

## Homework 2

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### Question 1

### Question 9

If  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$  with unknown  $\mu$  and  $\sigma^2$ , then a  $\gamma\%$  confidence interval for  $\mu$  is given by

$$\mathcal{I} = \left( \bar{X} - t_\gamma(n) \cdot S/\sqrt{n}, \bar{X} + t_\gamma(n) \cdot S/\sqrt{n} \right),$$

where  $t_\gamma(n) = T_{n-1}^{-1}((1+\gamma)/2)$  is the  $(1+\gamma)/2$ th quantile of the  $t$  distribution with  $\text{df} = n-1$  and  $S$  is the sample standard deviation. The length of this confidence interval is given by

$$\Delta = \max(\mathcal{I}) - \min(\mathcal{I}) = \left( \bar{X} + t_\gamma(n) \cdot S/\sqrt{n} \right) - \left( \bar{X} - t_\gamma(n) \cdot S/\sqrt{n} \right) = 2t_\gamma(n) \cdot S/\sqrt{n}.$$

The squared length is then given by  $\Delta^2 = 4t_\gamma^2(n) \cdot S^2/n$ . Because the sample variance is an unbiased estimator for  $\sigma^2$ , we have  $\mathbb{E}[\Delta^2] = \mathbb{E}[4t_\gamma^2(n) \cdot S^2/n] = 4t_\gamma^2(n) \cdot \sigma^2/n$ . We now set  $\mathbb{E}[\Delta^2] < \sigma^2/2$ , and after some cancellations, we see that we need  $t_\gamma^2(n)/n < 1/8$ . There is no way to find a closed-form expression for this, so we will have to check the value of  $t_\gamma^2(n)/n$  for increasing values of  $n$ . I set up a `while` loop in R to solve for it, and when  $\gamma = 0.9$ , we find that  $n = 24$  is the smallest value of  $n$  such that  $\mathbb{E}[\Delta^2] < \sigma^2/2$ .

### Question 10

Let  $\mathbf{X} \stackrel{\text{iid}}{\sim} N(\theta, \sigma^2)$ , where  $\theta$  is unknown and  $\sigma^2$  is known, and we assume prior that  $\theta \sim N(\mu, \nu^2)$ , where both  $\mu$  and  $\nu^2$  are known.

- (a) Since normal distributions are conjugate to normal sampling, it follows that  $\theta | \mathbf{x} \sim N(\tilde{\mu}, \tilde{\sigma}^2)$ , where

$$\tilde{\mu} = \frac{\sigma^2\mu + n\nu^2\bar{x}}{\sigma^2 + n\nu^2} \quad \text{and} \quad \tilde{\sigma}^2 = \frac{\sigma^2\nu^2}{\sigma^2 + n\nu^2}.$$

We also know that  $(\theta | \mathbf{x} - \tilde{\mu})/\tilde{\sigma} \sim N(0, 1)$ , and so a 95% confidence interval for  $\theta | \mathbf{x}$  is given by

$$\mathcal{I} = \left( \tilde{\mu} - \Phi^{-1}(0.975) \cdot \tilde{\sigma}, \tilde{\mu} + \Phi^{-1}(0.975) \cdot \tilde{\sigma} \right).$$

- (b) We can think of our interval  $\mathcal{I}$  as a function of  $\nu^2$ . To examine what happens to  $\mathcal{I}(\nu^2)$  as  $\nu^2 \rightarrow \infty$ , we will first look at  $\tilde{\mu}$  and  $\tilde{\sigma}$ . Using L'Hopital's rule, we have

$$\begin{aligned} \lim_{\nu^2 \rightarrow \infty} \tilde{\mu} &= \lim_{\nu^2 \rightarrow \infty} \frac{\sigma^2\mu + n\nu^2\bar{x}}{\sigma^2 + n\nu^2} = \lim_{\nu^2 \rightarrow \infty} \frac{n\bar{x}}{n} = \bar{x}, \\ \lim_{\nu^2 \rightarrow \infty} \tilde{\sigma} &= \lim_{\nu^2 \rightarrow \infty} \sqrt{\frac{\sigma^2\nu^2}{\sigma^2 + n\nu^2}} = \sqrt{\lim_{\nu^2 \rightarrow \infty} \frac{\sigma^2\nu^2}{\sigma^2 + n\nu^2}} = \sqrt{\lim_{\nu^2 \rightarrow \infty} \frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}, \end{aligned}$$

and so  $\mathcal{I}(\nu^2) \rightarrow (\bar{x} - \Phi^{-1}(0.975) \cdot \sigma/\sqrt{n}, \bar{x} + \Phi^{-1}(0.975) \cdot \sigma/\sqrt{n})$ , which is a 95% confidence interval for  $\theta$ .

## Question 11

Let  $\mathbf{X} \stackrel{\text{iid}}{\sim} \text{Unif}(0, \theta)$ , where  $\theta$  is unknown, and let  $Y = X_{(n)}$  be the  $n$ th order statistic.

- (a) Let  $F_i(x) = \Pr(X_i \leq x) = x/\theta$  for all  $i \in \{1, \dots, n\}$ . Then the cdf of  $Y$  is  $G(y) = \prod_{i=1}^n F_i(y) = (y/\theta)^n$ , since all of the  $X_i$ 's are independent. The density of  $Y$  is then given by  $g(y) = ny^{n-1}/\theta^n$  for  $y \in [0, \theta]$ . By letting  $W = Y/\theta$ , we have  $Y = \theta W$  and  $\partial Y/\partial W = \theta$ , so the density of  $Y/\theta$  is given by  $h(w) = g(y(w)) \cdot \theta = nw^{n-1}$  for  $w \in [0, 1]$ . The cdf is then seen to be  $H(w) = w^n$ , and so the quantile is given by  $H^{-1}(w) = \sqrt[n]{w}$ .

- (b) Since we must have  $y \leq \theta$ , it is natural that  $Y$  will underestimate  $\theta$ , and is therefore biased. We have

$$\mathbb{E}[Y] = \int_0^\theta y \cdot \frac{ny^{n-1}}{\theta^n} dy = \frac{n}{\theta^n} \int_0^\theta y^n dy = \frac{n}{\theta^n} \cdot \frac{y^{n+1}}{n+1} \Big|_0^\theta = \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} = \frac{n\theta}{n+1},$$

and so the bias is  $\text{bias}(Y) = \mathbb{E}[Y] - \theta = -\theta/(n+1)$ .

- (d) Using the cdf of  $Y/\theta$ , for any interval involving  $Y/\theta$ , we have  $\Pr(a \leq Y/\theta \leq b) = b^n - a^n$ , where  $a, b \in (0, 1]$ . Rearranging the terms inside the interval gives us  $\Pr(Y/b \leq \theta \leq Y/a) = b^n - a^n \stackrel{\text{set}}{=} \gamma$ . That is, as long as we impose the constraint that  $b^n - a^n = \gamma$ , any interval  $(Y/b, Y/a)$  is a  $\gamma\%$  confidence interval for  $\theta$ .

## Question 12