

| x | 1 | 2 | 3 | 4 |
|----------------|-------|------|------|------|
| $f_X(x)$ | 0.19 | 0.32 | 0.31 | 0.18 |
| $f_{X Y}(x 1)$ | 10/19 | 5/19 | 2/19 | 2/19 |

| y | 1 | 2 | 3 | 4 |
|----------------|-------|------|------|------|
| $f_Y(y)$ | 0.19 | 0.32 | 0.31 | 0.18 |
| $f_{Y X}(y 1)$ | 10/19 | 5/19 | 2/19 | 2/19 |

Table 1: Information for question 2.

Homework 2

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Question 1 Let X have a pdf of $f(x) = cx^2$ for $0 \leq x \leq 1$ and $f(x) = 0$ elsewhere.

- (a) For this to be a valid pdf, it must integrate to 1 over the support. So

$$\int_0^1 cx^2 dx = \frac{cx^3}{3} \Big|_0^1 = \frac{c}{3} \stackrel{\text{set}}{=} 1,$$

which leads to $c = 3$. So $f(x) = 3x^2$ for $0 \leq x \leq 1$.

- (b) The cdf is given by

$$F(x) = \int_{-\infty}^x f(t) dt = \int_0^x 3t^2 dt = t^3 \Big|_0^x = x^3$$

for $0 \leq x \leq 1$. We also have $F(x) = 0$ when $x < 0$ and $F(x) = 1$ when $x > 1$.

- (c) We have

$$\Pr\left(\frac{1}{10} \leq X \leq \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{10}\right) = \frac{1}{2^3} - \frac{1}{10^3} = \frac{31}{250}.$$

Question 2 Two discrete random variables X and Y are jointly distributed.

- (a) The marginal pmf for X is obtained by summing over every value of Y for each value of X . For example, to find the marginal probability that $X = 1$, we have $f_X(1) = 0.10 + 0.05 + 0.02 + 0.02 = 0.19$. The other values are obtained in the same way. Finding the marginal pmf for Y is done the exact same way, and it turns out that $f_X(j) = f_Y(j)$ for $j = \{1, 2, 3, 4\}$; they are both found in Table 1.
- (b) X and Y are not independent. For two random variables to be independent, we need $f(x, y) = f_X(x) \cdot f_Y(y)$ for all possible (x, y) pairs. Here we have $f(1, 1) = 0.10$ and $f_X(1) \cdot f_Y(1) = 0.19^2 \neq f(1, 1)$, meaning X and Y are dependent.
- (c) To find the conditional pmf of X given that $Y = 1$, we take each value of $f(x, 1)$ and divide by $f_Y(1)$, i.e. $f_{X|Y}(x|1) = f(x, 1)/f_Y(1)$. For example, we have $f_{X|Y}(1|1) = 0.10/0.19 = 10/19$. Finding the conditional pmf for Y given that $X = 1$ is done in a similar way, and again they are the same. Both pmfs can be found in Table 1.

Question 3 We are considering points (x, y) uniformly selected within an ellipse given by the equation $(x/a)^2 + (y/b)^2 = 1$, where $a, b > 0$. Therefore, the probability of selecting a point (x, y) from this region is $f(x, y) = c$ for $-a \leq x \leq a$, $-b \leq y \leq b$, and $(x/a)^2 + (y/b)^2 \leq 1$, and $f(x, y) = 0$ elsewhere. To find c , we use the fact that a joint pdf must integrate to 1 over all values of x and y , so

$$1 = \int_{-a}^a \int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} c \, dy \, dx = c \cdot 2 \int_{-a}^a b\sqrt{1-(x/a)^2} \, dx = c \cdot \pi ab,$$

which implies that $c = 1/\pi ab$. The last equality is from the fact that the area of an ellipse is πab , which is what the integral is computing. To find the marginal density of X (Y), we integrate over all possible values of Y (X):

$$\begin{aligned} f_X(x) &= \int_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} \frac{1}{\pi ab} \, dy = \frac{y}{\pi ab} \Big|_{-b\sqrt{1-(x/a)^2}}^{b\sqrt{1-(x/a)^2}} = \frac{2\sqrt{1-(x/a)^2}}{\pi a} \quad \text{for } -a \leq x \leq a, \\ f_Y(y) &= \int_{-a\sqrt{1-(y/b)^2}}^{a\sqrt{1-(y/b)^2}} \frac{1}{\pi ab} \, dx = \frac{x}{\pi ab} \Big|_{-a\sqrt{1-(y/b)^2}}^{a\sqrt{1-(y/b)^2}} = \frac{2\sqrt{1-(y/b)^2}}{\pi b} \quad \text{for } -b \leq y \leq b. \end{aligned}$$

These results make sense intuitively. For $f_X(x)$, we can see that the highest probability is obtained when $x = 0$. At $x = 0$, the height of the ellipse is the greatest, so there is more “width” for $x = 0$ to be chosen. And as $x \rightarrow \pm a$, $f_X(x) \rightarrow 0$, meaning that the closer you get to the end of the ellipse, the less likely that point is to be chosen.

Question 4 The joint cdf of X and Y is given by $F(x, y) = (1 - e^{-\alpha x})(1 - e^{-\beta y})$ for $x, y \geq 0$ and $F(x, y) = 0$ elsewhere, where $\alpha, \beta > 0$ are fixed constants.

- (a) X and Y are independent. This is because we can express the joint cdf as $F(x, y) = G(x) \cdot H(y)$, where $G(x) = 1 - e^{-\alpha x}$ and $H(y) = 1 - e^{-\beta y}$.
- (b) The joint pdf is given by

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x} (1 - e^{-\alpha x}) \cdot \frac{\partial}{\partial y} (1 - e^{-\beta y}) = \alpha e^{-\alpha x} \cdot \beta e^{-\beta y}$$

for $x, y \geq 0$ and $f(x, y) = 0$ elsewhere. The marginal densities are given by

$$\begin{aligned} f_X(x) &= \lim_{\phi \rightarrow \infty} \int_0^\phi \alpha e^{-\alpha x} \cdot \beta e^{-\beta y} \, dy = \alpha e^{-\alpha x} \cdot \lim_{\phi \rightarrow \infty} [-e^{-\beta y}]_0^\phi = \alpha e^{-\alpha x}, \\ f_Y(y) &= \lim_{\phi \rightarrow \infty} \int_0^\phi \alpha e^{-\alpha x} \cdot \beta e^{-\beta y} \, dx = \beta e^{-\beta y} \cdot \lim_{\phi \rightarrow \infty} [-e^{-\alpha x}]_0^\phi = \beta e^{-\beta y}. \end{aligned}$$

Here we see that $f_X(x) \cdot f_Y(y) = \alpha e^{-\alpha x} \cdot \beta e^{-\beta y} = f(x, y)$, another indication that X and Y are independent.

Question 5 Let X and Y have a joint pdf $f(x, y) = c(x^2 - y^2)e^{-x}$ for $x \leq 0$ and $-x \leq y \leq x$. To help with some of the integrations, this question makes use of the following result:

$$\Omega(\theta, n) := \int_\theta^\infty x^n e^{-x} \, dx = n! \cdot e^{-\theta} \sum_{k=0}^n \frac{\theta^k}{k!}$$

for all $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 0}$. This specific integral came up so many times in this question that I thought it was worth figuring out a general formula for it. A derivation of this result can be found in Appendix A.

- (a) Using the fact that the joint pdf must integrate to 1, we have

$$\begin{aligned} 1 &= \int_0^\infty \int_{-x}^x c(x^2 - y^2)e^{-x} \, dy \, dx = c \int_0^\infty e^{-x} \left[x^2 y - \frac{y^3}{3} \right]_{-x}^x \, dx \\ &= \frac{4c}{3} \cdot \int_0^\infty x^3 e^{-x} \, dx = \frac{4c \cdot \Omega(0, 3)}{3} = \frac{4c \cdot 3!}{3} = 8c, \end{aligned}$$

and so $c = 1/8$.

- (b) X and Y are not independent since the joint pdf does not have rectangular support.
- (c) When considering all possible $x \geq 0$ values, we see that we must have $-x \leq y \leq x$. So the marginal density of X is given by

$$f_X(x) = \int_{-x}^x \frac{(x^2 - y^2)e^{-x}}{8} \, dx = \frac{e^{-x}}{8} \left[x^2 y - \frac{y^3}{3} \right]_{-x}^x = \frac{e^{-x}}{8} \cdot \frac{4x^3}{3} = \frac{x^3 e^{-x}}{6}$$

for $x \geq 0$.