Homework 2

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Question 1 Let X have a pdf of $f(x) = cx^2$ for $0 \leq x \leq 1$ and $f(x) = 0$ elsewhere.

(a) For this to be a valid pdf, it must integrate to 1 over the support. So

$$\int_0^1 cx^2 \, \mathrm{d}x = \frac{cx^3}{3}\bigg|_0^1 = \frac{c}{3} \overset{\text{set}}{=} 1,$$

which leads to $c = 3$. So $f(x) = 3x^2$ for $0 \leq x \leq 1$.

(b) The cdf is given by

$$F(x) = \int_{-\infty}^{x} f(t) \, \mathrm{d}t = \int_0^x 3t^2 \, \mathrm{d}t = t^3\big|_0^x = x^3$$

for $0 \leq x \leq 1$. We also have $F(x) = 0$ when $x < 0$ and $F(x) = 1$ when $x > 1$.

(c) We have

$$\Pr\left(\frac{1}{10} \leq X \leq \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{10}\right) = \frac{1}{2^3} - \frac{1}{10^3} = \frac{31}{250}.$$

Question 2 Two discrete random variables X and Y are jointly distributed.

(a) The marginal pmf for X is obtained by summing over every value of Y for each value of X. For example, to find the marginal probability that $X = 1$, we have $f_X(x) = 0.10+0.05+0.02+0.02 = 0.19$. The other values are obtained in the same way. Finding the marginal pmf for Y is done the exact same way, and it turns out that $f_X(j) = f_Y(j)$ for $j = \{1, 2, 3, 4\}$; they are both found in Table 1.

(b) X and Y are not independent. For two random variables to be independent, we need $f(x, y) = f_X(x) \cdot f_Y(y)$ for all possible (x, y) pairs. Here we have $f(1, 1) = 0.10$ and $f_X(1) \cdot f_Y(1) = 0.19^2 \neq f(1, 1)$, meaing X and Y are dependent.

(c) To find the conditional pmf of X given that $Y = 1$, we take each value of $f(x, 1)$ and divide by $f_Y(1)$, i.e. $f_{X|Y}(x|1) = f(x, 1)/f_Y(1)$. For example, we have $f_{X|Y}(1|1) = 0.10/0.19 = 10/19$. Finding the conditional pmf for Y given that $X = 1$ is done in a similar way, and again they are the same. Both pmfs can be found in be found in Table 1.

$X \backslash Y$	1	2	3	4
1	0.10	0.05	0.02	0.02
2	0.05	0.20	0.05	0.02
3	0.02	0.05	0.20	0.04
4	0.02	0.02	0.04	0.10

x	1	2	3	4
$f_X(x)$	0.19	0.32	0.31	0.18
$f_{X\|Y}(x\|1)$	10/19	5/19	2/19	2/19

y	1	2	3	4
$f_Y(y)$	0.19	0.32	0.31	0.18
$f_{Y\|X}(y\|1)$	10/19	5/19	2/19	2/19

Table 1: Information for question 2.