

$x$	1	2	3	4
$f_X(x)$	0.19	0.32	0.31	0.18
$f_{X Y}(x 1)$	10/19	5/19	2/19	2/19

$y$	1	2	3	4
$f_Y(y)$	0.19	0.32	0.31	0.18
$f_{Y X}(y 1)$	10/19	5/19	2/19	2/19

Table 1: Information for question 2.

**Homework 2**

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**Question 1** Let  $X$  have a pdf of  $f(x) = cx^2$  for  $0 \leq x \leq 1$  and  $f(x) = 0$  elsewhere.

- (a) For this to be a valid pdf, it must integrate to 1 over the support. So

$$\int_0^1 cx^2 dx = \frac{cx^3}{3} \Big|_0^1 = \frac{c}{3} \stackrel{\text{set}}{=} 1,$$

which leads to  $c = 3$ . So  $f(x) = 3x^2$  for  $0 \leq x \leq 1$ .

- (b) The cdf is given by

$$F(x) = \int_{-\infty}^x f(t) dt = \int_0^x 3t^2 dt = t^3 \Big|_0^x = x^3$$

for  $0 \leq x \leq 1$ . We also have  $F(x) = 0$  when  $x < 0$  and  $F(x) = 1$  when  $x > 1$ .

- (c) We have

$$\Pr\left(\frac{1}{10} \leq X \leq \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{10}\right) = \frac{1}{2^3} - \frac{1}{10^3} = \frac{31}{250}.$$

**Question 2** Two discrete random variables  $X$  and  $Y$  are jointly distributed.

- (a) The marginal pmf for  $X$  is obtained by summing over every value of  $Y$  for each value of  $X$ . For example, to find the marginal probability that  $X = 1$ , we have  $f_X(x) = 0.10 + 0.05 + 0.02 + 0.02 = 0.19$ . The other values are obtained in the same way. Finding the marginal pmf for  $Y$  is done the exact same way, and it turns out that  $f_X(j) = f_Y(j)$  for  $j = \{1, 2, 3, 4\}$ ; they are both found in Table 1.
- (b)  $X$  and  $Y$  are not independent. For two random variables to be independent, we need  $f(x, y) = f_X(x) \cdot f_Y(y)$  for all possible  $(x, y)$  pairs. Here we have  $f(1, 1) = 0.10$  and  $f_X(1) \cdot f_Y(1) = 0.19^2 \neq f(1, 1)$ , meaning  $X$  and  $Y$  are dependent.
- (c) To find the conditional pmf of  $X$  given that  $Y = 1$ , we take each value of  $f(x, 1)$  and divide by  $f_Y(1)$ , i.e.  $f_{X|Y}(x|1) = f(x, 1)/f_Y(1)$ . For example, we have  $f_{X|Y}(1|1) = 0.10/0.19 = 10/19$ . Finding the conditional pmf for  $Y$  given that  $X = 1$  is done in a similar way, and again they are the same. Both pmfs can be found in Table 1.

**Question 3**

We are considering points  $(x, y)$  uniformly selected within an ellipse given by the equation  $(x/a)^2 + (y/b)^2 = 1$ , where  $a, b > 0$ . Therefore, the probability of selecting a point  $(x, y)$  from this region is  $f(x, y) = c$  for  $-a \leq x \leq a$ ,  $-b \leq y \leq b$ , and  $(x/a)^2 + (y/b)^2 \leq 1$ , and  $f(x, y) = 0$  elsewhere. To find  $c$ , we use the fact that a joint pdf must integrate to 1 over all values of  $x$  and  $y$ , so

$$1 = \int_{-a}^a \int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} c \, dy \, dx = c \cdot 2 \int_{-a}^a b\sqrt{1-(x/a)^2} \, dx = c \cdot \pi ab,$$

which implies that  $c = 1/\pi ab$ . The last equality is from the fact that the area of an ellipse is  $\pi ab$ , which is what the integral is computing. To find the marginal density of  $X$  ( $Y$ ), we integrate over all possible values of  $Y$  ( $X$ ):

$$\begin{aligned} f_X(x) &= \int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} \frac{1}{\pi ab} \, dy = \frac{y}{\pi ab} \Big|_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} = \frac{2\sqrt{1-(x/a)^2}}{\pi a} \quad \text{for } -a \leq x \leq a, \\ f_Y(y) &= \int_{-a\sqrt{1-(y/b)^2}}^{a\sqrt{1-(y/b)^2}} \frac{1}{\pi ab} \, dx = \frac{x}{\pi ab} \Big|_{-a\sqrt{1-(y/b)^2}}^{a\sqrt{1-(y/b)^2}} = \frac{2\sqrt{1-(y/b)^2}}{\pi b} \quad \text{for } -b \leq y \leq b. \end{aligned}$$

These results make sense intuitively. For  $f_X(x)$ , we can see that the highest probability is obtained when  $x = 0$ . At  $x = 0$ , the height of the ellipse is the greatest, so there is more “width” for  $x = 0$  to be chosen. And as  $x \rightarrow \pm a$ ,  $f_X(x) \rightarrow 0$ , meaning that the closer you get to the end of the ellipse, the less likely that point is to be chosen.

**Question 4**

The joint cdf of  $X$  and  $Y$  is given by  $F(x, y) = (1 - e^{-\alpha x})(1 - e^{-\beta y})$  for  $x, y \geq 0$  and  $F(x, y) = 0$  elsewhere, where  $\alpha, \beta > 0$  are fixed constants.

- (a)  $X$  and  $Y$  are independent. This is because we can express the joint cdf as  $F(x, y) = G(x) \cdot H(y)$ , where  $G(x) = 1 - e^{-\alpha x}$  and  $H(y) = 1 - e^{-\beta y}$ .
- (b) The joint pdf is given by

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x} (1 - e^{-\alpha x}) \cdot \frac{\partial}{\partial y} (1 - e^{-\beta y}) = \alpha e^{-\alpha x} \cdot \beta e^{-\beta y}$$

for  $x, y \geq 0$  and  $f(x, y) = 0$  elsewhere. The marginal densities are given by

$$\begin{aligned} f_X(x) &= \lim_{\phi \rightarrow \infty} \int_0^\phi \alpha e^{-\alpha x} \cdot \beta e^{-\beta y} \, dy = \alpha e^{-\alpha x} \cdot \lim_{\phi \rightarrow \infty} [-e^{-\beta y}]_0^\phi = \alpha e^{-\alpha x}, \\ f_Y(y) &= \lim_{\phi \rightarrow \infty} \int_0^\phi \alpha e^{-\alpha x} \cdot \beta e^{-\beta y} \, dx = \beta e^{-\beta y} \cdot \lim_{\phi \rightarrow \infty} [-e^{-\alpha x}]_0^\phi = \beta e^{-\beta y}. \end{aligned}$$

Here we see that  $f_X(x) \cdot f_Y(y) = \alpha e^{-\alpha x} \cdot \beta e^{-\beta y} = f(x, y)$ , another indication that  $X$  and  $Y$  are independent.

**Question 5**

Let  $X$  and  $Y$  have a joint pdf  $f(x, y) = c(x^2 - y^2)e^{-x}$  for  $x \leq 0$  and  $-x \leq y \leq x$ . To help with some of the integrations, this question makes use of the following result:

$$\Omega(\theta, n) := \int_{\theta}^{\infty} x^n e^{-x} \, dx = n! \cdot e^{-\theta} \sum_{k=0}^n \frac{\theta^k}{k!}$$

for all  $\theta \in \mathbb{R}$  and  $n \in \mathbb{Z}_{\geq 0}$ . This specific integral came up so many times in this question that I thought it was worth figuring out a general formula for it. A derivation of this result can be found in Appendix A.

- (a) Using the fact that the joint pdf must integrate to 1, we have

$$\begin{aligned} 1 &= \int_0^\infty \int_{-x}^x c(x^2 - y^2)e^{-x} dy dx = c \int_0^\infty e^{-x} \left[ x^2 y - \frac{y^3}{3} \right]_{-x}^x dx \\ &= \frac{4c}{3} \cdot \int_0^\infty x^3 e^{-x} dx = \frac{4c \cdot \Omega(0, 3)}{3} = \frac{4c \cdot 3!}{3} = 8c, \end{aligned}$$

and so  $c = 1/8$ .

- (b)  $X$  and  $Y$  are not independent since the joint pdf does not have rectangular support.  
 (c) When considering all possible  $x \geq 0$  values, we see that we must have  $-x \leq y \leq x$ . So the marginal density of  $X$  is given by

$$f_X(x) = \int_{-x}^x \frac{(x^2 - y^2)e^{-x}}{8} dx = \frac{e^{-x}}{8} \left[ x^2 y - \frac{y^3}{3} \right]_{-x}^x = \frac{e^{-x}}{8} \cdot \frac{4x^3}{3} = \frac{x^3 e^{-x}}{6}$$

for  $x \geq 0$ . When considering all possible  $-\infty < y < \infty$ , we must split the support into two regions: where  $y \geq 0$  and where  $y < 0$ . When  $y \geq 0$ , we have  $0 \leq y \leq x < \infty$ , so we have  $y \leq x < \infty$ . In this region, the marginal density is given by

$$\begin{aligned} f_Y(y) &= \int_y^\infty \int_{-x}^x \frac{(x^2 - y^2)e^{-x}}{8} dx = \frac{1}{8} \left( \int_y^\infty x^2 e^{-x} dx - y^2 \int_y^\infty e^{-x} dx \right) \\ &= \frac{1}{8} (\Omega(y, 2) - y^2 \cdot \Omega(y, 0)) = \frac{1}{8} (e^{-y}(2 + 2y + y^2) - y^2 e^{-y}) = \frac{(1 + y)e^{-y}}{8}. \end{aligned}$$

In the second region, where  $y < 0$ , we have  $0 > y \geq -x > -\infty$ , and so we can have  $-y < x < \infty$ . The marginal density in this region is given by

$$\begin{aligned} f_Y(y) &= \int_{-y}^\infty \frac{(x^2 - y^2)e^{-x}}{8} dx = \frac{1}{8} \left( \int_{-y}^\infty x^2 e^{-x} dx - y^2 \int_{-y}^\infty e^{-x} dx \right) \\ &= \frac{1}{8} (\Omega(-y, 2) - y^2 \cdot \Omega(-y, 0)) = \frac{1}{8} (e^y(2 - 2y + y^2) - y^2 e^y) = \frac{(1 - y)e^y}{8}. \end{aligned}$$

As a whole, the joint density is given by

$$f_Y(y) = \begin{cases} (1 + y)e^{-y}/8 & \text{when } y \geq 0, \\ (1 - y)e^y/8 & \text{when } y < 0. \end{cases}$$

- (d) The conditional distribution  $f_{X|Y}(x|y)$  is found by dividing  $f(x, y)$  by  $f_Y(y)$ :

$$f_{X|Y}(x|y) = \begin{cases} \frac{(x^2 - y^2)e^{y-x}}{2(1 + y)}, & y \geq 0, y \leq x < \infty \\ \frac{(x^2 - y^2)e^{-(y+x)}}{2(1 - y)}, & y < 0, -y < x < \infty \end{cases}$$

Similarly,

$$f_{Y|X}(y|x) = \frac{3(x^2 - y^2)}{4x^2}, \quad -x \leq y \leq x.$$