# Homework 2

Aiden Kenny
STAT GR5204: Statistical Inference
Columbia University

Novermber 26, 2020

## Question 1

## Question 9

If $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathrm{N}(\mu, \sigma^2)$ with unknown $\mu$ and $\sigma^2$, then a $\gamma\%$ confidence interval for $\mu$ is given by

$$\mathcal{I} = \Big( \ \bar{X} - t_\gamma(n) \cdot S/\sqrt{n} \ , \ \bar{X} + t_\gamma(n) \cdot S/\sqrt{n} \ \Big),$$

where $t_\gamma(n) = T_{n-1}^{-1}((1+\gamma)/2)$ is the $(1+\gamma)/2$th quantile of the $t$ distribution with df $= n-1$ and $S$ is the sample standard deviation. The length of this confidence interval is given by

$$\Delta = \max(\mathcal{I}) - \min(\mathcal{I}) = \Big(\bar{X} + t_\gamma(n) \cdot S/\sqrt{n}\Big) - \Big(\bar{X} - t_\gamma(n) \cdot S/\sqrt{n}\Big) = 2t_\gamma(n) \cdot S/\sqrt{n}.$$

The squared length is then given by $\Delta^2 = 4t_\gamma^2(n) \cdot S^2/n$. Because the sample variance is an unbiased estimator for $\sigma^2$, we have $\mathbb{E}[\Delta^2] = \mathbb{E}\big[4t_\gamma^2(n) \cdot S^2/n\big] = 4t_\gamma^2(n) \cdot \sigma^2/n$. We now set $\mathbb{E}[\Delta^2] < \sigma^2/2$, and after some cancellations, we see that we need $t_\gamma^2(n)/n < 1/8$. There is no way to find a closed-form expression for this, so we will have to check the value of $t_\gamma^2(n)/n$ for increasing values of $n$. I set up a `while` loop in `R` to solve for it, and when $\gamma = 0.9$, we find that $n = 24$ is the smallest value of $n$ such that $\mathbb{E}[\Delta^2] < \sigma^2/2$.

## Question 10

Let $\boldsymbol{X} \overset{\text{iid}}{\sim} \mathrm{N}(\theta, \sigma^2)$, where $\theta$ is unknown and $\sigma^2$ is known, and we assume prior that $\theta \sim \mathrm{N}(\mu, \nu^2$, where both $\mu$ and $\nu^2$ are known.

(a) Since normal distributions are are conjugate to normal sampling, it follows that $\theta \,|\, \mathbf{x} \sim \mathrm{N}(\tilde{\mu}, \tilde{\sigma}^2)$, where

$$\tilde{\mu} = \frac{\sigma^2\mu + n\nu^2\bar{x}}{\sigma^2 n\nu^2} \quad \text{and} \quad \tilde{\sigma}^2 = \frac{\sigma^2\nu^2}{\sigma^2 + n\mu^2}.$$

We also know that $(\theta \,|\, \mathbf{x} - \tilde{\mu})/\tilde{\sigma} \sim \mathrm{N}(0,1)$, and so a 95% confidence interval for $\theta \,|\, \mathbf{x}$ is given by

$$\mathcal{I} = \Big( \ \tilde{\mu} - \Phi^{-1}(0.975) \cdot \tilde{\sigma} \ , \ \tilde{\mu} + \Phi^{-1}(0.975) \cdot \tilde{\sigma} \ \Big).$$

(b) We can think of our interval $\mathcal{I}$ as a function of $\nu^2$. To examine what happens to $\mathcal{I}(\mu^2)$ as $\nu^2 \to \infty$, we will first look at $\tilde{\mu}$ and $\tilde{\sigma}$. Using L'Hopital's rule, we have

$$\lim_{\nu^2 \to \infty} \tilde{\mu} = \lim_{\nu^2 \to \infty} \frac{\sigma^2\mu + n\nu^2\bar{x}}{\sigma^2 n\nu^2} = \lim_{\nu^2 \to \infty} \frac{n\bar{x}}{n} = \bar{x},$$
$$\lim_{\nu^2 \to \infty} \tilde{\sigma} = \lim_{\nu^2 \to \infty} \sqrt{\frac{\sigma^2\nu^2}{\sigma^2 + n\mu^2}} = \sqrt{\lim_{\nu^2 \to \infty} \frac{\sigma^2\nu^2}{\sigma^2 + n\mu^2}} = \sqrt{\lim_{\nu^2 \to \infty} \frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}},$$

and so $\mathcal{I}(\nu^2) \to \big( \ \bar{x} - \Phi^{-1}(0.975) \cdot \sigma/\sqrt{n} \ , \ \bar{x} + \Phi^{-1}(0.975) \cdot \sigma/\sqrt{n} \ \big)$, which is a 95% confidence interval for $\theta$.