Homework 1

Aiden Kenny
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Columbia University

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Question 1

When rolling two dice, there are six possible ways for their total to sum up to seven: $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, and $(6,1)$, so the probability of the sum being seven is $6/36 = 1/6$. If X is the number of trials where the total of both rolls is seven, then we can think of $X \sim \text{Bin}(120, 1/6)$, and so $\mathbb{E}X = 20$ and $\text{Var}X = 50/3$. Using the Central Limit Theorem, we then have

$$\Pr\big(|X-20| \le k\big) = \Pr\left(\left|\frac{X-20}{\sqrt{50/3}}\right| \le k\sqrt{\frac{3}{50}}\right) = 2\Phi\left(k\sqrt{\frac{3}{50}}\right) - 1 \overset{\text{set}}{=} 0.95 \implies \Phi\left(k\sqrt{\frac{3}{50}}\right) = 0.975.$$

Using a table of values for $\Phi(z)$, we can see that $k\sqrt{3/50} = 1.96$, and so $k = 1.96\sqrt{50/3} \approx 8$.

Question 2

Let $X \sim \text{Pois}(10)$, and so $\mathbb{E}X = \text{Var}X = 10$. Using the CLT without any continuity correction, we have $(X-10)/\sqrt{10} \approx \text{N}(0,1)$, and so

$$\Pr(8 \le X \le 12) = \Pr\left(\frac{8-10}{\sqrt{10}} \le Z \le \frac{12-10}{\sqrt{10}}\right) = \Pr\big(|Z| \le \sqrt{2/5}\big) \approx 2\Phi\big(\sqrt{2/5}\big) - 1 = 0.4714.$$

If we do use continuity correction, then we have

$$\begin{aligned}\Pr(8 \le X \le 12) &\approx \Pr(7.5 \le X \le 12.5) \\ &= \Pr\left(\frac{7.5-10}{\sqrt{10}} \le Z \le \frac{12.5-10}{\sqrt{10}}\right) = \Pr\big(|Z| \le 2.5/\sqrt{10}\big) \approx 2\Phi\big(2.5/\sqrt{10}\big) - 1 = 0.5704.\end{aligned}$$

Question 3

We are assuming that when a program is run, an execution error will occur with probability $\theta \in [0,1]$. If X is whether or not an execution error occurs, we have $X \sim \text{Ber}(\theta)$, and $f(x \,|\, \theta) = \theta^x(1-\theta)^{1-x}$ for $x = \{0,1\}$. We also believe that $\theta \sim \text{Unif}(0,1)$, and so $\xi(\theta) = 1$ for $0 \le \theta \le 1$.

(a) After 25 runs of the program we have 10 erros, so $f(\mathbf{x} \,|\, \theta) = \theta^{10}(1-\theta)^{15}$. The marginal distribution of $\mathbf{x}$ is given by

$$g_{\boldsymbol{X}}(\mathbf{x}) = \int_\Theta f(\mathbf{x} \,|\, \theta) \cdot \xi(\theta) \,\mathrm{d}\theta = \int_0^1 \theta^{10}(1-\theta)^{15} \cdot 1 \,\mathrm{d}\theta = \int_0^1 \theta^{11-1}(1-\theta)^{16-1} \,\mathrm{d}\theta = \text{B}(11,16),$$

and so the posterior pdf of θ is

$$\xi(\theta \,|\, \mathbf{x}) = \frac{f(\mathbf{x} \,|\, \theta) \cdot \xi(\theta)}{g_{\boldsymbol{X}}(\mathbf{x})} = \frac{\theta^{10}(1-\theta)^{15} \cdot 1}{\text{B}(11,16)} = \frac{\theta^{11-1}(1-\theta)^{16-1}}{\text{B}(11,16)}.$$

That is, $\theta \sim \text{Beta}(11,16)$.

(b) If we are using squared error loss, then our Bayes' estimate is $\delta^*(\mathbf{x}) = \mathbb{E}(\theta \,|\, \mathbf{x}) = 11/27$.

Question 4

We believe that $\theta \sim \text{Beta}(3, 4)$, where $\theta \in [0, 1]$ is the proportion of bad apples in the lot. Choosing apples from the lot is essentially sampling from a Bernoulli distribution with parameter θ, and we know that Beta distributions are closed under sampling from a Bernoulli distribution. After choosing 10 apples, we find that three of them are bad, so our posterior distribution becomes $\theta \sim \text{Beta}(3 + 3, 4 + 7) = \text{Beta}(6, 11)$. If we use squared error loss, our Bayes' estimate is then $\delta^*(\mathbf{x}) = \mathbb{E}(\theta \,|\, \mathbf{x}) = 6/17$.

Question 5

Let $X_1, \ldots, X_n$ be a random sample from $X \sim \text{Unif}(\theta, 2\theta)$, where $\theta > 0$. The likelihood function is then given by $f(\mathbf{x} \,|\, \theta) = 1/\theta^n$ when $\theta \leq x_i \leq 2\theta$ for $i \in \{1, \ldots, n\}$. We can re-frame the boundaries of the likelihood function using order statistics. Since we need every observation $x_i \in [\theta, 2\theta]$, it follows that $\theta \leq x_{(1)} \leq \cdots \leq x_{(n)} \leq 2\theta$, where $x_{(j)}$ is the jth order statistics; namely, we have $\theta \leq x_{(1)}$ and $x_{(n)} \leq 2\theta$. From the second inequality, we have $x_{(n)}/2 \leq \theta$, and so the possible values of θ are $x_{(n)}/2 \leq \theta \leq x_{(1)}$. In other words, even though we had the original parameter space $\Theta = (0, \infty)$, because the bounds of the density functions depended on θ, we were able to restrict θ to a new parameter space $\tilde{\Theta} = [x_{(n)}/2, x_{(1)}]$. We can see that our likelihood function is monotone decreasing, and so it will be maximized by the smallest possible value of θ. Therefore, the MLE of θ is $\hat{\theta} = x_{(n)}/2$.