Computational Methods for Ideal Magnetohydrodynamics

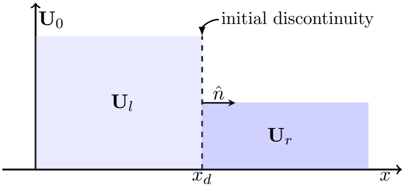
Andrew Kercher

A defense of the dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy
George Mason University

25 August 2014

Introduction

Riemann Problems

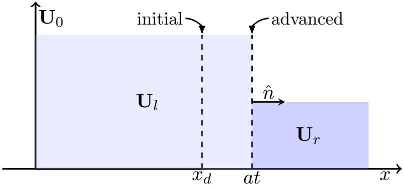


$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial \mathbf{x}} = \mathbf{0}\ ,$$

$$\mathbf{U}_0 = \begin{cases} \mathbf{U}_l & \text{if} \quad x < x_d, \\ \mathbf{U}_r & \text{if} \quad x > x_d, \end{cases}$$

- Initial discontinuity at x_d separates two constant states.

Riemann Problems



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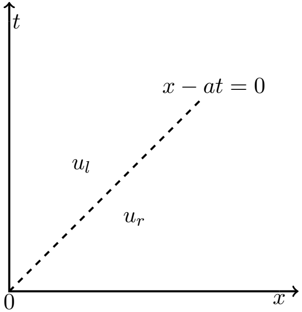
- Solution at time t.
- Wave speed: a.

$$\mathbf{U} = \begin{cases} \mathbf{U}_l & \text{if} \quad x < at, \\ \mathbf{U}_r & \text{if} \quad x > at, \end{cases}$$

Riemann Problems



$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \mathbf{0} \ ,$$

$$u_0 = \begin{cases} u_l & \text{if} \quad x < x_d, \\ u_r & \text{if} \quad x > x_d, \end{cases}$$

- Linear advection, $\mathbf{U}_l = u_l$ and $\mathbf{U}_r = u_r$.
- Self-similar solution in (x, t)-plane.

$$u = \begin{cases} u_l & \text{if} \quad x < at, \\ u_r & \text{if} \quad x > at, \end{cases}$$

Riemann Problems: Systems of linear equations

- Hyperbolic $\rightarrow$ real eigenvalues, $\lambda_1, \ldots, \lambda_n$, and linearly independent right eigenvectors, $\mathbf{r}^1, \ldots, \mathbf{r}^n$.
- Strictly hyperbolic $\rightarrow$ distinct eigenvalues.

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} = 0,$$

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- Strictly hyperbolic $\rightarrow$ distinct eigenvalues.
- Characteristic form, $\mathbf{W} = \mathbf{l}^i \mathbf{U}$, where $\mathbf{l}^i$ is the matrix of left eigenvectors.

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} = 0,$$

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{\Lambda} \frac{\partial \mathbf{W}}{\partial x} = 0,$$

Riemann Problems: Systems of linear equations

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- n scalar equations.

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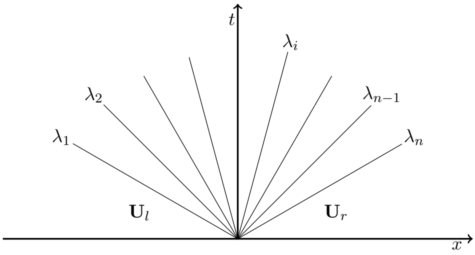
$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{\Lambda} \frac{\partial \mathbf{W}}{\partial x} = 0,$$

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0,$$

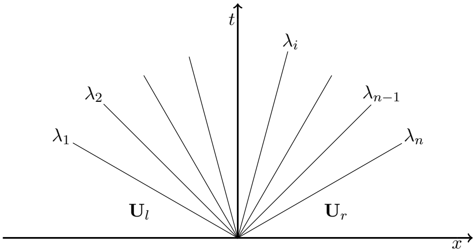
Riemann Problems: Systems of linear equations



- Self-similar solutions for a system of linear waves.

- Difference across each wave $\alpha_i = \mathbf{l}_i \cdot (\mathbf{U}_l - \mathbf{U}_r)$.

Riemann Problems: Systems of linear equations



- Self-similar solutions for a system of linear waves.
- Information is global, state across wave is known.

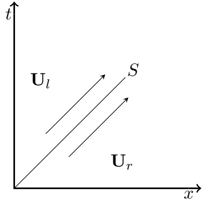
- Difference across each wave $\alpha_i = \mathbf{l}_i \cdot (\mathbf{U}_l - \mathbf{U}_r)$.
- In terms of conservative variables

$$\mathbf{U} = \mathbf{U}_l + \sum_{i=1}^{m} \alpha_i \mathbf{r}^i$$

or

$$\mathbf{U} = \mathbf{U}_r - \sum_{i=m+1}^{n} \alpha_i \mathbf{r}^i$$

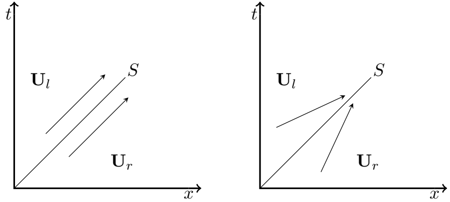
Riemann Problems: Systems of nonlinear equations



$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0,$$

- linear wave, characteristics parallel.

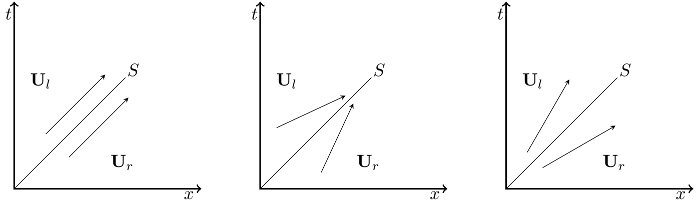
Riemann Problems: Systems of nonlinear equations



$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0,$$

- linear wave, characteristics parallel.
- Shock wave, compression, characteristics converge.

Riemann Problems: Systems of nonlinear equations



$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0,$$

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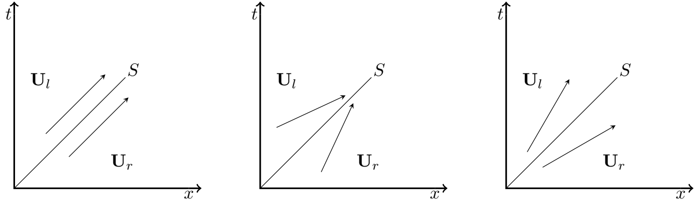
- Shock wave, compression, characteristics converge.
- Rarefaction wave, expansion, characteristics diverge.

Riemann Problems: Systems of nonlinear equations



$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0,$$

- linear wave, characteristics parallel.
- Shock wave, compression, characteristics converge.
- Rarefaction wave, expansion, characteristics diverge.
- Information only local, state across wave unknown.

Compressible hydrodynamics

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \ , \\
\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v} \otimes \mathbf{v} + p_g] = 0 \ , \\
\frac{\partial E}{\partial t} + \nabla \cdot [(E + p_g)\, \mathbf{v}] = 0 \ ,
\end{aligned}
$$

where the energy density is defined as

$$
E = \frac{p_g}{\gamma - 1} + \frac{\rho v^2}{2} \ ,
$$

Compressible hydrodynamics

The Euler equations are strictly hyperbolic.
1D conservative form

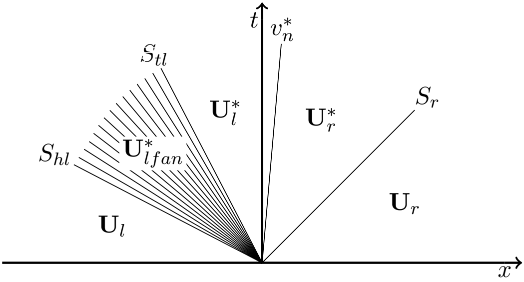
$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0.$$

Jacobian $\mathbf{J}(\mathbf{U}) = \partial \mathbf{F}/\partial \mathbf{U}$ has three real and distinct eigenvalues:

$$\begin{aligned}
\lambda_3 &= v_n + a : \text{rarefaction or shock}\\
\lambda_2 &= v_n \qquad : \text{contact discontinuity}\\
\lambda_1 &= v_n - a : \text{rarefaction or shock}
\end{aligned}$$

where $a = \sqrt{\gamma p_g/\rho}$ is the speed of sound.

Compressible hydrodynamics

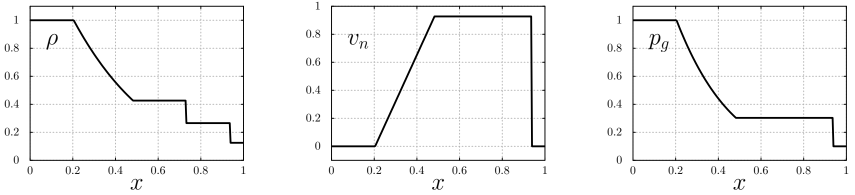


$\lambda_3 = S_l$: rarefaction

$\lambda_2 = v_n^*$: contact discontinuity

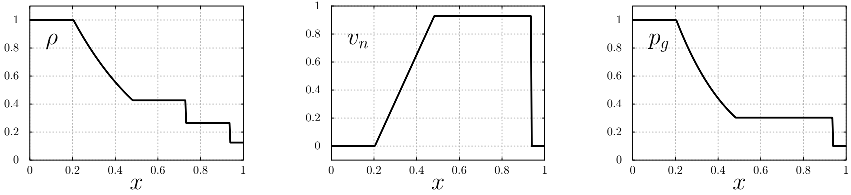
$\lambda_1 = S_r$: shock

Compressible hydrodynamics



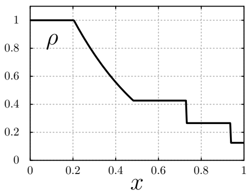
- Exact solution to Sod shock tube problem [4].

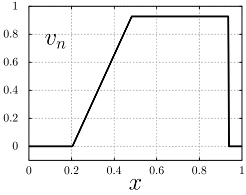
Compressible hydrodynamics

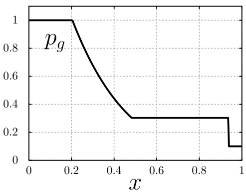


- Exact solution to Sod shock tube problem [4].
- Rarefaction head: $x \approx 0.2$.

Compressible hydrodynamics

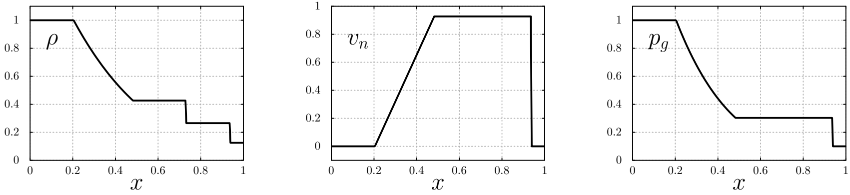






- Exact solution to Sod shock tube problem [4].
- Rarefaction head: $x \approx 0.2$.
- Rarefaction tail: $x \approx 0.45$.

Compressible hydrodynamics



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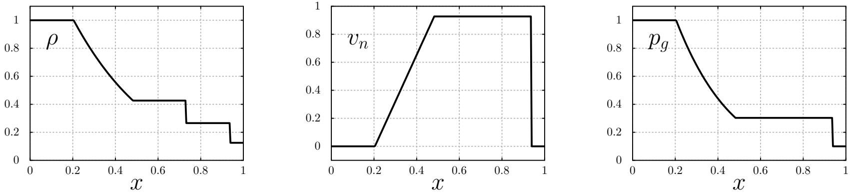
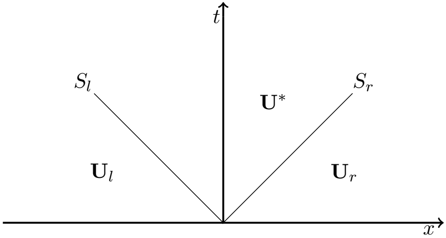
- Contact discontinuity: $x \approx 0.72$.

Compressible hydrodynamics



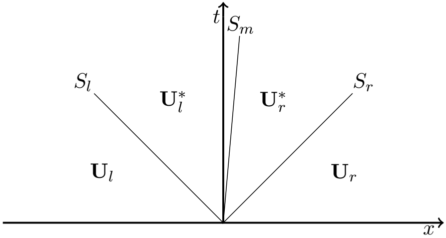
- Exact solution to Sod shock tube problem [4].
- Rarefaction head: $x \approx 0.2$.
- Rarefaction tail: $x \approx 0.45$.
- Contact discontinuity: $x \approx 0.72$.
- Shock: $x \approx 0.95$.

HLL family of approximate Riemann solvers



- The solver of Harden-Lax-van Leer-Einfeldt (HLLE) [1] assumes a two state solution.
- The solution approximated as weighted average of the left and right states.

HLL family of approximate Riemann solvers



- The solver of Harden-Lax-van Leer-Einfeldt (HLLE) [1] assumes a two state solution.
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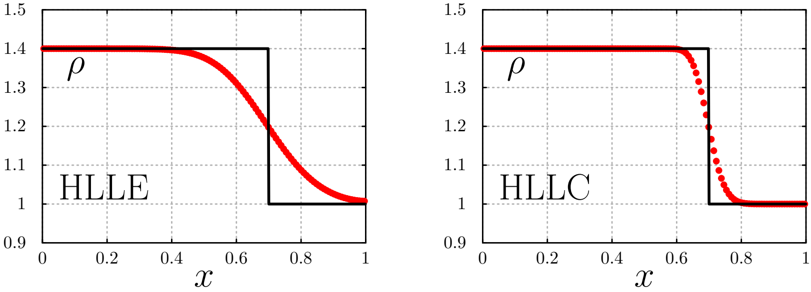
- Toro, Spruce, and Speares [5] included the contact discontinuity and termed the solver HLLC.

HLL family of approximate Riemann solvers



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- The solution approximated as weighted average of the left and right states.
- Toro, Spruce, and Speares [5] included the contact discontinuity and termed the solver HLLC.
- HLLC less diffuse at contact discontinuity.

Ideal magnetohydrodynamics

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \ ,$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \left[\rho \mathbf{v} \otimes \mathbf{v} + \left(p_g + \frac{B^2}{2} \right) \underline{\mathbf{I}} - \mathbf{B} \otimes \mathbf{B} \right] = 0 \ ,$$

$$\frac{\partial E}{\partial t} + \nabla \cdot \left[\left(E + p_g + \frac{B^2}{2} \right) \mathbf{v} - \mathbf{v} \cdot \mathbf{B} \otimes \mathbf{B} \right] = 0 \ , \text{ and}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot [\mathbf{v} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{v}] = 0 \ ,$$

where the energy density is defined as

$$E = \frac{p_g}{\gamma - 1} + \frac{\rho v^2}{2} + \frac{B^2}{2} \ ,$$

Ideal magnetohydrodynamics

The ideal MHD equations are non-strictly hyperbolic. 1D conservative form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0.$$

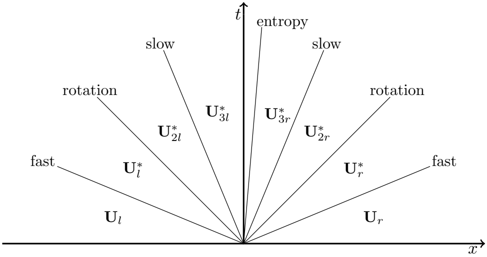
Jacobian $\mathbf{J}(\mathbf{U}) = \partial \mathbf{F}/\partial \mathbf{U}$ has seven real, but not necessarily distinct eigenvalues:

$$\begin{aligned}
v_n &: \text{contact or tangential discontinuity (entropy)},\\
v_n \pm c_s &: \text{slow rarefaction or shock},\\
v_n \pm c_a &: \text{rotational discontinuity (Alfvén), and}\\
v_n \pm c_f &: \text{fast rarefaction or shock},
\end{aligned}$$

$$c_{f,s}^2 = \frac{1}{2}\left[a^2 + c_a^2 + c_t^2 \pm \sqrt{\left(a^2 + c_a^2 + c_t^2\right)^2 - 4a^2 c_a^2}\right],$$

$$c_a^2 = \frac{B_n^2}{\rho}, \text{ and } c_t^2 = \frac{B_t^2}{\rho}.$$

Ideal magnetohydrodynamics



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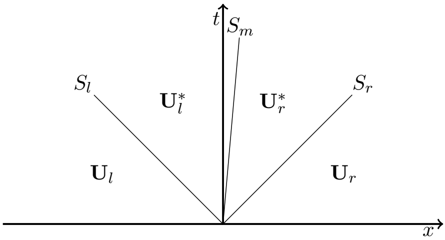
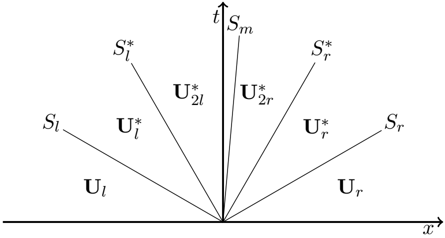
Extension of HLLC for MHD



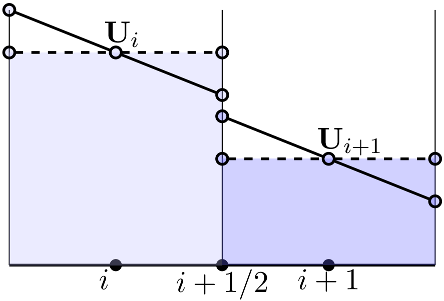
- HLLD [3] is an MHD extention to HLLC.

Extension of HLLC for MHD



- HLLD [3] is an MHD extention to HLLC.
- Captures all linear discontinuities, contact and rotational.

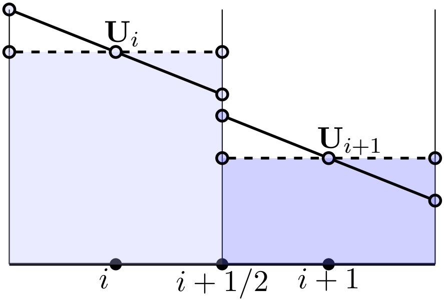
Finite volume discretization



- Conservative variables are located at the cell centers.
- Flux is evaluated at cell interface.
- Flux contribution of each face added to cell.

$$\mathbf{U}_{\text{cell}}^{n+1} = \mathbf{U}_{\text{cell}}^{n} - \frac{\delta t}{V_{\text{cell}}} \sum_{\text{faces}} \mathbf{F} \cdot \mathbf{s}$$

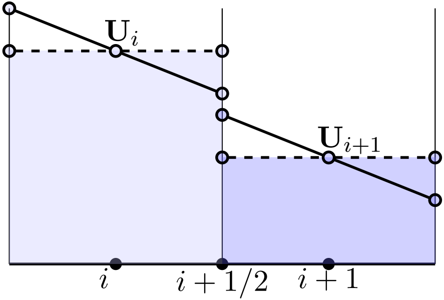
Finite volume discretization



- Conservative variables are located at the cell centers.
- Flux is evaluated at cell interface.
- In 1D

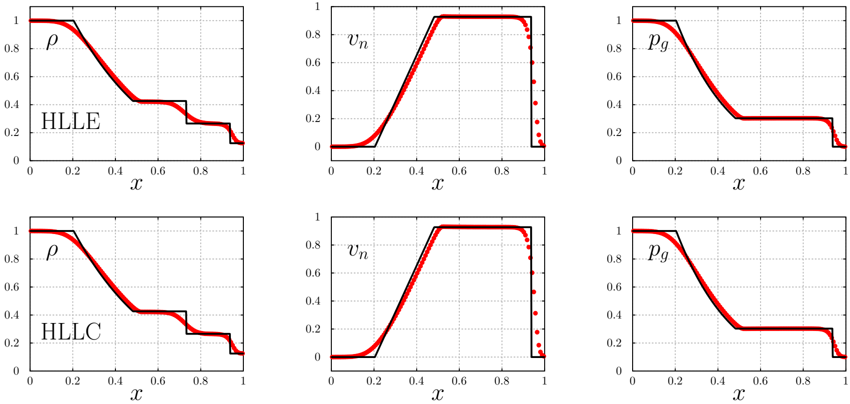
$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\delta t}{\delta x}\left(\mathbf{F}_{i+1/2}^n - \mathbf{F}_{i-1/2}^n\right)$$

Higher order extensions



- Increase accuracy and preserve monotonicity.
- Total variation diminishing (TVD).
- Limit slope to ensure no new extrema are created.

Higher order extensions



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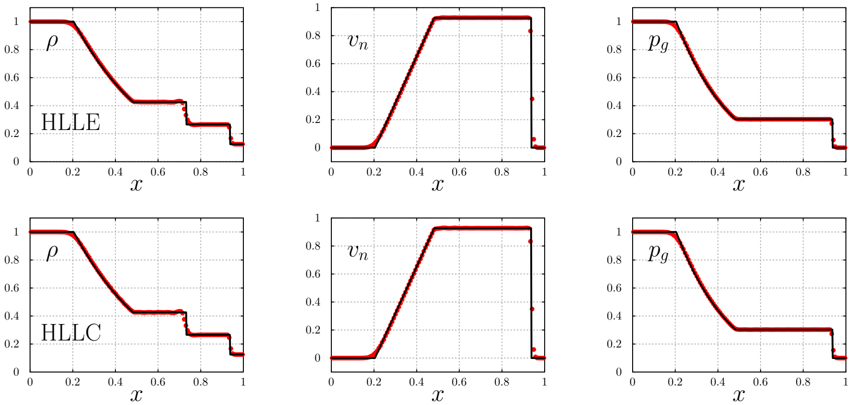

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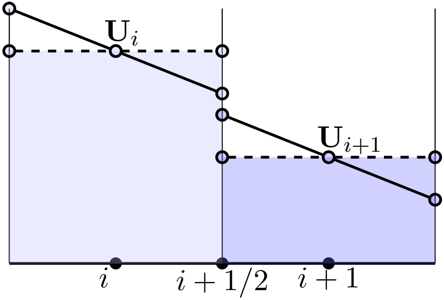
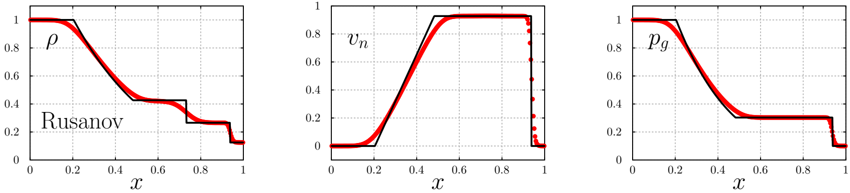


- Increase accuracy and preserve monotonicity.
- Flux corrected transport.
- Limit fluxes so that no new extrema are created.

Higher order extensions



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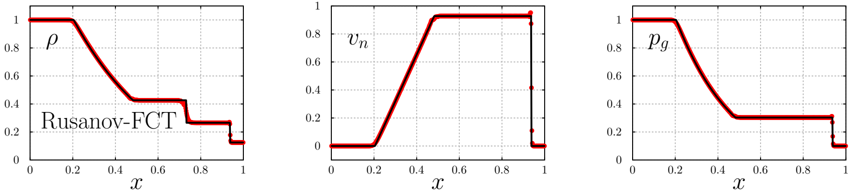

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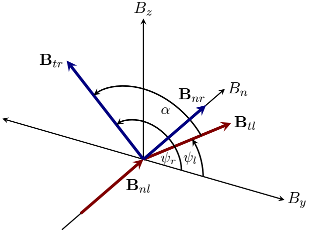
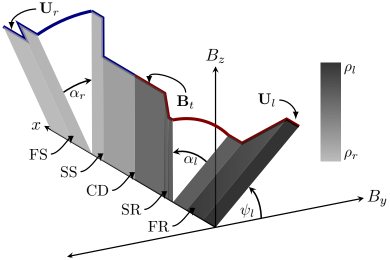
Riemann problems for ideal MHD



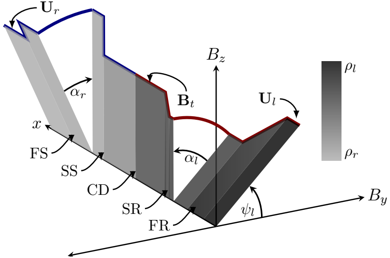
- Initial discontinuity separates two states.
- Rotation angle arctan B_z/B_y.
- The initial twist angle $\alpha = \psi_r - \psi_l$.

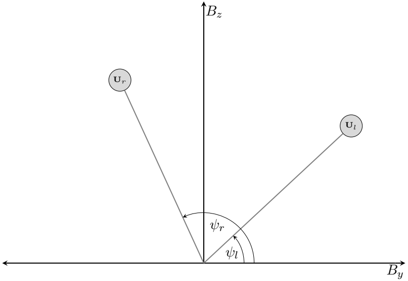
Riemann problems for ideal MHD



- Seven waves propagate away from initial discontinuity creating eight distinct states.

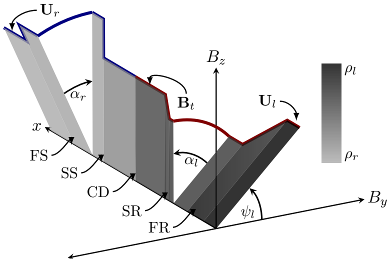
Riemann problems for ideal MHD

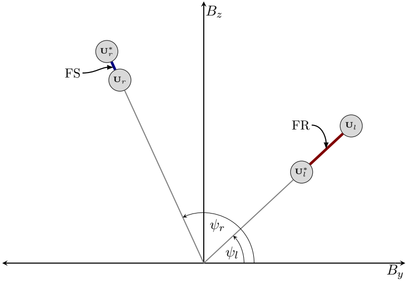




- Regular waves only alter the magnitude or orientation of B_t.

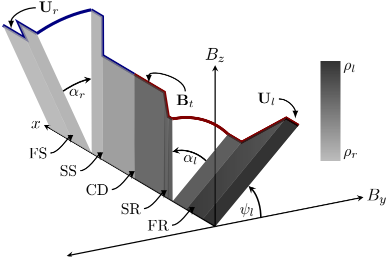
Riemann problems for ideal MHD

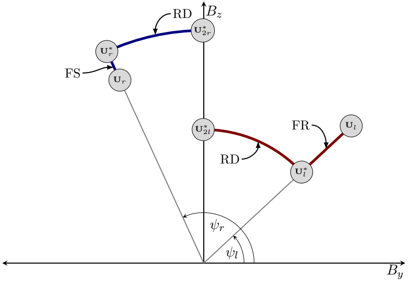




- Regular waves only alter the magnitude or orientation of B_t.
- B_t increases across fast shock, decreases across fast rarefaction.

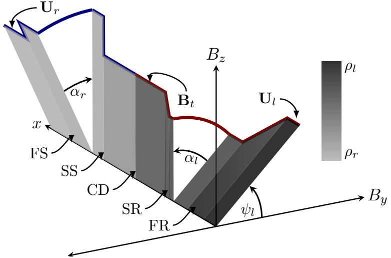
Riemann problems for ideal MHD

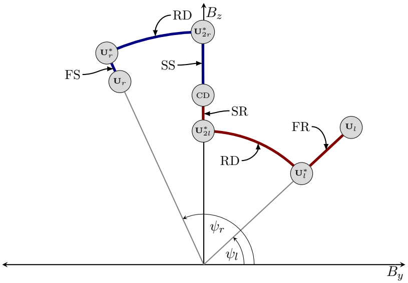




- Regular waves only alter the magnitude or orientation of B_t.
- B_t increases across fast shock, decreases across fast rarefaction.
- B_t changes orientation across a rotational discontinuity.

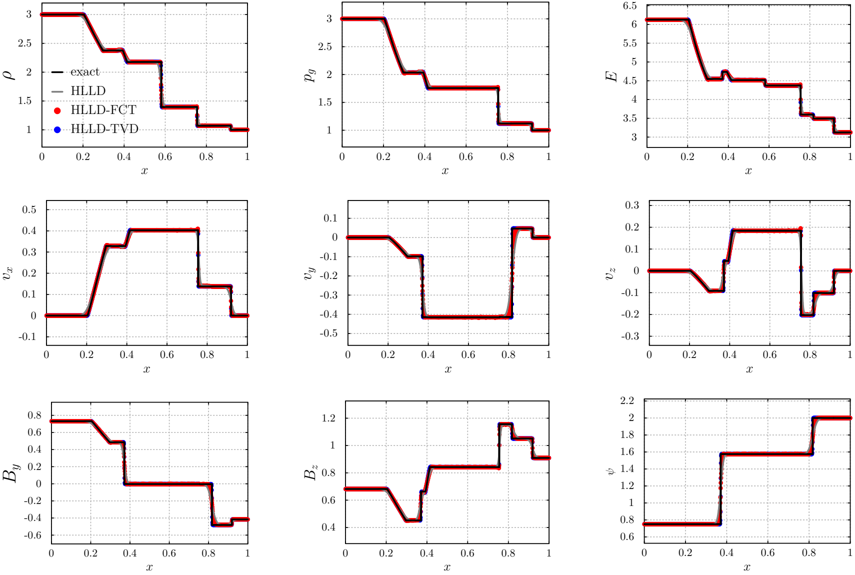
Riemann problems for ideal MHD



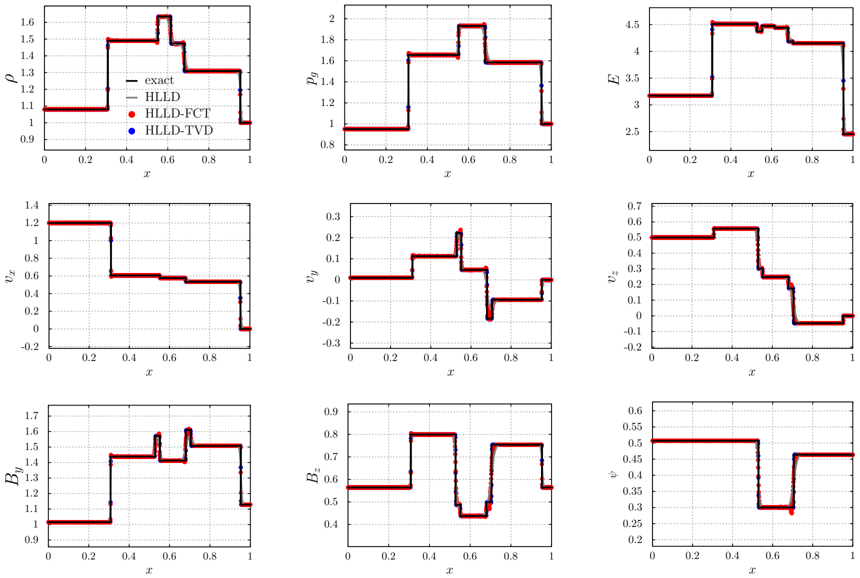


- Regular waves only alter the magnitude or orientation of B_t.
- B_t increases across fast shock, decreases across fast rarefaction.
- B_t changes orientation across a rotational discontinuity.
- B_t decreases across slow shock, increases across slow rarefaction.

Ideal MHD test problems



Ideal MHD test problems



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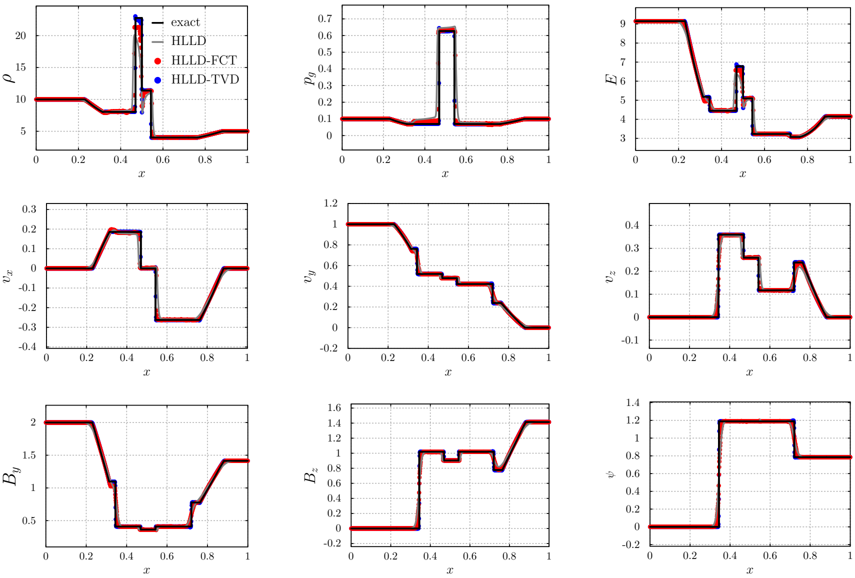

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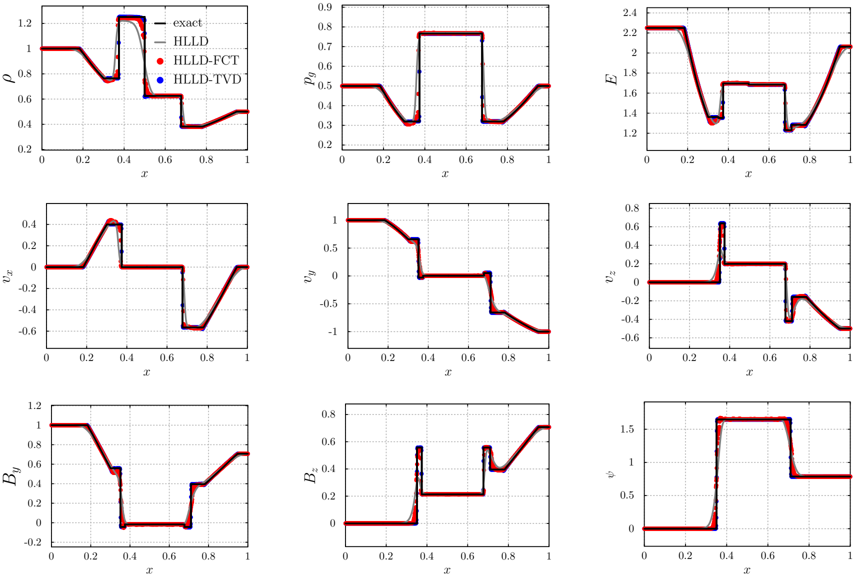

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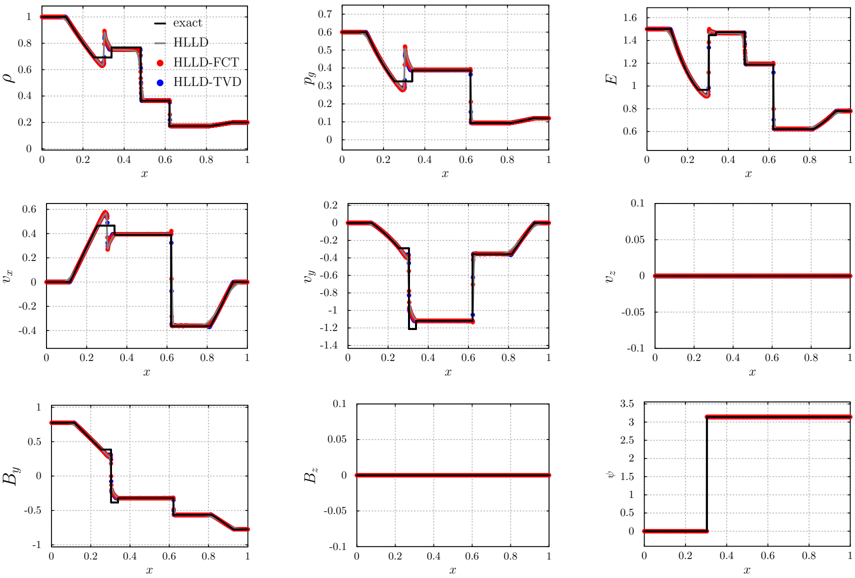

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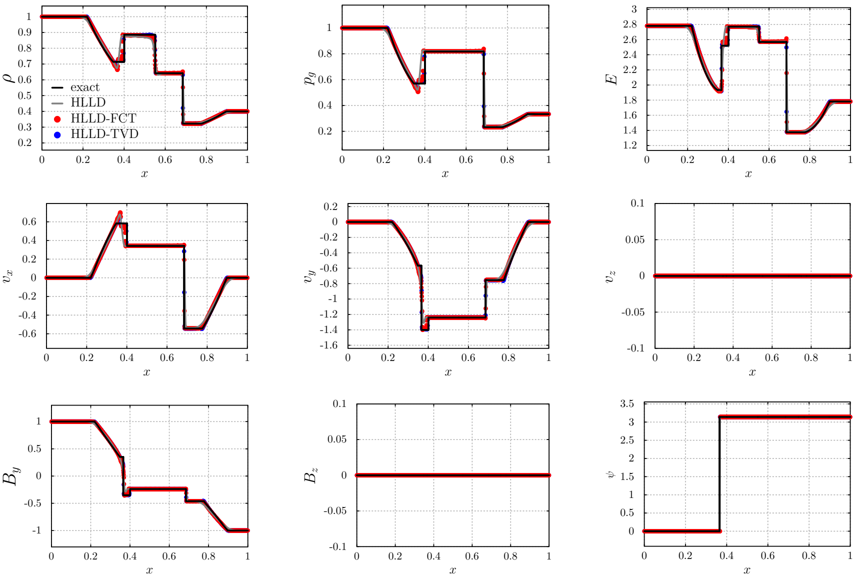

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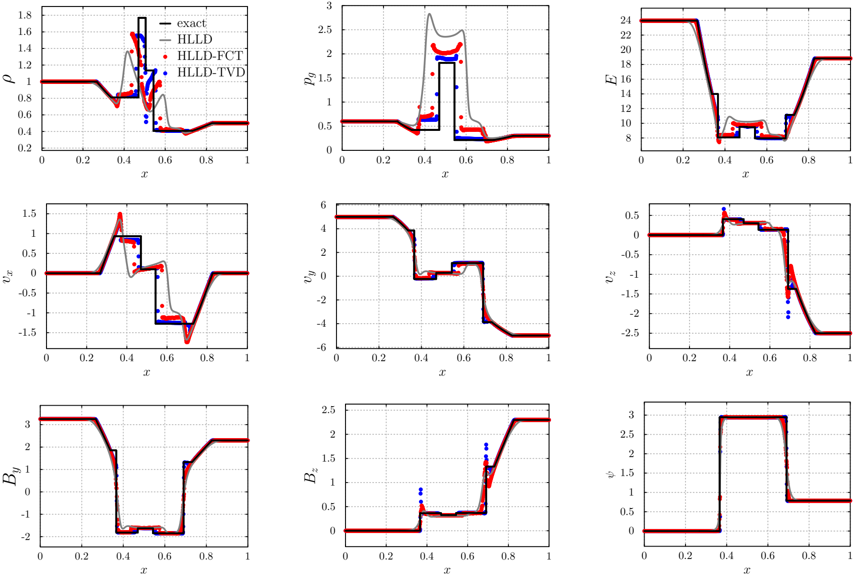
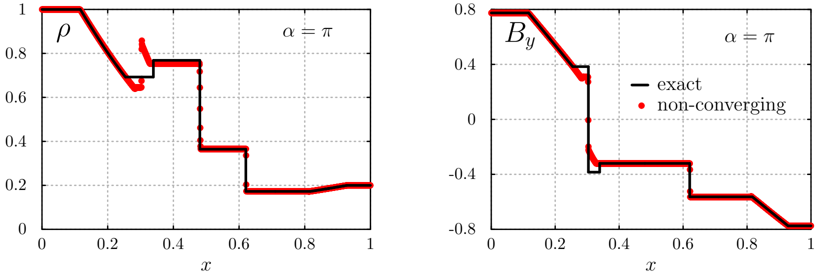
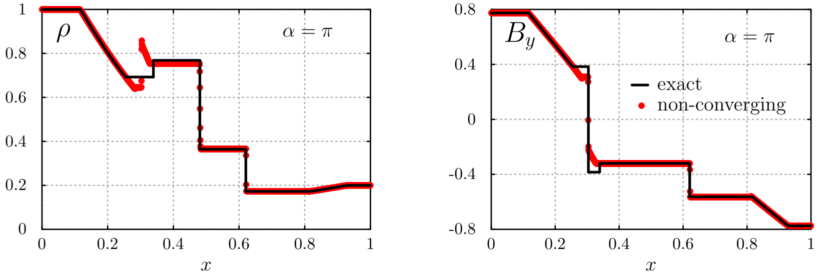


Non-unique solutions



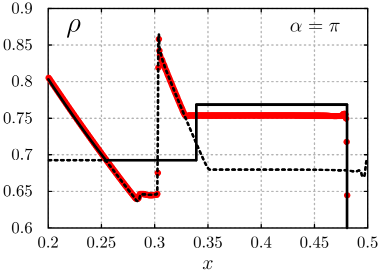
- Solutions to coplanar Riemann problems of ideal MHD are non-unique.
- At $x = 0.303$, rotational discontinuity $\rightarrow$ compound wave.

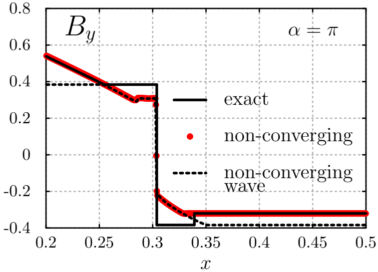
Non-unique solutions



- Solutions to coplanar Riemann problems of ideal MHD are non-unique.
- At $x = 0.303$, rotational discontinuity $\rightarrow$ compound wave.
- Compound wave is composed of an intermediate shock and a slow rarefaction.
- Physical?
- Unstable for under small perturbations [2].
- Satisfies jump conditions in the coplanar case.

Non-unique solutions





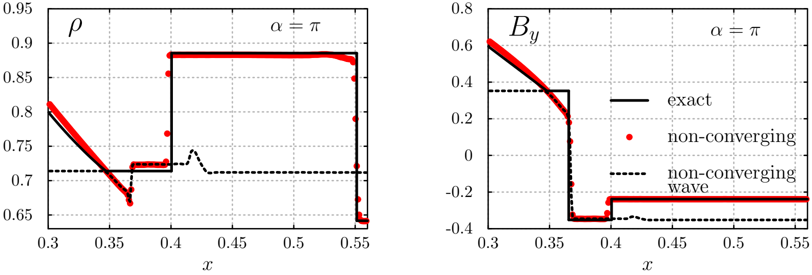
- An intermediate shock is a non-regular wave that changes the orientation of the magnetic field.
- Slow compound wave: super-Alfvénic $\rightarrow$ sub-slow.
- In shock frame upstream:

$$v_n = 1.0174, c_a = 0.9661, c_f = 1.1295$$

- In shock frame downstream:

$$v_n = 0.7341, c_s = 0.8033, c_a = 0.8222$$

Non-unique solutions



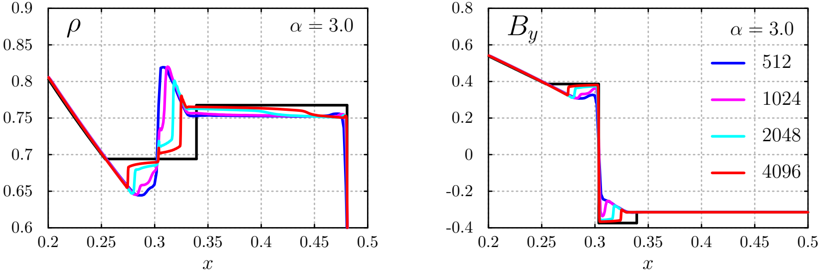
- An intermediate shock is a non-regular wave that changes the orientation of the magnetic field.
- Fast compound wave: super-fast $\rightarrow$ sub-Alfvénic.
- In shock frame upstream:

$$v_n = 1.5868, c_a = 1.5041, c_f = 1.5401$$

- In shock frame downstream:

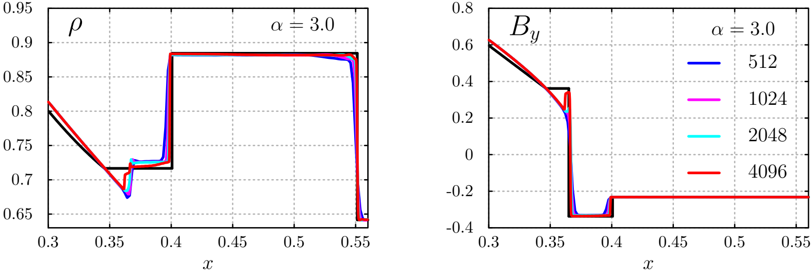
$$v_n = 1.4638, c_s = 1.0766, c_a = 1.4698$$

Psuedo-convergence



- Near-coplanar case, compound waves present at lower resolutions.
- Psuedo-convergence: increasing resolution produces convergence to regular wave solution [6].
- The compound waves begin to break apart between 1024 and 2048 grid points.
- As $\pi - \alpha$ decreases, a finer resolution is required to recover the regular wave solution.
- When $\pi = \alpha$, compound wave at all resolutions.

Psuedo-convergence



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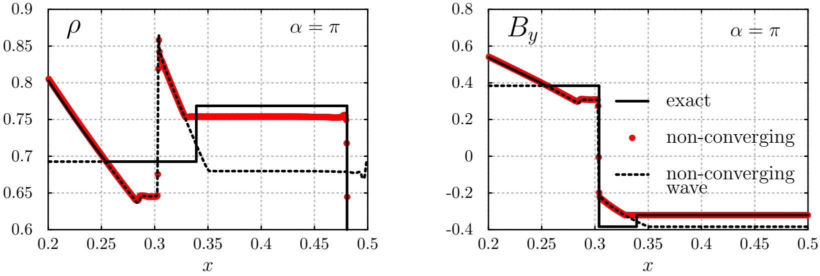
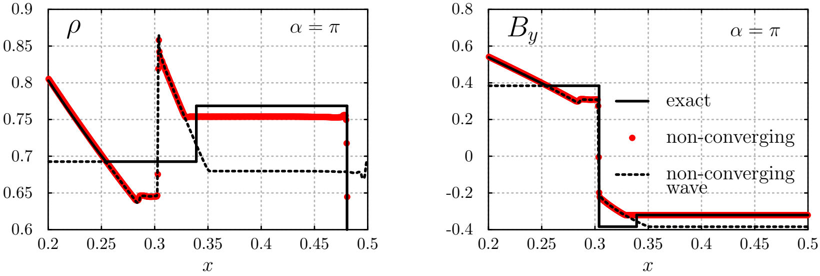
Compound Wave modification



- Compound waves are product of numerical diffusion for near coplanar problems.
- Reduce diffusion to recover correct solution. How?

Compound Wave modification



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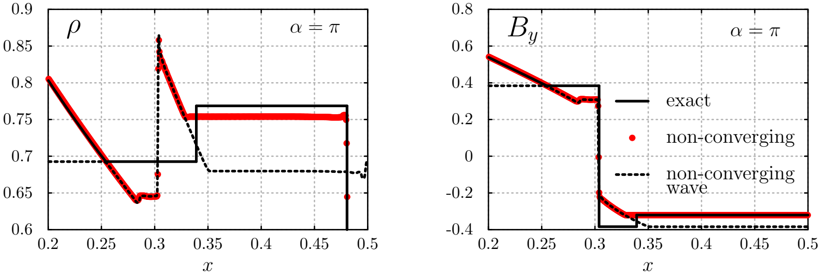

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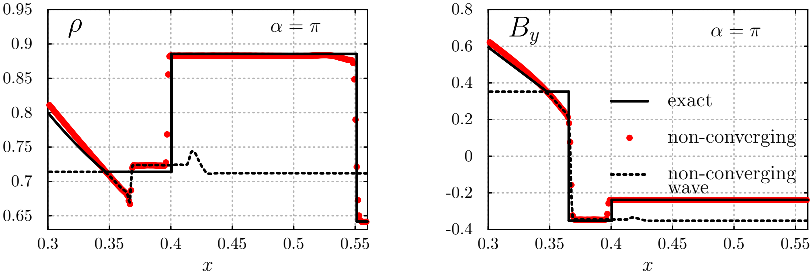

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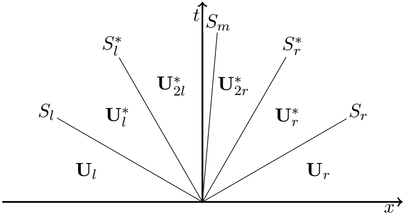
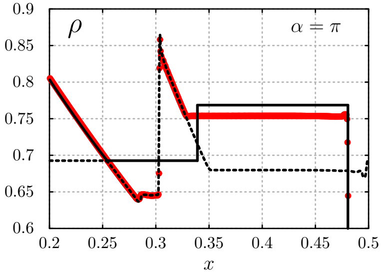
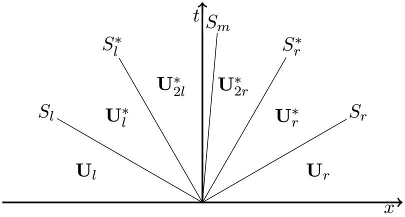
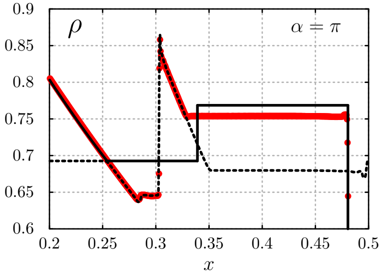


- Use HLLD to approximate intermediate states upstream and downstream of rotational discontinuity, i.e., $\mathbf{U}_l^*$ and $\mathbf{U}_{2l}^*$.



Compound Wave modification





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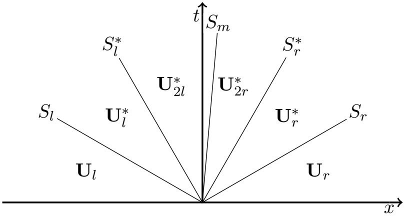
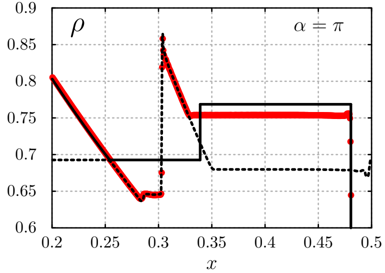
Compound Wave modification





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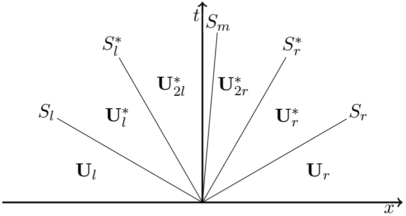
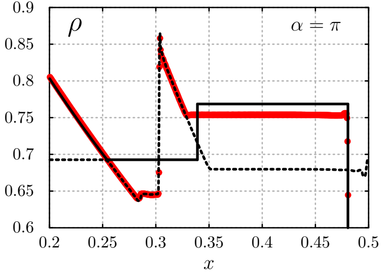
- Reduce the contribution of $\mathbf{F}^c = \mathbf{F}(\mathbf{U}_l^*, \mathbf{U}_{l2}^*)$ to the total flux.

Compound Wave modification





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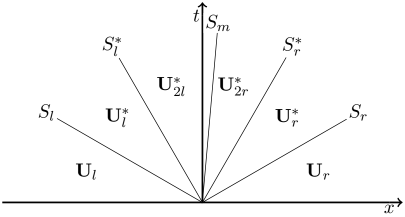
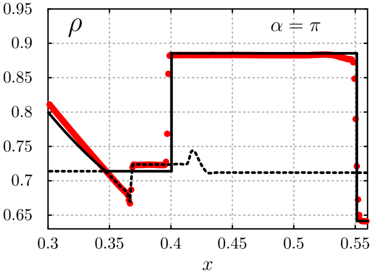
- $\mathbf{F} = \mathbf{F}(\mathbf{U}_l, \mathbf{U}_r) - A\mathbf{F}^c$, where $A < 0.5$.

Compound Wave modification





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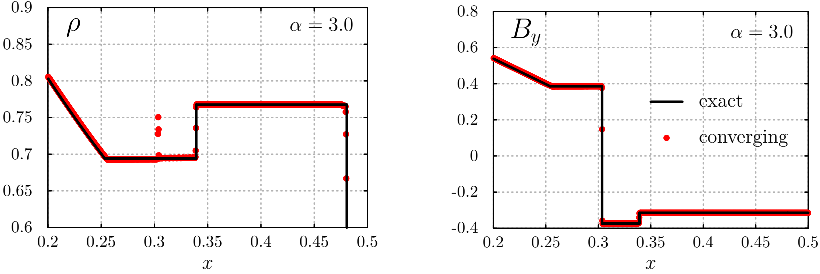
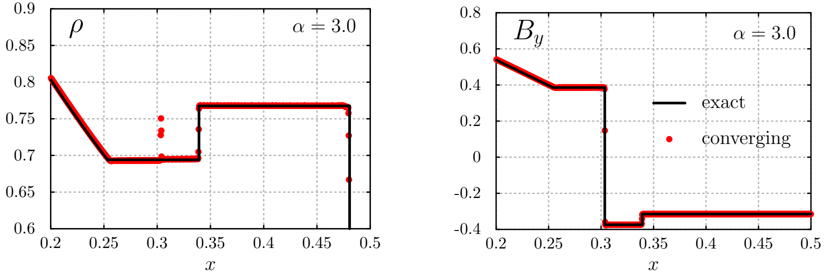
CWM Results



- CWM recovers the correct solution upstream and downstream of rotational discontinuity.
- Transition across rotational discontinuity is unresolved.
- Resolve it?

CWM Results



- CWM recovers the correct solution upstream and downstream of rotational discontinuity.
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- Simple but non-conservative.

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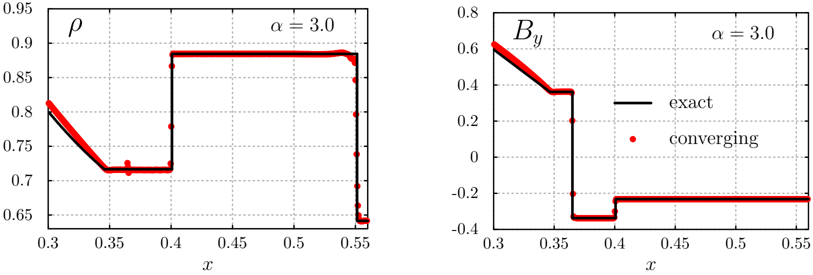

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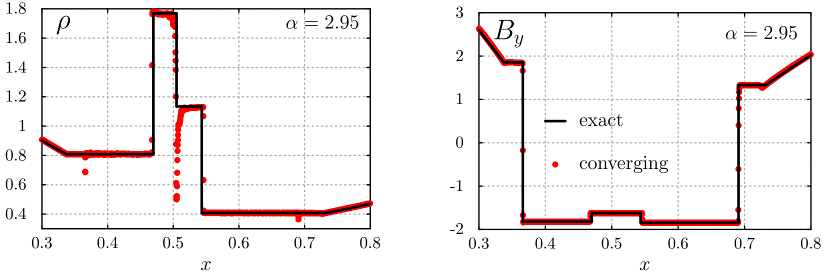

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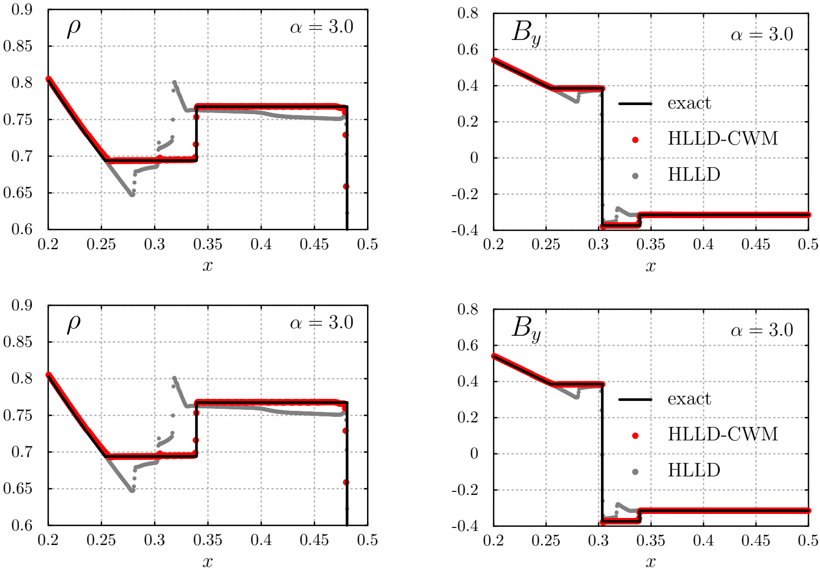
CWM Resolving the transition



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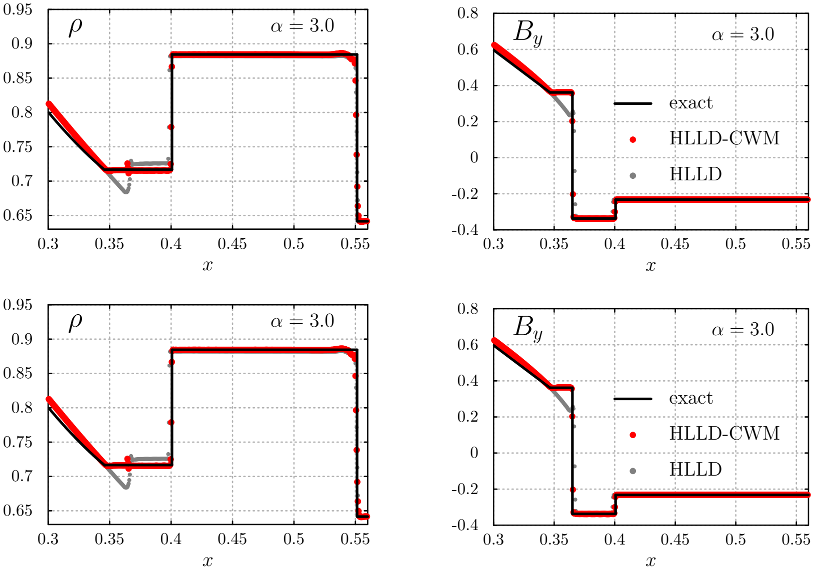

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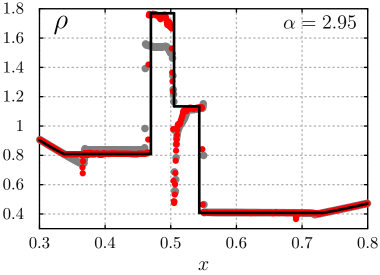

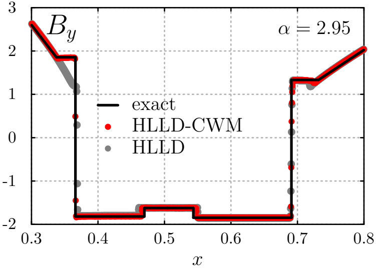

- In the case of weaker intermediate shocks, there is very little deviation at the rotational discontinuity.

Conclusion

- FAST! HLLD intermediate states are already calculated, flux evaluation nearly free.

Bibliography I

[1] B. Einfeldt. “On Godunov-Type Methods for Gas Dynamics”. In: *SIAM J. Numer. Anal.* 25 (1988), pp. 294–318.

[2] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii. *Electrodynamics of continuous media*. 3rd. 1984.

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[6] M. Torrilhon. “Non-uniform convergence of finite volume schemes for Riemann problems of ideal magnetohydrodynamics”. In: *J. Comp. Phys.* 192 (Nov. 2003), pp. 73–94.