

## Homework 2

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STAT GR5204: Statistical Inference

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December 1, 2020

### Question 1

Suppose that  $Y \sim \text{Bin}(100, p)$ , and we want to make an inference about the value of  $p$ . We test  $H_0 : p = 0.08$  against  $H_A : p < 0.08$ , and our test  $\delta$  will reject  $H_0$  if and only if  $Y = 6$ .

- (a) The significance  $\alpha$  is the probability of making a Type I error,

$$\alpha(\delta) = \Pr(Y = 6 | p = 0.08) = \binom{100}{6} (0.08)^6 (0.92)^{94} = 0.123.$$

- (b) Suppose that  $p = 0.04$ . The probability of a Type II error,  $\beta$ , is

$$\beta(\delta) = \Pr(Y \neq 6 | p = 0.04) = 1 - \Pr(Y = 6 | p = 0.04) = 1 - \binom{100}{6} (0.04)^6 (0.96)^{94} = 0.895.$$

### Question 2

For a random variable  $Y \sim \text{Binom}(n, p)$ , if  $n$  is large enough we can approximate it using a normal distribution with the same mean and variance, i.e.  $Y \sim N(np, np(1-p))$ . Then the sample proportion,  $\hat{p} = Y/n$ , is also normally distributed as  $\hat{p} \sim N(p, p(1-p)/n)$ , and standardizing  $\hat{p}$  gives us  $(\hat{p} - p)/\sqrt{p(1-p)/n} \sim N(0, 1)$ . It can then be shown that a  $100\gamma\%$  confidence interval for  $\hat{p}$  is given by

$$\mathcal{I} = \left( \frac{\hat{p} + z_\gamma^2/2n}{1 + z_\gamma^2/2n} - z_\gamma \cdot \frac{\sqrt{\hat{p}(1-\hat{p})/n + z_\gamma^2/4n^2}}{1 + z_\gamma^2/n}, \frac{\hat{p} + z_\gamma^2/2n}{1 + z_\gamma^2/2n} + z_\gamma \cdot \frac{\sqrt{\hat{p}(1-\hat{p})/n + z_\gamma^2/4n^2}}{1 + z_\gamma^2/n} \right),$$

where  $z_\gamma = \Phi^{-1}((1+\gamma)/2)$ . For this example, we have  $n = 300$  and  $\hat{p} = 75/300 = 1/4$ , and to get a 90% confidence interval, we have  $z_\gamma = 1.645$ . Therefore, a 90% confidence interval for  $p$  is  $(0.212, 0.294)$ .

### Question 3

Suppose we have a random sample  $\mathbf{X} \stackrel{\text{iid}}{\sim} \text{Gamma}(4, \beta)$ , so  $\mathbb{E}[X_i] = 4\beta$  and  $\text{Var}[X_i] = 4\beta^2$ . The expected value and variance of the sample mean  $\bar{X}$  is then given by  $\mathbb{E}[\bar{X}] = 4\beta$  and  $\text{Var}[\bar{X}] = 4\beta^2/n$ , and from the CLT we have  $\sqrt{n}(\bar{X} - 4\beta)/2\beta = \sqrt{n}(\bar{X}/2\beta - 2) \sim N(0, 1)$ . If  $z_\gamma = \Phi^{-1}((1+\gamma)/2)$ , then we have  $\gamma = \Pr(-z_\gamma \leq \sqrt{n}(\bar{X}/2\beta - 2) \leq z_\gamma)$ . Rearranging to get  $\beta$  in the middle gives us

$$\mathcal{I} = \left( \frac{2\bar{X}}{2 + z_\gamma/\sqrt{n}}, \frac{2\bar{X}}{2 - z_\gamma/\sqrt{n}} \right).$$

For this random sample,  $n = 25$ , and because we want a 95.4% confidence interval (oddly specific), we have  $z_\gamma = 2$ , so the confidence interval is given by  $\mathcal{I} = (5\bar{X}/6, 5\bar{X}/4)$ .

### Question 4

Suppose that  $X \sim \text{Binom}(100, p)$ , where  $p \in (1/4, 1/2)$  is unknown. We test  $H_0 : 1/2$  against  $H_A : p = 1/4$  using  $\delta$  : reject  $H_0$  if  $X \leq 3$ . That is, our rejection region is  $\mathcal{S}_X = \{0, 1, 2, 3\}$ , and so the power function for this test is

$$\pi(p | \delta) = \Pr(X \in \mathcal{S}_X | p) = \sum_{k=0}^3 \binom{100}{k} p^k (1-p)^{100-k}.$$

### Question 5

Suppose  $X_1, X_2 \stackrel{\text{iid}}{\sim} \text{Exp}(1/\theta)$ , where  $\theta > 0$ , which means  $\mathbb{E}[X_i] = \theta$  and  $\text{Var}[X_i] = \theta^2$ . The joint density is

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{e^{-x_1/\theta}}{\theta} \cdot \frac{e^{-x_2/\theta}}{\theta} = \frac{e^{-(x_1+x_2)/\theta}}{\theta^2}$$

Consider the transformation  $Y_1 = X_1 + X_2$  and  $Y_2 = X_2$ , which implies  $Y_1 \sim \text{Gamma}(2, 1/\theta)$  and  $Y_2 \sim \text{Exp}(1/\theta)$ . We note that because  $f_{\mathbf{X}}(\mathbf{x})$ , which is also the likelihood function of the random sample, is a function of  $y_1$  and  $\theta$ , so it is a sufficient statistic. In vector notation, the transformation is given by  $\mathbf{Y} = \mathbf{A}\mathbf{X}$ , where  $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ , and so  $\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y}$ , where  $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ . Therefore, the joint density of  $Y_1$  and  $Y_2$  is

$$g_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{x}) \cdot \left| \det \left( \frac{\partial \mathbf{A}\mathbf{x}}{\partial \mathbf{y}} \right) \right| = f_{\mathbf{X}}(\mathbf{A}^{-1}\mathbf{y}) \cdot |\det(\mathbf{A})| = \frac{e^{-(y_1 - y_2 + y_2)/\theta}}{\theta^2} \cdot 1 = \frac{e^{-y_1/\theta}}{\theta^2},$$

for  $0 < y_2 < y_1$ . Because  $Y_2 \sim \text{Exp}(1/\theta)$ , we have  $\mathbb{E}[Y_2] = \theta$ , and so  $Y_2$  is an unbiased estimator for  $\theta$ . The conditional density of  $Y_2$  given  $Y_1$  is

$$g_{Y_2|Y_1}(y_2 | y_1) = \frac{g_{\mathbf{Y}}(\mathbf{y})}{g_{Y_1}(y_1)} = \frac{e^{-y_1/\theta}/\theta^2}{y_1 e^{-y_1/\theta}} = \frac{1}{y_1},$$

and so the conditional expectation of  $Y_2$  given  $Y_1$  is

$$\mathbb{E}[Y_2 | y_1] = \int y_2 \cdot g_{Y_2|Y_1}(y_2 | y_1) \partial y_2 = \int_0^{y_1} \frac{y_2}{y_1} \partial y_2 = \frac{y_2^2}{2y_1} \Big|_0^{y_1} = \frac{y_1}{2} := \varphi(y_1).$$

As is standard, we can define the random variable  $\varphi(Y_1)$ , and immediately see that  $\mathbb{E}[\varphi(Y_1)] = \mathbb{E}[Y_2] = \theta$ . In addition, we have

$$\mathbb{E}[\varphi(Y_1)^2] = \int \varphi(y_1)^2 \cdot g_{Y_1}(y_1) \partial y_1 = \int_0^\infty \frac{y_1^2}{4} \cdot \frac{y_1 e^{-y_1/\theta}}{\theta^2} = \frac{3\theta^2}{2} \int_0^\infty \frac{y_1^{4-1} \cdot e^{-y_1/\theta}}{\theta^4 \Gamma(4)} \partial y_1 = \frac{3\theta^2}{2},$$

and so  $\text{Var}[\varphi(Y_1)] = \mathbb{E}[\varphi(Y_1)^2] - \mathbb{E}[\varphi(Y_1)]^2 = \theta^2/2$ .

### Question 6

Suppose that  $\mathbf{X} \stackrel{\text{iid}}{\sim} \text{Poisson}(\theta)$ . We test  $H_0 : \theta = 1/2$  against  $H_A : \theta < 1/2$  using our procedure  $\delta$  : reject  $H_0$  if the test statistic  $Y = \mathbf{1}^T \mathbf{X} = \sum X_i \leq 2$ . Since it is a sum of independent Poisson random variables, our test statistic is also a Poisson random variable, namely  $Y \sim \text{Poisson}(n\theta)$ , and the rejection region for  $Y$  is  $\mathcal{S}_Y = \{0, 1, 2\}$ . Therefore, for  $\theta \in (0, 1/2]$ , the power function of  $\delta$  is given by

$$\pi(\theta | \delta) = \Pr(Y \in \mathcal{S}_Y | \theta) = \sum_{k=0}^2 \frac{(n\theta)^k e^{-n\theta}}{k!}.$$

A graph of  $\pi(\theta | \delta)$  can be found in the left panel of Figure 1. The values of  $\pi(\theta | \delta)$  when  $\theta = \{1/2, 1/3, 1/4, 1/6, 1/12\}$  are given by 0.062, 0.238, 0.423, 0.677, 0.920, respectively, and have also been marked on Figure 1. The significance of this test is given by  $\alpha(\delta) = \pi(1/2 | \delta) = 0.062$ .

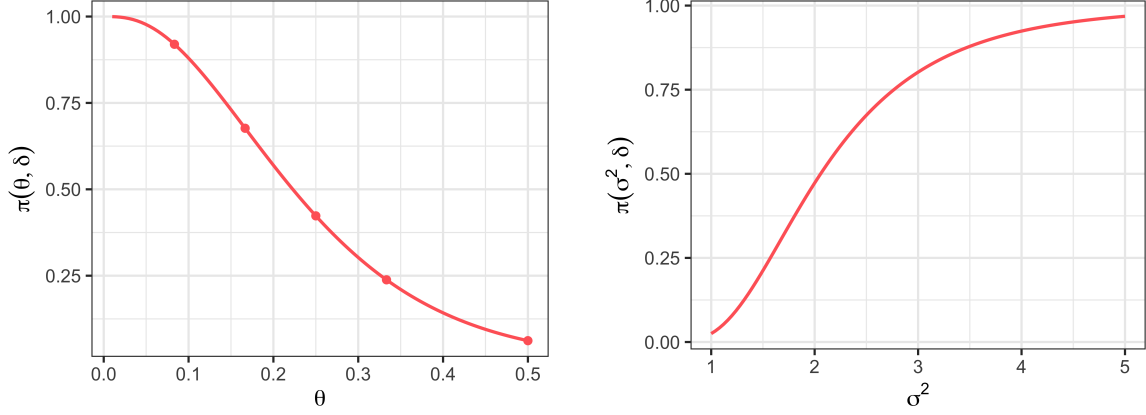


Figure 1: The power functions  $\pi(\theta | \delta)$  (question 6) and  $\pi(\sigma^2 | \delta)$  (question 8).

### Question 7

Let  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Unif}(0, \theta)$ , where  $\theta$  is unknown, and let  $Y_1, \dots, Y_n$  be the  $n$  order statistics of the random sample. We are testing  $H_0 : \theta = 1$  against  $H_A : \theta > 1$  with procedure  $\delta$  : reject  $H_0$  if  $Y_n \geq c$ , where  $c$  is an unknown constant that will be determined by the significance of  $\delta$ . If  $Y_n$  is the  $n$ th order statistic of  $\mathbf{X}$ , its cdf and density are respectively given by  $F_{Y_n}(y_n) = (y_n/\theta)^n$  and  $f_{Y_n}(y_n) = ny_n^{n-1}/\theta^n$  for  $y_n \in [0, \theta]$ .

(b) The rejection region of  $Y_n$  is  $\mathcal{S}_{Y_n} = [c, \theta]$ , and so the power function is given by

$$\pi(\theta | \delta) = \Pr(Y_n \in \mathcal{S}_{Y_n} | \theta) = F_{Y_n}(\theta) - F_{Y_n}(c) = 1 - \frac{c^n}{\theta^n}.$$

(a) To have a significance level of  $\alpha = 0.05$ , we must have  $\alpha(\delta) = \pi(1 | \delta) = 1 - c^n \stackrel{\text{set}}{=} 0.05$ , and solving for  $c$  gives us  $c = \sqrt[4]{0.95}$ , and the power function with this level of significance is  $\pi(\theta | \delta) = 1 - 0.95/\theta^4$ . The question specifically asks for  $n = 4$ , in which case  $c = \sqrt[4]{0.95}$  and  $\pi(\theta | \delta) = 1 - 0.95/\theta^4$ . More generally, for any specified significance level  $\alpha_*$ , we will have  $c = \sqrt[n]{1 - \alpha_*}$  and  $\pi(\theta | \delta) = 1 - (1 - \alpha_*)/\theta^n$ .

### Question 8

Let  $\mathbf{X} \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$ , where both  $\mu$  and  $\sigma^2$  are unknown. We want to test  $H_0 : \sigma^2 = \sigma_0^2$  against  $H_A : \sigma^2 > \sigma_0^2$  using procedure  $\delta$  : reject  $H_0$  if  $nS^2/\sigma_0^2 \geq c$ , where  $c$  will be determined by the significance of the test. Our test statistic is based off of  $S^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2$ , and we know that  $T(\mathbf{X}) := nS^2/\sigma_0^2 \sim \chi^2(n-1)$ . The rejection region for our test statistic is  $\mathcal{S}_{T(\mathbf{X})} = [c, \infty)$ , and we would like to find a value of  $c$  such that the significance is  $\alpha = 0.025$ .

$$\alpha(\delta) = \pi(\sigma_0^2 | \delta) = \Pr(T(\mathbf{X}) \in \mathcal{S}_{T(\mathbf{X})} | \sigma_0^2) = \int_c^\infty \frac{t^{(n-3)/2} \cdot e^{-t/2}}{2^{(n-1)/2} \Gamma((n-1)/2)} \partial t \stackrel{\text{set}}{=} 0.025.$$

There is no way to derive a closed form relationship between  $c$  and  $\alpha$  like we did in question 7, but using R we can write a function to find the value of  $c$  for a given significance value  $\alpha$ . When  $n = 13$  and  $\alpha = 0.025$ , we have  $c = 23.337$ . To find a general formula of  $\pi(\sigma^2 | \delta)$ , we notice that if  $T(\mathbf{X}) \geq c$ , then  $S^2 \geq c\sigma_0^2/n$ , which serves as a rejection region for  $S^2$ . Since  $nS^2/\sigma^2 \sim \chi^2(n-1)$  for any value of  $\sigma^2$ , it can be shown that  $S^2 \sim \text{Gamma}((n-1)/2, n/2\sigma^2)$ . Therefore, the power function is given by

$$\pi(\sigma^2 | \delta) = \Pr(S^2 \geq c\sigma_0^2/n | \sigma^2) = \int_{c\sigma_0^2/n}^\infty \frac{x^{(n-3)/2} \cdot e^{-nx/2\sigma^2}}{(n/2\sigma^2)^{(n-1)/2} \Gamma((n-1)/2)} \partial x,$$

where  $c$  is chosen to make  $\alpha(\delta) = 0.025$  (or any specified value, for that matter). Again, there is no closed form solution to this function, but using **R** we can find its values, and given a specified value of  $\sigma_0^2$ , we will always have  $\pi(\sigma_0^2 | \delta) = \alpha$  as desired. A plot of  $\pi(\sigma^2 | \delta)$  can be found in the right panel of Figure 1.

## Question 9

If  $\mathbf{X} \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$  with unknown  $\mu$  and  $\sigma^2$ , then a  $\gamma\%$  confidence interval for  $\mu$  is given by

$$\mathcal{I} = \left( \bar{X} - t_\gamma(n) \cdot S/\sqrt{n}, \bar{X} + t_\gamma(n) \cdot S/\sqrt{n} \right),$$

where  $t_\gamma(n) = T_{n-1}^{-1}((1 + \gamma)/2)$  is the  $(1 + \gamma)/2$ th quantile of the  $t$  distribution with  $\text{df} = n - 1$  and  $S$  is the sample standard deviation. The length of this confidence interval is given by

$$\Delta = \max(\mathcal{I}) - \min(\mathcal{I}) = \left( \bar{X} + t_\gamma(n) \cdot S/\sqrt{n} \right) - \left( \bar{X} - t_\gamma(n) \cdot S/\sqrt{n} \right) = 2t_\gamma(n) \cdot S/\sqrt{n}.$$

The squared length is then given by  $\Delta^2 = 4t_\gamma^2(n) \cdot S^2/n$ . Because the sample variance is an unbiased estimator for  $\sigma^2$ , we have  $\mathbb{E}[\Delta^2] = \mathbb{E}[4t_\gamma^2(n) \cdot S^2/n] = 4t_\gamma^2(n) \cdot \sigma^2/n$ . We now set  $\mathbb{E}[\Delta^2] < \sigma^2/2$ , and after some cancellations, we see that we need  $t_\gamma^2(n)/n < 1/8$ . There is no way to find a closed-form expression for this, so we will have to check the value of  $t_\gamma^2(n)/n$  for increasing values of  $n$ . I set up a **while** loop in **R** to solve for it, and when  $\gamma = 0.9$ , we find that  $n = 24$  is the smallest value of  $n$  such that  $\mathbb{E}[\Delta^2] < \sigma^2/2$ .

## Question 10

Let  $\mathbf{X} \stackrel{\text{iid}}{\sim} N(\theta, \sigma^2)$ , where  $\theta$  is unknown and  $\sigma^2$  is known, and we assume prior that  $\theta \sim N(\mu, \nu^2)$ , where both  $\mu$  and  $\nu^2$  are known.

- (a) Since normal distributions are conjugate to normal sampling, it follows that  $\theta | \mathbf{x} \sim N(\tilde{\mu}, \tilde{\sigma}^2)$ , where

$$\tilde{\mu} = \frac{\sigma^2 \mu + n \nu^2 \bar{x}}{\sigma^2 + n \nu^2} \quad \text{and} \quad \tilde{\sigma}^2 = \frac{\sigma^2 \nu^2}{\sigma^2 + n \nu^2}.$$

We also know that  $(\theta | \mathbf{x} - \tilde{\mu})/\tilde{\sigma} \sim N(0, 1)$ , and so a 95% confidence interval for  $\theta | \mathbf{x}$  is given by

$$\mathcal{I} = \left( \tilde{\mu} - \Phi^{-1}(0.975) \cdot \tilde{\sigma}, \tilde{\mu} + \Phi^{-1}(0.975) \cdot \tilde{\sigma} \right).$$

- (b) We can think of our interval  $\mathcal{I}$  as a function of  $\nu^2$ . To examine what happens to  $\mathcal{I}(\nu^2)$  as  $\nu^2 \rightarrow \infty$ , we will first look at  $\tilde{\mu}$  and  $\tilde{\sigma}$ . Using L'Hopital's rule, we have

$$\begin{aligned} \lim_{\nu^2 \rightarrow \infty} \tilde{\mu} &= \lim_{\nu^2 \rightarrow \infty} \frac{\sigma^2 \mu + n \nu^2 \bar{x}}{\sigma^2 + n \nu^2} = \lim_{\nu^2 \rightarrow \infty} \frac{n \bar{x}}{n} = \bar{x}, \\ \lim_{\nu^2 \rightarrow \infty} \tilde{\sigma} &= \lim_{\nu^2 \rightarrow \infty} \sqrt{\frac{\sigma^2 \nu^2}{\sigma^2 + n \nu^2}} = \sqrt{\lim_{\nu^2 \rightarrow \infty} \frac{\sigma^2 \nu^2}{\sigma^2 + n \nu^2}} = \sqrt{\lim_{\nu^2 \rightarrow \infty} \frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}, \end{aligned}$$

and so  $\mathcal{I}(\nu^2) \rightarrow (\bar{x} - \Phi^{-1}(0.975) \cdot \sigma/\sqrt{n}, \bar{x} + \Phi^{-1}(0.975) \cdot \sigma/\sqrt{n})$ , which is a 95% confidence interval for  $\theta$ .

## Question 11

Let  $\mathbf{X} \stackrel{\text{iid}}{\sim} \text{Unif}(0, \theta)$ , where  $\theta$  is unknown, and let  $Y = X_{(n)}$  be the  $n$ th order statistic.

- (a) Let  $F_i(x) = \Pr(X_i \leq x) = x/\theta$  for all  $i \in \{1, \dots, n\}$ . Then the cdf of  $Y$  is  $G(y) = \prod_{i=1}^n F_i(y) = (y/\theta)^n$ , since all of the  $X_i$ 's are independent. The density of  $Y$  is then given by  $g(y) = ny^{n-1}/\theta^n$  for  $y \in [0, \theta]$ . By letting  $W = Y/\theta$ , we have  $Y = \theta W$  and  $\partial Y/\partial W = \theta$ , so the density of  $Y/\theta$  is given by  $h(w) = g(y(w)) \cdot \theta = nw^{n-1}$  for  $w \in [0, 1]$ . The cdf is then seen to be  $H(w) = w^n$ , and so the quantile is given by  $H^{-1}(w) = \sqrt[n]{w}$ .

(b) Since we must have  $y \leq \theta$ , it is natural that  $Y$  will underestimate  $\theta$ , and is therefore biased. We have

$$\mathbb{E}[Y] = \int_0^\theta y \cdot \frac{ny^{n-1}}{\theta^n} dy = \frac{n}{\theta^n} \int_0^\theta y^n dy = \frac{n}{\theta^n} \cdot \frac{y^{n+1}}{n+1} \Big|_0^\theta = \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} = \frac{n\theta}{n+1},$$

and so the bias is  $\text{bias}(Y) = \mathbb{E}[Y] - \theta = -\theta/(n+1)$ .

(d) Using the cdf of  $Y/\theta$ , for any interval involving  $Y/\theta$ , we have  $\Pr(a \leq Y/\theta \leq b) = b^n - a^n$ , where  $a, b \in (0, 1]$ . Rearranging the terms inside the interval gives us  $\Pr(Y/b \leq \theta \leq Y/a) = b^n - a^n \stackrel{\text{set}}{=} \gamma$ . That is, as long as we impose the constraint that  $b^n - a^n = \gamma$ , any interval  $(Y/b, Y/a)$  is a 100% confidence interval for  $\theta$ .

## Question 12

Suppose that  $X \sim P$ , where  $P$  is an unknown distribution, and we want to test  $H_0 : P = \text{Unif}(0, 1)$  against  $H_A : P = N(0, 1)$ . The densities of both distributions (within their support) are given by  $f(x | P_0) = 1$  for  $x \in [0, 1]$  and  $f(x | P_A) = (2\pi)^{-1/2} \exp(-x^2/2)$  for  $x \in \mathbb{R}$ , respectively, and so the likelihood ratio is given by

$$\ell(x) = \frac{f(x | P_A)}{f(x | P_0)} = \begin{cases} \phi(x) & \text{if } x \in [0, 1] \\ \infty & \text{if } x \notin [0, 1], \end{cases}$$

where  $\phi(x)$  is the standard normal density. As a result, we will reject  $H_0$  if  $\ell(X) \geq k$ , and so the power function is given by  $\pi(P | \delta) = \Pr(\ell(X) \geq k | P)$ . We note that if  $H_0$  is true, then we will always have  $x \in [0, 1]$ , since we would be sampling from  $\text{Unif}(0, 1)$ , so in this case  $\ell(x) = \phi(x)$ . We also note that  $\phi(x)$  is a decreasing function on this interval. As a result, the significance of this test is

$$\alpha(\delta) = \pi(P_0 | \delta) = \Pr(\ell(X) \geq k | P_0) = \Pr(\phi(X) \geq k) = \Pr(X \leq c) = F_0(c) \stackrel{\text{set}}{=} 0.01$$

where  $c = \phi^{-1}(k)$  and  $F_0(c)$  is the cdf of  $\text{Unif}(0, 1)$ . As a result,  $c = 0.01$  gives us a test with significance 0.01. More generally, since  $\phi(X) > k$  for any value of  $k$  (and thus any level of significance), we will always reject  $H_0$  when  $X \notin [0, 1]$ , which makes the power function

$$\begin{aligned} \pi(P | \delta) &= \Pr(\phi(X) \geq \phi(0.01) | P) \\ &= \Pr(X < 0 \cup X \leq 0.01 \cup X > 1 | P) = \Pr(X \leq 0.01 | P) + \Pr(X > 1 | P) \end{aligned}$$

Therefore, when  $H_A$  is true, the power of  $\delta$  is

$$\pi(P_A | \delta) = \Pr(X \leq 0.01 | P_A) + \Pr(X > 1 | P_A) = \Phi(0.01) + 1 - \Phi(1) = 0.663.$$

## Question 13

Suppose that  $\mathbf{X} \stackrel{\text{iid}}{\sim} N(\theta, 1)$ , where  $\theta > 1$  is unknown, and we wish to test  $H_0 : \theta \leq \theta_*$  against  $H_A : \theta > \theta_*$  using some UMP test  $\delta^*$ . More specifically, we want our UMP test to have some pre-determined power  $\pi_*$  at a given mean  $\varphi$ , i.e.  $\pi(\varphi | \delta^*) = \pi_*$ . Since we are sampling from a normal distribution, the joint distribution of  $\mathbf{X}$  has increasing monotone likelihood ratio in the test statistic  $\bar{X}$ , where  $\bar{X} \sim N(\theta, 1/n)$ , so our UMP test is given by  $\delta^* : \text{reject } H_0 \text{ if } \bar{X} \geq c$ . We also notice that  $\sqrt{n}(\bar{X} - \theta) := Z_\theta \sim N(0, 1)$ . The rejection region for  $\bar{X} = \mathcal{S}_{\bar{X}} = [c, \infty)$ , so to find  $c$  we have

$$\pi(\varphi | \delta^*) = \Pr(\bar{X} \geq c | \theta = \varphi) = \Pr(Z_\varphi \geq \sqrt{n}(c - \varphi)) = 1 - \Phi(\sqrt{n}(c - \varphi)) \stackrel{\text{set}}{=} \pi_*.$$

From here, we have  $\sqrt{n}(c - \varphi) = \Phi^{-1}(1 - \pi_*) := z_{\pi_*}$ , and so  $c = \varphi + z_{\pi_*}/\sqrt{n}$ . With the value of  $c$  determined, the power function is then given by

$$\pi(\theta | \delta^*) = \Pr(\bar{X} \geq \varphi + z_{\pi_*}/\sqrt{n} | \theta) = \Pr(Z_\theta \geq \sqrt{n}(\varphi - \theta) + z_{\pi_*}) = 1 - \Phi(\sqrt{n}(\varphi - \theta) + z_{\pi_*}).$$

Since  $\pi(\theta | \delta^*)$  is a monotone increasing function of  $\theta$ , it will be maximized at the largest possible value over an interval. Therefore, the significance of this test is

$$\alpha(\delta^*) = \sup_{\theta \leq \theta_*} \pi(\theta | \delta^*) = \pi(\theta_* | \delta^*) = 1 - \Phi(\sqrt{n}(\varphi - \theta_*) + z_{\pi_*}).$$

The question specifically asks for  $c$  and  $\alpha$  when  $\theta_* = 0$ ,  $\varphi = 1$ ,  $\pi_* = 0.95$ , and  $n = 16$ . In this case, we have  $z_{\pi_*} = \Phi^{-1}(0.05) \approx -1.64$ , which means  $c = 1 - 1.64/4 = 0.589$  and  $\alpha = 1 - \Phi(4 - 1.64) = 0.00926$ . Put more formally, if we have a random sample  $\mathbf{X} \stackrel{\text{iid}}{\sim} N(\theta, 1)$ , and want to test  $H_0 : \theta \leq 0$  against  $H_A : \theta > 0$  using a UMP test with a power of 0.95 at  $\theta = 1$ , then our test is given by  $\delta^* : \text{reject } H_0 \text{ if } \bar{X} \geq 0.589$ . With 16 observations, the size of this test is  $\alpha = 0.00926$ .

## Question 14

Suppose  $X_1, \dots, X_8$  are randomly sampled from a random variable with density  $f(x | \theta) = \theta x^{\theta-1}$ , supported for all  $x \in (0, 1)$ , and we would like to test  $H_0 : \theta \leq 1$  against  $H_A : \theta > 1$  using a UMP test  $\delta^*$ . For two values  $\theta_1, \theta_2$  such that  $\theta_1 < \theta_2$ , the ratio of the joint densities (which are also the likelihood functions) is

$$\frac{f(\mathbf{x} | \theta_2)}{f(\mathbf{x} | \theta_1)} = \frac{\prod_{i=1}^8 \theta_2 x_i^{\theta_2-1}}{\prod_{i=1}^8 \theta_1 x_i^{\theta_1-1}} = \frac{\theta_2^8 \left( \prod_{i=1}^8 x_i \right)^{\theta_2-1}}{\theta_1^8 \left( \prod_{i=1}^8 x_i \right)^{\theta_1-1}} = \left( \frac{\theta_2}{\theta_1} \right)^8 \cdot \left( \prod_{i=1}^8 x_i \right)^{\theta_2-\theta_1}.$$

which is a function of the test statistic  $\prod_{i=1}^8 x_i$ . Because  $\theta_2 > \theta_1$ , we have  $\theta_2/\theta_1 > 1$  and  $\theta_2 - \theta_1 > 0$ , so the likelihood ratio is monotone and increasing in the parameter  $\prod_{i=1}^8 x_i$ . As a result, the UMP test is  $\delta^* : \text{reject } H_0 \text{ if } \prod_{i=1}^8 x_i \geq c$  for some value of  $c$ . For this test, we specify the level of significance as  $\alpha_* = 0.05$ , and since  $\pi(\theta | \delta^*)$  is a monotone increasing function, we have

$$\alpha_* = \sup_{\theta \leq 1} \pi(\theta | \delta^*) = \pi(1 | \delta^*) \stackrel{\text{set}}{=} 0.05.$$

When  $\theta = 1$ , we can see from the density function that  $X \sim \text{Unif}(0, 1)$ , with its cdf given by  $F(X) = X$ . It can be shown, then, that  $-2 \sum_{i=1}^8 \log F(X_i) \sim \chi^2(16)$ . Therefore,

$$\pi(1 | \delta^*) = \Pr \left( \prod_{i=1}^8 x_i \geq c | \theta = 1 \right) = \Pr \left( -2 \sum_{i=1}^8 \log X_i \leq -2 \log c \right) = C_{16}(-2 \log c) = 0.05$$

where  $C_{16}(x)$  is the cdf of  $\chi^2(16)$ . If  $k := -2 \log c$ , it follows that  $k = C_{16}^{-1}(0.05) = 7.962$ , and as a result,  $\sum_{i=1}^8 \log X_i \geq -k/2 = 3.981$ .

## Question 15

Suppose that  $\mathbf{X} \stackrel{\text{iid}}{\sim} \chi^2(\theta)$ , where  $\theta \in \mathbb{N}$  is unknown. We would like to test  $H_0 : \theta \leq 8$  against  $H_A : \theta > 8$ , using a UMP test  $\delta^*$  with a specified significance  $\alpha_* \in (0, 1)$ . The joint density of  $\mathbf{X}$  is given by

$$f(\mathbf{x} | \theta) = \prod_{i=1}^n \frac{x_i^{\theta/2-1} e^{-x_i/2}}{2^{\theta/2} \Gamma(\theta/2)} = 2^{-n\theta/2} \cdot \Gamma^{-n}(\theta/2) \cdot \left( \prod_{i=1}^n x_i \right)^{n(\theta/2-1)} \cdot \exp \left( -\frac{1}{2} \sum_{i=1}^n x_i \right)$$

To determine  $\delta^*$ , we will look at the likelihood ratio. If we have two values  $\theta_1, \theta_2$  such that  $\theta_1 < \theta_2$ , then then likelihood ratio is

$$\begin{aligned} \frac{f(\mathbf{x} | \theta_2)}{f(\mathbf{x} | \theta_1)} &= \frac{2^{-n\theta_1/2} \cdot \Gamma^{-n}(\theta_2/2) \cdot \left( \prod_i x_i \right)^{n(\theta_2/2-1)} \cdot \exp \left( -\frac{1}{2} \sum_i x_i \right)}{2^{-n\theta_1/2} \cdot \Gamma^{-n}(\theta_1/2) \cdot \left( \prod_i x_i \right)^{n(\theta_1/2-1)} \cdot \exp \left( -\frac{1}{2} \sum_i x_i \right)} \\ &= 2^{n(\theta_1-\theta_2)/2} \left( \frac{\Gamma(\theta_1/2)}{\Gamma(\theta_2/2)} \right)^n \left( \prod_{i=1}^n x_i \right)^{n(\theta_2-\theta_1)/2}, \end{aligned}$$

which is a monotone increasing function of the test statistic  $\prod_{i=1}^n x_i$ . Therefore, the UMP test is  $\delta^*$  : reject  $H_0$  if  $\prod_{i=1}^n X_i \geq c_*$ , where  $c_*$  is chosen such that the test has a significance level  $\alpha_*$ . Taking the log of both sides gives us  $\sum_{i=1}^n \log X_i \geq \log c_* := k$ .