GR5203: PROBABILITY

Autumn 2020

Homework 2

This homework is shorter and due Monday, October 12 at 11pm, EDT. You need to upload it to canvas assignments tab.

Read Sections 2.1-2.3 and 3.1-3.6 of DeGroot and Schervish. You may find solutions to some problems in various sources, including solution manual for the textbook. I encourage you to solve the problems yourself rather than obtain solutions, homework is your most valuable tool in studying and preparing for the exams. As I mentioned previously, you may work in groups but do write up on your own and mention the people you collaborated with (no points will be taken off for collaboration as long as there is no evidence of copying).

1. Suppose that X has the probability density function (pdf) $f(x) = cx^2$ for $0 \leq x \leq 1$ and $f(x) = 0$ otherwise.
   (a) Find c.
   (b) Find the cdf of X.
   (c) What is $P(.1 \leq X < .5)$?

2. The joint pmf of two random variables X and Y is given in the following table

	1	2	3	4
1	.10	.05	.02	.02
2	.05	.20	.05	.02
3	.02	.05	.20	.04
4	.02	.02	.04	.10

   (a) Find the marginal pmfs of X and Y.
   (b) Are X and Y independent?
   (c) Find the conditional pmf of X given $Y = 1$ and of Y given $X = 1$.

3. A point is chosen uniformly in the interior of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

   Find the marginal densities of the x and y coordinates of the point.

4. Let X and Y be random variables with the joint cdf

$$F(x, y) = (1 - e^{-\alpha x})(1 - e^{-\beta y}), \quad x \geq 0, \quad y \geq 0$$

   for some fixed $\alpha > 0$ and $\beta > 0$.
   (a) Are X and Y independent?

(b) Find the joint and marginal densities of X and Y.

5. Let X and Y be random variables with joint pdf

$$f(x,y) = c(x^2 - y^2)e^{-x}, \quad 0 \le x < \infty, \quad -x \le y \le x$$

   (a) Find c.
   (b) Are X and Y independent?
   (c) Find the marginal densities. item Find the conditional densities.

6. Show that if X has density f_X and $Y = aX + b$ for some $a \neq 0$ then

$$f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a})$$

   is the pdf of Y.

7. Suppose that F is the cdf of an integer-valued fandom variable, and let U be uniform on $[0,1]$ (that is, $U \sim \mathrm{Unif}[0,1]$.) Define a random variable $Y = k$ if $F(k-1) < U \le F(k)$. Show that Y has cdf F. (This result can be used to generate integer-valued random variables from a uniform random number generator.)

8. A civil engineer is studying a left turn lane that is long enough to hold 7 cars. Let X be the number of cars in the lane at the end of a randomly chosen red light. The engineer believes that the probability that $X = x$ is proportional to $(x+1)(8-x)$ for $x = 0, \ldots, 7$ (the possible values of X).

   (a) Find the p.m.f of X.
   (b) Find the probability that X will be at least 5.
   (c) Suppose that the p.d.f. of a random variable X is

$$f(x) = \frac{1}{8}x, \quad \text{for} 0 \le x \le 4.$$

   (d) Find the value of t such that $P(X \le t) = 1/4$.
   (e) Find the value of t such that $P(X \ge t) = 1/2$.

9. Suppose that X and Y are random variables such that (X,Y) must belong to the rectangle $0 \le x \le 3$ and $0 \le y \le 4$. Suppose also that the joint c.d.f. of X and Y for every (x,y) in the rectangle is:

$$F(x,y) = \frac{1}{156}xy(x^2 + y).$$

   Determine

   (a) $P(1 \le X \le 2 \text{ and } 1 \le Y \le 2)$;
   (b) $P(2 \le X \le 4 \text{ and } 2 \le Y \le 4)$;
   (c) the c.d.f. of Y;
   (d) the joint p.d.f. of X and Y;
   (e) $P(Y \le X)$.