

Homework 3

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Question 1

Suppose that $\mathbf{X} \stackrel{\text{iid}}{\sim} \chi^2(\theta)$, where $\theta \in \mathbb{N}$ is unknown. We would like to test $H_0 : \theta \leq 8$ against $H_A : \theta > 8$, using a UMP test δ^* with a specified significance $\alpha_* \in (0, 1)$. The joint density of \mathbf{X} is given by

$$f(\mathbf{x} | \theta) = \prod_{i=1}^n \frac{x_i^{\theta/2-1} e^{-x_i/2}}{2^{\theta/2} \Gamma(\theta/2)} = 2^{-n\theta/2} \cdot \Gamma^{-n}(\theta/2) \cdot \left(\prod_{i=1}^n x_i \right)^{n(\theta/2-1)} \cdot \exp \left(-\frac{1}{2} \sum_{i=1}^n x_i \right)$$

To determine δ^* , we will look at the likelihood ratio. If we have two values θ_1, θ_2 such that $\theta_1 < \theta_2$, then the likelihood ratio is

$$\begin{aligned} \frac{f(\mathbf{x} | \theta_2)}{f(\mathbf{x} | \theta_1)} &= \frac{2^{-n\theta_2/2} \cdot \Gamma^{-n}(\theta_2/2) \cdot (\prod_i x_i)^{n(\theta_2/2-1)} \cdot \exp(-\frac{1}{2} \sum_i x_i)}{2^{-n\theta_1/2} \cdot \Gamma^{-n}(\theta_1/2) \cdot (\prod_i x_i)^{n(\theta_1/2-1)} \cdot \exp(-\frac{1}{2} \sum_i x_i)} \\ &= 2^{n(\theta_1-\theta_2)/2} \left(\frac{\Gamma(\theta_1/2)}{\Gamma(\theta_2/2)} \right)^n \left(\prod_{i=1}^n x_i \right)^{n(\theta_2-\theta_1)/2}, \end{aligned}$$

which is a monotone increasing function of the test statistic $\prod_{i=1}^n x_i$. Therefore, the UMP test is δ^* : reject H_0 if $\prod_{i=1}^n X_i \geq c_*$, where c_* is chosen such that the test has a significance level α_* . Taking the log of both sides gives us $\sum_{i=1}^n \log X_i \geq \log c_* := k$.