Modeling Epidemics and the Effect of Social Distancing

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April 30, 2020

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We are interested in modeling the progress of these infectious diseases as they spread through the population during an epidemic.

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Compartmental models describe how the population moves through the compartments.

More on compartments

A very simple model breaks the population into *three* compartments:

- **S** : individuals who are susceptible to the disease.
- **I** : individuals who are infected with the disease.
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Let N denote the total population at a given time $t \geq 0$.

We have the general relationship

$$N(t) = S(t) + I(t) + R(t)$$

for all $t \geq 0$.

The SIR model

A very basic system of differential equations that models these compartments is the *SIR model*,

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= -\frac{\beta SI}{N}\\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\beta SI}{N} - \gamma I\\ \frac{\mathrm{d}R}{\mathrm{d}t} &= \gamma I.\end{aligned}$$

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$\beta > 0$ and $\gamma > 0$.

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Namely, we can account for how *social distancing* effects the progression of an epidemic.

Initial conditions

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- No one is removed from the epidemic at $t = 0$.
- $S_0 + I_0 = N$.

We also assume that $S_0 \approx N$, meaning:

- Most of the population is initially susceptible at $t = 0$.
- One a small number of individuals are infected at $t = 0$.

Nullclines and equilibria

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We can see that $\mathrm{d}S/\mathrm{d}t < 0$ for all values of t, so the number of susceptible individuals is *always decreasing*.

We have

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\beta S I}{N} - \gamma I = \gamma I \left(\frac{\beta}{\gamma} \frac{S}{N} - 1 \right).$$

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Setting $\mathrm{d}I/\mathrm{d}t = 0$ gives us $I = 0$ and $S = N/\mathcal{R}_0$ as the two I-nullclines.

The value of $\mathcal{R}_0$ effects the severity of the epidemic.

Phase portrait

We conclude that everywhere on the line $I = 0$ is an equilibrium point.

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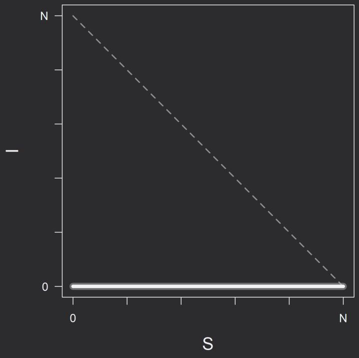

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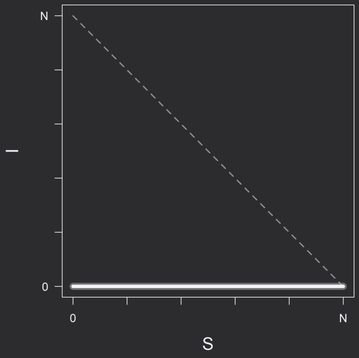

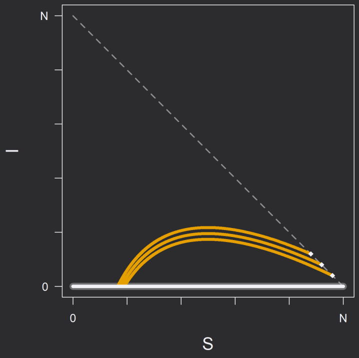

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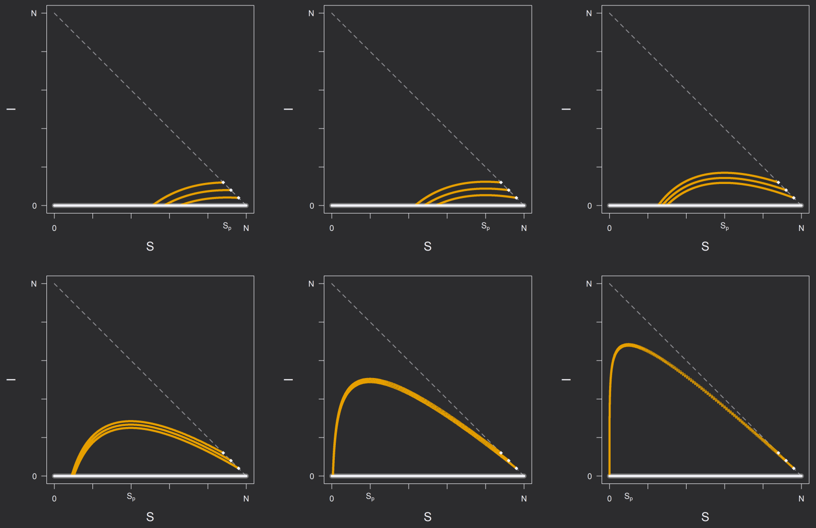
- The peak is located outside of the possible values of S.
- Will *always* have $S < N/\mathcal{R}_0 \rightarrow$ will *always* have $\mathrm{d}I/\mathrm{d}t < 0$.



N
I
0
0
S_p
N
S

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Social distancing

Our original model,

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\beta S I}{N}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\beta S I}{N} - \gamma I,$$

assumed that no changes in social behavior would occur.

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Our original model,

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We can add a *social distancing term* to the equations.

$k \geq 0$ is a behavior parameter.

- $k = 0$ is the original SIR model with no social distancing.
- As k increases, society becomes more sensitive to the prevalence of the epidemic.

Nullclines and equilibria (again)

It can be shown that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta I \left(\frac{S}{N}\right)^{k+1}$$

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However, the point of maximum infections S_p is now given by

$$S_p = \frac{N}{\mathcal{R}_0^{1/(k+1)}}.$$

Effect of increasing k

We can see that

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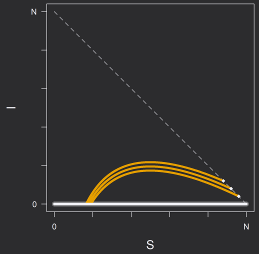
In the case of $\mathcal{R}_0 > 1$, as k increases, the peak will be pushed to the right.

As a result, *the epidemic will not be as severe*.

Effect of changing k



The left panel shows when $k = 0 \rightarrow$ no social distancing.

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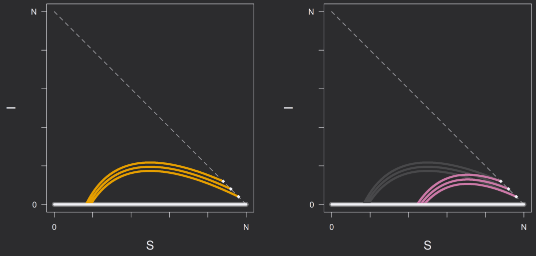

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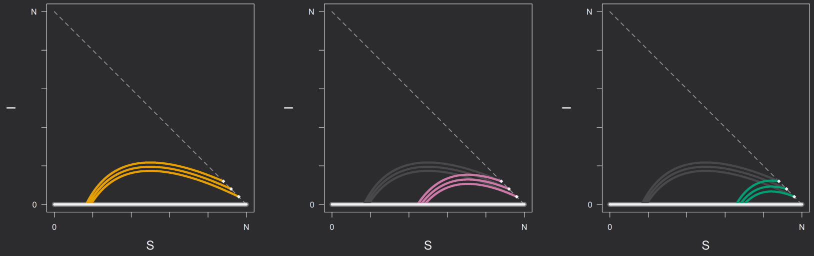

The left panel shows when $k = 0 \rightarrow$ no social distancing.

The middle panel shows when $k = 1 \rightarrow$ moderate social distancing.

The right panel shows when $k = 3 \rightarrow$ aggressive social distancing.

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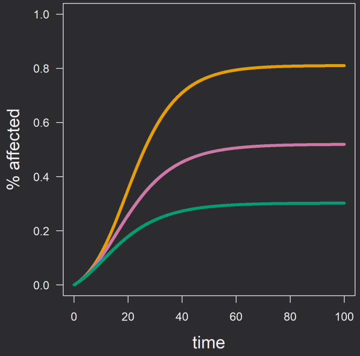

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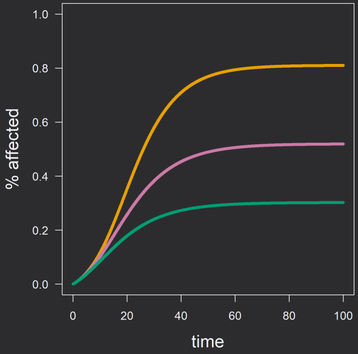

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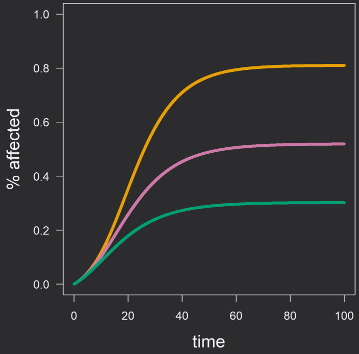

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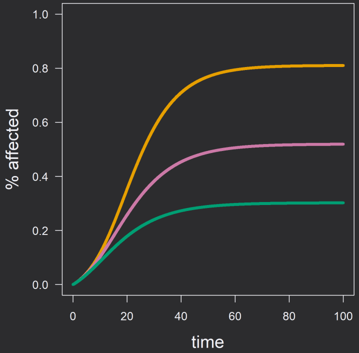

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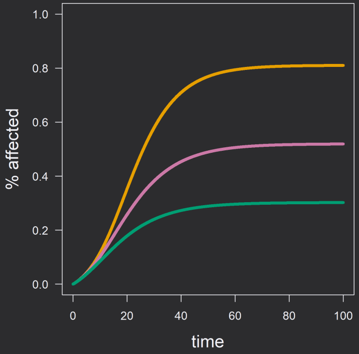

Things to notice:

- Long term: social distancing has huge affect.
- Initially, percent affected looks very similar between the three models.
- This shows pitfall of not accounting for changes in social behavior.