Quiz 1 - Aiden Kenny 09/24/2020

1. P = polygraph ~~is~~ positive (lying)
N = polygraph negative (truth)
T = person is actually telling truth
F = person is actually lying

$Pr(P \mid F) = 0.88$
$Pr(P \mid T) = 0.14$
$Pr(T \mid P) = ?$ (what we want)
Assume ~~$Pr(T) = Pr(F) = 0.5$~~
$Pr(T) = 0.99$, $Pr(F) = 0.01$

$Pr(T \mid P)$ ... use Bayes' Rule

$$Pr(T \mid P) = \frac{Pr(P \mid T)}{Pr(P \mid T)\,Pr(T) + Pr(P \mid F)\,Pr(F)} = \frac{0.14}{\cancel{0.5(0.14 + 0.88)}\ (0.99)(0.14) + (0.01)(0.88)} = \cancel{\frac{0.28}{1.02}}$$

~~So~~ $Pr(T \mid P) = \cancel{28/102} = \cancel{\boxed{14/51}}$

$$= \boxed{\frac{0.14}{(0.99)(0.14) + (0.01)(0.88)}}$$

2. HH, TT, HT. Coin randomly chosen, flipped, and comes up heads.

(a) What is probability that the chosen coin is the two-headed coin?

Let C_1 be 2-headed coin, C_2 be 2-tailed coin, and C_3 be fair coin. Then
$Pr(H \mid C_1) = 1$, $Pr(H \mid C_2) = 0$, and $Pr(H \mid C_3) = 0.5$.
Want $Pr(C_1 \mid H)$.
Also, since we are choosing coin randomly, we have
$Pr(C_1) = Pr(C_2) = Pr(C_3) = 1/3$.

$$\text{So } Pr(C_1 \mid H) = \frac{Pr(H \mid C_1) \cdot Pr(C_1)}{\sum_{i=1}^{3} Pr(H \mid C_i) \cdot Pr(C_i)} = \frac{\cancel{1/3}}{\cancel{1/3}(1 + 0 + 0.5)} = \frac{1}{1.5} = \boxed{\frac{2}{3}}$$

(b) Coin is thrown a second time and comes up heads again. What is the probability that the chosen coin is the two-headed coin?
i.e. getting two heads in a row
Let H_2 be the event that two heads are flipped in a row.
Note: we are not picking coin again, so $Pr(C_1) = Pr(C_2) = Pr(C_3) = 1/3$.
Then $Pr(H_2 \mid C_1) = 1$, $Pr(H_2 \mid C_2) = 0$, and $Pr(H_2 \mid C_3) = 0.25$.

$$\text{So } Pr(C_1 \mid H_2) = \frac{Pr(H_2 \mid C_1) \cdot Pr(C_1)}{\sum_{i=1}^{3} Pr(H_2 \mid C_i)\,Pr(C_i)} = \frac{\cancel{1/3}}{\cancel{1/3}(1 + 0 + 0.25)} = \frac{1}{1.25} = \boxed{\frac{4}{5}}$$

3. Player A shoots first, has probability p_1 of making basket.

If Player A misses, then Player B goes and has probability p_2 of scoring.

Probabilities never change, and shots are independent.

Game is over after first basket is made.

(a) Find the pmf for total number of attempts.

Theoretically, game could go on forever, so # trials $\in [1, \ldots)$

Let n be the number of trials.

Some example outcomes:

$n=1$	p_1	$n=5$	$p_1^3 p_2^2$	$n=9$	$p_1^5 p_1^4$
$n=2$	$p_1 p_2$	$n=6$	$p_1^3 p_2^3$	$n=10$	$p_1^5 p_2^5$
$n=3$	$p_1^2 p_2$	$n=7$	$p_1^4 p_2^3$		
$n=4$	$p_1^2 p_2^2$	$n=8$	$p_1^4 p_2^4$		

exponents add up to n, $Pr(N=n) = p_1^i p_2^j$, where $i+j=n$

have other constraints, $i \geq j$, but $i \leq 2j$

basically, either $i=j$ or $i=j+1$

$\uparrow$ n is even $\qquad \uparrow$ n is odd

<u>n even</u>

$i=j$

So $i+j=n \rightarrow i+i=n \rightarrow 2i=n$

$\rightarrow i = n/2$

So $Pr(N=n) = p_1^{n/2} p_2^{n/2} = (p_1 p_2)^{n/2}$

$= (p_1^n p_2^n)^{1/2}$

<u>n odd</u>

$i=j+1 \rightarrow j=i-1$

$i+j=n \rightarrow i+i-1=n \rightarrow 2i=n+1$

$\rightarrow i = \frac{n}{2}+\frac{1}{2}$

$j = \frac{n}{2}+\frac{1}{2}-1 = \frac{n}{2}-\frac{1}{2}$

So $Pr(N=n) = p_1^{(n/2+1/2)} p_2^{(n/2-1/2)}$

$= (p_1^{n+1} p_2^{n-1})^{1/2}$

So
$$f(n) = Pr(N=n) = \begin{cases} (p_1^{n+1} p_2^{n-1})^{1/2} & \text{if } n \text{ is odd} \\ (p_1 p_2)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{elsewhere} \end{cases} \quad \text{for } n \geq 1$$

(b) Probability Player A wins ... sum over all odd $n = \{1, 3, 5, \ldots\}$

Let $n = 2k+1$, for $k = 0, 1, \ldots, \infty$

$$Pr(n \text{ is odd}) = \sum_{k=0}^{\infty} \left(p_1^{2k+1+1} p_2^{2k+1-1}\right)^{1/2} = \sum_{k=0}^{\infty} \left(p_1^{2(k+1)} p_2^{2k}\right)^{1/2}$$

$$= \sum_{k=0}^{\infty} p_1^{k+1} p_2^{k} = \sum_{k=0}^{\infty} p_1 p_1^k p_2^k = p_1 \sum_{k=0}^{\infty} (p_1 p_2)^k = \boxed{\frac{p_1}{1-p_1 p_2}}$$

Geometric series!