0.$$

6. Which of the following functions are solutions to the differential equation $y'' + y = \sin(x)$?

   (a) $y = \sin(x)$ (b) $y = \cos(x)$

   (b) $y = \dfrac{x\sin(x)}{2}$ (d) $y = -\dfrac{x\cos(x)}{2}$

Series

1. Using any method that you find appropriate, determine if the following series converge or not (you do not have to determine *what* they converge to, if they do):

(a) $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ (c) $\sum_{n=0}^{\infty} \frac{1+\sin(n)}{10^n}$

(d) $\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$ (e) $\sum_{n=0}^{\infty} \sin^n(\pi/4)$ (f) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2}{n^3+1}$

(g) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ (h) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ (i) $\sum_{n=0}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+2}}$

2. In economics, the *money multiplier effect* is the maximum amount of commercial bank money that can be created from an initial deposit to a bank. In 2016, banks were required by law to keep 10% of any deposit; they will then trade the remaining 90% to another bank. In turn, this next bank will trade 90% of this new amount (so 90% of 90% of the original amount) to another bank. Theoretically, this process continues forever. If you invest \$1000 into your bank account, how much money can potentially be created? (Hint: if you don't see it, this is a geometric series with $a = 1000$ and $r = 0.9$).

3. The *Bessel function of order 0*, given by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2},$$

can be used to model various physical situations, including planetary motion, heat conduction, and bio-molecular diffraction. Find the radius of convergence of $J_0(x)$.

4. Many professors in the math department love to say "Taylor series are how calculators compute values of functions like $\sin(x)$" in attempt to make the topic seem somewhat relevant, but this is in fact not true (there are much more sophisticated algorithms in use). The problem is that Taylor series estimate poorly when you move away from the series' center.

   (a) Write out the first 4 terms of the Taylor series of $\sin(x)$, and use it to approximate $\sin(\pi/4)$. Compare your answer to the true value of $\sin(\pi/4)$.

   (b) Now use the same 4 terms to approximate $\sin(10\pi)$, and compare your answer to the true value. Is your approximation close?

5. Suppose that $f(x) = 10^x$. Find its Taylor series expansion about $x = 0$, and find its radius of convergence.

6. Determine the radius of convergence for the power series given by

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^2.$$

7. Using an appropriate Taylor series, find the value of the following sums:

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{3^{2n}(2n)!}$ (b) $\sum_{n=0}^{\infty} \frac{(-e)^n}{n!}$ (c) $\sum_{n=0}^{\infty} \frac{(1/2)^n}{n}$