x	1	2	3	4
$f_X(x)$	0.19	0.32	0.31	0.18
$f_{X\|Y}(x\|1)$	10/19	5/19	2/19	2/19

y	1	2	3	4
$f_Y(y)$	0.19	0.32	0.31	0.18
$f_{Y\|X}(y\|1)$	10/19	5/19	2/19	2/19

Table 1: Information for question 2.

Homework 2

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Question 1 Let X have a pdf of $f(x) = cx^2$ for $0 \leq x \leq 1$ and $f(x) = 0$ elsewhere.

(a) For this to be a valid pdf, it must integrate to 1 over the support. So

$$\int_0^1 cx^2 \, \mathrm{d}x = \frac{cx^3}{3}\bigg|_0^1 = \frac{c}{3} \overset{\text{set}}{=} 1,$$

which leads to $c = 3$. So $f(x) = 3x^2$ for $0 \leq x \leq 1$.

(b) The cdf is given by

$$F(x) = \int_{-\infty}^{x} f(t) \, \mathrm{d}t = \int_0^x 3t^2 \, \mathrm{d}t = t^3\big|_0^x = x^3$$

for $0 \leq x \leq 1$. We also have $F(x) = 0$ when $x < 0$ and $F(x) = 1$ when $x > 1$.

(c) We have

$$\Pr\left(\frac{1}{10} \leq X \leq \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{10}\right) = \frac{1}{2^3} - \frac{1}{10^3} = \frac{31}{250}.$$

Question 2 Two discrete random variables X and Y are jointly distributed.

(a) The marginal pmf for X is obtained by summing over every value of Y for each value of X. For example, to find the marginal probability that $X = 1$, we have $f_X(x) = 0.10+0.05+0.02+0.02 = 0.19$. The other values are obtained in the same way. Finding the marginal pmf for Y is done the exact same way, and it turns out that $f_X(j) = f_Y(j)$ for $j = \{1, 2, 3, 4\}$; they are both found in Table 1.

(b) X and Y are not independent. For two random variables to be independent, we need $f(x, y) = f_X(x) \cdot f_Y(y)$ for all possible (x, y) pairs. Here we have $f(1, 1) = 0.10$ and $f_X(1) \cdot f_Y(1) = 0.19^2 \neq f(1, 1)$, meaing X and Y are dependent.

(c) To find the conditional pmf of X given that $Y = 1$, we take each value of $f(x, 1)$ and divide by $f_Y(1)$, i.e. $f_{X|Y}(x|1) = f(x, 1)/f_Y(1)$. For example, we have $f_{X|Y}(1|1) = 0.10/0.19 = 10/19$. Finding the conditional pmf for Y given that $X = 1$ is done in a similar way, and again they are the same. Both pmfs can be found in be found in Table 1.

Question 3 We are considering points (x, y) uniformly selected within an ellipse given by the equation $(x/a)^2 + (y/b)^2 = 1$, where $a, b > 0$. Therefore, the probability of selecting a point (x, y) from this region is $f(x, y) = c$ for $-a \le x \le a$, $-b \le y \le b$, and $(x/a)^2 + (y/b)^2 \le 1$, and $f(x, y) = 0$ elsewhere. To find c, we use the fact that a joint pdf must integrate to 1 over all values of x and y, so

$$1 = \int_{-a}^{a} \int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} c \; \mathrm{d}y \; \mathrm{d}x = c \cdot 2 \int_{-a}^{a} b\sqrt{1-(x/a)^2} \; \mathrm{d}x = c \cdot \pi ab,$$

which implies that $c = 1/\pi ab$. The last equality is from the fact that the area of an ellipse is πab, which is what the integral is computing. To find the marginal density of X (Y), we integrate over all possible values of Y (X):

$$f_X(x) = \int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} \frac{1}{\pi ab} \; \mathrm{d}y = \frac{y}{\pi ab}\Big|_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} = \frac{2\sqrt{1-(x/a)^2}}{\pi a} \quad \text{for } -a \le x \le a,$$
$$f_Y(y) = \int_{-a\sqrt{1-(y/b)^2}}^{a\sqrt{1-(y/b)^2}} \frac{1}{\pi ab} \; \mathrm{d}x = \frac{x}{\pi ab}\Big|_{-a\sqrt{1-(y/b)^2}}^{a\sqrt{1-(y/b)^2}} = \frac{2\sqrt{1-(y/b)^2}}{\pi b} \quad \text{for } -b \le y \le b.$$

These results make sense intuitively. For $f_X(x)$, we can see that the highest probability is obtained when $x = 0$. At $x = 0$, the height of the ellipse is the greatest, so there is more "width" for $x = 0$ to be chosen. And as $x \to \pm a$, $f_X(x) \to 0$, meaning that the closer you get to the end of the ellipse, the less likely that point is to be chosen.

Question 4 The joint cdf of X and Y is given by $F(x, y) = \left(1 - \mathrm{e}^{-\alpha x}\right)\left(1 - \mathrm{e}^{-\beta y}\right)$ for $x, y \ge 0$ and $F(x, y) = 0$ elsewhere, where $\alpha, \beta > 0$ are fixed constants.

(a) X and Y are independent. This is because we can express the joing cdf as $F(x, y) = G(x) \cdot H(y)$, where $G(x) = 1 - \mathrm{e}^{-\alpha x}$ and $H(y) = 1 - \mathrm{e}^{-\beta y}$.

(b) The joint pdf is given by

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(1 - \mathrm{e}^{-\alpha x}\right) \cdot \frac{\partial}{\partial y}\left(1 - \mathrm{e}^{-\beta y}\right) = \alpha \mathrm{e}^{-\alpha x} \cdot \beta \mathrm{e}^{-\beta y}$$

for $x, y \ge 0$ and $f(x, y) = 0$ elsewhere. The marginal densities are given by

$$f_X(x) = \lim_{\phi \to \infty} \int_0^{\phi} \alpha \mathrm{e}^{-\alpha x} \cdot \beta \mathrm{e}^{-\beta y} \; \mathrm{d}y = \alpha \mathrm{e}^{-\alpha x} \cdot \lim_{\phi \to \infty} \left[-\mathrm{e}^{-\beta y}\right]_0^{\phi} = \alpha \mathrm{e}^{-\alpha x},$$
$$f_Y(y) = \lim_{\phi \to \infty} \int_0^{\phi} \alpha \mathrm{e}^{-\alpha x} \cdot \beta \mathrm{e}^{-\beta y} \; \mathrm{d}x = \beta \mathrm{e}^{-\beta y} \cdot \lim_{\phi \to \infty} \left[-\mathrm{e}^{-\alpha x}\right]_0^{\phi} = \beta \mathrm{e}^{-\beta y}.$$

Here we see that $f_X(x) \cdot f_Y(y) = \alpha \mathrm{e}^{-\alpha x} \cdot \beta \mathrm{e}^{-\beta y} = f(x, y)$, another indication that X and Y are independent.

Question 5 Let X and Y have a joint pdf $f(x, y) = c(x^2 - y^2)\mathrm{e}^{-x}$ for $x \le 0$ and $-x \le y \le x$. To help with some of the integrations, this question makes use of the following result:

$$\Omega(\theta, n) := \int_{\theta}^{\infty} x^n \mathrm{e}^{-x} \; \mathrm{d}x = n! \cdot \mathrm{e}^{-\theta} \sum_{k=0}^{n} \frac{\theta^k}{k!}$$

for all $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}_{\ge 0}$. This specific integral came up so many times in this question that I thought it was worth figuring out a general formula for it. A derivation of this result can be found in Appendix A.

(a) Using the fact that the joint pdf must integrate to 1, we have

$$
\begin{aligned}
1 &= \int_0^\infty \int_{-x}^{x} c\big(x^2 - y^2\big)\mathrm{e}^{-x}\,\mathrm{d}y\,\mathrm{d}x = c\int_0^\infty \mathrm{e}^{-x}\left[x^2y - \frac{y^3}{3}\right]_{-x}^{x}\,\mathrm{d}x \\
&= \frac{4c}{3}\cdot\int_0^\infty x^3\mathrm{e}^{-x}\,\mathrm{d}x = \frac{4c\cdot\Omega(0,3)}{3} = \frac{4c\cdot 3!}{3} = 8c,
\end{aligned}
$$

and so $c = 1/8$.

(b) X and Y are not independent since the joint pdf does not have rectangular support.

(c) When considering all possible $x \geq 0$ values, we see that we must have $-x \leq y \leq x$. So the marginal density of X is given by

$$
f_X(x) = \int_{-x}^{x} \frac{(x^2 - y^2)\mathrm{e}^{-x}}{8}\,\mathrm{d}x = \frac{\mathrm{e}^{-x}}{8}\left[x^2y - \frac{y^3}{3}\right]_{-x}^{x} = \frac{e^{-x}}{8}\cdot\frac{4x^3}{3} = \frac{x^3\mathrm{e}^{-x}}{6}
$$

for $x \geq 0$. When considering all possible $-\infty < y < \infty$, we must split the support into two regions: where $y \geq 0$ and where $y < 0$. When $y \geq 0$, we have $0 \leq y \leq x < \infty$, so we have $y \leq x < \infty$. In this region, the marginal density is given by

$$
\begin{aligned}
f_Y(y) &= \int_y^\infty \int_{-x}^{x} \frac{\big(x^2 - y^2\big)\mathrm{e}^{-x}}{8}\,\mathrm{d}x = \frac{1}{8}\left(\int_y^\infty x^2\mathrm{e}^{-x}\,\mathrm{d}x - y^2\int_y^\infty \mathrm{e}^{-x}\,\mathrm{d}x\right) \\
&= \frac{1}{8}\big(\Omega(y,2) - y^2\cdot\Omega(y,0)\big) = \frac{1}{8}\big(\mathrm{e}^{-y}(2 + 2y + y^2) - y^2\mathrm{e}^{-y}\big) = \frac{(1+y)e^{-y}}{8}.
\end{aligned}
$$

In the second region, where $y < 0$, we have $0 > y \geq -x > -\infty$, and so we can have $-y < x < \infty$. The marginal density is this region is given by

$$
\begin{aligned}
f_Y(y) &= \int_{-y}^\infty \frac{(x^2 - y^2)\mathrm{e}^{-x}}{8}\,\mathrm{d}x = \frac{1}{8}\left(\int_{-y}^\infty x^2\mathrm{e}^{-x}\,\mathrm{d}x - y^2\int_y^\infty \mathrm{e}^{-x}\,\mathrm{d}x\right) \\
&= \frac{1}{8}\big(\Omega(-y,2) - y^2\cdot\Omega(-y,0)\big) = \frac{1}{8}\big(\mathrm{e}^{y}(2 - 2y + y^2) - y^2\mathrm{e}^{y}\big) = \frac{(1-y)e^{y}}{8}.
\end{aligned}
$$

As a whole, the joint density is given by

$$
f_Y(y) = \begin{cases} (1+y)\mathrm{e}^{-y}/8 & \text{when } y \geq 0, \\ (1-y)\mathrm{e}^{y}/8 & \text{when } y < 0. \end{cases}
$$

(d) The conditional distrubution $f_{X|Y}(x|y)$ is found by dividing $f(x,y)$ by $f_Y(y)$:

$$
f_{X|Y}(x|y) = \begin{cases} \dfrac{(x^2 - y^2)\mathrm{e}^{y-x}}{2(1+y)}, & y \geq 0,\ y \leq x < \infty \\ \dfrac{(x^2 - y^2)\mathrm{e}^{-(y+x)}}{2(1-y)}, & y < 0,\ -y < x < \infty \end{cases}
$$

Similarly,

$$
f_{Y|X}(y|x) = \frac{3(x^2 - y^2)}{4x^2}, \quad -x \leq y \leq x.
$$

Question 6 Suppose the rv X has some density $f_X(x)$ and let $Y = aX + b$ for some $a \neq 0$, meaning Y is a linear combination of X. We see that $g(X) = aX + b$ is monotonic with inverse $h(Y) = \frac{Y-b}{a}$ and $h'(Y) = 1/a$. Then the density of Y is given by

$$
f_Y(y) = f_X\big(h(y)\big)\cdot\big|h'(y)\big| = f_X\left(\frac{y-a}{b}\right)\cdot\frac{1}{|a|} = \frac{y}{|a|}.
$$

Question 7

Question 8 Let X be the number of cars in the left lane at a randomly chosen red light, where $x = \{0, 1, \ldots, 7\}$. The engineer also believes $f(x) \propto (x+1)(8-x)$.

(a) The engineer's assumption implies that $f(x) = c(x+1)(8-x)$ for some c. So

$$\begin{aligned} 1 &= \sum_{x=0}^{7} c(x+1)(x-8) = c\sum_{i=0}^{7}\left(8+7x-x^2\right) = c\left(8\sum_{i=0}^{7}1 + 7\sum_{i=0}^{7}x - \sum_{i=0}^{7}x^2\right) \\ &= c\left(8\cdot 8 + \frac{7\cdot 7\cdot 8}{2} - \frac{7\cdot 8\cdot 15}{6}\right) = 120c, \end{aligned}$$

which means $c = 120$ and $f(x) = (x+1)(x-8)/120$.

(b) We have

$$\Pr(X \geq 5) = \sum_{x=5}^{7}\frac{(x+1)(8-x)}{120} = \frac{6\cdot 3 + 7\cdot 2 + 8\cdot 1}{120} = \frac{40}{120} = \frac{1}{3}.$$

Appendix A Here we will prove the result used throughout question 5. By using integration by parts with $u = x^n$ and $\mathrm{d}v = \mathrm{e}^{-x}$, we have

$$\int x^n \mathrm{e}^{-x}\,\mathrm{d}x = -x^n\mathrm{e}^{-x} + n\int x^{n-1}\mathrm{e}^{-x}\,\mathrm{d}x.$$

The next integral must also be solved via integration by parts, with $u = x^{n-1}$ and $\mathrm{d}v = \mathrm{e}^{-x}$:

$$\begin{aligned} \int x^n\mathrm{e}^{-x}\,\mathrm{d}x &= -x^n\mathrm{e}^{-x} + n\left(-x^{n-1}\mathrm{e}^{-x} + (n-1)\int x^{n-2}\mathrm{e}^{-x}\,\mathrm{d}x\right) \\ &= -x^n\mathrm{e}^{-x} - nx^{n-1}\mathrm{e}^{-x} - n(n-1)\int x^{n-2}\mathrm{e}^{-x}\,\mathrm{d}x. \end{aligned}$$

By now a pattern starts to appear. We have to repeat the integration by parts for the right most integral n times in total. Completing this pattern and factoring out an $n!$ from each term, we get

$$\begin{aligned} \int x^n\mathrm{e}^{-x}\,\mathrm{d}x &= -x^n\mathrm{e}^{-x} - nx^{n-1}\mathrm{e}^{-x} - n(n-1)x^{n-2}\mathrm{e}^{-x} - \cdots - n! + C \\ &= -\mathrm{e}^{-x}\left(\frac{n!}{n!}x^n + \frac{n!}{(n-1)!}x^{n-1} + \frac{n!}{(n-2)!}x^{n-2} + \cdot + \frac{n!}{0!}x^0\right) + C \\ &= -n!\cdot\mathrm{e}^{-x}\left(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \frac{x^{n-2}}{(n-2)!} + \cdot + \frac{x^0}{0!}\right) + C = -n!\cdot\mathrm{e}^{-x}\sum_{k=0}^{n}\frac{x^k}{k!} + C, \end{aligned}$$

where C is the integration constant. Evaluating this integral from θ to ∞ gives us

$$\int_\theta^\infty x^n\mathrm{e}^{-x}\,\mathrm{d} = \left[-n!\cdot\mathrm{e}^{-x}\sum_{k=0}^{n}\frac{x^k}{k!}\right]_\theta^\infty = n!\cdot\mathrm{e}^{-\theta}\sum_{k=0}^{n}\frac{\theta^k}{k!} - n!\lim_{\phi\to\infty}\mathrm{e}^{-\phi}\sum_{k=0}^{n}\frac{\phi^k}{k!},$$

and using L'Hopital's rule n times on the right limit shows that

$$\lim_{\phi\to\infty}\mathrm{e}^{-\phi}\sum_{k=0}^{n}\frac{\phi^k}{k!} = \lim_{\phi\to\infty}(-1)^n\mathrm{e}^{-\theta} = 0.$$

which gives us our desired result. $\square$