Brief Thoughts on Irrational Numbers

Adam Kercheval

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As far as history is aware, the first people to prove the existence of irrational numbers were the Greeks. This makes sense, because they had defined a number system that was ripe for glimmers of irrationality. They thought, and I would say reasonably (rationally?) so, that all numbers could relate to each other in a countable, integer-based way. Of course there are fractions, which are not integers, but those can be represented by some integer divided by another, like $\frac{1}{2}$ or $\frac{2}{5}$ or $\frac{587}{7}$ or even $\frac{1}{1000029}$. For every point along the number line, that is, for <u>all</u> numbers, there is some integer-over-integer fraction that must equal them. This was the Greeks' impression, and while it was not wrong, it was incomplete.

The story goes that a group of Pythagoreans, geometry-obsessed ancient Greek mathematicians, were in a boat, discussing numbers. The Pythagoreans were particularly interested in a piece of geometry called *comeasurability*, which can be thought of as being able to measure one thing in another thing's terms. For example, if there were a short length of rope and a long length of rope, and they were comeasurable, one could say that the shorter rope is some fraction of the longer rope's length (and vice versa). Maybe it's $\frac{59}{77}$ or $\frac{14}{83}$, but somewhere there must be a way to divide up the long rope and assemble enough pieces to exactly match the length of the short rope. Again, the rational idea: every number must be representable as a fraction.

Anyway, the Greeks were on their boat, and they were discussing the Pythagorean theorem: $a^2 + b^2 = c^2$, to represent the side lengths of right triangles. One inquisitive Pythagorean pointed out that if a and b are equal to 1, c is equal to $\sqrt{2}$, the square root of two. It's a tricky number, the Pythagorean said, but surely it must be representable as a reduced fraction, according to the rules of what numbers are. If

$$\sqrt{2} = \frac{x}{y}$$

then

$$2 = \frac{x^2}{y^2}$$

and

$$2y^2 = x^2$$

making x^2, and therefore x, an even number. Every even number is two times another number, so we can say that $x = 2z$. So,

$$\begin{aligned} 2y^2 &= (2z)^2 \\ 2y^2 &= 4z^2 \\ y^2 &= 2z^2 \end{aligned}$$

and we run into our problem. If $y^2 = 2z^2$, then y^2, and therefore y, must be an even number. But we have just shown x to be an even number as well. So we are insisting that $\sqrt{2} = \frac{x}{y}$, where $\frac{x}{y}$ is a fully-reduced fraction and where x and y are both even, walking us right into a contradiction: no fraction with an even numerator and an even denominator can be fully reduced, yet no matter how we might try to find a fraction that represents $\sqrt{2}$, we run into the same issue. Therefore, we resolve that while $\sqrt{2}$ is a real number, it cannot be represented as any fraction of integers. It exists outside the Pythagorean, "rational" view of numbers; it is irrational.

And, as the story goes, after the Pythagorean proved this to the other nautical geometricians, they picked him up and threw him over the side of the boat in outrage, and he drowned. *Q. E. D.*

Unfortunately for the murderous Greeks, irrational numbers do definitely exist, whether or not they are are compatible with a fraction-centric number system. And it gets better: not only are irrational numbers real, they're everywhere, too. The square root of every number that is not a perfect square is an irrational number. In fact, set theorists and their ilk have proved that the set of all rational numbers is countably infinite, while the corresponding set of irrational numbers is uncountably infinite. In other words, there are provably more irrational numbers than rational ones. The Greeks had a cool system, but it missed the mark on a whole lot.

The square roots are certainly an interesting bunch of irrational numbers, but within the irrationals exists an even more exclusive group: the transcendentals. These irrational numbers are numbers that, unlike many irrationals, cannot be represented as the solution to a polynomial equation. They are non-repeating, non-terminating numbers that cannot be represented as a fraction of integers, and no arithmetic or algebraic manipulation makes their nature any clearer. Two transcendental numbers that are widely known are π and e.

π is the ratio of a circle's diameter to its circumference, and while I personally have always been baffled by the mysterious nature of π I've been willing to accept it as something that is both readily observable in the real world and quite necessary to it. e, however, is more confounding. It seems to have appeared out of thin air and despite being nothing more than a number between two and three it acts like a peculiar key to all sorts of mathematical manipulations.

e first showed up in explorations of interest, as the solution to

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n$$

In English, that expression represents the return on a \$1 investment compounded by 100% n times. Compounding interest on $1 is fruitful if $n = 1$, more fruitful if $n = 2$, and more fruitful still if $n = 1000$, but as n approaches infinity the result of compounding interest infinitely in this way levels off, oddly enough, at exactly e dollars.

In calculus, the land of ever-changing everything, $f(x) = e^x$ pushes back: the first, second, and millionth derivative of e^x is e^x, every time. In complex analysis, the value of $e^{i\pi}$ is -1, a famous mashup of atypical numbers that line up perfectly. e is everywhere, we know that it is, but that doesn't necessarily mean it's any clearer what exactly e is or why it is that all these things are true.

And well, so what? The existence of irrational numbers, and transcendental numbers on top of them, doesn't change much about the day-to-day experience of most people. In fact, for everyday, commonplace use, the Greeks actually had an excellent system that to this day remains thorough and applicable to the human experience. The irrational numbers, then, aside from their many applications in mathematics, only exist because they must exist, and because they always have existed. That's the important part: just because we were only (relatively) recently able to see and discuss irrational and transcendental numbers does not mean they haven't always been there, patiently hiding, and in the case of e, quietly holding enormous and fundamental keys to mathematical relationships.

So in the memory of the famously drowned Pythagorean, we should all use the existence of irrational numbers as a reminder to hold ourselves to a higher standard, and to always seek out more information. No matter how complete our worldview may feel, it never is. Let us all keep looking.

Dedicated to Will Mairs