Homework 1
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Question 1 We will let H denote a heads and T denote a tails.

(a) The sample space is given by $\mathcal{S} = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTH}, \text{TTT}\}$.

(b) We have

1. $A =$ at least two heads $= \{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}\}$.
2. $B =$ the first two tosses are heads $= \{\text{HHH}, \text{HHT}\}$.
3. $C =$ the last toss is a tail $= \{\text{HHT}, \text{HTT}, \text{THT}, \text{TTT}\}$.

(c) We have

1. $A^c = \{\text{HTT}, \text{THT}, \text{TTH}, \text{TTT}\}$.
2. $A \cap B = \{\text{HHH}, \text{HHT}\} = B$ (since $B \subset A$).
3. $A \cup C = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTT}\}$.

Question 2 For each of the scenarios, we are pulling five cards from a well-shuffled deck without replacement. Since the order of the cards does not matter, there are $\binom{52}{5}$ possible combinations.

(a) *Royal flush*: since the order does not matter, for a given suite there is only one possible royal flush, so there are only four possible royal flushes (one for each suite). So the probability of getting a royal flush is $4/\binom{52}{5}$.

(b) *Straight flush*: for a given suite, not counting the royal flush, there are nine possible card combinations that fit this criteria, meaning there are 36 possible straight flushes. So the probability of getting a straight flush is $36/\binom{52}{5}$.

(c) *Four of a kind*: to select four cards of the same value, we must select all four suites of a given value, of which there are 13. Once this happens, four of the five cards in the hand have been determined, and we just have to select the last card. There are 12 possible card values and four possible suites. Therefore, there are $13 \cdot 12 \cdot 4 = 624$ plausible hands, and the probability of getting a four of a kind is $624/\binom{52}{5}$.

(d) *Flush*: each suite has 13 unique values, so there are $\binom{13}{5}$ ways to select five cards for a given suite. However, this is including the ten possible *consecutive* hands, which must be removed (as that hand would be either a straigh or royal flush). This can be done for each suite, so there are $4\left(\binom{13}{5} - 10\right)$ plausible hands, and so the probability of getting a flush is $4\left(\binom{13}{5} - 10\right)/\binom{52}{5}$.

(e) *Three of a kind*: to get three cards of the same value, for a given value we have to choose three cards from the four possible suites. There are $\binom{4}{3}$ ways to do this for each of the 13 values. For the remaining two cards to be chosen, there are 12 possible values to choose from (choosing the same value of the three matching cards would give us a four of a kind), and each of these two cards can be any of the four suites. That is, there are $\binom{12}{2} \cdot 4 \cdot 4$ ways to choose the last two cards. Therefore, there are $\binom{4}{3} \cdot 13 \cdot \binom{12}{2} \cdot 4 \cdot 4 = 208\binom{4}{3}\binom{12}{2}$ plausible hands, and so the probability of getting a three of a kind is $208\binom{4}{3}\binom{12}{2}/\binom{52}{5}$.

(f) *Two pairs*: to get two pairs of cards of the same value, we have to have two unique values to begin with, and there are $\binom{13}{2}$ ways to choose them. For each value, we are choosing two of the four possible suites, so there are $\binom{4}{2}$ choices *for each value*. For the last card, there are 11 possible values (choosing either of the previous values would result in a three of a kind), and from these there are four possible suites, so there are 44 ways to choose the last card. So there are $44\binom{13}{2}\binom{4}{2}^2$ plausible hands, and so the probability of getting two pairs is $44\binom{13}{2}\binom{4}{2}^2/\binom{52}{5}$.

Question 3 Let E be the event that the president is a woman, F be the event that the vice-president is a man, and G be the event that both leaders are of the same sex. Since we are choosing leaders without replacement and with regard to order, there are $48 \cdot 47$ possible leadership arrangements. We are also assuming committee choices are independent.

(a) For E to happen we only care that the president is a woman, the sex of the vice-president is irrelevant. So there are 16 possible choices for the president, and then any of the 47 remianing members can be chosen. So there are $16 \cdot 47$ outcomes in E, and $\Pr(E) = \frac{16 \cdot 47}{48 \cdot 47} = 1/3$. Similarly, for F we only care about the gender of the vice president. Here there are two possibilities: a man is chosen as the president or a woman is chosen as the president. For the former, there are $32 \cdot 31$ possibilities, and for the latter, there are $16 \cdot 32$ possibilities. Since the two situations are disjoint, we have $\Pr(F) = \frac{32 \cdot 31 + 16 \cdot 32}{48 \cdot 47} = 2/3$. Finally, there are $32 \cdot 31$ combinations of two male leaders and $16 \cdot 15$ combinations of female leaders, and since the two possibilities are disjoint, we have $\Pr(G) = \frac{32 \cdot 31 + 16 \cdot 15}{48 \cdot 47} = 77/141$.

(b) $E \cap F$ is the event that the president is female and the vice president is male. There are $16 \cdot 32$ ways this can happen, so $\Pr(E \cap F) = \frac{16 \cdot 32}{48 \cdot 47} = 32/141$. To find $\Pr(E \cup F)$ (the probability that the president is female or the vice-president is a male), we have

$$\Pr(E \cup F) - \Pr(E) + \Pr(F) - \Pr(E \cap F) = 1/3 + 2/3 - 32/141 = 109/141.$$

Finally, since the event $E \cap F$ requires the leaders to be of opposite sex, we know that $(E \cap F) \cap G = \emptyset$, which means $\Pr(E \cap F \cap G) = 0$.

(c) There are two possibilities when the event G occurs: both leaders are male, or both leaders are female. Since these possibilities are disjoint, we have $G = (G \cap E) \cup (G \cap F)$. Then, using the definition of conditional probability, we have

$$\Pr(G|E \cup F) = \frac{\Pr(G \cap (E \cup F))}{\Pr(E \cup F)} = \frac{\Pr\big((G \cap E) \cup (G \cap F)\big)}{\Pr(E \cup F)} = \frac{\Pr(G)}{\Pr(E \cup F)} = \frac{77}{109}.$$

Question 4 Since we want to consider the location of the four aces in the card deck, order does matter here, and so there are 52! ways of shuffling the deck. To solve this problem, we will first shuffle the four aces separately, then shuffle the remaining cards, then finally insert the four aces together into the rest of the deck. There are 4! ways to arrange the four aces and 48! ways of arranging the remaining cards, and there are 49 possible spaces where the four aces can be inserted into the rest of the deck. Therefore, there are $4! \cdot 48! \cdot 49 = 4! \cdot 49!$ plausible deck arrangements, and so the probability that the four aces are next to each other is $4! \cdot 49!/52! = 1/5525$.

Question 5 For this question, we only care about how the first 30 students are chosen, since that will automatically tell us the students in both classrooms, and there are $\binom{60}{30}$ ways to choose 30 of the 60 students.

(a) There is only 1 way to choose all five friends to be in the same class, and then there are $\binom{55}{25}$ ways to choose the remaining 25 studnets. So the probability of all five friends being in the same class is $\binom{55}{25}/\binom{60}{30}$.

(b) There are $\binom{5}{4}$ ways of choosing four of the five friends to be in the same class, and then $\binom{56}{26}$ ways to choose the remaining 26 students. So the probability that four of the five friends is in the same class is $\binom{5}{4}\binom{56}{26}/\binom{60}{30}$. We can see a more general pattern here: there are $\binom{5}{j}\binom{60-j}{30-j}$ ways to have j of the five friends be in the same class, for $j = 0, \ldots, 5$; in part (a), $j = 5$, and in part (b), $j = 4$.

(c) While there are $\binom{5}{4}$ ways for *any* of the four friends to be chosen in the class, there is only *one* way in which Marcelle is left out. Since there are $\binom{56}{26}$ ways to choose the other 26 students whenever Marcell is left out, the probability that Marcell is in class by herself is $\binom{56}{25}/\binom{60}{30}$.

Question 6 Let I_j be the event that the stock's price moves up on day j, and let D_j be the event that the stock's price moves down on day j. We have $\Pr(I_j) = p$ and $\Pr(D_j) = 1 - p$ for all j, the changes on each day are independent, and each unique sequence of events is disjoint.

(a) Let E be the event that the stock's price stays the same after two days. There are two ways this can happen: $I_1 \cap D_2$ or $D_1 \cap I_2$. So

$$\begin{aligned}\Pr(E) &= \Pr\Big((I_1 \cap D_2) \cup (D_1 \cap I_2)\Big) \\ &= \Pr(I_1)\Pr(D_2) + \Pr(D_1)\Pr(I_2) = p(1-p) + (1-p)p = 2p(1-p).\end{aligned}$$

(b) Let F be the event that the stock's price increases by 1 unit after three days. There are three ways this can happen: $I_1 \cap D_2 \cap I_3$, $D_1 \cap I_2 \cap I_3$, or $I_1 \cap I_2 \cap D_3$. So

$$\begin{aligned}\Pr(F) &= \Pr\Big((I_1 \cap D_2 \cap I_3) \cup (D_1 \cap I_2 \cap I_3) \cup (I_1 \cap I_2 \cap D_3)\Big) \\ &= \Pr(I_1)\Pr(D_2)\Pr(I_3) + \Pr(D_1)\Pr(I_2)\Pr(I_3) + \Pr(I_1)\Pr(I_2)\Pr(D_3) \\ &= p(1-p)p + (1-p)p^2 + p^2(1-p) = 3p^2(1-p).\end{aligned}$$

(c) We observe that of the three possibile scenarios in F, two of them begin with I_1, and so $\Pr(I_1 \cap F) = 2p^2(1-p)$ (to be more rigorous, we could show that $I_1 \cap F = (I_1 \cap D_2 \cap I_3) \cup (I_1 \cap I_2 \cap D_3)$ and then calculate the probability directly). We then have

$$\Pr(I_1|F) = \frac{\Pr(I_1 \cap F)}{\Pr(F)} = \frac{2p^2(1-p)}{3p^2(1-p)} = \frac{2}{3}.$$

Question 7 Let C be the event that the question is answered correctly, and let I be the event that the question is answered incorrectly. Both the husband and the wife answer the question correctly with probability p and incorrectly with probability $1 - p$.

(a) For the first strategy, let H be the event that the husband is chosen, and let W be the event that the wife is chosen. We have $\Pr(C|H) = \Pr(C|W) = p$, and since we are randomly choosing either the husband or wife, so $\Pr(H) = \Pr(W) = 1/2$. By the law of total probability, we have $\Pr(C) = \Pr(C|H)\Pr(H) + \Pr(C|W)\Pr(W) = p/2 + p/2 = p$.

(b) For the second strategy, there are three possible ways that they can be correct:

1. Both the husband and wife are correct and agree.
2. The husband is correct and the wife is incorrect, so they flip a coin and are correct.
3. The wife is correct and the husband is incorrect, so they flip a coin and are correct.

The probability of the first scenario is p^2. In both the second and third scenarios, the probability of them disagreeing is $p(1-p)$. When flipping a coin, the probability that the coin is correct is $1/2$, so the probability of the couple disagreeing and the coin being correct is $p(1-p)/2$, since the two events are independent. Finally, since all three scenarios are disjoint, we have

$$\Pr(C) = p^2 + \frac{p(1-p)}{2} + \frac{p(1-p)}{2} = p^2 + p - p^2 = p.$$

We have shown that *the probability of being correct is the same for both stragies.*

Question 8 Let $p = 0.6$, let A be the event that the couple agrees, and let D be the event that the couple disagrees. We note that $A \cap D = \emptyset$, since the couple cannot agree and disagree at the same time.

(a) We note that $\Pr(A \cap C) = p^2$ and $\Pr(A \cap I) = (1-p)^2$. Then using Bayes' theorem, we have

$$\Pr(C|A) = \frac{\Pr(A \cap C)}{\Pr(A)} = \frac{\Pr(A \cap C)}{\Pr(A \cap C) + \Pr(A \cap I)} = \frac{p^2}{p^2 + (1-p)^2}.$$

When $p = 0.6$, we have $\Pr(C|A) = 0.6923077$.

(b) When the couple disagrees, their answer being correct comes down to the coin flip, and so $\Pr(C|D) = 1/2$.

So, using the second strategy, the couple is more likely to be correct if they agree on the answer.

Question 9 There are n independent coint flips, and for each coin toss we have $\Pr(H) = p$. Let K denote the number of heads that show up throughout the n flips. We want to determine how large n should be such that $\Pr(K \geq 1) \geq 1/2$. First, we observe that $\Pr(K \geq 1) = 1 - \Pr(K = 0)$. Since $K = 0$ corresponds to observing n tails, we have $\Pr(K = 0) = (1-p)^n$, and so we have $1 - (1-p)^n \geq 1/2$. Slightly rearranging gives us $(1-p)^n \leq 1/2$, and taking the log of both sides gives us $n \ln(1-p) \leq \ln(1/2)$. Since $1 - p \leq 1$, we have $\ln(1-p) \leq 0$, and dividing by it will flip the inequality. Therefore, we need

$$n \geq \frac{\ln(1/2)}{\ln(1-p)} = -\frac{\ln 2}{\ln(1-p)}.$$

We note that as $p \to 1$, this lower bound approaches zero, and as $p \to 0$, the lower bound approaches infinity; both of these results make sense intuitively.

Question 10 Let S_1 be the event that the first coin is silver, and let S_2 be the event that the second coin is silver. Each of the three cabinets has an equal chance of being selected, so $\Pr(A) = \Pr(B) = \Pr(C) = 1/3$. Since the drawers are opened at random as well, we have $\Pr(S_1|A) = 0$, $\Pr(S_1|B) = 1$, and $\Pr(S_1|C) = 1/2$. We also note that since cabinet B is the only cabinet to have two silver coins, we have $\Pr(S_2 \cap S_1) = \Pr(B) = 1/3$. Using Bayes' theorem, we have

$$\begin{aligned} \Pr(S_2|S_1) = \frac{\Pr(S_2 \cap S_1)}{\Pr(\mathrm{S}_1)} &= \frac{\Pr(B)}{\Pr(S_1|A)\Pr(A) + \Pr(S_1|B)\Pr(B) + \Pr(S_1|C)\Pr(C)} \\ = \frac{1/3}{(1 + 0 + 1/2)/3} &= \frac{1}{3/2} = \frac{2}{3}. \end{aligned}$$

Question 11 For notational purposes, let α denote urn A and β denote urn B. Let $R_\alpha, B_\alpha, and G_\alpha$ be the events that a red, blue, and green ball is chosen from urn A, respectively, and $R_\beta, B_\beta, G_\beta$ be the corresponding events for urn B.

(a) Using the law of total probability, we have

$$\begin{aligned} \Pr(R_\beta) &= \Pr(R_\beta|R_\alpha) \cdot \Pr(R_\alpha) + \Pr(R_\beta|B_\alpha) \cdot \Pr(B_\alpha) + \Pr(R_\beta|G_\alpha) \cdot \Pr(G_\alpha) \\ &= \frac{3}{10} \cdot \frac{4}{9} + \frac{2}{10} \cdot \frac{3}{9} + \frac{2}{10} \cdot \frac{2}{9} = \frac{11}{45}. \end{aligned}$$

(b) By the definition of conditional probability, we have $\Pr(R_\alpha|R_\beta) = \Pr(R_\alpha \cap R_\beta)/\Pr(R_\beta)$. It is easy to see that $\Pr(R_\alpha \cap R_\beta) = 3/10 \cdot 4/9 = 6/45$, and so $\Pr(R_\alpha|R_\beta) = \frac{6/45}{11/45} = 6/11$.

Question 12 Let M be the number on the first dice, and let N be the number on the second dice. We want to determine $\Pr(|M - N| < 3)$. Using a brute force approach, we can see that there are 24 different combinations whose difference is less than three (e.g. $(3,3), (6,4), (5,4)$). Since there are 36 possible number combinations, each with an equal chance of occuring, we have $\Pr(|M - N| < 3) = 24/36 = 2/3$.

Question 13 The United States selects two senators from each of the 50 states, so there are 100 total members in the Senate.

(a) There are $\binom{100}{8}$ ways to choose eight senators for this committee. Let E be the event that one of the two senators from a specified state are selected, and so E^c is the event that neither senator from a specified state is on the eight-person committee. If these two senators are not considered to be chosen, then there are $\binom{98}{8}$ ways to choose the eight senators from the remainimg 48. That is, $\Pr(E^c) = \binom{98}{8}/\binom{100}{8}$, and so $\Pr(E) = 1 - \binom{98}{8}/\binom{100}{8}$.

(b) There are $\binom{100}{50}$ ways to choose 50 senators. There are two ways you can select one (and only one) senator from each state, and since the picks are independent, there are 2^{50} ways to choose one senator from each state. Therefore, the probability of having one senator from each state is $2^{50}/\binom{100}{50}$.

Question 14 There are 6^7 possible combinations of dice numbers, and note that we are considering the order in which the dice land. Since there are only 6 possible numbers that a single die can display, but seven different dice, then one of the six numbers will be repeated somewhere, and the other five numbers will appear only once. That is, we can think of splitting up the seven spots into one group of two and five groups of 1. So for a given number that repeats twice, there are $\binom{7}{2,1,1,1,1,1}$ ways to arrange the dice, and since there are six possible numbers that can repeat, we have $6\binom{7}{2,1,1,1,1,1}$ possible arrangements. Therefore, the probability of all numbers appearing at least once is $6\binom{7}{2,1,1,1,1,1}/6^7 = \binom{7}{2,1,1,1,1,1}/6^6$.

Question 15 There are $\binom{90}{10}$ ways to choose ten of the 90 balls without replacement. Let R, W, B denote the events that *no* red, white, or blue balls (respectively) are selected. We have

$$\begin{aligned}\Pr(R \cup W \cup B) = \Pr(R) + \Pr(W) + \Pr(B) \\ - \Pr(R \cap W) - \Pr(R \cap B) - \Pr(W \cap B) + \Pr(R \cap W \cap B).\end{aligned}$$

There are $\binom{60}{10}$ ways to choose ten balls with one color missing, there are $\binom{30}{10}$ ways to choose ten balls where two colors are missing, and there is no way to choose ten balls where all three colors are missing. Therefore,

$$\Pr(R \cup W \cup B) = 3\left(\frac{\binom{60}{10} + \binom{30}{10}}{\binom{90}{10}}\right).$$