Homework 3

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Question 1

Suppose that $\boldsymbol{X} \overset{\text{iid}}{\sim} \chi^2(\theta)$, where $\theta \in \mathbb{N}$ is unknown. We would like to test $H_0 : \theta \leq 8$ against $H_A : \theta > 8$, using a UMP test δ^* with a specified significance $\alpha_* \in (0, 1)$. The joint density of $\boldsymbol{X}$ is given by

$$f(\mathbf{x} \mid \theta) = \prod_{i=1}^{n} \frac{x_i^{\theta/2-1} \mathrm{e}^{-x/2}}{2^{\theta/2}\Gamma(\theta/2)} = 2^{-n\theta/2} \cdot \Gamma^{-n}(\theta/2) \cdot \left(\prod_{i=1}^{n} x_i \right)^{n(\theta/2-1)} \cdot \exp\left(-\frac{1}{2} \sum_{i=1}^{n} x_i \right)$$

To determine δ^*, we will look at the likelihood ratio. If we have two values θ_1, θ_2 such that $\theta_1 < \theta_2$, then then likelihood ratio is

$$\begin{aligned} \frac{f(\mathbf{x} \mid \theta_2)}{f(\mathbf{x} \mid \theta_1)} &= \frac{2^{-n\theta_1/2} \cdot \Gamma^{-n}(\theta_2/2) \cdot \left(\prod_i x_i\right)^{n(\theta_2/2-1)} \cdot \exp\left(-\frac{1}{2}\sum_i x_i\right)}{2^{-n\theta_1/2} \cdot \Gamma^{-n}(\theta_1/2) \cdot \left(\prod_i x_i\right)^{n(\theta_1/2-1)} \cdot \exp\left(-\frac{1}{2}\sum_i x_i\right)} \\ &= 2^{n(\theta_1-\theta_2)/2} \left(\frac{\Gamma(\theta_1/2)}{\Gamma(\theta_2/2)} \right)^n \left(\prod_{i=1}^{n} x_i \right)^{n(\theta_2-\theta_1)/2}, \end{aligned}$$

which is a monotone increasing function of the test statistic $\prod_{i=1}^{n} x_i$. Therefore, the UMP test is δ^* : reject H_0 if $\prod_{i=1}^{n} X_i \geq c_*$, where c_* is chosen such that the test has a significance level α_*. Taking the log of both sides gives us $\sum_{i=1}^{8} \log X_i \geq \log c_* := k$.

Question 2

Question 8

By definition, the least-squares estimate for the intercept is $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$. Rearranging gives us $\bar{y} = \hat{\beta}_0 + \hat{\beta}_1\bar{x}$, and so the least-squares line $y = \hat{\beta}_0 + \hat{\beta}_1 x$ will always pass through the point $(\bar{x}, \bar{y})$.

Question 9

(a) The least-squares coefficients for the model are given by $\hat{\beta}_0 = 40.9$ and $\hat{\beta}_1 = 0.548$.