# Pricing Options using the COS method

Dimitris Nikitas

January 15, 2025

**Abstract**

This document presents the mathematical derivation and formulation of the multivariate COS method for pricing European basket options, as outlined in the works of Fang and Oosterlee [1] and Ruijter and Oosterlee [2]. Equations (1) to (10) contain the complete development, including a new generalized expression for the COS coefficients (Equation (10)) which simplifies the summation structure by avoiding alternating sign terms ($\pm$) as in the original 2D COS paper. While the resulting expressions are more intricate, they provide a systematic and compact framework applicable to higher dimensions.

We consider a basket put option on assets modeled by Geometric Brownian Motion. Using the arithmetic mean of the asset prices and a strike price $K = 1$, we apply the derived COS expansion method to compute the option price. A numerical implementation in Python is provided separately, capable of handling arbitrary dimensions $N$, although computational cost becomes significant for $N > 10$, consistent with observations in [2]. When run with $N = 2$, the implementation also demonstrates probability density function reconstruction through COS expansion visualizations.

# References

[1] Fang, Fang and Oosterlee, Cornelis W. *A novel pricing method for European options based on Fourier cosine series expansions.* SIAM Journal on Scientific Computing, 2009.

[2] Ruijter, Marlies J. and Oosterlee, Cornelis W. *Two-dimensional Fourier cosine series expansion method for pricing financial options.* SIAM Journal on Scientific Computing, 2012.

# Part 1: Approximation of Discounted Expected Payoffs using COS method

For example, the value of a European option with a pay-off function $g(\cdot)$ is given by the risk neutral option evaluation formula:

$$v(t_0, x) = e^{-r\Delta t} \int_{\mathbb{R}^u} g(y) \cdot f(y \mid x)\, dy \tag{1}$$

Where:

- $r$: risk-free rate
- $\Delta t = T - t_0$: time to expiration
- $x = (x_1, x_2, \ldots, x_u)$: current state
- $f(y \mid x) = f(y_1, y_2, \ldots, y_u \mid x_1, x_2, \ldots, x_u)$: conditional density function to approximate
- $g(y)$: payoff for each possible outcome $y$

## Fourier Pair

- Forward integral:

$$\phi(\omega) = \int_{\mathbb{R}^u} e^{i\langle x,\omega\rangle} f(x)\, dx \tag{2}$$

(with $\langle x, \omega\rangle$ the dot product)

- Inverse integral:

$$f(x) = \frac{1}{(2\pi)^u} \int_{\mathbb{R}^u} e^{-i\langle \omega,x\rangle} \phi(\omega)\, d\omega \tag{3}$$

## COS Expansion

For a function supported on $[0, \pi]^u$, the cosine expansion reads:

$$f(\omega_1, \omega_2, \ldots, \omega_u) = \sum_{k=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_u=0}^{\infty} A_{k_1,k_2,\ldots,k_u} \cos(k_1\omega_1)\cos(k_2\omega_2)\cdots\cos(k_u\omega_u)$$

Here, the coefficients are given by:

$$A_{k_1,k_2,\ldots,k_u} = \left(\frac{2}{\pi}\right)^u \int_{[0,\pi]^u} f(\omega_1, \ldots, \omega_u) \cos(k_1\omega_1)\cdots\cos(k_u\omega_u)\, d\omega_1 \cdots d\omega_u \tag{4}$$

*Note:* $\sum^*$ indicates that the first term of the sum has half the weight.

## Change of Variables for Finite Domain

For a function supported on the interval $[a_k, b_k]$, the Fourier cosine expansion is obtained via the variable change:

$$\theta_k = \frac{x_k - a_k}{b_k - a_k}\pi, \quad \Rightarrow \quad d\theta_k = \frac{\pi}{b_k - a_k}\, dx_k$$

Hence, the function can be expanded as follows:

$$f(x_1, x_2, \ldots, x_u) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_u=0}^{\infty} A_{k_1,k_2,\ldots,k_u} \cos\left(k_1 \frac{x_1 - a_1}{b_1 - a_1}\pi\right) \cdots \cos\left(k_u \frac{x_u - a_u}{b_u - a_u}\pi\right) \tag{5}$$

With the cosine expansion coefficients given by:

$$A_{k_1,k_2,\ldots,k_u} = \prod_{k=1}^{u} \left(\frac{2}{b_k - a_k}\right) \int_{a_1}^{b_1} \cdots \int_{a_u}^{b_u} f(x_1, \ldots, x_u) \cos\left(k_1 \frac{x_1 - a_1}{b_1 - a_1}\pi\right) \cdots \cos\left(k_u \frac{x_u - a_u}{b_u - a_u}\pi\right) dx_1 \cdots dx_u \tag{6}$$

## Assumption 1

Suppose $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_u, b_u] \subset \mathbb{R}^u$ is chosen such that the truncated integral approximates the infinite counterpart very well, i.e.

$$\bar{\phi}(\omega) = \int_{a_1}^{b_1} \cdots \int_{a_u}^{b_u} e^{i\langle \omega, x \rangle} f(x)\, dx \approx \int_{\mathbb{R}^u} e^{i\langle \omega, x \rangle} f(x)\, dx = \phi(\omega) \tag{7}$$

## Let's denote:

$$\psi_i(x_i) = k_i \pi \frac{x_i - a_i}{b_i - a_i}$$

## Proposition 1: Product of Cosines to Sum

The following identity allows us to rewrite a product of cosines as a sum of cosines:

$$\prod_{i=1}^{u} \cos(q_i) = \frac{1}{2^{u-1}} \sum_{\substack{\varepsilon_i = \pm 1 \\ \varepsilon_1 = +1}} \cos\left(\sum_{i=1}^{u} \varepsilon_i q_i\right) \tag{8}$$

(Sketch of the proof is provided is provided in the end).

In our case, equation (8) corresponds to:

$$\prod_{i=1}^{u} \cos\left(\psi_i(x_i)\right) = \frac{1}{2^{u-1}} \sum_{\substack{\varepsilon_i = \pm 1 \\ i \geq 1 \\ \varepsilon_1 = +1}} \cos\left(\psi_1(x_1) + \sum_{i=2}^{u} \varepsilon_i \psi_i(x_i)\right) \tag{9}$$

—

**Thus, the coefficient expression (6) becomes:**

$$A_{k_1,\ldots,k_u} = \prod_{i=1}^{u}\left(\frac{2}{b_i-a_i}\right)\int_{a_1}^{b_1}\cdots\int_{a_u}^{b_u} f(x_1,\ldots,x_u)\cdot\frac{1}{2^{u-1}}\sum_{\substack{\varepsilon_i=\pm1\\ i\geq1\\ \varepsilon_1=+1}}\cos\left(\psi_1(x_1)+\sum_{i=2}^{u}\varepsilon_i\psi_i(x_i)\right)dx_1\cdots dx_u$$

Using Euler's identity and rearranging:

$$=\prod_{i=1}^{u}\left(\frac{2}{b_i-a_i}\right)\int_{a_1}^{b_1}\cdots\int_{a_u}^{b_u} f(x)\cdot\frac{1}{2^{u-1}}\sum_{\substack{\varepsilon_i=\pm1\\ i>1}}\Re\left[e^{i\left(\psi_1(x_1)+\sum_{i=2}^{u}\varepsilon_i\psi_i(x_i)\right)}\right]dx$$

Note that

$$\psi_i(x_i)=k_i\pi\cdot\frac{x_i-a_i}{b_i-a_i}\Rightarrow\psi_i(x_i)=\frac{k_i\pi x_i}{b_i-a_i}-\frac{k_i\pi a_i}{b_i-a_i}$$

So, by linearity of the exponential and the integral, constants (independent of $x$) can be factored out of the integral.

—

*Remark: Due to linearity and independence from $x$, some terms can be removed from the integral.*

Combining previous steps, we write:

$$\begin{aligned}A_{k_1,\ldots,k_u} &= \prod_{i=1}^{u}\left(\frac{2}{b_i-a_i}\right)\cdot\sum_{\substack{\varepsilon_i=\pm1\\ i>1}}\frac{1}{2^{u-1}}\\ &\cdot\Re\left[\int_{a_1}^{b_1}\cdots\int_{a_u}^{b_u} f(x)\cdot\exp\left(j\left(k_1\pi\frac{x_1}{b_1-a_1}+\sum_{i=2}^{u}\varepsilon_i k_i\pi\frac{x_i}{b_i-a_i}\right)\right)dx\right]\\ &\cdot\exp\left(-j\left(k_1\pi\frac{a_1}{b_1-a_1}+\sum_{i=2}^{u}\varepsilon_i k_i\pi\frac{a_i}{b_i-a_i}\right)\right)\end{aligned}$$

Using Assumption 1 from earlier:

$$\begin{aligned}A_{k_1,\ldots,k_u} \approx \prod_{i=1}^{u}\left(\frac{2}{b_i-a_i}\right)\cdot\frac{1}{2^{u-1}}\sum_{\substack{\varepsilon_i=\pm1\\ i>1}}\Re\left[\phi\left(\frac{k_1\pi}{b_1-a_1},\ldots,\frac{\varepsilon_u k_u\pi}{b_u-a_u}\right)\right]\\ \cdot\exp\left(-j\left(k_1\pi\frac{a_1}{b_1-a_1}+\sum_{i=2}^{u}\varepsilon_i k_i\pi\frac{a_i}{b_i-a_i}\right)\right)\end{aligned}\tag{10}$$

**Summary:** Equation (10) provides a compact formula for the coefficients $A_{k_1,\ldots,k_u}$.

Using these coefficients, we can approximate $f(x_1, x_2, \ldots, x_u)$ on the basis of the truncated series of cosines in Equation (5).

If $\phi$ takes real values, we only need the real part of the exponent in (10).

—

## Proposition 1: Sketch of Proof for Product of Cosines

**For** $u = 2$**:**

$$\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right)$$

**For** $u = 3$**:**

$$\cos(a)\cos(b)\cos(c) = \frac{1}{4}\left[\cos(a+b+c) + \cos(a+b-c) + \cos(a-b+c) + \cos(a-b-c)\right]$$

**For** $u = 4$**:**

$$\cos(q_1)\cos(q_2)\cos(q_3)\cos(q_4) = \frac{1}{8}\sum_{\substack{\varepsilon_i=\pm 1\\ i>1}} \cos(q_1 + \varepsilon_2 q_2 + \varepsilon_3 q_3 + q_4)$$

This result can be rigorously proved using mathematical induction and the basic identity for $\cos(a)\cos(b)$.

## Part 2: Application to Bucket Option Characteristic Function

We define the characteristic function $\varphi(t)$ of a bucket option as:

$$\varphi(t) = \mathbb{E}\left[e^{it\cdot\frac{1}{N}\sum_j S_j(\tau)}\right] = \int_{\mathbb{R}^N} e^{it\cdot\frac{1}{N}\sum_j S_j(\tau)} \cdot p_w(\omega_1,\ldots,\omega_N)\, d\omega_1\cdots d\omega_N \tag{14}$$

Now we use the COS method approximation for the multivariate density function $p(\cdot)$, which becomes:

$$p_w(\omega_1,\ldots,\omega_N) = \sum_{k_1,\ldots,k_N}{}' A_{k_1,\ldots,k_N} \prod_{j=1}^{N} \cos\left(k_j\pi\cdot\frac{\omega_j - a}{b-a}\right) \tag{12}$$

Thus, the characteristic function becomes the following.

$$\begin{aligned}\varphi(t) = \sum_{k_1,\ldots,k_N}{}' A_{k_1,\ldots,k_N} \int_{\mathbb{R}^N} \exp\left(it\cdot\frac{1}{N}\sum_{j=1}^{N}\left(\left(r-\frac{\sigma_j^2}{2}\right)T + \sigma_j\omega_j\right)\right)\\ \cdot\prod_{j=1}^{N}\cos\left(k_j\pi\cdot\frac{\omega_j - a}{b-a}\right)\, d\omega_1\cdots d\omega_N\end{aligned} \tag{13}$$

### Decoupling the Integral

Using the identity:

$$e^{q_1+q_2+\cdots+q_N} = \prod_{j=1}^{N} e^{q_j}$$

we can write:

$$\varphi(t) = \sum_{k_1,\ldots,k_N}{}' A_{k_1,\ldots,k_N} \int_{\mathbb{R}^N} \prod_{j=1}^{N}\left[e^{it\cdot\frac{1}{N}\left(\left(r-\frac{\sigma_j^2}{2}\right)T+\sigma_j\omega_j\right)}\cdot\cos\left(k_j\pi\cdot\frac{\omega_j - a}{b-a}\right)\right] d\omega_1\cdots d\omega_N \tag{1}$$

### 6. Splitting the Products into 1D Integrals

Now we combine the product terms into one:

$$\varphi(t) = \sum_{k_1,\ldots,k_N}{}' A_{k_1,\ldots,k_N} \int_{\mathbb{R}^N} \prod_{j=1}^{N}\left[e^{it\cdot\frac{1}{N}\left(\left(r-\frac{\sigma_j^2}{2}\right)T+\sigma_j\omega_j\right)}\cdot\cos\left(k_j\pi\cdot\frac{\omega_j - a}{b-a}\right)\right] d\omega_1\cdots d\omega_N \tag{2}$$

This becomes:

$$\varphi(t) = \sum_{k_1,\ldots,k_N}{}^{\prime} A_{k_1,\ldots,k_N} \prod_{j=1}^{N} \int_{\mathbb{R}} e^{it\cdot\frac{1}{N}\left(\left(r-\frac{\sigma_j^2}{2}\right)T+\sigma_j\omega_j\right)} \cdot \cos\left(k_j\pi\cdot\frac{\omega_j-a}{b-a}\right)\, d\omega_j \tag{3}$$

**Remark:** This is now a product of 1D integrals.

We approximate the integrals by bounding them over $[a,b]$:

$$\varphi(t) \approx \sum_{k_1,\ldots,k_N}{}^{\prime} A_{k_1,\ldots,k_N} \prod_{j=1}^{N} \int_{a}^{b} e^{it\cdot\frac{1}{N}\left(\left(r-\frac{\sigma_j^2}{2}\right)T+\sigma_j\omega_j\right)} \cdot \cos\left(k_j\pi\cdot\frac{\omega_j-a}{b-a}\right)\, d\omega_j \tag{4}$$

$$= \sum_{k_1,\ldots,k_N}{}^{\prime} A_{k_1,\ldots,k_N} \prod_{j=1}^{N} I_j(t) \tag{15}$$

## Numerical Evaluation of $I_j(t)$

For each fixed $t$, we can numerically evaluate $I_j(t)$ for $j = 1,\ldots,N$.

Since the integrand is complex-valued, we split it into real and imaginary parts:

$$I_j(t) = \int_a^b \Re[q(\omega_j)]\, d\omega_j \quad + \quad \int_a^b \Im[q(\omega_j)]\, d\omega_j \tag{15'}$$

## Alternative Formulation of $\varphi(t)$

We start from:

$$\varphi(t) = \mathbb{E}\left[e^{it\cdot\frac{1}{N}\sum_j S_j(\tau)}\right], \quad S_j(\tau) = S_{j,0}\cdot e^{\left(r-\frac{\sigma_j^2}{2}\right)T+\sigma_j W_j(\tau)} \tag{5}$$

$$= \mathbb{E}\left[e^{it\cdot\frac{1}{N}\sum_j S_{j,0}\cdot e^{\left(r-\frac{\sigma_j^2}{2}\right)T+\sigma_j W_j(\tau)}}\right] \tag{6}$$

$$= \int_{\mathbb{R}^N} e^{it\cdot\frac{1}{N}S_0 e^{\left(r-\frac{\sigma_j^2}{2}\right)T}\sum_j e^{\omega_j}} \cdot p_\omega(\omega_1,\ldots,\omega_N)\, d\omega_1\cdots d\omega_N \tag{7}$$

Here, $p_\omega$ is the multivariate density of the $\omega_j$, which we approximate using the COS method.

**Aggregating the Variables** We now reduce the dimensionality of the integral by considering the aggregated variable:

$$\bar{W}(\tau) = \sum_j e^{W_j(\tau)} \in \mathbb{R}_+^*$$

Then the characteristic function becomes:

$$\varphi(t) = \int_{\mathbb{R}_+} \exp\left[ it \cdot \frac{1}{N} S_0 \cdot e^{\left(r - \frac{\sigma^2}{2}\right)T} \cdot \bar{W} \right] \cdot p_{\bar{W}(\tau)}(\bar{W})\, d\bar{W} \tag{8}$$

This reduces the dimensionality but introduces an unknown distribution $p_{\bar{W}(\tau)}$.

## Notes

- $W_j \sim \mathcal{N} \Rightarrow e^{W_j} \sim$ Lognormal
- But: sum of dependent lognormals $\sum_j e^{W_j}$ is not lognormal
- We must estimate $p_{\bar{W}(\tau)}$ via:
  1. Monte Carlo simulation
  2. Assuming simplifications for $p_{\bar{W}(\tau)}$

## Pricing the Basket Option

To find the price/value of the basket option, we follow the reasoning of the paper:

$$\mathbb{P} = \frac{1}{2}(b' - a')e^{-rT} \sum_{k=0}^{N-1} F_k \cdot V_k \tag{16}$$

- $a', b'$ correspond to the truncation limits when using the 1-dimensional COS method with respect to the characteristic function $\varphi(t)$.
  - $a' = 0, \quad b' = 1$, selected after Monte Carlo estimation
- $V_k$ are the cosine series coefficients for the payoff function:

$$g(x) = \max(k - x, 0) \quad \text{(Put option)}$$

- $F_k$ are the cosine series coefficients for the COS method w.r.t. $\varphi(t)$, using the 1-dimensional COS method.

The coefficients $F_k$ are calculated numerically as:

$$f_k = \frac{2}{b' - a'} \Re \left\{ \varphi\left( \frac{k\pi}{b' - a'} \right) \cdot e^{-i \frac{k\pi a'}{b' - a'}} \right\} \tag{9}$$

$$= 2 \cdot \Re \left\{ \varphi\left(k\pi\right) \right\} \quad \text{if } b' = 1, a' = 0 \tag{17}$$

## COS Coefficients for Payoff Function and Monte Carlo Estimator

The coefficients $V_k$ are calculated as:

$$V_0 = 2\int_0^1 (k - x)\,dx \tag{10}$$

$$V_k = 2\int_0^1 (k - x)\cos(k\pi x)\,dx, \quad k \geq 1 \tag{18}$$

**Evaluation of** $V_0$:

$$V_0 = 2\left[kx - \frac{x^2}{2}\right]_0^1 = 2\left(k - \frac{1}{2}\right) = 2k - 1 \quad \text{with } k = 1 \Rightarrow V_0 = 1$$

**Evaluation of** $V_k$ for $k \geq 1$:

$$\begin{aligned} V_k &= 2\left(k\int_0^1 \cos(k\pi x)\,dx - \int_0^1 x\cos(k\pi x)\,dx\right) \\ &= -2\cdot\frac{\cos(k\pi) - 1}{(k\pi)^2} = -2\cdot\frac{(-1)^k - 1}{(k\pi)^2} = \frac{1 - (-1)^k}{(k\pi)^2} \end{aligned}$$

This uses the fact that:

$$\max(k - x, 0) = k - x \quad \text{for } x \in [0, k]$$

**Monte Carlo Estimator**

$$\mathbb{P}_{\mathrm{MC}} = e^{-rT}\cdot\frac{1}{N_{\mathrm{MC}}}\sum_{i=1}^{N_{\mathrm{MC}}}\max\{k - B_T^{(i)}, 0\} \tag{19}$$

Where:

$$B_T = \frac{1}{N}\sum_{j=1}^{N} S_j(T)$$

Alternatively (as a limit when $N_{\mathrm{MC}} \to \infty$):

$$\mathbb{P}_{\mathrm{MC}} \approx e^{-rT}\cdot\mathbb{E}\left[\max\{k - B_T, 0\}\right] \quad \text{(Payoff function)}$$

## Notation and Parameter Explanation

- $N$ – Number of assets in the basket.
- $S_j(T)$ – Price of asset $j$ at maturity time $T$.
- $S_{j,0}$ – Initial price of asset $j$.
- $\sigma$ – Volatility (assumed same for all assets in simplified case).
- $r$ – Risk-free interest rate.
- $T$ – Time to maturity.
- $K$ – Strike price of the option.
- $W_j(T)$ – Brownian motion (Wiener process) for asset $j$.
- $B_T$ – Value of the basket at time $T$: $B_T = \frac{1}{N} \sum_{j=1}^{N} S_j(T)$.
- $\phi(t)$ – Characteristic function of the basket payoff distribution.
- $f(x)$ – Probability density function of the basket value.
- $[a, b]$ – Truncation domain for COS expansion (chosen to contain most of the probability mass).
- $A_k$, $A_{k_1,\ldots,k_u}$ – COS coefficients in 1D and multivariate case, respectively.
- $\varepsilon_i \in \{-1, +1\}$ – Auxiliary sign variables used for multidimensional Fourier transform symmetry.
- $k, k_i$ – Frequency indices for cosine expansion (number of terms in Fourier series).
- $u$ – Dimension of the integral (equal to number of underlying assets).
- $j$ – Imaginary unit, $j = \sqrt{-1}$.
- $P_{\text{COS}}$ – Option price computed using the COS method.
- $F_k$ – Fourier-Cosine coefficients of the probability density function.
- $V_k$ – Fourier-Cosine coefficients of the payoff function.
- $P_{\text{MC}}$ – Option price estimated by Monte Carlo simulation.
- $N_{\text{MC}}$ – Number of Monte Carlo paths/simulations.
- $B_T^{(i)}$ – Basket value in the $i$-th Monte Carlo path.