Homework 1

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Question 1 We will let H denote a heads and T denote a tails.

(a) The sample space is given by $\mathcal{S} = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTH}, \text{TTT}\}$.

(b) We have

1. $A =$ at least two heads $= \{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}\}$.
2. $B =$ the first two tosses are heads $= \{\text{HHH}, \text{HHT}\}$.
3. $C =$ the last toss is a tail $= \{\text{HHT}, \text{HTT}, \text{THT}, \text{TTT}\}$.

(c) We have

1. $A^c = \{\text{HTT}, \text{THT}, \text{TTH}, \text{TTT}\}$.
2. $A \cap B = \{\text{HHH}, \text{HHT}\} = B$ (since $B \subset A$).
3. $A \cup C = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTT}\}$.

Question 2 For each of the scenarios, we are pulling five cards from a well-shuffled deck without replacement. Since the order of the cards does not matter, there are $\binom{52}{5}$ total possible combinations.

(a) *Royal flush*: since the order does not matter, for a given suite there is only one possible royal flush, so there are only four possible royal flushes (one for each suite). So the probability of getting a royal flush is $4/\binom{52}{5}$.

(b) *Straight flush*: for a given suite, not counting the royal flush, there are nine possible card combinations that fit this criteria, meaning there are 36 possible straight flushes. So the probability of getting a straight flush is $36/\binom{52}{5}$.

(c) *Four of a kind*: to select four cards of the same value, we must select all four suites of a given value, of which there are 13. Once this happens, four of the five cards in the hand have been determined, and we just have to select the last card. There are 12 possible card values and four possible suites. Therefore, there are $13 \cdot 12 \cdot 4 = 624$ plausible hands, and the probability of getting a four of a kind is $624/\binom{52}{5}$.

(d) *Flush*: each suite has 13 unique values, so there are $\binom{13}{5}$ ways to select five cards for a given suite. However, this is including the ten possible *consecutive* hands, which must be removed (as that hand would be either a straigh or royal flush). This can be done for each suite, so there are $4\left(\binom{13}{5} - 10\right)$ plausible hands, and so the probability of getting a flush is $4\left(\binom{13}{5} - 10\right)/\binom{52}{5}$.

(e) *Three of a kind*: to get three cards of the same value, for a given value we have to choose three cards from the four possible suites. There are $\binom{4}{3}$ ways to do this for each of the 13 values. For the remaining two cards to be chosen, there are 12 possible values to choose from (choosing the same value of the three matching cards would give us a four of a kind), and each of these two cards can be any of the four suites. That is, there are $\binom{12}{2} \cdot 4 \cdot 4$ ways to choose the last two cards. Therefore, there are $\binom{4}{3} \cdot 13 \cdot \binom{12}{2} \cdot 4 \cdot 4 = 208\binom{4}{3}\binom{12}{2}$ plausible hands, and so the probability of getting a three of a kind is $208\binom{4}{3}\binom{12}{2}/\binom{52}{5}$.

(f) *Two pairs*: to get two pairs of cards of the same value, we have to have two unique values to begin with, and there are $\binom{13}{2}$ ways to choose them. For each value, we are choosing two of the four possible suites, so there are $\binom{4}{2}$ choices *for each value*. For the last card, there are 11 possible values (choosing either of the previous values would result in a three of a kind), and from these there are four possible suites, so there are 44 ways to choose the last card. So there are $44\binom{13}{2}\binom{4}{2}^2$ plausible hands, and so the probability of getting two pairs is $44\binom{13}{2}\binom{4}{2}^2/\binom{52}{5}$.