

Homework 2

Aiden Kenny
 STAT GR5203: Probability
 Columbia University
 October 12, 2020

Question 1 Let X have a pdf of $f(x) = cx^2$ for $0 \leq x \leq 1$ and $f(x) = 0$ elsewhere.

(a) For this to be a valid pdf, it must integrate to 1 over the support. So

$$\int_0^1 cx^2 dx = \frac{cx^3}{3} \Big|_0^1 = \frac{c}{3} \stackrel{\text{set}}{=} 1,$$

which leads to $c = 3$. So $f(x) = 3x^2$ for $0 \leq x \leq 1$.

(b) The cdf is given by

$$F(x) = \int_{-\infty}^x f(t) dt = \int_0^x 3t^2 dt = t^3 \Big|_0^x = x^3$$

for $0 \leq x \leq 1$. We also have $F(x) = 0$ when $x < 0$ and $F(x) = 1$ when $x > 1$.

(c) We have

$$\Pr\left(\frac{1}{10} \leq X \leq \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{10}\right) = \frac{1}{2^3} - \frac{1}{10^3} = \frac{31}{250}.$$

Question 2 Two discrete random variables X and Y are jointly distributed.

- (a) The marginal pmf for X is obtained by summing over every value of Y for each value of X . For example, to find the marginal probability that $X = 1$, we have $f_X(x) = 0.10 + 0.05 + 0.02 + 0.02 = 0.19$. The other values are obtained in the same way. Finding the marginal pmf for Y is done the exact same way, and it turns out that $f_X(j) = f_Y(j)$ for $j = \{1, 2, 3, 4\}$; they are both found in Table 1.
- (b) X and Y are not independent. For two random variables to be independent, we need $f(x, y) = f_X(x) \cdot f_Y(y)$ for all possible (x, y) pairs. Here we have $f(1, 1) = 0.10$ and $f_X(1) \cdot f_Y(1) = 0.19^2 \neq f(1, 1)$, meaning X and Y are dependent.
- (c) To find the conditional pmf of X given that $Y = 1$, we take each value of $f(x, 1)$ and divide by $f_Y(1)$, i.e. $f_{X|Y}(x|1) = f(x, 1)/f_Y(1)$. For example, we have $f_{X|Y}(1|1) = 0.10/0.19 = 10/19$. Finding the conditional pmf for Y given that $X = 1$ is done in a similar way, and again they are the same. Both pmfs can be found in be found in Table 1.

Question 3 We are considering points (x, y) uniformly selected within an ellipse given by the equation $(x/a)^2 + (y/b)^2 = 1$, where $a, b > 0$. Therefore, the probability of selecting a point (x, y) from this region is $f(x, y) = c$ for $-a \leq x \leq a$, $-b \leq y \leq b$, and $(x/a)^2 + (y/b)^2 \leq 1$, and $f(x, y) = 0$ elsewhere. To find c , we use the fact that a joint pdf must integrate to 1 over all values of x and y , so

$$1 = \int_{-a}^a \int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} c dy dx = c \cdot 2 \int_{-a}^a b\sqrt{1-(x/a)^2} dx = c \cdot \pi ab,$$

which implies that $c = 1/\pi ab$. The last equality is from the fact that the area of an ellipse is πab , which is what the integral is computing. To find the marginal density of X (Y), we integrate over all

| x | 1 | 2 | 3 | 4 |
|----------------|-------|------|------|------|
| $f_X(x)$ | 0.19 | 0.32 | 0.31 | 0.18 |
| $f_{X Y}(x 1)$ | 10/19 | 5/19 | 2/19 | 2/19 |

| y | 1 | 2 | 3 | 4 |
|----------------|-------|------|------|------|
| $f_Y(y)$ | 0.19 | 0.32 | 0.31 | 0.18 |
| $f_{Y X}(y 1)$ | 10/19 | 5/19 | 2/19 | 2/19 |

Table 1: Information for question 2.

possible values of Y (X):

$$f_X(x) = \int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} \frac{1}{\pi ab} dy = \frac{y}{\pi ab} \Big|_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} = \frac{2\sqrt{1-(x/a)^2}}{\pi a} \quad \text{for } -a \leq x \leq a,$$

$$f_Y(y) = \int_{-a\sqrt{1-(y/b)^2}}^{a\sqrt{1-(y/b)^2}} \frac{1}{\pi ab} dx = \frac{x}{\pi ab} \Big|_{-a\sqrt{1-(y/b)^2}}^{a\sqrt{1-(y/b)^2}} = \frac{2\sqrt{1-(y/b)^2}}{\pi b} \quad \text{for } -b \leq y \leq b.$$

These results make sense intuitively. For $f_X(x)$, we can see that the highest probability is obtained when $x = 0$. At $x = 0$, the height of the ellipse is the greatest, so there is more “width” for $x = 0$ to be chosen. And as $x \rightarrow \pm a$, $f_X(x) \rightarrow 0$, meaning that the closer you get to the end of the ellipse, the less likely that point is to be chosen.

Question 4 The joint cdf of X and Y is given by $F(x, y) = (1 - e^{-\alpha x})(1 - e^{-\beta y})$ for $x, y \geq 0$ and $F(x, y) = 0$ elsewhere, where $\alpha, \beta > 0$ are fixed constants.

- (a) X and Y are independent. This is because we can express the joint cdf as $F(x, y) = G(x) \cdot H(y)$, where $G(x) = 1 - e^{-\alpha x}$ and $H(y) = 1 - e^{-\beta y}$.
- (b) The joint pdf is given by

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x} (1 - e^{-\alpha x}) \cdot \frac{\partial}{\partial y} (1 - e^{-\beta y}) = \alpha e^{-\alpha x} \cdot \beta e^{-\beta y}$$

for $x, y \geq 0$ and $f(x, y) = 0$ elsewhere. The marginal densities are given by

$$f_X(x) = \lim_{\phi \rightarrow \infty} \int_0^\phi \alpha e^{-\alpha x} \cdot \beta e^{-\beta y} dy = \alpha e^{-\alpha x} \cdot \lim_{\phi \rightarrow \infty} [-e^{-\beta y}]_0^\phi = \alpha e^{-\alpha x},$$

$$f_Y(y) = \lim_{\phi \rightarrow \infty} \int_0^\phi \alpha e^{-\alpha x} \cdot \beta e^{-\beta y} dx = \beta e^{-\beta y} \cdot \lim_{\phi \rightarrow \infty} [-e^{-\alpha x}]_0^\phi = \beta e^{-\beta y}.$$

Here we see that $f_X(x) \cdot f_Y(y) = \alpha e^{-\alpha x} \cdot \beta e^{-\beta y} = f(x, y)$, another indication that X and Y are independent.

Question 5 Let X and Y have a joint pdf $f(x, y) = c(x^2 - y^2)e^{-x}$ for $x \leq 0$ and $-x \leq y \leq x$. To help with some of the integrations, this question makes use of the following result:

$$\Omega(\theta, n) := \int_\theta^\infty x^n e^{-x} dx = n! \cdot e^{-\theta} \sum_{k=0}^n \frac{\theta^k}{k!}$$

for all $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 0}$. This specific integral came up so many times in this question that I thought it was worth figuring out a general formula for it. A derivation of this result can be found in Appendix A.

- (a) Using the fact that the joint pdf must integrate to 1, we have

$$\begin{aligned} 1 &= \int_0^\infty \int_{-x}^x c(x^2 - y^2)e^{-x} dy dx = c \int_0^\infty e^{-x} \left[x^2 y - \frac{y^3}{3} \right]_{-x}^x dx \\ &= \frac{4c}{3} \cdot \int_0^\infty x^3 e^{-x} dx = \frac{4c \cdot \Omega(0, 3)}{3} = \frac{4c \cdot 3!}{3} = 8c, \end{aligned}$$

and so $c = 1/8$.

- (b) X and Y are not independent since the joint pdf does not have rectangular support.
- (c) When considering all possible $x \geq 0$ values, we see that we must have $-x \leq y \leq x$. So the marginal density of X is given by

$$f_X(x) = \int_{-x}^x \frac{(x^2 - y^2)e^{-x}}{8} dx = \frac{e^{-x}}{8} \left[x^2 y - \frac{y^3}{3} \right]_{-x}^x = \frac{e^{-x}}{8} \cdot \frac{4x^3}{3} = \frac{x^3 e^{-x}}{6}$$

for $x \geq 0$. When considering all possible $-\infty < y < \infty$, we must split the support into two regions: where $y \geq 0$ and where $y < 0$. When $y \geq 0$, we have $0 \leq y \leq x < \infty$, so we have $y \leq x < \infty$. In this region, the marginal density is given by

$$\begin{aligned} f_Y(y) &= \int_y^\infty \int_{-x}^x \frac{(x^2 - y^2)e^{-x}}{8} dx = \frac{1}{8} \left(\int_y^\infty x^2 e^{-x} dx - y^2 \int_y^\infty e^{-x} dx \right) \\ &= \frac{1}{8} (\Omega(y, 2) - y^2 \cdot \Omega(y, 0)) = \frac{1}{8} (e^{-y}(2 + 2y + y^2) - y^2 e^{-y}) = \frac{(1 + y)e^{-y}}{4}. \end{aligned}$$

In the second region, where $y < 0$, we have $0 > y \geq -x > -\infty$, and so we can have $-y < x < \infty$. The marginal density in this region is given by

$$\begin{aligned} f_Y(y) &= \int_{-y}^\infty \frac{(x^2 - y^2)e^{-x}}{8} dx = \frac{1}{8} \left(\int_{-y}^\infty x^2 e^{-x} dx - y^2 \int_{-y}^\infty e^{-x} dx \right) \\ &= \frac{1}{8} (\Omega(-y, 2) - y^2 \cdot \Omega(-y, 0)) = \frac{1}{8} (e^y(2 - 2y + y^2) - y^2 e^y) = \frac{(1 - y)e^y}{4}. \end{aligned}$$

As a whole, the joint density is given by

$$f_Y(y) = \begin{cases} \frac{(1 + y)e^{-y}}{4} & \text{when } y \geq 0, \\ \frac{(1 - y)e^y}{4} & \text{when } y < 0. \end{cases}$$

- (d) The conditional distribution $f_{X|Y}(x|y)$ is found by dividing $f(x, y)$ by $f_Y(y)$:

$$f_{X|Y}(x|y) = \begin{cases} \frac{(x^2 - y^2)e^{y-x}}{2(1 + y)}, & y \geq 0, y \leq x < \infty \\ \frac{(x^2 - y^2)e^{-(y+x)}}{2(1 - y)}, & y < 0, -y < x < \infty \end{cases}$$

Similarly,

$$f_{Y|X}(y|x) = \frac{3(x^2 - y^2)}{4x^2}, \quad -x \leq y \leq x.$$

Question 6 Suppose the rv X has some density $f_X(x)$ and let $Y = aX + b$ for some $a \neq 0$, meaning Y is a linear combination of X . We see that $g(X) = aX + b$ is monotonic with inverse $h(Y) = \frac{Y-b}{a}$ and $h'(Y) = 1/a$. Then the density of Y is given by

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)| = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|} = \frac{f_X\left(\frac{y-b}{a}\right)}{|a|}.$$

Question 7 Let $F(x)$ be the cdf of some unspecified discrete random variable, and let $U \sim \text{Unif}[0, 1]$. The cdf for U is given by $\tilde{F}(u) = u$ for $0 \leq u \leq 1$. If we define the random variable $Y = k$ if $F(k-1) < U \leq F(k)$, then

$$\Pr(Y \leq k) = \Pr(U \leq F(k)) = \tilde{F}(F(k)) = F(k),$$

and so the cdf of Y is actually given by $F(k)$.

Question 8 Let X be the number of cars in the left lane at a randomly chosen red light, where $x = \{0, 1, \dots, 7\}$. The engineer also believes $f(x) \propto (x+1)(8-x)$.

(a) The engineer's assumption implies that $f(x) = c(x+1)(8-x)$ for some c . So

$$\begin{aligned} 1 &= \sum_{x=0}^7 c(x+1)(8-x) = c \sum_{i=0}^7 (8+7x-x^2) = c \left(8 \sum_{i=0}^7 1 + 7 \sum_{i=0}^7 x - \sum_{i=0}^7 x^2 \right) \\ &= c \left(8 \cdot 8 + 7 \cdot \frac{7 \cdot 8}{2} - \frac{7 \cdot 8 \cdot 15}{6} \right) = 120c, \end{aligned}$$

which means $c = 1/120$ and $f(x) = (x+1)(8-x)/120$.

(b) We have

$$\Pr(X \geq 5) = \sum_{x=5}^7 \frac{(x+1)(8-x)}{120} = \frac{6 \cdot 3 + 7 \cdot 2 + 8 \cdot 1}{120} = \frac{40}{120} = \frac{1}{3}.$$

Question 9 Let X have a pdf given by $f(x) = x/8$ for $0 \leq x \leq 4$. The cdf is then given by

$$F(x) = \int_0^x \frac{t}{8} dt = \frac{t^2}{16} \Big|_0^x = \frac{x^2}{16}$$

for $0 \leq x \leq 4$, $F(x) = 0$ if $x < 0$, and $F(x) = 1$ if $x > 4$.

(a) We have $\Pr(X \leq t) = F(t) = t^2/16 \stackrel{\text{set}}{=} 1/4$, and solving for t gives us $t = 2$.

(b) We have $\Pr(X \geq t) = 1 - F(t) = (16 - t^2)/16 \stackrel{\text{set}}{=} 1/2$, and solving for t gives us $t = 2\sqrt{2}$.

Question 10 Let X and Y be random variables such that $0 \leq x \leq 3$ and $0 \leq y \leq 4$. Within this rectangular region, let the joint cdf be $F(x, y) = xy(x^2 + y)/156$. From this we can also determine the joint cdf outside the region of joint support. To start we know that $F(x, y) = 0$ if $x < 0$ or $y < 0$. When $0 \leq X \leq 3$ and $y > 4$ (beyond the support of Y), we have $F(x, y) = F_X(x)$, and when $0 \leq Y \leq 4$ but $X > 3$ (beyond the support of X), we have $F(x, y) = F_Y(y)$; these quantities are given by

$$\begin{aligned} F_X(x) &= \lim_{y \rightarrow 4} F(x, y) = F(x, 4) = \frac{x(x^2 + 4)}{39}, \\ F_Y(y) &= \lim_{x \rightarrow 3} F(x, y) = F(3, y) = \frac{y(9 + y)}{52}. \end{aligned}$$

(a) We have

$$\begin{aligned} \Pr(1 \leq X \leq 2 \cap 1 \leq Y \leq 2) &= F(2, 2) - F(2, 1) - F(1, 2) + F(1, 1) \\ &= \frac{2 \cdot 2(4 + 2)}{156} + \frac{1 \cdot 2(1 + 2)}{156} + \frac{2 \cdot 1(4 + 1)}{156} + \frac{1 \cdot 1(1 + 1)}{156} = \frac{24 - 6 - 10 + 2}{156} = \frac{5}{78}. \end{aligned}$$

(b) We have

$$\begin{aligned} \Pr(2 \leq X \leq 4 \cap 2 \leq Y \leq 4) &= F(4, 4) - F(4, 2) - F(2, 4) + F(2, 2) \\ &= \frac{4(9 + 4)}{52} + \frac{2(9 + 2)}{52} + \frac{2 \cdot 4(4 + 4)}{156} + \frac{2 \cdot 2(4 + 2)}{156} = \frac{156 - 66 - 64 + 24}{156} = \frac{25}{78}. \end{aligned}$$

(c) The cdf of Y is given by $F_Y(y) = y(9 + y)/52$ for $0 \leq Y \leq 4$, $F_Y(y) = 0$ if $y < 0$, and $F_Y(y) = 1$ if $y > 4$.

(d) The joint pdf of X and Y is given by

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{xy(x^2 + y)}{156} = \frac{\partial}{\partial x} \frac{x(x^2 + y) + xy}{156} = \frac{3x^2 + 2y}{156}$$

when $0 \leq x \leq 3$ and $0 \leq y \leq 4$, and $f(x, y) = 0$ outside of the joint support.

(e) We have

$$\begin{aligned}\Pr(Y \leq X) &= \int_0^3 \int_0^x \frac{3x^2 + 2y}{156} dy dx = \frac{1}{156} \int_0^3 [3x^2 y + y^2]_0^x dx = \frac{1}{156} \int_0^3 (3x^3 + x^2) dx \\ &= \frac{1}{156} \left[\frac{3x^4}{4} + \frac{x^3}{3} \right]_0^3 = \frac{1}{156} \left(\frac{3 \cdot 81}{4} + \frac{27}{3} \right) = \frac{243 + 36}{156 \cdot 4} = \frac{93}{208}.\end{aligned}$$

Appendix A Here we will prove the result used throughout question 5. By using integration by parts with $u = x^n$ and $dv = e^{-x}$, we have

$$\int x^n e^{-x} dx = -x^n e^{-x} + n \int x^{n-1} e^{-x} dx + C.$$

The next integral must also be solved via integration by parts, with $u = x^{n-1}$ and $dv = e^{-x}$:

$$\begin{aligned}\int x^n e^{-x} dx &= -x^n e^{-x} + n \left(-x^{n-1} e^{-x} + (n-1) \int x^{n-2} e^{-x} dx \right) + C \\ &= -x^n e^{-x} - nx^{n-1} e^{-x} - n(n-1) \int x^{n-2} e^{-x} dx + C.\end{aligned}$$

By now a pattern starts to appear. We have to repeat the integration by parts for the right most integral n times in total. Completing this pattern and factoring out an $n!$ from each term yields

$$\begin{aligned}\int x^n e^{-x} dx &= -x^n e^{-x} - nx^{n-1} e^{-x} - n(n-1)x^{n-2} e^{-x} - \dots - n! + C \\ &= -e^{-x} \left(\frac{n!}{n!} x^n + \frac{n!}{(n-1)!} x^{n-1} + \frac{n!}{(n-2)!} x^{n-2} + \dots + \frac{n!}{0!} x^0 \right) + C \\ &= -n! \cdot e^{-x} \left(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \frac{x^{n-2}}{(n-2)!} + \dots + \frac{x^0}{0!} \right) + C = -n! \cdot e^{-x} \sum_{k=0}^n \frac{x^k}{k!} + C.\end{aligned}$$

Evaluating this integral from θ to ∞ gives us

$$\int_{\theta}^{\infty} x^n e^{-x} dx = \left[-n! \cdot e^{-x} \sum_{k=0}^n \frac{x^k}{k!} \right]_{\theta}^{\infty} = n! \cdot e^{-\theta} \sum_{k=0}^n \frac{\theta^k}{k!} - n! \lim_{\phi \rightarrow \infty} e^{-\phi} \sum_{k=0}^n \frac{\phi^k}{k!},$$

and using L'Hopital's rule n times on the right limit shows that

$$\lim_{\phi \rightarrow \infty} e^{-\phi} \sum_{k=0}^n \frac{\phi^k}{k!} = \lim_{\phi \rightarrow \infty} (-1)^n e^{-\phi} = 0.$$

which gives us our desired result. □