Homework 2

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Question 1

Question 9

If $X_1, \dots, X_n \overset{\text{iid}}{\sim} \mathrm{N}(\mu, \sigma^2)$ with unknown μ and σ^2, then a $\gamma\%$ confidence interval for μ is given by

$$\mathcal{I} = \Big(\ \bar{X} - t_\gamma(n) \cdot S/\sqrt{n} \ , \ \bar{X} + t_\gamma(n) \cdot S/\sqrt{n} \ \Big),$$

where $t_\gamma(n) = T_{n-1}^{-1}((1+\gamma)/2)$ is the $(1+\gamma)/2$th quantile of the t distribution with $\mathrm{df} = n-1$ and S is the sample standard deviation. The length of this confidence interval is given by

$$\Delta = \max(\mathcal{I}) - \min(\mathcal{I}) = \Big(\bar{X} + t_\gamma(n) \cdot S/\sqrt{n}\Big) - \Big(\bar{X} - t_\gamma(n) \cdot S/\sqrt{n}\Big) = 2t_\gamma(n) \cdot S/\sqrt{n}.$$

The squared length is then given by $\Delta^2 = 4t_\gamma^2(n) \cdot S^2/n$. Because the sample variance is an unbiased estimator for σ^2, we have $\mathbb{E}[\Delta^2] = \mathbb{E}\big[4t_\gamma^2(n) \cdot S^2/n\big] = 4t_\gamma^2(n) \cdot \sigma^2/n$. We now set $\mathbb{E}[\Delta^2] < \sigma^2/2$, and after some cancellations, we see that we need $t_\gamma^2(n)/n < 1/8$. There is no way to find a closed-form expression for this, so we will have to check the value of $t_\gamma^2(n)/n$ for increasing values of n. I set up a `while` loop in `R` to solve for it, and when $\gamma = 0.9$, we find that $n = 24$ is the smallest value of n such that $\mathbb{E}[\Delta^2] < \sigma^2/2$.