Final Presentation Follow-Up Homework Question

Aiden Kenny
MAT439: Nonlinear Dynamics
Franklin & Marshall College

May 5, 2020

In my presentation, we talked about the (relatively) basic SIR model, which was used to model the progression of *epidemics* through a population. Recall that an epidemic is an unusually large, short-term (less than a year) outbreak of a disease. The SIR model was given by

$$\begin{aligned} \mathrm{d}S/\mathrm{d}t &= -\beta SI, \\ \mathrm{d}I/\mathrm{d}t &= \beta SI - \gamma I, \\ \mathrm{d}R/\mathrm{d}t &= \gamma I. \end{aligned} \tag{1}$$

where β is the infection rate of the disease and γ is the healing rate from the disease. For this model, we have $S, I, R \geq 0$, and have a constant population $S + I + R = N$.

However, it is not always the case that we are dealing with epidemics. Some diseases, such as HIV/AIDS, infect their hosts for a long time (10+ years), sometimes for the duration of the lifespan of the host. As a result, we must account for deaths in the population that are not caused by the disease. In addition, we must account for new births in the population that will increase the number of susceptible. In this case, we can no longer assume that the total population is constant. Letting Λ be the birth rate of the population and μ be the death rate, we can modify the original SIR model as follows:

$$\begin{aligned} \mathrm{d}S/\mathrm{d}t &= \Lambda - \mu S - \beta SI, \\ \mathrm{d}I/\mathrm{d}t &= \beta SI - \gamma I - \mu I, \\ \mathrm{d}R/\mathrm{d}t &= \gamma I - \mu R. \end{aligned} \tag{2}$$

This model is known as the SIR model with *vital dynamics*.

(a) Find the two possible equilibrium points for this system. One of the points is the *disease-free equilibrium*, where the disease is completely eradicated from the population. The other is the *endemic equilibrium*, where the disease is not completely eradicated and remains in the population forever. Which one is which? (Hint: start with factoring I out of $\mathrm{d}I/\mathrm{d}t$ and go from there).

The two equilibrium points are given by

$$\left(\frac{\Lambda}{\mu}, 0, 0\right) \quad \text{and} \quad \left(\frac{\gamma+\mu}{\beta}, \frac{\mu}{\beta}\left(\frac{\beta\Lambda}{\mu(\gamma+\mu)} - 1\right), \frac{\gamma}{\beta}\left(\frac{\beta\Lambda}{\mu(\gamma+\mu)} - 1\right)\right).$$

The first one is the disease-free equilibrium point, while the second is the endemic equilibrium point.

(b) Determine the linearized system of this model at *only* the disease-free equilibrium point.

The Jacobian is generally given by

$$\mathbf{J} = \begin{pmatrix} -\mu - \beta I & -\beta S & 0 \\ \beta I & \beta S - \gamma - \mu & 0 \\ 0 & \gamma & -\mu \end{pmatrix}.$$

At the disease-free equilibrium point, $(\Lambda/\mu, 0, 0)$, the Jacobian is given by

$$\mathbf{J} = \begin{pmatrix} -\mu & -\beta\Lambda/\mu & 0 \\ 0 & \beta\Lambda/\mu - \gamma - \mu & 0 \\ 0 & \gamma & -\mu \end{pmatrix}.$$

(c) Determine the three eigenvalues of this system.

The characteristic polynomial is given by

$$f(\lambda) = (-\mu - \lambda)\left(\frac{\beta\Lambda}{\mu} - \gamma - \mu - \lambda\right)(-\mu - \lambda),$$

which implies that $\lambda_1 = -\mu$, $\lambda_2 = \beta\Lambda/\mu - \gamma$, and $\lambda_3 = -\mu$ are the three eigenvalues. Note that two of the eigenvalues, λ_1 and λ_3, are always negative.

(d) Prove that the disease-free equilibrium point is a sink if

$$\frac{\beta\Lambda}{\mu(\gamma+\mu)} < 1$$

and a saddle if

$$\frac{\beta\Lambda}{\mu(\gamma+\mu)} > 1.$$

(Hint: if part (c) was done correctly, two of the three eigenvalues will always be negative, so the sign of the third will determine the behavior of the linearized system).

From the last equation, for it to be a sink, we need $\lambda_2 < 0$. Plugging in λ_2 and re-arranging eventually gives you the result. Showing it is a source is the same process.

Aside for Professor Weaver: the quantity $\beta\Lambda/\mu(\gamma+\mu)$ is actually the basic reproductive number of this model, and is usually denoted as $\mathcal{R}_0$. We can also see that this quantity appears in the endemic equilibrium point:

- When $\mathcal{R}_0 < 1$, the case when the disease-free equilibrium point is a sink, we have I^* and R^* negative, so the endemic equilibrium point is not a possibility.
- When $\mathcal{R}_0 > 1$, the case when the disease-free equilibrium point is a source, we have I^* and R^* positive, so the endemic equilibrium point can be reached.

Practically, this means that if $\mathcal{R}_0$ is small, the disease will die out, but if $\mathcal{R}_0$ is large, it will remain in the population forever. I was happy to finally figure out where this result came from by myself, since the work to arrive at this process was not in any article or textbook, and this derivation is closer to what we learned in class, as opposed to the standard derivations in epidemiology textbooks.