Final Presentation Follow-Up Homework Question

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In my presentation, we talked about the (relatively) basic SIR model, which was used to model the progression of *epidemics* through a population. Recall that an epidemic is an unusually large, short-term (less than a year) outbreak of a disease. The SIR model was given by

$$\begin{aligned} \mathrm{d}S/\mathrm{d}t &= -\beta SI, \\ \mathrm{d}I/\mathrm{d}t &= \beta SI - \gamma I, \\ \mathrm{d}R/\mathrm{d}t &= \gamma I. \end{aligned} \tag{1}$$

where β is the infection rate of the disease and γ is the healing rate from the disease. For this model, we have $S, I, R \geq 0$, and have a constant population $S + I + R = N$.

However, it is not always the case that we are dealing with epidemics. Some diseases, such as HIV/AIDS, infect their hosts for a long time (10+ years), sometimes for the duration of the lifespan of the host. As a result, we must account for deaths in the population that are not caused by the disease. In addition, we must account for new births in the population that will increase the number of susceptible. In this case, we can no longer assume that the total population is constant. Letting Λ be the birth rate of the population and μ be the death rate, we can modify the original SIR model as follows:

$$\begin{aligned} \mathrm{d}S/\mathrm{d}t &= \Lambda - \mu S - \beta SI, \\ \mathrm{d}I/\mathrm{d}t &= \beta SI - \gamma I - \mu I, \\ \mathrm{d}R/\mathrm{d}t &= \gamma I - \mu R. \end{aligned} \tag{2}$$

This model is known as the SIR model with *vital dynamics*.

(a) Find the two possible equilibrium points for this system. One of the points is the *disease-free equilibrium*, where the disease is completely eradicated from the population. The other is the *endemic equilibrium*, where the disease is not completely eradicated and remains in the population forever. Which one is which? (Hint: start with factoring I out of $\mathrm{d}I/\mathrm{d}t$ and go from there).

(b) Determine the linearized system of this model at *only* the disease-free equilibrium point.

(c) Determine the three eigenvalues of this system.

(d) Prove that the disease-free equilibrium point is a sink if

$$\frac{\beta\Lambda}{\mu(\gamma + \mu)} < 1$$

and a saddle if

$$\frac{\beta\Lambda}{\mu(\gamma + \mu)} > 1.$$

(Hint: if part (c) was done correctly, two of the three eigenvalues will always be negative, so the sign of the third will determine the behavior of the linearized system).