

**Homework 2**

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**Question 1** Let  $X$  have a pdf of  $f(x) = cx^2$  for  $0 \leq x \leq 1$  and  $f(x) = 0$  elsewhere.

(a) For this to be a valid pdf, it must integrate to 1 over the support. So

$$1 = \int_0^1 cx^2 \, dx = \frac{cx^3}{3} \Big|_0^1 = \frac{c}{3},$$

which leads to  $c = 3$ . So  $f(x) = 3x^2$  for  $0 \leq x \leq 1$ .

(b) The cdf is given by

$$F(x) = \int_{-\infty}^x f(t) \, dt = \int_0^x 3t^2 \, dt = t^3 \Big|_0^x = x^3$$

for  $0 \leq x \leq 1$ . We also have  $F(x) = 0$  when  $x < 0$  and  $F(x) = 1$  when  $x > 1$ .

(c) We have

$$\Pr\left(\frac{1}{10} \leq X \leq \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{10}\right) = \frac{1}{2^3} - \frac{1}{10^3} = \frac{31}{250}.$$

**Question 2** Two discrete random variables  $X$  and  $Y$  are jointly distributed.

- (a) The marginal pmf for  $X$  is obtained by summing over every value of  $Y$  for each value of  $X$ . For example, to find the marginal probability that  $X = 1$ , we have  $f_X(1) = 0.10 + 0.05 + 0.02 + 0.02 = 0.19$ . The other values are obtained in the same way. Finding the marginal pmf for  $Y$  is done the exact same way, and it turns out that  $f_X(j) = f_Y(j)$  for  $j = \{1, 2, 3, 4\}$ ; they are both found in Table 1.
- (b)  $X$  and  $Y$  are not independent. For two random variables to be independent, we need  $f(x, y) = f_X(x) \cdot f_Y(y)$  for all possible  $(x, y)$  pairs. Here we have  $f(1, 1) = 0.10$  and  $f_X(1) \cdot f_Y(1) = 0.19^2 \neq f(1, 1)$ , meaning  $X$  and  $Y$  are dependent.
- (c) To find the conditional pmf of  $X$  given that  $Y = 1$ , we take each value of  $f(x, 1)$  and divide by  $f_Y(1)$ , i.e.  $f_{X|Y}(x|1) = f(x, 1)/f_Y(1)$ . For example, we have  $f_{X|Y}(1|1) = 0.10/0.19 = 10/19$ . Finding the conditional pmf for  $Y$  given that  $X = 1$  is done in a similar way, and again they are the same. Both pmfs can be found in Table 1.

**Question 3** We are considering points  $(x, y)$  uniformly selected within an ellipse given by the equation  $(x/a)^2 + (y/b)^2 = 1$ , where  $a, b > 0$ . Therefore, the probability of selecting a point  $(x, y)$  from this region is  $f(x, y) = c$  for  $-a \leq x \leq a$ ,  $-b \leq y \leq b$ , and  $(x/a)^2 + (y/b)^2 \leq 1$ , and  $f(x, y) = 0$  elsewhere. To find  $c$ , we use the fact that a joint pdf must integrate to 1 over all values of  $x$  and  $y$ , so

$$1 = \int_{-a}^a \int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} c \, dy \, dx = c \cdot 2 \int_{-a}^a b\sqrt{1-(x/a)^2} \, dx = c \cdot \pi ab,$$

which implies that  $c = 1/\pi ab$ . The last equality is from the fact that the area of an ellipse is  $\pi ab$ , which is what the integral is computing. To find the marginal density of  $X$  ( $Y$ ), we integrate over all

$x$	1	2	3	4
$f_X(x)$	0.19	0.32	0.31	0.18
$f_{X Y}(x 1)$	10/19	5/19	2/19	2/19

$y$	1	2	3	4
$f_Y(y)$	0.19	0.32	0.31	0.18
$f_{Y X}(y 1)$	10/19	5/19	2/19	2/19

Table 1: Information for question 2.

possible values of  $Y$  ( $X$ ):

$$f_X(x) = \int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} \frac{1}{\pi ab} dy = \frac{y}{\pi ab} \Big|_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} = \frac{2\sqrt{1-(x/a)^2}}{\pi a} \quad \text{for } -a \leq x \leq a,$$

$$f_Y(y) = \int_{-a\sqrt{1-(y/b)^2}}^{a\sqrt{1-(y/b)^2}} \frac{1}{\pi ab} dx = \frac{x}{\pi ab} \Big|_{-a\sqrt{1-(y/b)^2}}^{a\sqrt{1-(y/b)^2}} = \frac{2\sqrt{1-(y/b)^2}}{\pi b} \quad \text{for } -b \leq y \leq b.$$

These results make sense intuitively. For  $f_X(x)$ , we can see that the highest probability is obtained when  $x = 0$ . At  $x = 0$ , the height of the ellipse is the greatest, so there is more “width” for  $x = 0$  to be chosen. And as  $x \rightarrow \pm a$ ,  $f_X(x) \rightarrow 0$ , meaning that the closer you get to the end of the ellipse, the less likely that point is to be chosen.

**Question 4** The joint cdf of  $X$  and  $Y$  is given by  $F(x, y) = (1 - e^{-\alpha x})(1 - e^{-\beta y})$  for  $x, y \geq 0$  and  $F(x, y) = 0$  elsewhere, where  $\alpha, \beta > 0$  are fixed constants.

- (a)  $X$  and  $Y$  are independent. This is because we can express the joint cdf as  $F(x, y) = G(x) \cdot H(y)$ , where  $G(x) = 1 - e^{-\alpha x}$  and  $H(y) = 1 - e^{-\beta y}$ .
- (b) The joint pdf is given by

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x} (1 - e^{-\alpha x}) \cdot \frac{\partial}{\partial y} (1 - e^{-\beta y}) = \alpha e^{-\alpha x} \cdot \beta e^{-\beta y}$$

for  $x, y \geq 0$  and  $f(x, y) = 0$  elsewhere. The marginal densities are given by

$$f_X(x) = \int_0^\infty \alpha e^{-\alpha x} \cdot \beta e^{-\beta y} dy = \alpha e^{-\alpha x} \cdot \lim_{\phi \rightarrow \infty} [-e^{-\beta y}]_0^\phi = \alpha e^{-\alpha x},$$

$$f_Y(y) = \int_0^\infty \alpha e^{-\alpha x} \cdot \beta e^{-\beta y} dx = \beta e^{-\beta y} \cdot \lim_{\phi \rightarrow \infty} [-e^{-\alpha x}]_0^\phi = \beta e^{-\beta y}.$$

Here we see that  $f_X(x) \cdot f_Y(y) = \alpha e^{-\alpha x} \cdot \beta e^{-\beta y} = f(x, y)$ , another indication that  $X$  and  $Y$  are independent.

**Question 5** *Collaborators:* Melissa Wu, Mariko Sawada

Let  $X$  and  $Y$  have a joint pdf  $f(x, y) = c(x^2 - y^2)e^{-x}$  for  $x \leq 0$  and  $-x \leq y \leq x$ . To help with some of the integrations, we make use of the following result:

$$\Omega(\theta, n) := \int_\theta^\infty x^n e^{-x} dx = n! \cdot e^{-\theta} \sum_{k=0}^n \frac{\theta^k}{k!}$$

for all  $\theta \in \mathbb{R}$  and  $n \in \mathbb{Z}_{\geq 0}$ . This specific integral came up so many times in this question that I thought it was worth figuring out a general formula for it. A derivation of this result can be found in Appendix A.

- (a) Using the fact that the joint pdf must integrate to 1, we have

$$\begin{aligned} 1 &= \int_0^\infty \int_{-x}^x c(x^2 - y^2)e^{-x} dy dx = c \int_0^\infty e^{-x} \left[ x^2 y - \frac{y^3}{3} \right]_{-x}^x dx \\ &= \frac{4c}{3} \cdot \int_0^\infty x^3 e^{-x} dx = \frac{4c \cdot \Omega(0, 3)}{3} = \frac{4c \cdot 3!}{3} = 8c, \end{aligned}$$

and so  $c = 1/8$ .

- (b)  $X$  and  $Y$  are not independent since the joint pdf does not have rectangular support.
- (c) The support for  $X$  and  $Y$  can be viewed two ways: all points such that  $x \geq 0$  and  $-x \leq y \leq x$ , and all points such that  $y \in \mathbb{R}$  and  $|y| \leq x < \infty$ . The marginal density of  $X$  is given by

$$f_X(x) = \int_{-x}^x \frac{(x^2 - y^2)e^{-x}}{8} dx = \frac{e^{-x}}{8} \left[ x^2 y - \frac{y^3}{3} \right]_{-x}^x = \frac{e^{-x}}{8} \cdot \frac{4x^3}{3} = \frac{x^3 e^{-x}}{6}$$

for  $x \geq 0$  and  $f_X(x) = 0$  otherwise. The marginal density of  $Y$  is given by

$$\begin{aligned} f_Y(y) &= \int_{|y|}^{\infty} \frac{(x^2 - y^2)e^{-x}}{8} dx = \frac{1}{8} \left( \int_{|y|}^{\infty} x^2 e^{-x} dx - y^2 \int_{|y|}^{\infty} e^{-x} dx \right) \\ &= \frac{1}{8} \left( \Omega(|y|, 2) - y^2 \cdot \Omega(|y|, 0) \right) = \frac{1}{8} \left( 2e^{-|y|}(1 + |y| + |y|^2/2) - y^2 e^{-|y|} \right) \\ &= \frac{1}{8} \left( 2e^{-|y|} + 2|y|e^{-|y|} + y^2 e^{-|y|} - y^2 e^{-|y|} \right) = \frac{(1 + |y|)e^{-|y|}}{4} \end{aligned}$$

for all  $y \in \mathbb{R}$ .

- (d) The conditional distribution for  $X$  given  $Y$  is

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{\frac{1}{8}(x^2 - y^2)e^{-x}}{\frac{1}{4}(1 + |y|)e^{-|y|}} = \frac{(x^2 - y^2)e^{-(x+|y|)}}{2(1 + |y|)}$$

for  $|y| \leq x < \infty$  and  $f_{X|Y}(x|y) = 0$  elsewhere. Similarly, the conditional distribution for  $Y$  given  $X$  is

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{\frac{1}{8}(x^2 - y^2)e^{-x}}{\frac{1}{6}x^3 e^{-x}} = \frac{3(x^2 - y^2)}{4x^3}$$

for  $-x \leq y \leq x$  and  $f_{Y|X}(y|x) = 0$  elsewhere.

**Question 6** Suppose the random variable  $X$  has some density  $f_X(x)$  and let  $Y = aX + b$  for some  $a \neq 0$ , meaning  $Y$  is a linear combination of  $X$ . We see that  $g(X) = aX + b$  is monotonic with inverse  $h(Y) = \frac{Y-b}{a}$  and  $h'(Y) = 1/a$ . Then the density of  $Y$  is given by

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)| = f_X\left(\frac{y-a}{b}\right) \cdot \frac{1}{|a|} = \frac{y}{|a|}.$$

**Question 7** Let  $F(x)$  be the cdf of some unspecified discrete random variable, and let  $U \sim \text{Unif}[0, 1]$ . The cdf for  $U$  is given by  $\tilde{F}(u) = u$  for  $0 \leq u \leq 1$ . If we define the random variable  $Y = k$  if  $F(k-1) < U \leq F(k)$ , then

$$\Pr(Y \leq k) = \Pr(U \leq F(k)) = \tilde{F}(F(k)) = F(k),$$

and so the cdf of  $Y$  is actually given by  $F(k)$ .

**Question 8** Let  $X$  be the number of cars in the left lane at a randomly chosen red light, where  $x = \{0, 1, \dots, 7\}$ . The engineer also believes  $f(x) \propto (x+1)(8-x)$ .

- (a) The engineer's assumption implies that  $f(x) = c(x+1)(8-x)$  for some  $c$ . So

$$\begin{aligned} 1 &= \sum_{x=0}^7 c(x+1)(8-x) = c \sum_{i=0}^7 (8 + 7x - x^2) = c \left( 8 \sum_{i=0}^7 1 + 7 \sum_{i=0}^7 x - \sum_{i=0}^7 x^2 \right) \\ &= c \left( 8 \cdot 8 + 7 \cdot \frac{7 \cdot 8}{2} - \frac{7 \cdot 8 \cdot 15}{6} \right) = 120c, \end{aligned}$$

which means  $c = 1/120$  and  $f(x) = (x+1)(8-x)/120$ .

(b) We have

$$\Pr(X \geq 5) = \sum_{x=5}^7 \frac{(x+1)(8-x)}{120} = \frac{6 \cdot 3 + 7 \cdot 2 + 8 \cdot 1}{120} = \frac{40}{120} = \frac{1}{3}.$$

**Question 9** Let  $X$  have a pdf given by  $f(x) = x/8$  for  $0 \leq x \leq 4$ . The cdf is then given by

$$F(x) = \int_0^x \frac{t}{8} dt = \frac{t^2}{16} \Big|_0^x = \frac{x^2}{16}$$

for  $0 \leq x \leq 4$ ,  $F(x) = 0$  if  $x < 0$ , and  $F(x) = 1$  if  $x > 4$ .

(a) We have  $\Pr(X \leq t) = F(t) = t^2/16 \stackrel{\text{set}}{=} 1/4$ , and solving for  $t$  gives us  $t = 2$ .

(b) We have  $\Pr(X \geq t) = 1 - F(t) = (16 - t^2)/16 \stackrel{\text{set}}{=} 1/2$ , and solving for  $t$  gives us  $t = 2\sqrt{2}$ .

**Question 10** Let  $X$  and  $Y$  be random variables such that  $0 \leq x \leq 3$  and  $0 \leq y \leq 4$ . Within this rectangular region, let the joint cdf be  $F(x, y) = xy(x^2 + y)/156$ . From this we can also determine the joint cdf outside the region of joint support. To start we know that  $F(x, y) = 0$  if  $x < 0$  or  $y < 0$ . When  $0 \leq X \leq 3$  and  $y > 4$  (beyond the support of  $Y$ ), we have  $F(x, y) = F_X(x)$ , and when  $0 \leq Y \leq 4$  but  $X > 3$  (beyond the support of  $X$ ), we have  $F(x, y) = F_Y(y)$ ; these quantities are given by

$$F_X(x) = \lim_{y \rightarrow 4} F(x, y) = F(x, 4) = \frac{x(x^2 + y)}{39},$$

$$F_Y(y) = \lim_{x \rightarrow 3} F(x, y) = F(3, y) = \frac{y(9 + y)}{52}.$$

(a) We have

$$\begin{aligned} \Pr(1 \leq X \leq 2 \cap 1 \leq Y \leq 2) &= F(2, 2) - F(2, 1) - F(1, 2) + F(1, 1) \\ &= \frac{2 \cdot 2(4 + 2)}{156} + \frac{1 \cdot 2(1 + 2)}{156} + \frac{2 \cdot 1(4 + 1)}{156} + \frac{1 \cdot 1(1 + 1)}{156} = \frac{24 - 6 - 10 + 2}{156} = \frac{5}{78}. \end{aligned}$$

(b) We have

$$\begin{aligned} \Pr(2 \leq X \leq 4 \cap 2 \leq Y \leq 4) &= F(4, 4) - F(4, 2) - F(2, 4) + F(2, 2) \\ &= \frac{4(9 + 4)}{52} + \frac{2(9 + 2)}{52} + \frac{2 \cdot 4(4 + 4)}{156} + \frac{2 \cdot 2(4 + 2)}{156} = \frac{156 - 66 - 64 + 24}{156} = \frac{25}{78}. \end{aligned}$$

(c) The cdf of  $Y$  is given by  $F_Y(y) = y(9 + y)/52$  for  $0 \leq Y \leq 4$ ,  $F_Y(y) = 0$  if  $y < 0$ , and  $F_Y(y) = 1$  if  $y > 4$ .

(d) The joint pdf of  $X$  and  $Y$  is given by

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{xy(x^2 + y)}{156} = \frac{\partial}{\partial x} \frac{x(x^2 + y) + xy}{156} = \frac{3x^2 + 2y}{156}$$

when  $0 \leq x \leq 3$  and  $0 \leq y \leq 4$ , and  $f(x, y) = 0$  outside of the joint support.

(e) We have

$$\begin{aligned} \Pr(Y \leq X) &= \int_0^3 \int_0^x \frac{3x^2 + 2y}{156} dy dx = \frac{1}{156} \int_0^3 [3x^2 y + y^2]_0^x dx = \frac{1}{156} \int_0^3 (3x^3 + x^2) dx \\ &= \frac{1}{156} \left[ \frac{3x^4}{4} + \frac{x^3}{3} \right]_0^3 = \frac{1}{156} \left( \frac{3 \cdot 81}{4} + \frac{27}{3} \right) = \frac{243 + 36}{156 \cdot 4} = \frac{93}{208}. \end{aligned}$$

**Appendix A** Here we will prove the result used throughout question 5. By using integration by parts with  $u = x^n$  and  $dv = e^{-x}$ , we have

$$\int x^n e^{-x} dx = -x^n e^{-x} + n \int x^{n-1} e^{-x} dx + C.$$

The next integral must also be solved via integration by parts, with  $u = x^{n-1}$  and  $dv = e^{-x}$ :

$$\begin{aligned} \int x^n e^{-x} dx &= -x^n e^{-x} + n \left( -x^{n-1} e^{-x} + (n-1) \int x^{n-2} e^{-x} dx \right) + C \\ &= -x^n e^{-x} - nx^{n-1} e^{-x} - n(n-1) \int x^{n-2} e^{-x} dx + C. \end{aligned}$$

By now a pattern starts to appear. We have to repeat the integration by parts for the right most integral  $n$  times in total. Completing this pattern and factoring out an  $n!$  from each term yields

$$\begin{aligned} \int x^n e^{-x} dx &= -x^n e^{-x} - nx^{n-1} e^{-x} - n(n-1)x^{n-2} e^{-x} - \dots - n! + C \\ &= -e^{-x} \left( \frac{n!}{n!} x^n + \frac{n!}{(n-1)!} x^{n-1} + \frac{n!}{(n-2)!} x^{n-2} + \dots + \frac{n!}{0!} x^0 \right) + C \\ &= -n! \cdot e^{-x} \left( \frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \frac{x^{n-2}}{(n-2)!} + \dots + \frac{x^0}{0!} \right) + C = -n! \cdot e^{-x} \sum_{k=0}^n \frac{x^k}{k!} + C. \end{aligned}$$

Evaluating this integral from  $\theta$  to  $\infty$  gives us

$$\int_{\theta}^{\infty} x^n e^{-x} dx = \left[ -n! \cdot e^{-x} \sum_{k=0}^n \frac{x^k}{k!} \right]_{\theta}^{\infty} = n! \cdot e^{-\theta} \sum_{k=0}^n \frac{\theta^k}{k!} - n! \lim_{\phi \rightarrow \infty} e^{-\phi} \sum_{k=0}^n \frac{\phi^k}{k!},$$

and using L'Hopital's rule  $n$  times on the right limit shows that

$$\lim_{\phi \rightarrow \infty} e^{-\phi} \sum_{k=0}^n \frac{\phi^k}{k!} = \lim_{\phi \rightarrow \infty} (-1)^n e^{-\phi} = 0.$$

which gives us our desired result. □