x	1	2	3	4
$f_X(x)$	0.19	0.32	0.31	0.18
$f_{X\|Y}(x\|1)$	10/19	5/19	2/19	2/19

y	1	2	3	4
$f_Y(y)$	0.19	0.32	0.31	0.18
$f_{Y\|X}(y\|1)$	10/19	5/19	2/19	2/19

Table 1: Information for question 2.

Homework 2

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Question 1 Let X have a pdf of $f(x) = cx^2$ for $0 \le x \le 1$ and $f(x) = 0$ elsewhere.

(a) For this to be a valid pdf, it must integrate to 1 over the support. So

$$\int_0^1 cx^2 \, \mathrm{d}x = \frac{cx^3}{3}\bigg|_0^1 = \frac{c}{3} \overset{\text{set}}{=} 1,$$

which leads to $c = 3$. So $f(x) = 3x^2$ for $0 \le x \le 1$.

(b) The cdf is given by

$$F(x) = \int_{-\infty}^{x} f(t) \, \mathrm{d}t = \int_0^x 3t^2 \, \mathrm{d}t = t^3\big|_0^x = x^3$$

for $0 \le x \le 1$. We also have $F(x) = 0$ when $x < 0$ and $F(x) = 1$ when $x > 1$.

(c) We have

$$\Pr\left(\frac{1}{10} \le X \le \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{10}\right) = \frac{1}{2^3} - \frac{1}{10^3} = \frac{31}{250}.$$

Question 2 Two discrete random variables X and Y are jointly distributed.

(a) The marginal pmf for X is obtained by summing over every value of Y for each value of X. For example, to find the marginal probability that $X = 1$, we have $f_X(x) = 0.10+0.05+0.02+0.02 = 0.19$. The other values are obtained in the same way. Finding the marginal pmf for Y is done the exact same way, and it turns out that $f_X(j) = f_Y(j)$ for $j = \{1, 2, 3, 4\}$; they are both found in Table 1.

(b) X and Y are not independent. For two random variables to be independent, we need $f(x, y) = f_X(x) \cdot f_Y(y)$ for all possible (x, y) pairs. Here we have $f(1, 1) = 0.10$ and $f_X(1) \cdot f_Y(1) = 0.19^2 \ne f(1, 1)$, meaing X and Y are dependent.

(c) To find the conditional pmf of X given that $Y = 1$, we take each value of $f(x, 1)$ and divide by $f_Y(1)$, i.e. $f_{X|Y}(x|1) = f(x, 1)/f_Y(1)$. For example, we have $f_{X|Y}(1|1) = 0.10/0.19 = 10/19$. Finding the conditional pmf for Y given that $X = 1$ is done in a similar way, and again they are the same. Both pmfs can be found in be found in Table 1.

Question 3 We are considering points (x, y) uniformly selected within an ellipse given by the equation $(x/a)^2 + (y/b)^2 = 1$, where $a, b > 0$. Therefore, the probability of selecting a point (x, y) from this region is $f(x, y) = c$ for $-a \leq x \leq a$, $-b \leq y \leq b$, and $(x/a)^2 + (y/b)^2 \leq 1$, and $f(x, y) = 0$ elsewhere. To find c, we use the fact that a joint pdf must integrate to 1 over all values of x and y, so

$$1 = \int_{-a}^{a} \int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} c \, \mathrm{d}y \, \mathrm{d}x = c \cdot 2 \int_{-a}^{a} b\sqrt{1-(x/a)^2} \, \mathrm{d}x = c \cdot \pi ab,$$

which implies that $c = 1/\pi ab$. The last equality is from the fact that the area of an ellipse is πab, which is what the integral is computing. To find the marginal density of X (Y), we integrate over all possible values of Y (X):

$$f_X(x) = \int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} \frac{1}{\pi ab} \, \mathrm{d}y = \frac{y}{\pi ab}\bigg|_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} = \frac{2\sqrt{1-(x/a)^2}}{\pi a} \quad \text{for } -a \leq x \leq a,$$

$$f_Y(y) = \int_{-a\sqrt{1-(y/b)^2}}^{a\sqrt{1-(y/b)^2}} \frac{1}{\pi ab} \, \mathrm{d}x = \frac{x}{\pi ab}\bigg|_{-a\sqrt{1-(y/b)^2}}^{a\sqrt{1-(y/b)^2}} = \frac{2\sqrt{1-(y/b)^2}}{\pi b} \quad \text{for } -b \leq y \leq b.$$

These results make sense intuitively. For $f_X(x)$, we can see that the highest probability is obtained when $x = 0$. At $x = 0$, the height of the ellipse is the greatest, so there is more "width" for $x = 0$ to be chosen. And as $x \to \pm a$, $f_X(x) \to 0$, meaning that the closer you get to the end of the ellipse, the less likely that point is to be chosen.

Question 4 The joint cdf of X and Y is given by $F(x, y) = \left(1 - \mathrm{e}^{-\alpha x}\right)\left(1 - \mathrm{e}^{-\beta y}\right)$ for $x, y \geq 0$ and $F(x, y) = 0$ elsewhere, where $\alpha, \beta > 0$ are fixed constants.

(a) X and Y are independent. This is because we can express the joing cdf as $F(x, y) = G(x) \cdot H(y)$, where $G(x) = 1 - \mathrm{e}^{-\alpha x}$ and $H(y) = 1 - \mathrm{e}^{-\beta y}$.

(b) The joint pdf is given by

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(1 - \mathrm{e}^{-\alpha x}\right) \cdot \frac{\partial}{\partial y}\left(1 - \mathrm{e}^{-\beta y}\right) = \alpha \mathrm{e}^{-\alpha x} \cdot \beta \mathrm{e}^{-\beta y}$$

for $x, y \geq 0$ and $f(x, y) = 0$ elsewhere. The marginal densities are given by

$$f_X(x) = \lim_{\phi \to \infty} \int_0^{\phi} \alpha \mathrm{e}^{-\alpha x} \cdot \beta \mathrm{e}^{-\beta y} \, \mathrm{d}y = \alpha \mathrm{e}^{-\alpha x} \cdot \lim_{\phi \to \infty} \left[-\mathrm{e}^{-\beta y}\right]_0^{\phi} = \alpha \mathrm{e}^{-\alpha x},$$

$$f_Y(y) = \lim_{\phi \to \infty} \int_0^{\phi} \alpha \mathrm{e}^{-\alpha x} \cdot \beta \mathrm{e}^{-\beta y} \, \mathrm{d}x = \beta \mathrm{e}^{-\beta y} \cdot \lim_{\phi \to \infty} \left[-\mathrm{e}^{-\alpha x}\right]_0^{\phi} = \beta \mathrm{e}^{-\beta y}.$$

Here we see that $f_X(x) \cdot f_Y(y) = \alpha \mathrm{e}^{-\alpha x} \cdot \beta \mathrm{e}^{-\beta y} = f(x, y)$, another indication that X and Y are independent.