# Homework 2

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# Question 1

Suppose that  $Y \sim \text{Bin}(100, p)$ , and we want to make an inference about the value of p. We test  $H_0: p = 0.08$  against  $H_A: p < 0.08$ , and our test  $\delta$  will reject  $H_0$  if and only if Y = 6.

(a) The significance  $\alpha$  is the probability of making a Type I error,

$$\alpha(\delta) = \Pr(Y = 6 \mid p = 0.08) = {100 \choose 6} (0.08)^6 (0.92)^{94} = 0.123.$$

(b) Suppose that p = 0.04. The probability of a Type II error,  $\beta$ , is

$$\beta(\delta) = \Pr(Y \neq 6 \mid p = 0.04) = 1 - \Pr(Y = 6 \mid p = 0.04) = 1 - \binom{100}{6} (0.04)^6 (0.96)^{94} = 0.895.$$

#### Question 2

For a random variable  $Y \sim \text{Binom}(n,p)$ , if n is large enough we can approximate it using a normal distribution with the same mean and variance, i.e.  $Y \sim \text{N}(np, np(1-p))$ . Then the sample proportion,  $\hat{p} = Y/n$ , is also normally distributed as  $\hat{p} \sim \text{N}(p, p(1-p)/n)$ , and standardizing  $\hat{p}$  gives us  $(\hat{p}-p)/\sqrt{p(1-p)/n} \sim \text{N}(0,1)$ . It can then be shown that a  $100\gamma\%$  confidence interval for  $\hat{p}$  is given by

$$\mathcal{I} = \left( \frac{\hat{p} + z_{\gamma}^{2}/2n}{1 + z_{\gamma}^{2}/2n} - z_{\gamma} \cdot \frac{\sqrt{\hat{p}(1-\hat{p})/n + z_{\gamma}^{2}/4n^{2}}}{1 + z_{\gamma}^{2}/n} \right., \\ \left. \frac{\hat{p} + z_{\gamma}^{2}/2n}{1 + z_{\gamma}^{2}/2n} + z_{\gamma} \cdot \frac{\sqrt{\hat{p}(1-\hat{p})/n + z_{\gamma}^{2}/4n^{2}}}{1 + z_{\gamma}^{2}/n} \right),$$

where  $z_{\gamma} = \Phi^{-1}((1+\gamma)/2)$ . For this example, we have n = 300 and  $\hat{p} = 75/300 = 1/4$ , and to get a 90% confidence interval, we have  $z_{\gamma} = 1.645$ . Therefore, a 90% confidence interval for p is (0.212, 0.294).

#### Question 3

Suppose we have a random sample  $X \stackrel{\text{iid}}{\sim} \text{Gamma}(4,\beta)$ , so  $\mathbb{E}[X_i] = 4\beta$  and  $\text{Var}[X_i] = 4\beta^2$ . The expected value and variance of the sample mean  $\bar{X}$  is then given by  $\mathbb{E}[\bar{X}] = 4\beta$  and  $\text{Var}[\bar{X}] = 4\beta^2/n$ , and from the CLT we have  $\sqrt{n}(\bar{X} - 4\beta)/2\beta = \sqrt{n}(\bar{X}/2\beta - 2) \sim \text{N}(0,1)$ . If  $z_{\gamma} = \Phi^{-1}((1+\gamma)/2)$ , then we have  $\gamma = \text{Pr}(-z_{\gamma} \leq \sqrt{n}(\bar{X}/2\beta - 2 \leq z_{\gamma})$ . Rearranging to get  $\beta$  in the middle gives us

$$\mathcal{I} = \left( \frac{2\bar{X}}{2 + z_{\gamma}/\sqrt{n}} , \frac{2\bar{X}}{2 - z_{\gamma}/\sqrt{n}} \right).$$

For this random sample, n=25, and because we want a 95.4% confidence interval (oddly specific), we have  $z_{\gamma}=2$ , so the confidence interval is given by  $\mathcal{I}=\left(\ 5\bar{X}/6\ ,\ 5\bar{X}/4\ \right)$ .

### Question 4

Suppose that  $X \sim \text{Binom}(100, p)$ , where  $p \in (1/4, 1/2)$  is unknown. We test  $H_0: 1/2$  against  $H_A: p = 1/4$  using  $\delta$ : reject  $H_0$  if  $X \leq 3$ . That is, our rejection region is  $\mathcal{S}_X = \{0, 1, 2, 3\}$ , and so the power function for this test is

$$\pi(p \mid \delta) = \Pr(X \in \mathcal{S}_X \mid p) = \sum_{k=0}^{3} {10 \choose k} p^k (1-p)^{n-k}.$$

### Question 5

# Question 6

Suppose that  $X \stackrel{\text{iid}}{\sim} \text{Poisson}(\theta)$ . We test  $H_0: \theta = 1/2$  against  $H_A: \theta < 1/2$  using our procedure  $\delta:$  reject  $H_0$  if the test statistic  $Y = \mathbf{1}^T X = \sum X_i \leq 2$ . Since it is a sum of independent Poisson random variables, our test statistic is also a Poisson random variable, namely  $Y \sim \text{Poisson}(n\theta)$ , and the rejection region for Y is  $\mathcal{S}_Y = \{0,1,2\}$ . Therefore, for  $\theta \in (0,1/2]$ , the power function of  $\delta$  is given by

$$\pi(\theta \mid \delta) = \Pr(Y \in \mathcal{S}_Y \mid \theta) = \sum_{k=0}^{2} \frac{(n\theta)^k e^{-n\theta}}{k!}.$$

A graph of  $\pi(\theta \mid \delta)$  can be found in Figure 1. The values of  $\pi(\theta \mid \delta)$  when  $\theta = \{1/2, 1/3, 1/4, 1/6, 1/12\}$  are given by 0.062, 0.238, 0.423, 0.677, 0.920, respectively, and have also been marked on Figure 1. The significance of this test is given by  $\alpha(\delta) = \pi(1/2 \mid \delta) = 0.062$ .

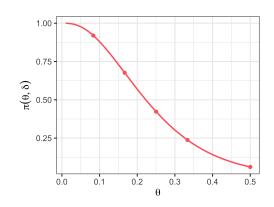


Figure 1: The power function  $\pi(\theta \mid \delta)$ .

### Question 7

Let  $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} \text{Unif}(0, \theta)$ , where  $\theta$  is unknown, and let  $Y_1, \ldots, Y_n$  be the n order statistics of the random sample. We are testing  $H_0: \theta = 1$  against  $H_A: \theta > 1$  with procedure  $\delta$ : reject  $H_0$  if  $Y_n \geq c$ , where c is an unknown constant that will be determined by the significance of  $\delta$ . If  $Y_n$  is the nth order statistic of X, its cdf and density are respectively given by  $F_{Y_n}(y_n) = (y_n/\theta)^n$  and  $f_{Y_n}(y_n) = ny_n^{n-1}/\theta^n$  for  $y_n \in [0, \theta]$ .

(b) The rejection region of  $Y_n$  is  $S_{Y_n} = [c, \theta]$ , and so the power function is given by

$$\pi(\theta \mid \delta) = \Pr(Y_n \in \mathcal{S}_{Y_n} \mid \theta) = F_{Y_n}(\theta) - F_{Y_n}(c) = 1 - \frac{c^n}{\theta^n}.$$

(a) To have a significance level of  $\alpha = 0.05$ , we must have  $\alpha(\delta) = \pi(1 \mid \delta) = 1 - c^n \stackrel{\text{set}}{=} 0.05$ , and solving for c gives us  $c = \sqrt[n]{0.95}$ , and the power function with this level of significance is  $\pi(\theta \mid \delta) = 1 - 0.95/\theta^n$ . The question specifically asks for n = 4, in which case  $c = \sqrt[4]{0.95}$  and  $\pi(\theta \mid \delta) = 1 - 0.95/\theta^4$ . More generally, for any specified significance level  $\alpha_*$ , we will have  $c = \sqrt[n]{1 - \alpha_*}$  and  $\pi(\theta \mid \delta) = 1 - (1 - \alpha_*)/\theta^n$ .

#### Question 9

If  $X \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$  with unknown  $\mu$  and  $\sigma^2$ , then a  $\gamma\%$  confidence interval for  $\mu$  is given by

$$\mathcal{I} = \left( \bar{X} - t_{\gamma}(n) \cdot S / \sqrt{n} , \bar{X} + t_{\gamma}(n) \cdot S / \sqrt{n} \right),$$

where  $t_{\gamma}(n) = T_{n-1}^{-1}((1+\gamma)/2)$  is the  $(1+\gamma)/2$ th quantile of the t distribution with df = n-1 and S is the sample standard deviation. The length of this confidence interval is given by

$$\Delta = \max(\mathcal{I}) - \min(\mathcal{I}) = \left(\bar{X} + t_{\gamma}(n) \cdot S/\sqrt{n}\right) - \left(\bar{X} - t_{\gamma}(n) \cdot S/\sqrt{n}\right) = 2t_{\gamma}(n) \cdot S/\sqrt{n}.$$

The squared length is then given by  $\Delta^2 = 4t_\gamma^2(n) \cdot S^2/n$ . Because the sample variance is an unbiased estimator for  $\sigma^2$ , we have  $\mathbb{E}[\Delta^2] = \mathbb{E}[4t_\gamma^2(n) \cdot S^2/n] = 4t_\gamma^2(n) \cdot \sigma^2/n$ . We now set  $\mathbb{E}[\Delta^2] < \sigma^2/2$ , and after some cancellations, we see that we need  $t_\gamma^2(n)/n < 1/8$ . There is no way to find a closed-form expression for this, so we will have to check the value of  $t_\gamma^2(n)/n$  for increasing values of n. I set up a while loop in R to solve for it, and when  $\gamma = 0.9$ , we find that n = 24 is the smallest value of n such that  $\mathbb{E}[\Delta^2] < \sigma^2/2$ .

# Question 10

Let  $X \stackrel{\text{iid}}{\sim} N(\theta, \sigma^2)$ , where  $\theta$  is unknown and  $\sigma^2$  is known, and we assume prior that  $\theta \sim N(\mu, \nu^2)$ , where both  $\mu$  and  $\nu^2$  are known.

(a) Since normal distributions are are conjugate to normal sampling, it follows that  $\theta \mid \mathbf{x} \sim N(\tilde{\mu}, \tilde{\sigma}^2)$ , where

$$\tilde{\mu} = \frac{\sigma^2 \mu + n \nu^2 \bar{x}}{\sigma^2 n \nu^2}$$
 and  $\tilde{\sigma}^2 = \frac{\sigma^2 \nu^2}{\sigma^2 + n \mu^2}$ .

We also know that  $(\theta \mid \mathbf{x} - \tilde{\mu})/\tilde{\sigma} \sim N(0,1)$ , and so a 95% confidence interval for  $\theta \mid \mathbf{x}$  is given by

$$\mathcal{I} = \Big( \ \tilde{\mu} - \Phi^{-1}(0.975) \cdot \tilde{\sigma} \ , \ \tilde{\mu} + \Phi^{-1}(0.975) \cdot \tilde{\sigma} \ \Big).$$

(b) We can think of our interval  $\mathcal{I}$  as a function of  $\nu^2$ . To examine what happens to  $\mathcal{I}(\nu^2)$  as  $\nu^2 \to \infty$ , we will first look at  $\tilde{\mu}$  and  $\tilde{\sigma}$ . Using L'Hopital's rule, we have

$$\lim_{\nu^2 \to \infty} \tilde{\mu} = \lim_{\nu^2 \to \infty} \frac{\sigma^2 \mu + n\nu^2 \bar{x}}{\sigma^2 n\nu^2} = \lim_{\nu^2 \to \infty} \frac{n\bar{x}}{n} = \bar{x},$$

$$\lim_{\nu^2 \to \infty} \tilde{\sigma} = \lim_{\nu^2 \to \infty} \sqrt{\frac{\sigma^2 \nu^2}{\sigma^2 + n\mu^2}} = \sqrt{\lim_{\nu^2 \to \infty} \frac{\sigma^2 \nu^2}{\sigma^2 + n\mu^2}} = \sqrt{\lim_{\nu^2 \to \infty} \frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}},$$

and so  $\mathcal{I}(\nu^2) \to (\bar{x} - \Phi^{-1}(0.975) \cdot \sigma/\sqrt{n} , \bar{x} + \Phi^{-1}(0.975) \cdot \sigma/\sqrt{n})$ , which is a 95% confidence interval for  $\theta$ .

# Question 11

Let  $X \stackrel{\text{iid}}{\sim} \text{Unif}(0,\theta)$ , where  $\theta$  is unknown, and let  $Y = X_{(n)}$  be the nth order statistic.

- (a) Let  $F_i(x) = \Pr(X_i \leq x) = x/\theta$  for all  $i \in \{1, \dots, n\}$ . Then the cdf of Y is  $G(y) = \prod_{i=1}^n F_i(y) = (y/\theta)^n$ , since all of the  $X_i$ 's are independent. The density of Y is then given by  $g(y) = ny^{n-1}/\theta^n$  for  $y \in [0, \theta]$ . By letting  $W = Y/\theta$ , we have  $Y = \theta W$  and  $\partial Y/\partial W = \theta$ , so the density of  $Y/\theta$  is given by  $h(w) = g(y(w)) \cdot \theta = nw^{n-1}$  for  $w \in [0, 1]$ . The cdf is then seen to be  $H(w) = w^n$ , and so the quantile is given by  $H^{-1}(w) = \sqrt[n]{w}$ .
- (b) Since we must have  $y \leq \theta$ , it is natural that Y will underestimate  $\theta$ , and is therefore biased. We have

$$\mathbb{E}[Y] = \int_0^\theta y \cdot \frac{ny^{n-1}}{\theta^n} \, \partial y = \frac{n}{\theta^n} \int_0^\theta y^n \, \partial y = \frac{n}{\theta^n} \cdot \frac{y^{n+1}}{n+1} \bigg|_0^\theta = \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} = \frac{n\theta}{n+1},$$

and so the bias is  $bias(Y) = \mathbb{E}[Y] - \theta = -\theta/(n+1)$ .

(d) Using the cdf of  $Y/\theta$ , for any interval involving  $Y/\theta$ , we have  $\Pr(a \leq Y/\theta \leq b) = b^n - a^n$ , where  $a, b \in (0, 1]$ . Rearranging the terms inside the interval gives us  $\Pr(Y/b \leq \theta \leq Y/a) = b^2 - a^2 \stackrel{\text{set}}{=} \gamma$ . That is, as long as we impose the constraint that  $b^2 - a^2 = \gamma$ , any interval (Y/b, Y/a) is a  $\gamma\%$  confidence interval for  $\theta$ .

# Question 12

Suppose that  $X \sim P$ , where P is an unknown distribution, and we want to test  $H_0: P = \text{Unif}(0,1)$  against  $H_1: P = N(0,1)$ . The densities of both distributions (within their support) are given by  $f_0(x) = 1$  for  $x \in [0,1]$  and  $f_2(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ , respectively, and so our test statistics is