Homework 2

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Question 1

Let X have a pdf of $f(x) = cx^2$ for $0 \leq x \leq 1$ and $f(x) = 0$ elsewhere.

(a) For this to be a valid pdf, it must integrate to 1 over the support. So

$$1 = \int_0^1 cx^2 \, \mathrm{d}x = \left. \frac{cx^3}{3} \right|_0^1 = \frac{c}{3},$$

which leads to $c = 3$. So $f(x) = 3x^2$ for $0 \leq x \leq 1$.

(b) The cdf is given by

$$F(x) = \int_{-\infty}^{x} f(t) \, \mathrm{d}t = \int_0^x 3t^2 \, \mathrm{d}t = \left. t^3 \right|_0^x = x^3$$

for $0 \leq x \leq 1$. We also have $F(x) = 0$ when $x < 0$ and $F(x) = 1$ when $x > 1$.

(c) We have

$$\Pr\left(\frac{1}{10} \leq X \leq \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{10}\right) = \frac{1}{2^3} - \frac{1}{10^3} = \frac{31}{250}.$$

Question 2

Two discrete random variables X and Y are jointly distributed.

(a) The marginal pmf for X is obtained by summing over every value of Y for each value of X. For example, to find the marginal probability that $X = 1$, we have $f_X(1) = 0.10 + 0.05 + 0.02 + 0.02 = 0.19$. The other values are obtained in the same way. Finding the marginal pmf for Y is done the exact same way, and it turns out that $f_X(j) = f_Y(j)$ for $j = \{1, 2, 3, 4\}$; they are both found in Table 1.

(b) X and Y are not independent. For two random variables to be independent, we need $f(x, y) = f_X(x) \cdot f_Y(y)$ for all possible (x, y) pairs. Here we have $f(1, 1) = 0.10$ and $f_X(1) \cdot f_Y(1) = 0.19^2 \neq f(1, 1)$, meaing X and Y are dependent.

(c) To find the conditional pmf of X given that $Y = 1$, we take each value of $f(x, 1)$ and divide by $f_Y(1)$, i.e. $f_{X|Y}(x|1) = f(x, 1)/f_Y(1)$. For example, we have $f_{X|Y}(1|1) = 0.10/0.19 = 10/19$. Finding the conditional pmf for Y given that $X = 1$ is done in a similar way, and again they are the same. Both pmfs can be found in be found in Table 1.

Question 3

We are considering points (x, y) uniformly selected within an ellipse given by the equation $(x/a)^2 + (y/b)^2 = 1$, where $a, b > 0$. Therefore, the probability of selecting a point (x, y) from this region is $f(x, y) = c$ for $-a \leq x \leq a$,

x	1	2	3	4
$f_X(x)$	0.19	0.32	0.31	0.18
$f_{X\|Y}(x\|1)$	10/19	5/19	2/19	2/19

y	1	2	3	4
$f_Y(y)$	0.19	0.32	0.31	0.18
$f_{Y\|X}(y\|1)$	10/19	5/19	2/19	2/19

Table 1: Information for question 2.

$-b \le y \le b$, and $(x/a)^2 + (y/b)^2 \le 1$, and $f(x,y) = 0$ elsewhere. To find c, we use the fact that a joint pdf must integrate to 1 over all values of x and y, so

$$1 = \int_{-a}^{a} \int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} c \,\mathrm{d}y \,\mathrm{d}x = c \cdot 2 \int_{-a}^{a} b\sqrt{1-(x/a)^2} \,\mathrm{d}x = c \cdot \pi ab,$$

which implies that $c = 1/\pi ab$. The last equality is from the fact that the area of an ellipse is πab, which is what the integral is computing. To find the marginal density of X (Y), we integrate over all possible values of Y (X):

$$f_X(x) = \int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} \frac{1}{\pi ab} \,\mathrm{d}y = \frac{y}{\pi ab}\bigg|_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} = \frac{2\sqrt{1-(x/a)^2}}{\pi a} \quad \text{for } -a \le x \le a,$$
$$f_Y(y) = \int_{-a\sqrt{1-(y/b)^2}}^{a\sqrt{1-(y/b)^2}} \frac{1}{\pi ab} \,\mathrm{d}x = \frac{x}{\pi ab}\bigg|_{-a\sqrt{1-(y/b)^2}}^{a\sqrt{1-(y/b)^2}} = \frac{2\sqrt{1-(y/b)^2}}{\pi b} \quad \text{for } -b \le y \le b.$$

These results make sense intuitively. For $f_X(x)$, we can see that the highest probability is obtained when $x = 0$. At $x = 0$, the height of the ellipse is the greatest, so there is more "width" for $x = 0$ to be chosen. And as $x \to \pm a$, $f_X(x) \to 0$, meaning that the closer you get to the end of the ellipse, the less likely that point is to be chosen.

Question 4

The joint cdf of X and Y is given by $F(x,y) = \left(1 - \mathrm{e}^{-\alpha x}\right)\left(1 - \mathrm{e}^{-\beta y}\right)$ for $x, y \ge 0$ and $F(x,y) = 0$ elsewhere, where $\alpha, \beta > 0$ are fixed constants.

(a) X and Y are independent. This is because we can express the joing cdf as $F(x,y) = G(x) \cdot H(y)$, where $G(x) = 1 - \mathrm{e}^{-\alpha x}$ and $H(y) = 1 - \mathrm{e}^{-\beta y}$.

(b) The joint pdf is given by

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(1 - \mathrm{e}^{-\alpha x}\right) \cdot \frac{\partial}{\partial y}\left(1 - \mathrm{e}^{-\beta y}\right) = \alpha \mathrm{e}^{-\alpha x} \cdot \beta \mathrm{e}^{-\beta y}$$

for $x, y \ge 0$ and $f(x,y) = 0$ elsewhere. The marginal densities are given by

$$f_X(x) = \int_0^\infty \alpha \mathrm{e}^{-\alpha x} \cdot \beta \mathrm{e}^{-\beta y} \,\mathrm{d}y = \alpha \mathrm{e}^{-\alpha x} \cdot \lim_{\phi \to \infty} \left[-\mathrm{e}^{-\beta y}\right]_0^\phi = \alpha \mathrm{e}^{-\alpha x},$$
$$f_Y(y) = \int_0^\infty \alpha \mathrm{e}^{-\alpha x} \cdot \beta \mathrm{e}^{-\beta y} \,\mathrm{d}x = \beta \mathrm{e}^{-\beta y} \cdot \lim_{\phi \to \infty} \left[-\mathrm{e}^{-\alpha x}\right]_0^\phi = \beta \mathrm{e}^{-\beta y}.$$

Here we see that $f_X(x) \cdot f_Y(y) = \alpha \mathrm{e}^{-\alpha x} \cdot \beta \mathrm{e}^{-\beta y} = f(x,y)$, another indication that X and Y are independent.

Question 5

Collaborators: Melissa Wu, Mariko Sawada.
Let X and Y have a joint pdf $f(x,y) = c(x^2 - y^2)\mathrm{e}^{-x}$ for $x \le 0$ and $-x \le y \le x$. To help with some of the integrations, we make use of the following result:

$$\Omega(\theta, n) \coloneqq \int_\theta^\infty x^n \mathrm{e}^{-x} \,\mathrm{d}x = n! \cdot \mathrm{e}^{-\theta} \sum_{k=0}^{n} \frac{\theta^k}{k!}$$

for all $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}_{\ge 0}$. This specific integral came up so many times in this question that I thought it was worth figuring out a general formula for it. A derivation of this result can be found in Appendix A.

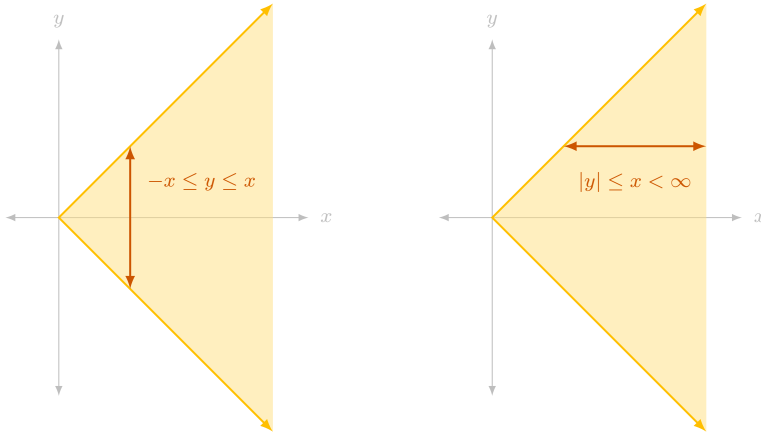

Figure 1: The region of support for $f(x, y) = (x^2 - y^2)\mathrm{e}^{-x}/8$.

(a) Using the fact that the joint pdf must integrate to 1, we have

$$\begin{aligned} 1 &= \int_0^\infty \int_{-x}^{x} c\big(x^2 - y^2\big)\mathrm{e}^{-x}\, \mathrm{d}y\, \mathrm{d}x = c \int_0^\infty \mathrm{e}^{-x} \left[x^2 y - \frac{y^3}{3}\right]_{-x}^{x} \mathrm{d}x \\ &= \frac{4c}{3} \cdot \int_0^\infty x^3 \mathrm{e}^{-x}\, \mathrm{d}x = \frac{4c \cdot \Omega(0,3)}{3} = \frac{4c \cdot 3!}{3} = 8c, \end{aligned}$$

and so $c = 1/8$.

(b) X and Y are not independent since the joint pdf does not have rectangular support.

(c) The region of support can be seen in Figure 1. The support for X and Y can be viewed two ways: all points such that $x \ge 0$ and $-x \le y \le x$, or all points such that $y \in \mathbb{R}$ and $|y| \le x < \infty$. The marginal density of X is given by

$$f_X(x) = \int_{-x}^{x} \frac{(x^2 - y^2)\mathrm{e}^{-x}}{8}\, \mathrm{d}y = \frac{\mathrm{e}^{-x}}{8} \left[x^2 y - \frac{y^3}{3}\right]_{-x}^{x} = \frac{\mathrm{e}^{-x}}{8} \cdot \frac{4x^3}{3} = \frac{x^3 \mathrm{e}^{-x}}{6}$$

for $x \ge 0$ and $f_X(x) = 0$ otherwise. The marginal density of Y is given by

$$\begin{aligned} f_Y(y) &= \int_{|y|}^{\infty} \frac{(x^2 - y^2)\mathrm{e}^{-x}}{8}\, \mathrm{d}x = \frac{1}{8}\left(\int_{|y|}^{\infty} x^2 \mathrm{e}^{-x}\, \mathrm{d}x - y^2 \int_{|y|}^{\infty} \mathrm{e}^{-x}\, \mathrm{d}x\right) \\ &= \frac{1}{8}\Big(\Omega(|y|, 2) - y^2 \cdot \Omega(|y|, 0)\Big) = \frac{1}{8}\Big(2\mathrm{e}^{-|y|}(1 + |y| + |y|^2/2) - y^2 \mathrm{e}^{-|y|}\Big) \\ &= \frac{1}{8}\Big(2\mathrm{e}^{-|y|} + 2|y|\mathrm{e}^{-|y|} + y^2 \mathrm{e}^{-|y|} - y^2 \mathrm{e}^{-|y|}\Big) = \frac{(1 + |y|)e^{-|y|}}{4} \end{aligned}$$

for all $y \in \mathbb{R}$.

(d) The conditional distrubution for X given Y is

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{\frac{1}{8}(x^2 - y^2)\mathrm{e}^{-x}}{\frac{1}{4}(1 + |y|)\mathrm{e}^{-|y|}} = \frac{(x^2 - y^2)\mathrm{e}^{-(x+|y|)}}{2(1 + |y|)}$$

for $|y| \le x < \infty$ and $f_{X|Y}(x|y) = 0$ elsewhere. Similarly, the conditional distribution for Y given X is

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{\frac{1}{8}(x^2 - y^2)\mathrm{e}^{-x}}{\frac{1}{6}x^3 \mathrm{e}^{-x}} = \frac{3(x^2 - y^2)}{4x^3}$$

for $-x \le y \le x$ and $f_{Y|X}(y|x) = 0$ elsewhere.

Question 6

Suppose the random variable X has some density $f_X(x)$ and let $Y = aX + b$ for some $a \neq 0$, meaning Y is a linear combination of X. We see that $g(X) = aX + b$ is monotonic with inverse $h(Y) = \frac{Y-b}{a}$ and $h'(Y) = 1/a$. Then the density of Y is given by

$$f_Y(y) = f_X\big(h(y)\big) \cdot \big|h'(y)\big| = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|} = \frac{y}{|a|}.$$

Question 7

Let $F(x)$ be the cdf of some unspecified discrete random variable, and let $U \sim \text{Unif}[0,1]$. The cdf for U is given by $\tilde{F}(u) = u$ for $0 \leq u \leq 1$. If we define the random variable $Y = k$ if $F(k-1) < U \leq F(k)$, then

$$\Pr(Y \leq k) = \Pr\big(U \leq F(k)\big) = \tilde{F}\big(F(k)\big) = F(k),$$

and so the cdf of Y is actually given by $F(k)$.

Question 8

Let X be the number of cars in the left lane at a randomly chosen red light, where $x = \{0, 1, \ldots, 7\}$. The engineer also believes $f(x) \propto (x+1)(8-x)$.

(a) The engineer's assumption implies that $f(x) = c(x+1)(8-x)$ for some c. So

$$\begin{aligned} 1 = \sum_{x=0}^{7} c(x+1)(x-8) &= c\sum_{i=0}^{7}\big(8 + 7x - x^2\big) = c\left(8\sum_{i=0}^{7} 1 + 7\sum_{i=0}^{7} x - \sum_{i=0}^{7} x^2\right) \\ &= c\left(8 \cdot 8 + 7 \cdot \frac{7 \cdot 8}{2} - \frac{7 \cdot 8 \cdot 15}{6}\right) = 120c, \end{aligned}$$

which means $c = 120$ and $f(x) = (x+1)(x-8)/120$.

(b) We have

$$\Pr(X \geq 5) = \sum_{x=5}^{7} \frac{(x+1)(8-x)}{120} = \frac{6 \cdot 3 + 7 \cdot 2 + 8 \cdot 1}{120} = \frac{40}{120} = \frac{1}{3}.$$

Question 9

Let X have a pdf given by $f(x) = x/8$ for $0 \leq x \leq 4$. The cdf is then given by

$$F(x) = \int_0^x \frac{t}{8}\,\mathrm{d}t = \frac{t^2}{16}\bigg|_0^x = \frac{x^2}{16}$$

for $0 \leq x \leq 4$, $F(x) = 0$ if $x < 0$, and $F(x) = 1$ if $x > 4$.

(a) We have $\Pr(X \leq t) = F(t) = t^2/16 \overset{\text{set}}{=} 1/4$, and solving for t gives us $t = 2$.

(b) We have $\Pr(X \geq t) = 1 - F(t) = (16 - t^2)/16 \overset{\text{set}}{=} 1/2$, and solving for t gives us $t = 2\sqrt{2}$.

Question 10

Let X and Y be random variables such that $0 \leq x \leq 3$ and $0 \leq y \leq 4$. Within this rectangular region, let the joint cdf be $F(x,y) = xy(x^2 + y)/156$. From this we can also determine the joint cdf outside the region of joint support. To start we know that $F(x,y) = 0$ if $x < 0$ or $y < 0$. When $0 \leq X \leq 3$ and $y > 4$ (beyond the support of Y), we have $F(x, y) = F_X(x)$, and when $0 \le Y \le 4$ but $X > 3$ (beyond the support of X), we have $F(x, y) = F_Y(y)$; these quantities are given by

$$F_X(x) = \lim_{y \to 4} F(x, y) = F(x, 4) = \frac{x(x^2 + y)}{39},$$
$$F_Y(y) = \lim_{x \to 3} F(x, y) = F(3, y) = \frac{y(9 + y)}{52}.$$

(a) We have

$$\Pr(1 \le X \le 2 \cap 1 \le Y \le 2) = F(2, 2) - F(2, 1) - F(1, 2) + F(1, 1)$$
$$= \frac{2 \cdot 2(4 + 2)}{156} + \frac{1 \cdot 2(1 + 2)}{156} + \frac{2 \cdot 1(4 + 1)}{156} + \frac{1 \cdot 1(1 + 1)}{156} = \frac{24 - 6 - 10 + 2}{156} = \frac{5}{78}.$$

(b) We have

$$\Pr(2 \le X \le 4 \cap 2 \le Y \le 4) = F(4, 4) - F(4, 2) - F(2, 4) + F(2, 2)$$
$$= \frac{4(9 + 4)}{52} + \frac{2(9 + 2)}{52} + \frac{2 \cdot 4(4 + 4)}{156} + \frac{2 \cdot 2(4 + 2)}{156} = \frac{156 - 66 - 64 + 24}{156} = \frac{25}{78}.$$

(c) The cdf of Y is given by $F_Y(y) = y(9 + y)/52$ for $0 \le Y \le 4$, $F_Y(y) = 0$ if $y < 0$, and $F_Y(y) = 1$ if $y > 4$.

(d) The joint pdf of X and Y is given by

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{xy(x^2 + y)}{156} = \frac{\partial}{\partial x}\frac{x(x^2 + y) + xy}{156} = \frac{3x^2 + 2y}{156}$$

when $0 \le x \le 3$ and $0 \le y \le 4$, and $f(x, y) = 0$ outside of the joint support.

(e) We have

$$\Pr(Y \le X) = \int_0^3 \int_0^x \frac{3x^2 + 2y}{156}\, \mathrm{d}y\, \mathrm{d}x = \frac{1}{156}\int_0^3 \left[3x^2 y + y^2\right]_0^x \mathrm{d}x = \frac{1}{156}\int_0^3 \left(3x^3 + x^2\right) \mathrm{d}x$$
$$= \frac{1}{156}\left[\frac{3x^4}{4} + \frac{x^3}{3}\right]_0^3 = \frac{1}{156}\left(\frac{3 \cdot 81}{4} + \frac{27}{3}\right) = \frac{243 + 36}{156 \cdot 4} = \frac{93}{208}.$$

Appendix A

Here we will prove the result used throughout question 5. By using integration by parts with $u = x^n$ and $\mathrm{d}v = \mathrm{e}^{-x}$, we have

$$\int x^n \mathrm{e}^{-x}\, \mathrm{d}x = -x^n \mathrm{e}^{-x} + n \int x^{n-1} \mathrm{e}^{-x}\, \mathrm{d}x + C.$$

The next integral must also be solved via integration by parts, with $u = x^{n-1}$ and $\mathrm{d}v = \mathrm{e}^{-x}$:

$$\int x^n \mathrm{e}^{-x}\, \mathrm{d}x = -x^n \mathrm{e}^{-x} + n\left(-x^{n-1}\mathrm{e}^{-x} + (n-1)\int x^{n-2}\mathrm{e}^{-x}\, \mathrm{d}x\right) + C$$
$$= -x^n \mathrm{e}^{-x} - n x^{n-1} \mathrm{e}^{-x} - n(n-1)\int x^{n-2}\mathrm{e}^{-x}\, \mathrm{d}x + C.$$

By now a pattern starts to appear. We have to repeat the integration by parts for the right most integral n times in total. Completing this pattern and factoring out an $n!$ from each term yields

$$\int x^n \mathrm{e}^{-x}\, \mathrm{d}x = -x^n \mathrm{e}^{-x} - n x^{n-1}\mathrm{e}^{-x} - n(n-1)x^{n-2}\mathrm{e}^{-x} - \cdots - n! + C$$
$$= -\mathrm{e}^{-x}\left(\frac{n!}{n!}x^n + \frac{n!}{(n-1)!}x^{n-1} + \frac{n!}{(n-2)!}x^{n-2} + \cdot + \frac{n!}{0!}x^0\right) + C$$
$$= -n! \cdot \mathrm{e}^{-x}\left(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \frac{x^{n-2}}{(n-2)!} + \cdot + \frac{x^0}{0!}\right) + C = -n! \cdot \mathrm{e}^{-x}\sum_{k=0}^{n}\frac{x^k}{k!} + C.$$

Evaluating this integral from θ to ∞ gives us

$$\int_{\theta}^{\infty} x^n \mathrm{e}^{-x} \, \mathrm{d} = \left[-n! \cdot \mathrm{e}^{-x} \sum_{k=0}^{n} \frac{x^k}{k!} \right]_{\theta}^{\infty} = n! \cdot \mathrm{e}^{-\theta} \sum_{k=0}^{n} \frac{\theta^k}{k!} - n! \lim_{\phi \to \infty} \mathrm{e}^{-\phi} \sum_{k=0}^{n} \frac{\phi^k}{k!},$$

and using L'Hopital's rule n times on the right limit shows that

$$\lim_{\phi \to \infty} \mathrm{e}^{-\phi} \sum_{k=0}^{n} \frac{\phi^k}{k!} = \lim_{\phi \to \infty} (-1)^n \mathrm{e}^{-\phi} = 0.$$

which gives us our desired result. $\square$