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apk2152

G5203: PROBABILITY

Fall 2020

Midterm

1. Please **print** your name and student ID number in the upper right corner of this page.
2. This is a closed book, closed-notes examination. You can refer to 2 two-sided pages of notes.
3. Please write the answers in the space provided. If you do not have enough space, use the back of a nearby page or ask for additional blank paper. Make sure you sign any loose pages.
4. In order to receive full credit for a problem, you should show all of your work and explain your reasoning. Good work can receive substantial partial credit even if the final answer is incorrect.

Question	Total Points	Credit
1	20	
2	20	
3	10	
4	10	
5	20	
6	20	
total	100	

1. The joint density of X and Y is given by

$$f(x,y) = k(x+y), \qquad 0 \le y \le 1 \text{ and } 0 \le x \le 1.$$

(a) Are X and Y independent? Find k.

$F(x,y) = k(x+y) = kx + ky$

$= g(x) + h(y)$

Cannot split up into form $g(x) \cdot h(y)$.

So X and Y are $\boxed{\text{not}}$ independent.

$$1 = \int_0^1 \int_0^1 k(x+y)\, dy\, dx = \int_0^1 \int_0^1 (kx + ky)\, dy\, dx$$

$$= \int_0^1 \left(kxy + \frac{k}{2}y^2 \Big|_0^1 \right) dx = \int_0^1 (kx + k/2)\, dx$$

$$= \frac{k}{2}x^2 + \frac{k}{2}x \Big|_0^1 = \frac{k}{2} + \frac{k}{2} = k$$

$\rightarrow \boxed{k=1}$ and so $f(x,y) = x+y$

$0 \le y \le 1$ and $0 \le x \le 1$

(b) What is the marginal density of X and the marginal density of Y?

$$f_X(x) = \int_0^1 (x+y)\, dy$$

$$= xy + \frac{y^2}{2} \Big|_0^1$$

$$= x + \frac{1}{2}$$

$$\boxed{f_X(x) = x + 1/2 \quad \text{for } 0 \le x \le 1}$$

Check: $\int_0^1 (x + 1/2)\, dx$

$= \frac{x^2}{2} + \frac{1}{2}x \Big|_0^1$

$= \frac{1}{2} + \frac{1}{2} = 1$ ✓

$$f_Y(y) = \int_0^1 (x+y)\, dx$$

$$= \frac{x^2}{2} + xy \Big|_0^1$$

$$= y + \frac{1}{2}$$

$$\boxed{f_Y(y) = y + 1/2 \quad \text{for } 0 \le y \le 1}$$

Check: same as $f_X(x)$.

$$\frac{f_{XY}(x,y)}{f_X(x)} \overset{?}{=} f_Y(y)$$

$$= \frac{x+y}{x+1/2} \overset{?}{=} y + 1/2$$

$$x + y \ne (x + 1/2)(y + 1/2)$$

$$= xy + \frac{1}{2}(x+y) + \frac{1}{4}.$$

Another check that X and Y are not independent.

Assume ~~student~~ awards cannot be shared by students

2. Five separate awards are to be presented to selected students from a class of 30. How many outcomes are possible if

(a) a student can receive any number of awards;

Choosing 5 students from 30

~~Without~~ replacement (~~don't~~ can choose same student again and again)

Order does matter (first student chosen → wins first award)

That is, for any given award, you can choose any of the 30 students.

So there are $\boxed{30^5}$ possible outcomes.

(b) each student can receive at most one award?

Without replacement (now student cannot win multiple awards)

Order does matter (same as before)

Once a student wins an award, they cannot win another and are ~~ ~~ removed from the selection pool.

So there are $\boxed{\frac{30!}{25!}} = 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26$ possible outcomes.

3. If A and B alternate rolling a pair of dice, stopping either when A rolls the sum of 9 or when B rolls the sum of 6. Assuming that A rolls first, what's the probability that the final roll is made by A?

Out of 36 outcomes:

A

$(3,6),(4,5),(5,4),(6,3) \rightarrow Pr(A) = 4/36 = 1/9$

B

$(1,5),(2,4),(3,3),(4,2),(5,1) \rightarrow Pr(B) = 5/36$

A: A rolls a 9 on a given turn

A_i: " " on roll i

A_i^c: not " " on roll i

for odd i

B: B rolls a 6 on a given turn

B_j: " " on roll j

B_j^c: not " " on roll j

for even j

Sloppy, sorry!

$\cancel{Pr(A_i) = (8/9)(31/36)\,1/9}$

W: final role is by player A $= \bigcup_{\text{odd } j} A_j$

A_i's are all disjoint, so

$$Pr(W) = Pr\left(\bigcup_{\text{odd } j} A_j\right) = \sum_{\text{odd } j} \frac{1}{9} \cdot \left(\frac{62}{81}\right)^{\frac{j-1}{2}}$$

geometric series

$j = 2k+1,\ k = 0,1,2,\ldots$

$$Pr(W) = \sum_{k=0}^{\infty} \frac{1}{9} \cdot \left(\frac{62}{81}\right)^{\frac{2k+1-1}{2}} = \sum_{k=0}^{\infty} \frac{1}{9} \cdot \left(\frac{62}{81}\right)^{k}$$

$$= \frac{1/9}{1-\frac{62}{81}} = \frac{1/9}{19/81} = \boxed{\frac{9}{19}}$$

$Pr(A_1) = \frac{1}{9}$

$Pr(A_3) = (8/9)(31/36)\frac{1}{9}$

$Pr(A_5) = (8/9)^2 (31/36)^2\, 1/9$

$Pr(A_7) = (8/9)^3 (31/36)^3\, 1/9$

$$Pr(A_j) = (8/9)^D (31/36)^D \cdot \frac{1}{9}$$

$$= \left(\frac{8}{9} \cdot \frac{31}{36}\right)^D \cdot \frac{1}{9}$$

$$= \left(\frac{62}{81}\right)^D \cdot \frac{1}{9}$$

$D = \frac{j-1}{2}$

* could also see we just add up each of these values

* remove 1 from j (from 1/9)

divide by 2 (one for 8/9 and one for 31/36)

$40(40+2)$

420
$\times 4$
1680

4. If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife.

without replacement
order matters.

e.g. $H_1 W_2 H_3 W_4 H_2 W_1 H_4 W_3$

$H_1 H_2 H_3 H_4 W_1 W_2 W_3 W_4$

There are 8!
total arrangements.

There are $\frac{8!}{~~(4!)~~}$ ~~4·8 40·5~~ ~~40·42~~ ~~1680~~ 8! total arrangements.

Say we start with H_1, he can sit next to anyone <u>except</u> W_1, so 6 possible people.

Next, this person, cannot choose H_1 (already seated) or their partner, or themselves, so there are 5 choices again.

8 initial choices.

After first choice, there are 6 choices (cannot choose themselves or wife).

Once we get to third person, there is a possibility that their partner was chosen first.

- If partner was first choice, then ~~third person has~~ there are 6 choices for person 3.
- If partner was not first choice, then there are 5 choices for person 3.

— I feel like this will be too complicated going forward.

Prob at least one husband sits next to his wife?

Permute other 6 couples such that the couples are not together.

Seems to ~~make another~~ harder problem.

- Can we structure it so that first person is <u>always</u> third person's partner? No probably not.

- Fourth person: chance first or second person is their partner
- 5th person: P1 or P2 or P3 could be partner

Maybe consider husbands and wives separately

4! ways to permute husbands

4! ways to permute wifes

permute then mix?

can't have them mix certain ways

6x4 + 3

4x2 + 10

will be only one ~~way to~~ arrangement here

(_ _)(_ _)(_ _)(_ _)

8x6 possible people here

Assume

$$\boxed{\frac{8 \times 6 \times 6 \times 4 \times 4 \times 2 \times 1 + 13}{8!}}$$

5. You arrive at a bus stop at 10am, knowing that the bus will arrive at some times uniformly distributed between 10 and 10:30. Continuous

(a) What is the probability that you will have to wait longer than 10 minutes?

Let X = time (min) waiting after 10 am, $X \in [0, 30]$

$$f(x) = \frac{1}{30} \quad \text{for} \quad 0 \le x \le 30$$

$$Pr(X > 10) = 1 - Pr(X \le 10) = 1 - \int_0^{10} \frac{1}{30}\, dx = 1 - \frac{x}{30}\Big|_0^{10} = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$$

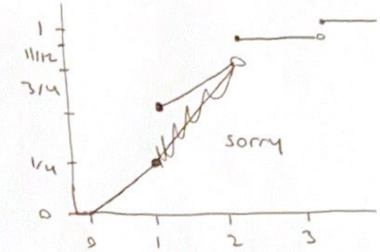
Makes sense intuitively

(b) If at 10:15 the bus has not arrived yet, what's the probability that you will have to wait at least an additional 10 minutes?

$$Pr(X > 25 \mid X > 15) = \frac{Pr(X > 25)}{Pr(X > 15)} = \frac{1 - Pr(X \le 25)}{1 - Pr(X \le 15)}$$

$$= \frac{1 - \int_0^{25} 1/30\, dx}{1 - \int_0^{15} 1/30\, dx} = \frac{1 - 25/30}{1 - 15/30} = \frac{5/30}{15/30} = \frac{5}{15} = \boxed{\frac{1}{3}}$$

So this arrival time does not have the memoryless property!



6. Suppose that the cumulative distribution function of X is given by

Continuous RV ?

(but not "usual")

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{4}, & 0 \le x < 1 \quad \leftarrow @1 = 1/4 \\ \frac{1}{2} + \frac{x-1}{4}, & 1 \le x < 2 \quad \leftarrow @2 = 3/4 \ne 11/12 \\ \frac{11}{12}, & 2 \le x < 3 \\ 1, & 3 \le x \end{cases}$$

This CDF jumps, so have to use technical definition.

(a) Find $P(X = i)$ for $i = 1, 2, 3$.

CDF is right-continuous (as always)

And this is a continuous RV

But CDF has a break at X=2. and X=3. and X=1

$$\Pr(X=j) = F(j) - \lim_{k \to j+} F(k)$$

~~$\Pr(X=i) = 0$~~

$$\Pr(X=1) = F(1) - \lim_{k\to 1+} F(k) = \frac{1}{2} - \lim_{k \to 1+} \frac{k}{4} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\Pr(X=2) = F(2) - \lim_{k\to 2+} F(k) = \frac{11}{12} - \lim_{k\to 2+}\left(\frac{1}{2} + \frac{x-1}{4}\right) = \frac{11}{12} - \frac{3}{4} = 1/6$$

$$\Pr(X=3) = F(3) - \lim_{k \to 3+} F(k) = 1 - \lim_{k\to 3+} \frac{11}{12} = 1 - \frac{11}{12} = \frac{1}{12}$$

$$\boxed{\begin{aligned} \Pr(X=1) &= \tfrac{1}{4} \\ \Pr(X=2) &= 1/6 \\ \Pr(X=3) &= 1/12 \end{aligned}}$$

(b) Find $P(\frac{1}{2} < X < \frac{3}{2})$.

$$\Pr(1/2 < X < 3/2) = \cancel{F(3/2) - F(1/2)}$$

Go with this

$$\cancel{\frac{1}{2} + \frac{3/2 - 1}{4} - \frac{1/2}{4}}$$

$$\cancel{= \frac{1}{2} + \frac{1/2}{4} - \frac{1/2}{4} = \boxed{\frac{1}{2}}}$$

~~Differentiate to get pdf between 0 and 2~~

$$\cancel{f(x) = \begin{cases} 1/4 & 0 \le x \le 1 \\ 1/4 & 1 \le x \le 2 \end{cases} = 1/4 \quad 0 \le x < 2}$$

$$\cancel{\Pr(1/2 < X < 3/2) = \int_{1/2}^{3/2} 1/4 \, dx = \frac{x}{4}\Big|_{1/2}^{3/2} = \frac{1}{4}\left(\frac{3}{2} - \frac{1}{2}\right) = \boxed{\frac{1}{4}}}$$

cannot be right

~~Can do this since we are not worried about breaks at X=2 or X=3.~~

Written on back of this page! →

$$\Pr(1/2 < X < 3/2) = F(3/2) - F(1/2)$$

$$= \frac{1}{2} + \frac{3/2 - 1}{4} - \frac{1/2}{4}$$

$$= \frac{1}{2} + \frac{1/2}{4} - \frac{1/2}{4} = \boxed{\frac{1}{2}}$$