

Homework 1

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Question 1

When rolling two dice, there are six possible ways for their total to sum up to seven: (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), and (6, 1), so the probability of the sum being seven is $6/36 = 1/6$. If X is the number of trials where the total of both rolls is seven, then we can think of $X \sim \text{Bin}(120, 1/6)$, and so $\mathbb{E}X = 20$ and $\text{Var}X = 50/3$. Using the Central Limit Theorem, we then have

$$\Pr(|X - 20| \leq k) = \Pr\left(\left|\frac{X - 20}{\sqrt{50/3}}\right| \leq k\sqrt{\frac{3}{50}}\right) = 2\Phi\left(k\sqrt{\frac{3}{50}}\right) - 1 \stackrel{\text{set}}{=} 0.95 \implies \Phi\left(k\sqrt{\frac{3}{50}}\right) = 0.975.$$

Using a table of values for $\Phi(z)$, we can see that $k\sqrt{3/50} = 1.96$, and so $k = 1.96\sqrt{50/3} \approx 8$.

Question 2

Let $X \sim \text{Pois}(10)$, and so $\mathbb{E}X = \text{Var}X = 10$. Using the CLT without any continuity correction, we have $(X - 10)/\sqrt{10} \approx N(0, 1)$, and so

$$\Pr(8 \leq X \leq 12) = \Pr\left(\frac{8 - 10}{\sqrt{10}} \leq Z \leq \frac{12 - 10}{\sqrt{10}}\right) = \Pr(|Z| \leq \sqrt{2/5}) \approx 2\Phi(\sqrt{2/5}) - 1 = 0.4714.$$

If we do use continuity correction, then we have

$$\begin{aligned}\Pr(8 \leq X \leq 12) &\approx \Pr(7.5 \leq X \leq 12.5) \\ &= \Pr\left(\frac{7.5 - 10}{\sqrt{10}} \leq Z \leq \frac{12.5 - 10}{\sqrt{10}}\right) = \Pr(|Z| \leq 2.5/\sqrt{10}) \approx 2\Phi(2.5/\sqrt{10}) - 1 = 0.5704.\end{aligned}$$

Question 3

We are assuming that when a program is run, an execution error will occur with probability $\theta \in [0, 1]$. If X is whether or not an execution error occurs, we have $X \sim \text{Ber}(\theta)$, and $f(x|\theta) = \theta^x(1-\theta)^{1-x}$ for $x = \{0, 1\}$. We also believe that $\theta \sim \text{Unif}(0, 1)$, and so $\xi(\theta) = 1$ for $0 \leq \theta \leq 1$.

- (a) After 25 runs of the program we have 10 erros, so $f(\mathbf{x}|\theta) = \theta^{10}(1-\theta)^{15}$. The marginal distribution of \mathbf{x} is given by

$$g_{\mathbf{X}}(\mathbf{x}) = \int_{\Omega} f(\mathbf{x}|\theta) \cdot \xi(\theta) \, d\theta = \int_0^1 \theta^{10}(1-\theta)^{15} \cdot 1 \, d\theta = \int_0^1 \theta^{11-1}(1-\theta)^{16-1} \, d\theta = B(11, 16),$$

and so the posterior pdf of θ is

$$\xi(\theta|\mathbf{x}) = \frac{f(\mathbf{x}|\theta) \cdot \xi(\theta)}{g_{\mathbf{X}}(\mathbf{x})} = \frac{\theta^{10}(1-\theta)^{15} \cdot 1}{B(11, 16)} = \frac{\theta^{11-1}(1-\theta)^{16-1}}{B(11, 16)}.$$

That is, $\theta \sim \text{Beta}(11, 16)$.

- (b) If we are using squared error loss, then our Bayes' estimate would be $\hat{\theta} = \mathbb{E}(\theta|\mathbf{x}) = 11/(11+16) = 11/27$.

Question 4

We believe that $\theta \sim \text{Beta}(3, 4)$, where $\theta \in [0, 1]$ is the proportion of bad apples in the lot. Choosing apples from the lot is essentially sampling from a Bernoulli distribution with parameter θ , and we know that Beta distributions are closed under sampling from a Bernoulli distribution. After choosing 10 apples, we find that three of them are bad, so our posterior distribution becomes $\theta \sim \text{Beta}(3 + 3, 4 + 7) = \text{Beta}(6, 11)$. If we use squared error loss, our Bayes' estimate is then $\delta^*(\mathbf{x}) = \mathbb{E}(\theta | \mathbf{x}) = 6/17$.