Summary of Probability Distributions

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Random variables

- Suppose you have an experiment with sample space $\mathcal{S}$.
- A *random variable* is a function $X : \mathcal{S} \to \mathbb{R}$.
- Assigns a numerical value to all outcomes of experiment.

- The *support* of X is defined as $\mathcal{D} = \mathrm{img}(X) \subseteq \mathbb{R}$.
- The support is all possible values that X can obtain.

- A random variable is *discrete* if $\mathrm{card}(\mathcal{D}) \leq \mathrm{card}(\mathbb{N})$.
- When $\mathrm{card}(\mathcal{D}) < \mathrm{card}(\mathbb{N})$, X has finitely many values.
- When $\mathrm{card}(\mathcal{D}) = \mathrm{card}(\mathbb{N})$, X has countably infinite values.

- A random variable is *continuous* if $\mathrm{card}(\mathcal{D}) > \mathrm{card}(\mathbb{N})$.

Discrete Distributions

Useful summations

Geometric series: for all $x \in \mathbb{R}$ and $|r| < 1$, we have

$$\frac{x(1-r^{n+1})}{1-r} = \sum_{k=0}^{n} x r^k$$

$$\frac{x}{1-r} = \sum_{k=0}^{\infty} x r^k$$

Binomial series: for all $x, y \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 0}$, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k}.$$

Taylor series: for all $x \in \mathbb{R}$, we have

$$\mathrm{e}^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Probability mass functions

The *probability mass function (pmf)* of a discrete random variable X with support $\mathcal{D}$ is a function $f : \mathcal{D} \to [0, 1]$ where

$$f(x) = \Pr(X = x).$$

In other words, it assigns a *probability* to each possible value of X.

If $\mathcal{C} \subseteq \mathcal{D}$, we have

$$\Pr(X \in \mathcal{C}) = \sum_{x \in \mathcal{C}} f(x).$$

A valid pdf has the following proparties:

- $f(x) \geq 0$ for all $x \in \mathcal{D}$,
- $f(x) = 0$ if $x \notin \mathcal{D}$,
- $\sum_{x \in \mathcal{D}} f(x) = 1$.

Geometric Distribution

A random variable X has a *geometric distribution* with pmf and cdf

$$f(x;p) = \begin{cases} (1-p)^x p & x = \{1, \infty) \\ 0 & \text{elsewhere} \end{cases}$$

$$F(x;p) = \begin{cases} 0 & x < 0 \\ 1-(1-p)^x & x = \{1, \infty) \end{cases}$$

where $p \in [0,1]$ is the *probability of success*.

- X represents the number of Bernoulli trials needed before a success occurs.
- X has the *memoryless property*: $\Pr(X > n \mid X > m) = \Pr(X > n - m)$.

We say $X \sim \text{Geo}(p)$, and we have:

- $\text{E}[X] = 1/p$
- $\text{Var}[X] = (1-p)/p^2$
- $\text{M}_X(t) = p\text{e}^t / \big(1-(1-p)\text{e}^t\big)$

Binomial Distribution

A random variable X has a *binomial distribution* with pmf and cdf

$$f(x;n,p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x \in \{0,n\} \\ 0 & \text{elsewhere} \end{cases}$$

$$F(x;n,p) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{x} f(k) & x \in \{0,n\} \\ 1 & x > n \end{cases}$$

where $p \in [0,1]$ is the *probability of success* and $n \in \mathbb{N}$ is the *number of trials.*

- X represents the number of successed oberved after n Bernoulli trials are conducted.

We say $X \sim \mathrm{Bin}(n,p)$, and we have:

- $\mathrm{E}[X] = np$
- $\mathrm{Var}[X] = np(1-p)$
- $\mathrm{M}_X(t) = \big(p(\mathrm{e}^t - 1) + 1\big)^n$