Homework 3

Aiden Kenny
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Columbia University

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Question 1

Suppose that $\boldsymbol{X} \overset{\text{iid}}{\sim} \chi^2(\theta)$, where $\theta \in \mathbb{N}$ is unknown. We would like to test $H_0 : \theta \leq 8$ against $H_A : \theta > 8$, using a UMP test δ^* with a specified significance $\alpha_* \in (0, 1)$. The joint density of $\boldsymbol{X}$ is given by

$$f(\mathbf{x} \,|\, \theta) = \prod_{i=1}^{n} \frac{x_i^{\theta/2-1} \mathrm{e}^{-x/2}}{2^{\theta/2}\Gamma(\theta/2)} = 2^{-n\theta/2} \cdot \Gamma^{-n}(\theta/2) \cdot \left(\prod_{i=1}^{n} x_i\right)^{n(\theta/2-1)} \cdot \exp\left(-\frac{1}{2}\sum_{i=1}^{n} x_i\right)$$

To determine δ^*, we will look at the likelihood ratio. If we have two values θ_1, θ_2 such that $\theta_1 < \theta_2$, then then likelihood ratio is

$$\begin{aligned} \frac{f(\mathbf{x} \,|\, \theta_2)}{f(\mathbf{x} \,|\, \theta_1)} &= \frac{2^{-n\theta_1/2} \cdot \Gamma^{-n}(\theta_2/2) \cdot \left(\prod_i x_i\right)^{n(\theta_2/2-1)} \cdot \exp\left(-\frac{1}{2}\sum_i x_i\right)}{2^{-n\theta_1/2} \cdot \Gamma^{-n}(\theta_1/2) \cdot \left(\prod_i x_i\right)^{n(\theta_1/2-1)} \cdot \exp\left(-\frac{1}{2}\sum_i x_i\right)} \\ &= 2^{n(\theta_1-\theta_2)/2} \left(\frac{\Gamma(\theta_1/2)}{\Gamma(\theta_2/2)}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{n(\theta_2-\theta_1)/2}, \end{aligned}$$

which is a monotone increasing function of the test statistic $\prod_{i=1}^{n} x_i$. Therefore, the UMP test is δ^* : reject H_0 if $\prod_{i=1}^{n} X_i \geq c_*$, where c_* is chosen such that the test has a significance level α_*. Taking the log of both sides gives us $\sum_{i=1}^{8} \log X_i \geq \log c_* := k$.

Question 2

Question 8

By definition, the least-squares estimate for the intercept is $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$. Rearranging gives us $\bar{y} = \hat{\beta}_0 + \hat{\beta}_1\bar{x}$, and so the least-squares line $y = \hat{\beta}_0 + \hat{\beta}_1 x$ will always pass through the point $(\bar{x}, \bar{y})$.

Question 9

(a) The least-squares coefficients for the model are given by $\hat{\beta}_0 = 40.9$ and $\hat{\beta}_1 = 0.548$.

Question 10

Let $(\mathbf{x}, \mathbf{y})$ be the vectors of observations for the predictor x and the response Y. From the data, we have $\bar{x} = 2.33$ and $\bar{y} = 0.81$. Here we are assuming that $Y = \beta_0 + \beta_1 x + \epsilon$, where $\epsilon \sim \mathrm{N}(0, \sigma^2)$.

(a) The MLEs for β_0, β_1, and σ^2 are given by

$$\hat{\beta}_1 = \frac{(\mathbf{y} - \bar{y}\mathbf{1})^T(\mathbf{x} - \bar{x}\mathbf{1})}{\|\mathbf{x} - \bar{x}\mathbf{1}\|^2} = 0.685 \ , \ \hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x} = -0.786 \ , \ \hat{\sigma}^2 = \frac{\|\mathbf{y} - \hat{\beta}_0\mathbf{1} - \hat{\beta}_1\mathbf{x}\|^2}{n} = 0.938.$$

(b) The variance of $\hat{\beta}_0$ and $\hat{\beta}_1$ is given by

$$\mathrm{Var}\big[\hat{\beta}_1\big] = \frac{\sigma^2}{\|\mathbf{x} - \bar{x}\mathbf{1}\|^2} = 0.0277\sigma^2 \ , \ \mathrm{Var}\big[\hat{\beta}_0\big] = \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\|\mathbf{x} - \bar{x}\mathbf{1}\|^2}\right) = 0.25\sigma^2.$$

(c) The covariance between $\hat{\beta}_0$ and $\hat{\beta}_1$, and therefore the correltaion, is

$$\mathrm{Cov}\big[\hat{\beta}_0, \hat{\beta}_1\big] = -\frac{\bar{x}\sigma^2}{\|\mathbf{x} - \bar{x}\mathbf{1}\|^2} = -0.0646\sigma^2 \ , \ \mathrm{Cor}\big[\hat{\beta}_0, \hat{\beta}_1\big] = \frac{\mathrm{Cov}\big[\hat{\beta}_0, \hat{\beta}_1\big]}{\sqrt{\mathrm{Var}\big[\hat{\beta}_0\big] \cdot \mathrm{Var}\big[\hat{\beta}_1\big]}} = -0.775.$$

Question 11

Suppose β_0, β_1 are the coefficients from the linear model in question 10, and we want to estimate $\theta = 3\beta_0 - 2\beta_1 + 5$. Because $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased estimators for the coefficients, we can estimate θ with $\hat{\theta} = 3\hat{\beta}_0 - 2\hat{\beta}_1 + 5$. The MSE of $\hat{\theta}$, which is just its variance, is given by

$$\mathbb{E}\Big[(\hat{\theta} - \theta)^2\Big] = \mathrm{Var}\big[3\hat{\beta}_0 - 2\hat{\beta}_1 + 5\big] = 9\mathrm{Var}\big[\hat{\beta}_0\big] + 4\mathrm{Var}\big[\hat{\beta}_1\big] - 6\mathrm{Cov}\big[\hat{\beta}_0, \hat{\beta}_1\big] = 10.549\sigma^2.$$

Question 12

We know that the MLE and least-squares estimates of β_0 and β_1 are the same, so $\hat{\beta}_0 = -0.786$ and $\hat{\beta}_1 = 0.685$. Using this linear model, when $x = 2$, we predict $\hat{Y} = -0.786_0.685 \cdot 2 = 0.584$. The variance of $\hat{Y}$ is given by

$$\mathbb{E}\Big[(\hat{Y} - Y)^2\Big] = \sigma^2\left(1 + \frac{1}{n} + \frac{(2 - \bar{x})^2}{\|\mathbf{x} - \bar{x}\mathbf{1}\|^2}\right) = 1.103\sigma^2.$$