Homework 2

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Question 1

Suppose that $Y \sim \text{Bin}(100, p)$, and we want to make an inference about the value of p. We test $H_0 : p = 0.08$ against $H_A : p < 0.08$, and our test δ will reject H_0 if and only if $Y = 6$.

(a) The significance α is the probability of making a Type I error,

$$\alpha(\delta) = \Pr(Y = 6 \,|\, p = 0.08) = \binom{100}{6}(0.08)^6(0.92)^{94} = 0.123.$$

(b) Suppose that $p = 0.04$. The probability of a Type II error, β, is

$$\beta(\delta) = \Pr(Y \neq 6 \,|\, p = 0.04) = 1 - \Pr(Y = 6 \,|\, p = 0.04) = 1 - \binom{100}{6}(0.04)^6(0.96)^{94} = 0.895.$$

Question 2

For a random variable $Y \sim \text{Binom}(n, p)$, if n is large enough we can approximate it using a normal distribution with the same mean and variance, i.e. $Y \sim \text{N}(np, np(1-p))$. Then the sample proportion, $\hat{p} = Y/n$, is also normally distributed as $\hat{p} \sim \text{N}(p, p(1-p)/n)$, and standardizing $\hat{p}$ gives us $(\hat{p} - p)/\sqrt{p(1-p)/n} \sim \text{N}(0, 1)$. It can then be shown that a $100\gamma\%$ confidence interval for $\hat{p}$ is given by

$$\mathcal{I} = \left(\frac{\hat{p} + z_\gamma^2/2n}{1 + z_\gamma^2/2n} - z_\gamma \cdot \frac{\sqrt{\hat{p}(1-\hat{p})/n + z_\gamma^2/4n^2}}{1 + z_\gamma^2/n} \;,\; \frac{\hat{p} + z_\gamma^2/2n}{1 + z_\gamma^2/2n} + z_\gamma \cdot \frac{\sqrt{\hat{p}(1-\hat{p})/n + z_\gamma^2/4n^2}}{1 + z_\gamma^2/n} \right),$$

where $z_\gamma = \Phi^{-1}((1+\gamma)/2)$. For this example, we have $n = 300$ and $\hat{p} = 75/300 = 1/4$, and to get a 90% confidence interval, we have $z_\gamma = 1.645$. Therefore, a 90% confidence interval for p is $(0.212, 0.294)$.

Question 3

Suppose we have a random sample $\boldsymbol{X} \overset{\text{iid}}{\sim} \text{Gamma}(4, \beta)$, so $\mathbb{E}[X_i] = 4\beta$ and $\text{Var}[X_i] = 4\beta^2$. The expected value and variance of the sample mean $\bar{X}$ is then given by $\mathbb{E}[\bar{X}] = 4\beta$ and $\text{Var}[\bar{X}] = 4\beta^2/n$, and from the CLT we have $\sqrt{n}(\bar{X} - 4\beta)/2\beta = \sqrt{n}(\bar{X}/2\beta - 2) \sim \text{N}(0, 1)$. If $z_\gamma = \Phi^{-1}((1+\gamma)/2)$, then we have $\gamma = \Pr(-z_\gamma \leq \sqrt{n}(\bar{X}/2\beta - 2) \leq z_\gamma)$. Rearranging to get β in the middle gives us

$$\mathcal{I} = \left(\frac{2\bar{X}}{2 + z_\gamma/\sqrt{n}} \;,\; \frac{2\bar{X}}{2 - z_\gamma/\sqrt{n}} \right).$$

For this random sample, $n = 25$, and because we want a 95.4% confidence interval (oddly specific), we have $z_\gamma = 2$, so the confidence interval is given by $\mathcal{I} = \left(\, 5\bar{X}/6 \;,\; 5\bar{X}/4 \,\right)$.

Question 4

Suppose that $X \sim \text{Binom}(100, p)$, where $p \in (1/4, 1/2)$ is unknown. We test $H_0 : 1/2$ against $H_A : p = 1/4$ using δ : reject H_0 if $X \leq 3$. That is, our rejection region is $\mathcal{S}_X = \{0, 1, 2, 3\}$, and so the power function for this test is

$$\pi(p \,|\, \delta) = \Pr(X \in \mathcal{S}_X \,|\, p) = \sum_{k=0}^{3} \binom{10}{k} p^k (1-p)^{n-k}.$$

Question 5

Suppose $X_1, X_2 \overset{\text{iid}}{\sim} \text{Exp}(1/\theta)$, where $\theta > 0$, which means $\mathbb{E}[X_i] = \theta$ and $\text{Var}[X_i] = \theta^2$. The joint density is

$$f_{\boldsymbol{X}}(\mathbf{x}) = \frac{\mathrm{e}^{-x_1/\theta}}{\theta} \cdot \frac{\mathrm{e}^{-x_2/\theta}}{\theta} = \frac{\mathrm{e}^{-(x_1+x_2)/\theta}}{\theta^2}$$

Consider the transformation $Y_1 = X_1 + X_2$ and $Y_2 = X_2$, which implies $Y_1 \sim \text{Gamma}(2, 1/\theta)$ and $Y_2 \sim \text{Exp}(1/\theta)$. We note that because $f_{\boldsymbol{X}}(\mathbf{x})$, which is also the likelihood function of the random sample, is a function of y_1 and θ, so it is a sufficinet statistic. In vector notation, the transformation is given by $\boldsymbol{Y} = \mathbf{A}\boldsymbol{X}$, where $\mathbf{A} = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, and so $\boldsymbol{X} = \mathbf{A}^{-1}\boldsymbol{Y}$, where $\mathbf{A}^{-1} = \left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]$. Therefore, the joint density of Y_1 and Y_2 is

$$g_{\boldsymbol{Y}}(\mathbf{y}) = f_{\boldsymbol{X}}(\mathbf{x}) \cdot \left| \det\left(\frac{\partial \mathbf{A}\mathbf{x}}{\partial \mathbf{x}}\right) \right| = f_{\boldsymbol{X}}\left(\mathbf{A}^{-1}\mathbf{y}\right) \cdot |\det(\mathbf{A})| = \frac{\mathrm{e}^{-(y_1 - y_2 + y_2)/\theta}}{\theta^2} \cdot 1 = \frac{\mathrm{e}^{-y_1/\theta}}{\theta^2},$$

for $0 < y_2 < y_1$. Because $Y_2 \sim \text{Exp}(1/\theta)$, we have $\mathbb{E}[Y_2] = \theta$, and so Y_2 is an unbiased estimator for θ. The conditional density of Y_2 given Y_1 is

$$g_{Y_2|Y_1}(y_2 \,|\, y_1) = \frac{g_{\boldsymbol{Y}(\mathbf{y})}}{g_{Y_1}(y_1)} = \frac{\mathrm{e}^{-y_1/\theta}/\theta^2}{y_1 \mathrm{e}^{-y_1/\theta}} = \frac{1}{y_1},$$

and so the conditional expectation of Y_2 given Y_1 is

$$\mathbb{E}[Y_2 \,|\, y_1] = \int y_2 \cdot g_{Y_2|Y_1}(y_2 \,|\, y_1)\, \partial y_2 = \int_0^{y_1} \frac{y_2}{y_1}\, \partial y_2 = \left.\frac{y_2^2}{2y_1}\right|_0^{y_1} = \frac{y_1}{2} \coloneqq \varphi(y_1).$$

As is standard, we can define the random variable $\varphi(Y_1)$, and immediately see that $\mathbb{E}[\varphi(Y_1)] = \mathbb{E}[Y_2] = \theta$. In addition, we have

$$\mathbb{E}[\varphi(Y_1)^2] = \int \varphi(y_1)^2 \cdot g_{Y_1}(y_1)\, \partial y_1 = \int_0^\infty \frac{y_1^2}{4} \cdot \frac{y_1 \mathrm{e}^{-y_1/\theta}}{\theta^2} = \frac{3\theta^2}{2} \int_0^\infty \frac{y_1^{4-1} \cdot \mathrm{e}^{-y_1/\theta}}{\theta^4 \Gamma(4)}\, \partial y_1 = \frac{3\theta^2}{2},$$

and so $\text{Var}[\varphi(Y_1)] = \mathbb{E}[\varphi(Y_1)^2] - \mathbb{E}[\varphi(Y_1)]^2 = \theta^2/2$.

Question 6

Suppose that $\boldsymbol{X} \overset{\text{iid}}{\sim} \text{Poisson}(\theta)$. We test $H_0 : \theta = 1/2$ against $H_A : \theta < 1/2$ using our procedure δ : reject H_0 if the test statistic $Y = \mathbf{1}^T \boldsymbol{X} = \sum X_i \leq 2$. Since it is a sum of independent Poisson random variables, our test statistic is also a Poisson random variable, namely $Y \sim \text{Poisson}(n\theta)$, and the rejection region for Y is $\mathcal{S}_Y = \{0, 1, 2\}$. Therefore, for $\theta \in (0, 1/2]$, the power function of δ is given by

$$\pi(\theta \,|\, \delta) = \Pr(Y \in \mathcal{S}_Y \,|\, \theta) = \sum_{k=0}^{2} \frac{(n\theta)^k \mathrm{e}^{-n\theta}}{k!}.$$

A graph of $\pi(\theta \,|\, \delta)$ can be found in the left panel of Figure 1 for when $n = 12$. The values of $\pi(\theta \,|\, \delta)$ when $\theta = \{1/2, 1/3, 1/4, 1/6, 1/12\}$ are given by $0.062, 0.238, 0.423, 0.677, 0.920$, respectively, and have also been marked on Figure 1. The significance of this test is given by $\alpha(\delta) = \pi(1/2 \,|\, \delta) = 0.062$.

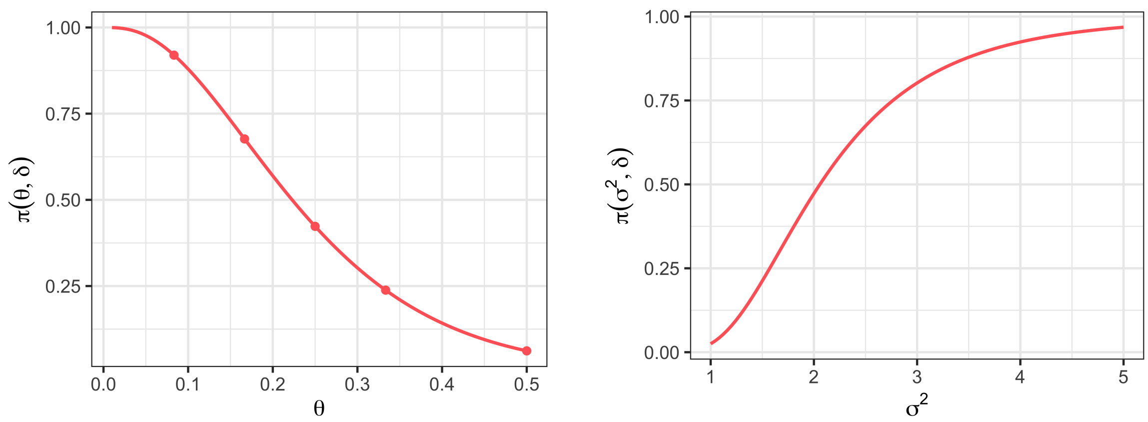

Figure 1: The power functions $\pi(\theta \mid \delta)$ (question 6) and $\pi(\sigma^2 \mid \delta)$ (question 8).

Question 7

Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Unif}(0, \theta)$, where θ is unknown, and let $Y_1, \ldots, Y_n$ be the n order statistics of the random sample. We are testing $H_0 : \theta = 1$ against $H_A : \theta > 1$ with procedure δ : reject H_0 if $Y_n \geq c$, where c is an unknown constant that will be determined by the significance of δ. If Y_n is the nth order statistic of $\boldsymbol{X}$, its cdf and density are respectively given by $F_{Y_n}(y_n) = (y_n/\theta)^n$ and $f_{Y_n}(y_n) = n y_n^{n-1}/\theta^n$ for $y_n \in [0, \theta]$.

(b) The rejection region of Y_n is $\mathcal{S}_{Y_n} = [c, \theta]$, and so the power function is given by

$$\pi(\theta \mid \delta) = \Pr(Y_n \in \mathcal{S}_{Y_n} \mid \theta) = F_{Y_n}(\theta) - F_{Y_n}(c) = 1 - \frac{c^n}{\theta^n}.$$

(a) To have a significance level of $\alpha = 0.05$, we must have $\alpha(\delta) = \pi(1 \mid \delta) = 1 - c^n \overset{\text{set}}{=} 0.05$, and solving for c gives us $c = \sqrt[n]{0.95}$, and the power function with this level of significance is $\pi(\theta \mid \delta) = 1 - 0.95/\theta^n$. The question specifically asks for $n = 4$, in which case $c = \sqrt[4]{0.95}$ and $\pi(\theta \mid \delta) = 1 - 0.95/\theta^4$. More generally, for any specified significance level α_*, we will have $c = \sqrt[n]{1 - \alpha_*}$ and $\pi(\theta \mid \delta) = 1 - (1 - \alpha_*)/\theta^n$.

Question 8

Let $\boldsymbol{X} \overset{\text{iid}}{\sim} \text{N}(\mu, \sigma^2)$, where both μ and σ^2 are unknown. We want to test $H_0 : \sigma^2 = \sigma_0^2$ against $H_A : \sigma^2 > \sigma_0^2$ using procedure δ : reject H_0 if $nS^2/\sigma_0^2 \geq c$, where c will be determined by the significance of the test. Our test statistic is based off of $S^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$, and we know that $T(\boldsymbol{X}) \coloneqq nS^2/\sigma_0^2 \sim \chi^2(n-1)$. The rejection region for our test statistic is $\mathcal{S}_{T(\boldsymbol{X})} = [c, \infty)$, and we would like to find a value of c such that the significance is $\alpha = 0.025$.

$$\alpha(\delta) = \pi(\sigma_0^2 \mid \delta) = \Pr\big(T(\boldsymbol{X}) \in \mathcal{S}_{T(\boldsymbol{X})} \mid \sigma_0^2\big) = \int_c^\infty \frac{t^{(n-3)/2} \cdot \mathrm{e}^{-t/2}}{2^{(n-1)/2}\Gamma\left((n-1)/2\right)}\, \partial t \overset{\text{set}}{=} 0.025.$$

There is no way to derive a closed form relationship between c and α like we did in question 7, but using `R` we can write a function to find the value of c for a given significance value α. When $n = 13$ and $\alpha = 0.025$, we have $c = 23.337$. To find a general formula of $\pi(\sigma^2 \mid \delta)$, we notice that if $T(\boldsymbol{X}) \geq c$, then $S^2 \geq c\sigma_0^2/n$, which serives as a rejection region for S^2. Since $nS^2/\sigma^2 \sim \chi^2(n-1)$ for any value of σ^2, it can be shown that $S^2 \sim \text{Gamma}\big((n-1)/2, n/2\sigma^2\big)$. Therefore, the power function is given by

$$\pi(\sigma^2 \mid \delta) = \Pr(S^2 \geq c\sigma_0^2/n \mid \sigma^2) = \int_{c\sigma_0^2/n}^\infty \frac{x^{(n-3)/2} \cdot \mathrm{e}^{-nx/2\sigma^2}}{\big(n/2\sigma^2\big)^{(n-1)/2}\Gamma\big((n-1)/2\big)}\, \partial x,$$

where c is chosen to make $\alpha(\delta) = 0.025$ (or any specified value, for that matter). Again, there is no closed form solution to this function, but using `R` we can find its values, and given a specified value of σ_0^2, we will always have $\pi(\sigma_0^2 \,|\, \delta) = \alpha$ as desired. A plot of $\pi(\sigma^2 \,|\, \delta)$ can be found in the right panel of Figure 1.

Question 9

If $\boldsymbol{X} \overset{\text{iid}}{\sim} \mathrm{N}(\mu, \sigma^2)$ with unknown μ and σ^2, then a γ% confidence interval for μ is given by

$$\mathcal{I} = \Big(\; \bar{X} - t_\gamma(n) \cdot S/\sqrt{n} \; , \; \bar{X} + t_\gamma(n) \cdot S/\sqrt{n} \; \Big),$$

where $t_\gamma(n) = T_{n-1}^{-1}((1+\gamma)/2)$ is the $(1+\gamma)/2$th quantile of the t distribution with $\mathrm{df} = n-1$ and S is the sample standard deviation. The length of this confidence interval is given by

$$\Delta = \max(\mathcal{I}) - \min(\mathcal{I}) = \Big(\bar{X} + t_\gamma(n) \cdot S/\sqrt{n}\Big) - \Big(\bar{X} - t_\gamma(n) \cdot S/\sqrt{n}\Big) = 2t_\gamma(n) \cdot S/\sqrt{n}.$$

The squared length is then given by $\Delta^2 = 4t_\gamma^2(n) \cdot S^2/n$. Because the sample variance is an unbiased estimator for σ^2, we have $\mathbb{E}[\Delta^2] = \mathbb{E}\big[4t_\gamma^2(n) \cdot S^2/n\big] = 4t_\gamma^2(n) \cdot \sigma^2/n$. We now set $\mathbb{E}[\Delta^2] < \sigma^2/2$, and after some cancellations, we see that we need $t_\gamma^2(n)/n < 1/8$. There is no way to find a closed-form expression for this, so we will have to check the value of $t_\gamma^2(n)/n$ for increasing values of n. I set up a `while` loop in `R` to solve for it, and when $\gamma = 0.9$, we find that $n = 24$ is the smallest value of n such that $\mathbb{E}[\Delta^2] < \sigma^2/2$.

Question 10

Let $\boldsymbol{X} \overset{\text{iid}}{\sim} \mathrm{N}(\theta, \sigma^2)$, where θ is unknown and σ^2 is known, and we assume prior that $\theta \sim \mathrm{N}(\mu, \nu^2)$, where both μ and ν^2 are known.

(a) Since normal distributions are are conjugate to normal sampling, it follows that $\theta \,|\, \mathbf{x} \sim \mathrm{N}(\tilde{\mu}, \tilde{\sigma}^2)$, where

$$\tilde{\mu} = \frac{\sigma^2\mu + n\nu^2\bar{x}}{\sigma^2 n\nu^2} \quad \text{and} \quad \tilde{\sigma}^2 = \frac{\sigma^2\nu^2}{\sigma^2 + n\mu^2}.$$

We also know that $(\theta \,|\, \mathbf{x} - \tilde{\mu})/\tilde{\sigma} \sim \mathrm{N}(0,1)$, and so a 95% confidence interval for $\theta \,|\, \mathbf{x}$ is given by

$$\mathcal{I} = \Big(\; \tilde{\mu} - \Phi^{-1}(0.975) \cdot \tilde{\sigma} \; , \; \tilde{\mu} + \Phi^{-1}(0.975) \cdot \tilde{\sigma} \; \Big).$$

(b) We can think of our interval $\mathcal{I}$ as a function of ν^2. To examine what happens to $\mathcal{I}(\nu^2)$ as $\nu^2 \to \infty$, we will first look at $\tilde{\mu}$ and $\tilde{\sigma}$. Using L'Hopital's rule, we have

$$\lim_{\nu^2\to\infty} \tilde{\mu} = \lim_{\nu^2\to\infty} \frac{\sigma^2\mu + n\nu^2\bar{x}}{\sigma^2 n\nu^2} = \lim_{\nu^2\to\infty} \frac{n\bar{x}}{n} = \bar{x},$$
$$\lim_{\nu^2\to\infty} \tilde{\sigma} = \lim_{\nu^2\to\infty} \sqrt{\frac{\sigma^2\nu^2}{\sigma^2 + n\mu^2}} = \sqrt{\lim_{\nu^2\to\infty} \frac{\sigma^2\nu^2}{\sigma^2 + n\mu^2}} = \sqrt{\lim_{\nu^2\to\infty} \frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}},$$

and so $\mathcal{I}(\nu^2) \to \big(\; \bar{x} - \Phi^{-1}(0.975) \cdot \sigma/\sqrt{n} \; , \; \bar{x} + \Phi^{-1}(0.975) \cdot \sigma/\sqrt{n} \; \big)$, which is a 95% confidence interval for θ.

Question 11

Let $\boldsymbol{X} \overset{\text{iid}}{\sim} \mathrm{Unif}(0, \theta)$, where θ is unknown, and let $Y = X_{(n)}$ be the nth order statistic.

(a) Let $F_i(x) = \Pr(X_i \leq x) = x/\theta$ for all $i \in \{1, \ldots, n\}$. Then the cdf of Y is $G(y) = \prod_{i=1}^n F_i(y) = (y/\theta)^n$, since all of the X_i's are independent. The density of Y is then given by $g(y) = ny^{n-1}/\theta^n$ for $y \in [0, \theta]$. By letting $W = Y/\theta$, we have $Y = \theta W$ and $\partial Y/\partial W = \theta$, so the density of Y/θ is given by $h(w) = g\big(y(w)\big) \cdot \theta = nw^{n-1}$ for $w \in [0,1]$. The cdf is then seen to be $H(w) = w^n$, and so the quantile is given by $H^{-1}(w) = \sqrt[n]{w}$.

(b) Since we must have $y \le \theta$, it is natural that Y will underestimate θ, and is therefore biased. We have

$$\mathbb{E}[Y] = \int_0^\theta y \cdot \frac{ny^{n-1}}{\theta^n}\,\partial y = \frac{n}{\theta^n}\int_0^\theta y^n\,\partial y = \frac{n}{\theta^n}\cdot\frac{y^{n+1}}{n+1}\bigg|_0^\theta = \frac{n}{\theta^n}\cdot\frac{\theta^{n+1}}{n+1} = \frac{n\theta}{n+1},$$

and so the bias is $\text{bias}(Y) = \mathbb{E}[Y] - \theta = -\theta/(n+1)$.

(d) Using the cdf of Y/θ, for any interval involving Y/θ, we have $\Pr(a \le Y/\theta \le b) = b^n - a^n$, where $a, b \in (0,1]$. Rearranging the terms inside the interval gives us $\Pr(Y/b \le \theta \le Y/a) = b^n - a^n \overset{\text{set}}{=} \gamma$. That is, as long as we impose the constraint that $b^n - a^n = \gamma$, any interval $(Y/b, Y/a)$ is a $100\gamma\%$ confidence interval for θ.

Question 12

Suppose that $X \sim \mathrm{P}$, where P is an unknown distribution, and we want to test $H_0 : \mathrm{P} = \mathrm{Unif}(0,1)$ against $H_A : \mathrm{P} = \mathrm{N}(0,1)$. The densities of both distributions (within their support) are given by $f(x \,|\, P_0) = 1$ for $x \in [0,1]$ and $f(x \,|\, P_A) = (2\pi)^{-1/2}\exp(-x^2/2)$ for $x \in \mathbb{R}$, respectively, and so the likelihood ratio is given by

$$\ell(x) = \frac{f(x \,|\, P_A)}{f(x \,|\, P_0)} = \begin{cases} \phi(x) & \text{if } x \in [0,1] \\ \infty & \text{if } x \notin [0,1], \end{cases}$$

where $\phi(x)$ is the standard normal density. As a result, we will reject H_0 if $\ell(X) \ge k$, and so the power function is given by $\pi(P \,|\, \delta) = \Pr(\ell(X) \ge k \,|\, P)$. We note that if H_0 is true, then we will always have $x \in [0,1]$, since we would be sampling from $\mathrm{Unif}(0,1)$, so in this case $\ell(x) = \phi(x)$. We also note that $\phi(x)$ is a decreasing function on this interval. As a result, the significance of this test is

$$\alpha(\delta) = \pi(P_0 \,|\, \delta) = \Pr(\ell(X) \ge k \,|\, P_0) = \Pr(\phi(X) \ge k) = \Pr(X \le c) = F_0(c) \overset{\text{set}}{=} 0.01$$

where $c = \phi^{-1}(k)$ and $F_0(c)$ is the cdf of $\mathrm{Unif}(0,1)$. As a result, $c = 0.01$ gives us a test with significance 0.01. More generally, since $\phi(X) > k$ for any value of k (and thus any level of significance), we will always reject H_0 when $X \notin [0,1]$, which makes the power function

$$\begin{aligned} \pi(P \,|\, \delta) &= \Pr\big(\phi(X) \ge \phi(0.01) \,|\, P\big) \\ &= \Pr\big(X < 0 \cup X \le 0.01 \cup X > 1 \,|\, P\big) = \Pr\big(X \le 0.01 \,|\, P\big) + \big(X > 1 \,|\, P\big) \end{aligned}$$

Therefore, when H_A is true, the power of δ is

$$\pi(P_A \,|\, \delta) = \Pr\big(X \le 0.01 \,|\, P_A\big) + \big(X > 1 \,|\, P_A\big) = \Phi(0.01) + 1 - \Phi(1) = 0.663.$$

Question 13

Suppose that $\boldsymbol{X} \overset{\text{iid}}{\sim} N(\theta, 1)$, where $\theta > 1$ is unknown, and we wish to test $H_0 : \theta \le \theta_*$ against $H_A : \theta > \theta_*$ using some UMP test δ^*. More specifically, we want our UMP test to have some pre-determined power π_* at a given mean φ, i.e. $\pi(\varphi \,|\, \delta^*) = \pi_*$. Since we are sampling from a normal distribution, the joint distribution of $\boldsymbol{X}$ has increasing montone likelihood ratio in the test statistic $\bar{X}$, where $\bar{X} \sim \mathrm{N}(\theta, 1/n)$, so our UMP test is given by δ^* : reject H_0 if $\bar{X} \ge c$. We also notice that $\sqrt{n}(\bar{X} - \theta) \coloneqq Z_\theta \sim \mathrm{N}(0,1)$. The rejection region for $\bar{X} = \mathcal{S}_{\bar{X}} = [c, \infty)$, so to find c we have

$$\pi(\varphi \,|\, \delta^*) = \Pr(\bar{X} \ge c \,|\, \theta = \varphi) = \Pr\big(Z_\varphi \ge \sqrt{n}(c - \varphi)\big) = 1 - \Phi\big(\sqrt{n}(c - \varphi)\big) \overset{\text{set}}{=} \pi_*.$$

From here, we have $\sqrt{n}(c - \varphi) = \Phi^{-1}(1 - \pi_*) \coloneqq z_{\pi_*}$, and so $c = \varphi + z_{\pi_*}/\sqrt{n}$. With the value of c determined, the power function is then given by

$$\pi(\theta \,|\, \delta^*) = \Pr\big(\bar{X} \ge \varphi + z_{\pi_*}/\sqrt{n} \,|\, \theta\big) = \Pr\big(Z_\theta \ge \sqrt{n}(\varphi - \theta) + z_{\pi_*}\big) = 1 - \Phi\big(\sqrt{n}(\varphi - \theta) + z_{\pi_*}\big).$$

Since $\pi(\theta \mid \delta^*)$ is a montone increasing function of θ, it will be maximized at the largest possible value over an interval. Therefore, the significance of this test is

$$\alpha(\delta^*) = \sup_{\theta \leq \theta_*} \pi(\theta \mid \delta^*) = \pi(\theta_* \mid \delta^*) = 1 - \Phi\big(\sqrt{n}(\varphi - \theta_*) + z_{\pi_*}\big).$$

The question specifically asks for c and α when $\theta_* = 0$, $\varphi = 1$, $\pi_* = 0.95$, and $n = 16$. In this case, we have $z_{\pi_*} = \Phi^{-1}(0.05) \approx -1.64$, which means $c = 1 - 1.64/4 = 0.589$ and $\alpha = 1 - \Phi(4 - 1.64) = 0.00926$. Put more formally, if we have a random sample $\boldsymbol{X} \overset{\text{iid}}{\sim} \mathrm{N}(\theta, 1)$, and want to test $H_0 : \theta \leq 0$ against $H_A : \theta > 0$ using a UMP test with a power of 0.95 at $\theta = 1$, then our test is given by δ^* : reject H_0 if $\bar{X} \geq 0.589$. With 16 observations, the size of this test is $\alpha = 0.00926$.

Question 14

Suppose $X_1, \ldots, X_8$ are randomly sampled from a random variable with density $f(x \mid \theta) = \theta x^{\theta - 1}$, supported for all $x \in (0, 1)$, and we would like to test $H_0 : \theta \leq 1$ against $H_A : \theta > 1$ using a UMP test δ^*. For two values θ_1, θ_2 such that $\theta_1 < \theta_2$, the ratio of the joint densities (which are also the likelihood functions) is

$$\frac{f(\mathbf{x} \mid \theta_2)}{f(\mathbf{x} \mid \theta_1)} = \frac{\prod_{i=1}^8 \theta_2 x_i^{\theta_2 - 1}}{\prod_{i=1}^8 \theta_1 x_i^{\theta_1 - 1}} = \frac{\theta_2^8 \left(\prod_{i=1}^8 x_i\right)^{\theta_2 - 1}}{\theta_1^8 \left(\prod_{i=1}^8 x_i\right)^{\theta_1 - 1}} = \left(\frac{\theta_2}{\theta_1}\right)^8 \cdot \left(\prod_{i=1}^8 x_i\right)^{\theta_2 - \theta_1}.$$

which is a function of the test statistic $\prod_{i=1}^8 x_i$. Because $\theta_2 > \theta_1$, we have $\theta_2/\theta_1 > 1$ and $\theta_2 - \theta_1 > 0$, so the likelihood ratio is monotone and increasing in the parameter $\prod_{i=1}^8 x_i$. As a result, the UMP test is δ^* : reject H_0 if $\prod_{i=1}^8 x_i \geq c$ for some value of c. For this test, we specify the level of significance as $\alpha_* = 0.05$, and since $\pi(\theta \mid \delta^*)$ is a monotone increasing function, we have

$$\alpha_* = \sup_{\theta \leq 1} \pi(\theta \mid \delta^*) = \pi(1 \mid \delta^*) \overset{\text{set}}{=} 0.05.$$

When $\theta = 1$, we can see from the density function that $X \sim \mathrm{Unif}(0, 1)$, with its cdf given by $F(X) = X$. It can be shown, then, that $-2 \sum_{i=1}^8 \log F(X_i) \sim \chi^2(16)$. Therefore,

$$\pi(1 \mid \delta^*) = \Pr\left(\prod_{i=1}^8 x_i \geq c \mid \theta = 1\right) = \Pr\left(-2 \sum_{i=1}^8 \log X_i \leq -2 \log c\right) = C_{16}(-2 \log c) = 0.05$$

where $C_{16}(x)$ is the cdf of $\chi^2(16)$. If $k \coloneqq -2 \log c$, it follows that $k = C_{16}^{-1}(0.05) = 7.962$, and as a result, $\sum_{i=1}^8 \log X_i \geq -k/2 = 3.981$.

Question 15

Suppose that $\boldsymbol{X} \overset{\text{iid}}{\sim} \chi^2(\theta)$, where $\theta \in \mathbb{N}$ is unknown. We would like to test $H_0 : \theta \leq 8$ against $H_A : \theta > 8$, using a UMP test δ^* with a specified significance $\alpha_* \in (0, 1)$. The joint density of $\boldsymbol{X}$ is given by

$$f(\mathbf{x} \mid \theta) = \prod_{i=1}^n \frac{x_i^{\theta/2 - 1} \mathrm{e}^{-x/2}}{2^{\theta/2} \Gamma(\theta/2)} = 2^{-n\theta/2} \cdot \Gamma^{-n}(\theta/2) \cdot \left(\prod_{i=1}^n x_i\right)^{n(\theta/2 - 1)} \cdot \exp\left(-\frac{1}{2} \sum_{i=1}^n x_i\right)$$

To determine δ^*, we will look at the likelihood ratio. If we have two values θ_1, θ_2 such that $\theta_1 < \theta_2$, then then likelihood ratio is

$$\begin{aligned} \frac{f(\mathbf{x} \mid \theta_2)}{f(\mathbf{x} \mid \theta_1)} &= \frac{2^{-n\theta_1/2} \cdot \Gamma^{-n}(\theta_2/2) \cdot \left(\prod_i x_i\right)^{n(\theta_2/2 - 1)} \cdot \exp\left(-\frac{1}{2} \sum_i x_i\right)}{2^{-n\theta_1/2} \cdot \Gamma^{-n}(\theta_1/2) \cdot \left(\prod_i x_i\right)^{n(\theta_1/2 - 1)} \cdot \exp\left(-\frac{1}{2} \sum_i x_i\right)} \\ &= 2^{n(\theta_1 - \theta_2)/2} \left(\frac{\Gamma(\theta_1/2)}{\Gamma(\theta_2/2)}\right)^n \left(\prod_{i=1}^n x_i\right)^{n(\theta_2 - \theta_1)/2}, \end{aligned}$$

which is a monotone increasing function of the test statistic $\prod_{i=1}^n x_i$. Therefore, the UMP test is δ^* : reject H_0 if $\prod_{i=1}^n X_i \geq c_*$, where c_* is chosen such that the test has a significance level α_*. Taking the log of both sides gives us $\sum_{i=1}^8 \log X_i \geq \log c_* := k$.