Homework 1

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Question 1 We will let H denote a heads and T denote a tails.

(a) The sample space is given by $\mathcal{S} = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTH}, \text{TTT}\}$.

(b) We have

1. $A =$ at least two heads $= \{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}\}$.
2. $B =$ the first two tosses are heads $= \{\text{HHH}, \text{HHT}\}$.
3. $C =$ the last toss is a tail $= \{\text{HHT}, \text{HTT}, \text{THT}, \text{TTT}\}$.

(c) We have

1. $A^c = \{\text{HTT}, \text{THT}, \text{TTH}, \text{TTT}\}$.
2. $A \cap B = \{\text{HHH}, \text{HHT}\} = B$ (since $B \subset A$).
3. $A \cup C = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTT}\}$.

Question 2 For each of the scenarios, we are pulling five cards from a well-shuffled deck without replacement. Since the order of the cards does not matter, there are $\binom{52}{5}$ possible combinations.

(a) *Royal flush*: since the order does not matter, for a given suite there is only one possible royal flush, so there are only four possible royal flushes (one for each suite). So the probability of getting a royal flush is $4/\binom{52}{5}$.

(b) *Straight flush*: for a given suite, not counting the royal flush, there are nine possible card combinations that fit this criteria, meaning there are 36 possible straight flushes. So the probability of getting a straight flush is $36/\binom{52}{5}$.

(c) *Four of a kind*: to select four cards of the same value, we must select all four suites of a given value, of which there are 13. Once this happens, four of the five cards in the hand have been determined, and we just have to select the last card. There are 12 possible card values and four possible suites. Therefore, there are $13 \cdot 12 \cdot 4 = 624$ plausible hands, and the probability of getting a four of a kind is $624/\binom{52}{5}$.

(d) *Flush*: each suite has 13 unique values, so there are $\binom{13}{5}$ ways to select five cards for a given suite. However, this is including the ten possible *consecutive* hands, which must be removed (as that hand would be either a straigh or royal flush). This can be done for each suite, so there are $4\left(\binom{13}{5} - 10\right)$ plausible hands, and so the probability of getting a flush is $4\left(\binom{13}{5} - 10\right)/\binom{52}{5}$.

(e) *Three of a kind*: to get three cards of the same value, for a given value we have to choose three cards from the four possible suites. There are $\binom{4}{3}$ ways to do this for each of the 13 values. For the remaining two cards to be chosen, there are 12 possible values to choose from (choosing the same value of the three matching cards would give us a four of a kind), and each of these two cards can be any of the four suites. That is, there are $\binom{12}{2} \cdot 4 \cdot 4$ ways to choose the last two cards. Therefore, there are $\binom{4}{3} \cdot 13 \cdot \binom{12}{2} \cdot 4 \cdot 4 = 208\binom{4}{3}\binom{12}{2}$ plausible hands, and so the probability of getting a three of a kind is $208\binom{4}{3}\binom{12}{2}/\binom{52}{5}$.

(f) *Two pairs*: to get two pairs of cards of the same value, we have to have two unique values to begin with, and there are $\binom{13}{2}$ ways to choose them. For each value, we are choosing two of the four possible suites, so there are $\binom{4}{2}$ choices *for each value*. For the last card, there are 11 possible values (choosing either of the previous values would result in a three of a kind), and from these there are four possible suites, so there are 44 ways to choose the last card. So there are $44\binom{13}{2}\binom{4}{2}^2$ plausible hands, and so the probability of getting two pairs is $44\binom{13}{2}\binom{4}{2}^2/\binom{52}{5}$.

Question 3 Let E be the event that the president is a woman, F be the event that the vice-president is a man, and G be the event that both leaders are of the same sex. Since we are choosing leaders without replacement and with regard to order, there are $48 \cdot 47$ possible leadership arrangements. We are also assuming committee choices are independent.

(a) For E to happen we only care that the president is a woman, the sex of the vice-president is irrelevant. So there are 16 possible choices for the president, and then any of the 47 remianing members can be chosen. So there are $16 \cdot 47$ outcomes in E, and $\Pr(E) = \frac{16 \cdot 47}{48 \cdot 47} = 1/3$. Similarly, for F we only care about the gender of the vice president. Here there are two possibilities: a man is chosen as the president or a woman is chosen as the president. For the former, there are $32 \cdot 31$ possibilities, and for the latter, there are $16 \cdot 32$ possibilities. Since the two situations are disjoint, we have $\Pr(F) = \frac{32 \cdot 31 + 16 \cdot 32}{48 \cdot 47} = 2/3$. Finally, there are $32 \cdot 31$ combinations of two male leaders and $16 \cdot 15$ combinations of female leaders, and since the two possibilities are disjoint, we have $\Pr(G) = \frac{32 \cdot 31 + 16 \cdot 15}{48 \cdot 47} = 77/141$.

(b) $E \cap F$ is the event that the president is female and the vice president is male. There are $16 \cdot 32$ ways this can happen, so $\Pr(E \cap F) = \frac{16 \cdot 32}{48 \cdot 47} = 32/141$. To find $\Pr(E \cup F)$ (the probability that the president is female or the vice-president is a male), we have

$$\Pr(E \cup F) - \Pr(E) + \Pr(F) - \Pr(E \cap F) = 1/3 + 2/3 - 32/141 = 109/141.$$

Finally, since the event $E \cap F$ requires the leaders to be of opposite sex, we know that $(E \cap F) \cap G = \emptyset$, which means $\Pr(E \cap F \cap G) = 0$.

(c) There are two possibilities when the event G occurs: both leaders are male, or both leaders are female. Since these possibilities are disjoint, we have $G = (G \cap E) \cup (G \cap F)$. Then, using the definition of conditional probability, we have

$$\Pr(G|E \cup F) = \frac{\Pr(G \cap (E \cup F))}{\Pr(E \cup F)} = \frac{\Pr\big((G \cap E) \cup (G \cap F)\big)}{\Pr(E \cup F)} = \frac{\Pr(G)}{\Pr(E \cup F)} = \frac{77}{109}.$$

Question 4 Since we want to consider the location of the four aces in the card deck, order does matter here, and so there are 52! ways of shuffling the deck. To solve this problem, we will first shuffle the four aces separately, then shuffle the remaining cards, then finally insert the four aces together into the rest of the deck. There are 4! ways to arrange the four aces and 48! ways of arranging the remaining cards, and there are 49 possible spaces where the four aces can be inserted into the rest of the deck. Therefore, there are $4! \cdot 48! \cdot 49 = 4! \cdot 49!$ plausible deck arrangements, and so the probability that the four aces are next to each other is $4! \cdot 49!/52! = 1/5525$.