STABILITY OF SOLIDS:
FROM STRUCTURES TO MATERIALS

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FOREWORD

Stability is a fascinating topic in solid mechanics that has its roots in the celebrated Euler column buckling problem, which first appeared in 1744. Over the years advances in technology have led to the study of ever more complicated structures first in civil and subsequently in mechanical engineering applications. Aerospace applications, most notably failure of solid propellant rockets, led the way in the 1950s. Problems associated with materials and electronics industries came on stage in the 1970s and 1980s, starting with instabilities associated with thin films and phase transformations in shape memory alloys (SMA's), just to name some of the most preeminent examples. In a parallel path, starting in the late 19th century, mathematicians studying nonlinear differential equations, developed the concept of a bifurcation (term coined by Poincare) and created powerful techniques to study the associated singularities. They have also recognized the close association between bifurcation and symmetry in structures. It was for Koiter, beginning with his famous thesis in 1945, to connect the two communities. Amazing progress has been made since the early days of structural buckling problems and continues to be made in this field, with applications ranging from atomistic to geological scales. With the advent of new materials, the number of applications in this area continues to progress with an ever increasing pace, making it a challenge to present a first course in this topic within the short time available in one semester. The notes that follow are the first attempt to present a comprehensive, modern introduction to the subject of stability of solids. Given the time constraints, only equilibrium configurations of conservative systems will be considered here. These notes start with the introduction of the concepts of stability and bifurcation for conservative elastic systems through finite degree of freedom examples. They continue with the general theory of Lyapunov-Schmidt-Koiter (LSK) asymptotics, followed by examples from continuum mechanics. The presentation subsequently addresses the issue of scale in the stability of solids. In particular the relation between instability at the microstructural level and macroscopic properties of the solid is studied for several types of applications involving different scales: composites (fiber and particle-reinforced), cellular solids and finally SMA's, where temperature- or stress-induced instabilities at the atomic level have macroscopic manifestations visible to the naked eye.

These notes are intended as a complement to lectures given in class. A first draft of these notes was started a long time ago, during my sabbatical leave at the Ecole Polytechnique in 1987. As a researcher in this field, I have learned a lot during the years and for countless times I kept adding, subtracting and modifying. Since better is the worst enemy of good, no comprehensive set of notes has ever been compiled for more than twenty years. Since I recently joined the faculty of the Ecole and wanted to give a new course in this fascinating subject, I finally had to write a set of course notes. I expect that with kind help these notes will soon be completed and relieved from the many mistakes that this first draft inevitably contains (the unbounded kindness of the reader for these deficiencies is greatly appreciated). However, and much more importantly, I hope that this course transmits to the students my enthusiasm about this exciting and vibrant field of solid mechanics.

Nicolas Triantafyllidis, Paris FRANCE, December 2013.

Chapter A

STABILITY AND BIFURCATION - EXAMPLES AND THEORY

Of interest in this chapter is the development of a general theory for the bifurcation and stability of solids. The issues of bifurcation in the equilibrium solutions of nonlinear solids and their stability are closely linked. In addition the stability of the equilibrium solutions near bifurcation points are of great importance to engineering applications. In the first two sections the notions of stability and bifurcation will be introduced with the help of simple finite d.o.f. examples. In the third section the general theory for the bifurcation and stability of the equilibrium solutions of nonlinear elastic systems (discrete or continuum) will be presented.

AA STABILITY OF EQUILIBRIA: DEFINITIONS AND EXAMPLES

The first issue to be addressed in this section is the stability of equilibrium solutions in discrete, nonlinear systems. Following the definition of stability, the two most useful general methods for stability analysis, i.e. the linearization method and Lyapunov's direct method, will be presented along with some simple examples. In addition the case of conservative systems with finite degrees of freedom will be discussed and the energy criterion will be introduced.

AA-1 STABILITY OF AN EQUILIBRIUM - DEFINITIONS

Consider a mechanical system defined by a finite set of real numbers $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ [1] where $\mathbf{p} \in \mathbb{R}^n$. The motion of the system is described by $\mathbf{p}(t)$ and is governed by a set of evolution equations, with respect to time t, of the form:

$$\dot{\mathbf{p}} = \mathbf{f}(\mathbf{p}, t); \quad \text{in component form}: \quad \dot{p}_i = f_i(\mathbf{p}, t), \; i = 1, \cdots, n. \tag{AA-1.1}$$

Here a superimposed dot $\dot{(\;)} \equiv d(\;)/dt$ denotes the differentiation with respect to time t. In addition to the evolution equations, one has also to supply the initial conditions at $t = 0$

$$\mathbf{p}(0) \equiv \mathbf{p}_0. \tag{AA-1.2}$$

By definition, the system is said to be in equilibrium at $\mathbf{p}_e$ if the constant (independent of time) vector $\mathbf{p}_e$ satisfies the evolution equation (AA-1.1) for all times t, i.e. $\mathbf{f}(\mathbf{p}_e, t) = \mathbf{0}$.

Of interest is the notion of stability of an equilibrium solution. Intuitively, an equilibrium solution is stable if small initial perturbations will generate only small perturbed motions away from it which will remain small for all time. A classical illustration of this concept is given in Fig. AA-1.1, where a ball, under the influence of gravity, is at equilibrium at the bottom of a well or the top of a ridge. The stable equilibrium corresponds to the ball sitting at the bottom of the well, since a small deviation from this position induces small amplitude oscillations about equilibrium. The unstable equilibrium corresponds to the ball resting at the top of the ridge, since a small deviation from the equilibrium position leads to the ball's falling away from the top.

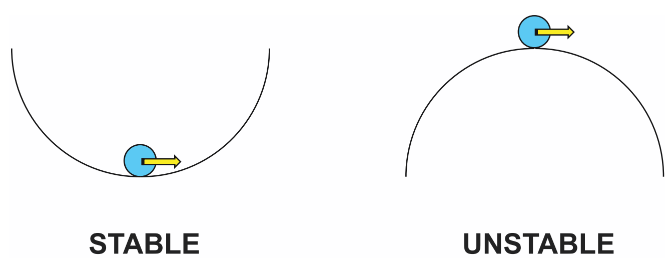

Figure AA-1.1: Intuitive explanation of stability.

In mathematical terms, the stability definition can be stated as follows: Given any small number $\varepsilon > 0$, there exists a number $\eta(\varepsilon) > 0$ such that if the initial conditions are within η from equilibrium, the subsequent motion for all time $t \geq 0$ is within ε from equilibrium

$$\forall \varepsilon > 0 \; \exists \eta(\varepsilon) > 0 \text{ such that}: \quad \| \mathbf{p}(0) - \mathbf{p}_e \| \leq \eta(\varepsilon) \quad \Longrightarrow \quad \| \mathbf{p}(t) - \mathbf{p}_e \| \leq \varepsilon \tag{AA-1.3}$$

where $\| \cdot \|$ denotes the usual Euclidean norm. For the discrete case considered here the norm choice is inconsequential since all norms are equivalent to the Euclidean norm in finite

[1]NOTE Here and subsequently bold symbols are used for vectors or tensors and script symbols for scalars.

dimensional spaces. From the above definition of stability, it is also clear that stability is a local property, since one deals only with small initial perturbations from the equilibrium state in consideration.

A stronger concept of stability, i.e. one that implies (AA-1.3), is that of an asymptotically stable equilibrium, which states that for an adequately small initial perturbation, the perturbed solution tends, for an adequately large time t, to the equilibrium state

$$\exists \eta > 0 \text{ such that : } \quad \| \mathbf{p}(0) - \mathbf{p}_e \| < \eta \quad \Longrightarrow \quad \lim_{t \longrightarrow \infty} \mathbf{p}(t) = \mathbf{p}_e. \tag{AA-1.4}$$

AA-2 LINEARIZATION METHOD OF STABILITY

According to this general method, the governing equations (AA-1.1) are linearized near the equilibrium solution $\mathbf{p}_e$. The analysis of the resulting linearized system is the basis for the study of its stability. The method proceeds in three steps:

- i) Linearization of the evolution equations about $\mathbf{p}_e$.
- ii) Stability analysis of the linearized perturbed motions.
- iii) Justification of the results with respect to the actual motion of the initial system.

In the interest of simplicity, it will be assumed here that the the system's motion does not depend explicitly on time, i.e. $\partial \mathbf{f}/\partial t = 0$.

• i) The Linearization of (AA-1.1) gives

$$\Delta\dot{\mathbf{p}} = \mathbf{A}\bullet\Delta\mathbf{p}; \quad \Delta\mathbf{p} \equiv \mathbf{p}(t) - \mathbf{p}_e, \quad \mathbf{A} \equiv (\partial\mathbf{f}/\partial\mathbf{p})_{\mathbf{p}_e}, \tag{AA-2.1}$$

where $\bullet$ denotes the contraction of $\partial\mathbf{f}/\partial\mathbf{p}$ and $\Delta\mathbf{p}$, i.e. $(\mathbf{A}\bullet\Delta\mathbf{p})_i = \sum_{j=1}^{n}(\partial f_i/\partial p_j)\Delta p_j$. Notice that the matrix $\mathbf{A}$ is constant and recall that $\mathbf{f}(\mathbf{p}_e) = \mathbf{0}$.

• ii) The stability of the linearized system (called autonomous since $\mathbf{A}$ does not depend on time t) depends on the real part of the eigenvalues a_i of $\mathbf{A}$. Lyapunov's theorem states that the linearized system is

a) stable, if $\Re(a_i) \leq 0, \ \forall i \ 1 \leq i \leq n$

b) unstable, if $\exists i, \ \Re(a_i) > 0$

The proof of this statement is straightforward and based on the fact that the solution of the linearized system $\Delta\dot{\mathbf{p}} = \mathbf{A}\bullet\Delta\mathbf{p}$ is given by $\Delta\mathbf{p}(t) = \exp(t\mathbf{A})\bullet\Delta\mathbf{p}(0)$. Since the solution $\Delta\mathbf{p}(t)$ is a linear combination of the functions $\exp(ta_i)$, the above theorem follows.

• iii) The justification of the stability criterion with respect to the actual motion of the initial system comes next. More specifically we will show that if $\Re(a_i) < 0, \ \forall i \ 1 \leq i \leq n$, the perturbation $\Delta\mathbf{p}(t)$ is asymptotically stable. The proof requires an additional assumption about $\mathbf{g}(\Delta\mathbf{p})$, the remainder from the linear expansion of $\mathbf{f}$ about the equilibrium state $\mathbf{p}_e$, defined in (AA-2.1). It is assumed that for small $\| \Delta\mathbf{p} \|$, the remainder grows faster than linearly, i.e.

$$\mathbf{g}(\Delta\mathbf{p}) = \mathrm{o}(\| \Delta\mathbf{p} \|), \quad \mathbf{g}(\Delta\mathbf{p}) \equiv \mathbf{f}(\mathbf{p}_e + \Delta\mathbf{p}) - \mathbf{A}\bullet\Delta\mathbf{p}, \tag{AA-2.2}$$

and works as follows: From the hypothesis that all eigenvalues of $\mathbf{A}$ have strictly negative real parts, one can find constants $c > 1, \ a > 0$ such that

$$\| \exp(t\mathbf{A}) \| \leq c\exp(-at), \quad \forall t > 0 \tag{AA-2.3}$$

Also following the hypothesis about the growth of the remainder in (AA-2.2), one can always find a small positive number $\varepsilon > 0$ such that

$$\| \Delta\mathbf{p} \| \leq \varepsilon \quad \Longrightarrow \quad \| \mathbf{g}(\Delta\mathbf{p}) \| \leq \frac{a}{2c} \| \Delta\mathbf{p} \| \tag{AA-2.4}$$

It will first be shown that

$$\| \Delta\mathbf{p}(0) \| \leq \frac{\varepsilon}{2c} \quad \Longrightarrow \quad \| \Delta\mathbf{p}(t) \| < \varepsilon. \tag{AA-2.5}$$

The proof works by contradiction, in which case by time continuity of $\Delta\mathbf{p}(t)$, one could find a time t_ε such that

$$\| \Delta\mathbf{p}(t_\varepsilon) \| = \varepsilon \quad \| \Delta\mathbf{p}(t) \| < \varepsilon, \quad 0 \leq t < t_\varepsilon. \tag{AA-2.6}$$

The general expression for the solution of (AA-2.1) can be shown (by direct substitution) to be

$$\Delta\mathbf{p}(t) = \exp(t\mathbf{A}) \bullet \Delta\mathbf{p}(0) + \int_0^t \exp[(t-s)\mathbf{A}] \bullet \mathbf{g}(\Delta\mathbf{p}(s))ds \tag{AA-2.7}$$

Taking the norm of both sides of (AA-2.7) and using of the inequalities in (AA-2.2) - (AA-2.4) one obtains the following estimate

$$\| \Delta\mathbf{p}(t_\varepsilon) \| \leq \frac{\varepsilon}{2} \exp(-at_\varepsilon) + c \int_0^{t_\varepsilon} \exp[-a(t_\varepsilon - s)] \frac{a}{2c} \varepsilon ds = \frac{\varepsilon}{2}, \tag{AA-2.8}$$

which is a contradiction of (AA-2.6).

Taking norms of both sides of the solution for $\Delta\mathbf{p}(t)$ in (AA-2.7) and recalling the second inequality in (AA-2.4) one obtains the following estimate

$$\exp(at) \| \Delta\mathbf{p}(t) \| \leq \frac{\varepsilon}{2} + \frac{a}{2} \int_0^t \exp(as) \| \Delta\mathbf{p}(s) \| ds, \quad \forall t > 0. \tag{AA-2.9}$$

The last piece of the proof follows by rewriting the above inequality (AA-2.9) in terms of an auxiliary function $F(t)$

$$\dot{F}(t) - \frac{a}{2} F(t) \leq \frac{\varepsilon}{2}, \quad \forall t > 0; \quad F(t) \equiv \int_0^t \exp(as) \| \Delta\mathbf{p}(s) \| ds. \tag{AA-2.10}$$

Multiplying both sides of the above equation (AA-2.10) by $\exp(-at/2)$ and integrating in the interval $[0, t]$ one obtains the following inequality for $F(t)$

$$F(t) \leq \frac{\varepsilon}{a} [\exp(at/2) - 1], \tag{AA-2.11}$$

which combined with the previous inequality for $F(t)$ in (AA-2.10) yields

$$\dot{F}(t) \leq \frac{\varepsilon}{2} \exp(at/2). \tag{AA-2.12}$$

By substituting in (AA-2.12) the definition for $F(t)$ from (AA-2.10) one finally has

$$\| \Delta\mathbf{p}(t) \| \leq \frac{\varepsilon}{2} \exp(-at/2), \quad \forall t > 0, \tag{AA-2.13}$$

thus proving the asymptotic stability of the system.

A word of caution: It is always possible to linearize any system. For the case of autonomous systems, it is not difficult to find the eigenvalues of $\mathbf{A}$ and check stability of the linearized perturbations. However, guaranteeing stability for the initial, nonlinear system is not trivial, as we have seen. Without this final step, the stability analysis is incomplete.

AA-3 LYAPUNOV'S DIRECT METHOD OF STABILITY

For certain mechanical systems, and especially the conservative systems or systems with simple dissipative mechanisms (exactly the cases that are of interest in this work) the stability issue can be assessed directly without the employment of a linearization process. The method consists of finding a functional $L(\mathbf{p}(t))$, termed Lyapunov's functional since its independent variable is the function $\mathbf{p}(t)$, and which in general depends on the history of the motion from time $t = 0$ up to the present time and which has the following properties:

- $L(\mathbf{p}(t))$ is non increasing function of t:

$$dL/dt \leq 0. \tag{AA-3.1}$$

- $L(\mathbf{p}(t))$ is a measure of the distance from the equilibrium for each t:

$$L(\mathbf{p}(t)) \geq c \parallel \mathbf{p}(t) - \mathbf{p}_e \parallel^2 . \tag{AA-3.2}$$

- $L(\mathbf{p}(0))$ is a measure of the initial perturbation at $t = 0$:

$$L(\mathbf{p}(0)) \leq d \parallel \mathbf{p}(0) - \mathbf{p}_e \parallel^2, \tag{AA-3.3}$$

where c and d are arbitrary positive constants and $\mathbf{p}(t)$ satisfies the evolution equations (AA-1.1).

The existence of such a functional ensures the stability of the equilibrium solution $\mathbf{p}_e$. Indeed from Eqs. (AA-3.1) - (AA-3.3) one has:

$$c \parallel \mathbf{p}(t) - \mathbf{p}_e \parallel^2 \leq L(\mathbf{p}(t)) \leq L(\mathbf{p}(0)) \leq d \parallel \mathbf{p}(0) - \mathbf{p}_e \parallel^2, \tag{AA-3.4}$$

which ensures that the inequality $\parallel \mathbf{p}(t) - \mathbf{p}_e \parallel \leq \varepsilon$ is satisfied if η is chosen to be $\eta^2 \leq c\varepsilon^2/d$ thus ensuring stability according to the definition given in (AA-1.3).

AA-4 FINITE D.O.F. EXAMPLES

Analyzing the finite d.o.f. examples given here will need some elements of Lagrangian mechanics. It will be assumed that the mechanical system in question can be fully described by $m = n/2$ generalized displacement coordinates $\mathbf{q} = (q_1,\ q_2, ...,\ q_m)$, while the system's complete d.o.f. consist of the generalized displacement coordinates $\mathbf{q}$ and their derivatives $\dot{\mathbf{q}}$. This way we are consistent with the notation of subsection AA-1 by noting that $\mathbf{p} \equiv (\mathbf{q}, \dot{\mathbf{q}})$.

It is assumed that the system has a kinetic energy $\mathcal{K}$ and an internal (elastic) energy $\mathcal{E}$. The Lagrangian of the system is defined as their difference, i.e.

$$\mathcal{L} = \mathcal{K} - \mathcal{E}. \tag{AA-4.1}$$

The system's equation of motion takes the form

$$\mathbf{F} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}}; \quad \text{in component form :} \quad F_i = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i},\ i = 1, \cdots, m \tag{AA-4.2}$$

where $F_i(\mathbf{q}, \dot{\mathbf{q}})$[2] are the generalized external forces associated with the generalized coordinate q_i. As mentioned before, the second order system of (AA-4.1) - (AA-4.2) is amenable to the general first order form considered in (AA-1.1) by defining $\mathbf{p} \equiv (\mathbf{q}, \dot{\mathbf{q}})$.

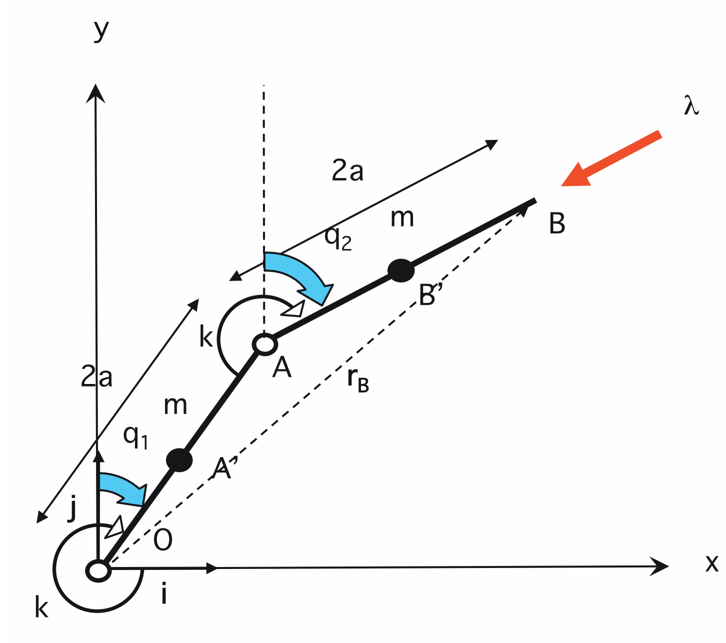

Figure AA-4.1: Two-bar model with follower force.

To illustrate the linearization method of stability for the study of finite degree of freedom mechanical systems, one can consider the mechanism in Fig. AA-4.1. Two rigid bars OA and AB each of length $2a$ have a mass m attached to their respective mid points A' and B'.

[2] NOTE For the case of conservative systems, the external forces are derivable from a potential, which can be added to the internal (elastic) energy. In this case $F_i = 0$ and $\mathcal{E}$ is the system's total (potential) energy

Two identical torsional springs of stiffness k are attached at the frictionless hinges O and A. The system is loaded by a follower force of magnitude λ acting always on the direction of the rigid bar AB. The two degrees of freedom q_1 and q_2 are the angles between OA, AB and the Oy axis respectively.

The system's kinetic energy $\mathcal{K}$ is given by:

$$\mathcal{K} = \frac{1}{2}\ m v_{_{A'}}^2 + \frac{1}{2}\ m v_{_{B'}}^2 = \frac{1}{2}\ m(\dot{\mathbf{r}}_{_{A'}} \bullet \dot{\mathbf{r}}_{_{A'}} + \dot{\mathbf{r}}_{_{B'}} \bullet \dot{\mathbf{r}}_{_{B'}}) \tag{AA-4.3}$$

where the position vectors of points A' and B' are:

$$\begin{aligned} \mathbf{r}_{_{A'}} &= (a \sin q_1)\mathbf{i} + (a \cos q_1)\mathbf{j} \\ \mathbf{r}_{_{B'}} &= (2a \sin q_1 + a \sin q_2)\mathbf{i} + (2a \cos q_1 + a \cos q_2)\mathbf{j} \end{aligned} \tag{AA-4.4}$$

By using (AA-4.4) into (AA-4.3) the total kinetic energy of the system is:

$$\mathcal{K} = ma^2\ [\frac{5}{2}(\dot{q}_1)^2 + \frac{1}{2}(\dot{q}_2)^2 + 2\dot{q}_1\dot{q}_2 \cos(q_1 - q_2)] \tag{AA-4.5}$$

For the calculation of the external forces, one can proceed through the principle of virtual work. Consequently

$$-\lambda\ (\sin q_2\ \mathbf{i} + \cos q_2\ \mathbf{j}) \bullet \delta\mathbf{r}_{_B} = F_1 \delta q_1 + F_2 \delta q_2 \tag{AA-4.6}$$

Since from geometry the position vector of point B is found to be:

$$\mathbf{r}_{_B} = 2a\ [(\sin\ \ q_1 + \sin\ \ q_2)\mathbf{i} + (\cos\ \ q_1 + \cos\ \ q_2)\mathbf{j}] \tag{AA-4.7}$$

which combined with (AA-4.6) gives for the generalized external forces F_1 and F_2:

$$F_1 = 2a\lambda \sin(q_1 - q_2)\ , \quad F_2 = 0 \tag{AA-4.8}$$

The system's internal energy $\mathcal{E}$ is the elastic energy stored in the springs at O and A

$$\mathcal{E} = \frac{1}{2}k(q_1)^2 + \frac{1}{2}k(q_1 - q_2)^2 \tag{AA-4.9}$$

Consequently, by employing (AA-4.9), (AA-4.8) and (AA-4.5) into (AA-4.2) one obtains the following nonlinear equations governing the motion of the mechanism in Fig. AA-4.1:

$$\begin{aligned} &2kq_1 - kq_2 - 2a\lambda \sin(q_1 - q_2) + ma^2[5\ddot{q}_1 + 2\ddot{q}_2 \cos(q_1 - q_2) \\ &-2\dot{q}_2(\dot{q}_1 - \dot{q}_2)\sin(q_1 - q_2) + 2\dot{q}_1\dot{q}_2 \sin(q_1 - q_2)] = 0\ , \\ &-kq_1 + kq_2 + ma^2[\ddot{q}_2 + 2\ddot{q}_1 \cos(q_1 - q_2) - 2\dot{q}_1(\dot{q}_1 - \dot{q}_2)\sin(q_1 - q_2) \\ &-2\dot{q}_1\dot{q}_2 \sin(q_1 - q_2)] = 0\ . \end{aligned} \tag{AA-4.10}$$

It is not difficult to see that $\mathbf{q}(t) = \mathbf{0} \equiv \mathbf{q}_e$ is an equilibrium solution. To investigate its stability, one obtains from the linearization of (AA-4.10), i.e. by substituting $\mathbf{q}(t) = \Delta\mathbf{q}(t) + \mathbf{q}_e$ into (AA-4.10) and keeping only the linear terms in $\Delta\mathbf{q}$

$$\mathbf{M} \bullet \Delta\ddot{\mathbf{q}} + \mathbf{K} \bullet \Delta\mathbf{q} = \mathbf{0}; \quad \mathbf{M} \equiv ma^2 \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{K} \equiv \begin{bmatrix} 2k - 2a\lambda & 2a\lambda - k \\ -k & k \end{bmatrix}. \tag{AA-4.11}$$

In (AA-4.11), $\mathbf{M}$ is the structure's mass matrix and $\mathbf{K}$ is the structure's stiffness matrix. The solution of the above, constant coefficient linear system is:

$$\Delta \mathbf{q}(t) = \sum_{I=1}^{4} \Delta \mathbf{Q}^I \exp(s_I t), \tag{AA-4.12}$$

where $\Delta \mathbf{Q}^I$ are constants and s_I are the four roots of the characteristic equation which results by substituting (AA-4.12) into (AA-4.11), namely

$$\det[\mathbf{M}(s_I)^2 + \mathbf{K}] = 0, \quad \Longrightarrow \quad m^2a^4(s_I)^4 + ma^2(11k - 6a\lambda)(s_I)^2 + k^2 = 0, \; I = 1 \cdots 4. \tag{AA-4.13}$$

The study of the four roots of (AA-4.13) yields the following results:

• If $\lambda \leq 3k/2a$, then $\Re(s_I) = 0, \; (I = 1, 2, 3, 4)$, i.e. all four roots of the biquadratic are purely imaginary. The linearized system is stable.

• If $3k/2a < \lambda < 13k/6a$, then $\Re(s_I) > 0, \; \Im(s_I) \neq 0, \; (I = 1, 2)$, i.e. two of the four roots of the biquadratic have positive real part. Hence the linearized system is unstable. By definition the instability where $\Re(s_I) > 0, \; \Im(s_I) \neq 0$ is called a flutter type instability.

• If $13k/6a \leq \lambda$, then $\Re(s_I) > 0, \; \Im(s_I) = 0, \; (I = 1, 2, 3, 4)$, i.e. all four roots of the biquadratic are real and positive. Hence the linearized system is unstable. By definition the instability where $\Re(s_I) > 0, \; \Im(s_I) = 0$ is called a divergence type instability.

When $0 \leq \lambda < 3k/2a$, any initial perturbation produces a constant amplitude periodic motion in the linearized system, since no dissipation is considered. Consequently the system is not asymptotically stable according to the definition in (AA-1.4). If a small linear viscous damping is introduced, a viscous term $\mathbf{C} \bullet \Delta \mathbf{q}$ – where $\mathbf{C}$ is a small positive definite viscosity matrix – is added to the linearized perturbation (AA-4.11). Consequently one can show that for $\lambda \leq 3k/2a$ all roots have negative real parts, i.e. $\Re(s_I) < 0, \; (I = 1, 2, 3, 4)$ and hence that the more realistic, linearized viscous system is also asymptotically stable.

Some final remarks are in order for the mechanism in Fig. AA-4.1. In view of its follower force loading, a potential energy does not exist, i.e. external forces cannot be derived from a potential, and the only equilibrium solution is the trivial one $q_1(t) = q_2(t) = 0$.

The second example to be studied is one that does have a potential energy. For this a small change of the external loading of the mechanism in Fig. AA-4.1 is considered, namely a load λ acting in the **-j** direction as shown in Fig. AA-4.2. The only change in the model pertains to the generalized external forces F_1 and F_2 which in this case are:

$$(-\lambda \; \mathbf{j}) \bullet \delta \mathbf{r}_B = F_1 \delta q_1 + F_2 \delta q_2 \tag{AA-4.14}$$

which in conjunction with (AA-4.7) yields for the external forces:

$$F_1 = 2a\lambda \sin q_1, \quad F_2 = 2a\lambda \sin q_2 \tag{AA-4.15}$$

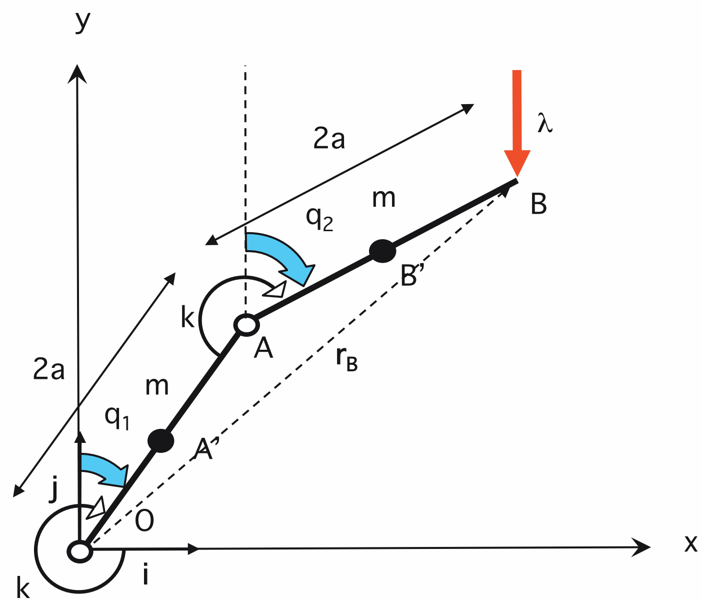

Figure AA-4.2: Two-bar model with conservative force.

Consequently from (AA-4.1), (AA-4.2), (AA-4.5), (AA-4.9) and (AA-4.15), the corresponding nonlinear equation of motion takes the form:

$$\begin{aligned}
&2kq_1 - kq_2 - 2a\lambda\sin q_1 + ma^2[5\ddot{q}_1 + 2\ddot{q}_2\cos(q_1-q_2) \\
&-2\dot{q}_2(\dot{q}_1-\dot{q}_2)\sin(q_1-q_2) + 2\dot{q}_1\dot{q}_2\sin(q_1-q_2)] = 0 \\
&-kq_1 + kq_2 - 2a\lambda\sin q_2 + ma^2[\ddot{q}_2 + 2\ddot{q}_1(q_1-q_2)\cos(q_1-q_2) \\
&-2\dot{q}_1(\dot{q}_1-\dot{q}_2)\sin(q_1-q_2) - 2\dot{q}_1\dot{q}_2\sin(q_1-q_2)] = 0
\end{aligned} \tag{AA-4.16}$$

One can easily verify that $q_1(t) = q_2(t) = 0$ is again an equilibrium solution. Upon linearization, one obtains from (AA-4.16):

$$\mathbf{M}\bullet\Delta\ddot{\mathbf{q}} + \mathbf{K}\bullet\Delta\mathbf{q} = \mathbf{0}; \quad \mathbf{M} \equiv ma^2\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{K} \equiv \begin{bmatrix} 2k-2a\lambda & -k \\ -k & k-2a\lambda \end{bmatrix}, \tag{AA-4.17}$$

which is similar to (AA-4.11), save for a new stiffness matrix $\mathbf{K}$. Once again, the solution to the above constant coefficient linear system is given by (AA-4.12), thus leading to the following characteristic equation for s_I

$$\det[\mathbf{M}(s_I)^2 + \mathbf{K}] = 0, \quad \Longrightarrow \quad m^2a^4(s_I)^4 + ma^2(11k - 12a\lambda)(s_I)^2 + k^2 - 6a\lambda k + 4a^2\lambda^2 = 0, \;\; I = 1\cdots 4. \tag{AA-4.18}$$

It is not difficult to verify that:

• If $\lambda \le (3-\sqrt{5})k/4a$, then $\Re(s_I) = 0,\ \Im(s_I) \neq 0,\ (I = 1,2,3,4)$ and hence the linearized system is stable.

- If $(3-\sqrt{5})k/4a < \lambda < (3+\sqrt{5})k/4a$, then $\Re(s_I) > 0$, $\Im(s_I) = 0$, $(I = 1, 2)$ and hence the linearized system exhibits a divergence type instability.
- If $(3+\sqrt{5})k/4a \leq \lambda$, then $\Re(s_I) > 0$, $\Im(s_I) = 0$, $(I = 1, 2, 3, 4)$ and hence the linearized system again exhibits a divergence type instability.

When $0 \leq \lambda < (3-\sqrt{5})k/4a$, any initial perturbation produces a constant amplitude periodic motion in the linearized system, since no dissipation is considered. Consequently the system is not asymptotically stable according to the definition in (AA-1.4). If a small linear viscous damping is introduced, a viscous term $\mathbf{C} \bullet \Delta\mathbf{q}$ – where $\mathbf{C}$ is a small positive definite viscosity matrix – is added to the linearized perturbation (AA-4.17). Consequently one can show that for $\lambda \leq (3-\sqrt{5})k/4a$ all roots have negative real parts, i.e. $\Re(s_I) < 0$, $(I = 1, 2, 3, 4)$ and hence that the more realistic, linearized viscous system is also asymptotically stable.

Some final remarks are in order for the mechanism in Fig. AA-4.2. In view of its fixed direction loading, a potential energy does exist, i.e. external forces can be derived from a potential. Moreover, for loads $0 \leq \lambda < (3-\sqrt{5})k/4a$, it can be shown that the equilibrium solution $\mathbf{q}_e = \mathbf{0}$ minimizes the system's potential energy $\mathcal{E}$, which is the sum of the internal (elastic) energy in (AA-4.9) and the potential energy of the applied load $(-\lambda\mathbf{j}) \bullet \mathbf{r}_B$, namely

$$\mathcal{E} = \frac{1}{2}\ k(q_1)^2 + \frac{1}{2}\ k(q_1 - q_2)^2 + 2\lambda a(\cos q_1 + \cos q_2). \tag{AA-4.19}$$

It follows from (AA-4.19), by taking the second derivative of $\mathcal{E}$ with respect to $\mathbf{q}$ on the equilibrium state $\mathbf{q}_e$ that

$$\left[\frac{\partial^2 \mathcal{E}}{\partial\mathbf{q}\partial\mathbf{q}}\right]_{\mathbf{q}_e} = \mathbf{K}, \tag{AA-4.20}$$

where $\mathbf{K}$ is the linearized conservative system's stiffness matrix defined in (AA-4.17). It can easily be checked that for loads $0 \leq \lambda < (3-\sqrt{5})k/4a$, the stiffness matrix $\mathbf{K}$ is positive definite, thus showing that loads for which the linearized system is stable, also minimize the structure's potential energy. The connection found in this example between stability and minimization of potential energy, is a general property of conservative systems and will be discussed in a more general setting in the next subsection.

AA-5 ENERGY CRITERION OF STABILITY

By definition, a conservative system is a system with a constant total energy which, for the case of the mechanical systems considered here, means that the sum of its potential energy $\mathcal{E}$ and its kinetic energy $\mathcal{K}$ is a constant. We will show that for a finite d.o.f. conservative system, stability of an equilibrium is equivalent to minimization of the potential energy by the equilibrium solution in question. To achieve this, we will show that if an equilibrium solution $\mathbf{q}_e$ minimizes the potential energy of a system, then one can define a Lyapunov functional by

$$L(\mathbf{p}) \equiv \mathcal{E}(\mathbf{q}) + \mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) - \mathcal{E}(\mathbf{q}_e), \quad \mathbf{p} \equiv (\mathbf{q}, \dot{\mathbf{q}}). \tag{AA-5.1}$$

The above defined functional is a constant (since the system is conservative) and hence non-increasing, thus satisfying the first requirement for a Lyapunov functional according to (AA-3.1).

The fact that $\mathcal{E}(\mathbf{q}_e)$ is a strict local minimum of the function $\mathcal{E}(\mathbf{q})$ implies the existence of a constant c_1 such that $\mathcal{E}(\mathbf{q}) - \mathcal{E}(\mathbf{q}_e) \geq c_1 \parallel \mathbf{q} - \mathbf{q}_e \parallel^2$ $(c_1 > 0)$. For any realistic system the kinetic energy $\mathcal{K} \geq 0$, being a sum of squares of velocities, implies that $\mathcal{K} \geq c_2 \parallel \dot{\mathbf{q}} \parallel^2$ $(c_2 \geq 0)$ and hence one can find $c > 0$ such that:

$$\mathcal{E}(\mathbf{q}) - \mathcal{E}(\mathbf{q}_e) + \mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) \geq c \parallel (\mathbf{q} - \mathbf{q}_e, \dot{\mathbf{q}} - \dot{\mathbf{q}}_e) \parallel^2 = c \parallel \mathbf{p} - \mathbf{p}_e \parallel^2, \tag{AA-5.2}$$

hence ensuring the second property required of a Lyapunov functional according to (AA-3.2).

Similarly, the continuity of $\mathcal{E}(\mathbf{q})$ as well as the finite dimensionality of the space of $\mathbf{q}$ ensure a $d_1 > 0$ such that $\mathcal{E}(\mathbf{q}) - \mathcal{E}(\mathbf{q}_e) \leq d_1 \parallel \mathbf{q} - \mathbf{q}_e \parallel^2$ while for the kinetic energy $\mathcal{K}$ one can also find a $d_2 > 0$ such that $\mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) \leq d_2 \parallel \dot{\mathbf{q}} \parallel^2$. Consequently, and for the same reasons as before one can find a $d > 0$ with the property:

$$\mathcal{E}(\mathbf{q}) - \mathcal{E}(\mathbf{q}_e) + \mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) \leq d \parallel \mathbf{q} - \mathbf{q}_e, \dot{\mathbf{q}} - \dot{\mathbf{q}}_e \parallel^2 = d \parallel \mathbf{p} - \mathbf{p}_e \parallel^2, \tag{AA-5.3}$$

thus ensuring the third property of a Lyapunov functional according to (AA-3.3) when above inequality is evaluated at $t = 0$

The above discussion proves that a local minimum of the potential energy at equilibrium is a sufficient condition for stability, a statement which is known in the literature under the name of Lejeune-Dirichlet (or minimum potential energy) stability theorem.

For the case of finite d.o.f. conservative systems, the strict minimum of the potential energy is also a necessary condition for stability. It is not difficult to see that if $\mathbf{q}_e$ is not a minimum of the potential energy $\mathcal{E}(\mathbf{q})$ then in view of the energy conservation $\mathcal{E}(\mathbf{q}_e) - \mathcal{E}(\mathbf{q}) = \mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) > 0$ for some $\mathbf{q}$ in the neighborhood of $\mathbf{q}_e$. Moreover if the initial perturbation is $\mathbf{q}(0) \neq \mathbf{0}, \dot{\mathbf{q}}(0) = \mathbf{0}$ with $\parallel \mathbf{q}(0) \parallel < \eta$ for any small η, one can show that $\parallel \mathbf{p} - \mathbf{p}_e \parallel$ will be an increasing function of time in view of the energy conservation equation mentioned above, thus proving instability. Consequently, (assuming appropriate continuity conditions), for a finite d.o.f., conservative system to be stable, a local minimum of the system's energy $\mathcal{E}$ at an equilibrium state $\mathbf{p}_e$ is equivalent with:

$$\left[\frac{\partial^2 \mathcal{E}}{\partial \mathbf{q} \partial \mathbf{q}}\right]_{\mathbf{q}_e} \quad \text{positive definite} \tag{AA-5.4}$$

The above necessary and sufficient criterion for stability in finite d.o.f. conservative systems will be employed in the rest of this work for the stability discussion of all discrete elastic systems. A generalization of the above criterion for continuous conservative systems will be discussed subsequently. It is interesting to note that although the criterion of stability is a dynamical concept and involves the equations of motion and hence the mass distribution in the system, for the case of conservative systems it has just been shown that the stability criterion involves only the properties of the potential energy, thus considerably simplifying the task of stability investigations.

AB BIFURCATION CONCEPTS AND FINITE D.O.F. SIMPLE EXAMPLES

In this chapter the notion of a *bifurcation* in the solution of a nonlinear system is presented and illustrated by means of simple examples that permit analytical solutions.

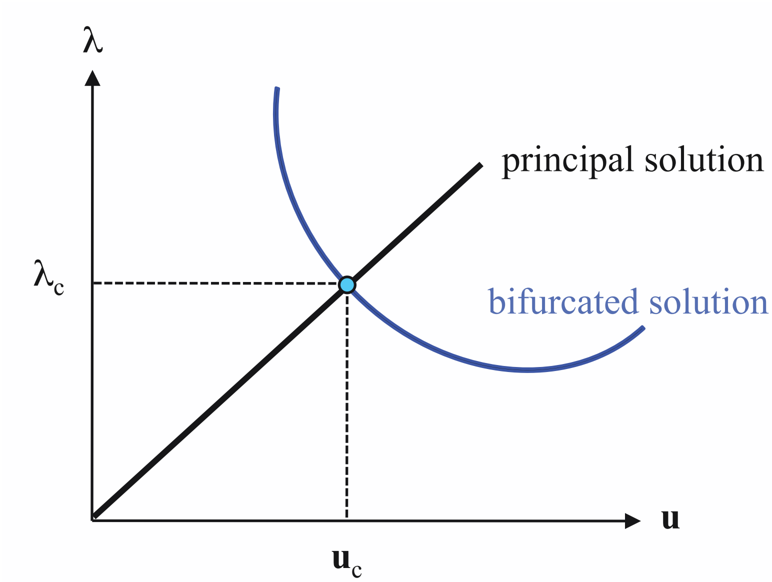

Figure AB-0.1: Schematic representation of a bifurcation.

The concept is explained with the help of Fig. AB-0.1. Assume that we want to solve the nonlinear system (representing the equilibrium of a finite d.o.f. structure)

$$\mathbf{f}(\mathbf{u}, \lambda) = \mathbf{0}, \tag{AB-0.1}$$

where $\mathbf{u} \in \mathbb{R}^n$ are the system's unknowns (typically displacements) and λ a scalar parameter characterizing the system's load. To indicate that the system's d.o.f. are time-independent, from here on use the symbol $\mathbf{u}$ instead of $\mathbf{p}$ that has been used before. Since we deal here with conservative systems that have a potential $\mathcal{E}$, the above equilibrium equations are the stationarity conditions for $\mathcal{E}$, namely:

$$\mathbf{f}(\mathbf{u}, \lambda) = \frac{\partial \mathcal{E}(\mathbf{u}, \lambda)}{\partial \mathbf{u}} \equiv \mathcal{E}_{,\mathbf{u}}\,. \tag{AB-0.2}$$

We are interested in the solutions $\mathbf{u}(\lambda)$ of (AB-0.1) as functions of the load parameter λ. One solution, termed "*principal*" solution is the (usually straightforward) solution of (AB-0.1) which starts at zero load with $\mathbf{u} = \mathbf{0}$ at $\lambda = 0$. Due to the system's nonlinearity the principal solution is not necessarily unique and as the load increases away from zero there is a certain point in the load versus displacement graph shown in Fig. AB-0.1 where another solution, termed "*bifurcated*" solution (in view of the fork shape of the graph) emerges at point $(\lambda_c, \mathbf{u}_c)$ termed respectively "*critical*" load and displacement.

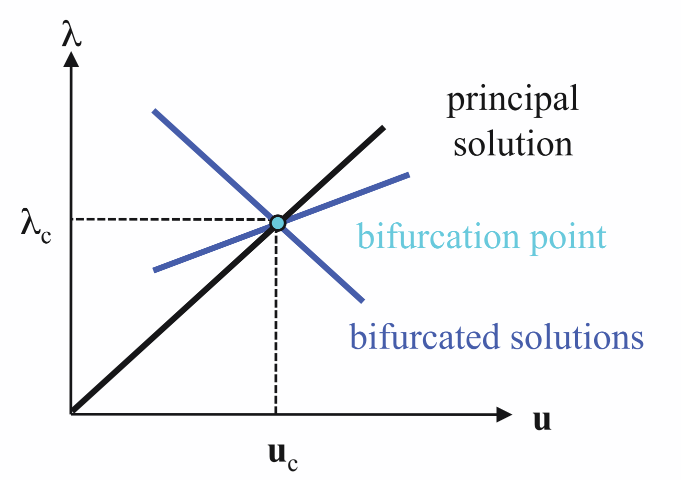

Figure AB-0.2: Schematic representation of a multiple bifurcation.

A rigid T model with two degrees of freedom will provide the example of a "*simple bifurcation*", the case where just one branch emerges from the principal path, as seen in Fig. AB-0.1. Many applications exist where several equilibrium branches emerge from the principal path, in which case we talk about a "*multiple bifurcation*", a situation depicted in Fig. AB-0.2. As such an example the study a rigid plate with three degrees of freedom will provide the example of a multiple (double) bifurcation.

An important feature of bifurcation is that it is non-robust, i.e. changes its character, under perturbations. By means of the simple examples introduced here, it will be shown that the bifurcation point either becomes a *limit load* (or *limit point*) (see Fig. AB-0.3) or a bifurcation point of lower order in the presence of imperfections.

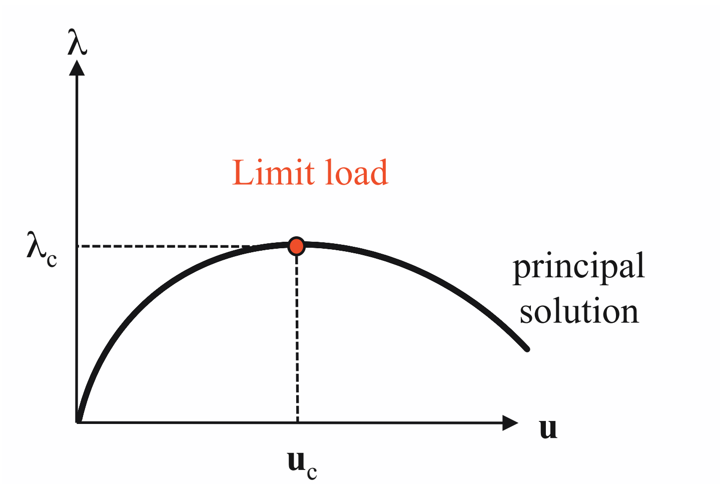

Figure AB-0.3: Schematic representation of a limit point.

Finally to illustrate the concept of a limit load, a two bar truss model is presented.

AB-1 PERFECT RIGID T MODEL

To illustrate the concept of bifurcation, we use a two degrees of freedom rigid T model shown in FIG. AB-1.1. For the perfect model the part OC of length L is attached perpendicularly to the middle of the segment AB of length $2l$. The midpoint O of the segment AB can only move vertically by a distance v while the entire structure can rotate about O by a small angle θ. Two identical, vertical linear springs with restoring force f proportional to their change of length d, ($f = -Ed$, $E > 0$, are attached to ends A and B. At the end C a horizontal nonlinear spring is attached, with a force-displacement relation given by $F = -[kd + md^2 + nd^3]$. The structure is subjected to a vertical load $\lambda \geq 0$ at C.

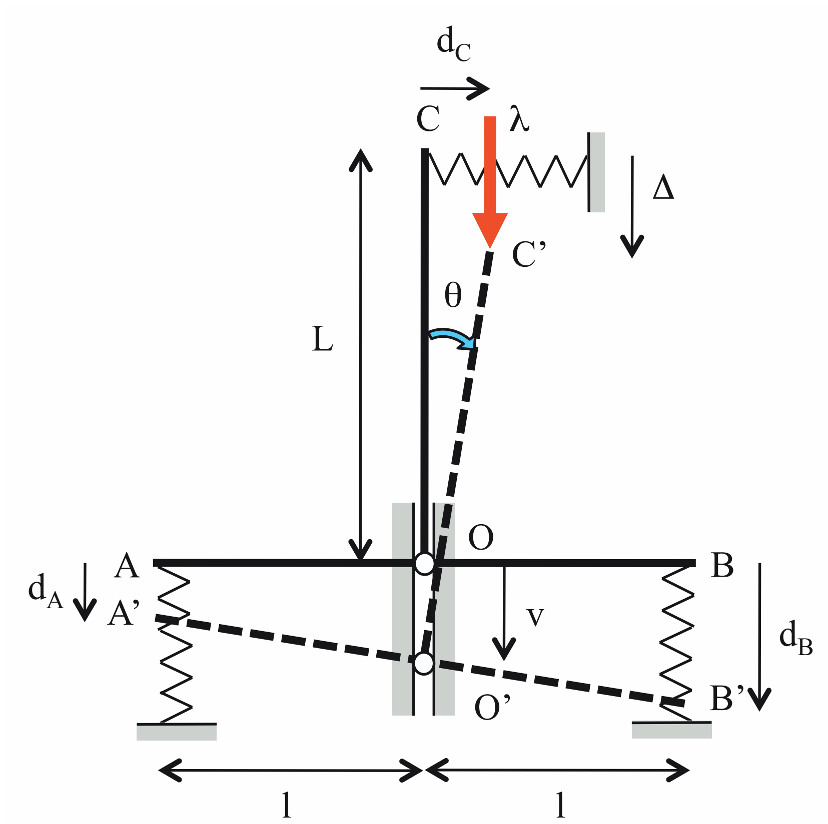

Figure AB-1.1: Perfect rigid T model.

From kinematics d_A, d_B, the vertical displacements at points A and B, d_C, Δ the horizontal and vertical displacements at point C, are:

$$d_A = v - l\theta, \quad d_B = v + l\theta, \quad d_C = L\theta, \quad \Delta = v + L(1 - \cos\theta) \approx v + L\frac{\theta^2}{2}, \tag{AB-1.1}$$

where the dependence of all the displacements on θ are correct up to $O(\theta^2)$.

The total potential energy $\mathcal{E}$ of the system, consisting of the energies stored in springs at

A, B and C, plus the potential energy of the applied load, is found to be:

$$\mathcal{E}(v,\theta,\lambda) = E[v^2 + (l\theta)^2] + \frac{k}{2}\ (L\theta)^2 + \frac{m}{3}\ (L\theta)^3 + \frac{n}{4}\ (L\theta)^4 - \lambda\ [v + L\frac{\theta^2}{2}]. \tag{AB-1.2}$$

Extremizing $\mathcal{E}$ with respect to the degrees of freedom v and θ, one obtains two equilibrium equations (respectively, the force equilibrium along the vertical direction and the moment equilibrium about point O, as one can also verify by direct calculations)

$$\begin{aligned} &\mathcal{E}_{,v} = 2Ev - \lambda = 0,\\ &\mathcal{E}_{,\theta} = (2El^2 + kL^2)\theta + mL^3\theta^2 + nL^4\theta^3 - \lambda L\theta = 0. \end{aligned} \tag{AB-1.3}$$

One solution to the above system is obviously:

$$\overset{0}{v}(\lambda) = \lambda/2E, \quad \overset{0}{\theta}(\lambda) = 0, \tag{AB-1.4}$$

which is the principal solution, for it satisfies equilibrium at the unloaded state, i.e. for $\lambda = 0$ the displacements $(v,\theta) = (0,0)$.

For $\theta \neq 0$, the same system of equilibrium equations admits the solutions:

$$\begin{aligned} &v(\lambda) = \lambda/2E, \quad \lambda = \lambda_c + \theta^2 nL^3 \quad \text{if } n \neq 0,\ m = 0, \quad \text{symmetric},\\ &v(\lambda) = \lambda/2E, \quad \lambda = \lambda_c + \theta mL^2 \quad \text{if } n = 0,\ m \neq 0, \quad \text{asymmetric},\\ &\text{where}: \lambda_c \equiv (2El^2 + kL^2)/L. \end{aligned} \tag{AB-1.5}$$

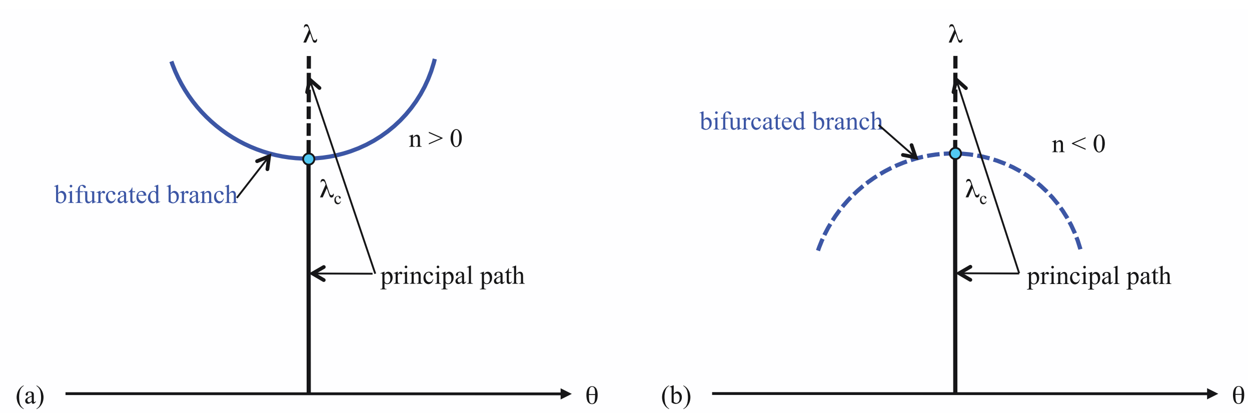

Figure AB-1.2: Symmetric bifurcation of perfect rigid T model. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

The above solutions are the bifurcated solutions and pass at $\theta = 0$ through the load $\lambda = \lambda_c$, which, according to the definition at the beginning of this section, is the critical load. The bifurcation shown for the symmetric case $(m = 0, n \neq 0)$ in Fig. AB-1.2 where the bifurcated branch emerges perpendicular to the principal branch is called "*symmetric*" bifurcation. The bifurcation shown for the asymmetric case $(m \neq 0, n = 0)$ in Fig. AB-1.3, where the bifurcated branch intersects the principal one at an angle different from a right angle, is called "*transverse*" or "*asymmetric*" bifurcation.

Notice that for the symmetric bifurcation in Fig. AB-1.2, the load λ of the bifurcated solution is higher (if $n > 0$) or lower (if $n < 0$) than the critical load in the neighborhood of λ_c. The corresponding bifurcations are termed "*supercritical*" and "*subcritical*" respectively. For the asymmetric bifurcation in Fig. AB-1.3, the load λ of the bifurcated solution can be either higher or lower than λ_c in the neighborhood of the critical load depending on the sign of θ and the corresponding bifurcation is called "*transcritical*".

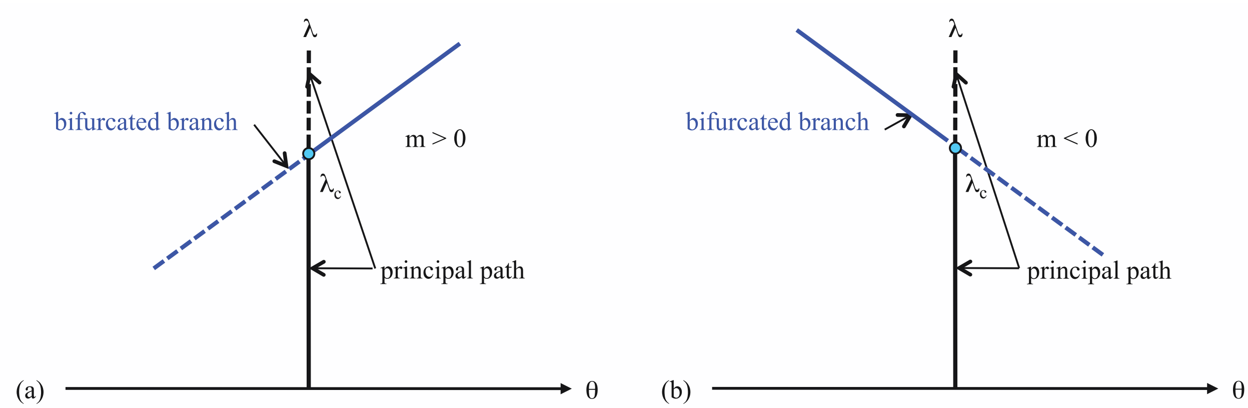

Figure AB-1.3: Transverse (or asymmetric) bifurcation of perfect rigid T model. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

An important feature of the problem is the stability of the different equilibrium paths. According to the discussion in subsection AA-5, an equilibrium path is stable if it corresponds to a local minimum of the system's potential energy. For the conservative two degree of freedom system here, a point $v(\lambda), \theta(\lambda)$ of an equilibrium path is a local minimum of the energy if, according to (AA-5.4), the matrix $\mathcal{E}_{,\mathbf{uu}}$ defined below is positive definite:

$$\mathcal{E}_{,\mathbf{uu}} = \begin{bmatrix} \mathcal{E}_{,vv} & \mathcal{E}_{,v\theta} \\ \mathcal{E}_{,\theta v} & \mathcal{E}_{,\theta\theta} \end{bmatrix} = \begin{bmatrix} 2E & 0 \\ 0 & (\lambda_c - \lambda)L + 2mL^3\theta + 3nL^4\theta^2 \end{bmatrix}. \qquad \text{(AB-1.6)}$$

It is not difficult to see from (AB-1.6) that the principal branch $\theta = 0$ is stable for $0 \leq \lambda < \lambda_c$ and unstable for $\lambda \geq \lambda_c$ (since $E > 0$ and the positive definiteness of the diagonal matrix $\mathcal{E}_{,\mathbf{uu}}$ is determined by the sign of $\mathcal{E}_{,\theta\theta}$). For the symmetric bifurcation case $(m = 0, n \neq 0)$ substituting (AB-1.5) into (AB-1.6) one obtains $\mathcal{E}_{,\theta\theta} = 2(\lambda - \lambda_c)L$. Hence, for the supercritical bifurcation where $\lambda > \lambda_c$ the bifurcated branch is stable, while for the subcritical bifurcation where $\lambda < \lambda_c$, the bifurcated branch is unstable. For the asymmetric bifurcation $(m \neq 0, n = 0)$, the same reasoning as before leads to $\mathcal{E}_{,\theta\theta} = (\lambda - \lambda_c)L$ which shows that one part of the bifurcated branch is stable (the one with $\lambda > \lambda_c$), while the other (in which $\lambda < \lambda_c$) is not, regardless of the sign of m. In Fig. AB-1.2 and Fig. AB-1.3 the stable equilibrium paths are drawn using a continuous line while the unstable equilibrium paths are drawn using a dashed line.

Since at any given load the system has several possible equilibrium paths, it is also of interest to compare the energy levels associated with these different solutions.

Using (AB-1.4), (AB-1.5) into (AB-1.1) the total potential energy on the principal branch, is found to be:

$$\mathcal{E} = -\frac{1}{4E}\lambda^2, \tag{AB-1.7}$$

while the energy associated with each bifurcated branch is given by:

$$\begin{aligned} \mathcal{E} &= -\frac{1}{4E}\lambda^2 - \frac{(\lambda-\lambda_c)^2}{4nL^2} \quad \text{if } m = 0, \quad n \neq 0, \\ \mathcal{E} &= -\frac{1}{4E}\lambda^2 - \frac{(\lambda-\lambda_c)^3}{6m^2L^3} \quad \text{if } m \neq 0, \quad n = 0. \end{aligned} \tag{AB-1.8}$$

Given a load level λ, one can see from (AB-1.8) that for a symmetric bifurcation $(m = 0, n \neq 0)$, the stable bifurcation branch of the supercritical case $(n > 0)$ has less energy than the principal branch, while for the subcritical case $(n < 0)$ the situation is reversed. For the asymmetric bifurcation $(m \neq 0, n = 0)$ one observes that for loads $\lambda > \lambda_c$ the bifurcated branch has lower energy than the principal branch, while for loads $\lambda < \lambda_c$ the situation is reversed.

This simple example shows that for a given load level λ, the minimum energy always corresponds to the stable equilibrium solution (principal or bifurcated). Also note that for the transcritical asymmetric or the supercritical symmetric bifurcation, one can always find a stable equilibrium branch for any load level λ, while for the subcritical symmetric bifurcation a stable equilibrium solution exists only for $\lambda < \lambda_c$.

AB-2 IMPERFECT RIGID T MODEL

The introduction of an imperfection into the rigid T model investigated in the previous section, provides the physically plausible mechanism which determines uniquely the structure's equilibrium path in a loading process starting from $\lambda = 0$. Of all the many possible ways to introduce an imperfection, the one considered here is geometric and is in the form of a slight defect in the normality of the part OC to the part OA by angle δ, as seen in Fig. AB-2.1. All the other elements of the model (dimensions, stiffnesses of linear and nonlinear springs, etc.) remain the same as in the perfect model. The imperfect model reduces to the perfect model when the imperfection angle $\delta = 0$.

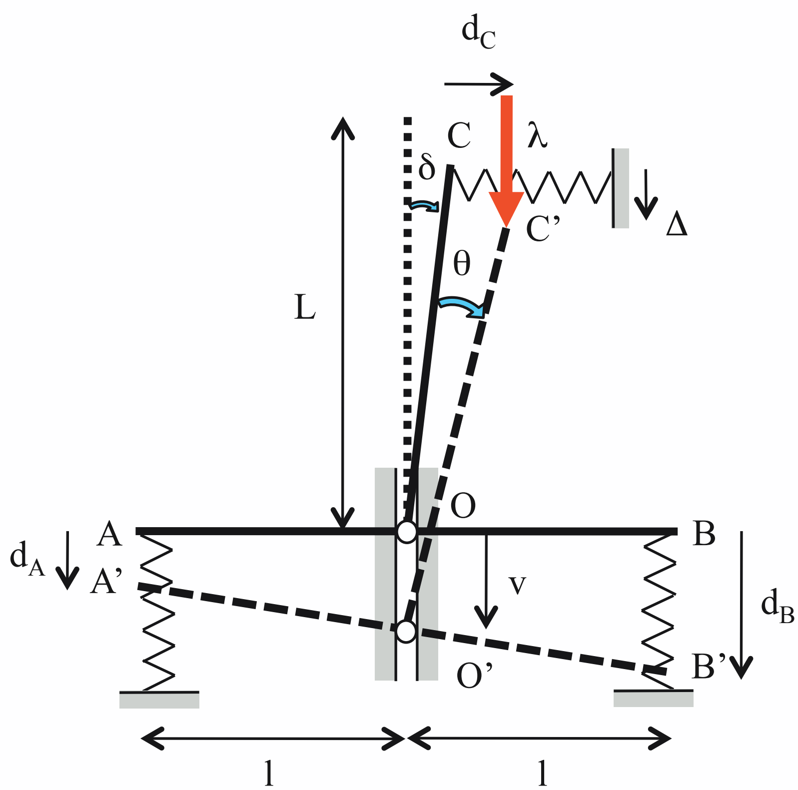

Figure AB-2.1: Imperfect rigid T model.

The kinematics of the imperfect rigid T model are the same as for its perfect counterpart (see (AB-1.1)), save for the vertical displacement Δ of point C, which, due to the small values of the angles θ and δ, is now given by

$$\Delta = v + L[\cos\delta - \cos(\theta + \delta)] \approx v + L(\frac{\theta^2}{2} + \theta\delta). \qquad \text{(AB-2.1)}$$

The potential energy $\overline{\mathcal{E}}$ of the imperfect system is:

$$\overline{\mathcal{E}}(v, \theta, \lambda, \delta) = E[v^2 + (l\theta)^2] + \frac{k}{2}(L\theta)^2 + \frac{m}{3}(L\theta)^3 + \frac{n}{4}(L\theta)^4 - \lambda[v + L(\frac{\theta^2}{2} + \theta\delta)]. \qquad \text{(AB-2.2)}$$

As expected the imperfect energy reduces to its perfect counterpart when the imperfection vanishes, i.e. $\overline{\mathcal{E}}(v, \theta, \lambda, 0) = \mathcal{E}(v, \theta, \lambda)$.

Extremizing $\overline{\mathcal{E}}$ with respect to the two degrees of freedom of the system v and θ, the two equilibrium equations obtained are:

$$\begin{aligned} \overline{\mathcal{E}}_{,v} &= 2Ev - \lambda = 0 \\ \overline{\mathcal{E}}_{,\theta} &= (2El^2 + kL^2)\theta + mL^3\theta^2 + nL^4\theta^3 - \lambda L(\theta + \delta) = 0 \end{aligned} \tag{AB-2.3}$$

Recalling from (AB-1.5) that $\lambda_c \equiv (2El^2 + kL^2)/L$, the solution of the above system is:

$$\begin{aligned} v(\lambda, \delta) &= \lambda/2E, \quad \lambda = [\lambda_c\theta + n(L\theta)^3]/(\theta + \delta) \qquad \text{if } n \neq 0,\ m = 0, \\ v(\lambda, \delta) &= \lambda/2E, \quad \lambda = [\lambda_c\theta + m(L\theta)^2]/(\theta + \delta) \qquad \text{if } n = 0,\ m \neq 0. \end{aligned} \tag{AB-2.4}$$

and the results are depicted in Fig. AB-2.2 and Fig. AB-2.3.

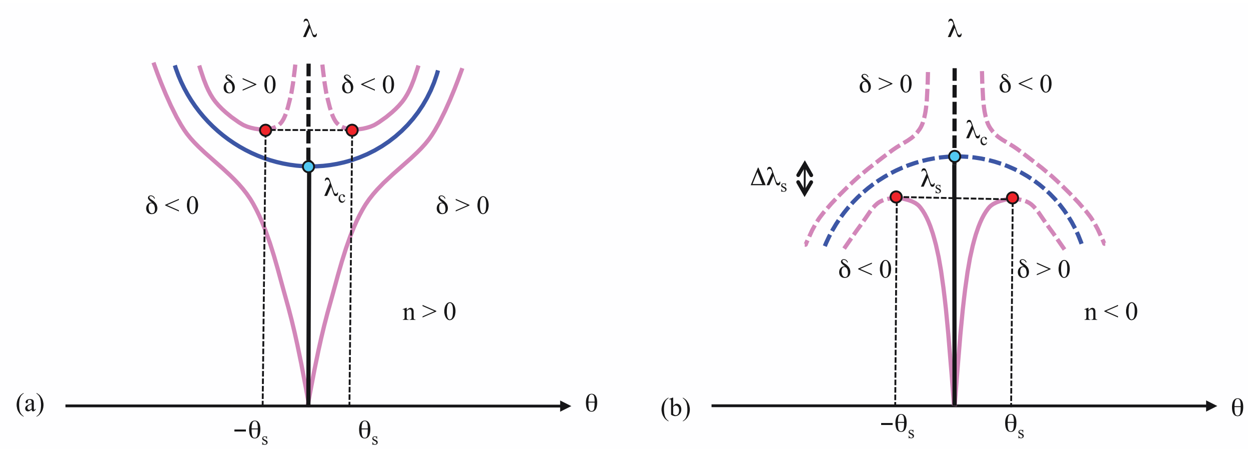

Figure AB-2.2: Equilibrium solutions for symmetric imperfect rigid T model. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

The most important observation made about the imperfect structure is the absence of bifurcation. Notice in Fig. AB-2.2 and Fig. AB-2.3 that the equilibrium branches no longer intersect as in the perfect case. Moreover, for a given value of the imperfection parameter δ, only one equilibrium branch passes through the initial, stress free state $(v, \theta, \lambda) = (0, 0, 0)$. The behavior of this simple imperfect model is representative of what occurs in a real structure. For values of the externally applied load parameter sufficiently lower than the perfect structure's critical load λ_c, the equilibrium solution of the real imperfect structure differs little from the principal solution of the perfect one. For load values near the critical load λ_c of the perfect structure, the equilibrium solution of the imperfect structure deviates considerably from the principal solution of its perfect counterpart, since it starts following one of the perfect structures bifurcated equilibrium paths.

Only the equilibrium paths passing through the origin have a physical meaning, for they are the ones accessible to the structure starting from rest $(v, \theta, \lambda) = (0, 0, 0)$. The other equi-

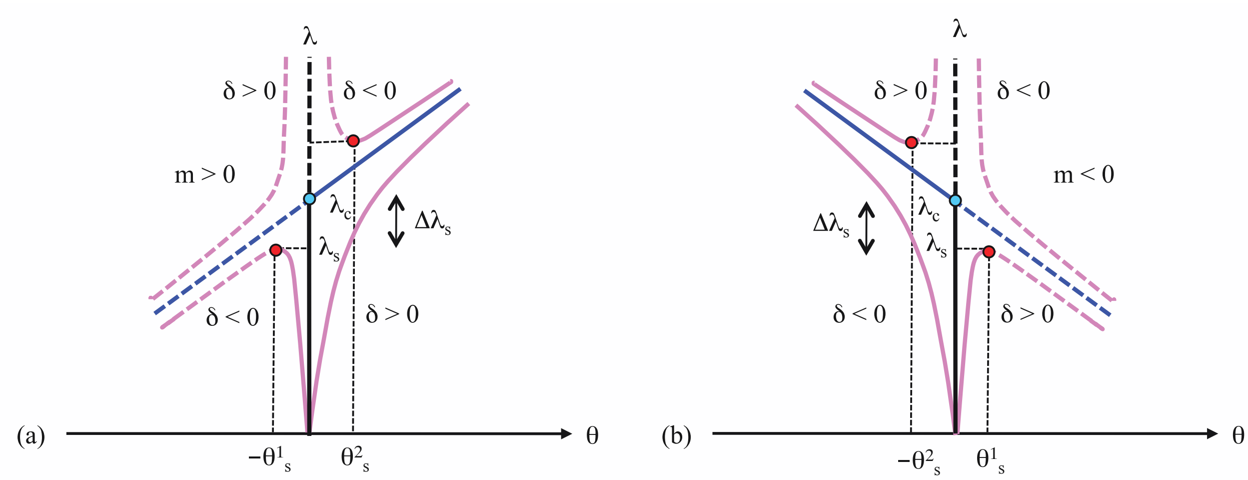

Figure AB-2.3: Equilibrium solutions for asymmetric imperfect rigid T model. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

librium paths cannot be reached in a continuing loading process starting from the unloaded state.

The next important question to be addressed concerns the stability of the aforedescribed equilibrium branches. Reasoning in a similar fashion as in the perfect case, an equilibrium point $v(\lambda, \delta)$, $\theta(\lambda, \delta)$ is stable if it is a local minimum of the system's energy $\overline{\mathcal{E}}$, which according to (AA-5.4), requires the matrix $\overline{\mathcal{E}}_{,\mathbf{uu}}$ to be positive definite:

$$\overline{\mathcal{E}}_{,\mathbf{uu}} = \begin{bmatrix} \overline{\mathcal{E}}_{,vv}\,, & \overline{\mathcal{E}}_{,v\theta} \\ \overline{\mathcal{E}}_{,\theta v}\,, & \overline{\mathcal{E}}_{,\theta\theta} \end{bmatrix} = \begin{bmatrix} 2E, & 0 \\ 0, & (\lambda_c - \lambda)L + 2mL^3\theta + 3nL^4\theta^2 \end{bmatrix}, \tag{AB-2.5}$$

or equivalently (since $E > 0$) if $\overline{\mathcal{E}}_{,\theta\theta} > 0$. Two cases are distinguished:

In case $m = 0, n \neq 0$ substituting (AB-2.4)$_1$ into the expression for $\overline{\mathcal{E}}_{,\theta\theta}$ in (AB-2.5) one obtains $\overline{\mathcal{E}}_{,\theta\theta} = [nL^4\theta^2(2\theta + 3\delta) + \lambda_c L\delta]/(\theta + \delta)$. When $n > 0$, $\overline{\mathcal{E}}_{,\theta\theta} > 0$ for all θ if $\theta\delta > 0$, and $\overline{\mathcal{E}}_{,\theta\theta} < 0$ for $|\delta| < |\theta| < \theta_s$, $\overline{\mathcal{E}}_{,\theta\theta} > 0$ for $|\theta| < |\delta|$ or $|\theta| > \theta_s$ if $\theta\delta < 0$ where θ_s is the positive root of $2nL^3(\theta_s)^3 + 3nL^3(\theta_s)^2\delta + \lambda_c\delta = 0$ for $\delta > 0$. (It can easily be shown that only one admissible root exists for adequately small δ). When $n < 0$, $\overline{\mathcal{E}}_{,\theta\theta} < 0$ for $|\theta| > |\delta|$ and $\overline{\mathcal{E}}_{,\theta\theta} > 0$ for $|\theta| < |\delta|$ if $\theta\delta < 0$ and $\overline{\mathcal{E}}_{,\theta\theta} > 0$ for $|\theta| < \theta_s$, $\overline{\mathcal{E}}_{,\theta\theta} < 0$ for $|\theta| > \theta_s$ if $\theta\delta > 0$ where θ_s is again the unique positive root of the cubic equation $2nL^3(\theta_s)^3 + 3nL^3(\theta_s)^2\delta + \lambda_c\delta = 0$ for $\delta > 0$.

It is interesting to notice that at $\theta = \theta_s$ for $\delta > 0$ (or $\theta = -\theta_s$ for $\delta < 0$), the $\lambda - \theta$ curves have an extremum of λ. Indeed from (AB-2.4)$_1$:

$$\frac{d\lambda}{d\theta} = \frac{2nL^3\theta^3 + 3nL^3\theta^2\delta + \lambda_c\delta}{(\theta + \delta)^2} \tag{AB-2.6}$$

which shows that the extremum of λ is reached for $|\theta| = \theta_s$.

For the case $m \neq 0, n = 0$ substituting (AB-2.4)$_2$ into the expression for $\overline{\mathcal{E}}_{,\theta\theta}$ from (AB-2.5) one obtains $\overline{\mathcal{E}}_{,\theta\theta} = [mL^3\theta(\theta + 2\delta) + \lambda_c L\delta]/(\theta + \delta)$. When $m\delta > 0$, $\overline{\mathcal{E}}_{,\theta\theta} > 0$ if $\theta\delta > 0$ and $\overline{\mathcal{E}}_{,\theta\theta} < 0$ for

$|\theta| > |\delta|$ if $\theta\delta < 0$ (assuming adequately small values of δ). When $m\delta < 0$, $\overline{\mathcal{E}}_{,\theta\theta} < 0$ for $|\theta| > \theta_s^1$ or $|\delta| < |\theta| < \theta_s^2$ and $\overline{\mathcal{E}}_{,\theta\theta} > 0$ for $|\theta| < \theta_s^1$ or $|\theta| > \theta_s^2$ where $\theta_s^1 > 0$ and $-\theta_s^2 < 0$ are the two roots of the quadratic $mL^2(\theta_s)^2 + 2mL^2\theta_s\delta + \lambda_c\delta = 0$ with $\theta_s^1 < \theta_s^2$.

Once more the points $\theta = -\theta_s^1,\ \theta_s^2$ for $\delta < 0$ and $\theta = -\theta_s^2,\ \theta_s^1$ for $\delta > 0$ are the extrema of the corresponding $\lambda - \theta$ curves. Indeed from (AB-2.4)$_2$ one has:

$$\frac{d\lambda}{d\theta} = \frac{mL^2\theta^2 + 2mL^2\theta\delta + \lambda_c\delta}{(\theta+\delta)^2} \tag{AB-2.7}$$

In Fig. AB-2.2 and Fig. AB-2.3 the stable equilibrium branches of the imperfect structure are drawn using a solid line while the unstable equilibrium branches are drawn using a dotted line.

Only for the case where the perfect structure exhibits a supercritical bifurcation $(m = 0, n > 0)$ the (physically admissible) equilibrium paths of the corresponding imperfect structure are stable for all possible values of the imperfection δ (assumed small). In all the other cases at least one (physically admissible) equilibrium path of the imperfect structure exhibits a load maximum beyond which the equilibrium is unstable. Such an instability is called in the literature a "*snap through*" instability. The snap through instability that happens in realistic imperfect structures explains the experimental observation that the critical load calculated for the perfect structure is always higher than the one actually measured in a lab test.

Using (AB-2.4) in (AB-2.6) and (AB-2.7) evaluated at θ_s one finds for $\Delta\lambda_s \equiv \lambda(\theta_s) - \lambda_c$:

$$\begin{aligned} &\Delta\lambda_s = 3nL^3(\theta_s)^2 &&\text{if } m = 0,\ n < 0 \\ &\Delta\lambda_s = 2mL^2\theta_s &&\text{if } m \neq 0,\ n = 0 \end{aligned} \tag{AB-2.8}$$

For small values of δ, by expanding (AB-2.6) and (AB-2.7) in terms of powers of δ one obtains the following expressions for θ_s and $\Delta\lambda_s$:

$$\begin{aligned} \theta_s &= \frac{1}{2^{1/3}}\ \frac{(\lambda_c)^{1/3}(-n)^{-1/3}}{L}\ |\delta|^{1/3} + O(|\delta|^{2/3}), \\ && \text{if } m = 0,\ n < 0 \\ \Delta\lambda_s &= \frac{3}{2^{2/3}} L(\lambda_c)^{2/3}(n)^{1/3}\ |\delta|^{2/3} + O(|\delta|) \\ \theta_s &= \frac{(\lambda_c)^{1/2}(-m\ \mathrm{sgn}\ \delta)^{-1/2}}{L}(\mathrm{sgn}\ \delta)\ |\delta|^{1/2} + O(|\delta|), \\ && \text{if } m \neq 0,\ n = 0 \\ \Delta\lambda_s &= -2L(\lambda_c)^{1/2}(-m\ \mathrm{sgn}\ \delta)^{1/2}\ |\delta|^{1/2} + O(|\delta|) \end{aligned} \tag{AB-2.9}$$

which shows that even a small imperfection δ can account for a rather substantial load reduction $\Delta\lambda_s$ given that $\Delta\lambda_s$ is proportional to the fractional powers $|\delta|^{1/2}$ or $|\delta|^{2/3}$.

AB-3 PERFECT SQUARE PLATE MODEL

The perfect rigid T model examined in subsection AB-1 has only one bifurcated equilibrium branch emerging from the principal path at $\lambda = \lambda_c$. This situation occurs in structures where the matrix $\mathcal{E}_{,\mathbf{uu}}$ has a simple eigenvalue at the critical load λ_c, as seen in (AB-1.6). Often in applications one encounters structures whose $\mathcal{E}_{,\mathbf{uu}}$ matrix has a multiple eigenvalue at λ_c. In such structures one typically finds more than one bifurcated equilibrium emerging from the critical load. This case is illustrated by the simple rigid plate model given below.

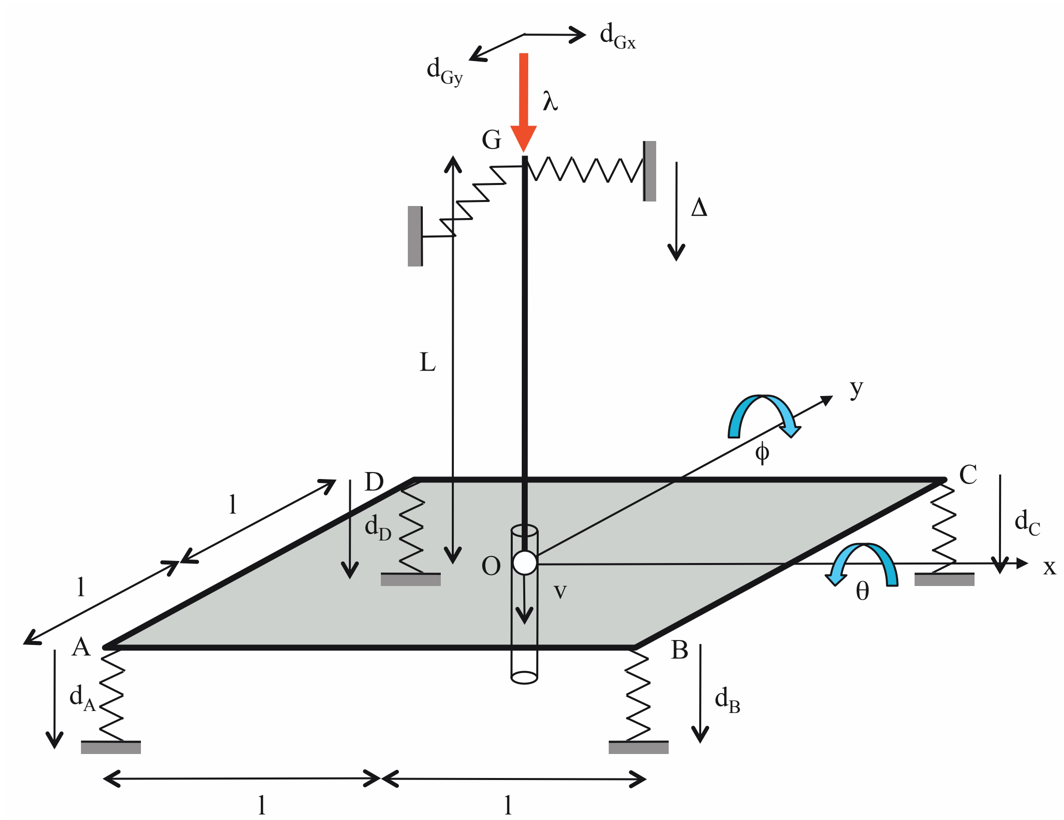

Figure AB-3.1: Perfect rigid plate model.

The rigid square plate ABCD shown in Fig. AB-3.1 has dimensions $2l \times 2l$. A rigid rod OG of length L is attached perpendicularly to the plate at its midpoint O. The center O of the rigid plate can only move vertically by a distance v, while the entire structure can rotate about the axes x and y by small angles θ and ϕ respectively. Four identical linear springs with restoring force f proportional to their length change d, ($f = -Ed$) are attached to the ends A, B, C, and D. At the end G two nonlinear springs are attached in the x and y directions with corresponding force–displacement relations $F_x = -[kd_x + m(d_x^2 + d_y^2) + n(2d_x^3 + 6d_x^2 d_y - 3d_x d_y^2 + 2d_y^3)]$ and $F_y = -[kd_y + 2md_x d_y + n(2d_y^3 + 6d_y^2 d_x - 3d_y d_x^2 + 2d_x^3)]$ respectively, where $d_x = d_{Gx}$ and $d_y = d_{Gy}$ are the two horizontal displacements of G. A vertical load λ is applied on OG at point G and the corresponding vertical displacement of G is Δ.

From kinematics $d_A, \ d_B, \ d_C, \ d_D$, the vertical displacements at points A, B, C and B, $d_{Gx}, \ d_{Gy}, \ \Delta$ the horizontal and vertical displacements at point G, are for small values of angles θ and ϕ (again taking approximations that are correct up to $O(\theta^2)$ and $O(\phi^2)$):

$$
\begin{gathered}
d_A = v + l\theta - l\phi, \quad d_B = v + l\theta + l\phi, \quad d_C = v - l\theta + l\phi, \quad d_D = v - l\theta - l\phi, \\
d_{Gx} = L\phi, \quad d_{Gy} = L\theta, \quad \Delta = v + L[1 - (1 - \sin^2\theta - \sin^2\phi)^{1/2}] \approx v + \frac{L}{2}(\theta^2 + \phi^2).
\end{gathered}
\tag{AB-3.1}
$$

The total energy $\mathcal{E}$ of the system, consisting of the energy stored in the springs and the potential energy of the applied load, is found to be:

$$
\begin{aligned}
\mathcal{E}(v,\theta,\phi,\lambda) = {}& 2E[v^2 + l^2(\theta^2+\phi^2)] + \frac{kL^2}{2}(\theta^2+\phi^2) + \frac{mL^3}{3}(3\theta^2\phi + \phi^3) + \\
&+ \frac{nL^4}{4}(2\theta^4 + 8\theta^3\phi - 6\theta^2\phi^2 + 8\theta\phi^3 + 2\phi^4) - \lambda[v + \frac{L}{2}(\theta^2+\phi^2)].
\end{aligned}
\tag{AB-3.2}
$$

By extremizing $\mathcal{E}$ with respect to its degrees of freedom, the three equilibrium equations of the system are:

$$
\begin{aligned}
&\mathcal{E}_{,v} = 4Ev - \lambda = 0, \\
&\mathcal{E}_{,\theta} = (4El^2 + kL^2 - \lambda L)\theta + 2mL^3\phi\theta + nL^4(2\theta^3 + 6\theta^2\phi - 3\theta\phi^2 + 2\phi^3) = 0, \\
&\mathcal{E}_{,\phi} = (4El^2 + kL^2 - \lambda L)\phi + mL^3(\phi^2 + \theta^2) + nL^4(2\phi^3 + 6\phi^2\theta - 3\phi\theta^2 + 2\theta^3) = 0.
\end{aligned}
\tag{AB-3.3}
$$

The principal solution of the above system, i.e. the one passing at zero load $\lambda = 0$ through the origin $(v,\theta,\phi) = (0,0,0)$ is:

$$
\overset{0}{v}(\lambda) = \lambda/4E, \quad \overset{0}{\theta}(\lambda) = \overset{0}{\phi}(\lambda) = 0.
\tag{AB-3.4}
$$

For $\phi \neq 0$, the same system of equilibrium equations has the following solutions:

$$
\begin{array}{lllll}
N1: & v(\lambda) = \lambda/4E, & \theta(\lambda) = \phi(\lambda), & \lambda = \lambda_c + 7nL^3\phi^2 & \\
N2: & v(\lambda) = \lambda/4E, & \theta(\lambda) = -\phi(\lambda), & \lambda = \lambda_c - 9nL^3\phi^2 & \\
N3: & v(\lambda) = \lambda/4E, & \theta(\lambda) = 2\phi(\lambda), & \lambda = \lambda_c + 18nL^3\phi^2 & m = 0, \ n \neq 0, \\
N4: & v(\lambda) = \lambda/4E, & \theta(\lambda) = \phi(\lambda)/2, & \lambda = \lambda_c + (9/2)nL^3\phi^2 & \\
 & & & & \\
M1: & v(\lambda) = \lambda/4E, & \theta(\lambda) = 0, & \lambda = \lambda_c + mL^2\phi & \\
M2: & v(\lambda) = \lambda/4E, & \theta(\lambda) = \phi(\lambda), & \lambda = \lambda_c + 2mL^2\phi & m \neq 0, \ n = 0, \\
M3: & v(\lambda) = \lambda/4E, & \theta(\lambda) = -\phi(\lambda), & \lambda = \lambda_c + 2mL^2\phi &
\end{array}
\tag{AB-3.5}
$$

with $\quad \lambda_c \equiv (4El^2 + kL^2)/L.$

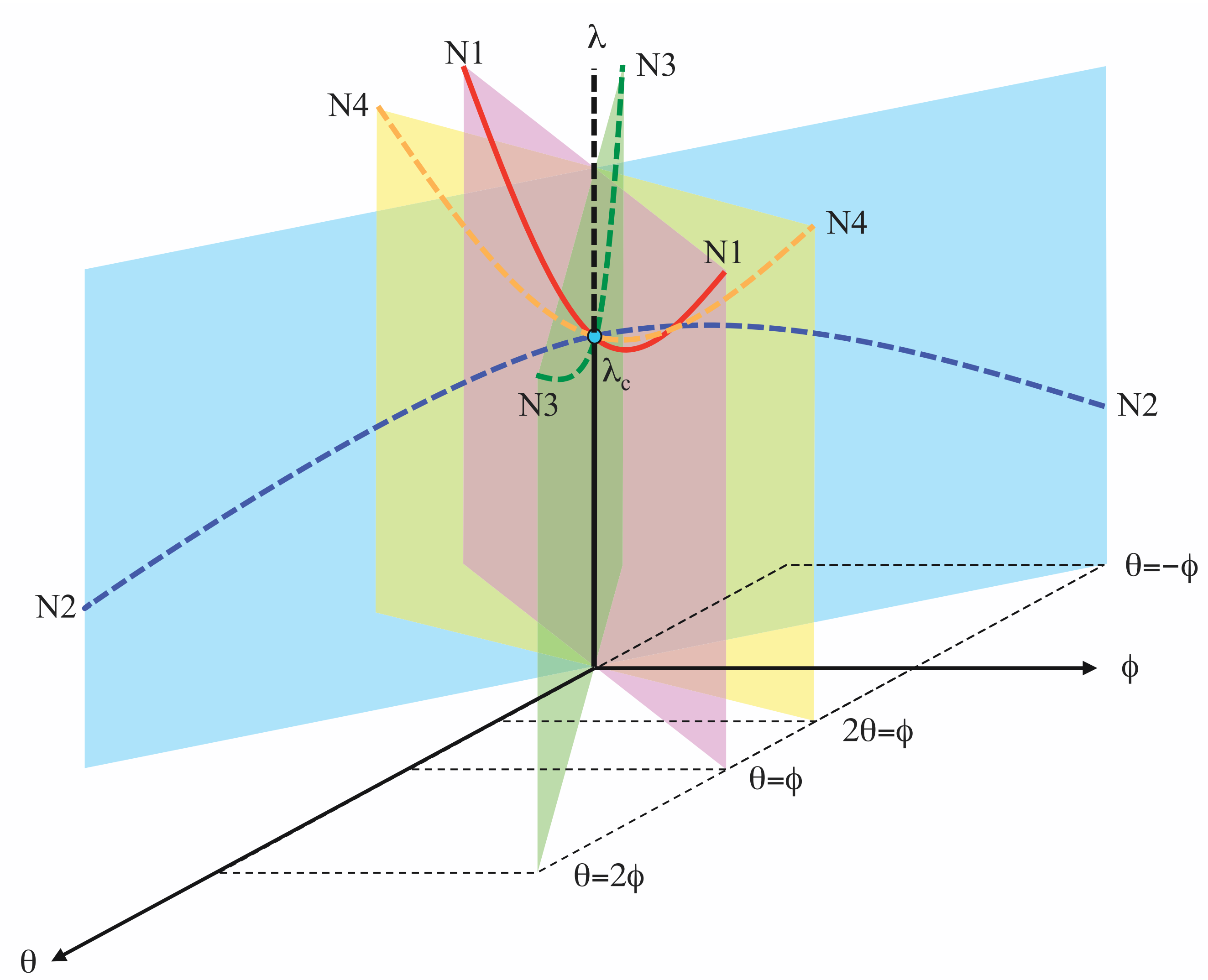

Figure AB-3.2: Bifurcated solutions for symmetric perfect rigid plate model. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

Note that in the case $m = 0, n \neq 0$ four equilibrium paths go through the critical load $\lambda = \lambda_c$ while for the case $m \neq 0, n = 0$ three equilibrium paths go through the critical load. Also notice that all the bifurcated equilibrium paths in case $m = 0, n \neq 0$ are symmetric (i.e. they intersect the λ axis at a right angle) while the corresponding equilibrium paths for case $m \neq 0, n = 0$ are asymmetric (i.e. they intersect the λ axis obliquely). One can also observe from (AB-3.5), where it was tacitly assumed that $n > 0$, that in the symmetric bifurcation case some bifurcated solutions (N1, N3, N4) are supercritical (i.e. $\lambda > \lambda_c$) while the remaining one (N2) is subcritical (i.e. $\lambda < \lambda_c$). The bifurcated paths for the asymmetric bifurcation are found from (AB-3.5) to be all transcritical. The equilibrium solutions in $\lambda - \theta - \phi$ space are depicted in Fig. AB-3.2 and Fig. AB-3.3 for the symmetric and asymmetric bifurcations respectively. Only $n > 0$ and $m > 0$ have been considered here. The discussion of $n < 0$ and

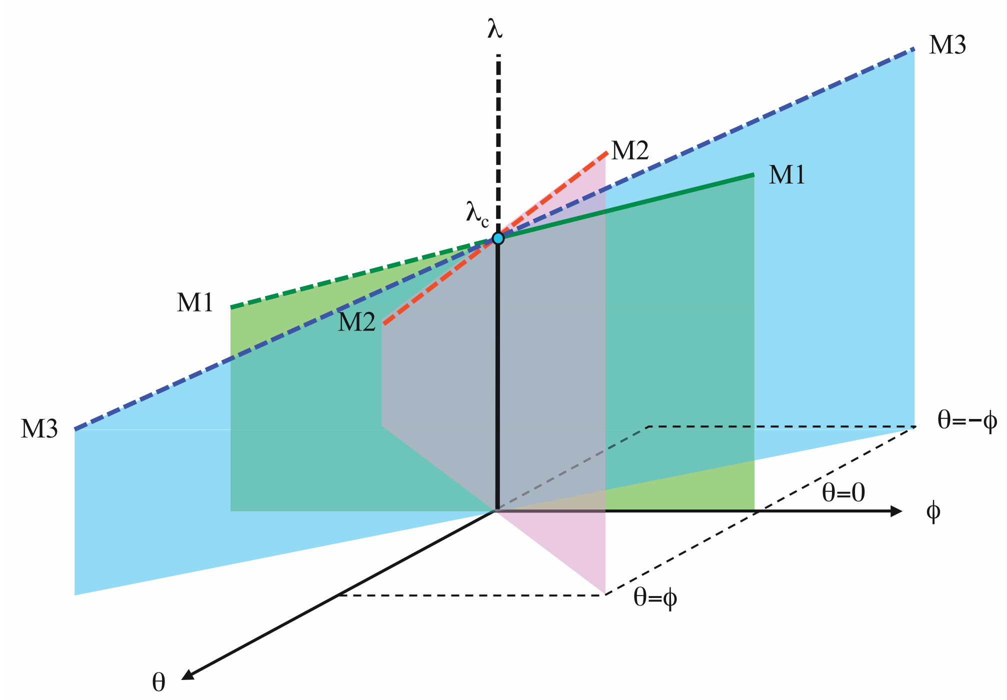

Figure AB-3.3: Bifurcated solutions for asymmetric perfect rigid plate model. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

$m < 0$ will be omitted as completely analogous.

Similar to the discussion of the perfect rigid T model of subsection AB-1, the next issue to be addressed is that of stability of the principal and bifurcated equilibrium paths of the plate given respectively by (AB-3.4) and (AB-3.5). Recall that an equilibrium path is stable if it corresponds to a local minimum of the system's potential energy. For the present three degree of freedom system, an equilibrium point $v(\lambda), \theta(\lambda), \phi(\lambda)$ is stable if it corresponds to local minimum of the energy $\mathcal{E}$, which according to (AA-5.4) requires the matrix $\mathcal{E}_{,\mathbf{uu}}$ to be positive definite:

$$\mathcal{E}_{,\mathbf{uu}} = \begin{bmatrix} \mathcal{E}_{vv}, & \mathcal{E}_{v\theta}, & \mathcal{E}_{,v\phi} \\ \mathcal{E}_{,\theta v}, & \mathcal{E}_{,\theta\theta}, & \mathcal{E}_{,\theta\phi} \\ \mathcal{E}_{,\phi v}, & \mathcal{E}_{,\phi\theta}, & \mathcal{E}_{,\phi\phi} \end{bmatrix} =$$

$$\begin{bmatrix} 4E, & 0, & 0 \\ 0, & (\lambda_c - \lambda)L + 2mL^3\phi + nL^4(6\theta^2 + 12\phi\theta - 3\phi^2), & 2mL^3\theta + nL^4(6\theta^2 - 6\phi\theta + 6\phi^2) \\ 0, & 2mL^3\theta + nL^4(6\phi^2 - 6\theta\phi + 6\theta^2), & (\lambda_c - \lambda)L + 2mL^3\phi + nL^4(6\phi^2 + 12\theta\phi - 3\theta^2) \end{bmatrix} \tag{AB-3.6}$$

One can see from (AB-3.4) that on the principal branch $\mathcal{E}_{,\mathbf{uu}} = \mathrm{diag}[4E, (\lambda_c - \lambda)L, (\lambda_c - \lambda)L]$ and hence the principal solution is stable for $0 \leq \lambda < \lambda_c$ and unstable for $\lambda \geq \lambda_c$.

For the symmetric bifurcation case $(m = 0, n > 0)$ substitution of (AB-3.5)$_1$ into (AB-3.6) yields:

$$
\begin{aligned}
\text{For } N1: \quad \mathcal{E}_{,\mathbf{uu}} &= \begin{bmatrix} 4E, & 0, & 0 \\ 0, & (8/7)(\lambda-\lambda_c)L, & (6/7)(\lambda-\lambda_c)L \\ 0, & (6/7)(\lambda-\lambda_c)L, & (8/7)(\lambda-\lambda_c)L \end{bmatrix}, \\
\text{For } N2: \quad \mathcal{E}_{,\mathbf{uu}} &= \begin{bmatrix} 4E, & 0, & 0 \\ 0, & 0, & 2(\lambda_c-\lambda)L \\ 0, & 2(\lambda_c-\lambda)L, & 0 \end{bmatrix}, \\
\text{For } N3: \quad \mathcal{E}_{,\mathbf{uu}} &= \begin{bmatrix} 4E, & 0, & 0 \\ 0, & (3/2)(\lambda-\lambda_c)L, & (\lambda-\lambda_c)L \\ 0, & (\lambda-\lambda_c)L, & 0 \end{bmatrix}, \\
\text{For } N4: \quad \mathcal{E}_{,\mathbf{uu}} &= \begin{bmatrix} 4E, & 0, & 0 \\ 0, & 0, & (\lambda-\lambda_c)L \\ 0, & (\lambda-\lambda_c)L, & (3/2)(\lambda-\lambda_c)L \end{bmatrix}.
\end{aligned}
\tag{AB-3.7}
$$

From (AB-3.7) and (AB-3.5) follows that the only positive definite matrix $\mathcal{E}_{,\mathbf{uu}}$ is found for the N1 equilibrium solution. Thus of the four possible equilibrium paths in the symmetric multiple bifurcation only one supercritical path (N1) is stable.

For the asymmetric bifurcation case $(m > 0, n = 0)$, substitution of (AB-3.5)$_2$ into (AB-3.6) yields:

$$
\begin{aligned}
\text{For } M1: \quad \mathcal{E}_{,\mathbf{uu}} &= \begin{bmatrix} 4E, & 0, & 0 \\ 0, & (\lambda-\lambda_c)L, & 0 \\ 0, & 0, & (\lambda-\lambda_c)L \end{bmatrix}, \\
\text{For } M2: \quad \mathcal{E}_{,\mathbf{uu}} &= \begin{bmatrix} 4E, & 0, & 0 \\ 0, & 0, & (\lambda-\lambda_c)L \\ 0, & (\lambda-\lambda_c)L, & 0 \end{bmatrix}, \\
\text{For } M3: \quad \mathcal{E}_{,\mathbf{uu}} &= \begin{bmatrix} 4E, & 0, & 0 \\ 0, & 0, & (\lambda_c-\lambda)L \\ 0, & (\lambda_c-\lambda)L, & 0 \end{bmatrix}.
\end{aligned}
\tag{AB-3.8}
$$

From (AB-3.8) and (AB-3.5) follows that the only positive definite matrix $\mathcal{E}_{,\mathbf{uu}}$ is found for the supercritical $(\lambda > \lambda_c)$ part of the M1 equilibrium solution.

For both the symmetric $(m = 0, n > 0)$ and asymmetric $(m > 0, n = 0)$ models, the stable and unstable equilibrium paths, denoted by a solid line and a dashed line respectively, are depicted in Fig. AB-3.2 and Fig. AB-3.3. It is interesting to note the exchange of stability of the principal branch at the critical load λ_c exactly as in the rigid T model analyzed in subsection AB-1 (see Fig. AB-1.2 and Fig. AB-1.3).

Finally of interest is the total potential energy $\mathcal{E}$, for a given load parameter λ, associated with each equilibrium path. Using (AB-3.4) into (AB-3.2) the potential energy of the principal branch is:

$$\mathcal{E} = -\frac{\lambda^2}{8E} \tag{AB-3.9}$$

For the bifurcated equilibrium branches of the symmetric structure $m = 0, n > 0$ the corresponding potential energies are from (AB-3.2) and (AB-3.5):

$$\begin{aligned}
\text{For } N1: \quad \mathcal{E} &= -\frac{\lambda^2}{8E} - \frac{(\lambda-\lambda_c)^2}{14nL^2}, \\
\text{For } N2: \quad \mathcal{E} &= -\frac{\lambda^2}{8E} + \frac{(\lambda-\lambda_c)^2}{18nL^2}, \\
\text{For } N3: \quad \mathcal{E} &= -\frac{\lambda^2}{8E} - \frac{5(\lambda-\lambda_c)^2}{72nL^2}, \\
\text{For } N4: \quad \mathcal{E} &= -\frac{\lambda^2}{8E} - \frac{5(\lambda-\lambda_c)^2}{72nL^2}.
\end{aligned} \tag{AB-3.10}$$

Notice that for $\lambda > \lambda_c$ the equilibrium branch with the least energy is the stable bifurcated branch N1. On the other hand for $\lambda < \lambda_c$ the stable principal equilibrium branch has less energy than the bifurcated subcritical branch. Hence for all values of the load parameter λ, the stable equilibrium path is the one with least energy. This situation is similar to the behavior of the simple perfect rigid T model. Indeed recall from (AB-1.7) and (AB-1.8) that for a given λ the stable equilibrium branch always corresponds to the lowest potential energy.

For the bifurcated equilibrium branches of the asymmetric structure $(m > 0, n = 0)$ the corresponding potential energies are from (AB-3.2) and (AB-3.5)$_2$

$$\begin{aligned}
\text{For } M1: \quad \mathcal{E}_b &= -\frac{\lambda^2}{8E} - \frac{(\lambda-\lambda_c)^3}{6m^2L^3}, \\
\text{For } M2: \quad \mathcal{E}_b &= -\frac{\lambda^2}{8E} - \frac{(\lambda-\lambda_c)^3}{12m^2L^3}, \\
\text{For } M3: \quad \mathcal{E}_b &= -\frac{\lambda^2}{8E} - \frac{(\lambda-\lambda_c)^3}{12m^2L^3}.
\end{aligned} \tag{AB-3.11}$$

From (AB-3.11) follows that for $\lambda < \lambda_c$ the minimum potential energy corresponds to the stable principal branch while for $\lambda > \lambda_c$ the minimum potential energy corresponds to the stable bifurcated branch M1. This situation is again similar to the behavior of the asymmetric rigid T model (see (AB-1.7) and (AB-1.8)).

AB-4 IMPERFECT SQUARE PLATE MODEL

As in the case of the rigid T model, the introduction of imperfections in the perfect rigid plate model investigated in subsection AB-3 provides a physically realistic model for the determination of the plate's actual equilibrium path in a loading process starting at $\lambda = 0$.

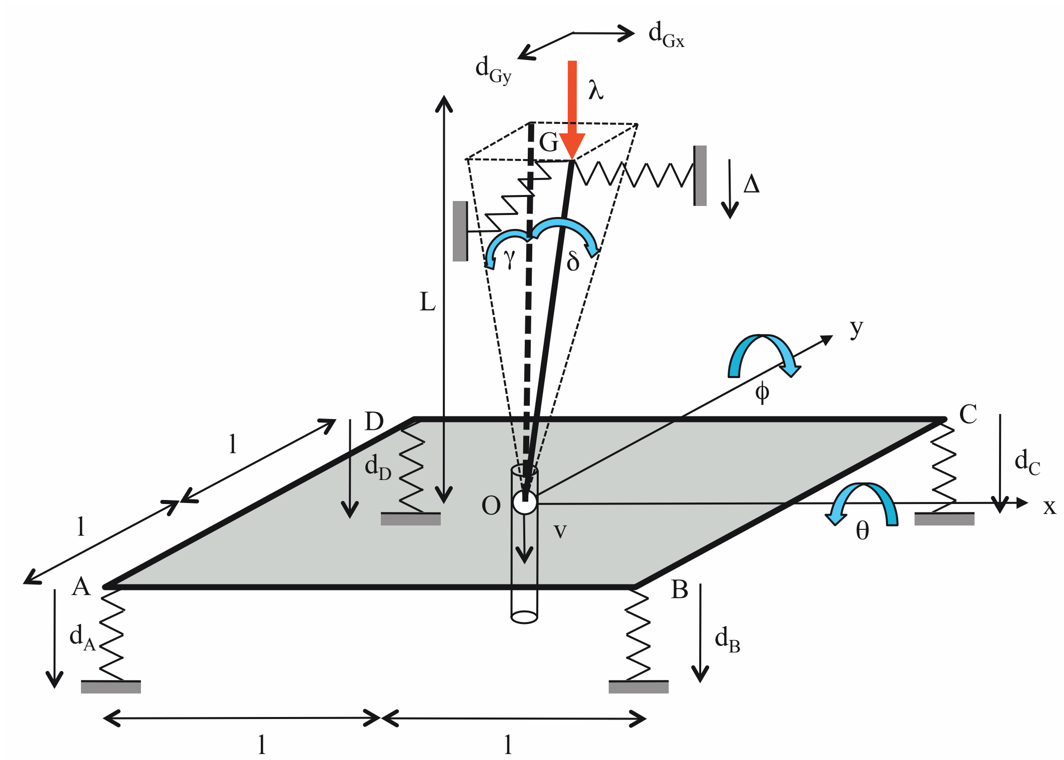

Figure AB-4.1: Imperfect square plate model.

Once again, a geometric type imperfection is considered. The rod OG is no longer normal to the plate but deviates from the ideal normal by the small angles γ and δ in the θ and ϕ directions respectively as shown in Fig. AB-4.1. All the remaining elements of the model (dimensions, stiffness of springs, etc.) remain the same as for the perfect structure shown in Fig. AB-3.1.

The kinematics of the imperfect rigid plate model are the same as for its perfect counterpart (see (AB-3.1)), save for the vertical displacement Δ of point G, which, due to the small values of the angles $\gamma,\ \delta,\ \theta,\ \phi$, is now given by

$$\Delta = v + L[(1 - \sin^2\gamma - \sin^2\delta)^{1/2} - (1 - \sin^2(\theta+\gamma) - \sin^2(\phi+\delta))^{1/2}] \approx v + L(\frac{\theta^2+\phi^2}{2} + \theta\gamma + \phi\delta), \quad \text{(AB-4.1)}$$

where, as in the previous sections, only terms up to the second order with respect to the small angles $\theta,\ \phi,\ \gamma,\ \delta$ are kept in the kinematic relations.

The total potential energy $\overline{\mathcal{E}}$ for the imperfect plate is:

$$\begin{aligned}\overline{\mathcal{E}}(v,\theta,\phi\lambda\gamma,\delta) &= 2E[v^2+l^2(\theta^2+\phi^2)]+\frac{kL^2}{2}(\theta^2+\phi^2)+\frac{mL^3}{3}(3\theta^2\phi+\phi^3)+\\ &+\frac{nL^4}{4}(2\theta^4+8\theta^3\phi-6\theta^2\phi^2+8\theta\phi^3+2\phi^4)-\lambda[v+\frac{L}{2}(\theta^2+\phi^2+2\gamma\theta+2\delta\phi)].\end{aligned} \tag{AB-4.2}$$

As expected, for the case of zero values for the imperfections, the imperfect energy yields back its perfect counterpart, i.e. $\overline{\mathcal{E}}(v,\theta,\phi,\lambda,0,0)=\mathcal{E}(v,\theta,\phi,\lambda)$ (see (AB-3.2)).

Extremizing $\overline{\mathcal{E}}$ with respect to its degrees of freedom, one obtains the following equilibrium equations:

$$\begin{aligned}&\overline{\mathcal{E}}_{,v}=4Ev-\lambda=0,\\ &\overline{\mathcal{E}}_{,\theta}=-\lambda L\gamma+(4El^2+kL^2-\lambda L)\theta+2mL^3\theta\phi+nL^4(2\theta^3+6\theta^2\phi-3\theta\phi^2+2\phi^3)=0,\\ &\overline{\mathcal{E}}_{,\phi}=-\lambda L\delta+(4El^2+kL^2-\lambda L)\phi+mL^3(\phi^2+\theta^2)+nL^4(2\phi^3+6\phi^2\theta-3\phi\theta^2+2\theta^3)=0.\end{aligned} \tag{AB-4.3}$$

Recalling from (AB-3.5) the definition of the critical load λ_c, the solution of the above system is found to be:

$$\begin{aligned}&v=\lambda/4E\\ &\lambda=[\lambda_c\theta+nL^3(2\theta^3+6\theta^2\phi-3\theta\phi^2+2\phi^3)]/(\theta+\gamma) && m=0,\ n\neq 0,\\ &\lambda=[\lambda_c\phi+nL^3(2\phi^3+6\phi^2\theta-3\phi\theta^2+2\theta^3)]/(\phi+\delta)\\ \\ &v=\lambda/4E\\ &\lambda=[\lambda_c\theta+2mL^2\theta\phi]/(\theta+\gamma) && m\neq 0,\ n=0.\\ &\lambda=[\lambda_c\phi+mL^2(\phi^2+\theta^2)]/(\phi+\delta)\end{aligned} \tag{AB-4.4}$$

Determining the imperfect structure's equilibrium paths is a rather cumbersome matter that will not be pursued here. Suffices to say that of all possible imperfections with a given amplitude $\varepsilon\equiv(\delta^2+\gamma^2)^{1/2}$, the imperfection with the "*worst*" shape, i.e. the one with the maximum load drop from the critical load, is the one with equilibrium paths that are on the the N2 plane for the symmetric case or on the M1 plane for the asymmetric one (see Fig. AB-1.2 and Fig. AB-1.3).

AB-5 TWO-BAR PLANAR TRUSS MODEL

The perfect rigid T and plate models discussed in the previous subsections all share the common feature of a trivial principal solution with a linear force-displacement response. Many applications of interest have non-trivial principal solutions with non-monotonic force-displacement responses, which exhibit limit loads. In these applications bifurcated equilibrium paths can also emerge from the non-monotonic principal solutions. The discrete two-bar plan truss model presented below is an illustrative example of structures with a non-trivial principal solution that exhibit snap-trough instabilities (associated with limit loads) as well as buckling instabilities (associated with bifurcations).

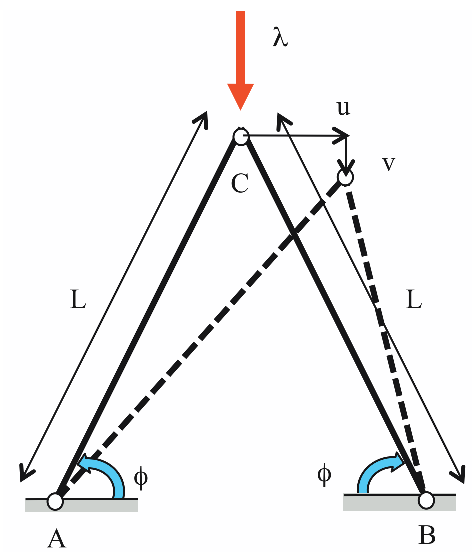

Figure AB-5.1: Two-bar truss model.

The two-bar planar truss shown in FIG. AB-5.1 consists of two elastic bars, AC (bar 1) and BC (bar 2) each of cross-sectional area A, Young's modulus E and initial length L. A vertical load λ is applied at node C which can move by u and v respectively along the horizontal and vertical directions. The response of each bar is linearly elastic ($\sigma = E\varepsilon$) where σ and ε are the stress and strain in the bar, which deforms only in the axial direction (no bending occurs in the bars).

The axial strain in each bar is given in terms of its final (ℓ) and initial (L) lengths by

$$\varepsilon = \frac{1}{2}\left[\left(\frac{\ell}{L}\right)^2 - 1\right]. \tag{AB-5.1}$$

From the geometry of the deformed configuration, one can see that the deformed lengths of the two bars are

$$\begin{aligned}(\ell_1)^2 &= (L\cos\phi + u)^2 + (L\sin\phi - v)^2,\\(\ell_2)^2 &= (L\cos\phi - u)^2 + (L\sin\phi - v)^2.\end{aligned} \tag{AB-5.2}$$

and hence the potential energy $\mathcal{E}$ of the truss, which consists of the elastic energy stored in the two bars plus the potential energy of the external load, takes the form

$$\mathcal{E}(u, v; \lambda) = \frac{1}{2} EAL[(\varepsilon_1)^2 + (\varepsilon_2)^2] - \lambda v. \tag{AB-5.3}$$

By extremizing $\mathcal{E}$ with respect to its degrees of freedom, the two equilibrium equations are

$$\begin{aligned} \mathcal{E}_{,u} &= \frac{EA}{L}[\varepsilon_1(u + L\cos\phi) + \varepsilon_2(u - L\cos\phi)] = 0, \\ \mathcal{E}_{,v} &= \frac{EA}{L}[\varepsilon_1(v - L\sin\phi) + \varepsilon_2(v - L\sin\phi)] - \lambda = 0. \end{aligned} \tag{AB-5.4}$$

The principal solution of this model, i.e. the one starting with zero displacements at zero load, is the symmetric solution with horizontal displacement $\overset{0}{u}(\lambda) = 0$, since as one can easily check, (AB-5.4) is always satisfied for $u = 0$. However this principal solution has a vertical displacement $\overset{0}{v}(\lambda)$ which is a non-monotonic function of λ, given from (AB-5.4)

$$\frac{EA}{L^3}\overset{0}{v}(\lambda)[\overset{0}{v}(\lambda) - 2L\sin\phi][\overset{0}{v}(\lambda) - L\sin\phi] - \lambda = 0. \tag{AB-5.5}$$

The dimensionless displacement $\overset{0}{v}(\lambda)/L$ versus dimensionless load $\lambda/(EA)$ for the principal solution is given by the cubic in $\overset{0}{v}/L$ equation (AB-5.5) and plotted in Fig. AB-5.2.

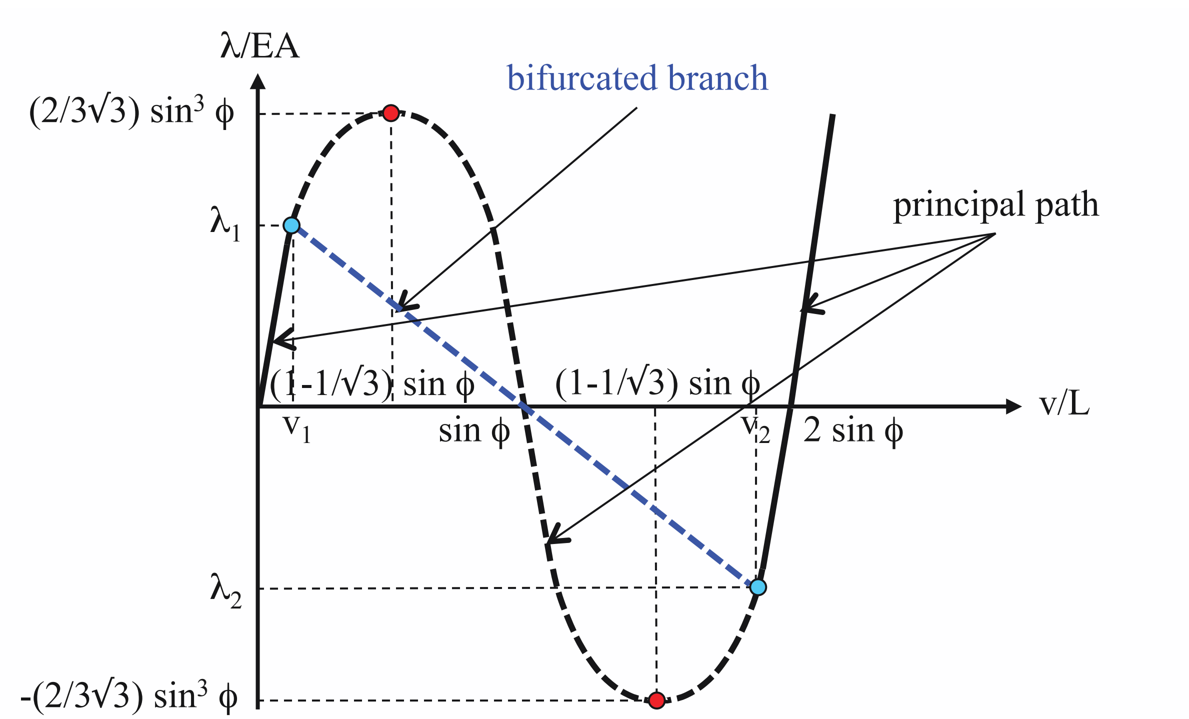

Figure AB-5.2: Principal and bifurcated solutions of the two–bar truss model in dimensionless load λ/EA vs. dimensionless vertical displacement v/L space. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

In addition to the above described principal solution, the structure also admits a bifurcated equilibrium branch with $u(\lambda) \neq 0$ which can also be found by solving (AB-5.4). Indeed by using (AB-5.1) and (AB-5.2) into (AB-5.4), one obtains for the bifurcated equilibrium branch

$$\begin{aligned} &u^2 + v^2 - 2Lv\sin\phi + 2L^2\cos^2\phi = 0, \\ &\frac{EA}{L^3}(u^2 + v^2 - 2vL\sin\phi)(v - L\sin\phi) = \lambda. \end{aligned} \tag{AB-5.6}$$

Notice that the equilibrium equation $(\text{AB-5.6})_1$ can be restated as

$$u^2 + (v - L\sin\phi)^2 = L^2(3\sin^2\phi - 2), \tag{AB-5.7}$$

which admits a real solution for $\sin\phi > (2/3)^{\frac{1}{2}}$, i.e. for angles $\phi > 54.7°$. In this case the bifurcated solution is a circle in the dimensionless displacements (u/L) vs (v/L) plane with center at $v/L = \sin\phi$ and radius $(3\sin^2\phi - 2)^{1/2}$ as shown in Fig. AB-5.3.

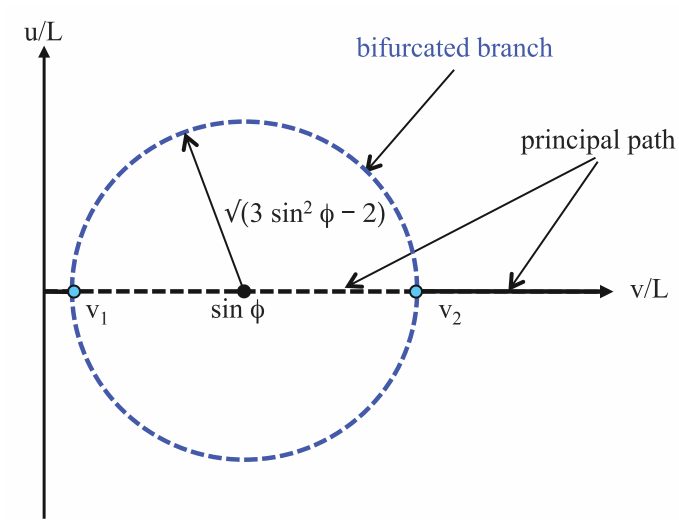

Figure AB-5.3: Principal and bifurcated solutions of the two–bar truss model in dimensionless horizontal displacement v/L vs. dimensionless vertical displacement v/L space. Stable solutions are drawn in continuous lines while unstable solutions are drawn in dashed lines.

The bifurcated solution in the (λ/EA) vs (v/L) plane is found, by substituting (AB-5.7) into the equilibrium equation $(\text{AB-5.6})_2$, to be

$$\frac{\lambda}{EA} = 2\cos^2\phi(\sin\phi - \frac{v}{L}), \tag{AB-5.8}$$

which appears as the straight line with negative slope in Fig. AB-5.2.

Notice that the straight line described by (AB-5.8) is the projection of the bifurcated equilibrium path given by (AB-5.6) into the $(\lambda/(EA))$ vs (v/L) plane, while the projection of the same bifurcated equilibrium path in the (u/L) vs (v/L) space is the circle depicted in Fig. AB-5.3. The bifurcated solution connects the with the principal branch of the two points depicted by a small blue circle in Fig. AB-5.2 and Fig. AB-5.3. The points have coordinates

$$
\begin{aligned}
(\lambda_1/(EA), v_1/L) &= (2\cos^2\phi(3\sin^2\phi - 2)^{\frac{1}{2}}, \sin\phi - (3\sin^2\phi - 2)^{\frac{1}{2}}) \\
(\lambda_2/(EA), v_2/L) &= (-2\cos^2\phi(3\sin^2\phi - 2)^{\frac{1}{2}}, \sin\phi + (3\sin^2\phi - 2)^{\frac{1}{2}}).
\end{aligned}
\tag{AB-5.9}
$$

Having found the principal as well as the bifurcated equilibrium paths of the two-bar planar truss model, the next topic to be addressed pertains to the stability of these solutions, which is given by analyzing the matrix $\mathcal{E}_{,\mathbf{uu}}$, found with the help of (AB-5.4) to be

$$
\mathcal{E}_{,\mathbf{uu}} = \begin{bmatrix} \mathcal{E}_{,uu} & \mathcal{E}_{,uv} \\ \mathcal{E}_{,vu} & \mathcal{E}_{,vv} \end{bmatrix}. \tag{AB-5.10}
$$

By evaluating $\mathcal{E}_{,\mathbf{uu}}$ on the principal solution $\overset{0}{u}(\lambda) = 0$ (AB-5.10) yields principal solution.

$$
\mathcal{E}_{,\mathbf{uu}} = \frac{EA}{L}\begin{bmatrix} (\overset{0}{v}/L)^2 - 2(\overset{0}{v}/L)\sin\phi + 2\cos^2\phi, & 0 \\ 0, & 3[(\overset{0}{v}/L)^2 - 2(\overset{0}{v}/L)\sin\phi + \frac{2}{3}\sin^2\phi] \end{bmatrix} \tag{AB-5.11}
$$

The roots of $\mathcal{E}_{,uu}$ are $\overset{0}{v}/L = \sin\phi \pm (3\sin^2\phi - 2)^{1/2}$ which correspond to the displacements $(\overset{0}{v}/L)_1$ and $(\overset{0}{v}/L)_2$ given in (AB-5.9) which are the displacements corresponding to the start and finish of the bifurcated equilibrium branch as seen in Fig. AB-5.2. The roots of $\mathcal{E}_{,vv}$ are $\overset{0}{v}/L = \sin\phi(1 \pm 1/\sqrt{3})$ which are the displacements $(\overset{0}{v}/L)_{max}$ and $(\overset{0}{v}/L)_{min}$ corresponding to the maximum and minimum loads of the principal solution as seen in Fig. AB-5.2. Hence in the absence of a bifurcated solution, i.e. for $\phi \leq 54.7°$. $\mathcal{E}_{,uu} > 0$ and the stability of the principal solution is determined solely by the sign of $\mathcal{E}_{,vv}$ which is < 0 in the interval of $\overset{0}{v}/L \in [(\overset{0}{v}/L)_{max}, (\overset{0}{v}/L)_{min}]$ and > 0 outside. Consequently, in the absence of a bifurcated solution, the principal branch is stable in the two load increasing branches, i.e. from zero load up to the maximum load and from the minimum load and beyond, while it is unstable in the load decreasing branch.

In the presence of a bifurcation ($\phi > 54.7°$) the principal solution is stable when both $\mathcal{E}_{,uu} > 0$ and $\mathcal{E}_{,vv} > 0$. Notice that the roots of $\mathcal{E}_{,uu}$ from (AB-5.11) are $(\overset{0}{v}/L) = \sin\phi \pm (3\sin^2\phi - 2)^{1/2}$, which are the displacements $(\overset{0}{v}/L)_1$ and $(\overset{0}{v}/L)_2$ where the bifurcated path connects with the principal solution. Since $\mathcal{E}_{,uu} < 0$ in the interval $\overset{0}{v}/L \in [(\overset{0}{v}/L)_1, (\overset{0}{v}/L)_2]$, the principal solution, in the case of existence of a bifurcated path, as seen in FIG. AB-5.2, is unstable (dashed line) for those displacements $(\overset{0}{v}/L)$ for which a bifurcated solution exists and is stable (solid line) otherwise. It should be noted at this point that in plotting Fig. AB-5.2 it was tacitly assumed that $(\overset{0}{v}/L)_1 < (\overset{0}{v}/L)_{max}$, or equivalently from (AB-5.9) $\sin^2\phi > 3/4$, ($\phi > 60°$), in which case bifurcation occurs prior to reaching maximum load, as load increases away from zero. For $60° < \phi < 54.7°$, the first bifurcation occurs past the maximum load on the descending part of the principal branch and ends prior to reaching the minimum load, also on the descending part of the principal branch.

Finally, the stability of the bifurcated path is investigated. Evaluating $\mathcal{E},_{\mathbf{uu}}$ on the bifurcated equilibrium branch, one obtains from (AB-5.10) with the help of (AB-5.7)

$$\mathcal{E},_{\mathbf{uu}} = \frac{EA}{L}\begin{bmatrix} 2(u/L)^2 & 2(u/L)[(v/L)-\sin\phi] \\ 2(u/L)[(v/L)-\sin\phi] & 2[-(u/L)^2+4\sin^2\phi-3] \end{bmatrix}, \tag{AB-5.12}$$

The determinant of the above matrix evaluated on the bifurcated equilibrium path (AB-5.7) is found to be

$$\text{Det } \mathcal{E},_{\mathbf{uu}} = \frac{4EA}{L}\left(\frac{u}{L}\right)^2[\sin^2\phi-1] < 0, \tag{AB-5.13}$$

thus establishing that the entire bifurcated equilibrium path is unstable.

AC BIFURCATION AND STABILITY - LSK ASYMPTOTICS FOR ELASTIC CONTINUA

In the first two sections of this chapter, the notions of stability and bifurcation of equilibrium solutions in nonlinear elastic solids have been introduced and illustrated by simple examples, which admitted analytical solutions. Unfortunately this is not the case for more realistic models, discrete as well as continuum.

To this end, an asymptotic method comes to rescue. As it turns out, a powerful asymptotic technique, termed "*Lyapunov – Schmidt – Koiter*" method (LSK) can be applied to track the equilibrium solutions of perfect or imperfect systems near critical points and check their stability. This method is presented in this section for the case of continuum, conservative elastic systems (structures or solids).

AC-1 FUNCTIONALS AND THEIR DERIVATIVES

The treatment of the bifurcation, post-bifurcation and imperfection sensitivity behavior of continuous elastic systems requires some elements from the calculus of variations. The purpose of this brief section is not the presentation of a mathematically oriented introduction to the subject. The goal here is to illustrate the technique for calculating the required derivatives of the system's potential energy involved in the corresponding bifurcation and stability analyses.

For an elastic system, its potential energy is given by a real valued functional $\mathcal{E}(u, \lambda)$, where the term "*functional*" is used to indicate that the function's independent variable is itself a function defined at each point of the elastic system in question. For elastic systems, the independent variable is $u \equiv \mathbf{u}(\mathbf{x})$, typically the displacement field $\mathbf{u}$ at point $\mathbf{x}$ of the solid in question. The independent variable u is a scalar or vector valued function of position in the system depending on the application. The scalar parameter λ, usually termed the "*load parameter*", controls the externally applied loads or displacements to the system. In general, a continuous structure has (countably) infinite possibilities to deform and hence the displacement field $u \in U$, with U some appropriate, infinite dimensional vector space. Often in applications U is the simplest possible such space, namely a Hilbert space (or a Cartesian product of such spaces). For the case of a discrete structure $U = \mathbb{R}^n$ where n is the total number of the degrees of freedom. Henceforth, it will be assumed that U possesses an inner product. If $u_1, u_2 \in U$ their inner product is denoted by (u_1, u_2). The norm of an element $u \in U$ will be the inner product induced one, namely $\parallel u \parallel \equiv (u, u)^{1/2}$.

Of interest is the notion of the derivative of a real valued functional, say $\mathcal{E}(u)$, with respect to its argument u. To this end one defines the first derivative of $\mathcal{E}$ with respect to u, denoted by $\mathcal{E},_u$, to be a linear operator on U, i.e. a linear function assigning a real scalar $\mathcal{E},_u \delta u$ to every element $\delta u \in U$. The mathematical definition for this derivative, also termed the "*Frechet derivative*" is:

$$\lim_{\parallel \delta u \parallel \to 0} \mid \mathcal{E}(u + \delta u) - \mathcal{E}(u) - \mathcal{E},_u \delta u \mid / \parallel \delta u \parallel = 0 \qquad \text{(AC-1.1)}$$

Since the δu in the above definition is arbitrary, one can fix in U the direction of δu and consider in the definition (AC-1.1) the special case $\delta u = \epsilon v$ with $\parallel v \parallel = 1$ but $\epsilon \to 0$. Thus, if $\mathcal{E},_u$ exists it will satisfy:

$$\lim_{\epsilon \to 0} \mid \mathcal{E}(u + \epsilon v) - \mathcal{E}(u) - \epsilon \mathcal{E},_u v \mid / \epsilon = 0 \qquad \text{(AC-1.2)}$$

Recalling the standard definition for the partial derivative, the above equation can alternately be rewritten:

$$\mathcal{E},_u v = [\partial \mathcal{E}(u + \epsilon v) / \partial \epsilon]_{\epsilon = 0} \qquad \text{(AC-1.3)}$$

thus providing the practical method for the computation of the derivative $\mathcal{E},_u$ of $\mathcal{E}$. Strictly speaking the existence of the special (weaker) derivative defined in (AC-1.2) or (AC-1.3), also termed the "*Gateau derivative*" of the functional, does not imply the existence of the stronger Frechet derivative defined in (AC-1.1). In all the applications of interest, however, such pathological cases will not arise and the computationally simpler Gateau definition will be used for calculations.

One can in a similar fashion define the derivative of the linear operator $\mathcal{E}_{,u}$, considered as a function of u, with respect to u. The result $\mathcal{E}_{,uu}$ is a symmetric, bilinear operator, operating on arbitrary elements $v, w \in U$ which, in analogy to (AC-1.3), is given by:

$$(\mathcal{E}_{,uu}\, v)w = (\mathcal{E}_{,uu}\, w)v = [\partial^2 \mathcal{E}(u + \epsilon v + \zeta w)/\partial\epsilon\partial\zeta]_{\epsilon=\zeta=0} \tag{AC-1.4}$$

The p^{th} order derivative of $\mathcal{E}(u)$ is a completely symmetric, with respect to any two arguments, p-linear operator, computed by a straightforward generalization of (AC-1.4). It should be noticed for future use that, like in the case of a real valued function of a real argument, one can have under suitable conditions a converging Taylor series expansion for the functional $\mathcal{E}(u)$, i.e.:

$$\mathcal{E}(u + \delta u) = \mathcal{E}(u) + \frac{1}{1!}\mathcal{E}_{,u}\, \delta u + \frac{1}{2!}(\mathcal{E}_{,uu}\, \delta u)\delta u + \frac{1}{3!}((\mathcal{E}_{,uuu}\, \delta u)\delta u)\delta u + ... \tag{AC-1.5}$$

It is perhaps useful to illustrate the above described definitions by applying them to a concrete example, namely the functional:

$$\mathcal{E}(u) = \int_0^1 [\exp(w) + (2 + \cos(x))w^2 + 3(v_{,x}^2 + v^2)]dx \tag{AC-1.6}$$

In this example $u \equiv (v(x), w(x))$. Moreover, for the above integral to make sense v and w are required to belong to some appropriate functional space. Although from the mathematical standpoint it is important to provide the spaces to which the displacement functions belong, in the engineering applications considered in this work it is tacitly assumed that all the integrals are finite and that all the displacement functions are adequately smooth.

Employing (AC-1.3), the first derivative of $\mathcal{E}$ is found to be:

$$\begin{aligned}\mathcal{E}_{,u}\, u_1 &= \frac{\partial}{\partial\epsilon_1}\Big\{\int_0^1 [\exp(w + \epsilon_1 w_1) + (2 + \cos(x))(w + \epsilon_1 w_1)^2 \\ &\quad + 3((v_{,x} + \epsilon_1 v_{1,x})^2 + (v + \epsilon_1 v_1)^2)]dx\Big\}_{\epsilon_1=0} \\ &= \int_0^1 [\exp(w)w_1 + 2(2 + \cos(x))ww_1 + 6(v_{,x}\, v_{1,x} + vv_1)]dx\end{aligned} \tag{AC-1.7}$$

where $u_1 = (v_1, w_1)$ and v_1, w_1 are arbitrary functions belonging to the same spaces as v, w respectively on which the linear operator $\mathcal{E}_{,u}$ acts. The linearity of $\mathcal{E}_{,u}$ is easily checked.

Similarly, from (AC-1.4) the second derivative of $\mathcal{E}$ is:

$$\begin{aligned}(\mathcal{E},_{uu} u_1)u_2 &= \frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}\Big\{\int_0^1[\exp(w+\epsilon_1 w_1+\epsilon_2 w_2)+(2+\cos(x))(w+\epsilon_1 w_1+\epsilon_2 w_2)^2 \\ &+3((v,_x+\epsilon_1 v_{1,x}+\epsilon_2 v_{2,x})^2+(v+\epsilon_1 v_1+\epsilon_2 v_2)^2)]dx\Big\}_{\epsilon_1=\epsilon_2=0} \qquad \text{(AC-1.8)} \\ &= \int_0^1[\exp(w)w_1 w_2+2(2+\cos(x))w_1 w_2+6(v_{1,x}v_{2,x}+v_1 v_2)]dx\end{aligned}$$

It is not difficult to verify that $(\mathcal{E},_{uu} u_1)u_2$ is linear with respect to both arguments u_1 and u_2 as well as symmetric i.e. $(\mathcal{E},_{uu} u_1)u_2 = (\mathcal{E},_{uu} u_2)u_1$.

One can continue in a similar fashion with the higher order of derivatives of $\mathcal{E}$. Thus the third derivative of $\mathcal{E}$ is a completely symmetric trilinear operator namely:

$$((\mathcal{E},_{uuu} u_1)u_2)u_3 = \int_0^1[\exp(w)w_1 w_2 w_3]dx \qquad \text{(AC-1.9)}$$

In general, the p^{th} order derivative of $\mathcal{E}$ can be computed by:

$$(((\mathcal{E},_{uuu...u})u_1)u_2...)u_p = \frac{\partial^p}{\partial\epsilon_1\partial\epsilon_2...\partial\epsilon_p}\Big[\mathcal{E}\big(u+\sum_{i=1}^{p}\epsilon_i u_i\big)\Big]_{\epsilon_1=\epsilon_2=...\epsilon_p=0} \qquad \text{(AC-1.10)}$$

For the special case where the displacement space is finite dimensional i.e. $U=\mathbb{R}^n$ and noting that $\{\mathbf{e}_i\}_{i=1}^n$ is the corresponding orthonormal basis ($u=\sum_{i=1}^{n} u_i\ \mathbf{e}_i$), from (AC-1.3) the first derivative of $\mathcal{E}$ is the vector:

$$\mathcal{E},_u = \sum_{i=1}^{n}\frac{\partial\mathcal{E}}{\partial u_i}\mathbf{e}_i \qquad \Big(\mathcal{E},_u v = \sum_{i=1}^{n}\frac{\partial\mathcal{E}}{\partial u_i}v_i\Big) \qquad \text{(AC-1.11)}$$

From (AC-1.4) the second derivative of $\mathcal{E}$ is the symmetric matrix:

$$\mathcal{E},_{uu} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\mathcal{E}}{\partial u_i\partial u_j}\mathbf{e}_i\mathbf{e}_j \qquad \Big((\mathcal{E},_{uu} v)w = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\mathcal{E}}{\partial u_i\partial u_j}v_i w_j\Big) \qquad \text{(AC-1.12)}$$

In general the p^{th} derivative of $\mathcal{E}$ is the symmetric rank p tensor:

$$\mathcal{E},_{uu..u} = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^p\mathcal{E}}{\partial u_i\partial u_j...\partial u_k}\mathbf{e}_i\mathbf{e}_j...\mathbf{e}_k \qquad \text{(AC-1.13)}$$

AC-2 CRITICAL POINTS: LIMIT VERSUS BIFURCATION POINTS

As discussed in the previous section, an elastic system is completely described by its potential energy $\mathcal{E}(u,\lambda)$, where the admissible displacement $u \in U$ and where λ is the load parameter. The potential energy is arbitrarily set to zero for zero displacements:

$$\mathcal{E}(0,\lambda) = 0 \tag{AC-2.1}$$

The system evolves from its initial stress-free equilibrium configuration at which $\lambda = 0$ and $u = 0$, to a loaded configuration with $\lambda \neq 0,\ u \neq 0$. For a given load level, the corresponding equilibrium equations are found by extremizing $\mathcal{E}$ with respect to u[3], namely:

$$\mathcal{E}_{,u}(u,\lambda)\delta u = 0, \quad \forall\ \delta u \in U \tag{AC-2.2}$$

For all physically realistic problems, for a given load parameter λ one expects the solution to the equilibrium equation (AC-2.2) to be unique in some neighborhood of $(u,\lambda) = (0,0)$. This solution denoted by $\overset{0}{u}(\lambda)$ and termed the "*principal branch*", in addition to identically satisfying the equilibrium equation for all values of λ, is the only equilibrium branch that passes trough the origin $(u,\lambda) = (0,0)$, i.e.

$$\mathcal{E}_{,u}(\overset{0}{u}(\lambda),\lambda)\delta u = 0, \quad \overset{0}{u}(0) = 0 \tag{AC-2.3}$$

One further assumes that for a physically meaningful problem the principal equilibrium solution $\overset{0}{u}(\lambda)$ has to be stable in some neighborhood of $\lambda = 0$, which implies the existence of a positive number $\overset{0}{\beta}(\lambda)$[4] such that:

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u \geq \overset{0}{\beta}(\lambda) \parallel \delta u \parallel^2, \quad \overset{0}{\beta}(\lambda) > 0 \tag{AC-2.4}$$

A symmetric operator satisfying the above property is termed in mathematics a "*strongly elliptic*" operator. Recalling that for the finite dimensional case, strong ellipticity coincides with positive definiteness of the matrix corresponding to the operator in question, by abuse of language a symmetric operator satisfying the above stability condition will subsequently be called "*positive definite*". It will further be assumed that the quantity $\overset{0}{\beta}(\lambda)$ is the minimum eigenvalue of the stability operator $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)$ and satisfies

$$\overset{0}{\beta}(\lambda) = \min_{\|\delta u\|=1} [(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u]. \tag{AC-2.5}$$

This eigenvalue will play an important role in the subsequent stability investigations.

Assuming that $\overset{0}{u}(\lambda)$ is an adequately smooth function of λ, one obtains by differentiating the equilibrium equation (AC-2.3) with respect to λ:

$$\left[\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)(d\overset{0}{u}/d\lambda) + \mathcal{E}_{,u\lambda}(\overset{0}{u}(\lambda),\lambda)\right]\delta u = 0 \tag{AC-2.6}$$

[3]NOTE: From here and subsequently δu denotes an arbitrary element of the space of admissible functions U.

[4]NOTE: From here on all quantities associated with the principal solution will be surmounted by a $\overset{0}{(\ )}$.

For as long as the stability operator $\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)$ is positive definite, it is also invertible with inverse $(\mathcal{E},_{uu})^{-1}$. Hence, equation (AC-2.6) is solvable and produces a unique solution $d\overset{0}{u}/d\lambda$. This way, by starting at load $\lambda = 0$, one can visualize a constructive method for finding the equilibrium path $\overset{0}{u}(\lambda)$. Assuming that at a given load λ the solution $\overset{0}{u}(\lambda)$ is known, then at $\lambda + \Delta\lambda$ (AC-2.6) gives $\overset{0}{u}(\lambda+\Delta\lambda) \approx \overset{0}{u}(\lambda) + (d\overset{0}{u}/d\lambda)\Delta\lambda$ since the existence of $d\overset{0}{u}/d\lambda = -(\mathcal{E},_{uu})^{-1}\mathcal{E},_{u\lambda}$ is guaranteed by (AC-2.4). This is exactly the approach that forms the basis for most numerical algorithms (variations of the incremental Newton - Raphson method) that calculate the equilibrium paths in nonlinear elastic systems.

Suppose now that as the load λ increases (without loss of generality it is tacitly assumed that $\lambda \geq 0$), it reaches a value λ_c termed "*critical load*" for which the stability operator loses its positive definiteness and becomes singular. More specifically there exist a unit vector $\overset{1}{u} \in U$ such that:

$$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda_c),\lambda_c)\overset{1}{u})\delta u = 0, \quad \| \overset{1}{u} \| = 1 \tag{AC-2.7}$$

At this point in the interest of simplicity it will be assumed that the eigenmode $\overset{1}{u}$ is unique (up to sign), a condition which will be relaxed subsequently when structures with multiple bifurcation points are considered. Under this assumption $\overset{1}{u}$ is the only direction in which $\mathcal{E},^c_{uu} \equiv \mathcal{E},_{uu}(\overset{0}{u}(\lambda_c),\lambda_c)$ loses its positive definiteness while in all other directions δv of the space U, the operator $\mathcal{E},^c_{uu}$ continues to be positive definite. Hence, it is assumed that a constant $c > 0$ exists such that:

$$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda_c),\lambda_c)\delta v)\delta v \geq c \| \delta v \|^2, \quad c > 0, \quad \forall \delta v \in \mathcal{N}^\perp \tag{AC-2.8}$$

where the set $\mathcal{N} \equiv \{u \in U \mid u = \alpha\overset{1}{u},\ \forall\alpha \in \mathbb{R}\}$ is called the "*null space*" of the stability operator $\mathcal{E},^c_{uu}$ and is defined as the set containing all elements $u \in U$ for which the expression $((\mathcal{E},^c_{uu})u)u = 0$. The set $\mathcal{N}^\perp$ is called the "*orthogonal complement of $\mathcal{N}$ with respect to U*" and is the subset of all elements δv [5] of U whose projection on $\overset{1}{u}$ is zero, i.e. $\mathcal{N}^\perp \equiv \{v \in U | (v, \overset{1}{u}) = 0\}$. Thus any element $u \in U$ can be uniquely decomposed into a sum of two parts, one in $\mathcal{N}$ and the other in $\mathcal{N}^\perp$ ($\mathcal{N} \oplus \mathcal{N}^\perp = U$). From continuity, it is expected that for $0 \leq \lambda < \lambda_c$, the stability operator $\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)$ is strictly positive definite, i.e. satisfies (AC-2.4).

To investigate the equilibrium around the critical point (u_c, λ_c), where $u_c \equiv \overset{0}{u}(\lambda_c)$, one employs the "*Lyapunov – Schmidt – Koiter*" decomposition method. According to this method, illustrated in Fig. AC-2.1, the increment of the displacement $u - u_c$ due to an increment in the load $\Delta\lambda \equiv \lambda - \lambda_c$ is decomposed in two components: One component $\xi\overset{1}{u}$ is on the null space $\mathcal{N}$ of $\mathcal{E},^c_{uu}$ and the other component v is in $\mathcal{N}^\perp$, i.e. orthogonal to the first, namely:

$$u = u_c + \xi\overset{1}{u} + v, \quad v \in \mathcal{N}^\perp, \quad \xi \in \mathbb{R} \tag{AC-2.9}$$

Since the unknown displacement u is in essence replaced by the equivalent pair (ξ, v), the equilibrium equation (AC-2.2) can also be replaced by the following two ones: An equilibrium

[5] NOTE: Here and subsequently δv denotes any element of $\mathcal{N}^\perp$.

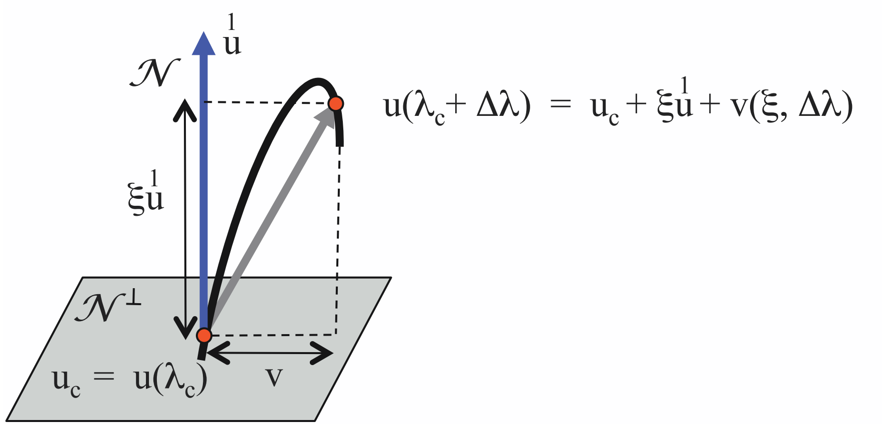

Figure AC-2.1: Schematics of the Lyapunov–Schmidt-Koiter (LSK) decomposition.

equation in $\mathcal{N}^\perp$, obtained by extremizing $\mathcal{E}$ with respect to v:

$$\mathcal{E},_v \delta v = 0 \implies \mathcal{E},_u (u_c + \xi\overset{1}{u} + v, \lambda_c + \Delta\lambda)\delta v = 0, \quad \forall \delta v \in \mathcal{N}^\perp, \tag{AC-2.10}$$

and an equilibrium equation in $\mathcal{N}$, obtained by extremizing $\mathcal{E}$ with respect to ξ:

$$\mathcal{E},_\xi = 0 \implies \mathcal{E},_u (u_c + \xi\overset{1}{u} + v, \lambda_c + \Delta\lambda)\overset{1}{u} = 0. \tag{AC-2.11}$$

Both equilibrium equations will be expanded about the critical point (u_c, λ_c). Starting with (AC-2.10), one first observes that this equation will provide v as a function of ξ and $\Delta\lambda$. Without loss of generality it can be assumed that $v(\xi, \Delta\lambda)$ defined in (AC-2.9) has a regular expansion about $(\xi, \Delta\lambda) = (0, 0)$:

$$v(\xi, \Delta\lambda) = \xi v_\xi + \Delta\lambda v_\lambda + \frac{1}{2!}[\xi^2 v_{\xi\xi} + 2\xi\Delta\lambda v_{\xi\lambda} + (\Delta\lambda)^2 v_{\lambda\lambda}] + ... \tag{AC-2.12}$$

Upon substitution of (AC-2.12) into the equilibrium equation (AC-2.10) and expansion of the result about $(\xi, \Delta\lambda) = (0, 0)$, one obtains the following results: the $O(1)$ term gives $\mathcal{E},^c_u \delta v = 0$ [6], which is automatically satisfied in view of (AC-2.2) since $\delta v \in U$. Continuing with the linear in ξ and $\Delta\lambda$ terms and recalling (AC-2.7) one obtains:

$$O(\xi) \ : \ (\mathcal{E},^c_{uu} v_\xi)\delta v = 0 \tag{AC-2.13}$$

In view of the positive definiteness of $\mathcal{E},^c_{uu}$ on $\mathcal{N}^\perp$ according to (AC-2.8), the only solution to (AC-2.13) is:

$$v_\xi = 0 \tag{AC-2.14}$$

for had this not been the case, (AC-2.13) for $\delta v = v_\xi$ would have been in contradiction with (AC-2.8).

[6] NOTE: From here and subsequently, a superscript $(^c)$ or a subscript $(_c)$ denotes evaluation of the quantity in question at the critical point (u_c, λ_c), following the convention introduced for (AC-2.8, AC-2.9).

The fact that the operator $\mathcal{E}_{,vv}^{c}$, which is defined as the operator $\mathcal{E}_{,uu}^{c}$ restricted on $\mathcal{N}^{\perp}$ is positive definite, assures the existence of a unique solution a to any equation of the type $(\mathcal{E}_{,vv}^{c}\, a)\delta v = (b, \delta v)$ for $a, b, \delta v \in \mathcal{N}^{\perp}$. The reason for the existence of such a unique solution $a = (\mathcal{E}_{,vv}^{c})^{-1} b$ is the existence of the unique inverse $(\mathcal{E}_{,vv}^{c})^{-1}$ on $\mathcal{N}^{\perp}$, a property guaranteed by the positive definiteness condition satisfied by $\mathcal{E}_{,vv}^{c}$ on the same space $\mathcal{N}^{\perp}$.

The $O(\Delta\lambda)$ term of the equilibrium equation (AC-2.10) yields:

$$O(\Delta\lambda) \;:\; (\mathcal{E}_{,uu}^{c}\, v_{\lambda} + \mathcal{E}_{,u\lambda}^{c})\delta v = 0 \tag{AC-2.15}$$

which in view of (AC-2.6) admits a unique solution $v_{\lambda} \in \mathcal{N}^{\perp}$.

The quadratic in $\xi, \Delta\lambda$ terms in the expansion of the equilibrium equation (AC-2.10) are found, with the help of (AC-2.14) to be:

$$O(\xi^2) \;:\; (\mathcal{E}_{,uu}^{c}\, v_{\xi\xi} + (\mathcal{E}_{,uuu}^{c}\, \overset{1}{u})\overset{1}{u})\delta v = 0 \tag{AC-2.16}$$

$$O(\xi\Delta\lambda) \;:\; (\mathcal{E}_{,uu}^{c}\, v_{\xi\lambda} + (\mathcal{E}_{,uuu}^{c}\, v_{\lambda} + \mathcal{E}_{,uu\lambda}^{c})\overset{1}{u})\delta v = 0 \tag{AC-2.17}$$

$$O((\Delta\lambda)^2) \;:\; (\mathcal{E}_{,uu}^{c}\, v_{\lambda\lambda} + (\mathcal{E}_{,uuu}^{c}\, v_{\lambda})v_{\lambda} + 2\mathcal{E}_{,uu\lambda}^{c}\, v_{\lambda} + \mathcal{E}_{,u\lambda\lambda}^{c})\delta v = 0 \tag{AC-2.18}$$

The existence and uniqueness of $v_{\xi\xi}, v_{\xi\lambda}, v_{\lambda\lambda}$ is assured from (AC-2.8) for the reasons already explained in detail in the discussion of (AC-2.13). The calculation of any term in the expansion (AC-2.12) of v proceeds in the straightforward fashion illustrated above.

Having thus constructed the solution $v(\xi, \Delta\lambda)$ of the equilibrium equation (AC-2.10), attention is next focused on the remaining equilibrium equation (AC-2.11), which in turn is expanded about the critical point (u_c, λ_c), or equivalently about $(\xi, \Delta\lambda) = (0, 0)$, yielding with the help of (AC-2.2), (AC-2.7) and (AC-2.14):

$$\begin{aligned} 0 = \Delta\lambda(\mathcal{E}_{,u\lambda}^{c}\, \overset{1}{u}) + \quad & \frac{1}{2}[\xi^2((\mathcal{E}_{,uuu}^{c}\, \overset{1}{u})\overset{1}{u})\overset{1}{u} + 2\xi\Delta\lambda((\mathcal{E}_{,uuu}^{c}\, v_{\lambda} + \mathcal{E}_{,uu\lambda}^{c})\overset{1}{u})\overset{1}{u} + \\ & (\Delta\lambda)^2((\mathcal{E}_{,uuu}^{c}\, v_{\lambda})v_{\lambda} + 2\mathcal{E}_{,uu\lambda}^{c}\, v_{\lambda} + \mathcal{E}_{,u\lambda\lambda})^{c}\overset{1}{u}] + \ldots \end{aligned} \tag{AC-2.19}$$

The above equation provides the wanted relation between ξ and $\Delta\lambda$ along the equilibrium path (or paths) through the critical point (u_c, λ_c). Assume that $\Delta\lambda$, in the neighborhood of the critical point, can be put in a Taylor series expansion in terms of ξ:

$$\Delta\lambda = \lambda_1 \xi + \lambda_2 \frac{\xi^2}{2!} + \lambda_3 \frac{\xi^3}{3!} + \ldots \tag{AC-2.20}$$

By introducing the above expansion into (AC-2.19) one can find the wanted coefficients λ_i.

Two cases are distinguished:

$$\text{case (i)} \;:\; \quad \mathcal{E}_{,u\lambda}^{c}\, \overset{1}{u} \neq 0 \tag{AC-2.21}$$

in which case all the coefficients λ_i in (AC-2.20) can be determined uniquely; the first two coefficients are:

$$\lambda_1 = 0, \quad \lambda_2 = -((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} / \mathcal{E},^c_{u\lambda} \overset{1}{u} \tag{AC-2.22}$$

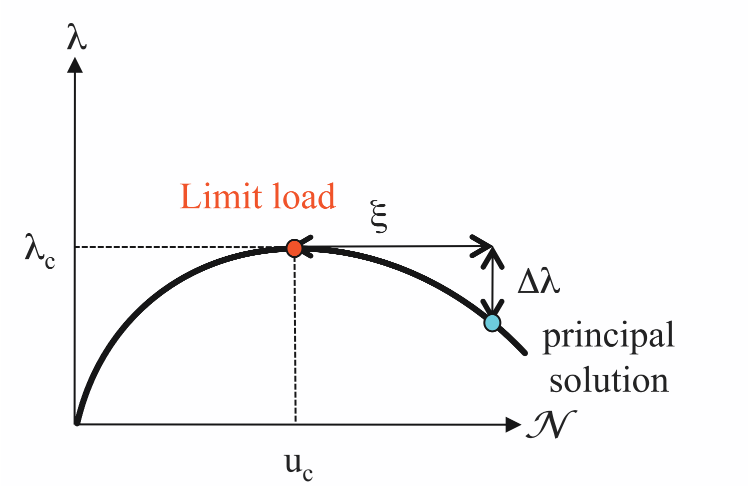

Figure AC-2.2: Schematics of the limit load case, where the solution about the critical load is unique.

One can conclude that if (AC-2.21) is satisfied, there is only one equilibrium branch through the critical point (u_c, λ_c). Moreover, and under the tacit assumption that $((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0$, (assumption satisfied in most applications) one also concludes from (AC-2.22) that the unique equilibrium branch through the critical point has a load extremum there, since the sign of $\Delta\lambda$ is independent on the sign of ξ. A schematic representation of this situation is given in Fig. AC-2.2

Once the uniqueness of the equilibrium path through the critical point has been established, attention is focused on its stability. To this end assume that $\beta(\xi)$ denotes the minimum eigenvalue of the stability matrix $\mathcal{E},_{uu}(u(\xi), \lambda(\xi))$ where ξ has been employed as a convenient parameter to describe the above found unique equilibrium path. The eigenvector corresponding to $\beta(\xi)$ is denoted by $x(\xi)$. Since β is an eigenvalue of the stability operator, by definition:

$$(\mathcal{E},_{uu}(u(\xi), \lambda(\xi)x(\xi))\delta u = \beta(\xi)(x(\xi), \delta u), \quad \| x(\xi) \| = 1 \tag{AC-2.23}$$

Since $\beta(\xi)$ is the lowest eigenvalue of $\mathcal{E},^c_{uu}$, as expected from (AC-2.7), at the critical point $\xi = 0$:

$$\beta_c = \beta(0) = 0, \quad x_c = x(0) = \overset{1}{u} \tag{AC-2.24}$$

Expansion of (AC-2.23) about $\xi = 0$ yields the following results: The $O(1)$ term reduces in view of (AC-2.24) to (AC-2.7). The next term is the $O(\xi)$ term, which in conjunction with (AC-2.9), (AC-2.12), (AC-2.14), (AC-2.20), (AC-2.22) and (AC-2.24) gives:

$$O(\xi) \ : \ ((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u} + \mathcal{E},^c_{uu}(dx/d\xi)_c)\delta u = (d\beta/d\xi)_c(\overset{1}{u}, \delta u) \tag{AC-2.25}$$

which for $\delta u = \overset{1}{u}$, and recalling (AC-2.7), implies:

$$(d\beta/d\xi)_c = ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0 \tag{AC-2.26}$$

The above result provides a physical explanation for the hypothesis $((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0$ adopted in the discussion of (AC-2.22), since it links it with the stability of the equilibrium path near the critical point. Since $\beta(\xi) = \xi(d\beta/d\xi)_c + O(\xi^2) = \xi((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} + O(\xi^2)$ in the neighborhood of the critical point, a stability change for the equilibrium branch is implied by crossing λ_c in view of the dependence of the sign of β on the sign of ξ. It has thus been shown that when (AC-2.21) holds (and $((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0$), the critical point (u_c, λ_c) corresponds to a load extremum of the equilibrium path $\overset{0}{u}(\lambda)$ and that the stability of the path changes by crossing λ_c. Given that the limit point in question is the first one encountered under increasing load , one concludes from continuity that this critical point should be a load maximum. Moreover, and since the principal equilibrium path is initially stable, one also concludes from continuity that the principal branch becomes unstable after it crosses the limit point.

The second possibility for the $\Delta\lambda - \xi$ relation in (AC-2.19) is:

$$\text{case (ii)} \quad : \quad \mathcal{E},^c_{u\lambda}\overset{1}{u} = 0 \tag{AC-2.27}$$

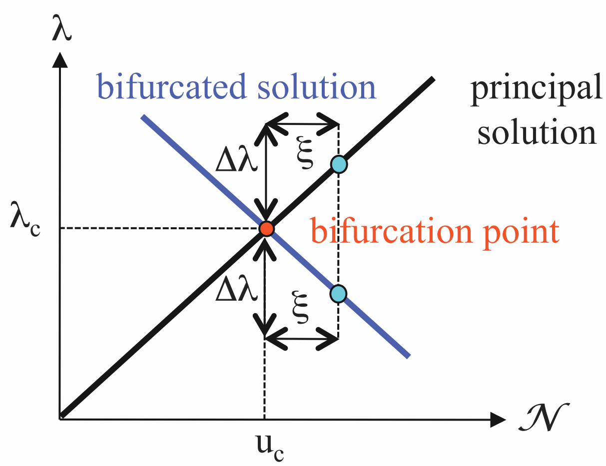

Figure AC-2.3: Schematics of the simple bifurcation case, where the solution about the critical load is not unique.

By introducing (AC-2.20) into (AC-2.19) one observes that, in view of (AC-2.27), the $\Delta\lambda - \xi$ relationship is no longer unique. Indeed, λ_1, the first term in the expansion of $\Delta\lambda$, satisfies:

$$(\lambda_1)^2((\mathcal{E},^c_{uuu} v_\lambda)v_\lambda + 2\mathcal{E},^c_{uu\lambda} v_\lambda + \mathcal{E},^c_{u\lambda\lambda})\overset{1}{u} + 2\lambda_1((\mathcal{E},^c_{uuu} v_\lambda + \mathcal{E},^c_{uu\lambda})\overset{1}{u})\overset{1}{u} + ((\mathcal{E},^c_{uuu})\overset{1}{u})\overset{1}{u})\overset{1}{u} = 0 \tag{AC-2.28}$$

The above equation has in general two real solutions in λ_1 and hence from (AC-2.19) and (AC-2.20) one can construct two functions $\Delta\lambda(\xi)$, as seen in the schematics of Fig. AC-2.3.

The fact that there already exists the principal equilibrium path through (u_c, λ_c) guarantees the existence of at least one real root λ_1 of (AC-2.28). Consequently two real roots for the quadratic in λ_1 equation (AC-2.28) must exist and which in general are different. Thus the condition (AC-2.27) implies that the critical point (u_c, λ_c) of the stability operator $\mathcal{E}_{,uu}$ is a bifurcation point (some additional conditions are also needed in order to ensure the existence of such a bifurcated path, as it will be seen in the next section). Since in most applications one equilibrium branch, the principal branch $\overset{0}{u}(\lambda)$ is known explicitly, this information facilitates enormously the resulting algebraic manipulations. In the remaining parts of this section, it is assumed that a principal solution $\overset{0}{u}(\lambda)$ is known and that the critical point (u_c, λ_c) satisfies (AC-2.27).

AC-3 PERFECT SYSTEM - SIMPLE MODE

Of interest in this section is the behavior of elastic systems near critical points which are simple bifurcation points, i.e. have a unique eigenvalue $\overset{1}{u}$ at λ_c. To this end it is assumed that the system's potential energy satisfies Eqs. (AC-2.1), (AC-2.3), (AC-2.4), (AC-2.7), (AC-2.8) and (AC-2.27), and that its principal solution $\overset{0}{u}(\lambda)$ is a known and adequately smooth function of λ.

At the neighborhood of the critical point (u_c, λ_c) the solution u of the equilibrium (AC-2.2) can be written with the help of an LSK decomposition:

$$u = \overset{0}{u}(\lambda) + \xi \overset{1}{u} + v, \quad v \in \mathcal{N}^{\perp}, \quad \xi \in \mathbb{R}, \tag{AC-3.1}$$

where the space $\mathcal{N}^{\perp}$ is defined in the discussion of (AC-2.8). The projection ξ of $u - \overset{0}{u}$ along the eigenmode $\overset{1}{u}$, defined as $\xi \equiv (u - \overset{0}{u}, \overset{1}{u})$, is termed the "*(bifurcation) amplitude parameter*".

Splitting as before the equilibrium equation (AC-2.2) into two components on the subspaces $\mathcal{N}^{\perp}$ and $\mathcal{N}$, one obtains by extremizing $\mathcal{E}$ with respect to v the equilibrium equation along $\mathcal{N}^{\perp}$:

$$\mathcal{E}_{,v}\,\delta v = 0 \implies \mathcal{E}_{,u}(\overset{0}{u}(\lambda) + \xi \overset{1}{u} + v, \lambda_c + \Delta\lambda)\delta v = 0 \quad \forall \delta v \in \mathcal{N}^{\perp}, \tag{AC-3.2}$$

while by extremizing $\mathcal{E}$ with respect to ξ one has the equilibrium equation in the null space $\mathcal{N}$:

$$\mathcal{E}_{,\xi} = 0 \implies \mathcal{E}_{,u}(\overset{0}{u}(\lambda) + \xi \overset{1}{u} + v, \lambda_c + \Delta\lambda)\overset{1}{u} = 0. \tag{AC-3.3}$$

As discussed in the previous section, the positive definiteness of $\mathcal{E}^{c}_{,uu}$ on $\mathcal{N}^{\perp}$ assumed in (AC-2.8) implies its invertibility there and hence ensures the existence of a unique solution $v(\xi, \Delta\lambda)$ to (AC-3.2). Consider once more the Taylor series expansion of v about $(\xi, \Delta\lambda) = (0, 0)$ according to (AC-2.12). Upon substitution of (AC-2.12) into (AC-3.2) and expansion of the result in a Taylor series about $(\xi, \Delta\lambda) = (0, 0)$ one obtains the following results: The $O(1)$ term gives $\mathcal{E}^{c}_{,u}\,\delta v = 0$ which is automatically satisfied in view of the equilibrium equation (AC-2.2). The $O(\xi)$ term yields in view of (AC-2.27) the same results as in (AC-2.13):

$$O(\xi) \ : \ \mathcal{E}^{c}_{,uu}\, v_{\xi} \delta v = 0. \tag{AC-3.4}$$

which for the same reasons admits as a unique solution:

$$v_{\xi} = 0. \tag{AC-3.5}$$

On the other hand, the $O(\Delta\lambda)$ term in (AC-3.2) gives:

$$O(\Delta\lambda) \ : \ (\mathcal{E}^{c}_{,uu}\, v_{\lambda} + \mathcal{E}^{c}_{,uu}(d\overset{0}{u}/d\lambda)_c + \mathcal{E}^{c}_{,u\lambda})\delta v = (\mathcal{E}^{c}_{,uu}\, v_{\lambda})\delta v = 0. \tag{AC-3.6}$$

In deriving (AC-3.6) use was made of the fact that $(\mathcal{E}^{c}_{,uu}(d\overset{0}{u}/d\lambda)_c + \mathcal{E}^{c}_{,u\lambda})\delta u = 0$, for it is obtained by differentiation with respect to λ of the equilibrium equation (AC-2.3) along the principal branch. Since (AC-3.6) is the same with (AC-3.4), it admits as its unique solution:

$$v_\lambda = 0. \tag{AC-3.7}$$

Continuing with the quadratic order terms in the expansion of the equilibrium (AC-3.2):

$$O(\xi^2) \;:\; (\mathcal{E},^c_{uu} v_{\xi\xi} + (\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\delta v = 0, \tag{AC-3.8}$$

$$O(\xi\Delta\lambda) \;:\; (\mathcal{E},^c_{uu} v_{\xi\lambda} + (\mathcal{E},^c_{uuu} (d\overset{0}{u}/d\lambda)_c + \mathcal{E},^c_{uu\lambda})\overset{1}{u})\delta v = 0, \tag{AC-3.9}$$

$$\begin{aligned} O((\Delta\lambda)^2) \;:\; & (\mathcal{E},^c_{uu} v_{\lambda\lambda} + (\mathcal{E},^c_{uuu} (d\overset{0}{u}/d\lambda)_c)(d\overset{0}{u}/d\lambda)_c + 2\mathcal{E},^c_{uu\lambda} (d\overset{0}{u}/d\lambda)_c + \\ & \mathcal{E},^c_{u\lambda\lambda} + \mathcal{E},^c_{uu} (d^2\overset{0}{u}/d\lambda^2)_c)\delta v = (\mathcal{E},^c_{uu} v_{\lambda\lambda})\delta v = 0. \end{aligned} \tag{AC-3.10}$$

Note that as in (AC-3.6), (AC-3.10) contains the second derivative with respect to λ of the equilibrium equation (AC-2.3) evaluated on the principal branch and hence it simplifies to $(\mathcal{E},^c_{uu} v_{\lambda\lambda})\delta v = 0$. For the same reasons as in (AC-3.4) and (AC-3.6) one obtains from (AC-3.10):

$$v_{\lambda\lambda} = 0. \tag{AC-3.11}$$

It is worth mentioning at this point that all the coefficients of $(\Delta\lambda)^n$ in the expansion of $v(\xi, \Delta\lambda)$ (see (AC-2.12)), satisfy $v,_\lambda = v,_{\lambda\lambda} = v,_{\lambda\lambda\lambda} ... = 0$. This result had to be expected, since by the definition of the LSK decomposition of u in (AC-3.1), $\xi = 0$ corresponds to the principal solution and hence one should have $v(0, \Delta\lambda) = 0$.

The results from the solution of (AC-3.2) for $v(\xi, \Delta\lambda)$ are employed in the remaining equilibrium equation (AC-3.3) which expanded about $(\xi, \Delta\lambda) = (0, 0)$ yields with the help of Eqs. (AC-2.3), (AC-2.7), (AC-2.27), (AC-3.5), (AC-3.7):

$$\begin{aligned} 0 = \; & \frac{1}{2}[\xi^2((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} + 2\xi\Delta\lambda((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}] + \\ & \frac{1}{6}[\xi^3((((\mathcal{E},^c_{uuuu} \overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E},^c_{uuu} v_{\xi\xi})\overset{1}{u})\overset{1}{u}) + ...] + ... \end{aligned} \tag{AC-3.12}$$

In the above equation, which provides the $\Delta\lambda - \xi$ relation for both the principal and the bifurcated equilibrium paths through the critical point (u_c, λ_c), one can easily verify that all the $O((\Delta\lambda)^n)$ terms vanish identically. Consequently, and as expected from (AC-3.1) and (AC-3.12) admits two different solutions: One solution with $\xi = 0,\ \Delta\lambda \neq 0$ corresponds to the principal branch $\overset{0}{u}(\lambda)$, while the other solution with $\xi \neq 0,\ \Delta\lambda = \Delta\lambda(\xi)$ corresponds to the bifurcated path.

For the bifurcated path, by employing the same Taylor series expansion of $\Delta\lambda$ about $\xi = 0$ as in (AC-2.20), one obtains for the asymmetric bifurcation case, defined as $((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0$:

$$\lambda_1 = -\frac{1}{2}((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u}/((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}, \tag{AC-3.13}$$

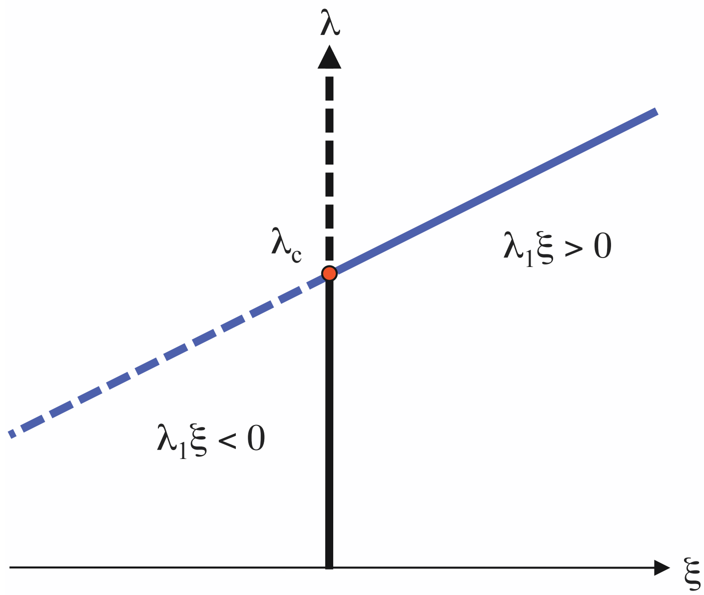

Figure AC-3.1: Case of an asymmetric bifurcation for a problem with a single eigenmode at the critical point. Stable paths are drawn in continuous lines while unstable paths are drawn in dashed lines.

while for the symmetric bifurcation case defined as $((\mathcal{E},^{c}_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} = 0$ one has:

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{1}{3}[((((\mathcal{E},^{c}_{uuuu} \overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E},^{c}_{uuu} v_{\xi\xi})\overset{1}{u})\overset{1}{u}]/((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}. \tag{AC-3.14}$$

where $v_{\xi\xi}$ is the unique solution of (AC-3.8). It should be noted at this point that in the above derivations it is tacitly assumed that the denominator in the expressions for λ_1 and λ_2 is nonzero [7]. A physical interpretation of this assumption will be given immediately below.

The next question of interest, pertains to the stability of the equilibrium branches through the critical point. To this end one has to investigate the sign of the minimum eigenvalue β of the stability operator $\mathcal{E},_{uu}$ evaluated on the equilibrium path in question. For the principal branch it is reasonable to assume in view of (AC-2.4), that $\overset{0}{\beta}(\lambda)$ the minimum eigenvalue of $\mathcal{E},_{uu}(\overset{0}{u}(\lambda), \lambda)$ has a strict crossing of zero at λ_c, i.e.:

$$\overset{0}{\beta}(\lambda_c) = 0 \quad , \quad (d\overset{0}{\beta}/d\lambda)_c < 0, \tag{AC-3.15}$$

with the negative sign of $(d\overset{0}{\beta}/d\lambda)_c$ explained by the fact that $\overset{0}{\beta} > 0$ for $\lambda < \lambda_c$ according to the definition of λ_c (see the discussion following (AC-2.8)). Assuming that the eigenvector corresponding to $\overset{0}{\beta}(\lambda)$ is $\overset{0}{x}(\lambda)$ one has from the definition of the eigenvalue:

$$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda), \lambda)\overset{0}{x}(\lambda))\delta u = \overset{0}{\beta}(\lambda)(\overset{0}{x}(\lambda), \delta u), \tag{AC-3.16}$$

[7] NOTE: From here and subsequently $(\ ),_\lambda$ denotes partial differentiation of the quantity in question with respect to the load parameter, while $d(\ )/d\lambda$ denotes differentiation of the quantity in question evaluated on the principal equilibrium branch $\overset{0}{u}(\lambda)$ with respect to the load parameter.

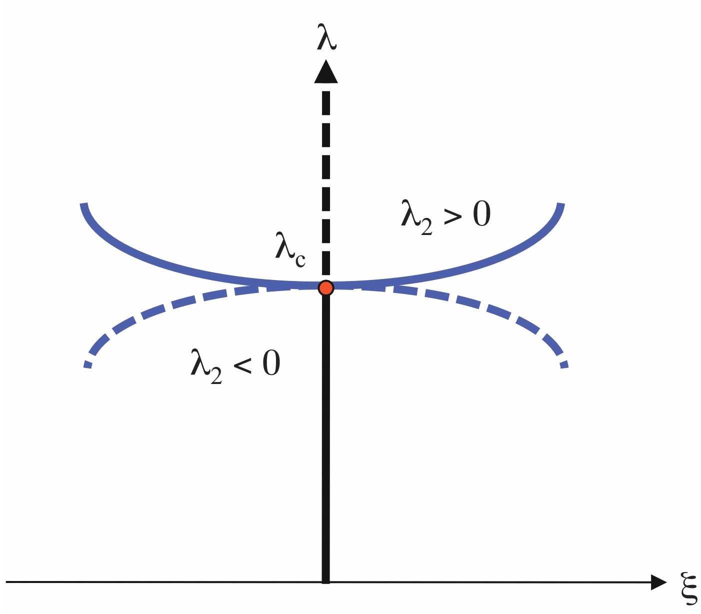

Figure AC-3.2: Case of a symmetric bifurcation for a problem with a single eigenmode at the critical point. Stable paths are drawn in continuous lines while unstable paths are drawn in dashed lines.

while to ensure uniqueness (up to sign) of the eigenvector $\overset{0}{x}$ one also requires it to have a unit norm:

$$(\overset{0}{x}(\lambda), \overset{0}{x}(\lambda)) = 1. \tag{AC-3.17}$$

Evaluating (AC-3.16), (AC-3.17) at λ_c and recalling from (AC-2.7) the uniqueness of the eigenvector of $\mathcal{E},^c_{uu}$:

$$(\mathcal{E},^c_{uu} \overset{0}{x}(\lambda_c))\delta u = 0, \quad (\overset{0}{x}(\lambda_c), \overset{0}{x}(\lambda_c)) = 1 \implies \overset{0}{x}(\lambda_c) = \overset{1}{u}. \tag{AC-3.18}$$

By differentiating (AC-3.16) with respect to λ and evaluating the resulting expression at λ_c, one obtains with the help of (AC-3.18):

$$((\mathcal{E},^c_{uuu} (d \overset{0}{u} /d\lambda)_c + \mathcal{E},^c_{uu\lambda})\overset{1}{u} + \mathcal{E},^c_{uu} (d\overset{0}{x}/d\lambda)_c)\delta u = (d\overset{0}{\beta}/d\lambda)_c(\overset{1}{u}, \delta u). \tag{AC-3.19}$$

By choosing $\delta u = \overset{1}{u}$ and recalling (AC-2.7) as well as (AC-3.15)$_1$ one finally has:

$$((d\mathcal{E},_{uu} /d\lambda)_c\overset{1}{u})\overset{1}{u} = (d\overset{0}{\beta}/d\lambda)_c. \tag{AC-3.20}$$

The above result, in conjunction with (AC-3.15)$_2$ ensures a non-zero denominator in (AC-3.13), (AC-3.14) as well in all the subsequent terms λ_n in the Taylor series expansion of $\Delta\lambda(\xi)$, and hence guarantees the existence of a bifurcated path through the critical load.

For the thus constructed bifurcated solution, a parameterization with respect to ξ is the most convenient one to study the path's stability. The starting point for the corresponding stability calculation is the definition of the minimum eigenvalue $\beta(\xi)$ of the stability operator evaluated on the bifurcated path, namely:

$$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda_c+\Delta\lambda(\xi))+\xi\overset{1}{u}+v(\xi,\Delta\lambda(\xi)),\lambda_c+\Delta\lambda(\xi))x(\xi))\delta u=\beta(\xi)(x(\xi),\delta u), \tag{AC-3.21}$$

which to ensure uniqueness of the eigenmode $x(\xi)$ has to be complemented by the normalization requirement:

$$(x(\xi),x(\xi))=1. \tag{AC-3.22}$$

By assuming a regular Taylor series expansion of β and x about $\xi=0$, namely:

$$\begin{aligned}\beta(\xi) &= \xi\beta_1+\frac{\xi^2}{2}\beta_2+... \\ x(\xi) &= x_0+\xi x_1+\frac{\xi^2}{2}x_2+...\end{aligned} \tag{AC-3.23}$$

and employing them together with (AC-3.13), (AC-3.14) and (AC-3.22) into (AC-3.21) one obtains by expanding about $\xi=0$ the following results:

For the $O(1)$ term, in view of (AC-3.22) and the assumed uniqueness of the eigenmode of $\mathcal{E},^c_{uu}$ (see (AC-2.7)) one has:

$$O(1)\ :\ (\mathcal{E},^c_{uu}x_0)\delta u=0,\quad (x_0,x_0)=1,\implies x_0=\overset{1}{u}. \tag{AC-3.24}$$

Continuing with the $O(\xi)$ term of the expansion of (AC-3.21):

$$O(\xi)\ :\ ((\mathcal{E},^c_{uuu}(\lambda_1(d\overset{0}{u}/d\lambda)_c+\overset{1}{u}))\overset{1}{u}+\lambda_1\mathcal{E},^c_{uu\lambda}\overset{1}{u}+\mathcal{E},^c_{uu}x_1)\delta u=\beta_1(\overset{1}{u},\delta u). \tag{AC-3.25}$$

By taking $\delta u=\overset{1}{u}$ and recalling (AC-2.7) as well as (AC-3.13), one obtains for the asymmetric bifurcation case, i.e. for $((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}\neq 0$, the following result for β_1:

$$\beta_1=\lambda_1((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}+((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}=\lambda_1[-((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}]. \tag{AC-3.26}$$

For the symmetric bifurcation case, i.e. for $((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}=0$, the $O(\xi)$ term in the expansion of (AC-3.21) gives according to (AC-3.13), (AC-3.24) and (AC-3.26) that $\beta_1=\lambda_1=0$. For this case, (AC-3.24), (AC-3.25) and the $O(\xi)$ term in (AC-3.22) give in view of (AC-3.8):

$$O(\xi)\ :\ ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u}+\mathcal{E},^c_{uu}x_1)\delta v=0,\quad (x_1,\overset{1}{u})=0,\implies x_1=v_{\xi\xi}. \tag{AC-3.27}$$

Continuing with the $O(\xi^2)$ term in the expansion of (AC-3.21) and making use of (AC-3.23), (AC-3.24), (AC-3.25) as well as (AC-3.14) one obtains:

$$\begin{aligned}O(\xi^2)\ :\ &((\frac{1}{2}((\mathcal{E},^c_{uuuu}\overset{1}{u})\overset{1}{u}+\mathcal{E},^c_{uuu}(\lambda_2(d\overset{0}{u}/d\lambda)_c+v_{\xi\xi})+\lambda_2\mathcal{E},^c_{uu\lambda})\overset{1}{u})x_1\\ &(\mathcal{E},^c_{uuu}\overset{1}{u})x_1+\frac{1}{2}\mathcal{E},^c_{uu}x_2)\delta u=\frac{1}{2}\beta_2(\overset{1}{u},\delta u).\end{aligned} \tag{AC-3.28}$$

Once again by taking $\delta u=\overset{1}{u}$ and recalling (AC-2.7), (AC-3.14), (AC-3.24) as well as (AC-3.27) one finds β_2 to be:

$$\begin{aligned}\beta_2 =\ & (((\mathcal{E},^c_{uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u}+\lambda_2((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}+((\mathcal{E},^c_{uuu}v_{\xi\xi})\overset{1}{u})\overset{1}{u}+2((\mathcal{E},^c_{uuu}x_1)\overset{1}{u})\overset{1}{u}\\ =\ & 2\lambda_2(-((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}).\end{aligned} \tag{AC-3.29}$$

Consequently the stability of the bifurcated equilibrium path in the neighborhood of the critical point (u_c, λ_c) is determined by the sign of the minimum eigenvalue $\beta(\xi)$ of the corresponding stability operator which from (AC-3.20), (AC-3.26), (AC-3.29) takes the form:

$$\beta(\xi) = \begin{cases} \lambda_1 \xi[-(d\overset{0}{\beta}/d\lambda)_c] + O(\xi^2) & \text{for asymmetric bifurcation } ((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0, \\ \\ \lambda_2 \xi^2[-(d\overset{0}{\beta}/d\lambda)_c] + O(\xi^3) & \text{for symmetric bifurcation } ((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} = 0. \end{cases} \tag{AC-3.30}$$

From the hypothesis $(d\overset{0}{\beta}/d\lambda)_c < 0$ one concludes the following: For the transcritical asymmetric bifurcation $\lambda_1 \neq 0$, the $\lambda > \lambda_c$ branch of the solution satisfies $\lambda_1 \xi > 0$ and hence the corresponding part of the bifurcated branch is stable, while for the $\lambda < \lambda_c$ part of the bifurcated branch $\lambda_1 \xi < 0$ and hence the corresponding part of the bifurcated branch is unstable. For the symmetric bifurcation, the supercritical $\lambda > \lambda_c$ branch corresponds to $\lambda_2 > 0$ and it is stable while the subcritical $\lambda < \lambda_c$ branch corresponds to $\lambda_2 < 0$ and is unstable. Hence the general stability analysis for any elastic system exhibiting a simple bifurcation gives the same results with the simple rigid T model analyzed in subection AB-1.

One can also compute $\Delta\mathcal{E}$, the difference between the potential energies on the bifurcated and principal branches of the system, for a fixed value of the load parameter λ. Since by definition:

$$\Delta\mathcal{E} = \mathcal{E}(\overset{0}{u}(\lambda_c + \Delta\lambda) + \xi\overset{1}{u} + v(\xi, \Delta\lambda), \lambda_c + \Delta\lambda) - \mathcal{E}(\overset{0}{u}(\lambda_c + \Delta\lambda), \lambda_c + \Delta\lambda), \tag{AC-3.31}$$

one can expand $\Delta\mathcal{E}$ about the critical point (u_c, λ_c) and obtain a power series expansion in ξ. For the asymmetric $(\lambda_1 \neq 0)$ bifurcation case, (AC-3.31) with the help of (AC-2.7), (AC-2.27) and (AC-3.13) yields:

$$\Delta\mathcal{E} = \frac{\xi^3}{6}\lambda_1((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} + O(\xi^4), \tag{AC-3.32}$$

while for the symmetric $(\lambda_1 = 0,\ \lambda_2 \neq 0)$ bifurcation case, (AC-3.31) with the help of (AC-2.27) and (AC-3.14) yields:

$$\Delta\mathcal{E} = \frac{\xi^4}{8}\lambda_2((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} + O(\xi^5). \tag{AC-3.33}$$

These results are similar to the the one obtained for the rigid T model (see (AB-1.6), (AB-1.7).

Example

As a simple first application of the general theory developed in this section, the perfect rigid T example will be revisited. In this case the space of admissible displacement functions $u = (v, \theta)$ is $U = \mathbb{R}^2$, whose Cartesian basis is $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. From (AB-1.1) the stability operator evaluated on the principal branch $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda)$ is the rank two tensor:

$$\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda) = 2E\mathbf{e}_1\mathbf{e}_1 + (\lambda_c - \lambda)L\mathbf{e}_2\mathbf{e}_2. \tag{AC-3.34}$$

It is easy to see that $\lambda = \lambda_c$ is a singular point for $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda)$ with the corresponding unique eigenvector:

$$\overset{1}{u} = \mathbf{e}_2. \tag{AC-3.35}$$

The critical point (u_c, λ_c) is not a limit point for (AC-2.27) is satisfied, as one can easily see from (AC-3.34) and (AC-3.35):

$$\mathcal{E}^{c}_{,u\lambda}\overset{1}{u} = (-\mathbf{e}_1) \bullet \mathbf{e}_2 = 0, \tag{AC-3.36}$$

where $(\bullet)$ is the standard dyadic notation for the single dot (inner) product in finite dimension vector and tensor calculus.

Note that the rigid T model satisfies the strict crossing at zero of the minimum eigenvalue condition (AC-3.15) (see also (AC-3.20)) namely:

$$((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = ((-L\mathbf{e}_2\mathbf{e}_2) \bullet \mathbf{e}) \bullet \mathbf{e}_2 = -L < 0. \tag{AC-3.37}$$

Also note that the bifurcation amplitude parameter ξ from (AC-3.1) is given by:

$$\xi = (u - \overset{0}{u}(\lambda)) \bullet \overset{1}{u} = [(v - (\lambda/2E))\mathbf{e}_1 + \theta\mathbf{e}_2] \bullet \mathbf{e}_2 = \theta, \tag{AC-3.38}$$

which identifies θ as the bifurcation amplitude parameter ξ in this example.

For the asymmetric rigid T model $(m \neq 0, n = 0)$ the first nontrivial term λ_1 in the $\lambda - \xi$ expansion for the bifurcation branch is calculated from (AC-3.13). Noticing that in this example from (AB-1.1)

$$\mathcal{E}^{c}_{,uuu} = 2mL^3\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2, \tag{AC-3.39}$$

the value of λ_1 is found from (AC-3.13) with the help of (AC-3.35), (AC-3.37) and (AC-3.38), to be:

$$\lambda_1 = -\frac{1}{2}(2mL^3)/(-L) = mL^2, \tag{AC-3.40}$$

exactly as expected from (AB-1.4).

For symmetric model $(m = 0, n \neq 0)$, the first nontrivial term in the $\lambda - \xi$ expansion is calculated from (AC-3.14). Notice that in this case from (AB-1.1):

$$\mathcal{E}^{c}_{,uuuu} = 6nL^4\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2, \tag{AC-3.41}$$

while $\mathcal{E},^c_{uuu} = 0$ and hence from (AC-3.14) with the help of (AC-3.35), (AC-3.37) and (AC-3.39) one obtains:

$$\lambda_2 = -\frac{1}{2}(6nL^4)/(-L) = 2nL^3, \tag{AC-3.42}$$

exactly as expected from (AB-1.4).

As it turns out, the above found asymptotic results describe completely the bifurcated solutions of the simple rigid T example, since one can verify that all the higher order terms λ_n in the expansion of $\lambda(\xi)$ are identically zero. For the same reason, the stability discussion based on the general asymptotic analysis of this section gives exactly the same results to the ones obtained before in the analysis of the rigid T.

The preceding general theory for simple bifurcations of elastic systems although not complete from the mathematical standpoint, is sufficient for the vast majority of engineering applications of interest. Pathological cases where $(d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = (d\overset{0}{\beta}/d\lambda)_c = 0$ do exist and require a modification of the preceding general analysis. This modification is not at all difficult since the LSK decomposition in (AC-3.1) can be used once more to obtain the new equilibrium equation (counterpart to (AC-3.12)) along the null space $\mathcal{N}$ which will automatically suggest the required parameterization of the equilibrium paths near the critical point. Depending on some additional conditions, $(d\overset{0}{\beta}/d\lambda)_c = 0$ can lead to a bifurcated branch emerging tangently from the principal path (in contrast to the results of this section where the bifurcated branches cut the principal path at a finite angle in $\lambda - \xi$ space) or might even be an isolated singular point with no bifurcation branch emerging from λ_c.

AC-4 IMPERFECT SYSTEM - SIMPLE MODE

Of interest in this section is the influence of imperfections in systems whose perfect counterparts exhibit a simple bifurcation. To this end, it is assumed that the potential energy of the imperfect system is $\overline{\mathcal{E}}(u,\lambda,w)$ where w is the imperfection field of the system while u and λ denote the displacement field and the load parameter as before. Without loss of generality, it is assumed that $w \in U$, the space which also contains all the admissible displacement functions u. If the imperfection function w vanishes, the system is reduced to its perfect counterpart and hence:

$$\overline{\mathcal{E}}(u,\lambda,0) = \mathcal{E}(u,\lambda). \tag{AC-4.1}$$

Similarly to the perfect case, the potential energy $\overline{\mathcal{E}}$ is arbitrarily set to zero for zero displacements:

$$\overline{\mathcal{E}}(0,\lambda,w) = 0. \tag{AC-4.2}$$

The system evolves from its stress and displacement-free configuration at which $\lambda = 0$ and $u = 0$ to a loaded configuration with $\lambda \neq 0,\ u \neq 0$. For a given load level, the corresponding equilibrium solutions are found by extremizing $\overline{\mathcal{E}}$ with respect to u, namely:

$$\overline{\mathcal{E}},_u(u,\lambda,w)\delta u = 0. \tag{AC-4.3}$$

For all physically realistic problems, for a given load parameter λ and a given imperfection w, one expects the equilibrium solution $u(\lambda,w)$ to be unique in a neighborhood of $\lambda = 0$ and coincide with $\overset{0}{u}(\lambda)$ in the absence of imperfections, in agreement with (AC-2.3) and (AC-4.1):

$$\overline{\mathcal{E}},_u(u(\lambda,w),\lambda,w)\delta u = 0, \quad u(0,w) = 0, \quad u(\lambda,0) = \overset{0}{u}(\lambda). \tag{AC-4.4}$$

Unlike $\overset{0}{u}(\lambda)$ however, $u(\lambda,w)$ is not as easy to find in applications, since the presence of imperfections destroys the symmetry of the system.

For imperfect systems, it is important to distinguish between the "*imperfection amplitude*" denoted by ϵ, and the "*imperfection shape*" denoted by $\overline{w}$. The corresponding definitions are:

$$\epsilon \equiv \parallel w \parallel, \quad \overline{w} \equiv w/\epsilon, \quad (\parallel \overline{w} \parallel = 1). \tag{AC-4.5}$$

Of interest here is the behavior of the imperfect system near the critical point $(\lambda_c,\ u_c)$ of its perfect counterpart for small imperfections, i.e. for small amplitudes ϵ of the imperfection w but for arbitrary imperfection shapes $\overline{w}$. To this end, one adopts the same LSK decomposition of the displacement $u = \overset{0}{u}(\lambda) + \xi\overset{1}{u} + v$ as in the perfect case (see (AC-3.1)). Splitting as before the equilibrium equation (AC-4.3) into two components on the subspaces $\mathcal{N}^{\perp}$ and $\mathcal{N}$ (see (AC-2.8) for the pertaining definitions), one obtains with the help of (AC-3.1) (in analogy to (AC-3.2), (AC-3.3)), that the equilibrium equation along $\mathcal{N}^{\perp}$ is:

$$\overline{\mathcal{E}},_v\delta v = 0 \Longrightarrow \overline{\mathcal{E}},_u(\overset{0}{u}(\lambda_c + \Delta\lambda) + \xi\overset{1}{u} + v, \lambda_c + \Delta\lambda, \epsilon\overline{w})\delta v = 0 \quad \forall \delta v \in \mathcal{N}^{\perp}, \tag{AC-4.6}$$

while the equilibrium equation along the null space $\mathcal{N}$ is:

$$\overline{\mathcal{E}},_\xi = 0 \Longrightarrow \overline{\mathcal{E}},_u (\overset{0}{u} (\lambda_c + \Delta\lambda) + \xi\overset{1}{u} + v, \lambda_c + \Delta\lambda, \epsilon\overline{w})\overset{1}{u} = 0. \tag{AC-4.7}$$

As discussed in subection AC-2, the invertibility of $\mathcal{E},^c_{vv}$ on $\mathcal{N}^\perp$ which follows from its positive definiteness according to (AC-2.8), ensures with the help of (AC-4.1) the existence of a unique solution $v(\xi, \Delta\lambda, \epsilon)$ to (AC-4.6). By assuming that $v(\xi, \Delta\lambda, \epsilon)$ has a Taylor series expansion about $(\xi, \Delta\lambda, \epsilon) = (0, 0, 0)$, i.e.:

$$\begin{aligned} v(\xi, \Delta\lambda, \epsilon) = \quad & \xi v_\xi + \Delta\lambda v_\lambda + \epsilon v_\epsilon + \\ & \frac{1}{2}[\xi^2 v_{\xi\xi} + 2\xi\Delta\lambda v_{\xi\lambda} + 2\xi\epsilon v_{\xi\epsilon} + (\Delta\lambda)^2 v_{\lambda\lambda} + 2\Delta\lambda\epsilon v_{\lambda\epsilon} + \epsilon^2 v_{\epsilon\epsilon}] + ... \end{aligned} \tag{AC-4.8}$$

and expanding in a Taylor series the equilibrium equation (AC-4.8) about $(\xi, \Delta\lambda, \epsilon) = (0, 0, 0)$ one obtains the following results: The terms involving only powers of ξ and $\Delta\lambda$ produce exactly the same results as the perfect system, since for imperfection amplitudes $\epsilon = \parallel w \parallel = 0$ the imperfect system reduces to its perfect counterpart in according to (AC-4.1). Hence for the imperfect system, in addition to Eqs. (AC-3.4) – (AC-3.11) one needs following terms to complete the Taylor series expansion of v up to the second order:

$$O(\epsilon) \ : \ (\mathcal{E},^c_{uu} v_\epsilon + \overline{\mathcal{E}},^c_{uw} \overline{w})\delta v = 0, \tag{AC-4.9}$$

$$O(\xi\epsilon) \ : \ (\mathcal{E},^c_{uu} v_{\xi\epsilon} + (\mathcal{E},^c_{uuu} v_\epsilon + \overline{\mathcal{E}},^c_{uuw} \overline{w})\overset{1}{u})\delta v = 0, \tag{AC-4.10}$$

$$O(\epsilon\Delta\lambda) \ : \ (\mathcal{E},^c_{uu} v_{\lambda\epsilon} + (d\mathcal{E},_{uu}/d\lambda)_c v_\epsilon + (d\overline{\mathcal{E}},_{uw}/d\lambda)_c\overline{w})\delta v = 0, \tag{AC-4.11}$$

$$O(\epsilon^2) \ : \ (\mathcal{E},^c_{uu} v_{\epsilon\epsilon} + (\mathcal{E},^c_{uuu} v_\epsilon)v_\epsilon + 2(\overline{\mathcal{E}},^c_{uuw} \overline{w})v_\epsilon + (\overline{\mathcal{E}},^c_{uww} \overline{w})\overline{w})\delta v = 0. \tag{AC-4.12}$$

All the above equations admit unique solutions in terms of the unknowns $v_\epsilon, v_{\xi\epsilon}, v_{\lambda\epsilon}, v_{\epsilon\epsilon}$ in view of the existence of $(\mathcal{E},^c_{vv})^{-1}$ in $\mathcal{N}^\perp$. By introducing the thus constructed expansion for v into the remaining equilibrium equation (AC-4.7), one obtains with the help of (AC-2.7), (AC-2.27), (AC-3.5) and (AC-3.7):

$$\begin{aligned} 0 = \quad & \epsilon(\overline{\mathcal{E}},^c_{uw} \overline{w})\overset{1}{u} + \frac{1}{2}[\xi^2((\mathcal{E},^c_{uuu} \overset{1}{u})\overset{1}{u})\overset{1}{u} + 2\xi\Delta\lambda((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} + 2\xi\epsilon((\mathcal{E}^c_{uuu} v_\epsilon + \overline{\mathcal{E}},_{uuw})\overline{w}\overset{1}{u})\overset{1}{u} + \\ & 2\epsilon\Delta\lambda((d\mathcal{E},_{uu}/d\lambda)_c v_\epsilon + (d\overline{\mathcal{E}},_{uw}/d\lambda)_c\overline{w})\overset{1}{u} + \epsilon^2((\mathcal{E}^c_{uuu} v_\epsilon)v_\epsilon + 2(\overline{\mathcal{E}},^c_{uuw} \overline{w})v_\epsilon + (\overline{\mathcal{E}},^c_{uww} \overline{w})\overline{w})\overset{1}{u}] + \\ & \frac{1}{6}[\xi^3(((\mathcal{E},^c_{uuuu} \overset{1}{u})\overset{1}{u} + 3\mathcal{E},^c_{uu} v_{\xi\xi})\overset{1}{u})\overset{1}{u} + ...] + ... \end{aligned} \tag{AC-4.13}$$

The above equation, which relates the load $\Delta\lambda = \lambda - \lambda_c$, the $\overset{1}{u}$ component of the deformation ξ and the imperfection amplitude ϵ, can be solved with respect to ϵ if $(\overline{\mathcal{E}},^c_{uw} \overline{w})\overset{1}{u} \neq 0$. Consequently, by considering the Taylor series expansion of ϵ in terms of $\Delta\lambda$ and ξ, one obtains

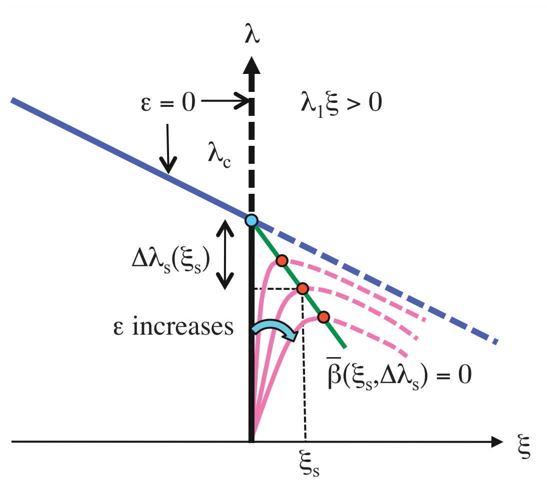

Figure AC-4.1: Case of an asymmetric bifurcation for an imperfect problem with a single eigenmode at the critical point. Stable paths are drawn in continuous lines while unstable paths are drawn in dashed lines.

from (AC-4.13), with the help of (AC-3.13), (AC-3.14) for the asymmetric and symmetric perfect systems respectively:

$$\epsilon(\xi,\Delta\lambda) = [((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}/(\overline{\mathcal{E}}^{\,c}_{,uw}\,\overline{w})\overset{1}{u}]\begin{cases}(\lambda_1\xi^2-\xi\Delta\lambda)+\ldots & \text{for} \quad ((\mathcal{E}^{\,c}_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}\neq 0,\\ (\dfrac{\lambda_2}{2}\xi^3-\xi\Delta\lambda)+\ldots & \text{for} \quad ((\mathcal{E}^{\,c}_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}=0.\end{cases} \tag{AC-4.14}$$

The above relation gives the magnitude of the imperfection amplitude ϵ for an equilibrium path of the imperfect system passing through the point $(\Delta\lambda,\xi)$ in the $\lambda-\xi$ space. Hence, for a given imperfection one can find the $\Delta\lambda-\xi$ relationship along the corresponding equilibrium path through (AC-4.14).

The next question of interest pertains to the stability of the aforedescribed equilibrium paths in the neighborhood of the perfect system's critical point (λ_c,u_c) and for small imperfection amplitudes. To this end one needs to investigate the minimum eigenvalue $\overline{\beta}$ of the corresponding stability operator $\overline{\mathcal{E}}_{,uu}$ evaluated on the imperfect system's equilibrium solution. The defining equation for $\overline{\beta}$ and the corresponding eigenvector $\overline{x}$ is:

$$\begin{aligned}&(\overline{\mathcal{E}}_{,uu}\,(\overset{0}{u}\,(\lambda_c+\Delta\lambda)+\xi\overset{1}{u}+v(\xi,\Delta\lambda,\epsilon(\xi,\Delta\lambda)),\lambda_c+\Delta\lambda,\epsilon(\xi,\Delta\lambda)\overline{w})\overline{x}(\xi,\Delta\lambda))\delta u=\\ &\overline{\beta}(\xi,\Delta\lambda)(\overline{x}(\xi,\delta\lambda),\delta u).\end{aligned} \tag{AC-4.15}$$

In addition, the normalization requirement for the eigenmode $\overline{x}(\xi,\Delta\lambda)$ is (compare with (AC-3.22)):

$$(\overline{x}(\xi,\Delta\lambda),\overline{x}(\xi,\Delta\lambda))=1. \tag{AC-4.16}$$

Notice in the above definition of $\overline{\beta}$, that the stability of all possible equilibrium paths in the neighborhood of the perfect system's critical load are examined, since $\Delta\lambda$ and ξ are considered as independent variables in this analysis.

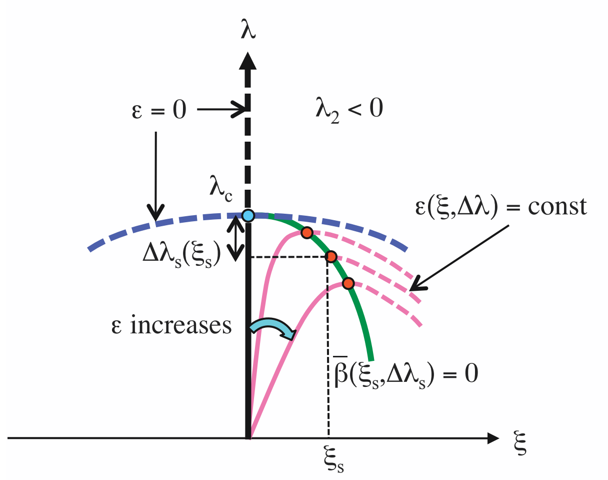

Figure AC-4.2: Case of an symmetric bifurcation for an imperfect problem with a single eigenmode at the critical point. Stable paths are drawn in continuous lines while unstable paths are drawn in dashed lines.

The minimum eigenvalue $\overline{\beta}(\xi, \Delta\lambda)$ and the corresponding eigenvector $\overline{x}(\xi, \Delta\lambda)$ are expanded in a Taylor series of their arguments in the neighborhood of $(\xi, \Delta\lambda) = (0, 0)$, i.e.:

$$\overline{\beta}(\xi, \Delta\lambda) = \xi\overline{\beta}_\xi + \Delta\lambda\overline{\beta}_\lambda + \frac{1}{2}(\xi^2\overline{\beta}_{\xi\xi} + 2\xi\Delta\lambda\overline{\beta}_{\xi\lambda} + (\Delta\lambda)^2\overline{\beta}_{\lambda\lambda}) + ... \tag{AC-4.17}$$

$$\overline{x}(\xi, \Delta\lambda) = \overline{x}_0 + \xi\overline{x}_\xi + \Delta\lambda\overline{x}_\lambda + \frac{1}{2}(\xi^2\overline{x}_{\xi\xi} + 2\xi\Delta\lambda\overline{x}_{\xi\lambda} + (\Delta\lambda)^2\overline{x}_{\lambda\lambda}) + ... \tag{AC-4.18}$$

By introducing (AC-4.17) and (AC-4.18) into (AC-4.15), expanding about $(\xi, \Delta\lambda) = (0, 0)$ and collecting the terms of the like order in ξ and $\Delta\lambda$ one obtains the following results. From the $O(1)$ term, in view of (AC-4.1), the assumed uniqueness of the eigenmode (AC-2.7) and (AC-4.16) one deduces:

$$O(1) \ : \ (\mathcal{E}^{\,c}_{,uu}\,\overline{x}_0)\delta u = 0, \quad (\overline{x}_0, \overline{x}_0) = 1 \implies \overline{x}_0 = \overset{1}{u}. \tag{AC-4.19}$$

Continuing with the $O(\xi)$ term in the Taylor series expansion of (AC-4.15) one has:

$$O(\xi) \ : \ (\mathcal{E}^{\,c}_{,uu}\,\overline{x}_\xi + (\mathcal{E}^{\,c}_{,uuu}\,\overset{1}{u})\overset{1}{u})\delta u = \overline{\beta}_\xi(\overset{1}{u}, \delta u). \tag{AC-4.20}$$

By taking $\delta u = \overset{1}{u}$, and recalling (AC-2.7) one obtains for $\overline{\beta}_\xi$:

$$\overline{\beta}_\xi = ((\mathcal{E}^{\,c}_{,uuu}\,\overset{1}{u})\overset{1}{u})\overset{1}{u}. \tag{AC-4.21}$$

while for $\delta u = \delta v$ and noticing from the $O(\xi)$ term of (AC-4.16) and (AC-4.19) that $(\overset{1}{u}, \overline{x}_\xi) = 0$, one obtains by comparing (AC-4.20) with (AC-3.8) that:

$$(\mathcal{E}^{\,c}_{,uu}\,\overline{x}_\xi + (\mathcal{E}^{\,c}_{,uuu}\,\overset{1}{u})\overset{1}{u})\delta v = 0 \implies \overline{x}_\xi = v_{\xi\xi}. \tag{AC-4.22}$$

Continuing with the $O(\Delta\lambda)$ term in the Taylor series expansion of (AC-4.15), one has:

$$O(\Delta\lambda) \ : \ (\mathcal{E}^{\,c}_{,uu}\,\overline{x}_\lambda + (d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\delta u = \overline{\beta}_\lambda(\overset{1}{u}, \delta u). \tag{AC-4.23}$$

By successively taking $\delta u = \overset{1}{u}$ into (AC-4.23) and recalling (AC-2.7) one finds that:

$$\overline{\beta}_\lambda = ((d\mathcal{E},_{uu}/d\lambda)_c \overset{1}{u})\overset{1}{u}. \tag{AC-4.24}$$

while by taking $\delta u = \delta v$, and noticing from the $O(\xi)$ term of (AC-4.16) and (AC-4.19) that $(\overset{1}{u}, \overline{x}_\lambda) = 0$, one obtains by comparing (AC-4.23) with (AC-3.9) that:

$$(\mathcal{E},^c_{uu}\overline{x}_\lambda + (d\mathcal{E},_{uu}/d\lambda)_c \overset{1}{u})\delta u = 0 \implies \overline{x}_\lambda = v_{\xi\lambda}. \tag{AC-4.25}$$

For the symmetric perfect system $((\mathcal{E},^c_{uuu})\overset{1}{u})\overset{1}{u} = 0$, the $O(\xi^2)$ term in the Taylor series expansion of (AC-4.15) is also required:

$$O(\xi^2) \ : \ (\frac{1}{2}(((\mathcal{E},^c_{uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} + \mathcal{E},^c_{uuu}v_{\xi\xi})\overset{1}{u} + (\mathcal{E},^c_{uuu}v_{\xi\xi})\overset{1}{u} + \frac{1}{2}\mathcal{E},^c_{uu}\overline{x}_{\xi\xi})\delta u = \frac{1}{2}\overline{\beta}_{\xi\xi}(\overset{1}{u}, \delta u). \tag{AC-4.26}$$

In the derivation of the above equation use was also made of (AC-4.17), (AC-4.19), (AC-4.21) and (AC-4.22). Once more by taking $\delta u = \overset{1}{u}$ and recalling (AC-2.7) one obtains:

$$\overline{\beta}_{\xi\xi} = (((\mathcal{E}^c_{uuuu}\overset{1}{u})\overset{1}{u} + 3\mathcal{E}^c_{uuu}v_{\xi\xi})\overset{1}{u})\overset{1}{u}. \tag{AC-4.27}$$

Consequently, from (AC-4.17), (AC-4.21), (AC-4.24), (AC-4.27) and (AC-3.13), (AC-3.14) the minimum eigenvalue $\overline{\beta}$ of the imperfect system's stability operator assumes the form:

$$\overline{\beta}(\xi, \Delta\lambda) = [((d\mathcal{E},_{uu}/d\lambda)_c \overset{1}{u})\overset{1}{u}] \begin{cases} (-2\lambda_1\xi + \Delta\lambda) + ... & \text{for} \quad ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0, \\ (-\frac{3}{2}\lambda_2\xi^2 + \Delta\lambda) + ... & \text{for} \quad ((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = 0. \end{cases} \tag{AC-4.28}$$

In the neighborhood of the perfect system's critical point $(\xi, \Delta\lambda) = (0, 0)$, one can find the load extrema of the equilibrium paths by setting $\overline{\beta} = 0$. The fact that the points with $\overline{\beta} = 0$, which are by definition the critical points of the imperfect system's stability operator $\overline{\mathcal{E}},_{uu}$, are load extrema of the corresponding equilibrium paths and not bifurcation points follows from the assumption $(\overline{\mathcal{E}}^c_{uw}\overline{w})\overset{1}{u} \neq 0$ as one can see from (AC-4.13).

Denote by $\Delta\lambda_s(\xi) = \lambda_s(\xi) - \lambda_c$ the difference between the load extremum points λ_s of the imperfect system's equilibrium paths corresponding to a given ξ and the perfect system's critical load λ_c. Assuming that $\Delta\lambda_s$ admits a Taylor series representation in terms of ξ, at least in the neighborhood of $\xi = 0$:

$$\Delta\lambda_s(\xi) = s_1\xi + \frac{s_2}{2!}\xi^2 + \frac{s_3}{3!}\xi^3 + O(\xi^4). \tag{AC-4.29}$$

Since $\overline{\beta}(\xi, \Delta\lambda_s(\xi)) = 0$, one can easily find the coefficients s_n in the Taylor series expansion of $\Delta\lambda_s$, by introducing (AC-4.29) into (AC-4.28):

$$\Delta\lambda_s(\xi) = \begin{cases} 2\lambda_1\xi + O(\xi^2) & \text{for} \quad ((\mathcal{E},^c_{uuu})\overset{1}{u})\overset{1}{u} \neq 0, \\ \frac{3}{2}\lambda_2\xi^2 + O(\xi^3) & \text{for} \quad ((\mathcal{E},^c_{uuu})\overset{1}{u})\overset{1}{u} = 0. \end{cases} \tag{AC-4.30}$$

Notice that the curve $\Delta\lambda_s(\xi)$ connects the load extrema of all the equilibrium paths of the imperfect system, each one of which corresponds to the same imperfection shape $\overline{w}$ but to a different imperfection amplitude ϵ, as shown in Fig. AC-4.1 and Fig. AC-4.2.

In applications, where imperfection w is known, i.e. for ϵ and $\overline{w}$ given, one is interested in finding $\Delta\lambda_s$ as a function of ϵ. The finding of $\Delta\lambda_s < 0$, the reduction from the critical load corresponding to the perfect system of the maximum load corresponding to the imperfect system is of particular interest, for it quantifies the critical load drop due to the presence of unavoidable imperfections in the system under investigation. Solutions for $\Delta\lambda_s > 0$, can be easily shown to correspond to equilibrium branches that do not pass through $\lambda = 0$ and hence are of no interest here.

Substitution of (AC-4.30) into (AC-4.14) and subsequent solution for $\Delta\lambda_s$ in terms of ϵ, gives to the leading order in ϵ for the asymmetric and symmetric perfect system respectively:

$$\Delta\lambda_s(\epsilon) = \begin{cases} -2[\epsilon\lambda_1(\overline{\mathcal{E}},^c_{uw}\,\overline{w})\overset{1}{u}/((-d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}]^{1/2} + O(\epsilon) \text{ for } ((\mathcal{E},^c_{uuu}\,\overset{1}{u})\overset{1}{u})\overset{1}{u} \neq 0,\ \epsilon\lambda_1(\overline{\mathcal{E}},^c_{uw}\,\overline{w})\overset{1}{u} > 0, \\ \dfrac{3}{2}(\lambda_2)^{1/3}[\epsilon(\overline{\mathcal{E}},^c_{uw}\,\overline{w})\overset{1}{u}/((-d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}]^{2/3} + O(\epsilon) \text{ for } ((\mathcal{E},^c_{uuu}\,\overset{1}{u})\overset{1}{u})\overset{1}{u} = 0,\ \lambda_2 < 0. \end{cases} \tag{AC-4.31}$$

Since in most practical applications the amplitude of the imperfection can be controlled but not its shape, it is important to find the imperfection shape $\overline{w}$ that maximizes the load drop $(\Delta\lambda_s)$ for a given ϵ. It is not difficult to see from (AC-4.31) that $(\Delta\lambda_s)$ is maximized when $\mid(\overline{\mathcal{E}},^c_{uw}\,\overline{w})\overset{1}{u}\mid$ is maximized, a rather straightforward problem in linear algebra.

Example

As a simple first application of the general theory developed here, the imperfect rigid T example will be revisited. In this case one can easily deduce by inspection of (AB-2.1) that $w = (0, \delta)$, $\overline{w} = (0, 1) = \mathbf{e}_2$ and $\epsilon \equiv \parallel w \parallel = \delta$. Also, from (AB-2.1) one obtains with the help of (AC-3.32):

$$\overline{\mathcal{E}},^c_{uw} = -\lambda_c L\mathbf{e}_2\mathbf{e}_2, \quad (\overline{\mathcal{E}},^c_{uw}\,\overline{w})\overset{1}{u} = -\lambda_c L. \tag{AC-4.32}$$

Recalling (AC-3.37), (AC-3.38), (AC-3.40) as well as (AC-3.32) one obtains from (AC-4.14) the result:

$$\epsilon = \begin{cases} (mL^2\xi^2 - \xi\Delta\lambda)/\lambda_c & \Longrightarrow\ \lambda_c\epsilon = mL^2\theta^2 - \theta(\lambda - \lambda_c) \quad \text{for} \quad m \neq 0,\ n = 0, \\ (nL^3\xi^3 - \xi\Delta\lambda)/\lambda_c & \Longrightarrow\ \lambda_c\epsilon = \ nL^3\theta^3 - \theta(\lambda - \lambda_c) \quad \text{for} \quad m = 0,\ n \neq 0. \end{cases} \tag{AC-4.33}$$

exactly as found in (AB-2.4).

Having independently rederived from the general theory the equilibrium equations for the rigid T model, attention is turned next in finding the minimum eigenvalue of the model's stability matrix. Recalling once more (AC-3.37), (AC-3.38) and (AC-3.40) one obtains from

(AC-4.28):

$$\overline{\beta} = \begin{cases} -L(\Delta\lambda - 2mL^2\xi) = (\lambda_c - \lambda)L + 2mL^3\theta & \text{for} \quad m \neq 0,\ n = 0, \\ \\ -L(\Delta\lambda - 3nL^3\xi^2) = (\lambda_c - \lambda)L + 3nL^4\theta^2 & \text{for} \quad m = 0,\ n \neq 0. \end{cases} \tag{AC-4.34}$$

As expected, both the above expressions coincide with the results obtained in (AC-2.6) for the minimum eigenvalue of the stability matrix of the imperfect rigid T. More realistic examples from structural and solid mechanics requiring the application the general theory developed in this section will be given in the next chapters.

AC-5 PERFECT SYSTEM - SIMULTANEOUS MULTIPLE MODES

Up until now, a fundamental assumption in the bifurcation, post-bifurcation and imperfection sensitivity analyses of elastic systems was the uniqueness of the eigenmode $\overset{1}{u}$ associated with the lowest critical load λ_c (see (AC-2.7)). For the remaining sections of this section, this assumption is to be generalized as to include systems with a finite number of eigenvectors $\overset{i}{u},\ 1 \le i \le m$ corresponding to λ_c.

The system considered is still characterized by a potential energy $\mathcal{E}(u,\lambda)$ which obeys Eqs. (AC-2.1) – (AC-2.4). The stability of the principal branch $\overset{0}{u}(\lambda)$ is lost for the first time, as λ increases away from zero, at λ_c where:

$$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda_c),\lambda_c)\overset{i}{u})\delta u = 0, \quad (\overset{i}{u},\overset{j}{u}) = \delta_{ij}; \quad i,j = 1,...,m. \tag{AC-5.1}$$

The mode normalization condition in the above equation [8] ensures that the eigenmodes form an orthonormal set. This property will facilitate some of the subsequent calculations. Further simplification can be achieved by choosing a specific inner product, as will be described later.

In analogy to the simple bifurcation case it is also assumed that the critical point $(\overset{0}{u}(\lambda_c),\lambda_c)$ is a true multiple bifurcation point for the system, i.e. it satisfies the m-dimensional version of (AC-2.27) namely:

$$\mathcal{E},^c_{u\lambda}\overset{i}{u} = 0, \quad i = 1,...,m. \tag{AC-5.2}$$

A criticality, the stability operator $\mathcal{E},^c_{uu}$ loses its positive definiteness only for those directions that are linear combinations of its eigenvectors $\overset{i}{u}$, i.e. for directions belonging to its null space $\mathcal{N}$. This implies the strict positive definiteness of $\mathcal{E},^c_{vv}$ on $\mathcal{N}^\perp$. Consequently (AC-2.8) continues to hold, but with the updated definition for the null space, namely $\mathcal{N} \equiv \{u \in U \mid u = \sum_{i=1}^m \xi_i \overset{i}{u},\ \xi_i \in \mathbb{R}\}$. The corresponding definition for the orthogonal complement to the null space is $\mathcal{N}^\perp \equiv \{v \in U \mid (v,\overset{i}{u}) = 0,\ i = 1,...,m\}$.

At the neighborhood of the critical point (u_c,λ_c), the solution u to the equilibrium (AC-2.2) can be written with the help of the LSK decomposition (compare to (AC-3.1)):

$$u = \overset{0}{u}(\lambda) + \sum_{i=1}^{m} \xi_i \overset{i}{u} + v; \quad \xi_i \in \mathbb{R}, \quad v \in \mathcal{N}^\perp, \tag{AC-5.3}$$

where ξ_i is the projection of $u - \overset{0}{u}$ on $\overset{i}{u}$.

The sought displacement u is thus replaced by an equivalent set of unknowns (v,ξ_i) and the solution to the equilibrium equation $\mathcal{E},_u \delta u = 0$ proceeds in two steps: First v is determined as a function of $\Delta\lambda$ $(\Delta\lambda \equiv \lambda - \lambda_c)$ and ξ_i from the equilibrium equation in $\mathcal{N}^\perp$, namely:

$$\mathcal{E},_v \delta v = 0 \implies \mathcal{E},_u(\overset{0}{u}(\lambda_c + \Delta\lambda) + \sum_{i=1}^{m} \xi_i \overset{i}{u} + v, \lambda_c + \Delta\lambda)\delta v = 0 \quad \forall \delta v \in \mathcal{N}^\perp. \tag{AC-5.4}$$

[8] Here δ_{ij} denotes the Kronecker delta symbol defined such that $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise

The resulting v is used in the remaining equilibrium equations on $\mathcal{N}$. This provides the relation between $\Delta\lambda$ and ξ_i. The m equilibrium equations that have to be solved on $\mathcal{N}$ are:

$$\mathcal{E},_{\xi_i} = 0 \implies \mathcal{E},_u (\overset{0}{u} (\lambda_c + \Delta\lambda) + \sum_{i=1}^{m} \xi_i \overset{i}{u} + v, \lambda_c + \Delta\lambda)\overset{i}{u} = 0 \tag{AC-5.5}$$

From the assumed positive definiteness of $\mathcal{E},^c_{vv}$ on $\mathcal{N}^\perp$ follows that (AC-5.4) has a unique and adequately smooth solution $v(\xi_i, \Delta\lambda)$, at least in the neighborhood of the critical point (u_c, λ_c). The Taylor series expansion of this solution is:

$$v(\xi_i, \Delta\lambda) = \sum_{i=1}^{m} \xi_i v_i + \Delta\lambda v_\lambda + \frac{1}{2}(\sum_{i=1}^{m}\sum_{j=1}^{m} \xi_i \xi_j v_{ij} + 2\Delta\lambda \sum_{i=1}^{m} \xi_i v_{i\lambda} + (\Delta\lambda)^2 v_{\lambda\lambda}) + ... \tag{AC-5.6}$$

Upon substitution of (AC-5.6) into (AC-5.4) and subsequent evaluation of its Taylor series expansion about $(\xi_i, \Delta\lambda) = (0, ..., 0)$ one obtains the following results: The $O(1)$ term of the expansion gives $\mathcal{E},^c_u \delta v = 0$ which is automatically satisfied in view of the equilibrium equation (AC-2.2). The $O(\xi_i)$ terms yield, with the help of (AC-5.2), the result: $(\mathcal{E},^c_{uu} v_i)\delta v = 0$ which in view of (AC-2.8) implies the generalization of (AC-3.5):

$$v_i = 0. \tag{AC-5.7}$$

By taking $\xi_i = 0$ into (AC-5.4), comparing the result to the equilibrium condition on the fundamental solution (AC-2.3), and invoking the uniqueness of the solution to (AC-5.4) for $v(\xi_i, \Delta\lambda)$, it is readily seen that $v(0, \Delta\lambda) = 0$. This implies once again the results already seen for the simple mode (see (AC-3.7) and (AC-3.11)):

$$v_\lambda = v_{\lambda\lambda} = ... = 0. \tag{AC-5.8}$$

This result could have also been obtained directly from the $O(\Delta\lambda)^n$ terms in the expansion of (AC-5.4).

One can similarly continue with the quadratic order terms in the expansion of the equilibrium equation (AC-5.4) to find:

$$O(\xi_i \xi_j) \ : \ (\mathcal{E},^c_{uu} v_{ij} + (\mathcal{E},^c_{uuu} \overset{i}{u})\overset{j}{u})\delta v = 0, \tag{AC-5.9}$$

$$O(\xi_i \Delta\lambda) \ : \ (\mathcal{E},^c_{uu} v_{i\lambda} + (d\mathcal{E},_{uu} /d\lambda)_c \overset{i}{u})\delta v = 0. \tag{AC-5.10}$$

The above equations have unique solutions $v_{ij}, v_{i\lambda}$ in view of (AC-2.8). In a similar way one can find the higher order terms in the expansion of $v(\xi_i, \Delta\lambda)$ and hence uniquely determine the solution to (AC-5.4). Upon substitution of the thus found $v(\xi_i, \Delta\lambda)$ into the remaining equilibrium equation (AC-5.5), and after using (AC-5.2), (AC-5.7) and (AC-5.8) one obtains the following m equations relating $\Delta\lambda$ and ξ_i:

$$\begin{aligned}
&\frac{1}{2}[\sum_{j=1}^{m}\sum_{k=1}^{m}\xi_j\xi_k\mathcal{E}_{ijk} + 2\Delta\lambda\sum_{j=1}^{m}\xi_j\mathcal{E}_{ij\lambda}\ ] + \frac{1}{6}[\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_j\xi_k\xi_l\mathcal{E}_{ijkl} + ...\ ]\ + ... = 0,\\
&\mathcal{E}_{ijk} \equiv ((\mathcal{E},^c_{uuu}\overset{i}{u})\overset{j}{u})\overset{k}{u},\\
&\mathcal{E}_{ijkl} \equiv (((\mathcal{E},^c_{uuuu}\overset{j}{u})\overset{k}{u})\overset{l}{u} + (\mathcal{E},^c_{uuu}v_{jk})\overset{l}{u} + (\mathcal{E},^c_{uuu}v_{kl})\overset{j}{u} + (\mathcal{E},^c_{uuu}v_{lj})\overset{k}{u})\overset{i}{u},\\
&\mathcal{E}_{ij\lambda} \equiv ((d\mathcal{E},_{uu}/d\lambda)_c)\overset{i}{u})\overset{j}{u} = ((\mathcal{E},^c_{uuu}(d\overset{0}{u}/d\lambda)_c + \mathcal{E},^c_{uu\lambda})\overset{i}{u})\overset{j}{u}.
\end{aligned} \tag{AC-5.11}$$

As expected, an obvious solution to (AC-5.11) is the principal equilibrium branch for which $\xi_i = 0$, $v = 0$ but $\Delta\lambda \neq 0$ according to (AC-5.3). The determination of the remaining equilibrium paths through the critical point, i.e. the determination of the curves $\xi_i(\Delta\lambda)$, is facilitated by introducing the "*bifurcation amplitude parameter*" ξ, defined as the projection of $u- \overset{0}{u}$ on the unit tangent of the equilibrium path at λ_c. For a neighborhood of the critical point, assuming an adequately smooth dependence of $\xi_i, \Delta\lambda$ on ξ one has:

$$\begin{aligned}
\xi_i(\xi) &= \alpha_i^1\xi + \alpha_i^2\frac{\xi^2}{2} + ...\\
\Delta\lambda(\xi) &= \lambda_1\xi + \lambda_2\frac{\xi^2}{2} + ...
\end{aligned} \qquad \text{where :} \quad \xi \equiv (u- \overset{0}{u}, \sum_{i=1}^{m}\alpha_i^1\overset{i}{u}). \tag{AC-5.12}$$

Two cases are to be distinguished at this point: First the asymmetric case for which $\mathcal{E}_{ijk} \neq 0$ at least for one triplet of indexes (i, j, k). By inserting (AC-5.12) into to (AC-5.11) and collecting the terms of the like order in ξ, one obtains the following system of m quadratic equations from the lowest order nontrivial term in this expansion (the $O(\xi^2)$ term):

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\alpha_j^1\alpha_k^1\mathcal{E}_{ijk} + 2\lambda_1\sum_{j=1}^{m}\alpha_j^1\mathcal{E}_{ij\lambda} = 0, \qquad \sum_{i=1}^{m}(\alpha_i^1)^2 = 1. \tag{AC-5.13}$$

where the second equation results from the unit norm of the tangent $\sum_{i=1}^{m}\alpha_i^1\overset{i}{u}$.

The above algebraic system of $m+1$ equations for the $m+1$ unknowns α_i^1, λ_1 has at most $2^m - 1$ pairs of real solutions (α_i^1, λ_1) and $(-\alpha_i^1, -\lambda_1)$, with each pair corresponding to a bifurcated equilibrium path through the critical point. Each one of these equilibrium paths can be constructed by computing its Taylor series expansion as indicated in (AC-5.12). Higher order terms in the expansion of (AC-5.11) show that the series can be continued to any desired order if the following condition is satisfied:

$$\text{Det }[B_{ij}] \neq 0, \qquad B_{ij} \equiv \sum_{k=1}^{m}\alpha_k^1\mathcal{E}_{ijk} + \lambda_1\mathcal{E}_{ij\lambda}. \tag{AC-5.14}$$

The second case to be investigated will be the symmetric case for which $\mathcal{E}_{ijk} = 0$ for all triplets of indexes (i, j, k). In this case (AC-5.13) implies that $\lambda_1 = 0$. By inserting (AC-5.12) into (AC-5.11) and collecting the terms of the like order in ξ, one obtains the following system of m cubic equations from the lowest order nontrivial term in this expansion (the $O(\xi^3)$ term):

$$\lambda_1 = 0, \quad \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}\alpha_j^1\alpha_k^1\alpha_l^1\mathcal{E}_{ijkl} + 3\lambda_2\sum_{j=1}^{m}\alpha_j^1\mathcal{E}_{ij\lambda} = 0, \quad \sum_{i=1}^{m}(\alpha_i^1)^2 = 1. \tag{AC-5.15}$$

where again the second equation results from the unit norm of the tangent $\sum_{i=1}^{m}\alpha_i^1\overset{i}{u}$.

This algebraic system of $m+1$ equations for the $m+1$ unknowns α_i^1, λ_2 has at most $(3^m-1)/2$ pairs of real solutions (α_i^1, λ_2) and $(-\alpha_i^1, \lambda_2)$, each corresponding to a bifurcated equilibrium path through the critical point. Each one of these equilibrium paths can be constructed by computing its Taylor series expansion as indicated in (AC-5.12). The continuation of the expansion of (AC-5.11) to terms of $O(\xi^4)$ and higher for each particular branch is assured when:

$$\text{Det }[B_{ij}] \neq 0, \quad B_{ij} \equiv \sum_{k=1}^{m}\sum_{l=1}^{m}\alpha_k^1\alpha_l^1\mathcal{E}_{ijkl} + \lambda_2\mathcal{E}_{ij\lambda}. \tag{AC-5.16}$$

To complete the study of the above found bifurcated equilibrium branches, one has to investigate their stability. To this end one needs to find the sign of the minimum eigenvalue β_{min} of the stability operator $\mathcal{E}_{,uu}(u,\lambda)$ evaluated on the equilibrium path whose stability is under investigation. First the stability of the principal branch is to be investigated. To this end, assume that $\overset{0}{\beta}(\lambda)$ is any one of the m eigenvalues of $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)$ that vanish at λ_c, while $\overset{0}{x}(\lambda)$ is the corresponding normalized eigenvector. A strict crossing of zero at the critical load will again be assumed for $\overset{0}{\beta}(\lambda)$ which will again have to satisfy Eqs. (AC-3.15) – (AC-3.17). Notice that in this case each one of these equations actually represents m different equations, each one of which corresponds to one of the m different functions $\overset{0}{\beta}_i(\lambda)$ and corresponding eigenvectors $\overset{0}{x}_i(\lambda)$ 0where the subscript i has been avoided to alleviate notation.

Evaluating (AC-3.16) at the critical load one has in view of (AC-2.8) and (AC-5.1)

$$\overset{0}{x}_c = \sum_{i=1}^{m}\overset{0}{\chi}_i\overset{i}{u}. \tag{AC-5.17}$$

Assuming that each one of the $\overset{0}{\beta}(\lambda)$ and $\overset{0}{x}(\lambda)$ are smooth functions of their argument, (AC-3.16) can be differentiated with respect to λ. Evaluating the result at the critical point, and recalling from (AC-3.15) that $\overset{0}{\beta}(\lambda_c) = 0$ one has:

$$((d\mathcal{E}_{,uu}/d\lambda)_c\overset{0}{x}_c + \mathcal{E}_{,uu}^{c}(d\overset{0}{x}/d\lambda)_c)\delta u = (d\overset{0}{\beta}/d\lambda)_c(\overset{0}{x}_c, \delta u). \tag{AC-5.18}$$

Taking $\delta u = \overset{i}{u}$ and recalling (AC-5.1) and (AC-5.17), equation (AC-5.18) yields:

$$\sum_{j=1}^{m}\mathcal{E}_{ij\lambda}\overset{0}{\chi}_j = (d\overset{0}{\beta}/d\lambda)_c\overset{0}{\chi}_i, \tag{AC-5.19}$$

which shows that the m derivatives $(d\overset{0}{\beta}/d\lambda)_c$ are the eigenvalues of $\mathcal{E}_{ij\lambda}$ and the m vectors $\overset{0}{\chi}_i$ are the corresponding eigenvectors. Since from (AC-3.15) each such eigenvalue satisfies

$(d\overset{0}{\beta}/d\lambda)_c < 0$, one concludes that $\mathcal{E}_{ij\lambda}$ is a negative definite matrix. This condition is satisfied in the majority of applications of interest.

The stability of each bifurcated equilibrium branch trough the critical point depends on the sign of the minimum eigenvalue $\beta_{min}(\xi)$ of the corresponding stability operator $\mathcal{E}_{,uu}$. In analogy to (AC-3.21), the definitions for each one of the m lowest eigenvalues $\beta(\xi)$ and the corresponding normalized eigenvectors $x(\xi)$ are:

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c + \Delta\lambda) + \sum_{i=1}^{m} \xi_i \overset{i}{u} + v(\xi_i, \Delta\lambda), \lambda_c + \Delta\lambda)x)\delta u = \beta(x, \delta u), \tag{AC-5.20}$$

while the normalization condition for $x(\xi)$ is still given by (AC-3.22). In the above definition $\xi_i, \Delta\lambda, \beta, x$ are functions of the parameter ξ. Recall that for every bifurcated equilibrium path the m lowest eigenvalues $\beta(\xi)$ of the corresponding stability operator have to vanish at $\xi = 0$. In addition, for each bifurcated equilibrium path, $\beta(\xi), x(\xi)$ are assumed smooth functions of their argument and their Taylor series expansions as still given by (AC-3.23).

By introducing (AC-3.23) into (AC-5.20) and recalling (AC-5.1), Eqs. (AC-5.6) – (AC-5.8) and (AC-5.12), one obtains by expanding about $\xi = 0$ and collecting the terms of the like order in ξ the following results: The $O(1)$ term yields:

$$x_0 = \sum_{i=1}^{m} \chi_i \overset{i}{u}. \tag{AC-5.21}$$

Continuing with the $O(\xi)$ term in the expansion of (AC-5.20) one has:

$$O(\xi) \ : \ ((\mathcal{E}^{\ c}_{,uuu}(\lambda_1(d\overset{0}{u}/d\lambda)_c + \sum_{k=1}^{m} \alpha_k^1 \overset{k}{u}) + \lambda_1 \mathcal{E}^{\ c}_{,uu\lambda})(\sum_{j=1}^{m} \chi_j \overset{j}{u}) + \mathcal{E}^{\ c}_{,uu} x_1)\delta u = \beta_1((\sum_{j=1}^{m} \chi_j \overset{j}{u}), \delta u). \tag{AC-5.22}$$

Taking $\delta u = \overset{i}{u}$ and recalling from $(\text{AC-5.1})_2$ the orthogonality of the eigenmodes, the above equation yields:

$$\sum_{j=1}^{m} B_{ij} \chi_j = \beta_1 \chi_i, \tag{AC-5.23}$$

where B_{ij} is defined in (AC-5.14). For the asymmetric bifurcation case, this matrix is non-singular, which ensures that all its eigenvalues β_1 are nonzero, as well as real, in view of the symmetry of the matrix B_{ij}. It follows from (AC-5.23) that the constants χ_i introduced in (AC-5.21) are the components of the eigenvector of B_{ij} corresponding to the eigenvalue β_1.

For the symmetric bifurcation case, since $\mathcal{E}_{ijk} = \lambda_1 = 0$, $B_{ij} = 0$ as seen from (AC-5.14), which in view of (AC-5.23) also implies that $\beta_1 = 0$. Consequently, the $O(\xi)$ term in the expansion of (AC-5.20), with the help of the definition of v_{ij} given in (AC-5.9) and the normalization condition from the eigenmode $(x, x) = 1$, results in the following expression for x_1:

$$O(\xi) \ : \ (\sum_{j=1}^{m}\sum_{k=1}^{m} \chi_j \alpha_k^1 (\mathcal{E}^{\ c}_{,uuu} \overset{j}{u})\overset{k}{u} + \mathcal{E}^{\ c}_{,uu} x_1)\delta u = 0, \quad (x_1, x_0) = 0, \implies x_1 = \sum_{i=1}^{m}\sum_{j=1}^{m} \chi_i \alpha_j^1 v_{ij}. \tag{AC-5.24}$$

Continuing with the $O(\xi^2)$ term in the expansion of (AC-5.20) and recalling that $\lambda_1 = \beta_1 = 0$ as well as (AC-5.1), Eqs. (AC-5.6) – (AC-5.8) and (AC-5.12), one has:

$$
\begin{aligned}
O(\xi^2) \ : \ & (((\mathcal{E},^c_{uuuu} (\sum_{k=1}^{m} \alpha_k^1 \overset{k}{u}))(\sum_{l=1}^{m} \alpha_l^1 \overset{l}{u}) + \mathcal{E},^c_{uuu} (\lambda_2 (d\overset{0}{u}/d\lambda)_c + \sum_{k=1}^{m}\sum_{l=1}^{m} \alpha_k^1 \alpha_l^1 v_{kl} + \sum_{k=1}^{m} \alpha_k^2 \overset{k}{u}) \\
& + \lambda_2 \mathcal{E},^c_{uu\lambda})(\sum_{j=1}^{m} \chi_j \overset{j}{u}) + 2(\mathcal{E},^c_{uuu} (\sum_{k=1}^{m} \alpha_k^1 \overset{k}{u}))x_1 + \mathcal{E},^c_{uu} x_2)\delta u = \beta_2((\sum_{j=1}^{m} \chi_j \overset{j}{u}), \delta u).
\end{aligned}
\tag{AC-5.25}
$$

By subsequently taking $\delta u = \overset{i}{u}$ and using (AC-5.24) into (AC-5.25) one obtains:

$$\sum_{j=1}^{m} B_{ij} \chi_j = \beta_2 \chi_i, \tag{AC-5.26}$$

where the matrix B_{ij} is now given by (AC-5.16). For the symmetric bifurcation case, this matrix is nonsingular, which ensures that all its eigenvalues β_2 are nonzero (as well as real in view of the symmetry of the matrix). It also follows from (AC-5.26) that χ_i, the components of x_0 introduced in (AC-5.21), are the components of the eigenvector of B_{ij} corresponding to the eigenvalue β_2.

Hence the wanted minimum eigenvalue $\beta_{min}(\xi)$ of $\mathcal{E},_{uu}$ for the bifurcated equilibrium path in question is:

$$\beta_{min}(\xi) = \begin{cases} \xi \beta_1^{max} + O(\xi^2) \text{ if } \xi < 0, \quad \xi \beta_1^{min} + O(\xi^2) \text{ if } \xi > 0 & \text{for asymmetric bifurcation,} \\ (\xi^2/2)\beta_2^{min} + O(\xi^3) & \text{for symmetric bifurcation.} \end{cases} \tag{AC-5.27}$$

For the asymmetric bifurcation case, if for a certain bifurcated branch the maximum and minimum eigenvalues of B_{ij} respectively β_1^{max} and β_1^{min} are of the same sign, then the bifurcated branch in question changes stability as it crosses the critical point, while if the two extremal eigenvalues are of opposite sign the bifurcated branch in question is always unstable. For the symmetric bifurcation case, if the minimum eigenvalue β_2^{min} of B_{ij} is positive, the bifurcation branch in question is stable. Any negative eigenvalues render it unstable.

One can also compute $\Delta\mathcal{E}$ the difference for a fixed value of the load parameter λ, between the potential energies on a bifurcated and on the principal branch. By definition:

$$\Delta\mathcal{E} = \mathcal{E}(\overset{0}{u}(\lambda_c + \Delta\lambda) + \sum_{i=1}^{m} \xi_i \overset{i}{u} + v(\xi_i, \Delta\lambda), \lambda_c + \Delta\lambda) - \mathcal{E}(\overset{0}{u}(\lambda_c + \Delta\lambda), \lambda_c + \Delta\lambda). \tag{AC-5.28}$$

Expanding $\Delta\mathcal{E}$ about $\xi = 0$, and collecting terms of the like order of ξ, one obtains with the help of (AC-2.2), (AC-5.1) and (AC-5.12), (AC-5.13) for the asymmetric bifurcation case ($\mathcal{E}_{ijk} \neq 0$):

$$\Delta\mathcal{E} = \frac{\xi^3}{6}\lambda_1 \sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i^1 \alpha_j^1 \mathcal{E}_{ij\lambda} + O(\xi^4), \tag{AC-5.29}$$

while following a similar procedure for the symmetric bifurcation case ($\mathcal{E}_{ijk} = 0$) and with the additional help of (AC-5.9) and (AC-5.15) one has:

$$\Delta\mathcal{E} = \frac{\xi^4}{8}\lambda_2 \sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i^1 \alpha_j^1 \mathcal{E}_{ij\lambda} + O(\xi^5). \tag{AC-5.30}$$

Recall from the discussion of (AC-5.19) that $\mathcal{E}_{ij\lambda}$ is negative definite and hence $\sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i^1 \alpha_j^1 \mathcal{E}_{ij\lambda} < 0$. Our conclusions about the energies of the bifurcated equilibrium branches through λ_c are the same to the ones reached for the single mode case (see (AC-3.32), (AC-3.33)). Given a load level λ, one can see from (AC-5.29), (AC-5.30) that the supercritical branches $\lambda > \lambda_c$ (which correspond to $\lambda_1 \xi > 0$ for the asymmetric case or to $\lambda_2 > 0$ for the symmetric case) have lower energy than the principal branch. For the subcritical branches $\lambda < \lambda_c$ (which correspond to $\lambda_1 \xi < 0$ for the asymmetric case or to $\lambda_2 < 0$ for the symmetric case) the situation is reversed and the principal branch has lower energy than the bifurcated one.

Example

As a simple first application of the general theory developed in this section, the perfect rigid plate example discussed in subection AB-3 will be revisited. In this case the space of admissible displacements $u = (v, \theta, \phi)$ is $U = \mathbb{R}^3$ whose Cartesian basis is $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$. From (AB-3.1), the stability operator evaluated on the principal branch $\mathcal{E},_{uu}(\overset{0}{u}(\lambda), \lambda)$ is the rank two tensor:

$$\mathcal{E},_{uu}(\overset{0}{u}(\lambda), \lambda) = 4E\mathbf{e}_1\mathbf{e}_1 + (\lambda_c - \lambda)L(\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3). \tag{AC-5.31}$$

It is obvious from (AC-5.31) that at $\lambda = \lambda_c$ the stability operator $\mathcal{E},^c_{uu}$ has zero as a double eigenvalue and that a corresponding pair of eigenvectors are:

$$\overset{1}{u} = \mathbf{e}_2, \quad \overset{2}{u} = \mathbf{e}_3. \tag{AC-5.32}$$

Notice that the eigenvectors constitute an orthonormal basis to the null space $\mathcal{N} = \{(\theta, \phi) |\ \theta, \phi \in \mathbb{R}\}$. Hence the space $\mathcal{N}^\perp = \{v |\ v \in \mathbb{R}\}$.

The critical point (u_c, λ_c) is a double bifurcation point since it satisfies (AC-5.2), as one can see from (AC-5.31), (AC-5.32):

$$\mathcal{E},^c_{u\lambda}\overset{1}{u} = (-\mathbf{e}_1) \bullet \mathbf{e}_2 = 0, \quad \mathcal{E},^c_{u\lambda}\overset{2}{u} = (-\mathbf{e}_1) \bullet \mathbf{e}_3 = 0. \tag{AC-5.33}$$

Note that the two lowest eigenvalues of the stability matrix evaluated on the principal branch $\mathcal{E},_{uu}(\overset{0}{u}(\lambda), \lambda)$ have a strict crossing of zero at λ_c and hence the matrix $\mathcal{E}_{ij\lambda}$ is negative definite. Indeed from (AC-5.31), (AC-5.32) and recalling the definition of $\mathcal{E}_{ij\lambda}$ in (AC-5.11):

$$\mathcal{E}_{ij\lambda} \equiv ((d\mathcal{E},_{uu}/d\lambda)_c \overset{i}{u})\overset{j}{u} = -L((\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3) \bullet \overset{i}{u}) \bullet \overset{j}{u} = -L\delta_{ij}. \tag{AC-5.34}$$

The two eigenvalues $(d\overset{0}{\beta}/d\lambda)_c$ of $\mathcal{E}_{ij\lambda}$ (see (AC-5.19)) are from (AC-5.34) found to be $(d\overset{0}{\beta}/d\lambda)_c = -L < 0$.

Also note that the bifurcation amplitude parameters ξ_1 and ξ_2 are found with the help of (AC-5.3) and (AC-5.32) to be:

$$
\begin{aligned}
\xi_1 &= (u - \overset{0}{u}(\lambda)) \bullet \overset{1}{u} = [(v - (\lambda/4E))\mathbf{e}_1 + \theta\mathbf{e}_2 + \phi\mathbf{e}_3] \bullet \mathbf{e}_2 = \theta, \\
\xi_2 &= (u - \overset{0}{u}(\lambda)) \bullet \overset{2}{u} = [(v - (\lambda/4E))\mathbf{e}_1 + \theta\mathbf{e}_2 + \phi\mathbf{e}_3] \bullet \mathbf{e}_3 = \phi,
\end{aligned}
\tag{AC-5.35}
$$

which gives a meaningful physical interpretation to the bifurcation amplitude parameters ξ_i in this example.

For the asymmetric rigid plate model $m \neq 0, n = 0$ the determination of the first order coefficients $\{\alpha_i^1\}$ in the expansion of the bifurcation amplitudes ξ_i and the first order coefficient λ_1 in the expansion of the load λ for each bifurcated equilibrium path, requires in addition to (AC-5.34) the evaluation of the rank three tensor $\mathcal{E},^c_{uuu}$. Hence from (AB-3.1) one has:

$$
\mathcal{E},^c_{uuu} = 2mL^3[(\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2)\mathbf{e}_2 + (\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3)\mathbf{e}_3]. \tag{AC-5.36}
$$

Using (AC-5.32), (AC-5.34) and recalling (AC-5.36) into the definition of the coefficients $\mathcal{E}_{ijk}$ (see (AC-5.11)), (AC-5.13) gives the following system for $\{\alpha_i^1\}$ and λ_1:

$$
\begin{aligned}
4mL^3\alpha_1^1\alpha_2^1 - 2\lambda_1 L\alpha_1^1 &= 0 \\
2mL^3[(\alpha_1^1)^2 + (\alpha_2^1)^2] - 2\lambda_1 L\alpha_2^1 &= 0
\end{aligned}
\qquad (\alpha_1^1)^2 + (\alpha_2^1)^2 = 1. \tag{AC-5.37}
$$

The above system admits three different solutions $M1$, $M2$, $M3$ namely:

$$
\begin{aligned}
&M1 \ : \ \alpha_1^1 = 0, && \alpha_2^1 = 1, && \lambda_1 = mL^2, \\
&M2 \ : \ \alpha_1^1 = 1/\sqrt{2}, && \alpha_2^1 = 1/\sqrt{2}, && \lambda_1 = \sqrt{2}mL^2, \\
&M3 \ : \ \alpha_1^1 = -1/\sqrt{2}, && \alpha_2^1 = 1/\sqrt{2}, && \lambda_1 = -\sqrt{2}mL^2.
\end{aligned}
\tag{AC-5.38}
$$

By continuing with the higher order coefficients $\{\alpha_i^n\}$, $n > 1$ in the expansion of $\xi_i(\xi)$ (see (AC-5.12) as well as with the component v of $u - \overset{0}{u}$ on $N^\perp$ ($v = (u - \overset{0}{u}) \bullet \mathbf{e}_1$) one easily finds $\alpha_i^n = 0$ for $n > 1$ and $v = 0$. Hence from (AC-5.38) the $2^m - 1 = 3$ bifurcated equilibrium branches for the asymmetric rigid plate model are:

$$
\begin{aligned}
&M1 \ : \ u = (\lambda/4E)\mathbf{e}_1 + \theta\mathbf{e}_3 + \phi\mathbf{e}_3, \quad \lambda = \lambda_c + \xi mL^2; && \theta = 0, \ \phi = \xi, \\
&M2 \ : \ u = (\lambda/4E)\mathbf{e}_1 + \theta\mathbf{e}_3 + \phi\mathbf{e}_3, \quad \lambda = \lambda_c + \xi\sqrt{2}mL^2; && \theta = \phi = \xi/\sqrt{2}, \\
&M3 \ : \ u = (\lambda/4E)\mathbf{e}_1 + \theta\mathbf{e}_3 + \phi\mathbf{e}_3, \quad \lambda = \lambda_c - \xi\sqrt{2}mL^2; && -\theta = \phi = \xi/\sqrt{2}
\end{aligned}
\tag{AC-5.39}
$$

which coincide with the solutions found in (AC-3.4)$_2$ as expected.

For the stability of the above bifurcated equilibrium branches one has to investigate the eigenvalues of the stability matrix B_{ij} defined in (AC-5.14), which with the help of (AC-5.34) and (AC-5.36) is found to be:

$$
[B_{ij}] = \begin{bmatrix} 2mL^3\alpha_2^1 - \lambda_1 L & 2mL^3\alpha_1^1 \\ 2mL^3\alpha_1^1 & 2mL^3\alpha_2^1 - \lambda_1 L \end{bmatrix} \tag{AC-5.40}
$$

For each one of the equilibrium paths in (AC-5.39), the above stability matrix takes the form:

$$
\begin{aligned}
&M1 \;:\; [B_{ij}] = \begin{bmatrix} mL^3 & 0 \\ 0 & mL^3 \end{bmatrix}, \qquad \beta_1^{max} = \beta_1^{min} = mL^3 \implies M1 \begin{cases} m\xi > 0 & \text{stable} \\ m\xi < 0 & \text{unstable} \end{cases} \\
&M2 \;:\; [B_{ij}] = \begin{bmatrix} 0 & \sqrt{2}mL^3 \\ \sqrt{2}mL^3 & 0 \end{bmatrix}, \quad \beta_1^{max} = \sqrt{2}mL^3,\; \beta_1^{min} = -\sqrt{2}mL^3 \implies M2 \text{ unstable} \\
&M3 \;:\; [B_{ij}] = \begin{bmatrix} 0 & \sqrt{2}mL^3 \\ \sqrt{2}mL^3 & 0 \end{bmatrix}, \quad \beta_1^{max} = \sqrt{2}mL^3,\; \beta_1^{min} = -\sqrt{2}mL^3 \implies M3 \text{ unstable}
\end{aligned} \tag{AC-5.41}
$$

which coincide with the results obtained in Section AB-3 (see the discussion of $(\text{AB-3.6})_2$).

For the symmetric rigid plate model $m = 0, n \neq 0$ as seen from (AC-5.14) one would require the additional evaluation of the rank four tensor $\mathcal{E}^{c}_{,uuuu}$ which from (AC-3.1) is found to be:

$$
\begin{aligned}
\mathcal{E}^{c}_{,uuuu} = \quad & 12nL^4[(\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2 - \frac{1}{2}\mathbf{e}_3\mathbf{e}_3)\mathbf{e}_2\mathbf{e}_2 + (-\frac{1}{2}\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3)\mathbf{e}_3\mathbf{e}_3 + \\
& +(\mathbf{e}_2\mathbf{e}_2 - \frac{1}{2}\mathbf{e}_2\mathbf{e}_3 - \frac{1}{2}\mathbf{e}_3\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3)(\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2)],
\end{aligned} \tag{AC-5.42}
$$

Substitution of (AC-5.32), (AC-5.34), (AC-5.36), (AC-5.42) into (AC-5.15) gives the following system for $\{\alpha_1^1\}$ and λ_2:

$$
\begin{aligned}
&nL^4[12(\alpha_1^1)^3 + 36(\alpha_1^1)^2\alpha_2^1 - 18\alpha_1^1(\alpha_2^1)^2 + 12(\alpha_2^1)^3] - 3\lambda_2 L\alpha_1^1 = 0 \\
&nL^4[12(\alpha_1^1)^3 - 18(\alpha_1^1)^2\alpha_2^1 + 36\alpha_1^1(\alpha_2^1)^2 + 12(\alpha_2^1)^3] - 3\lambda_2 L\alpha_2^1 = 0
\end{aligned}
\qquad (\alpha_1^1)^2 + (\alpha_2^1)^2 = 1. \tag{AC-5.43}
$$

The above system admits four different real solutions $N1$, $N2$, $N3$, $N4$ namely:

$$
\begin{aligned}
&N1 \;:\; \alpha_1^1 = 1/\sqrt{2}, \quad \alpha_2^1 = 1/\sqrt{2}, \quad \lambda_1 = 0, \quad \lambda_2 = 7nL^3 \\
&N2 \;:\; \alpha_1^1 = -1/\sqrt{2}, \quad \alpha_2^1 = 1/\sqrt{2}, \quad \lambda_1 = 0, \quad \lambda_2 = -9nL^3 \\
&N3 \;:\; \alpha_1^1 = 2/\sqrt{5}, \quad \alpha_2^1 = 1/\sqrt{5}, \quad \lambda_1 = 0, \quad \lambda_2 = (36/5)nL^3 \\
&N4 \;:\; \alpha_1^1 = 1/\sqrt{5}, \quad \alpha_2^1 = 2/\sqrt{5}, \quad \lambda_1 = 0, \quad \lambda_2 = (36/5)nL^3
\end{aligned} \tag{AC-5.44}
$$

By continuing with higher order terms $\{\alpha_i^n\}, n > 1$ in the expansion of $\xi_i(\xi)$ as well as with the component v of $u - \overset{0}{u}$ on $N^{\perp}$ $(v = (u - \overset{0}{u}) \bullet \mathbf{e}_1)$ one easily finds $\alpha_i^n = 0$ for $n > 1$ and $v = 0$. Hence from (AC-5.44) the $(3^m - 1)/2 = 4$ bifurcated equilibrium branches for the symmetric plate model are:

$$
\begin{aligned}
&N1 \;:\; u = (\lambda/4E)\mathbf{e}_1 + \theta\mathbf{e}_3 + \phi\mathbf{e}_3, \quad \lambda = \lambda_c + \xi^2(7nL^3/2), \quad \theta = \phi = \xi/\sqrt{2} \\
&N2 \;:\; u = (\lambda/4E)\mathbf{e}_1 + \theta\mathbf{e}_3 + \phi\mathbf{e}_3, \quad \lambda = \lambda_c - \xi^2(9nL^3/2), \quad -\theta = \phi = \xi/\sqrt{2} \\
&N3 \;:\; u = (\lambda/4E)\mathbf{e}_1 + \theta\mathbf{e}_3 + \phi\mathbf{e}_3, \quad \lambda = \lambda_c + \xi^2(18nL^3/5), \quad \theta = 2\phi = 2\xi/\sqrt{5} \\
&N4 \;:\; u = (\lambda/4E)\mathbf{e}_1 + \theta\mathbf{e}_3 + \phi\mathbf{e}_3, \quad \lambda = \lambda_c + \xi^2(18nL^3/5), \quad 2\theta = \phi = 2\xi/\sqrt{5}
\end{aligned} \tag{AC-5.45}
$$

For the stability of the above bifurcated equilibrium branches, one has to investigate the eigenvalues of the stability matrix B_{ij} defined in (AC-5.16) which with the help of (AC-5.37), (AC-5.38), (AC-5.40) and (AC-5.42) can be written as:

$$[B_{ij}] = \left[\begin{array}{cc} 6nL^4[2(\alpha_1^1)^2 + 4\alpha_1^1\alpha_2^1 - (\alpha_2^1)^2] - \lambda_2 L & 12nL^4[(\alpha_1^1)^2 - \alpha_1^1\alpha_2^1 + (\alpha_2^1)^2] \\ 12nL^4[(\alpha_1^1)^2 - \alpha_1^1\alpha_2^1 + (\alpha_2^1)^2] & 6nL^4[-(\alpha_1^1)^2 + 4\alpha_1^1\alpha_2^1 + 2(\alpha_2^1)^2] - \lambda_2 L \end{array} \right] \quad \text{(AC-5.46)}$$

For each one of the bifurcated equilibrium paths in (AC-5.45), the stability matrix in (AC-5.46) takes the form:

$$\begin{array}{llll} N1 & : & [B_{ij}] = \left[\begin{array}{cc} 8nL^4 & 6nL^4 \\ 6nL^4 & 8nL^4 \end{array} \right], & \beta_2^{min} = 2nL^4 \implies N1 \text{ stable} \\ N2 & : & [B_{ij}] = \left[\begin{array}{cc} 0 & 18nL^4 \\ 18nL^4 & 0 \end{array} \right], & \beta_2^{min} = -18nL^4 \implies N2 \text{ unstable} \\ N3 & : & [B_{ij}] = \left[\begin{array}{cc} (54/5)nL^4 & (36/5)nL^4 \\ (36/5)nL^4 & 0 \end{array} \right], & \beta_2^{min} = -(18/5)nL^4 \implies N3 \text{ unstable} \\ N4 & : & [B_{ij}] = \left[\begin{array}{cc} 0 & (36/5)nL^4 \\ (36/5)nL^4 & (54/5)nL^4 \end{array} \right], & \beta_2^{min} = -(18/5)nL^4 \implies N4 \text{ unstable} \end{array} \quad \text{(AC-5.47)}$$

which coincide with the results obtained in Section AB-3 (see the discussion of $(\text{AB-3.6})_2$).

AC-6 IMPERFECT SYSTEM - NEARLY SIMULTANEOUS MULTIPLE MODES

Having analyzed the bifurcation and stability behavior of a perfect structure with a multiple bifurcation point, attention is next turned to the effect of imperfections. The term "*imperfections*" is employed in a broad engineering sense here, as opposed to the more riguorous mathematical term of "*control parameters* " of singularity theory. In solid and structural mechanics applications two types of such imperfections (or control parameters) are distinguished: The first type of imperfection is a "*geometric imperfection* ", denoted by w, where w is a function belonging to the set of admissible displacements and which characterizes the departure of the unstressed configuration of the system from its perfect shape. More generally, a geometric imperfection can be due to the eccentricity of the applied loads, thickness variations, errors in the boundary conditions, residual stresses e.t.c. This type of imperfection is responsible for the deviation of the equilibrium solution from its counterpart in the perfect system for all loads λ. The presence of a geometric imperfection changes drastically the system's behavior near the critical point of its perfect counterpart: Bifurcation points in the equilibrium solution are usually replaced by limit points as seen in the rigid plate example of Section AB-4. The second type of imperfection is the "*mode separation parameter*" (also referred to as the "*mode splitting*" parameter), denoted by ζ. This type of imperfection is responsible for separating the critical loads on the principal branch by transforming the m-tuple bifurcation point into m different but closely spaced simple bifurcation points. The mode separation parameter ζ quantifies the average distance of the nearly simultaneous critical loads, as seen in the rectangular plate example of Section AB-4. Note that mode separation can occur in perfect structures, such as rectangular plates under in-plane loading, axially loaded cylinders or externally pressurized spheres that exhibit no geometric imperfections of any kind. Although one can find examples of strongly interacting modes with critical loads that are far apart, attention is focused here on the interaction of almost simultaneous modes. In the presentation of the general theory for imperfect structures whose perfect counterparts have multiple eigenmodes, both these types of imperfections will be considered simultaneously.

The structure with the nearly simultaneous eigenmodes (but no geometric imperfections) has a potential energy $\widehat{\mathcal{E}}(u, \lambda, \zeta)$ which coincides with $\mathcal{E}(u, \lambda)$ for $\zeta = 0$:

$$\widehat{\mathcal{E}}(u, \lambda, 0) = \mathcal{E}(u, \lambda) \tag{AC-6.1}$$

Similarly to the simple mode case (see (AC-4.2)), the potential energy $\widehat{\mathcal{E}}$ is set to zero at zero displacements:

$$\widehat{\mathcal{E}}(0, \lambda, \zeta) = 0 \tag{AC-6.2}$$

The system characterized by $\widehat{\mathcal{E}}$ admits a principal solution $\overset{0}{\widehat{u}}(\lambda, \zeta)$ which is unique in the neighborhood of $\lambda = 0$ and which in view of (AC-6.1) should coincide with $\overset{0}{u}$ for $\zeta = 0$, namely:

$$\widehat{\mathcal{E}},_u (\overset{0}{\widehat{u}}(\lambda,\zeta),\lambda,\zeta)\delta u = 0, \quad \overset{0}{\widehat{u}}(0,\zeta) = 0, \quad \overset{0}{\widehat{u}}(\lambda,0) = \overset{0}{u}(\lambda) \tag{AC-6.3}$$

Moreover,the structure with the nearly simultaneous modes has m distinct critical values $\overset{i}{\widehat{\lambda}}(\zeta)$ of the load parameter at the neighborhood of λ_c. Without loss of generality, we can assume that each such critical value corresponds to a simple bifurcation point of the stability operator $\widehat{\mathcal{E}},_{uu} (\overset{0}{\widehat{u}}(\lambda,\zeta),\lambda,\zeta)$ with unique eigenvector $\overset{i}{\widehat{u}}(\zeta)$. All these critical values are close to λ_c in the sense that $\overset{i}{\widehat{\lambda}}(\zeta) - \lambda_c = O(\zeta)$. More specifically for $i = 1, ..., m$ and in view of (AC-6.1) and (AC-5.1):

$$(\widehat{\mathcal{E}},_{uu} (\overset{0}{\widehat{u}}(\overset{i}{\widehat{\lambda}}(\zeta),\zeta),\overset{i}{\widehat{\lambda}}(\zeta),\zeta)\overset{i}{\widehat{u}}(\zeta))\delta u = 0; \quad \overset{i}{\widehat{\lambda}}(0) = \lambda_c, \quad \overset{i}{\widehat{u}}(0) = \overset{i}{u} \tag{AC-6.4}$$

The fact that $\overset{i}{\widehat{\lambda}}(\zeta)$ are simple bifurcation points of the system described by $\widehat{\mathcal{E}}$ is ensured by:

$$\widehat{\mathcal{E}},_{u\lambda} (\overset{0}{\widehat{u}}(\overset{i}{\widehat{\lambda}}(\zeta),\zeta),\overset{i}{\widehat{\lambda}}(\zeta),\zeta)\overset{i}{\widehat{u}}(\zeta) = 0 \tag{AC-6.5}$$

which as expected from (AC-6.1) reduces to its perfect structure counterpart relation (AC-5.2) for $\zeta = 0$.

From the above properties of $\widehat{\mathcal{E}}(u,\lambda,\zeta)$ one can deduce the following useful relations: By evaluating the derivative with respect to ζ of the equilibrium equation (AC-6.3) at $\zeta = 0$ and making use of (AC-6.1) one has:

$$(\mathcal{E},^c_{uu} (\partial\overset{0}{\widehat{u}}/\partial\zeta)_c + \widehat{\mathcal{E}},^c_{u\zeta})\delta u = 0 \tag{AC-6.6}$$

Upon taking $\delta u = \overset{i}{u}$ and recalling (AC-5.1) and (AC-6.1) the above equation yields:

$$\widehat{\mathcal{E}},^c_{u\zeta} \overset{i}{u} = 0 \tag{AC-6.7}$$

By evaluating the derivative with respect to ζ of (AC-6.4) at $\zeta = 0$ and recalling once again (AC-6.1) one obtains:

$$(\mathcal{E},^c_{uuu} ((d\overset{i}{\widehat{\lambda}}/d\zeta)_c(d\overset{0}{u}/d\lambda)_c + (\partial\overset{0}{\widehat{u}}/\partial\zeta)_c) + (d\overset{i}{\widehat{\lambda}}/d\zeta)_c\mathcal{E},^c_{uu\lambda} + \widehat{\mathcal{E}},^c_{uu\zeta})\overset{i}{u} + \mathcal{E},^c_{uu} (d\overset{i}{\widehat{u}}/d\zeta)_c)\delta u = 0 \tag{AC-6.8}$$

Upon taking $\delta u = \overset{i}{u}$ and recalling (AC-5.1) as well as the definition of $\mathcal{E}_{ij\lambda}$ given in (AC-5.11) the above equation yields:

$$\begin{aligned} &(d\overset{i}{\widehat{\lambda}}/d\zeta)_c\mathcal{E}_{ij\lambda} + \widehat{\mathcal{E}}_{ij\zeta} = 0 \\ &\widehat{\mathcal{E}}_{ij\zeta} \equiv ((\partial\widehat{\mathcal{E}},_{uu}/\partial\zeta)_c\overset{i}{u})\overset{j}{u} = ((\mathcal{E},^c_{uuu} (\partial\overset{0}{\widehat{u}}/\partial\zeta)_c + \widehat{\mathcal{E}},^c_{uu\zeta})\overset{i}{u})\overset{j}{u} \end{aligned} \tag{AC-6.9}$$

As will be subsequently discussed, there exist appropriate choices of the inner product in U such that the $m \times m$ matrix $\mathcal{E}_{ij\lambda} = -\delta_{ij}$. Hence from (AC-6.9) one can conclude that for this convenient inner product choice, $\widehat{\mathcal{E}}_{ij\zeta}$ is also diagonal.

The potential energy of an imperfect structure with nearly simultaneous eigenmodes is given by $\overline{\mathcal{E}}(u,\lambda,\zeta,w)$ where w denotes the geometric imperfection of the structure. Without loss of generality, it will be assumed that $u, w \in U$. In the absence of the geometric imperfection $w=0$, the system reduces to the previously introduced one with the nearly simultaneous modes, namely:

$$\overline{\mathcal{E}}(u,\lambda,\zeta,0) = \widehat{\mathcal{E}}(u,\lambda,\zeta) \tag{AC-6.10}$$

The energy of the imperfect system characterized by $\overline{\mathcal{E}}$ is also set to zero in the absence of displacements, similarly to (AC-6.2), namely $\overline{\mathcal{E}}(0,\lambda,\zeta,w)=0$. Following the same steps as in the simple mode case, it is convenient to distinguish between the imperfection amplitude $\epsilon \equiv \|w\|$ and the imperfection shape $\overline{w} \equiv w/\|w\|$, exactly as in (AC-4.5).

The Lyapunov - Schmidt decomposition of the displacement field u introduced in (AC-5.3), but with $\overset{0}{\widehat{u}}(\lambda,\zeta)$ replacing $\overset{0}{u}(\lambda)$, is again going to be employed for the solution of the equilibrium equations of the imperfect structure. The unknown displacement u is thus replaced by an equivalent set (v,ξ_i) and the solution to the equilibrium equation $\overline{\mathcal{E}}_{,u}\,\delta u = 0$ proceeds in two steps: First v is determined as a function of $\xi_i, \Delta\lambda, \zeta, \epsilon$ from the equilibrium equation in $\mathcal{N}^{\perp}$.

$$\overline{\mathcal{E}}_{,v}\,\delta v = 0 \implies \overline{\mathcal{E}}_{,u}\,(\overset{0}{\widehat{u}}(\lambda_c+\Delta\lambda,\zeta) + \sum_{i=1}^{m}\xi_i\overset{i}{u} + v, \lambda_c+\Delta\lambda,\zeta,\epsilon\overline{w})\delta v = 0 \tag{AC-6.11}$$

Then the resulting v is used in the remaining equilibrium equation on the null space $\mathcal{N}$ thus providing the relation between $\xi_i, \Delta\lambda, \zeta$ and ϵ.

$$\overline{\mathcal{E}}_{,\xi_i} = 0 \implies \overline{\mathcal{E}}_{,u}\,(\overset{0}{\widehat{u}}(\lambda_c+\Delta\lambda,\zeta) + \sum_{i=1}^{m}\xi_i\overset{i}{u} + v, \lambda_c+\Delta\lambda,\zeta,\epsilon\overline{w})\overset{i}{u} = 0 \tag{AC-6.12}$$

From the assumed positive definitness of $\mathcal{E}^c_{uu}$ on $\mathcal{N}^{\perp}$ follows that (AC-6.11) has a unique and adequately smooth solution $v(\xi_i,\Delta\lambda,\zeta,\epsilon)$, at least in the neighborhood of criticality, whose Taylor series expansion is:

$$\begin{aligned} v(\xi_i,\Delta\lambda,\zeta,\epsilon) = \quad & \sum_{i=1}^{m}\xi_i v_i + \Delta\lambda v_\lambda + \zeta v_\zeta + \epsilon v_\epsilon + \frac{1}{2}(\sum_{i=1}^{m}\sum_{j=1}^{m}\xi_i\xi_j v_{ij} + 2\Delta\lambda\sum_{i=1}^{m}\xi_i v_{i\lambda} 2\zeta\sum_{i=1}^{m}\xi_i v_{i\zeta} \\ & +2\epsilon\sum_{i=1}^{m}\xi_i v_{i\epsilon} + (\Delta\lambda)^2 v_{\lambda\lambda} + 2\zeta\Delta\lambda v_{\lambda\zeta} + 2\epsilon\Delta\lambda v_{\lambda\epsilon} + \zeta^2 v_{\zeta\zeta} + 2\zeta\epsilon v_{\zeta\epsilon} + \epsilon^2 v_{\epsilon\epsilon}) + ... \end{aligned} \tag{AC-6.13}$$

Upon substitution of (AC-6.13) into (AC-6.12) and after a Taylor series expansion about $(\xi_i,\Delta\lambda,\zeta,\epsilon)=0$ one obtains by collecting the terms of the like orders in these variables the following results: Since from (AC-6.1) and (AC-6.10), $\overline{\mathcal{E}}(u,\lambda,0,0) = \mathcal{E}(u,\lambda)$ all the terms in the expansion of (AC-6.12) of $O(\xi^p(\Delta\lambda)^q)$, i.e. the terms which do not contain any powers of ϵ or ζ, coincide with those found in the previous section for the perfect structure (see equations (AC-5.7)-(AC-5.10)). In addition, by taking $\xi_i=\epsilon=0$ in (AC-6.11) comparing the result to the equilibrium equation in (AC-6.3) and invoking the uniquness of the solution to (AC-6.11) for $v(\xi_i, \Delta\lambda, \zeta, \epsilon)$, it is readily seen that $v(0, \Delta\lambda, \zeta, 0) = 0$. This implies:

$$v_\zeta = v_{\lambda\zeta} = v_{\zeta\zeta} = ... = 0 \tag{AC-6.14}$$

The same result could have also been obtained in a straightforward way from the $O(\zeta^p(\Delta\lambda)^q)$ terms in the expansion of (AC-6.11).

The remaining terms in the expansion for v, i.e. $v_\epsilon, v_{i\zeta}, v_{i\epsilon}, v_{\zeta\epsilon}, v_{\epsilon\epsilon}$ etc, are given from equations of the type $(\mathcal{E},^c_{uu} v_\epsilon + \overline{\mathcal{E}},^c_{uw} \overline{w})\delta v = 0$ and can all be determined uniquely since the operator operating on the unkown function is always $\mathcal{E},^c_{uu}$ which is invertible in view of the positive definitness of $\mathcal{E}^c_{uu}$ on $\mathcal{N}^\perp$. Upon substitution of the thus found $v(\xi_i, \Delta\lambda, \zeta, \epsilon)$ into the remaining equilibrium equation (AC-6.12) and after using (AC-5.1), (AC-5.7)-(AC-5.10), (AC-6.1), (AC-6.7), (AC-6.10) and (AC-6.14) one obtains the following m equations between $\xi_i, \Delta\lambda, \zeta$ and ϵ:

$$\begin{aligned}
&\epsilon\overline{\mathcal{E}}_{i\epsilon} + \frac{1}{2}[\sum_{j=1}^{m}\sum_{k=1}^{m}\xi_j\xi_k\mathcal{E}_{ijk} + 2\Delta\lambda\sum_{j=1}^{m}\xi_j\mathcal{E}_{ij\lambda} + 2\zeta\sum_{j=1}^{m}\xi_j\widehat{\mathcal{E}}_{ij\zeta} + ...\ ] + \frac{1}{6}[\ \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}\xi_j\xi_k\xi_l\mathcal{E}_{ijkl} + ...\ ]\ + ... = 0 \\
&\overline{\mathcal{E}}_{i\epsilon} \equiv (\overline{\mathcal{E}},^c_{uw} \overline{w})\overset{i}{u}
\end{aligned} \tag{AC-6.15}$$

The coefficients $\mathcal{E}_{ijk}$, $\mathcal{E}_{ij\lambda}$ and $\mathcal{E}_{ijkl}$ appearing in (AC-6.15) have already been defined in the corresponding equilibrium equation for the perfect case (see (AC-5.11) while $\widehat{\mathcal{E}}_{ij\zeta}$ has been defined in (AC-6.9). As expected, for $\zeta = \epsilon = 0$ the equilibrium equation for the imperfect structure with the nearly simultaneous eigenmodes (AC-6.15) reduces to its perfect structure counterpart (AC-5.11).

The solution for the m equations in (AC-6.15) for the $m+3$ unknowns ξ_i, $\Delta\lambda$, ζ and ϵ can be expressed as a function of three parameters ξ, Λ and Z. This parametrization is by no means unique, but it turns out to be convenient. In analogy to the perfect case, two cases will be distinguished: First the asymmetric case, for which $\mathcal{E}_{ijk} \neq 0$ at least for some triplet of indexes (i, j, k). In this case the adopted parametrization is:

$$\begin{aligned}
\xi_i(\xi, \Delta\lambda, \zeta) &= \overline{\alpha}^1_i(\Lambda, Z)\xi + \overline{\alpha}^2_i(\Lambda, Z)\frac{\xi^2}{2} + ... \\
\epsilon(\xi, \Delta\lambda, \zeta) &= \xi^2\ [\overline{\epsilon}_0(\Lambda, Z) + \overline{\epsilon}_1(\Lambda, Z)\xi + ...\ ]
\end{aligned} \qquad \Delta\lambda \equiv \xi\Lambda, \quad \zeta \equiv \xi Z, \quad \xi^2 \equiv \sum_{i=1}^{m}(\xi_i)^2 \tag{AC-6.16}$$

By inserting (AC-6.16) into (AC-6.15) and subsequently collecting all powers of the like orders in ξ, one obtains from the lowest order nontrivial term in this expansion:

$$\overline{\epsilon}_0\overline{\mathcal{E}}_{i\epsilon} + \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}\overline{\alpha}^1_j\overline{\alpha}^1_k\mathcal{E}_{ijk} + \Lambda\sum_{j=1}^{m}\overline{\alpha}^1_j\mathcal{E}_{ij\lambda} + Z\sum_{j=1}^{m}\overline{\alpha}^1_j\widehat{\mathcal{E}}_{ij\zeta} = 0, \quad \sum_{i=1}^{m}(\overline{\alpha}^1_i)^2 = 1 \tag{AC-6.17}$$

For fixed values of Λ and Z the above system of $m+1$ equations for the $m+1$ unknowns $\overline{\alpha}^1_i$, $\overline{\epsilon}_0$ has at most 2^m real solutions. A tedious but straightforward calculation shows that the unique continuation of each solution found in (AC-6.17) for the $O(\xi^3)$ and higher terms in (AC-6.15) is assured when:

$$\mathrm{Det}[\overline{B}_{ij}] \neq 0, \quad \overline{B}_{ij}(\Lambda, Z) \equiv \sum_{k=1}^{m} \overline{\alpha}_k^1 \mathcal{E}_{ijk} + \Lambda \mathcal{E}_{ij\lambda} + Z \widehat{\mathcal{E}}_{ij\zeta} \tag{AC-6.18}$$

Of all the possible solutions of (AC-6.17) there is one of particular interest. It is the equilibrium branch that goes through the initial unloaded configuration $u = 0$, $\lambda = 0$. This solution is the unique and stable equilibrium solution that the system is going to follow during the early stages of the loading process. For a nontrivial imperfection i.e. when $\|\overline{\mathcal{E}}_{i\epsilon}\| \neq 0$ this solution corresponds to large negative values of Λ (recall that $\Lambda = (\lambda - \lambda_c)/\xi$ and that ξ is a small positive parameter). It is not difficult to see that for $\Lambda \to -\infty$ the only real solution to (AC-6.17) is

$$\overline{\alpha}_i^1 = \sum_{j=1}^{m} (\mathcal{E}_{ij\lambda})^{-1} \overline{\mathcal{E}}_{j\epsilon} / \| \sum_{j=1}^{m} (\mathcal{E}_{ij\lambda})^{-1} \overline{\mathcal{E}}_{j\epsilon} \|, \quad \overline{\epsilon}_0 = -\Lambda / \| \sum_{j=1}^{m} (\mathcal{E}_{ij\lambda})^{-1} \overline{\mathcal{E}}_{j\epsilon} \| \tag{AC-6.19}$$

The second case to be investigated is the symmetric one, for which $\mathcal{E}_{ijk} = 0$ for all triplets of indexes (i, j, k). The adopted parametrization for the unknowns ξ_i, $\Delta\lambda$, ζ, ϵ in terms of the parameters ξ, Λ, Z is:

$$\begin{aligned} \xi_i(\xi, \Delta\lambda, \zeta) &= \overline{\alpha}_i^1(\Lambda, Z)\xi + \overline{\alpha}_i^2(\Lambda, Z)\frac{\xi^2}{2} + ... \\ \epsilon(\xi, \Delta\lambda, \zeta) &= \frac{\xi^3}{2} \left[\overline{\epsilon}_0(\Lambda, Z) + \overline{\epsilon}_1(\Lambda, Z)\xi + ... \right] \end{aligned} \qquad \Delta\lambda \equiv \frac{\xi^2}{2}\Lambda, \quad \zeta \equiv \frac{\xi^2}{2} Z, \quad xi^2 \equiv \sum_{i=1}^{m} (\xi_i)^2 \tag{AC-6.20}$$

By inserting (AC-6.20) into (AC-6.15) and subsequently collecting all powers of the like orders in ξ, one obtains from the lowest order nontrivial term in this expansion:

$$\overline{\epsilon}_0 \overline{\mathcal{E}}_{i\epsilon} + \frac{1}{3} \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{l=1}^{m} \overline{\alpha}_j^1 \overline{\alpha}_k^1 \overline{\alpha}_l^1 \mathcal{E}_{ijkl} + \Lambda \sum_{j=1}^{m} \overline{\alpha}_j^1 \mathcal{E}_{ij\lambda} + Z \sum_{j=1}^{m} \overline{\alpha}_j^1 \widehat{\mathcal{E}}_{ij\zeta} = 0, \quad \sum_{i=1}^{m} (\overline{\alpha}_i^1)^2 = 1 \tag{AC-6.21}$$

For fixed values of Λ and Z the above system of $m + 1$ equations for the $m + 1$ unknowns $\overline{\alpha}_i^1$, $\overline{\epsilon}_0$ has at most 3^m real solutions. A straightforward calculation shows that the unique continuation of each solution found in (AC-6.21) for the $O(\xi^4)$ and higher terms in (AC-6.15) is assured when:

$$\mathrm{Det}\ [\overline{B}_{ij}] \neq 0, \quad \overline{B}_{ij}(\Lambda, Z) \equiv \sum_{k=1}^{m} \sum_{l=1}^{m} \overline{\alpha}_k^1 \overline{\alpha}_l^1 \mathcal{E}_{ijkl} + \Lambda \mathcal{E}_{ij\lambda} + Z \widehat{\mathcal{E}}_{ij\zeta} \tag{AC-6.22}$$

Of all the possible solutions of (AC-6.21) there is one of particular interest. It is the equilibrium branch that goes through the initial unstressed configuration $u = 0$, $\lambda = 0$. This solution is the unique and stable equilibrium solution that the system is going to follow during the early stages of the loading process. For a nontrivial imperfection this solution is identical to the one found in (AC-6.19) for the asymmetric structure. This result is hardly surprising, given that the symmetry or not of the bifurcation point refers to properties of the structure near the critical load and does not influence the behavior of the structure near the zero load.

To complete the study of the imperfect structure with the nearly simultaneous buckling eigenmodes one has also to investigate the stability of the equilibrim paths. Of all the equilibrium paths found near criticallity, only the one that passes trough the initial unloaded state $u = 0$, $\lambda = 0$ will be investigated, since it is the actual path that the imperfect structure is going to follow during the loading process. It will be shown that the actual equilibrium path is unique and stable near $\lambda = 0$ (or equivalently as $\Lambda \to -\infty$). As the loading progresses, i.e. as λ increases away from zero (or equivalently as Λ increases away from $-\infty$), there will be a load λ_s, which is called the "*snap - through load*", for which the equilibrium branch in question will loose its stability, usually due to the attaining of a local load maximum. The value of λ_s, which obviously depends on ζ, ϵ and $\overline{w}$ is very significant for the design of structures that exhibit multiple eigenvalues at criticality, for it represents the maximum load attained by the actual stucture before becoming unstable.

As for the perfect structure, the stability of an equilibrium solution will depend on the sign of the minimum eigenvalue $\overline{\beta}_{min}(\xi, \Lambda, Z)$ of the corresponding stability operator $\overline{\mathcal{E}}_{,uu}(u, \lambda, \zeta, \epsilon\overline{w})$. In analogy to (AC-5.20), the definititions for each one of the m lowest eigenvalues $\overline{\beta}(\xi, \Lambda, Z)$ of $\overline{\mathcal{E}}_{,uu}$ and their corresponding eigenvectors $\overline{x}(\xi, \Lambda, Z)$ are:

$$(\overline{\mathcal{E}}_{,uu}(\overset{0}{\widehat{u}}(\lambda_c + \Delta\lambda, \zeta) + \sum_{i=1}^{m} \xi_i \overset{i}{u} + v(\xi_i, \Delta\lambda, \zeta, \epsilon), \lambda_c + \Delta\lambda, \zeta, \epsilon\overline{w})\overline{x})\delta u = \overline{\beta}(\overline{x}, \delta u) \tag{AC-6.23}$$

The normalization condition for the eigenvectors is again given by $(\overline{x}, \overline{x}) = 1$ as in all the previous cases (see (AC-3.22), (AC-4.16)). Recall that in the above definition, $\Delta\lambda(\xi, \Lambda)$, $\zeta(\xi, Z)$, $\xi_i(\xi, \Lambda, Z)$ and $\epsilon(\xi, \Lambda, Z)$ are given by (AC-6.16) or (AC-6.20) for the asymmetric or the symmetric case respectively. Assuming adequate smoothness with respect to ξ, the Taylor series expansions for $\overline{\beta}(\xi, \Lambda, Z)$ and $\overline{x}(\xi, \Lambda, Z)$ are:

$$\begin{aligned} \overline{\beta}(\xi, \Lambda, Z) &= \overline{\beta}_1(\Lambda, Z)\xi + \overline{\beta}_2(\Lambda, Z)\frac{\xi^2}{2} + ... \\ \overline{x}(\xi, \Lambda, Z) &= \overline{x}_0(\Lambda, Z) + \overline{x}_1(\Lambda, Z)\xi + \overline{x}_2(\Lambda, Z)\frac{\xi^2}{2} \end{aligned} \tag{AC-6.24}$$

By introducing (AC-6.24) into (AC-6.23) and recalling (AC-5.1), (AC-5.7), (AC-5.8), (AC-6.1), (AC-6.10), (AC-6.13) and (AC-6.14), one obtains (for fixed values of Λ and Z) by expanding about $\xi = 0$ and collecting the terms of the like order in ξ, the following results: The $O(1)$ term yields (compare with (AC-5.21)):

$$\overline{x}_0 = \sum_{i=1}^{m} \overline{\chi}_i \overset{i}{u} \tag{AC-6.25}$$

Continuing with the $O(\xi)$ term in the expansion of (AC-6.23) one has:

$$O(\xi)\ :\ ((\mathcal{E},^c_{uuu}(\Lambda(d\overset{0}{u}/d\lambda)_c + Z(\partial\overset{0}{\widehat{u}}/\partial\zeta)_c + \sum_{k=1}^{m}\overline{\alpha}^1_k\overset{k}{u}) + \Lambda\mathcal{E},^c_{uu\lambda} + Z\widehat{\mathcal{E}},^c_{uu\zeta})\overline{x}_0 + \mathcal{E},^c_{uu}\overline{x}_1)\delta u = \overline{\beta}_1(\overline{x}_0, \delta u) \tag{AC-6.26}$$

Taking $\delta u = \overset{i}{u}$ and recalling (AC-5.1) the above equation yields:

$$\sum_{j=1}^{m}\overline{B}_{ij}\overline{\chi}_j = \overline{\beta}_1\overline{\chi}_i \tag{AC-6.27}$$

In the above equation the $m \times m$ matrix $\overline{B}_{ij}$ is given by (AC-6.18) and for the asymmetric bifurcation case, this matrix is a nonsingular one which ensures (in conjunction with the symmetry of this matrix) that all the eigenvalues $\overline{\beta}_1$ are real and nonzero. Consequently, if $\overline{\beta}_1^{min}(\Lambda, Z)$ is the minimum eigenvalue of $\overline{B}_{ij}$, then the wanted lowest eigenvalue $\overline{\beta}_{min}(\xi, \Lambda, Z)$ of the stability operator $\overline{\mathcal{E}},_{uu}$ is given for $\xi > 0$ by:

$$\overline{\beta}^{min}(\xi, \Lambda, Z) = \xi\overline{\beta}_1^{min}(\Lambda, Z) + O(\xi^2) \tag{AC-6.28}$$

The choice of $\xi > 0$ is in agreement with the assumption that $\Lambda \to -\infty$ as $\lambda \to 0$. Moreover, for the actual equilibrium path one can easily show that it is stable as $\Lambda \to -\infty$. Indeed, in view of the boundedness of $\overline{\alpha}^1_i(\Lambda, Z)$ and the assumption that $\mathcal{E}_{ij\lambda}$ is a negative definite matrix, as $\Lambda \to -\infty$ then $\overline{B}_{ij} \to \Lambda\mathcal{E}_{ij\lambda}$, which is a positive definite matrix. As the loading progresses, (as λ increases away from zero) it reaches a value, say $\lambda_s = \lambda_c + \Delta\lambda_s$, for which the actual equilibrium path becomes unstable, i.e. $\overline{\beta}_{min}(\xi, \Lambda_s, Z) = 0$. For a fixed value of Z, the following relations hold between the load drop corresponding to the first instability of the actual equilibrium path $\Delta\lambda_s(\xi, Z)$ in the case of an asymmetric bifurcation (see also (AC-6.16)):

$$\overline{\beta}_{min}(\xi, \Lambda_s, Z) = 0; \quad \lambda_s - \lambda_c = \Delta\lambda_s = \xi\Lambda_s(\xi, Z), \quad \Lambda_s(\xi, Z) = \Lambda^0_s(Z) + \xi\Lambda^1_s(Z) + \ldots \tag{AC-6.29}$$

Thus, in view of (AC-6.27)-(AC-6.29), $\Lambda^0_s(Z)$ is the first value of Λ as it increases from $-\infty$ (expected to be negative, since only structures with $\Delta\lambda_s < 0$ are of interest) at which $\overline{B}_{ij}(\Lambda, Z)$ loses its positive definiteness, i.e.:

$$\sum_{j=1}^{m}\overline{B}_{ij}(\Lambda^0_s(Z), Z)\overline{\chi}_j(\Lambda^0_s(Z), Z) = 0 \tag{AC-6.30}$$

For any other vector not colinear with $\overline{\chi}_i$ say N_i, one has:

$$\sum_{i=1}^{m}\sum_{j=1}^{m}N_i\overline{B}_{ij}N_j > 0 \tag{AC-6.31}$$

The above condition, in conjunction with (AC-6.29) gives us the lowest load at which the first instability in the actual equilibrium solution of the asymmetric imperfect structure will occur. For Λ^0_s to correspond to a maximum load and not a bifurcation point, one should satisfy according to the general theory (see (AC-2.21)) $\overline{\mathcal{E}}_{,u\lambda}(\lambda_s)\overline{x}_0 \neq 0$ which gives to the leading order in ξ:

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\mathcal{E}_{ij\lambda}\overline{\alpha}_j^1(\Lambda_s^0, Z)\overline{\chi}_i(\Lambda_s^0, Z) \neq 0 \tag{AC-6.32}$$

For the symmetric case, the $O(\xi^2)$ term in the expansion of (AC-6.23) gives $\overline{\beta}_1 = 0$ while the $O(\xi^3)$ term gives after some lengthy but straightforward manipulations:

$$\overline{\beta}_{min}(\xi, \Lambda, Z) = \frac{\xi^2}{2}\overline{\beta}_2^{min}(\Lambda, Z) + O(\xi^3) \tag{AC-6.33}$$

where $\overline{\beta}_2^{min}$ is the lowest eigenvalue of the matrix $\overline{B}_{ij}$ which is given in this case by (AC-6.22). The first instability load of the actual equilibrium solution satisfies:

$$\overline{\beta}_{min}(\xi, \Lambda_s, Z) = 0; \quad \lambda_s - \lambda_c = \Delta\lambda_s = \frac{\xi^2}{2}\Lambda_s(\xi, Z), \quad \Lambda_s(\xi, Z) = \Lambda_s^0(Z) + \xi\Lambda_s^1(Z) + ... \tag{AC-6.34}$$

Note that $\Lambda_s^0(Z)$ is also given in this case by (AC-6.30) and satisfies (AC-6.31), (AC-6.32) but with $\overline{B}_{ij}$ taken from the definition in (AC-6.22).

An alternative parametrization of the equilibrium equations in $\mathcal{N}$ of the imperfect structure i.e. (AC-6.15) which has an attractive physical interpretation can also be considered. For the asymmetric case this alternative parametrization is:

$$\epsilon \equiv \eta^2, \quad \Delta\lambda \equiv \eta\Lambda, \quad \zeta \equiv \eta Z, \quad \xi_i(\eta, \Delta\lambda, \zeta) = \overline{\alpha}_i^1(\Lambda, Z)\eta + \overline{\alpha}_i^2(\Lambda, Z)\frac{\eta^2}{2} + ... \tag{AC-6.35}$$

while for the symmetric case this alternative parametrization takes the form:

$$\epsilon \equiv \frac{\eta^3}{2}, \quad \Delta\lambda \equiv \frac{\eta^2}{2}\Lambda, \quad \zeta \equiv \frac{\eta^2}{2}Z, \quad \xi_i(\eta, \Delta\lambda, \zeta) = \overline{\alpha}_i^1(\Lambda, Z)\eta + \overline{\alpha}_i^2(\Lambda, Z)\frac{\eta^2}{2} + ... \tag{AC-6.36}$$

This parametrization is equivalent to controlling the size of the imperfection ϵ (or equivalently η) instead of controlling the amplitude ξ of the projection of the solution on the null space. The equilibrium and stability results for this new parametrization can be obtained from the equations given in this section if one sets $\overline{\epsilon}_0 = 1$ and drops the constraint $\sum_{i=1}^{m}(\overline{\alpha}_i^1)^2 = 1$. The advantage of this second parametrization is that by varying only Λ, while keeping all the other parameters fixed, one can follow the actual equilibrium path of a given imperfect structure. The advantage of the first parametrization is that it includes the perfect structure, in addition to providing a simpler solution to the corresponding worst imperfection shape problem, as it will be seen in the next section.

Finally a remark of practical importance is in order. In many applications, the functions $\overset{i}{\widehat{\lambda}}(\zeta)$ are not known explicitly, thus posing problems to the calculation of the matrix $\widehat{\mathcal{E}}_{ij\zeta}$ (see (AC-6.9)). For these cases, the following definition of the mode separation parameter ζ gives an approximation of the required derivative of $\overset{i}{\widehat{\lambda}}(\zeta)$ at the critical load within an $O(\zeta)$ error:

$$\zeta^2 \equiv \sum_{i=1}^{m}(\overset{i}{\widehat{\lambda}} - \lambda_c)^2, \quad (d\overset{i}{\widehat{\lambda}}/d\zeta)_c = (\overset{i}{\widehat{\lambda}} - \lambda_c)/\ [\sum_{i=1}^{m}(\overset{i}{\widehat{\lambda}} - \lambda_c)^2]^{1/2} \tag{AC-6.37}$$

AC-7 WORST GEOMETRIC IMPERFECTION SHAPE - MULTIPLE MODES

The results obtained in the previous section provide the load drop $\Delta\lambda_s$ corresponding to the first load maximum encountered during a monotonic (with respect to the load parameter λ) loading of a given imperfect structure. In practical applications pertaining to the design of such structures, the control of the geometric imperfection amplitude ϵ is often easier than the control of the imperfection shape $\overline{w}$. It is thus very important to identify the worst geometric imperfection shape, i.e. the shape that maximizes $\mid \Delta\lambda_s \mid$.

For small values of ξ, the worst geometric imperfection shape is found by minimizing Λ_s^0 (since one is interested only for $\Lambda_s^0 < 0$) which satisfies (AC-6.30), (AC-6.31), over all unit vectors $\overline{\alpha}_i^1$. Consideration of all possible unit vectors $\overline{\alpha}_i^1$ covers all possible equilibrium paths, as seen from (AC-6.17) and (AC-6.21). In order to facilitate the subsequent algebra, it will be further assumed in this section that the inner product choice in U corresponds to $\mathcal{E}_{ij\lambda} = -\delta_{ij}$. Such an inner product can be easily found in most of the applications of interest and this extra assumption does not impair the generality of the analysis.

Two alternative formulations of the worst imperfection problem will be considered. In both the solution to the equilibrium equations (AC-6.17) or (AC-6.21) as well as the limiting stability condition (AC-6.30) a geometric imperfection shape $\overline{w}$ is sought for which $\Lambda_s^0(Z)$ is minimized. However different constraints are used: In the first case (Formulation A), the amplitude ξ of the projection of $u- \overset{0}{u}(\lambda)$ onto the null space $\mathcal{N}$ is held fixed (by enforcing the constraint $\|\overline{\alpha}_i^1\| = 1$), whereas the magnitude of the imperfection $\overline{\epsilon}_0 \|\overline{\mathcal{E}}_{i\epsilon}\|$ is allowed to vary. For the second case (Formulation B) the opposite is done: the amplitude ξ of the projection of $u- \overset{0}{u}(\lambda)$ on the null space $\mathcal{N}$ is allowed to vary (by relaxing the constraint $\|\overline{\alpha}_i^1\| = 1$), whereas the magnitude of the imperfection is fixed as $\overline{\epsilon}_0 = \|\overline{\mathcal{E}}_{i\epsilon}\| = 1$.

Formulation A

Since from (AC-6.30), (AC-6.31), $\Lambda_s^0(Z)$ is the minimum eigenvalue of $\sum_{k=1}^{m} \mathcal{E}_{ijk}\overline{\alpha}_k^1 + Z\widehat{\mathcal{E}}_{ij\zeta}$ for the asymmetric case or of $\sum_{k=1}^{m}\sum_{l=1}^{m}\mathcal{E}_{ijkl}\overline{\alpha}_k^1\overline{\alpha}_l^1 + Z\widehat{\mathcal{E}}_{ij\zeta}$ for the symmetric case, it follows that wanted minimum of $\Lambda_s^0(Z)$ is:

$$\Lambda_s^0(Z)_{min} = \min \left\{ \begin{array}{ll} \sum_{i=1}^{m}\sum_{j=1}^{m}(\sum_{k=1}^{m}\mathcal{E}_{ijk}\overline{\alpha}_k^1 + Z\widehat{\mathcal{E}}_{ij\zeta})\overline{\chi}_i\overline{\chi}_j & \text{asym.} \\ \sum_{i=1}^{m}\sum_{j=1}^{m}(\sum_{k=1}^{m}\sum_{l=1}^{m}\mathcal{E}_{ijkl}\overline{\alpha}_k^1\overline{\alpha}_l^1 + Z\widehat{\mathcal{E}}_{ij\zeta})\overline{\chi}_i\overline{\chi}_j & \text{sym.} \end{array} \right\}, \quad \forall\ \|\overline{\chi}_i\| = \|\ ab_i^1\| = 1 \qquad \text{(AC-7.1)}$$

Since the functions to be minimized are continuous functions of their arguments and the minimization takes place over a compact set (the m-dimensional unit sphere), the minimum is attained for vectors $\overline{\alpha}_i^1$ and $\overline{\chi}_i$ which satisfy the following system of $2(m+1)$ equations for the $2(m+1)$ unknowns $\overline{\alpha}_i^1,\ \overline{\chi}_i,\ \Lambda_s^0,\ M$:

$$\left\{\begin{array}{ll} \sum_{j=1}^{m}(\sum_{k=1}^{m}\mathcal{E}_{ijk}\overline{\alpha}_k^1 + Z\widehat{\mathcal{E}}_{ij\zeta})\overline{\chi}_j = \Lambda_s^0\overline{\chi}_i & \sum_{j=1}^{m}\sum_{k=1}^{m}\mathcal{E}_{ijk}\overline{\chi}_j\overline{\chi}_k = M\overline{\alpha}_i^1 \quad \text{asym.} \\ \sum_{j=1}^{m}(\sum_{k=1}^{m}\sum_{l=1}^{m}\mathcal{E}_{ijkl}\overline{\alpha}_k^1\overline{\alpha}_l^1 + Z\widehat{\mathcal{E}}_{ij\zeta})\overline{\chi}_j = \Lambda_s^0\overline{\chi}_i & \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}\mathcal{E}_{ijkl}\overline{\alpha}_j^1\overline{\chi}_k\overline{\chi}_l = M\overline{\alpha}_i^1 \quad \text{sym.} \end{array}\right\} \tag{AC-7.2}$$

After finding the minimum solution of (AC-7.2), the corresponding worst imperfection shape $\overline{\mathcal{E}}_{i\epsilon}$ can be found from the equilibrium equations (AC-6.17) or (AC-6.21) for the asymmetric or the symmetric case respectively. If the solution of (AC-7.2) satisfies also (AC-6.32) the worst imperfection shape corresponds to a load maximum.

Of particular interest is the case of $Z = 0$, for which the solution of (AC-7.1) is simplified considerably. Indeed from (AC-7.1) one has

$$\Lambda_s^0(0)_{min} \ = \min \left\{\begin{array}{ll} \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\mathcal{E}_{ijk}\overline{\alpha}_k^1\overline{\chi}_i\overline{\chi}_j & \text{asym.} \\ \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}\mathcal{E}_{ijkl}\overline{\alpha}_k^1\overline{\alpha}_l^1\overline{\chi}_i\overline{\chi}_j & \text{sym.} \end{array}\right\} \tag{AC-7.3}$$

over all unit vectors $\overline{\alpha}_i^1, \overline{\chi}_i$. It can be easily shown that the minimum exists and is achieved for $\overline{\alpha}_i^1 = \overline{\chi}_i$. Under this condition, by comparing (AC-7.2) to (AC-5.13), (AC-5.15), one concludes that the worst imperfection shape is the one for which $\overline{\mathcal{E}}_{i\epsilon} \propto (\alpha_k^1)_{min}$, where $(\alpha_k^1)_{min}$ is the direction that minimizes λ_1 or λ_2 in the perfect structure. The wanted minimum $\Lambda_s^0(0)$ is equal to the minimum possible solution $2\lambda_1$ of (AC-5.13) for the asymmetric case or to the minimum possible solution $3\lambda_2$ of (AC-5.15) for the symmetric one. In addition, (AC-6.32) is automatically satisfied, thus ensuring that for $Z = 0$ the worst imperfection corresponds to a maximum load.

Notice that the polynomial system of equations in (AC-7.2), whose solution is required for the determination of the worst imperfection shape, in general has a number of solutions that increases exponentially with m. Exhaustive searches to find the wanted solution are numerically feasible for relatively low numbers m of interacting modes. Fortunately, by minimizing separately each one of the two terms of the sum appearing (AC-7.1), and with the help of (AC-6.9) one can deduce the following lower bound for $\Lambda_s^0(Z)_{min}$:

$$\left\{\begin{array}{ll} 2(\lambda_1)_{min} + Z((d\overset{i}{\widehat{\lambda}}/d\zeta)_c)_{min} & \text{asym.} \\ 3(\lambda_2)_{min} + Z((d\overset{i}{\widehat{\lambda}}/d\zeta)_c)_{min} & \text{sym.} \end{array}\right\} \le \Lambda_s^0(Z)_{min} \tag{AC-7.4}$$

Formulation B

The results in (AC-7.1), (AC-7.2) have been obtained for the case of a fixed projection amplitude parameter ξ. It is interesting to ask the same question about the worst imperfection for the case where the amplitude of the imperfection is fixed, i.e. $\bar{\epsilon}_0 = 1$ (see (AC-6.35), (AC-6.36)). For this case one seeks to minimize $\Lambda_s^0(Z)$ satisfying the equilibrium conditions (AC-6.17) or (AC-6.21) as well as the conditions (AC-6.30)-(AC-6.32) under the constraint $\|\overline{\mathcal{E}}_{i\epsilon}\| = 1$ instead of the constraint $\|\overline{\alpha}_i^1\| = 1$. Hence, the extremality condition that has to be satisfied by $\Lambda_s^0(Z)$ is:

$$(\partial\Lambda_s^0/\partial\overline{\mathcal{E}}_{i\epsilon}) + \psi\overline{\mathcal{E}}_{i\epsilon} = 0 \tag{AC-7.5}$$

where ψ is a scalar Lagrange multiplier. For the either the asymmetric or the symmetric case, differentiation of (AC-3.13) or (AC-3.16) with respect to $\overline{\mathcal{E}}_{i\epsilon}$, multiplication by $\overline{\chi}_i$ and summation with respect to the index i gives in view of (AC-6.30):

$$\overline{\chi}_l + (\sum_{i=1}^{m}\sum_{j=1}^{m}\mathcal{E}_{ij\lambda}\overline{\alpha}_j^1\overline{\chi}_i)(\partial\Lambda_s^0/\partial\overline{\mathcal{E}}_{l\epsilon}) = 0 \tag{AC-7.6}$$

Since only load extrema are of interest, (AC-6.32) is satisfied. From (AC-7.5), (AC-7.6) one concludes that at extremality $\overline{\mathcal{E}}_{i\epsilon} \propto \overline{\chi}_i$ and consequently from (AC-6.17) or (AC-6.21) and (AC-6.30) the following condition has to be satisfied at the minimum $\Lambda_s^0(Z)$:

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\overline{B}_{ij}(\overline{B}_{jk} + \phi(\Lambda_s^0\mathcal{E}_{jk\lambda} + Z\widehat{\mathcal{E}}_{jk\zeta}))\overline{\alpha}_k^1 = 0 \tag{AC-7.7}$$

where $\phi = 1$ for the asymmetric case while $\phi = 2$ for the symmetric one.

To find the wanted worst geometric imperfection shape, i.e. to find $\overline{\alpha}_i^1, \Lambda_s^0$, one has to solve the system of $m+1$ equations (AC-7.7) complemented by the constraint $\|\sum_{k=1}^{m}(\overline{B}_{jk} + \phi(\Lambda_s^0\mathcal{E}_{jk\lambda} + Z\widehat{\mathcal{E}}_{jk\zeta}))\overline{\alpha}_k^1\| = \phi + 1$. Notice however that the polynomial system in (AC-7.7) has a rather high number of possible nontrivial real solutions which increases exponentially with the number of interacting modes m. Exhaustive searches to find the wanted solution are numerically feasible for relatively low numbers m of interacting modes.

Once again, of particular interest is the solution of the worst imperfection shape problem for $Z = 0$. It will be shown that for this case, the problem is once more reduced to the solution of (AC-5.13) or (AC-5.15) and the wanted minimum $\Lambda_s^0(0)$ is equal to the minimum possible solution $2\lambda_1$ of (AC-5.13) for the asymmetric case or to the minimum possible solution $3\lambda_2$ of (AC-5.15) for the symmetric one.

Indeed, for $Z = 0$ and for the inner product choice that gives $\mathcal{E}_{ij\lambda} = -\delta_{ij}$ one can rewrite the extremality condition (AC-7.7) as:

$$\sum_{j=1}^{m}\sum_{k=1}^{m}(\overline{B}_{ij} + \phi\Lambda_s^0\mathcal{E}_{ij\lambda})\overline{B}_{jk}\overline{\alpha}_k^1 = 0 \tag{AC-7.8}$$

The wanted result $\sum_{k=1}^{m}\overline{B}_{jk}\overline{\alpha}_k^1 = 0$ follows by establishing that the factor in parenthesis in (AC-7.8) is a positive definite matrix. For this purpose, note that $\overline{B}_{ij}$ is positive semidefinite by (AC-6.30), (AC-6.31) and that the second term in the parenthesis is a positive multiple of the identity. Hence the lowest eigenvalue of the factor in parenthesis is $-\phi\Lambda_s^0$ which is positive.

Notice that when $Z = 0$, the worst imperfection shape problem has the same answer, irrespective on whether ξ or ϵ is controlled. For $Z \neq 0$, one expects in general different solutions to (AC-7.2) or (AC-7.7). To find the minimum $\Lambda_s^0(Z)$ for $Z \neq 0$ and the corresponding worst imperfection shape, an incremental Newton - Raphson method based on (AC-7.2) or (AC-7.7) can be used, starting with the perfect structure solution obtained for $Z = 0$, and tracking the solution for increasing values of $|Z|$. This procedure is computationally more appealing than finding directly all solutions of (AC-7.2) or (AC-7.7). However, there may be some value of Z beyond which this incremental procedure no longer produces an absolute minimum for Λ_s^0.

Chapter B

APPLICATIONS IN ELASTIC STRUCTURES AND SOLIDS

Having developed the general theory for bifurcation, stability and imperfection sensitivity of elastic structures, attention is now turned in applying it to solve problems in mechanics. This Chapter deals with structural applications.

BA ONE-DIMENSIONAL STRUCTURES

This section deals with one-dimensional structural applications. The most classical application is the inextensible beam under axial compression, the celebrated "*elastica*" problem, which was presented by Euler in 1744.

BA-1 ELASTICA

The first application to be given for the case of one-dimensional elastic structures is the well known "*elastica*" problem due to Euler. It consists of an inextensible, slender planar elastic beam of bending stiffness EI and total length L. The beam is fully clamped at end A while its end B can move freely along the y direction with a slope that remains fixed, as shown in Fig. BA-1.1. The beam is loaded by a compressive force λ which acts along the x axis.

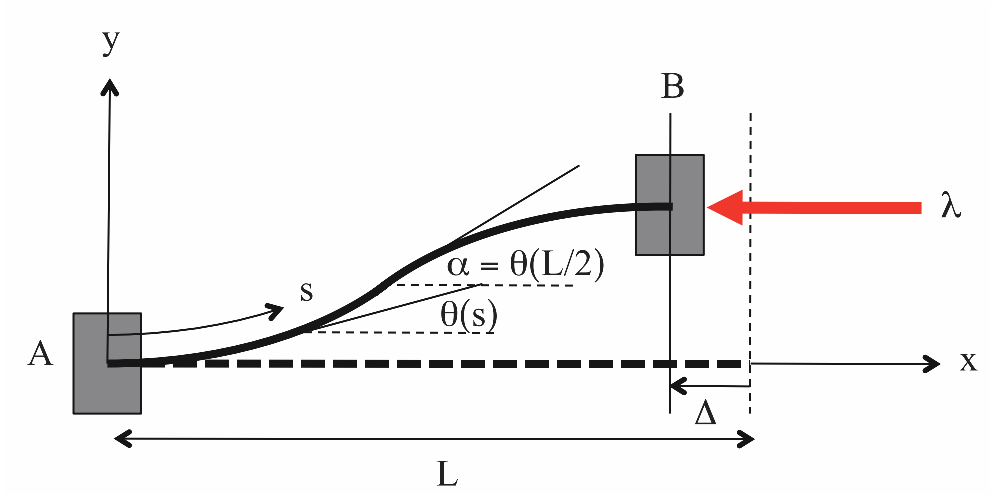

Figure BA-1.1: Undeformed and deformed configurations of Euler's elastica beam.

The role of displacement variable u for this structure is played in this problem by the scalar function $\theta(s)$ where θ denotes the rotation from its initial position of the tangent at a point with arc length coordinate s. The curvature at point s is $d\theta/ds$ and consequently the strain energy of the beam per unit length is $(1/2)EI(d\theta/ds)^2$. The point of application B of the compressive force λ displaces by $\Delta = \int_0^L (ds - dx) = \int_0^L (1 - \cos\theta)ds$ and hence the beam's potential energy is given by:

$$\mathcal{E}(\theta, \lambda) = \int_0^L [\frac{1}{2}EI(\frac{d\theta}{ds})^2 - \lambda(1 - \cos\theta)]ds. \tag{BA-1.1}$$

The kinematically admissible functions $\theta(s)$ are all continuous functions in the interval $[0, L]$ which vanish at $s = 0, L$ and for which the potential energy in (BA-1.1) exists and is finite.[1] One can also define an inner product for the admissible displacement functions $\theta(s)$. The simplest possible such choice is:

$$(\theta_1, \theta_2) \equiv \frac{1}{L}\int_0^L \theta_1(s)\theta_2(s)ds. \tag{BA-1.2}$$

[1]Note: The appropriate space U for $\theta(s)$ turns out to be the space $H_0^1[0, L]$, i.e. the set of all $\theta(s)$ such that $\{\int_0^L [(d\theta/ds)^2 + \theta^2]ds\}^{1/2} < \infty$ and which in addition satisfy $\theta(0) = \theta(L) = 0$.

From (BA-1.1) the equilibrium equation and boundary conditions of the problem are (see (AC-2.2)):

$$\frac{d^2\theta}{ds^2} + \frac{\lambda}{EI}\sin\theta = 0, \quad \theta(0) = \theta(L) = 0. \tag{BA-1.3}$$

This boundary value problem has one obvious solution, which is valid for arbitrary loads λ, namely:

$$\overset{0}{\theta}(\lambda) = 0. \tag{BA-1.4}$$

The above trivial solution is obviously the principal equilibrium solution of the beam, for it satisfies $\overset{0}{\theta}(0) = 0$ (see (AC-2.3)) and corresponds to the straight initial configuration. The principal solution is also stable, at least for adequately small values of the load λ i.e. it satisfies (AC-2.4). To prove this assertion, one has to show that for adequately small values of λ, a positive constant $\overset{0}{\beta}(\lambda)$ exists such:

$$\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u = \int_0^L [EI\left(\frac{d(\delta\theta)}{ds}\right)^2 - \lambda(\delta\theta)^2]ds \geq \overset{0}{\beta}(\lambda)\frac{1}{L}\int_0^L (\delta\theta)^2 ds, \quad \overset{0}{\beta}(\lambda) > 0, \quad \forall \delta\theta \in U. \tag{BA-1.5}$$

Any admissible $\delta\theta(s)$ can be expanded in a Fourier sine series:

$$\delta\theta = \sum_{n=1}^{\infty}[\delta\theta_n \sin(\frac{n\pi s}{L})]. \tag{BA-1.6}$$

Upon substitution of (BA-1.6) into (BA-1.5) one obtains:

$$\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u = \frac{L}{2}\sum_{n=1}^{\infty}\{(\delta\theta_n)^2[EI(\frac{n\pi}{L})^2 - \lambda]\} \geq \frac{\overset{0}{\beta}}{2}\sum_{n=1}^{\infty}(\delta\theta_n)^2. \tag{BA-1.7}$$

The above result proves that for $0 \leq \lambda < EI(\pi/L)^2$ one has $\overset{0}{\beta}(\lambda) = L(EI(\pi/L)^2 - \lambda) > 0$ and hence the straight principal configuration is stable. The $EI(\pi/L)^2$ value of the load, above which the beam loses its stability will be identified subsequently with the lowest critical load, λ_c at which the first bifurcation occurs.

Having discussed the principal solution and its stability, attention is turned next to the bifurcated solutions for the elastica, and in particular the bifurcated equilibrium solution that passes through the lowest critical load. An analytical solution for the wanted bifurcated equilibrium path does exist and is given in terms of elliptic functions. Following the general methodology developed in Section AC an asymptotic solution for the same bifurcated equilibrium path will also be derived. The results from the asymptotic method will be shown to agree with the exact solution.

i) *Exact Solution*

The exact solution for the bifurcated equilibrium path of the elastica is obtained from (BA-1.3) as follows: After multiplying the equilibrium equation by $(d\theta/ds)$ and integrating with respect to s one gets:

$$\frac{d\theta}{ds} = \mu[2(\cos\theta - \cos\alpha)]^{1/2}, \quad \mu^2 \equiv \lambda/EI, \tag{BA-1.8}$$

where α is the rotation of the beam at its inflexion point, i.e. the point for which $d\theta/ds = 0$. Introducing the change of variable $y = \sin(\theta/2)/\sin(\alpha/2)$ and the boundary condition $\theta(0) = 0$, one obtains by integration of (BA-1.8):

$$\sin\Big(\frac{\theta(s)}{2}\Big) = k \ \mathrm{sn}(\mu s), \quad k \equiv \sin(\frac{\alpha}{2}), \tag{BA-1.9}$$

where $\mathrm{sn}(x)$ denotes the Jacobian elliptic function "*sine-amplitude of* x" defined by:

$$x = \int_0^{\mathrm{sn}(x)} \frac{dy}{[(1-y^2)(1-k^2y^2)]^{1/2}}. \tag{BA-1.10}$$

Let $K(k)$ denote the quarter period of this periodic function, i.e. $\mathrm{sn}(x) = \mathrm{sn}(x + 4nK)$ for any integer n:

$$K(k) = \int_0^1 \frac{dy}{[(1-y^2)(1-k^2y^2)]^{1/2}}, \quad \mathrm{sn}(K) = -\mathrm{sn}(3K) = 1, \quad \mathrm{sn}(2K) = \mathrm{sn}(4K) = 0. \tag{BA-1.11}$$

From the remaining boundary condition $\theta(L) = 0$, (BA-1.9) and the property $\mathrm{sn}(2K) = 0$ (although $\sin(2nK) = 0$, the value $n = 1$ is considered since of interest is the lowest corresponding value of the applied load λ) one obtains the following relation between the load λ, the material and geometric properties of the beam EI, L and the rotation α at the inflexion point of the beam:

$$\mu L = 2K(k) \quad \text{or} \quad L(\frac{\lambda}{EI})^{1/2} = 2K(\sin(\frac{\alpha}{2})). \tag{BA-1.12}$$

For small values of $k = \sin(\alpha/2)$ by expanding K about $k = 0$ one obtains up to $O(k^4)$ in accuracy:

$$\lambda = EI(\frac{\pi}{L})^2[1 + \frac{1}{2}\sin^2(\frac{\alpha}{2}) + \ldots]. \tag{BA-1.13}$$

The above relation between the applied load λ and the resulting rotation α at the middle point (note for $s = L/2$ in (BA-1.9) $\mathrm{sn}(\mu L/2) = \mathrm{sn}(K) = 1$ and so $\theta(L/2) = \alpha$) of the beam shows that a bifurcated solution is possible when the load $\lambda > \lambda_c \equiv EI(\pi/L)^2$ where λ_c is the lowest load at which instability of the principal solution occurs.

ii) *Asymptotic Solution*

The bifurcation problem for the elastica beam will now be solved using the general asymptotic methodology developed in Section AC. The starting point for these calculations is the determination of the lowest critical load λ_c which from (AC-2.7) and (BA-1.1) is found by solving the following system:

$$\frac{d^2\overset{1}{\theta}}{ds^2} + \frac{\lambda_c}{EI}\overset{1}{\theta} = 0, \quad \overset{1}{\theta}(0) = \overset{1}{\theta}(L) = 0. \tag{BA-1.14}$$

The solution to the above problem corresponding to the lowest eigenvalue is:

$$\lambda_c = EI(\pi/L)^2, \quad \overset{1}{\theta}(s) = \sqrt{2}\sin(\pi s/L), \tag{BA-1.15}$$

where $\overset{1}{\theta}(s)$ is the unique eigenmode corresponding to λ_c. The $\sqrt{2}$ term in $\overset{1}{\theta}$ comes from normalization $(\overset{1}{\theta}, \overset{1}{\theta}) = 1$.

The above value for λ_c coincides as expected with the lowest bifurcation load of the elastica found from the exact solution. The fact that λ_c is a bifurcation point can be independently verified from the general theory which gives from (AC-2.27) with the help of (BA-1.1) and (BA-1.15):

$$\mathcal{E},^c_{u\lambda} \overset{1}{u} = -\int_0^L \sin(\overset{0}{\theta}(s))\overset{1}{\theta}(s)ds = 0. \tag{BA-1.16}$$

Proceeding with the calculation of the expansion of the load λ in terms of the bifurcation amplitude ξ, one obtains from (BA-1.1) for $((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}$:

$$((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = \lambda_c \int_0^L \sin(\overset{0}{\theta}(s))[\overset{1}{\theta}(s)]^3 ds = 0, \tag{BA-1.17}$$

which shows that the bifurcation is a symmetric one and $\lambda_1 = 0$ (see (AC-1.13)).

The calculation of λ_2, the next higher order term in the ξ expansion of the load λ_1, requires the evaluation of the various quantities appearing in (AC-3.14). Starting with $((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$ one obtains from (BA-1.1) and (BA-1.15):

$$((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = -\int_0^L \cos(\overset{0}{\theta}(s))[\overset{1}{\theta}(s)]^2 ds = -L. \tag{BA-1.18}$$

In view of (AC-3.20) the above result means that the lowest eigenvalue of the stability operator evaluated at the principal branch has a strict zero crossing at λ_c with $(d\overset{0}{\beta}/d\lambda)_c = -L$ exactly as expected from the general theory (see discussion of (AC-3.15)). The result could have also been obtained directly from (BA-1.7) which gives $\overset{0}{\beta}(\lambda) = L(EI(\pi/L)^2 - \lambda)$.

Of the two expressions appearing in the numerator of λ_2 in (AC-3.14), $((\mathcal{E}^c_{,uuu}v_{\xi\xi})\overset{1}{u})\overset{1}{u} = 0$ while $(((\mathcal{E}^c_{,uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u}$ yields from (BA-1.1) and (BA-1.15):

$$(((\mathcal{E}^c_{,uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} \quad = \lambda_c \int_0^L \cos(\overset{0}{\theta}(s))[\overset{1}{\theta}(s)]^4 ds = \frac{3}{2}L\lambda_c = \frac{3}{2}EI\frac{\pi^2}{L}. \tag{BA-1.19}$$

Hence using the results in (BA-1.18), (BA-1.19) into (AC-3.14) one has:

$$\lambda_2 = -\frac{1}{3}(\frac{3}{2}L\lambda_c)/(-L) = \frac{\lambda_c}{2}. \tag{BA-1.20}$$

Thus the asymptotic expansion for the load λ in terms of the bifurcation amplitude parameter ξ is:

$$\lambda = \lambda_c[1 + \frac{\xi^2}{4} + O(\xi^4)) = EI(\frac{\pi}{L})^2[1 + \frac{\xi^2}{4} + \cdots]. \tag{BA-1.21}$$

Similarly the expansions for $\theta(s)$ are in view of (BA-1.4) and (BA-1.15):

$$\theta(s) = \xi\sqrt{2}\sin(\frac{\pi s}{L}) + O(\xi^3). \tag{BA-1.22}$$

It is of interest to compare the above obtained asymptotic results for the bifurcated equilibrium solution with their exact counterparts in (BA-1.13) and (BA-1.9). Since the exact solution is given in terms of $k = \sin(\alpha/2)$, while the asymptotic one is given in terms of ξ, the starting point of the comparison is the relation between those two quantities. Recalling from (AC-3.1) the definition of $\xi = ((u - \overset{0}{u}), \overset{1}{u})$, one obtains up to $O(k^2)$ from (BA-1.4), (BA-1.9) and (BA-1.15):

$$\xi = \frac{1}{L}\int_0^L \theta(s)\overset{1}{\theta}(s)ds = \frac{\sqrt{2}}{L}\int_0^L 2\sin^{-1}[k\mathrm{sn}(\mu s)]\sin(\frac{\pi s}{L})ds = k\sqrt{2} + O(k^2). \tag{BA-1.23}$$

By introducing (BA-1.23) into (BA-1.9), (BA-1.13) one recovers as expected (BA-1.22) and (BA-1.21).

BA-2 THIN-WALLED BEAM

An interesting practical application of the general theory presented in Section AC is the study of buckling of thin walled beams. For reasons of completeness of the presentation, a brief derivation of the model's potential energy will precede the bifurcation and stability analyses of the simply supported and clamped axially compresed thin walled beam.

i) *Model Derivation – Perfect Case*

Consider a thin walled right prismatic beam whose cross-section at point x is shown in FIG. BA-2.1. The term "*prismatic*" means that the beam' s cross-section is the same for any point along the generator direction x, while the term "*right*" means that the two end sections of the beam are normal to the generator direction x. Two are the fundamental kinematic assumptions for the deformation of the beam: The first assumption dictates that in its own plane the motion of each cross-section x is a rigid body motion, i.e. a parallel translation $\mathbf{j}v(x)+\mathbf{k}w(x)$ of point S plus a rotation $\phi(x)$ about point S, with coordinates e_y and e_z as shown in FIG. BA-2.1. A physically convenient choice for the center of rotation S will be given subsequently. The second kinematic assumption is about the out of plane displacement d_x of each point M of the cross-section – also termed "*warping displacement*" – which postulates that the resulting shear strain at the lateral surface of the beam γ_{xs} is negligible.

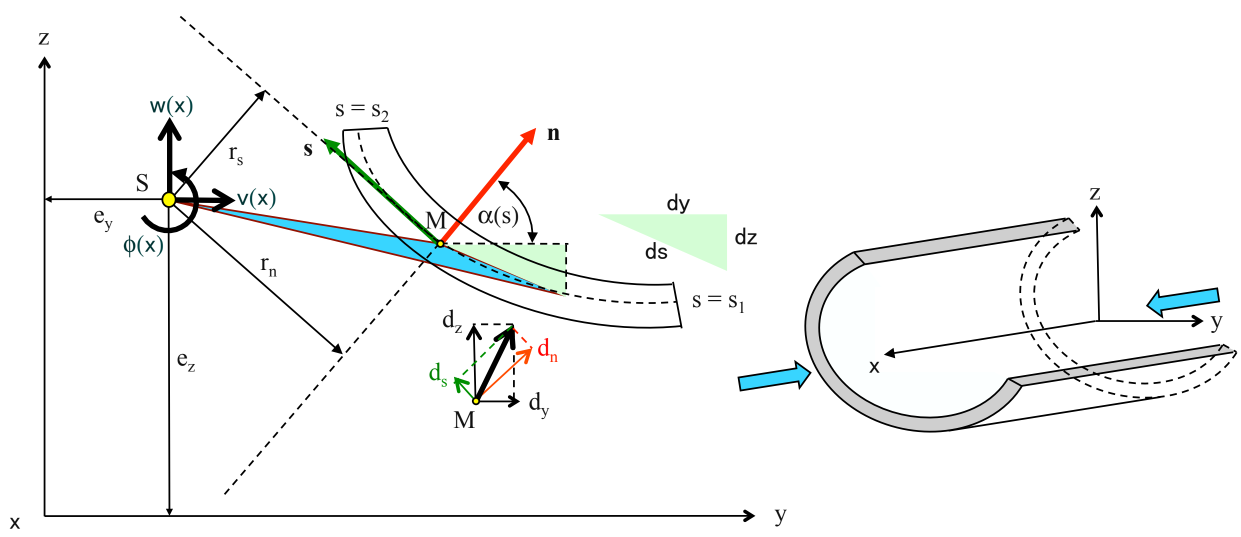

Figure BA-2.1: Kinematics of the cross-section at x of a thin-walled beam; section displaces in tis own plane as a rigid disc by $v(x)$, $w(x)$ along the y and z directions respectively and rotates about point S by an angle $\phi(x)$. Schematic 3D representation of the open-section thin-walled beam is shown in the right.

From the first kinematic assumption, the displacements $d_y(s)$ and $d_z(s)$ of an arbitrary point M of the cross-section are given by:

$$d_y(s) = v - (z(s) - e_z)\phi, \quad d_z(s) = w + (y(s) - e_y)\phi \tag{BA-2.1}$$

From here on s denotes the arc length coordinate along the middle line of the cross section that corresponds to an arbitrary point M. Decomposing the in-plane displacement $\mathbf{j}d_y(s)+\mathbf{k}d_z(s)$ along the tangent $\mathbf{s}$ and the normal $\mathbf{n}$ to the middle line of the cross-section at point M, one finds from (BA-2.1) the corresponding components d_s and d_n to be:

$$\begin{aligned} d_s &= -d_y(s)\sin\alpha(s)+d_z(s)\cos\alpha(s) &&= -v\sin\alpha(s)+w\cos\alpha(s)+r_s(s)\phi \\ d_n &= d_y(s)\cos\alpha(s)+d_z(s)\sin\alpha(s) &&= v\cos\alpha(s)+w\sin\alpha(s)+r_n(s)\phi \end{aligned} \tag{BA-2.2}$$

where $r_n(s)$ and $r_s(s)$ are the distances from S to the normal and tangential directions respectively of the cross-section's middle line at M as shown in FIG. BA-2.1. On the other hand, the zero shear assumption for the lateral surface of the beam gives with the help of (BA-2.2):

$$\gamma_{xs}=\frac{\partial d_x}{\partial s}+\frac{\partial d_s}{\partial x}=0 \quad\Longrightarrow\quad \frac{\partial d_x}{\partial s}=\frac{dv}{dx}\sin\alpha-\frac{dw}{dx}\cos\alpha-r_s\frac{d\phi}{dx} \tag{BA-2.3}$$

The following geometric properties of the cross-section's middle line are recorded here, where $\alpha(s)$ is the angle formed between the normal to the section's mid-line and the y-direction (see FIG. BA-2.1):

$$\sin\alpha(s)=-\frac{dy(s)}{ds},\quad \cos\alpha(s)=\frac{dz(s)}{ds},\quad \omega(s)=\frac{1}{2}\int_0^s r_s(s)ds \tag{BA-2.4}$$

where $\omega(s)$ is the cross-hatched sectorial area corresponding to point M as indicated in FIG. BA-2.1. One obtains by integrating (BA-2.3) that d_x, the x (warping) displacement of point M, is given by:

$$d_x=u-y\frac{dv}{dx}-z\frac{dw}{dx}-2\omega\frac{d\phi}{dx} \tag{BA-2.5}$$

Hence the displacements d_x, d_s, d_n of point M along the $\mathbf{i}$, $\mathbf{s}$ and $\mathbf{n}$ directions respectively are given in terms of the three displacements $u(x)$, $v(x)$, $w(x)$ of S and the rotation $\phi(x)$ about S by (BA-2.5) and (BA-2.2).

In addition to the kinematic assumptions made thus far, a dynamic assumption is also made about the stress state in the beam. According to this assumption, the only nontrivial stress components in the beam are the axial stress σ_{xx} acting along the x direction and the shear stress τ_{xs} acting on the cross-section. The axial stress σ_{xx} is related to the axial strain $\epsilon_{xx}\equiv\epsilon$ by $\sigma_{xx}=E\epsilon$, where E is the material's Young modulus. The shear stress τ_{xs} has an average over the thickness of the cross-section – termed the "*shear flow*" – which is zero (the shear flow at each point M should vanish because of the kinematic assumption that the lateral shear strain is zero). However, the shear stress contributes an elementary couple m_x per unit cross-sectional length ds in the x direction at point M which is related to the axial rate of twist γ of the tangent $\mathbf{s}$ and the normal $\mathbf{n}$ to the middle line at M about $\mathbf{i}$ by $m_x=(G/3)t^2\gamma$, where G is the material's shear modulus. The relation between m_x and γ comes from elementary strength of materials and relates the axial moment m_x of a prism with rectangular cross section $ds \times t$ due to a rotation per unit length γ of the prism along the x direction. In other words, each elementary parallelepiped of dimensions $ds \times t \times dx$ is storing in addition to an extensional energy $(1/2)E\epsilon^2 tdsdx$, a shear energy $(1/2)(G/3)\gamma^2 t^3 dsdx$.

The strain-displacement relations required for the thin walled beam model are:

$$\epsilon(x,s) = \frac{\partial d_x}{\partial x} + \frac{1}{2}\left(\frac{\partial d_s}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial d_n}{\partial x}\right)^2, \quad \gamma(x,s) = \frac{d\phi}{dx} \tag{BA-2.6}$$

The axial strain-displacement relation is nonlinear and contains the square of the rotations in the xy and xz planes of an element $d\mathbf{x}$ initially aligned with the x axis. Its justification is simple and comes from the definition of strain ϵ for a fiber element $d\mathbf{x}$ initially in the x direction that changes to $d\mathbf{x} + d\mathbf{u}$ due to the total displacement $\mathbf{u} = \mathbf{i}d_x + \mathbf{j}d_y + \mathbf{k}d_z$ of point M. For small strains $\epsilon \ll 1$, the strain $\epsilon \equiv (\|d\mathbf{x} + d\mathbf{u}\| - \|d\mathbf{x}\|)/\|d\mathbf{x}\|$ is to an $O(\epsilon^2)$ approximated by:

$$\epsilon = (\|d\mathbf{x} + d\mathbf{u}\|^2 - \|d\mathbf{x}\|^2)/2\|d\mathbf{x}\|^2; \quad d\mathbf{u} = \mathbf{i}\frac{\partial d_x}{\partial x}dx + \mathbf{n}\frac{\partial d_n}{\partial x}dx + \mathbf{s}\frac{\partial d_s}{\partial x}dx, \quad d\mathbf{x} = \mathbf{i}dx \tag{BA-2.7}$$

where $\mathbf{n}$ and $\mathbf{s}$ are the unit vectors along the normal and tangential directions to the middle line of the cross-section at point M. Notice that in the derivation of (BA-2.6)$_1$ from (BA-2.7) the assumption of small strain $\epsilon \ll 1$ implies $|\partial d_x/\partial x| \ll 1$ and hence the square of this term is neglected. On the other hand, since the cross-section rotates in its own plane as a rigid disc by an angle ϕ, the rate of twist γ is the same for all points s and is given by (BA-2.6)$_2$. Consequently by employing in (BA-2.6) the relations (BA-2.2) and (BA-2.5) one obtains:

$$\begin{aligned}
\epsilon &= \frac{du}{dx} - y\frac{d^2v}{dx^2} - z\frac{d^2w}{dx^2} - 2\omega\frac{d^2\phi}{dx^2} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + \frac{1}{2}[(y-e_y)^2 + (z-e_z)^2]\left(\frac{d\phi}{dx}\right)^2 \\
&\quad -(z-e_z)\frac{dv}{dx}\frac{d\phi}{dx} + (y-e_y)\frac{dw}{dx}\frac{d\phi}{dx} \\
\gamma &= \frac{d\phi}{dx}
\end{aligned} \tag{BA-2.8}$$

In the derivation of (BA-2.8) use was made of the following geometric relations: $r^2 = r_s^2 + r_n^2 = (y-e_y)^2 + (z-e_z)^2$, $r_s \sin\alpha - r_n \cos\alpha = z - e_z$, $r_s \cos\alpha + r_n \sin\alpha = y - e_y$ as one can easily verify by inspection of FIG. BA-2.1.

Recalling the assumption made earlier in this section about the linear elastic response of the structure, the total internal elastic energy stored in a beam of length L is:

$$\mathcal{E}_{int} = \frac{1}{2}\int_0^L \{\int_{s_1}^{s_2} [E\epsilon^2 + \frac{G}{3}t^2\gamma^2]t \; ds\} \; dx \tag{BA-2.9}$$

In the interest of simplification of the final expression for the elastic energy, it will be assumed that the point with coordinates $(y,z) = (0,0)$ is the center of mass of the cross section and that y,z are its principal axes of inertia. In this case the following geometric properties of the cross-section can be used:

$$\int_{s_1}^{s_2} y\ t\ ds = \int_{s_1}^{s_2} z\ t\ ds = 0, \quad \int_{s_1}^{s_2} y\ z\ t\ ds = 0 \tag{BA-2.10}$$

Moreover S is taken to be the shear center of the cross-section which implies the additional geometric properties:

$$\int_{s_1}^{s_2} \omega\ y\ t\ ds = \int_{s_1}^{s_2} \omega\ z\ t\ ds = 0 \tag{BA-2.11}$$

Further simplification is achieved when the origin of the arc length coordinate s is chosen so that:

$$\int_{s_1}^{s_2} \omega\ t\ ds = 0 \tag{BA-2.12}$$

In the evaluation of the internal elastic energy of the thin walled beam, the following geometric quantities of the cross-section will be needed:

$$A = \int_{s_1}^{s_2} t\ ds, \quad I_{yy} = \int_{s_1}^{s_2} z^2 t\ ds, \quad I_{zz} = \int_{s_1}^{s_2} y^2 t\ ds, \quad \Gamma = \int_{s_1}^{s_2} (2\omega)^2 t\ ds, \quad I_p = \int_{s_1}^{s_2} r^2 t\ ds$$

$$J = \frac{1}{3}\int_{s_1}^{s_2} t^3 ds, \quad I_{yp} = \int_{s_1}^{s_2} r^2 z\ t\ ds, \quad I_{zp} = \int_{s_1}^{s_2} r^2 y\ t\ ds, \quad I_{\omega p} = \int_{s_1}^{s_2} r^2(-2\omega)t\ ds, \quad I_{pp} = \int_{s_1}^{s_2} r^4 t\ ds \tag{BA-2.13}$$

where r is the distance between the shear center S and the point M. Some of the above defined geometric quantities are easily recognizable: A is the cross-sectional "*area*" of the beam, I_{yy}, I_{zz} are the section's "*moments of inertia*" with respect to the y and z axes respectively, Γ is the section's "*warping constant*" and J is the section's "*torsional constant*". The remaining geometric quantities in (BA-2.13) do not have universally accepted names.

Substituting (BA-2.8) into (BA-2.9) and using also (BA-2.10)-(BA-2.13), one obtains the following expression for the internal elastic energy of a thin walled beam:

$$\begin{aligned} \mathcal{E}_{int} &= \frac{1}{2}\int_0^L [EA(e)^2 + EI_{zz}(k_y)^2 + EI_{yy}(k_z)^2 + E\Gamma\left(\frac{d^2\phi}{dx^2}\right)^2 + 2GJ\psi + \\ &\quad 2EI_p e\psi + 2EI_{zp}k_y\psi + 2EI_{yp}k_z\psi + 2EI_{\omega p}\frac{d^2\phi}{dx^2}\psi + EI_{pp}(\psi)^2]dx \end{aligned} \tag{BA-2.14}$$

with the following definitions introduced for the beam's strain measures:

$$\begin{aligned} e &\equiv \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + \left(e_z\frac{dv}{dx} - e_y\frac{dw}{dx}\right)\frac{d\phi}{dx} \\ k_y &\equiv -\frac{d^2v}{dx^2} + \frac{dw}{dx}\frac{d\phi}{dx}, \quad k_z \equiv -\frac{d^2w}{dx^2} - \frac{dv}{dx}\frac{d\phi}{dx}, \quad \psi \equiv \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 \end{aligned} \tag{BA-2.15}$$

In the above introduced definitions, e is termed the "*axial strain*" measure of the beam, k_y and k_z are termed the "*bending strain*" measures of the beam while $d^2\phi/dx^2$ is the beam's "*warping strain*" and $d\phi/dx$ is the beam's "*torsional strain*" measure.

The completion of the model derivation requires the expression for the external potential energy due to the applied loads. If p_x, p_y, p_z and m are the distributed body forces and twisting moment about the shear center axis per unit length, the external potential energy of the thin walled beam is:

$$\mathcal{E}_{ext} = -\int_0^L [p_x u + p_y v + p_z w + m\phi]dx - \text{ (Boundary Terms at } x = 0, x = L) \tag{BA-2.16}$$

where the boundary terms depend on the end conditions applied to the particular problem under consideration.

Hence the total potential energy for the thin walled beam model is $\mathcal{E}(u, \lambda) = \mathcal{E}_{int} + \mathcal{E}_{ext}$. The space of the admissible displacement functions $u \equiv (u(x), v(x), w(x), \phi(x))$ depends on the boundary conditions imposed on the displacements as will be discussed in the examples below.

Finally, the question of inner product selection has to be addressed. Of the many possible choices, the following inner product for arbitrary $u_1, u_2 \in U$ is to be adopted, namely:

$$(u_1, u_2) \equiv \int_0^L [\frac{du_1}{dx}\frac{du_2}{dx} + \frac{dv_1}{dx}\frac{dv_2}{dx} + \frac{dw_1}{dx}\frac{dw_2}{dx} + \frac{d\phi_1}{dx}\frac{d\phi_2}{dx}]dx \tag{BA-2.17}$$

At this point it should be noted that for different definitions of the inner product one obtains different asymptotic expansions for the same bifurcated path. This is due to the different values of the resulting bifurcation amplitude ξ which obviously depends on the inner product selection.

ia) *Simply Supported, Axially Compressed Thin Walled Beam*

The first example is concerned with the buckling of a simply supported thin walled beam, subjected to a compressive axial force λ at end $x = L$. For this case the total potential energy of the beam is given, according to (BA-2.14)-(BA-2.16), by:

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\int_0^L [EAe^2 + EI_{zz}(k_y)^2 + EI_{yy}(k_z)^2 + E\Gamma\left(\frac{d^2\phi}{dx^2}\right)^2 + 2GJ\psi + \\ &\quad 2EI_p e\psi + 2EI_{zp}k_y\psi + 2EI_{yp}k_z\psi + 2EI_{\omega p}\frac{d^2\phi}{dx^2}\psi + EI_{pp}\psi^2]dx + \lambda u(L) \\ &\quad u(0) = 0, \quad v(0) = v(L) = 0, \quad w(0) = w(L) = 0, \quad \phi(0) = \phi(L) = 0 \end{aligned} \tag{BA-2.18}$$

where the strain-displacment relations for e, k_y, k_z, ψ in terms of u, v, w, ϕ are given in (BA-2.15). Hence as one can see from (BA-2.18) the space of admissible functions $u \in U$ for the energy functional of the problem is the set of functions $u \equiv (u(x), v(x), w(x), \phi(x))$ that have adequate continuity as to give finite values to the integral of the energy and that satisfy the essential boundary conditions indicated above.

According to the general theory, from (AC-2.2) the equilibrium equations for the beam are found by extremizing its potential energy $\mathcal{E}$ with respect to the displacements u. From (BA-2.18), (BA-2.15) we obtain:

$$\begin{aligned}\mathcal{E}_u\delta u &= \int_0^L [EAe\delta e + EI_{yy}k_z\delta k_z + EI_{zz}k_y\delta k_y + E\Gamma\frac{d^2\phi}{dx^2}\frac{d^2\delta\phi}{dx^2} + GJ\delta\psi + \\ &\quad EI_p(\psi\delta e + e\delta\psi) + EI_{zp}(\psi\delta k_y + k_y\delta\psi) + EI_{yp}(\psi\delta k_z + k_z\delta\psi) + \\ &\quad EI_{\omega p}(\psi\frac{d^2\delta\phi}{dx^2} + \frac{d^2\phi}{dx^2}\delta\psi) + EI_{pp}\psi\delta\psi]dx + \lambda\delta u(L) = 0\end{aligned} \tag{BA-2.19}$$

$$\begin{aligned}\delta e &= \frac{d\delta u}{dx} + \frac{dv}{dx}\frac{d\delta v}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} + (e_z\frac{dv}{dx} - e_y\frac{dw}{dx})\frac{d\delta\phi}{dx} + (e_z\frac{d\delta v}{dx} - e_y\frac{d\delta w}{dx})\frac{d\phi}{dx} \\ \delta k_y &= -\frac{d^2\delta v}{dx^2} + \frac{dw}{dx}\frac{d\delta\phi}{dx} + \frac{d\delta w}{dx}\frac{d\phi}{dx}, \quad \delta k_z = -\frac{d^2\delta w}{dx^2} - \frac{dv}{dx}\frac{d\delta\phi}{dx} - \frac{d\delta v}{dx}\frac{d\phi}{dx}, \quad \delta\psi = \frac{d\phi}{dx}\frac{d\delta\phi}{dx}\end{aligned}$$

where the admissible functions $\delta u, \delta v, \delta w, \delta\phi$ satisfy the boundary conditions in (BA 2.18).

It is not difficult to see that the principal solution $\overset{0}{u}$ to (BA-2.19) is given by:

$$\overset{0}{u}(x,\lambda) = -\frac{\lambda x}{EA}, \quad \overset{0}{v}(x,\lambda) = \overset{0}{w}(x,\lambda) = \overset{0}{\phi}(x,\lambda) = 0 \tag{BA-2.20}$$

Indeed a substitution of (BA-2.20) into (BA-2.19) proves that the equilibrium equations and boundary conditions are satisfied and that all the corresponding displacements vanish for $\lambda = 0$.

Of interest is the lowest critical load λ_c corresponding to the above found principal solution. According to the general theory, λ_c must satisfy (AC-2.7), namely: $(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c),\lambda_c)\overset{1}{u})\delta u = 0$. Upon taking an additional derivative of (BA-2.19) with respect to u and after accounting for (BA-2.20), one obtains the following variational equation at criticality:

$$\begin{aligned}(\mathcal{E}^c_{,uu}\overset{1}{u})\delta u &= \int_0^L [EA\frac{d\overset{1}{u}}{dx}\frac{d\delta u}{dx} + EI_{yy}\frac{d^2\overset{1}{w}}{dx^2}\frac{d^2\delta w}{dx^2} + EI_{zz}\frac{d^2\overset{1}{v}}{dx^2}\frac{d^2\delta v}{dx^2} + E\Gamma\frac{d^2\overset{1}{\phi}}{dx^2}\frac{d^2\delta\phi}{dx^2} + GJ\frac{d\overset{1}{\phi}}{dx}\frac{d\delta\phi}{dx} \\ &-\lambda_c\left(\frac{d\overset{1}{v}}{dx}\frac{d\delta v}{dx} + \frac{d\overset{1}{w}}{dx}\frac{d\delta w}{dx} + e_z\left(\frac{d\overset{1}{\phi}}{dx}\frac{d\delta v}{dx} + \frac{d\overset{1}{v}}{dx}\frac{d\delta\phi}{dx}\right) - e_y\left(\frac{d\overset{1}{\phi}}{dx}\frac{d\delta w}{dx} + \frac{d\overset{1}{w}}{dx}\frac{d\delta\phi}{dx}\right) + \frac{I_p}{A}\frac{d\overset{1}{\phi}}{dx}\frac{d\delta\phi}{dx}\right)]dx = 0\end{aligned} \tag{BA-2.21}$$

where the mode $\overset{1}{u} \equiv (\overset{1}{u}(x),\ \overset{1}{v}(x),\ \overset{1}{w}(x),\ \overset{1}{\phi}(x))$ has to satisfy the kinematic admissibility boundary conditions given in (BA-2.18). Integration of (BA-2.21) by parts gives the following four

Euler-Lagrange equations:

$$
\begin{aligned}
\delta u &: \quad \frac{d}{dx}(EA\frac{d\overset{1}{u}}{dx}) = 0, \quad \overset{1}{u}(0) = 0, \quad \frac{d\overset{1}{u}}{dx}(L) = 0 \\
\delta v &: \quad EI_{zz}\frac{d^4\overset{1}{v}}{dx^4} + \lambda_c\frac{d^2\overset{1}{v}}{dx^2} + \lambda_c e_z\frac{d^2\overset{1}{\phi}}{dx^2} = 0, \quad \overset{1}{v}(0) = \overset{1}{v}(L) = 0, \quad \frac{d^2\overset{1}{v}}{dx^2}(0) = \frac{d^2\overset{1}{v}}{dx^2}(L) = 0 \\
\delta w &: \quad EI_{yy}\frac{d^4\overset{1}{w}}{dx^4} + \lambda_c\frac{d^2\overset{1}{w}}{dx^2} - \lambda_c e_y\frac{d^2\overset{1}{\phi}}{dx^2} = 0, \quad \overset{1}{w}(0) = \overset{1}{w}(L) = 0, \quad \frac{d^2\overset{1}{w}}{dx^2}(0) = \frac{d^2\overset{1}{w}}{dx^2}(L) = 0 \\
\delta \phi &: \quad E\Gamma\frac{d^4\overset{1}{\phi}}{dx^4} + \lambda_c e_z\frac{d^2\overset{1}{v}}{dx^2} - \lambda_c e_y\frac{d^2\overset{1}{w}}{dx^2} + (\lambda_c\frac{I_p}{A} - GJ)\frac{d^2\overset{1}{\phi}}{dx^2} = 0, \quad \overset{1}{\phi}(0) = \overset{1}{\phi}(L) = 0, \quad \frac{d^2\overset{1}{\phi}}{dx^2}(0) = \frac{d^2\phi}{dx^2}(L) = 0
\end{aligned}
\tag{BA-2.22}
$$

By inspection, the nontrivial solution of interest to the above system is:

$$
\overset{1}{u}(x) = 0, \quad \overset{1}{v}(x) = V\sin\left(\frac{n\pi x}{L}\right), \quad \overset{1}{w}(x) = W\sin\left(\frac{n\pi x}{L}\right), \quad \overset{1}{\phi}(x) = \Phi\sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N} \tag{BA-2.23}
$$

which upon substitution into (BA-2.22) gives the following eigenvalue problem for λ_c:

$$
\begin{aligned}
&\mathbf{S}(n,\lambda_c)\mathbf{U} = 0 \\
&\mathbf{S}(n,\lambda_c) \equiv \begin{bmatrix} EI_{zz}\left(\frac{n\pi}{L}\right)^2 - \lambda_c & 0 & -\lambda_c e_z \\ 0 & EI_{yy}\left(\frac{n\pi}{L}\right)^2 - \lambda_c & \lambda_c e_y \\ -\lambda_c e_z & \lambda_c e_y & E\Gamma\left(\frac{n\pi}{L}\right)^2 + GJ - \lambda_c\frac{I_p}{A} \end{bmatrix}, \quad \mathbf{U} \equiv \begin{bmatrix} V \\ W \\ \Phi \end{bmatrix}
\end{aligned}
\tag{BA-2.24}
$$

The critical load of interest is the lowest value of λ_c, taken over all positive integers n, for which the above matrix $\mathbf{S}(n,\lambda_c)$ loses its positive definiteness. Consequently, the wanted critical load corresponds to $n = 1$ and is the lowest positive real root of the cubic in λ_c equation $\mathrm{Det}\ \mathbf{S}(1,\lambda_c) = 0$:

$$
\begin{aligned}
&(E\Gamma(\pi/L)^2 + GJ - \lambda_c I_p/A)(EI_{zz}(\pi/L)^2 - \lambda_c)(EI_{yy}(\pi/L)^2 - \lambda_c) \\
&-\lambda_c^2[e_y^2(EI_{zz}(\pi/L)^2 - \lambda_c) + e_z^2(EI_{yy}(\pi/L)^2 - \lambda_c)] = 0
\end{aligned}
\tag{BA-2.25}
$$

From (BA-2.23), (BA-2.24) and the normalization requirement for the eigenmode $(\overset{1}{u},\overset{1}{u}) = 1$, according to (AC-2.7), one obtains by recalling the inner product definition for u in (BA-2.17) the following result for the eigenmode $\overset{1}{u}$ that corresponds to the lowest critical load λ_c:

$$
\begin{aligned}
&\overset{1}{u} = (\overset{1}{u}(x), \overset{1}{v}(x), \overset{1}{w}(x), \overset{1}{\phi}(x)) = (0, V, W, \Phi)\sin(\pi x/L) \\
&(V^2 + W^2 + \Phi^2) = 2L/\pi^2, \\
&V = \Phi[\lambda_c e_z/(EI_{zz}(\pi/L)^2 - \lambda_c)], \\
&W = \Phi[-\lambda_c e_y/(EI_{yy}(\pi/L)^2 - \lambda_c)]
\end{aligned}
\tag{BA-2.26}
$$

At this point we would like to show that λ_c is a true bifurcation point and that the principal equilibrium path (BA-2.20) is stable in the interval $[0, \lambda_c)$. To show that λ_c in(BA-2.25) and the eigenmode (BA-2.26) correspond to a bifurcation point (and not to a maximum load) one has first to verify (AC-2.27) i.e. that $\mathcal{E},^c_{u\lambda} \overset{1}{u} = 0$. It is tacitly assumed here that $\overset{1}{u}$ is the unique such eigenmode, or equivalently, that λ_c is a simple root of (BA-2.25). The remaining conditions that guarantee the existance of the bifurcated branch through λ_c will be verified subsequently. From (BA-2.19) and (BA-2.26) one easily shows that:

$$\mathcal{E}^c,_{u\lambda} \overset{1}{u} = \overset{1}{u}(L) = 0 \tag{BA-2.27}$$

The stability of the principal solution for loads $0 \leq \lambda < \lambda_c$ can be shown as follows: in view of the kinematic boundary conditions imposed on the admissible displacement functions in (AC-2.19), any admissible set $\delta v,\ \delta w,\ \delta\phi$ admits the Fourier sine series representation:

$$\delta v = \sum_{n=1}^{\infty} \delta V_n \sin\left(\frac{n\pi x}{L}\right), \quad \delta w = \sum_{n=1}^{\infty} \delta W_n \sin\left(\frac{n\pi x}{L}\right), \quad \delta\phi = \sum_{n=1}^{\infty} \delta \Phi_n \sin\left(\frac{n\pi x}{L}\right) \tag{BA-2.28}$$

Consequently from (BA-2.18), (BA-2.15) and (BA-2.28), the operator $(\mathcal{E},_{uu} (\overset{0}{u}(\lambda), \lambda)\delta u)\delta u$ which determines the stability of the principal branch is found to satisfy:

$$\begin{aligned}
&(\mathcal{E},_{uu} (\overset{0}{u}(\lambda), \lambda)\delta u)\delta u = \int_0^L [EA\left(\frac{d\delta u}{dx}\right)^2]dx + \frac{L}{2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2[(EI_{zz}\left(\frac{n\pi}{L}\right)^2 - \lambda)(\delta V_n)^2 + \\
&(EI_{yy}\left(\frac{n\pi}{L}\right)^2 - \lambda)(\delta W_n)^2 + (E\Gamma\left(\frac{n\pi}{L}\right)^2 + GJ - \lambda\frac{I_p}{A})(\delta\Phi_n)^2 - 2\lambda e_z \delta V_n \delta\Phi_n + 2\lambda e_y \delta W_n \delta \Phi_n] \geq \\
&\overset{0}{\beta}(\lambda)\{\int_0^L [\left(\frac{d\delta u}{dx}\right)^2]dx + \frac{L}{2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2[(\delta V_n)^2 + (\delta W_n)^2 + (\delta\Phi_n)^2]\}
\end{aligned} \tag{BA-2.29}$$

where $\overset{0}{\beta}(\lambda)$ is the lowest eigenvalue of $\mathbf{S}(1, \lambda)$. This inequality can be established by observing first that $(\delta\mathbf{U}_n)^T\mathbf{S}(n, \lambda)\delta\mathbf{U}_n \geq \beta(n, \lambda)(\delta\mathbf{U}_n)^T\delta\mathbf{U}_n$, where $\beta(n, \lambda)$ is the minimum eigenvalue of $\mathbf{S}(n, \lambda)$, and subsequently noting from (BA-2.25) that $\overset{0}{\beta}(\lambda) = \min_{n\in\mathbb{N}}\{\beta(n, \lambda), EA\} = \min\{\beta(1, \lambda), EA\}$. Since for $0 \leq \lambda < \lambda_c$, $\mathbf{S}(1, \lambda)$ is positive definite, $\overset{0}{\beta}(\lambda) > 0$, and the principal solution is a local minimum of the potential energy and hence stable.

Having established that λ_c defined in (BA-2.25) is the lowest critical load and (BA-2.26) is the corresponding eigenmode of the simply supported axially compressed beam, attention is focused on showing by construction the existence of the bifurcated equilibrium branch through λ_c. According to the general theory, the first term λ_1 in the asymptotic expansion of the bifurcated equilibrium path is given by $(AC\ 3.13)$. Starting with the evaluation of the denominator $((d\mathcal{E},_{uu})/d\lambda)_c\overset{1}{u})\overset{1}{u}$, one obtains from (BA-2.21) and (BA-2.26):

$$((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = -\frac{\pi^2}{2L}[V^2 + W^2 + 2e_z\Phi V - 2e_y\Phi W + (I_p/A)\Phi^2] \tag{BA-2.30}$$

The above result could have also been obtained independently from (AC-3.2) and (BA-2.29). Indeed notice the relations $((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = (d\overset{0}{\beta}/d\lambda)_c$ and $(d\overset{0}{\beta}/d\lambda)_c = (d\beta(1, \lambda)/d\lambda)_c$ where $\beta(1, \lambda)$

is the minimum eigenvalue of $\mathbf{S}(1,\lambda)$ defined in (BA-2.24). The fact that $((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} < 0$ can be proved as follows: Recall from the geometry of the cross-section (see FIG. BA-2.1) the relations $r^2 = r_s^2 + r_n^2 = (y-e_y)^2 + (z-e_z)^2$, $r_s \sin\alpha - r_n \cos\alpha = z - e_z$, $r_s \cos\alpha + r_n \sin\alpha = y - e_y$. With the additional help of the definitions in (BA-2.10)-(BA-2.12), one has that for every set of nonzero real constants $a, b, c,$ (independent of y and z):

$$0 < \int_{s_1}^{s_2} [a\cos\alpha + b\sin\alpha + cr_n)^2 + (-a\sin\alpha + b\cos\alpha + cr_s)^2 t ds = A[a^2 + b^2 + 2e_z ac - 2e_y bc + \frac{I_p}{A}c^2] \quad \text{(BA-2.31)}$$

which proves the wanted result.

Finally the numerator $((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}$ in (AC-3.13) is found by evaluating at the critical point $\overset{0}{u}(\lambda_c), \lambda_c$ the third derivative of the potential energy (BA-2.18), (BA-2.15) and subsequently considering the eigenmode (BA-2.26). Combining the thus found result with (BA-2.30), we obtain for λ_1:

$$\lambda_1 = -2\frac{\pi}{L^2}\frac{[2(EI_{yy} - EI_{zz})VW\Phi - EI_{zp}V\Phi^2 - EI_{yp}W\Phi^2 + EI_{\omega p}\Phi^3]}{[V^2 + W^2 + 2e_z\Phi V - 2e_y\Phi W + (I_p/A)\Phi^2]} \quad \text{(BA-2.32)}$$

The fact that for arbitrary cross-sections $\lambda_1 \neq 0$ shows that the bifurcation of the simply supported axially compressed thin walled beam is a transcritical one. Moreover, when λ_c is a simple root of (BA-2.25), the eigenmode is unique. Hence for $\lambda_1\xi > 0$ the bifurcated equilibrium branch is stable, according to the general thory presented in Section AC-3, and unstable for $\lambda_1\xi < 0$.

It should be emphasized at this point that a transcritical bifurcation is possible only for the case of completely asymmetric cross-sections. As long as the cross-section has at least one axis of symmetry, say the z axis, $e_y = I_{zp} = I_{\omega p} = 0$ and hence from (BA-2.26) and (BA-2.32) follows that $\lambda_1 = 0$. For a cross-section with a point of symmetry, one has $e_y = e_z = I_{yp} = I_{zp} = I_{\omega p} = 0$ which also implies from from (BA-2.26) and (BA-2.32) that $\lambda_1 = 0$.

ib) *Clamped, Axially Compressed Thin Walled Beam*

The change of boundary conditions for the problem of the axially compressed beam alters drastically the bifurcation and post bifurcation behavior of the structure. It will be shown that the change of end conditions in the beam from simply supported to fully clamped, not only increases significantly the critical load but it also changes the character of the bifurcation from an asymmetric one (in the simply supported case) to a symmetric one (for the clamped case).

The potential energy for the doubly clamped thin walled beam is given by $(\text{BA-2.18})_1$ but the admissibility conditions for the displacements are different due to new end conditions:

$$\begin{aligned} &u(0) = 0, \quad v(0) = v(L) = 0, \quad w(0) = w(L) = 0, \quad \phi(0) = \phi(L) = 0 \\ &\frac{dv}{dx}(0) = \frac{dv}{dx}(L) = 0, \quad \frac{dw}{dx}(0) = \frac{dw}{dx}(L) = 0, \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0 \end{aligned} \quad \text{(BA-2.33)}$$

Note that the space of admissible displacement functions for the clamped problem is a smaller subset of the corresponding space for the simply supported case. The principal solution is still given by (BA-2.20) for it satisfies the variational form of the equilibrium equations (BA-2.19) subjected to the new boundary conditions (BA-2.33).

Since the expression for the potential energy is identical to the one for the simply supported case, the Euler-Lagrange differential equations determining the critical load and mode are still given by (BA-2.22). In view of the new boundary conditions (BA-2.33), the zero boundary conditions for the second derivatives of $\overset{1}{v}(x), \overset{1}{w}(x), \overset{1}{\phi}(x)$ at $x = 0, L$ are replaced by zero boundary conditions on their corresponding first derivatives.

By inspection the nonzero solution to the aforementioned system of ordinary differential equations (BA-2.22) subjected to the boundary conditions (BA-2.33) is

$$\overset{1}{u}(x) = 0,\ \overset{1}{v}(x) = V(1-\cos\frac{2n\pi x}{L}),\ \overset{1}{w}(x) = W(1-\cos\frac{2n\pi x}{L}),\ \overset{1}{\phi}(x) = \Phi(1-\cos\frac{2n\pi x}{L}),\ n \in \mathbb{N} \quad \text{(BA-2.34)}$$

which upon substitution into the differential equations in (BA-2.22) yields

$$\mathbf{S}(2n, \lambda_c)\mathbf{U} = 0 \quad \text{(BA-2.35)}$$

where the matrix $\mathbf{S}(n, \lambda)$ and the vector $\mathbf{U}$ are defined in (BA-2.24).

The critical load is the lowest positive value λ_c for which the matrix $\mathbf{S}(2n, \lambda)$ looses its positive definiteness when all possible integers n are considered. Consequently the wanted λ_c corresponds to $n = 1$ and is the lowest positive root of the cubic in λ_c equation Det $\mathbf{S}(2, \lambda_c) = 0$:

$$\begin{aligned}&(E\Gamma(2\pi/L)^2 + GJ - \lambda_c I_p/A)(EI_{zz}(2\pi/L)^2 - \lambda_c)(EI_{yy}(2\pi/L)^2 - \lambda_c)\\&-\lambda_c^2[e_y^2(EI_{zz}(2\pi/L)^2 - \lambda_c) + e_z^2(EI_{yy}(2\pi/L)^2 - \lambda_c)] = 0\end{aligned} \quad \text{(BA-2.36)}$$

From (BA-2.34)-(BA-2.36) and the normalization requirement for the eigenmode $(\overset{1}{u}, \overset{1}{u}) = 1$ one obtains, using the inner product definition in (BA-2.17), the following results that correspond to the lowest critical load λ_c:

$$\begin{aligned}&\overset{1}{u} = (\overset{1}{u}(x), \overset{1}{v}(x), \overset{1}{w}(x), \overset{1}{\phi}(x)) = (0, V, W, \Phi)(1 - \cos(2\pi x/L))\\&V^2 + W^2 + \Phi^2 = L/2\pi^2,\\&V = \Phi[\lambda e_z/(EI_{zz}(2\pi/L)^2 - \lambda_c)],\\&W = \Phi[-\lambda e_y/(EI_{yy}(2\pi/L)^2 - \lambda_c)]\end{aligned} \quad \text{(BA-2.37)}$$

Since $\overset{1}{u}(x) = 0$, the necessary condition (AC-2.27) for λ_c to be a bifurcation point is again verified and coincides with (BA-2.27).

The fact that the principal solution is stable in the interval $[0, \lambda_c)$ can be proved in a similar way as for the simply supported case. To this end, notice that any admissible set $\delta v, \delta w, \delta\phi$ can be put in the Fourier series form:

$$\delta v = \sum_{n=1}^{\infty} \delta V_n(1-\cos\frac{2n\pi x}{L}), \quad \delta w = \sum_{n=1}^{\infty} \delta W_n(1-\cos\frac{2\pi nx}{L}), \quad \delta\phi = \sum_{n=1}^{\infty} \delta\Phi_n(1-\cos\frac{2\pi nx}{L}) \quad \text{(BA-2.38)}$$

Upon substitution of (BA-2.38) in $(\mathcal{E},^c_{uu}\delta u)\delta u$ one obtains again the inequality (BA-2.29) but where the quantity $(n\pi/L)$ is replaced by $(2n\pi/L)$. A similar argument to the one presented for (BA-2.29) gives for the quantity $\overset{0}{\beta}(\lambda) = \min_{n\in\mathbb{N}}\{\beta(2n,\lambda), EA\} = \min\{\beta(2,\lambda), EA\}$ where $\beta(n,\lambda)$ is the smallest eigenvalue of $\mathbf{S}(n,\lambda)$ as defined in (BA-2.29). Consequently $\overset{0}{\beta}(\lambda) > 0$ for $0 \le \lambda < \lambda_c$ and hence stability of the principal branch up to the critical load is once again established as expected.

The construction of the bifurcated branch at λ_c is our next goal. For this notice that $((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$ can be found from (BA-2.21) and (BA-2.37):

$$((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = -\frac{2\pi^2}{L}[V^2 + W^2 + 2e_z\Phi V - 2e_y\Phi W + (I_p/A)\Phi^2] \tag{BA-2.39}$$

For the reasons explained in (BA-2.30)-(BA-2.31) we have $(d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} < 0$. Moreover one can also verify that $(d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = (d\overset{0}{\beta}/d\lambda)_c = (d\beta(2,\lambda)/d\lambda)_c$ where $\beta(2,\lambda)$ is the smallest eigenvalue of $\mathbf{S}(2,\lambda)$.

According to the general theory, the first term λ_1 in the asymptotic expansion of the bifurcated equilibrium path trough λ_c is given by (AC-3.13). The denominator in the expression for λ_1 is given in (BA-2.39) while the numerator $((\mathcal{E},^c_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u}$ is found from (BA-2.18) and (BA-2.37) to be:

$$\begin{aligned}((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} \;&=\; 3[2(EI_{yy} - EI_{zz})VW\Phi - EI_{zp}V\Phi^2 - EI_{yp}W\Phi^2 + \\ &\qquad EI_{\omega p}\Phi^3](2\pi/L)^3\textstyle\int_0^L[\sin^2(2\pi x/L)\cos(2\phi x/L)]dx = 0\end{aligned} \tag{BA-2.40}$$

which implies that $\lambda_1 = 0$ and hence the structure will undergo a symmetric bifurcation at λ_c.

The calculation of λ_2 requires the second order term in the asymptotic expansion of the bifurcated displacements which for the symmetric bifurcation case is given by (AC-3.14). The determination of λ_2 requires finding the term $v_{\xi\xi}$ which is given by (AC-3.8). With the help of (BA-2.18), (BA-2.21) and (BA-2.38) one obtains the following variational equation for $v_{\xi\xi} = (\overset{2}{u}(x), \overset{2}{v}(x), \overset{2}{w}(x), \overset{2}{\phi}(x))$:

$$\begin{aligned}
\int_0^L \Big[\quad & EA\frac{d\overset{2}{u}}{dx}\frac{d\delta u}{dx} - \lambda_c\left(\frac{d\overset{2}{v}}{dx}\frac{d\delta v}{dx} + \frac{d\overset{2}{w}}{dx}\frac{d\delta w}{dx} + e_z\left(\frac{d\overset{2}{\phi}}{dx}\frac{d\delta v}{dx} + \frac{d\overset{2}{v}}{dx}\frac{d\delta\phi}{dx}\right) - e_y\left(\frac{d\overset{2}{\phi}}{dx}\frac{d\delta w}{dx} + \frac{d\overset{2}{w}}{dx}\frac{d\delta\phi}{dx}\right)\right) \\
+ \quad & EI_{yy}\frac{d^2\overset{2}{w}}{dx^2}\frac{d^2\delta w}{dx^2} + EI_{zz}\frac{d^2\overset{2}{v}}{dx^2}\frac{d^2\delta v}{dx^2} + E\Gamma\frac{d^2\overset{2}{\phi}}{dx^2}\frac{d^2\delta\phi}{dx^2} - \lambda_c\frac{I_p}{A}\frac{d\overset{2}{\phi}}{dx}\frac{d\delta\phi}{dx} + GJ\frac{d\overset{2}{\phi}}{dx}\frac{d\delta\phi}{dx}\Big]dx \; + \\
\int_0^L \Big[\quad & 2EA\frac{d\overset{1}{u}}{dx}\left(\frac{d\overset{1}{v}}{dx}\frac{d\delta v}{dx} + \frac{d\overset{1}{w}}{dx}\frac{d\delta w}{dx} + e_z\left(\frac{d\overset{1}{\phi}}{dx}\frac{d\delta v}{dx} + \frac{d\overset{1}{v}}{dx}\frac{d\delta\phi}{dx}\right) - e_y\left(\frac{d\overset{1}{\phi}}{dx}\frac{d\delta w}{dx} + \frac{d\overset{1}{w}}{dx}\frac{d\delta\phi}{dx}\right)\right) \\
+ \quad & EA\left(\left(\frac{d\overset{1}{v}}{dx}\right)^2 + \left(\frac{d\overset{1}{w}}{dx}\right)^2 + 2e_z\frac{d\overset{1}{\phi}}{dx}\frac{d\overset{1}{v}}{dx} - 2e_y\frac{d\overset{1}{\phi}}{dx}\frac{d\overset{1}{w}}{dx}\right)\frac{d\delta u}{dx} + 2EI_{yy}\frac{d^2\overset{1}{w}}{dx^2}\left(\frac{d\overset{1}{v}}{dx}\frac{d\delta\phi}{dx} + \frac{d\overset{1}{\phi}}{dx}\frac{d\delta v}{dx}\right) \\
+ \quad & EI_{yy}\left(2\frac{d\overset{1}{v}}{dx}\frac{d\overset{1}{\phi}}{dx}\right)\frac{d^2\delta w}{dx^2} - 2EI_{zz}\frac{d^2\overset{1}{v}}{dx^2}\left(\frac{d\overset{1}{w}}{dx}\frac{d\delta\phi}{dx} + \frac{d\overset{1}{\phi}}{dx}\frac{d\delta w}{dx}\right) - EI_{zz}\left(2\frac{d\overset{1}{w}}{dx}\frac{d\overset{1}{\phi}}{dx}\right)\frac{d^2\delta v}{dx^2} \\
+ \quad & 2EI_p\frac{d\overset{1}{u}}{dx}\frac{d\overset{1}{\phi}}{dx}\frac{d\delta\phi}{dx} + EI_p\left(\frac{d\overset{1}{\phi}}{dx}\right)^2\frac{d\delta u}{dx} - 2EI_{yp}\frac{d^2\overset{1}{w}}{dx^2}\frac{d\overset{1}{\phi}}{dx}\frac{d\delta\phi}{dx} - EI_{yp}\left(\frac{d\overset{1}{\phi}}{dx}\right)^2\frac{d^2\delta w}{dx^2} \\
- \quad & 2EI_{zp}\frac{d^2\overset{1}{v}}{dx^2}\frac{d\overset{1}{\phi}}{dx}\frac{d\delta\phi}{dx} - EI_{zp}\left(\frac{d\overset{1}{\phi}}{dx}\right)^2\frac{d^2\delta v}{dx^2} + EI_{\omega p}\frac{d^2\overset{1}{\phi}}{dx^2}\frac{d\overset{1}{\phi}}{dx}\frac{d\delta\phi}{dx} + EI_{\omega p}\left(\frac{d\overset{1}{\phi}}{dx}\right)^2\frac{d^2\delta\phi}{dx^2}\Big]dx = 0
\end{aligned} \tag{BA-2.41}$$

Integration of (BA-2.41) by parts and use of (BA-2.37) results in the following Euler-Lagrange equations and boundary conditions for $\overset{2}{u}(x)$, $\overset{2}{v}(x)$, $\overset{2}{w}(x)$ and $\overset{2}{\phi}(x)$:

$$\begin{aligned}
\delta u \;:\quad & EA\left(\frac{d\overset{2}{u}}{dx} + \left(\frac{d\overset{1}{w}}{dx}\right)^2 + \left(\frac{d\overset{1}{v}}{dx}\right)^2 + 2e_z\frac{d\overset{1}{\phi}}{dx}\frac{d\overset{1}{v}}{dx} - 2e_y\frac{d\overset{1}{\phi}}{dx}\frac{d\overset{1}{w}}{dx} + \frac{I_p}{A}\left(\frac{d\overset{1}{\phi}}{dx}\right)^2\right) = 0\;; \quad \overset{2}{u}(0) = 0 \\
\delta v \;:\quad & EI_{zz}\frac{d^4\overset{2}{v}}{dx^2} + \lambda_c\frac{d^2\overset{2}{v}}{dx^2} + \lambda_c e_z\frac{d^2\overset{2}{\phi}}{dx^2} = 2EI_{yy}\frac{d}{dx}\left(\frac{d^2\overset{1}{w}}{dx^2}\frac{d\overset{1}{\phi}}{dx}\right) + 2EI_{zz}\frac{d^2}{dx^2}\left(\frac{d\overset{1}{w}}{dx}\frac{d\overset{1}{\phi}}{dx}\right) + EI_{zp}\frac{d^2}{dx^2}\left(\frac{d\overset{1}{\phi}}{dx}\right)^2\;; \\
& \overset{2}{v}(0) = \overset{2}{v}(L) = \frac{d\overset{2}{v}}{dx}(0) = \frac{d\overset{2}{v}}{dx}(L) = 0 \\
\delta w \;:\quad & EI_{yy}\frac{d^4\overset{2}{w}}{dx^4} + \lambda_c\frac{d^2\overset{2}{w}}{dx^2} - \lambda_c e_y\frac{d^2\overset{2}{\phi}}{dx^2} = -2EI_{zz}\frac{d}{dx}\left(\frac{d^2\overset{1}{v}}{dx^2}\frac{d\overset{1}{\phi}}{dx}\right) - 2EI_{yy}\frac{d^2}{dx^2}\left(\frac{d\overset{1}{v}}{dx}\frac{d\overset{1}{\phi}}{dx}\right) + EI_{yp}\frac{d^2}{dx^2}\left(\frac{d\overset{1}{\phi}}{dx}\right)^2\;; \\
& \overset{2}{w}(0) = \overset{2}{w}(L) = \frac{d\overset{2}{w}}{dx}(0) = \frac{d\overset{2}{w}}{dx}(L) = 0 \\
\delta\phi \;:\quad & E\Gamma\frac{d^4\overset{2}{\phi}}{dx^4} + \lambda_c e_z\frac{d^2\overset{2}{v}}{dx^2} - \lambda_c e_y\frac{d^2\overset{2}{w}}{dx^2} + \left(\lambda_c\frac{I_p}{A} - GJ\right)\frac{d^2\overset{2}{\phi}}{dx^2} = 2EI_{yy}\frac{d}{dx}\left(\frac{d^2\overset{1}{w}}{dx^2}\frac{d\overset{1}{v}}{dx}\right) - 2EI_{zz}\frac{d}{dx}\left(\frac{d^2\overset{1}{v}}{dx^2}\frac{d\overset{1}{w}}{dx}\right) \\
& -2EI_{yp}\frac{d}{dx}\left(\frac{d^2\overset{1}{w}}{dx^2}\frac{d\overset{1}{\phi}}{dx}\right) - 2EI_{zp}\frac{d}{dx}\left(\frac{d^2\overset{1}{v}}{dx^2}\frac{d\overset{1}{\phi}}{dx}\right)\;; \quad \overset{2}{\phi}(0) = \overset{2}{\phi}(L) = \frac{d\overset{2}{\phi}}{dx}(0) = \frac{d\overset{2}{\phi}}{dx}(L) = 0
\end{aligned} \tag{BA-2.42}$$

Noting from (BA-2.26) that the right hand sides of the last three differential equations in

(BA-2.42) are proportional to $\cos(4\pi x/l)$, $\overset{2}{u}$, $\overset{2}{v}$, $\overset{2}{w}$ and $\overset{2}{\phi}$ are found to be:

$$\begin{aligned}&\overset{2}{u}(x) = -[V^2 + W^2 + 2e_z\Phi V - 2e_y\Phi W + (I_p/A)\Phi^2](\pi/2L)[(4\pi x/L) - \sin(4\pi x/L)] \\ &\overset{2}{v}(x) = \widehat{V}(1 - \cos(4\pi x/L)), \quad \overset{2}{w}(x) = \widehat{W}(1 - \cos(4\pi x/L)), \quad \overset{2}{\phi}(x) = \widehat{\Phi}(1 - \cos(4\pi x/L))\end{aligned} \tag{BA-2.43}$$

where the constants $\widehat{V}$, $\widehat{W}$ and $\widehat{\Phi}$ can be found in terms of V, W, Φ by using (BA-2.37) and (BA-2.43) into (BA-2.42):

$$\mathbf{S}(4n, \lambda_c)\widehat{\mathbf{U}} = \mathbf{F}; \quad \widehat{\mathbf{U}} \equiv \begin{bmatrix} \widehat{V} \\ \widehat{W} \\ \widehat{\Phi} \end{bmatrix}, \quad \mathbf{F} \equiv 2\left(\frac{\pi}{L}\right)^2 \begin{bmatrix} (EI_{yy} + 2EI_{zz})W\Phi + EI_{zp}\Phi^2 \\ -(EI_{zz} + 2EI_{yy})V\Phi + EI_{yp}\Phi^2 \\ (EI_{yy} - EI_{zz})VW - EI_{yp}W\Phi - EI_{zp}V\Phi \end{bmatrix} \tag{BA-2.44}$$

In the above equation V, W, Φ are given by (BA-2.37) and $\mathbf{S}(n, \lambda)$ is defined in (BA-2.24). Notice that the solution to (BA-2.44) always exists and is unique since the matrix $\mathbf{S}(4n, \lambda_c)$ is positive definite.

Note that (BA-2.43)-(BA-2.44) give the wanted $v_{\xi\xi}$ term which is required for calculating the numerator of λ_2 in (AC-3.14). Moreover, one should also remark that $v_{\xi\xi}$ is orthogonal to the eigenmode $\overset{1}{u}$, as one can easily verify from (BA-2.37) and (BA-2.43) according to the requirement of the general theory that $v_{\xi\xi} \in \mathcal{N}^\perp$ i.e. $(v_{\xi\xi}, \overset{1}{u}) = 0$. By using (BA-2.37) and (BA-2.43) into the general expression (AC-3.14) for λ_2 one has:

$$\begin{aligned}\lambda_2 \; &= 8(\pi/L)^2[(3/8)(EI_{pp} - E((I_p)^2/A)\Phi^4 + EI_{zz}((3/2)W^2\Phi^2 - VW\widehat{\Phi} - V\Phi\widehat{W} + 2W\Phi\widehat{V}) \\ &+EI_{yy}((3/2)V^2\Phi^2 + VW\widehat{\Phi} + W\Phi\widehat{V} - 2V\Phi\widehat{W}) - EI_{yp}((3/4)V\Phi^3 + W\Phi\widehat{\Phi} - \Phi^2\widehat{W}) \\ &-EI_{zp}(-(3/4)W\Phi^3 + V\Phi\widehat{\Phi} - \Phi^2\widehat{V})] \;/\; [V^2 + W^2 + 2e_zV\Phi - 2e_yW\Phi + (I_p/A)\Phi^2]\end{aligned} \tag{BA-2.45}$$

where the coefficients $\widehat{V}$, $\widehat{W}$, $\widehat{\Phi}$ and V, W, Φ are given by (BA-2.37) and (BA-2.44) respectively.

According to the general theory (see Section AC-3) and assuming that λ_2 is nonzero, the symmetric bifurcated branch is stable if $\lambda_2 > 0$ and unstable for $\lambda_2 < 0$.

ii) *Model Derivation - Imperfect Case*

All the results obtained so far in this Section concern the perfect thin walled beam. In two instances, namely for the simply supported beam where $\lambda_1 \neq 0$, and for the clamped beam when $\lambda_2 < 0$, a snap-through instability has to be expected according to general theory (see Section AC-3) in the presence of imperfections. To find the corresponding load reduction $\Delta\lambda_s$ which is of practical interest for imperfection sensitive structures, we start by presenting the imperfect thin walled beam model.

To this end it will be assumed that in the stress-free state of the thin walled beam, the line joining the shear centers of all the cross sections is displaced, with respect to its perfect counterpart, by $\overline{v}(x)$ and $\overline{w}(x)$ in the y and z direction while the principal axes of the cross section at x are rotated with respect to their position in the perfect structure by $\overline{\phi}(x)$. Consequently, the net axial strain $\overline{\epsilon}$ which is defined as the difference of the axial elongations in the current and reference state, is found from (BA-2.8) to be

$$\begin{aligned}
\overline{\epsilon} \quad &\equiv \epsilon(u, v+\overline{v}, w+\overline{w}, \phi+\overline{\phi}) - \epsilon(0, \overline{v}, \overline{w}, \overline{\phi}) = \overline{e} + y\overline{k}_y + z\overline{k}_z - 2\omega\frac{d^2\phi}{dx^2} + [(y-e_y)^2 + (z-e_z)^2]\overline{\psi} \\
\overline{e} \quad &\equiv \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + \frac{dv}{dx}\frac{d\overline{v}}{dx} + \frac{dw}{dx}\frac{d\overline{w}}{dx} \\
&+e_z\left(\frac{dv}{dx}\frac{d\phi}{dx} + \frac{dv}{dx}\frac{d\overline{\phi}}{dx} + \frac{d\phi}{dx}\frac{d\overline{v}}{dx}\right) - e_y\left(\frac{dw}{dx}\frac{d\phi}{dx} + \frac{dw}{dx}\frac{d\overline{\phi}}{dx} + \frac{d\phi}{dx}\frac{d\overline{w}}{dx}\right), \quad \overline{\psi} \equiv \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{d\phi}{dx}\frac{d\overline{\phi}}{dx} \\
\overline{k}_y \quad &\equiv -\frac{d^2v}{dx^2} + \frac{dw}{dx}\frac{d\phi}{dx} + \frac{dw}{dx}\frac{d\overline{\phi}}{dx} + \frac{d\phi}{dx}\frac{d\overline{w}}{dx}, \quad \overline{k}_z \equiv -\frac{d^2w}{dx^2} - \frac{dv}{dx}\frac{d\phi}{dx} - \frac{dv}{dx}\frac{d\overline{\phi}}{dx} - \frac{d\phi}{dx}\frac{d\overline{v}}{dx}
\end{aligned} \tag{BA-2.46}$$

where $u(x),\ v(x),\ w(x),\ \phi(x)$ are the net displacements and rotation of each section with respect to the imperfect stress-free configuration.

The internal strain energy of the imperfect beam is still given by (BA-2.9) in which the perfect structure's axial strain ϵ is replaced by its imperfect counterpart $\overline{\epsilon}$ given in (BA-2.46). Substitution of (BA-2.46), $(\text{BA-2.8})_2$ into (BA-2.9) and after also taking again into account (BA-2.10)-(BA-2.13), yields the following expression for the imperfect structure's internal energy:

$$\begin{aligned}
\overline{\mathcal{E}}_{int} = \frac{1}{2}\int_0^L \quad &[EA(\overline{e})^2 + EI_{zz}(\overline{k}_y)^2 + EI_{yy}(\overline{k}_z)^2 + E\Gamma\left(\frac{d^2\phi}{dx^2}\right)^2 + 2GJ\psi + \\
&2EI_p\overline{e}\overline{\psi} + 2EI_{zp}\overline{k}_y\overline{\psi} + 2EI_{yp}\overline{k}_z\overline{\psi} + 2EI_{\omega p}\frac{d^2\phi}{dx^2}\overline{\psi} + EI_{pp}(\overline{\psi})^2]dx
\end{aligned} \tag{BA-2.47}$$

where $\overline{e},\ \overline{k}_y,\ \overline{k}_z,\ \overline{\psi}$ are given in $(BA\ 2.46)$. As expected for $\overline{v} = \overline{w} = \overline{\phi} = 0$, the imperfect structure's strains $\overline{e},\ \overline{k}_y,\ \overline{k}_z,\ \overline{\psi}$ reduce to (BA-2.15) and hence one recovers the perfect structure's potential energy in (BA-2.14).

The external potential energy in view of its linear dependent on $u,\ v,\ w$, and ϕ remains the same as for the perfect case (see (BA-2.16)). One should also mention at this point that the imperfection functions $\overline{v},\ \overline{w},\ \overline{\phi}$ have to satisfy the same admissibility conditions of v, w, and ϕ.

iia) *Simply Supported Imperfect Beam.*

Of interest in this case, where $\lambda_1 \neq 0$, is the critical load reduction $\Delta\lambda_s$ whose general expression is given by $(\text{AC-4.31})_1$. Since λ_1 and $((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$ have already been calculated in (BA-2.32) and (BA-2.30), the only quantity that remains to be computed is $(\overline{\mathcal{E}}^{\,c}_{,uw}\,\overline{w})\overset{1}{u}$, which from (BA-2.47), (BA-2.46) and after making also use of the principal solution (BA-

2.20), is found to be

$$(\overline{\mathcal{E}},^c_{uw}\,\overline{w})\overset{1}{u} = -\lambda_c \int_0^L [\frac{d\overset{1}{v}}{dx}\frac{d\overline{v}}{dx} + \frac{d\overset{1}{w}}{dx}\frac{d\overline{w}}{dx} + e_z\left(\frac{d\overset{1}{v}}{dx}\frac{d\overline{\phi}}{dx} + \frac{d\overset{1}{\phi}}{dx}\frac{d\overline{v}}{dx}\right) - e_y\left(\frac{d\overset{1}{w}}{dx}\frac{d\overline{\phi}}{dx} + \frac{d\overset{1}{\phi}}{dx}\frac{d\overline{w}}{dx}\right) + \frac{I_p}{A}\frac{d\overset{1}{\phi}}{dx}\frac{d\overline{\phi}}{dx}]dx \quad \text{(BA-2.48)}$$

with the mode $(\overset{1}{u}, \overset{1}{v}, \overset{1}{w}, \overset{1}{\phi})$ given for the case of the simply supported beam by (BA-2.26).

The obvious question that comes to mind is to find among all the possible imperfection shapes $\overline{v}, \overline{w}, \overline{\phi}$ the one that maximizes the expression $|(\overline{\mathcal{E}}^2_{,uw}\overline{w})\overset{1}{u}|$ in (BA-2.48) and which consequently produces the maximum possible critical load reduction $\Delta\lambda_s$ according to $(\text{AC-4.31})_1$.

To this end, one remarks the bilinear form $-(\overline{\mathcal{E}},^c_{uw}\,\Delta u)\delta u$ in (BA-2.48) is an inner product in the space of all admissible functions $\delta u = (\delta u, \delta v, \delta w, \delta\phi)$ since $-(\overline{\mathcal{E}},^c_{uw}\,\delta u)\delta u > 0$, according to (BA-2.31) - easy to verify if $a,\ b,\ c$ are replaced by $d(\delta v)/dx,\ d(\delta w)/dx,\ d(\delta\phi)/dx$. Consequently, and since from (AC-4.5) the norm of the imprefection $\overline{w}$ is assumed to be unity, the absolute value of the inner product in (BA-2.48) is maximized when $\overline{w} = \overset{1}{u}$ (recall from (AC-2.7) that the eigenmode $\overset{1}{u}$ is also taken to have a unit norm). Noting from (BA-2.48) for $\overline{w} = \overset{1}{u}$ and from (BA-2.26), (BA-2.30) $(\overline{\mathcal{E}}^c_{,uw}\overline{w})\overset{1}{u} = \lambda_c((d\,\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$, one concludes according to (AC-4.31) that the maximum possible load drop for the simply supported beam is

$$(\Delta\lambda_s)_{\max} = -2(-\lambda_1\lambda_c\epsilon)^{1/2} + O(\epsilon) \quad \text{(BA-2.49)}$$

where λ_c and λ_1 are given by (BA-2.25) and (BA-2.32) respectively. In the above expression, it is tacitly assumed that $\lambda_2\epsilon < 0$.

iib) *Fully clamped imperfect beam*

The asymptotic analysis of the fully clamped imperfect thin walled beam parallels closely the corresponding analysis of the simply supported imperfect case. Following exactly the same reasoning, one can show that $(\overline{\mathcal{E}}^c_{,uw}\overline{w})\overset{1}{u}$ is still given by (BA-2.48) (with the only difference this time being that the eigenmode $\overset{1}{u}$ is found in (BA-2.37)) and that consequently this inner product is again maximized for $\overline{w} = \overset{1}{u}$.

Consequently once again $(\overline{\mathcal{E}}^c_{,uw}\overset{1}{u})\overset{1}{u} = \lambda_c((d\ \mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$ and from $(\text{AC-4.31})_2$ the maximum load reduction due to the presence of an imperfection of amplitude ϵ is given by

$$(\Delta\lambda_s)_{\max} = \frac{3}{2}(\lambda_2)^{1/3}(-\lambda_c\epsilon)^{2/3} + O(\epsilon) \quad \text{(BA-2.50)}$$

where it is tacitly assumed that $\lambda_2 < 0$ for the structure to be imperfection sensitive.

iii) *Examples*

The general theory developed in this section for thin walled beams of arbitrary cross-section will be applied to two beams: one with an asymmetric L section, the other with a symmetric open circular section.

iiia) *Beam with an L section*

The beam is made up of two rectangular flanges perpendicular to each other, one with cross-section $a \times 2t$ and the other with cross-section $2a \times t$ as shown in FIG. BA-2.2. Since the flanges are thin, i.e. $t \ll a$, only the lowest term in t will be considered in the geometric quantities of the section defined in (BA-2.13).

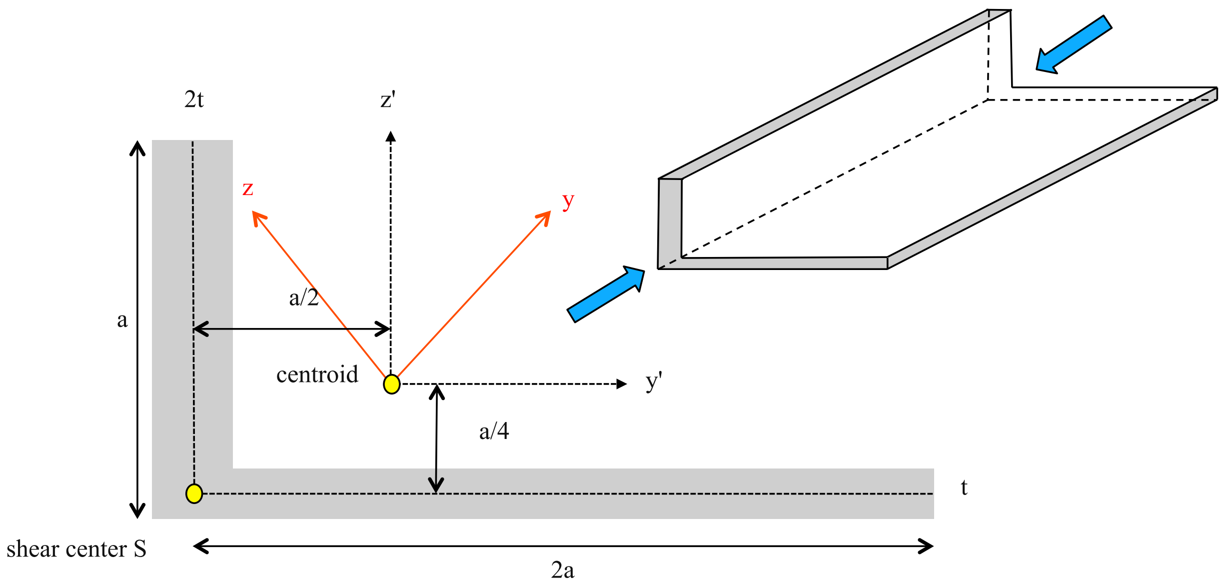

Figure BA-2.2: L-shaped thin walled cross-section. Schematic 3D representation of the axially compressed L-shaped thin walled section beam is shown in the right.

The shear center of the cross-section is obviously the corner point S, since $\omega(s) = 0$ with respect to S and consequently (BA-2.11) is automatically satisfied. The various geometric constants of the section will be calculated first with respect to the (y', z') system of axes which is parallel to the flanges and will be denoted by an additional prime ($'$) in their respective symbols. The same quantities without the prime correspond to the principal axes (y, z) which will be subsequently determined.

Starting with finding the centroid of the cross-section one can easily show that

$$e'_y = -a/2, \quad e'_z = a/4 \tag{BA-2.51}$$

Using the definitions given in $(BA\ 2.13)$ for the various geometric quantities of the cross-section and recalling that $t \ll a$, we obtain the following results for the (y', z') coordinate system

$$I'_{yy} = (5/12)a^3 t, \quad I'_{zz} = (5/3)a^3 t, \quad I'_{yz} = (1/2)a^3 t, \quad I'_{yp} = (1/3)a^4 t, \quad I'_{zp} = (7/3)a^4 t \tag{BA-2.52}$$

The angle θ formed between y' and y, where (y, z) is the principal axes system, is found from the requirement $I_{yz} = 0$, which gives

$$\tan 2\theta = 2I'_{yz}/(I'_{zz} - I'_{yy}) = 4/5 \tag{BA-2.53}$$

and hence consequently by a simple change of coordinate transformation which gives $e_y = e'_y \cos\theta + e'_z \sin\theta$, $e_z = -e'_y \sin\theta + e'_z \cos\theta$ and $I_{yy} = (1/2)(I'_{yy} + I'_{zz}) + (1/2)(I'_{yy} - I'_{zz})\cos 2\theta - I'_{yz}\sin 2\theta$, $I_{zz} = (1/2)(I'_{yy} + I'_{zz}) + (1/2)(I'_{zz} - I'_{yy})\cos 2\theta + I'_{yz}\sin 2\theta$, $I_{yp} = I'_{yp}\cos\theta - I'_{zp}\sin\theta$, $I_{zp} = I'_{yp}\sin\theta + I'_{zp}\cos\theta$, we obtain in the principal axes system

$$e_y = -0.389062a, \quad e_z = 0.401411a \tag{BA-2.54}$$

$$I_{yy} = 0.241276a^3t, \quad I_{zz} = 1.842057a^3t, \quad I_{yp} = -0.457807a^4t, \quad I_{zp} = 2.312134a^4t \tag{BA-2.55}$$

The remaining geometric quantities of the cross-section are easily computed from their respective definitions in (BA-2.13) and are found to be

$$A = 4at, \quad \Gamma = I_{wp} = 0, \quad J = (10/3)at^3, \quad I_p = (10/3)a^3t, \quad I_{pp} = (34/5)a^5t \tag{BA-2.56}$$

The beam is of length $L = 10\pi a$ and the flange thickness of the thinner flange $t = a/10$. The material's Poisson ratio $\nu = 1/4$ and consequently the shear modulus G is given in terms of Young's modulus E by $G = E/[2(1+\nu)] = (2/5)E$

$$L = 10\pi a, \quad t = a/10; \quad \nu = 1/4, \quad G = (2/5)E \tag{BA-2.57}$$

Two cases are distinguished, according to the boundary conditions at the ends $x = 0, L$ of the beam.

iiia.a) *Simply supported boundary conditions*

According to (BA-2.25), the critical load λ_c is the minimum positive root of the cubic equation which results by substitution of (BA-2.55)-(BA-2.57) into (BA-2.25)

$$\lambda_c = 0.233962 \times 10^{-3} Ea^2 = 0.584905 \times 10^{-3} EA \tag{BA-2.58}$$

The above found critical load corresponds to a simple root of the cubic equation. Also notice that $\lambda_c/EA = e_c$ is the axial strain of the beam at bifurcation. In this example, $e_c = 0.584905 \times 10^{-3}$ falls within the elastic range of most structural metals (e.g., steel, aluminum). From (BA-2.26) the amplitudes V, W, Φ of the eigenmode $\overset{1}{u}$ are found to be

$$V = 5.840135 \times 10^{-2}\Phi\alpha, \quad W = 12.445409\,\Phi\alpha, \quad \Phi = 1.09946 \tag{BA-2.59}$$

The calculation of Φ in $(BA\ 2.59)$ was based on the normalization condition $-((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = L$ (see (BA-2.30)) instead on the inner product (BA-2.17) (see (BA-2.26)). Hence the first term λ_1 in the asymptotic expansion of the bifurcated equilibrium load is found to be from (BA-2.32) with the help of (BA-2.55) and (BA-2.59)

$$\lambda_1 = -0.136879 \times 10^{-4} Ea^2 \tag{BA-2.60}$$

From the design perspective, the maximum possible load drop for an imperfection of amplitude ϵ is important. Using (BA-2.58), (BA-2.60) into (BA-2.49) we find

$$(\Delta\lambda_s)_{\max}/\lambda_c = -0.483756\,\epsilon^{1/2} + O(\epsilon) \tag{BA-2.61}$$

The above result shows the sensitivity of the maximum attainable load of the structure to the size of imperfections. For a small imperfection $\epsilon = 10^{-2}$, well within the order encountered in structural applications $(\Delta\lambda_c)_{\max}/\lambda_c \approx -5\%$ while for $\epsilon = 0.1$ we find $(\Delta\lambda_s)_{\max}/\lambda_c \approx -15\%$. Since the numerical values in this example are typical of structural applications, the imperfection sensitivity results obtained give a quantitative picture of the importance of imperfections.

iiia.b) *Clamped boundary conditions*

For this case, the critical load λ_c is the minimum positive root of the cubic in λ equation which results upon substitution of (BA-2.55)-(BA-2.57) into (BA-2.36)

$$\lambda_c = 0.81029 \times 10^{-3} Ea^2 = 2.02572 \times 10^{-3} EA \tag{BA-2.62}$$

The above critical load also corresponds to a simple root of the cubic equation. As expected, in view of the stiffer boundary conditons the critical load is higher than its counterpart for the simply supported case found in (BA-2.58). In addition the critical strain $e_c = \lambda_c/EA = 2.02572 \times 10^{-3}$, which is a typical value for elastic limit strain in most structural metals (usually around 0.2%). From (BA-2.37), the amplitudes (V, W, Φ) of the eigenmode $\overset{1}{u}$ are found to be

$$V = 4.95977 \times 10^{-2} \Phi a, \quad W = 2.03633\,\Phi a, \quad \Phi = 2.751 \tag{BA-2.63}$$

where once more the normalization condition $-((d\mathcal{E}_{,uu}\ \ d\lambda)_c \overset{1}{u})\overset{1}{u} = L$ has been used for the calculation of Φ (see (BA-2.39)) instead of the one based on the inner product (BA-2.17) (see (BA-2.37)). From (BA-2.44) and with the help of (BA-2.55)-(BA-2.57), (BA-2.62)-(BA-2.63) one finds

$$\widehat{V} = 0.745137\,\Phi^2 a, \quad \widehat{W} = -0.13856\,\Phi^2 a, \quad \widehat{\Phi} = 2.29523\,\Phi^2 \tag{BA-2.64}$$

Thus the first non-zero term λ_2 in the expansion of the bifurcated equilibrium load is calculated from (BA-2.45) with the help of (BA-2.55)-(BA-2.57) and (BA-2.63)-(BA-2.64) to be

$$\lambda_2 = 0.236236 \times 10^{-4} Ea^2 \tag{BA-2.65}$$

since $\lambda_2 > 0$, the clamped L beam is imperfection insensitive.

iiib) *Beam with an open circular section*

In this example the cross-section of the beam is a circular one of radius a and the thickness t with a small cut at $\theta = \pi$ as shown in FIG. BA-2.3. Again in all calculations $t \ll a$. Both ends will be assumed clamped. From the symmetry of the cross-section with respect to the z axis (the axis passing through the cut as shown in FIG. BA-2.3) follows that (y, z) are the principal axes with z . The shear center lies on the symmetry axis $(e_y = 0)$ at a distance e_z from the center of mass which coincides with the center of the cross section.

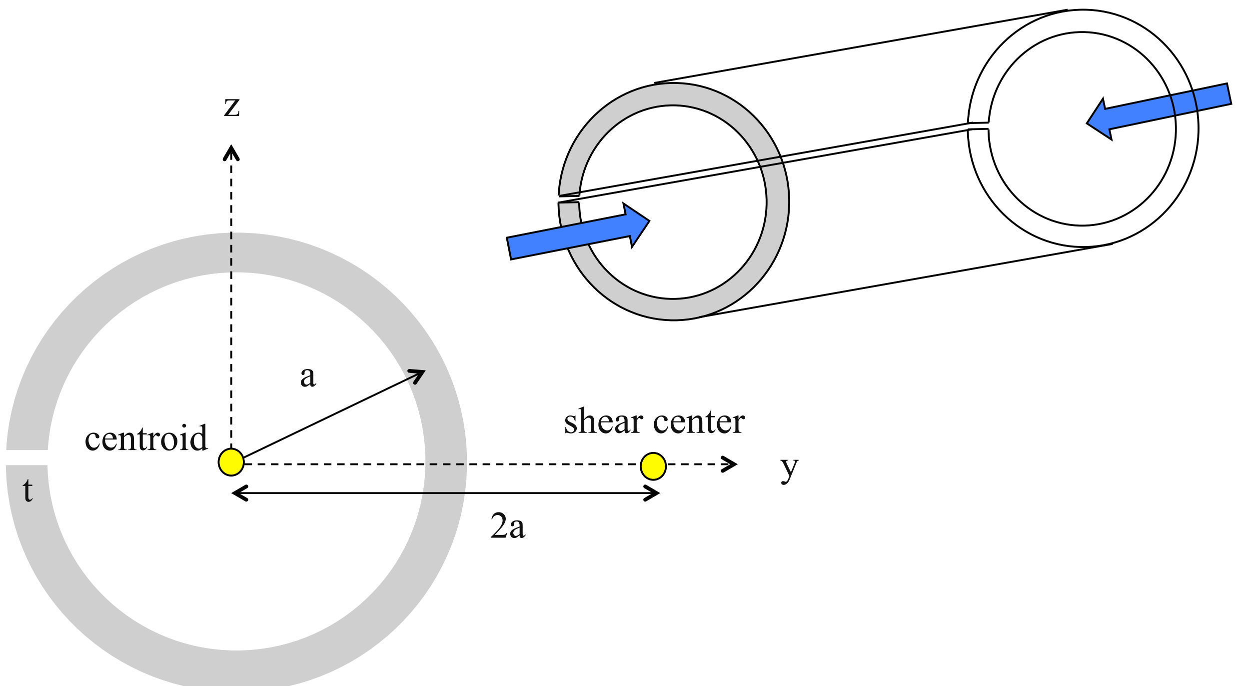

Figure BA-2.3: Open circular thin walled section. Schematic 3D representation of the axially compressed open circular thin walled section beam is shown in the right.

From the definition of the shear center in (BA-2.11) and noticing from geometry $2\omega = a(a\theta - e_z \sin\theta)$

$$\int_{-\pi}^{\pi} (-a\sin\theta)(a^2\theta - ae_z \sin\theta)tad\theta = 0 \quad \Longrightarrow \quad e_z = 2a \tag{BA-2.66}$$

The various geometric quantities of the cross-section defined in (BA-2.13) are found by a straightforward evaluation of the related integrals to be (notice $y = -a\sin\theta$, $z = a\cos\theta$, $2\omega = a^2(\theta - 2\sin\theta)$, $r^2 = y^2 + (z - e_z)^2$)

$$\begin{aligned} &A = 2\pi at, \quad I_{yy} = I_{zz} = \pi a^3 t, \quad \Gamma = (2/3)\pi(\pi^2 - 6)a^5 t, \quad I_p = 10\pi a^3 t \\ &J = (2/3)\pi a t^3, \quad I_{yp} = -4\pi a^4 t, \quad I_{zp} = I_{\omega p} = 0, \quad I_{pp} = 66\pi a^5 t \end{aligned} \tag{BA-2.67}$$

The circular beam is taken to be made of the same material and have the same length as the L section beam but has smaller thickness

$$L = 10\pi a, \quad t = a/10\pi; \quad \nu = 1/4, \quad G = (2/5)E \tag{BA-2.68}$$

Substitution of (BA-2.66)-(BA-2.68) into the critical load equation for the clamped beam (BA-2.36) gives the following result for the lowest positive root of the cubic in λ_c equation

$$\lambda_c = 1.431238 \times 10^{-3} Ea^2 \tag{BA-2.69}$$

From (BA-2.37) the amplitudes V, W, Φ of the corresponding eigenmode are found with the help of (BA-2.67)-(BA-2.69) to be

$$V = 1.114341\,\Phi a, \quad W = 0, \quad \Phi = 2.161777 \tag{BA-2.70}$$

where, as for the L section beam the normalization condition $-((d\mathcal{E}_{,uu}/d\lambda)_c \overset{1}{u})\overset{1}{u} = L$ has been employed in determining Φ (see (BA-2.39)). Notice that the lowest critical mode is a mixed torsion-bending mode since $V, \Phi \neq 0$ from $(BA\ \ 2.70)$ with bending in the (x, y) plane only. From (BA-2.44) and with the help of (BA-2.66)-(BA-2.69) one finds

$$\widehat{V} = 0, \quad \widehat{W} = -1.00805\,\Phi^2 a, \quad \widehat{\Phi} = 0 \tag{BA-2.71}$$

Consequently, the first non-ten term λ_2 in the asymptotic expansion of the bifurcated equilibrium load is found by substituting (BA-2.67)-(BA-2.68) and (BA-2.70)-(BA-2.71) into (BA-2.45) to be

$$\lambda_2 = 0.065022\,Ea^2 \tag{BA-2.72}$$

Since $\lambda_2 > 0$, the clamped open circular section beam is imperfection insensitive.

BA-3 CIRCULAR ARCH

The third application involving one dimensional structures pertains to the bifurcation of a doubly clamped planar beam in the form of a circular arch which is subjected to a uniform hydrostatic pressure. A particular feature of this problem which distinguishes it from the previous examples is that its principal solution possesses limit points. Depending on the slenderness of the arch, a bifurcated solution is also possible at loads inferior to the first limit load. A complete analytical solution of this problem can be found both for the principal as well as for the bifurcated solution which will be compared with the asymptotic solution. Unlike the previous applications, the prebifurcation solution has spatially varying axial and bending strains, a fact that complicates the investigation of stability of the solution. Following the description of the perfect arch model the corresponding principal and bifurcated solutions will be derived in analytical form. The asymptotic theory results will be subsequently derived and compared to the analytical solutions. The imperfect arch model's exact analytical and asymptotic solutions will be subsequently discussed.

i) *Perfect Model Description*

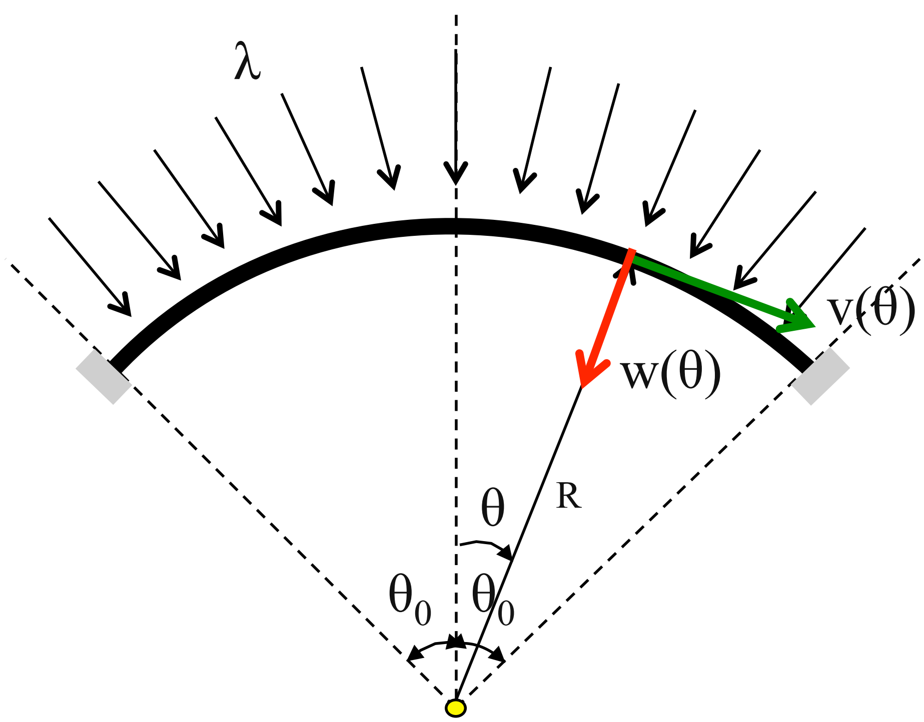

Figure BA-3.1: Shallow arch of radius R with total angle $2\theta_0$, clamped at both ends and subjected to a uniform pressure λ. The kinematic variables are the tangential $v(\theta)$ and normal $w(\theta)$ displacements.

Consider a planar circular beam of constant thickness h, whose center line has a radius of curvature R and whose length is $2R\theta_0$. The beam's cross section has an area A and a moment of inertia I. Small elastic strains and moderate rotations will be assumed. Consequently, if $v(\theta)$ and $w(\theta)$ are the tangential and normal displacements respectively of the arch shown in FIG. BA-3.1, the axial strain ϵ and the bending strain κ are related to those displacements by:

$$\epsilon = \Big(\frac{dv}{d\theta} - w\Big)/R + \Big(\frac{dw}{d\theta}\Big)^2/2R^2, \quad \kappa = \frac{d^2w}{d\theta^2}/R^2 \tag{BA-3.1}$$

The axial force resultant N and the bending moment M depend linearly on the corresponding small strains ϵ and k

$$N = EA\epsilon \ , \quad M = EI\kappa \tag{BA-3.2}$$

where as usual E denotes the material's Young modulus.

The potential energy $\mathcal{E}$ of the pressurized circular arc shown in FIG. BA-3.1 is therefore given by

$$\begin{gathered}\mathcal{E} = \tfrac{1}{2} \int_{-\theta_0}^{\theta_0} (EA\epsilon^2 + EIk^2)Rd\theta - \lambda \int_{-\theta_0}^{\theta_0} wRd\theta \\ v(-\theta_0) = v(\theta_0) = 0, \quad w(-\theta_0) = w(\theta_0) = 0, \quad \frac{dw}{d\theta}(-\theta_0) = \frac{dw}{d\theta}(\theta_0) = 0\end{gathered} \tag{BA-3.3}$$

where λ is the hydrostatic pressure acting on the beam.

In the experiment considered here, unless otherwise stated, the prescribed load parameter is the hydrostatic pressure λ. Its work conjugate quantity Λ is the displaced volume of the arch, which according to (BA-3.3) is

$$\Lambda = \int_{-\theta_0}^{\theta_0} wRd\theta \tag{BA-3.4}$$

A relation between λ and Λ will be established for the principal solution of the arch problem, which is presented immediately below.

ia) *Exact Principal Solution*

The equilibrium equations for the arch, which in their weak form are given by $\mathcal{E},_u \delta_u = 0$ (see (AC-2.2)) can be written for the case of the potential energy given by (BA-3.3), after the customary integrations by part, in the following form:

$$\begin{aligned}\delta v \ &: \quad \frac{d}{d\theta}\big(EA\epsilon\big) = 0 \\ \delta w \ &: \quad \frac{EI}{R^3}\frac{d^4w}{d\theta^4} - \frac{d}{d\theta}\Big(\frac{EA\epsilon}{R}\frac{dw}{d\theta}\Big) - EA\epsilon - \lambda R = 0\end{aligned} \tag{BA-3.5}$$

Of course the above equilibrium equations are complemented by (BA-3.1) and the boundary conditions $(\text{BA-3.3})_2$ thus forming a well posed, nonlinear in v and w boundary value problem. The above system can be solved analytically. To this end the following dimensionless quantities are introduced:

$$\overline{\lambda} \equiv \lambda \frac{R^2 h}{EI}, \quad k \equiv \frac{R\theta_0^2}{h}, \quad s \equiv \frac{I}{Ah^2}, \quad \mu^2 \equiv -\frac{NR^2}{EI}, \quad g \equiv \frac{\overline{\lambda} k}{(\mu\theta_0)^2}, \quad \overline{\Lambda} \equiv \frac{\Lambda}{R^2\theta_0^3} \tag{BA-3.6}$$

From (BA-3.5)$_1$ and (BA-3.2)$_1$ the axial force N is a constant independent of θ. Consequently by employing (BA-3.2)$_1$ as well as (BA-3.6) into the second equilibrium equation (BA-3.5)$_2$, one finds that the most general expression for the normal displacement $w(\theta)$ is given by

$$w(\theta) = A_1 + A_2\theta + A_3 \sin(\mu\theta) + A_4 \cos(\mu\theta) + \frac{R\theta^2}{2}(g-1) \tag{BA-3.7}$$

The boundary conditions (BA-3.3)$_2$, applied to (BA-3.7) give the following system for the four constants A_i

$$\begin{aligned} w(\pm\theta_0) = 0 \ : \quad & A_1 + A_4\cos(\mu\theta_0) + \frac{R\theta_0^2}{2}(g-1) = 0, \quad & A_2\theta_0 + A_3\sin(\mu\theta_0) = 0 \\ \frac{dw}{d\theta}(\pm\theta_0) = \ : \quad & -A_4\mu\sin(\mu\theta_0) + R\theta_0(g-1) = 0, \quad & A_2 + A_3\mu\cos(\mu\theta_0) = 0 \end{aligned} \tag{BA-3.8}$$

Since the principal branch of the equilibrium solution is of interest, one expects a symmetric solution for w with respect to θ in view of the same symmetry of the structure and its loading. Hence the coefficients of the asymmetric terms in (BA-3.7) should vanish, i.e. $A_2 = A_3 = 0$. Consequently from (BA-3.7) and (BA-3.8) the normal displacement $\overset{0}{w}$ corresponding to the principal solution of the arch is given by

$$\overset{0}{w}(\theta) = R\theta_0^2(g-1)\Big[\frac{\cos(\mu\theta) - \cos(\mu\theta_0)}{\mu\theta_0\sin(\mu\theta_0)} + \frac{1}{2}\Big(\frac{(\mu\theta)^2}{(\mu\theta_0)^2} - 1\Big)\Big] \tag{BA-3.9}$$

To completely specify the principal solution one also needs to establish the relation between the imposed pressure λ and the resulting axial force N, or equivalently from (BA-3.6) the relation between $\overline{\lambda}$ (or g) and μ (or $\mu\theta_0$). To this end, one has to exploit the remaining equilibrium equation (BA-3.5)$_1$, together with (BA-3.1)$_1$, and (BA-3.2)$_1$ to obtain

$$EA\Big[\Big(\frac{d\overset{0}{v}}{d\theta} - \overset{0}{w}\Big)/R + \Big(\frac{d\overset{0}{w}}{d\theta}\Big)^2/2R^2\Big] = N = -\mu^2\frac{EI}{R^2} \tag{BA-3.10}$$

where the superscript zero denotes the association of the field quantity in question with the principal solution. Expressing $\overset{0}{v}$ as a function of $\overset{0}{w}$ from (BA-3.10) and taking also into account the boundary conditions in v from (BA-3.3)$_2$ as well as the definitions in (BA-3.6), one finds that $\overset{0}{w}$ has to satisfy

$$0 = \overset{0}{v}(\theta_0) - \overset{0}{v}(-\theta_0) = \int_{-\theta_0}^{\theta_0} \frac{d\overset{0}{v}}{d\theta}d\theta = \int_{-\theta_0}^{\theta_0} \Big[\overset{0}{w} - \frac{1}{2R}\Big(\frac{d\overset{0}{w}}{d\theta}\Big)^2 - (\mu\theta_0)^2\frac{s}{k^2}R\theta_0^2\Big]d\theta \tag{BA-3.11}$$

Upon introducing (BA-3.9) into (BA-3.11), one finds the following $g - \mu\theta_0$ relation, or equivalently the relation between the applied external pressure λ and the compressive axial force $N < 0$

$$(g-1)^2\Big[\frac{5}{3}(\mu\theta_0)^2 + \Big(\frac{\mu\theta_0}{\tan(\mu\theta_0)}\Big)^2 + 3\frac{\mu\theta_0}{\tan(\mu\theta_0)} - 4\Big] + 4(g-1)\Big[\frac{(\mu\theta_0)^2}{3} + \frac{\mu\theta_0}{\tan(\mu\theta_0)} - 1\Big] + 4(\mu\theta_0)^4\frac{s}{k^2} = 0 \tag{BA-3.12}$$

where the pressure λ and the axial force N on the beam are functions of the dimensionless parameters g and μ as indicated in (BA-3.6). To verify that (BA-3.9) and (BA-3.12) constitute indeed the principal solution of the problem, one has to make sure that for $\lambda = 0$ the corresponding principal displacements vanish, i.e., $\overset{0}{v} = \overset{0}{w} = 0$ as should also the corresponding axial force and bending moment, i.e., $N = M = 0$. This is a rather straightforward task that involves the careful taking of limits in (BA-3.9) and (BA-3.12) as $\mu\theta_0 \to 0$.

At this point, one has all the information to relate the pressure λ with its work conjugate quantity Λ, which represents the displaced area under the arch. From the definition (BA-3.4), and the expression for $\overset{0}{w}$ in (BA-3.9) one obtains, after making also use of the definitions of the dimensionless quantities in (BA-3.6)

$$\overline{\Lambda} = \frac{2(g-1)}{(\mu\theta_0)^2}\Big[1 - \frac{\mu\theta_0}{\tan(\mu\theta_0)} - \frac{(\mu\theta_0)^2}{3}\Big] \tag{BA-3.13}$$

Combining (BA-3.12) and (BA-3.13) one can in principle obtain the relation between $\overline{\lambda}$ and $\overline{\Lambda}$ by eliminating $\mu\theta_0$. By using (BA-3.13) into (BA-3.12) one obtains the relation between Λ and μ for the case in which the area Λ is the load parameter of the problem. A graphic representation of this relation is depicted in FIG. **??** which has been calculated for $s/k^2 = 5 \times 10^3$.

It is worth noticing that the dimensionless pressure $\overline{\lambda}$ reaches a maximum $\overline{\lambda}_U$ and a minimum $\overline{\lambda}_L$ as shown in FIG. **??** before going to infinity as $\overline{\Lambda}$ increases monotonically without bound. These extremal values of the dimensionless pressure $\overline{\lambda}$ can be found from (BA-3.12)- (BA-3.13) as follows: By considering $\overline{\lambda}k$ and $\mu\theta_0$ as functions of Λ one obtains from the chain rule of differentiation applied to $f(\overline{\lambda}k, \mu\theta_0) = 0$ – where f denotes for simplicity the left hand side of (BA-3.12) – that $df/d\Lambda = [\partial f/\partial(\overline{\lambda}k)][d(\overline{\lambda}k)/d\Lambda] + [\partial f/\partial(\mu\theta_0)][d(\mu\theta_0)/d\Lambda] = 0$. Since at the load extrema $d(\overline{\lambda}k)/d\Lambda = 0$ and $d(\mu\theta_0)/d\Lambda \neq 0$ (one can easily verify this assertion) we conclude that the load extrema $\overline{\lambda}_m$ (where m stands for U or L) have to satisfy, in addition to (BA-3.12), the relation $\partial f/\partial(\mu\theta_0) = 0$. After some manipulations one finds that $\partial f/\partial(\mu\theta_0) = 0$ is equivalent to

$$\begin{aligned}&(g-1)^2\Big[-2\Big(\frac{\mu\theta_0}{\tan(\mu\theta_0)}\Big)^3 - \Big(\frac{\mu\theta_0}{\tan(\mu\theta_0)}\Big)^2 - 2\frac{(\mu\theta_0)^3}{\tan(\mu\theta_0)} + 3\frac{\mu\theta_0}{\tan(\mu\theta_0)} + \frac{(\mu\theta_0)^2}{3}\Big] + \\ &8(g-1)\Big[-\frac{2}{3}(\mu\theta_0)^2 - \Big(\frac{\mu\theta_0}{\tan(\mu\theta_0)}\Big)^2 + 1\Big] - 8\Big[\frac{(\mu\theta_0)^2}{3} + \frac{\mu\theta_0}{\tan(\mu\theta_0)} - 1\Big] + 32(\mu\theta_0)^4\frac{s}{k^2} = 0\end{aligned} \tag{BA-3.14}$$

By solving (BA-3.12), (BA-3.14) for $g = \overline{\lambda}k/(\mu\theta_0)^2$ and $\mu\theta_0$ one obtains the wanted values of $\overline{\lambda}_U$, $\overline{\lambda}_L$ in terms of the dimensionless parameters s and k only (see (BA-3.6) for their definitions).

Having established the principal solution for the arch, the next issue to be addressed is about its stability. In contrast to the elastica and the thin walled beam applications studied in the previous two subsections, the principal solution is a nonlinear function of the position variable θ and hence the corresponding stability functional $(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u$ has θ - dependent coefficients in its integrand. Consequently a simple Fourier decomposition argument for the admissible displacement $\delta u = (\delta v(\theta), \delta w(\theta))$ of the type employed for the previous applications (see (BA-1.6) and (BA-2.29), (BA-2.38)) is no longer applicable in this case. Although stability of the principal branch in the sense defined by (AC-2.4) can also be proved here, finding the minimum eigenvalue $\overset{0}{\beta}(\lambda)$ of $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda),\lambda)$ is an analytically cumbersome and lengthy procedure and will thus be omitted in this presentation.

ib) *Exact Bifurcated Solution*

After presenting the principal solution for the pressurized circular arch problem, attention is next focussed on the bifurcated solution of the arch problem, which can also be obtained analytically. To this end notice that the equilibrium equations are still given by (BA-3.5) and hence the general solution for $w(\theta)$ is of the form (BA-3.7) and has to satisfy the boundary conditions (BA-3.8). Since the bifurcated solution does not have to be symmetric with respect to θ, $A_2, A_3 \neq 0$ and thus from (BA-3.8) non zero values for A_2, A_3 are possible if

$$\tan(\mu_b\theta_0) = \mu_b\theta_0 = x_b \approx 4.4394 \tag{BA-3.15}$$

where x_b is the smallest positive root of $\tan x = x$. The fact that $\mu\theta_0$ is taken to be positive does not in any way affect the results of the analysis in view of (BA-3.6) which implies that our results should be independent on the sign of μ, a fact easily verified in (BA-3.9), (BA-3.12)-(BA-3.14). Hence from (BA-3.7), (BA-3.8) and (BA-3.15) the bifurcated solution $\overset{b}{w}(\theta)$ assumes the form

$$\overset{b}{w}(\theta) = R\theta_0^2\Big[\frac{\overline{\lambda}k}{(\mu_b\theta_0)^2} - 1\Big]\Big[\frac{\cos(\mu_b\theta) - \cos(\mu_b\theta_0)}{\mu_b\theta_0\sin(\mu_b\theta_0)} + \frac{1}{2}\Big(\frac{(\mu_b\theta)^2}{(\mu_b\theta_0)^2} - 1\Big)\Big] + \xi[\sin(\mu_b\theta) - \mu_b\theta\cos(\mu_b\theta_0)] \tag{BA-3.16}$$

where ξ is for the time being an unspecified constant. To determine this constant one has to exploit the axial equilibrium $(\text{BA-3.5})_1$ which in conjunction with $(\text{BA-3.1})_1$ and $(\text{BA-3.2})_1$ gives, in analogy to (BA-3.10)

$$EA\Big[\Big(\frac{d\overset{b}{v}}{d\theta} - \overset{b}{w}\Big)/R + \Big(\frac{d\overset{b}{w}}{d\theta}\Big)^2/2R^2\Big] = N = -\mu_b^2\frac{EI}{R^2} \tag{BA-3.17}$$

where the superscript b denotes the association of the field quantity in question with the bifurcated solution. Expressing $\overset{b}{v}$ as a function of $\overset{b}{w}$ and taking also into account the boundary conditions in v from $(\text{BA-3.3})_2$ as well as the definitions in (BA-3.6) one finds, in analogy to (BA-3.11), that $\overset{b}{w}$ has to satisfy

$$0 = \overset{b}{v}(\theta_0) - \overset{b}{v}(-\theta_0) = \int_{-\theta_0}^{\theta_0}\frac{d\overset{b}{v}}{d\theta}d\theta = \int_{-\theta_0}^{\theta_0}\Big[\overset{b}{w} - \frac{1}{2R}\Big(\frac{d\overset{b}{w}}{d\theta}\Big)^2 - (\mu_b\theta_0)^2\frac{s}{k^2}R\theta_0^2\Big]d\theta \tag{BA-3.18}$$

Upon introducing (BA-3.16) into (BA-3.18) one finds the following relation between ξ and the pressure $\overline{\lambda}$

$$\xi^2 = -(R\theta_0^2)^2 \frac{(5/3)(\overline{\lambda}k/x_b^2 - 1)^2 + (4/3)(\overline{\lambda}k/x_b^2 - 1) + 4x_b^2 s/k^2}{x_b^4/(1+x_b^2)} \tag{BA-3.19}$$

The principal solution $(\overset{0}{v}, \overset{0}{w})$ and the bifurcated solution $(\overset{b}{v}, \overset{b}{w})$ should coincide at the onset of bifurcation. Moreover, the corresponding dimensionless bifurcation pressure $\overline{\lambda}_b$ must satisfy (BA-3.12) and (BA-3.15) from which one can easily conclude that

$$\frac{5}{3}(g_b - 1)^2 + \frac{4}{3}(g_b - 1) + 4x_b^2 \frac{s}{k^2} = 0, \quad g_b \equiv \frac{\overline{\lambda}_b k}{(x_b)^2} \tag{BA-3.20}$$

Hence from (BA-3.20) and (BA-3.13) the solution for the dimensionless pressures at bifurcation $\overline{\lambda}_b = \overline{\lambda}_{I,II}$ and their corresponding work conjugate displacements $\overline{\Lambda}_{I,II}$ are found to be

$$\begin{aligned} \overline{\lambda}_I &= \frac{x_b^2}{5k}\Big\{3 + 2\Big[1 - 15\,x_b^2 \frac{s}{k^2}\Big]^{\frac{1}{2}}\Big\}, \quad & \overline{\Lambda}_I &= \frac{4}{15}\Big\{1 - \Big[1 - 15\,x_b^2 \frac{s}{k^2}\Big]^{\frac{1}{2}}\Big\} \\ \overline{\lambda}_{II} &= \frac{x_b^2}{5k}\Big\{3 - 2\Big[1 - 15\,x_b^2 \frac{s}{k^2}\Big]^{\frac{1}{2}}\Big\}, \quad & \overline{\Lambda}_{II} &= \frac{4}{15}\Big\{1 + \Big[1 - 15\,x_b^2 \frac{s}{k^2}\Big]^{\frac{1}{2}}\Big\} \end{aligned} \tag{BA-3.21}$$

The above points are the two end points where the bifurcated solution (BA-3.16), (BA-3.19) joins the principal solution (BA-3.9), (BA-3.12). On the bifurcated branch, the relation between the dimensionless load $\overline{\lambda}$ and its work conjugate dimensionless displacement $\overline{\Lambda}$ is found from (BA-3.4), (BA-3.6), (BA-3.15) and (BA-3.16) as

$$\overline{\Lambda} = \frac{2}{3}\Big[1 - \frac{\overline{\lambda}k}{x_b^2}\Big] \tag{BA-3.22}$$

which is the analogous relation to (BA-3.13) which holds for the principal branch and which reduces to (BA-3.22) by virtue of (BA-3.15). Notice that at bifurcation, for $\overline{\lambda}_b = \overline{\lambda}_I$, or $\overline{\lambda}_b = \overline{\lambda}_{II}$ the parameter ξ introduced in (BA-3.16), which will be identified subsequently with the bifurcation amplitude parameter of the problem, vanishes in view of (BA-3.20). Moreover, for a bifurcated solution to exist (BA-3.20) must admit real roots which implies from the required positive discriminant of the quadratic in $(g_b - 1)$ equation (BA-3.20)

$$\frac{k^2}{s} \geq 15\,x_b^2 \approx 302.86 \tag{BA-3.23}$$

The principal and bifurcated solution relations between the dimensionless load $\overline{\lambda}$ and its work conjugate dimensionless displacement $\overline{\Lambda}$ are plotted in FIG. ??. The bifurcated branch starts at point $(\overline{\lambda}_I, \overline{\Lambda}_I)$ and ends at point $(\overline{\lambda}_{II}, \overline{\Lambda}_{II})$. Notice from (BA-3.22) that bifurcation takes place under monotonically decreasing load.

The stability analysis for the bifurcated equilibrium branch presents exactly the same analytical difficulties as the stability analysis of the principal branch discussed previously. Nevertheless, after some laborious algebra one can show that the bifurcated branch is unstable. This result is consistent with our expectations from the general theory in Section AC-3 according to which if in a simple bifurcation $\Delta\lambda < 0$ the corresponding branch is unstable, at least in a neighborhood of the bifurcation point.

ic) *Asymptotic Solution. Limit and Bifurcation Points*

Having presented the analytically available principal and bifurcated solutions for the pressurized arch problem, it is instructive to apply the general theory presented in Section AC-3 and compare the asymptotic results with the exact solution.

The starting point of the asymptotic analysis is the determination of the critical points of the structure which according to (AC-2.7) are given by $(\mathcal{E}^c_{,uu}\overset{1}{u})\delta u = 0$. Consequently specializing this equation to the potential energy given by (BA-3.3) and after taking into account (BA-3.3) one has

$$
\begin{aligned}
(\mathcal{E}^c_{,uu}\overset{1}{u})\delta u \; = \; \int_{-\theta_0}^{\theta_0} \Big[& EA\Big(\Big(\frac{d\overset{1}{v}}{d\theta} - \overset{1}{w}\Big)/R + \frac{d\overset{0}{w}}{d\theta}\frac{d\overset{1}{w}}{d\theta}/R^2\Big)\Big(\Big(\frac{d\delta v}{d\theta} - \delta w\Big)/R + \frac{d\overset{0}{w}}{d\theta}\frac{d\delta w}{d\theta}/R^2\Big) + \\
& EA\Big(\Big(\frac{d\overset{0}{v}}{d\theta} - \overset{0}{w}\Big)/R + \Big(\frac{d\overset{0}{w}}{d\theta}\Big)^2/2R^2\Big)\frac{d\overset{1}{w}}{d\theta}\frac{d\delta w}{d\theta}/R^2 + EI\frac{d^2\overset{1}{w}}{d\theta^2}\frac{d^2\delta w}{d\theta^2}/R^4\Big] R d\theta = 0
\end{aligned}
\tag{BA-3.24}
$$

where of course $\delta u = (\delta v, \delta w)$ and the mode $\overset{1}{u} = (\overset{1}{v}, \overset{1}{w})$ have to satisfy the boundary conditions given in $(\text{BA-3.3})_2$. Upon integration by parts (BA-3.24) gives the following pointwise form of the equation

$$
\begin{aligned}
\delta v \; &: \quad EA\Big(\Big(\frac{d\overset{1}{v}}{d\theta} - \overset{1}{w}\Big)/R + \frac{d\overset{0}{w}}{d\theta}\frac{d\overset{1}{w}}{d\theta}/R^2\Big) = C; \quad \overset{1}{v}(\pm\theta_0) = 0 \\
\delta w \; &: \quad \frac{d^4\overset{1}{w}}{d\theta^4} + \mu^2\frac{d^2\overset{1}{w}}{d\theta^2} = \frac{CR^3}{EI}\Big(1 + \frac{d^2\overset{0}{w}}{d\theta^2}/R\Big); \quad \overset{1}{w}(\pm\theta_0) = 0, \; \frac{d\overset{1}{w}}{d\theta}(\pm\theta_0) = 0
\end{aligned}
\tag{BA-3.25}
$$

where in the derivation of $(\text{BA-3.25})_2$ use was made of $(\text{BA-3.25})_1$ and of the fact that the axial force N of the prebifurcated solution is given from (BA-3.6) in terms of μ by $N = -\mu^2 EI/R^2$ where μ is related to the applied pressure λ by (BA-3.12). Note also that in (BA-3.25), $\overset{0}{w}$ is the principal solution given by (BA-3.9) while C is an unspecified, for the time being, constant.

The general solution to $(\text{BA-3.25})_2$ is found, with the help of (BA-3.9), to be

$$
\overset{1}{w}(\theta) = B_1 + B_2\theta + B_3\sin(\mu\theta) + B_4\cos(\mu\theta) + \frac{1}{2}\frac{CR^3}{EI\mu^4}\Big[g(\mu\theta)^2 + (g-1)(\mu\theta)\sin(\mu\theta)\frac{\mu\theta_0}{\sin(\mu\theta_0)}\Big] \tag{BA-3.26}
$$

Applying to (BA-3.26) the boundary conditions for $\overset{1}{w}$ from (BA-3.25) one obtains the following system for the constants B_i, $(i = 1, 4)$ and C:

$$
\begin{aligned}
\overset{1}{w}(\pm\theta_0) = 0 \; &: \quad B_1 + B_4\cos(\mu\theta_0) + \frac{1}{2}\frac{CR^3}{EI\mu^4}\Big[2(g-1) + 1\Big](\mu\theta_0)^2 = 0, \; B_2\theta_0 + B_3\sin(\mu\theta_0) = 0 \\
\frac{d\overset{1}{w}}{d\theta}(\pm\theta_0) = 0 \; &: \quad -B_4\sin(\mu\theta_0) + \frac{1}{2} + \frac{CR^3}{EI\mu^4}\Big[(g-1)\Big(3 + +\frac{\mu\theta_0}{\tan(\mu\theta_0)}\Big) + 2\Big](\mu\theta_0) = 0, \; B_2 + B_3\mu\cos(\mu\theta_0) = 0
\end{aligned}
\tag{BA-3.27}
$$

At this point two cases are distinguished:

ic.a) *Limit point* ($\tan\mu\theta_0 \neq \mu\theta_0$)

For the case where $\tan\mu\theta_0 \neq \mu\theta_0$, one concludes from (BA-3.25) that $B_2 = B_3 = 0$ and $B_1,\ B_4,\ C \neq 0$. The justification that this case corresponds to a limit point will follow shortly. First however one has to determine the dimensionless critical load $\overline{\lambda}_c$. To this end, one has to exploit the remaining equilibrium equation (BA-3.25)$_1$, from which one obtains, after using the boundary condition $\overset{1}{w}(\pm\theta_0) = 0$

$$0 = \overset{1}{v}(\theta_0) - \overset{1}{v}(-\theta_0) = \int_{-\theta_0}^{\theta_0} \frac{d\overset{1}{v}}{d\theta} d\theta = \int_{-\theta_0}^{\theta_0} \Big[\frac{CR}{EA} + \Big(1 + \frac{d^2\overset{0}{w}}{d\theta^2}/R\Big)\overset{1}{w}\Big] d\theta \tag{BA-3.28}$$

Since $B_2 = B_3 = 0$, from (BA-3.26) and (BA-3.27), the required for (BA-3.28) $\overset{1}{w}(\theta)$ is given by

$$\begin{aligned} \overset{1}{w}(\theta) \;=\; & \frac{1}{2}\frac{CR^3}{EI\mu^4}\Big\{(g-1)\Big[\frac{\mu\theta_0}{\tan(\mu\theta_0)}\Big(\frac{\mu\theta_0}{\tan(\mu\theta_0)} + 3\Big)\Big(\frac{\cos(\mu\theta)}{\cos(\mu\theta_0)} - 1\Big) + \frac{\mu\theta_0}{\sin(\mu\theta_0)}\mu\theta\sin(\mu\theta) \;+ \\ & (\mu\theta)^2 - 2(\mu\theta_0)^2\Big] + \Big[2\frac{\mu\theta_0}{\tan(\mu\theta_0)}\Big(\frac{\cos(\mu\theta)}{\cos(\mu\theta_0)} - 1\Big) + (\mu\theta)^2 - (\mu\theta_0)^2\Big]\Big\} \end{aligned} \tag{BA-3.29}$$

Upon substitution of $\overset{1}{w}(\theta)$ and $\overset{0}{w}(\theta)$ from (BA-3.29) and (BA-3.9) respectively into (BA-3.28) one has with the help also of (BA-3.6)

$$\begin{aligned} & (g-1)^2\Big[-\Big(\frac{\mu\theta_0}{\tan(\mu\theta_0)}\Big)^3 - \frac{7}{2}\Big(\frac{\mu\theta_0}{\tan(\mu\theta_0)}\Big)^2 - \frac{(\mu\theta_0)^3}{\tan(\mu\theta_0)} - \frac{15}{2}\frac{\mu\theta_0}{\tan(\mu\theta_0)} - \frac{29}{6}(\mu\theta_0)^2 + 12\Big] \;+ \\ & (g-1)\Big[-4\Big(\frac{\mu\theta_0}{\tan(\mu\theta_0)}\Big)^2 - 12\frac{\mu\theta_0}{\tan(\mu\theta_0)} - \frac{20}{3}(\mu\theta_0)^2 + 16\Big] - 4\Big[\frac{\mu\theta_0}{\tan(\mu\theta_0)} + \frac{(\mu\theta_0)^2}{3} - 1\Big] + 4\frac{s}{k^2}(\mu\theta_0)^4 = 0 \end{aligned} \tag{BA-3.30}$$

Consequently the critical load $\overline{\lambda}_c$ (recall from (BA-3.6) that $\overline{\lambda}_c = g_c(\mu\theta_0)_c^2/k$ and $\lambda_c = \overline{\lambda}_c EI/R^2h$) is given by solving for $g,\ \mu\theta$ the system of (BA-3.30) and (BA-3.12), since the critical point is on the principal branch of equilibrium.

Notice that (BA-3.30) is a linear combination of (BA-3.12) and (BA-3.14) and thus the system (BA-3.30) and (BA-3.12) has exactly the same solutions with the system (BA-3.12) and (BA-3.14). Consequently, the critical load $\overline{\lambda}_c$ in this case coincides with either one of the two load extrema $\overline{\lambda}_U, \overline{\lambda}_L$ of the principal solution and is therefore a limit point. According to the general theory from (AC-2.21) at a limit point $\mathcal{E}^c_{,u\lambda}\overset{1}{u} \neq 0$. Indeed for this problem from (BA-3.3) and (BA-3.29) one obtains

$$\begin{aligned} \mathcal{E}^c_{,u\lambda}\overset{1}{u} \;=\; & \int_{-\theta_0}^{\theta_0} \overset{1}{w}R d\theta = -\frac{CR^4}{EI\mu^5}(\mu\theta_0)_c\Big\{(g_c-1)\Big[\Big(\frac{(\mu\theta_0)_c}{\tan(\mu\theta_0)_c}\Big)^2 + 3\frac{(\mu\theta_0)_c}{\tan(\mu\theta_0)_c} + \frac{5}{3}(\mu\theta_0)_c^2 - 4\Big] \;+ \\ & 2\Big[\frac{(\mu\theta_0)_c}{\tan(\mu\theta_0)_c} + \frac{(\mu\theta_0)_c^2}{3} - 1\Big]\Big\} \neq 0 \end{aligned} \tag{BA-3.31}$$

The above quantity is in general different from zero, as one can verify from (BA-3.12) and (BA-3.14), thus proving that the critical points for the case $\tan(\mu\theta_0) \neq \mu\theta_0$ are indeed limit points.

ic.b) *Bifurcation Point* $(\tan(\mu\theta_0) = \mu\theta_0 = x_b)$

For this case it will first be shown that $B_1 = B_4 = C = 0$ while $B_2,\ B_3 \neq 0$ in the expression (BA-3.26) for $\overset{1}{w}$. Assuming momentarily the contrary, i.e. that $B_1,\ B_4,\ C \neq 0$ one concludes by using (BA-3.26)-(BA-3.27) that $\mathcal{E}^c_{,u\lambda}\overset{1}{u}$ is still given by (BA-3.31) where $(\mu\theta_0)_c,\ g_c$ are replaced by x_b and g_b; the antisymmetric terms $B_2\theta$ and $B_3\sin(\mu\theta)$ give no contribution in $\mathcal{E}^c_{,u\lambda}\overset{1}{u}$. Since at bifurcation one expects from the general theory according to (AC-2.27) that $\mathcal{E}^c_{,u\lambda}\overset{1}{u} = 0$, the only possible way to enforce this condition requires $C = 0$. Consequently from (BA-3.26)-(BA-3.27) we must have $B_1 = B_4 = C = 0$ and corresponding eigenmode is (see also (BA-3.25)$_1$)

$$\overset{1}{w}(\theta) = \sin(\mu_b\theta) - (\mu_b\theta)\cos(\mu_b\theta_0), \quad \overset{1}{v}(\theta) = \int_{-\theta_0}^{\theta_0} \Big(\overset{1}{w} - \frac{d\overset{0}{w}}{d\theta}\frac{d\overset{1}{w}}{d\theta}/R\Big) d\theta \tag{BA-3.32}$$

while the bifurcation load is given by (BA-3.20)-(BA-3.21). The subsequent asymptotic analysis is valid for both bifurcation points $\overline{\lambda}_1,\ \overline{\lambda}_2$.

Continuing with the calculation of the next order terms in the asymptotic expansions of the load λ and the displacement u, one has from the general theory that λ_1, the first term in the asymptotic expansion of the load, is given by (BA-3.13). The denominator in the aforementioned equation $((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} \neq 0$ as it will be shown shortly, while its numerator is from (BA-3.1), (BA-3.3) and (BA-3.25)$_1$ is

$$((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = \int_{-\theta_0}^{\theta_0} \Big[3EA\frac{1}{R^2}\Big(\frac{d\overset{1}{w}}{d\theta}\Big)^2\Big(\frac{1}{R}\Big(\frac{d\overset{1}{v}}{d\theta} - \overset{1}{w}\Big) + \frac{1}{R^2}\frac{d\overset{0}{w}}{d\theta}\frac{d\overset{1}{w}}{d\theta}\Big)\Big] R d\theta = \int_{-\theta_0}^{\theta_0} \Big[\Big(\frac{3C}{R}\Big(\frac{d\overset{1}{w}}{d\theta}\Big)^2\Big] d\theta \tag{BA-3.33}$$

Since for the bifurcated solution $C = 0$,the integrand in (BA-3.33) vanishes and hence $\lambda_1 = 0$. This result comes to no surprise since as we know from (BA-3.19) the bifurcation of the arch is a symmetric one since it is independent of the sign of the amplitude ξ.

The determination of $v_{\xi\xi} = (\overset{2}{v}, \overset{2}{w})$ is based on (AC-3.8) of the general theory which for the arch problem from (BA-3.1), (BA-3.3) takes the form

$$
\begin{aligned}
((\mathcal{E}^c_{,uu}\overset{1}{u})\overset{1}{u}+\mathcal{E}^c_{,uu}v_{\xi\xi})\delta v \;=\; & \int_{-\theta_0}^{\theta_0}\Big[2EA\Big(\frac{1}{R}\Big(\frac{d\overset{1}{v}}{d\theta}-\overset{1}{w}\Big)+\frac{1}{R^2}\frac{d\overset{0}{w}}{d\theta}\frac{d\overset{1}{w}}{d\theta}\Big)\frac{1}{R^2}\frac{d\overset{1}{w}}{d\theta}\frac{d\delta w}{d\theta}\;+\\
& EA\frac{1}{R^2}\Big(\frac{d\overset{1}{w}}{d\theta}\Big)^2\Big(\frac{1}{R}\Big(\frac{d\delta v}{d\theta}-\delta w\Big)+\frac{1}{R^2}\frac{d\overset{0}{w}}{d\theta}\frac{d\delta w}{d\theta}\Big)\;+\\
& EA\Big(\frac{1}{R}\Big(\frac{d\overset{2}{v}}{d\theta}-\overset{2}{w}\Big)+\frac{1}{R^2}\frac{d\overset{0}{w}}{d\theta}\frac{d\overset{2}{w}}{d\theta}\Big)\Big(\frac{1}{R}\Big(\frac{d\delta v}{d\theta}-\delta w\Big)+\frac{1}{R^2}\frac{d\overset{0}{w}}{d\theta}\frac{d\delta w}{d\theta}\Big)\;+\\
& EA\Big(\frac{1}{R}\Big(\frac{d\overset{0}{v}}{d\theta}-\overset{0}{w}\Big)+\frac{1}{2R^2}\Big(\frac{d\overset{0}{w}}{d\theta}\Big)^2\Big)\frac{1}{R^2}\frac{d\overset{2}{w}}{d\theta}\frac{d\delta w}{d\theta}+EI\frac{1}{R^4}\frac{d^2\overset{2}{w}}{d\theta^2}\frac{d^2\delta w}{d\theta^2}\Big]Rd\theta=0
\end{aligned}
\tag{BA-3.34}
$$

In addition since $\overset{2}{v}$, $\overset{2}{w}$, δv, δw are kinematically admissible functions they have to satisfy the clamping boundary conditions in (BA-3.3)$_2$. By employing (BA-3.10) as well as (BA-3.25)$_1$, recalling that for the bifurcated solution the constant $C=0$ and integrating (BA-3.34) by parts, one obtains the following governing equations for $\overset{2}{v}(\theta)$ and $\overset{2}{w}(\theta)$

$$
\begin{aligned}
\delta v \;:\;\; & EA\Big(\frac{1}{R^2}\Big(\frac{d\overset{1}{w}}{d\theta}\Big)^2+\frac{1}{R}\Big(\frac{d\overset{2}{v}}{d\theta}-\overset{2}{w}\Big)+\frac{1}{R^2}\frac{d\overset{0}{w}}{d\theta}\frac{d\overset{2}{w}}{d\theta}\Big)=\overline{C}; \quad \overset{2}{v}(\pm\theta_0)=0\\
\delta w \;:\;\; & \frac{d^4\overset{2}{w}}{d\theta^4}+(\mu_b)^2\frac{d^2\overset{2}{w}}{d\theta^2}=\frac{\overline{C}R^3}{EI}\Big(1+\frac{1}{R}\frac{d^2\overset{0}{w}}{d\theta^2}\Big); \quad \overset{2}{w}(\pm\theta_0)=0,\;\; \frac{d\overset{2}{w}}{d\theta}(\pm\theta_0)=0
\end{aligned}
\tag{BA-3.35}
$$

where $\overline{C}$ is a constant to be subsequently specfied. Noticing that $\overset{2}{w}(\theta)$ satisfies the same equation and boundary conditions with $\overset{1}{w}(\theta)$ in (BA-3.25)$_2$ (with C replaced by $\overline{C}$), and recalling that $\overset{2}{w}(\theta)$ is orthogonal to the eigenmode $\overset{1}{w}(\theta)$ in (BA-3.32) and that at bifurcation $\tan(\mu_b\theta_0)=\mu_b\theta_0$, one obtains from (BA-3.29)

$$
\begin{aligned}
\overset{2}{w}(\theta) \;=\; & \frac{1}{2}\frac{\overline{C}R^3}{EI(\mu_b)^4}\Big\{(g_b-1)\Big[4(\frac{\cos(\mu_b\theta)}{\cos(\mu_b\theta_0)}-1)+\frac{(\mu_b\theta)\sin(\mu_b\theta)}{\cos(\mu_b\theta_0)}+(\mu_b\theta)^2-2(\mu_b\theta_0)^2\Big]\;+\\
& \Big[2\Big(\frac{\cos(\mu_b\theta)}{\cos(\mu_b\theta_0)}-1\Big)+(\mu_b\theta)^2-(\mu_b\theta_0)^2\Big]\Big\}
\end{aligned}
\tag{BA-3.36}
$$

where g_b is given in terms of the critical load $\overline{\lambda}_c=\overline{\lambda}_b$ by (BA-3.20). The determination of the unknown constant $\overline{C}$ in (BA-3.35), (BA-3.36) can be achieved by exploiting the remaining equilibrium equation (BA-3.35)$_1$ which gives

$$
0=\overset{2}{v}(\theta_0)-\overset{2}{v}(-\theta_0)=\int_{-\theta_0}^{\theta_0}\frac{d\overset{2}{v}}{d\theta}=\int_{-\theta_0}^{\theta_0}\Big[\frac{\overline{C}R}{EA}+\overset{2}{w}-\frac{1}{R}\frac{d\overset{0}{w}}{d\theta}\frac{d\overset{2}{w}}{d\theta}-\frac{1}{R}\Big(\frac{d\overset{1}{w}}{d\theta}\Big)^2\Big]d\theta \tag{BA-3.37}
$$

Upon substitution of (BA-3.9), (BA-3.32) and (BA-3.36) into (BA-3.37) and after taking also into account (BA-3.6) as well as (BA-3.20), one obtains for $\overline{C}$ after a lengthy but straightforward calculation

$$
\overline{C}=-\frac{12\,EI}{R^4(\theta_0)^6}\frac{x_b^6/[1+x_b^2]}{45(g_b-1)^2+48(g_b-1)+8} \tag{BA-3.38}
$$

Hence $\overset{2}{w}(\theta)$ has been completely specified from (BA-3.36) and (BA-3.38) while $\overset{2}{v}(\theta)$ can be subsequently determined form $(\text{BA-3.35})_2$.

At this step, all the ingredients are available for the calculation of the first non zero term λ_2 in the expansion of the critical load whose general expression is given in (AC-3.14). From (BA-3.1) and (BA-3.3) one finds (using also $(\text{BA-3.32})_2$ and $(\text{BA-3.35})_1$

$$
\begin{aligned}
(((\mathcal{E}^c_{,uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} &= \int_{-\theta_0}^{\theta_0}\Big[3EA\frac{1}{R^4}\Big(\frac{d\overset{1}{w}}{d\theta}\Big)^4\Big]Rd\theta \\
((\mathcal{E}^c_{,uuu}\, v_{\xi\xi}\overset{1}{u})\overset{1}{u}) &= \int_{-\theta_0}^{\theta_0}\Big[EA\Big(\frac{1}{R}\Big(\frac{d\overset{2}{v}}{d\theta}-\overset{2}{w}\Big)+\frac{1}{R^2}\frac{d\overset{0}{w}}{d\theta}\frac{d\overset{2}{w}}{d\theta}\Big)\frac{1}{R^2}\Big(\frac{d\overset{1}{w}}{d\theta}\Big)^2\Big]Rd\theta \\
((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} &= -\Big(\frac{d\mu^2}{d\lambda}\Big)_b\int_{-\theta_0}^{\theta_0}\Big[\frac{EI}{R^4}\Big(\frac{d\overset{1}{w}}{d\theta}\Big)^2\Big]Rd\theta
\end{aligned}
\tag{BA-3.39}
$$

$$
\lambda_2 = -\frac{1}{3}\Big[(((\mathcal{E}^c_{,uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u}+3((\mathcal{E}^c_{,uuu}\, v_{\xi\xi}\overset{1}{u})\overset{1}{u})\Big]/((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = \frac{\overline{C}}{R}/\Big[\frac{d(\mu\theta_0)^2}{d(\overline{\lambda}k)}\Big]_b
$$

The derivative $[d(\mu\theta_0)^2/d(\overline{\lambda}k)]_b$ at the bifurcation point $\mu_b\theta_0 = x_b$ is found from (BA-3.12) to be

$$
\Big[\frac{d(\mu\theta_0)^2}{d(\overline{\lambda}k)}\Big]_b = 4\frac{5(g_b-1)+2}{45(g_b-1)^2+48(g_b-1)+8} \tag{BA-3.40}
$$

and consequently λ_2 from (BA-3.38) and (BA-3.40) is found to be

$$
\lambda_2 = -\frac{3}{\theta_0}\frac{EI}{(R\theta_0)^5}\frac{x_b^6/[1+x_b^2]}{5(g_b-1)+2} \tag{BA-3.41}
$$

For the first instability encountered during a monotonically increasing pressure loading $\overline{\lambda}_c = \overline{\lambda}_I$ given from $(\text{BA-3.21})_1$ which substituted into (BA-3.41) yields

$$
\lambda_2 = -\frac{3}{2\theta_0}\frac{EI}{(R\theta_0)^5}\frac{x_b^6}{1+x_b^2}\Big[1-15x_b^2\frac{s}{k^2}\Big]^{\frac{1}{2}} \tag{BA-3.42}
$$

Consequently, since for the first critical load encountered at $\overline{\lambda}_c = \overline{\lambda}_I$ we have $\lambda_2 < 0$ (we can easily show that $((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} < 0$ as expected from the general theory), the bifurcated branch is unstable near the critical load. The fact that the entire bifurcated branch is unstable, has already been stated without proof in the discussion following the analytical derivation of that solution.

As an independent check of the correctness of the above derived asymptotic results λ_2 will be computed directly from the exact solution(BA-3.19). Using (BA-3.19), (BA-3.20) as well as (BA-3.6) one obtains

$$
\lambda_2 = \Big[\frac{d^2\lambda}{d\xi^2}\Big]_b = \frac{1}{R}\frac{EI}{(R\theta_0)^2}\Big[\frac{d^2(\overline{\lambda}k)}{d\xi^2}\Big]_b = -\frac{3}{\theta_0}\frac{EI}{(R\theta_0)^5}\frac{x_b^6/[1+x_b^2]}{5(g_b-1)+2} \tag{BA-3.43}
$$

which coincides as expected with (BA-3.41).

ii) *Imperfect Model Description*

So far we have found that for slender enough arches ($k^2/s \geq 15x_b^2 \approx 302.86$ - see (BA-3.23)) a symmetric bifurcation with $\lambda_2 < 0$ precedes the maximum pressure in a monotonically increasing pressure loading. Consequently, according to the general theory in Section AC-4 the structure is sensitive to geometric imperfections. The geometrically imperfect arch model can be easily derived from its perfect counterpart following an approach similar to the one used in the derivation of the imperfect thin walled beam model in Section BA-2 ii.

It is again assumed that in its stress-free state, the center line of the arch is displaced by $\overline{v}(\theta)$, $\overline{w}(\theta)$ with respect to its position in the perfect case. Consequently the axial $\overline{\epsilon}$ and bending $\overline{\kappa}$ strain measures are given by

$$\begin{aligned} \overline{\epsilon} &= \epsilon(v+\overline{v}, w+\overline{w}) - \epsilon(\overline{v}, \overline{w}) = \Big(\frac{dv}{d\theta} - w\Big)/R + \Big(\frac{dw}{d\theta}\Big)^2/2R^2 + \frac{dw}{d\theta}\frac{d\overline{w}}{d\theta}/R^2 \\ \overline{\kappa} &= \kappa(w+\overline{w}) - \kappa(\overline{w}) = \frac{d^2w}{d\theta^2}/R^2 \end{aligned} \tag{BA-3.44}$$

where $v(\theta), w(\theta)$ are the net tangential and normal displacement of the imperfect arch's centerline from its unstressed configuration. Notice that since the strain measures of the perfect structure in (BA-3.1) depend linearly on $v(\theta)$, the imperfect structure's strain measures depend solely on $\overline{w}(\theta)$.

The stress - strain relations for the imperfect structure are still given by (BA-3.2) while the corresponding potential energy is given by (BA-3.3) where ϵ, κ are replaced by $\overline{\epsilon}$, $\overline{\kappa}$ respectively (notice $\kappa = \overline{\kappa}$)

$$\overline{\mathcal{E}} = \frac{1}{2}\int_{-\theta_0}^{\theta_0} (EA\overline{\epsilon}^2 + EI\overline{\kappa}^2)Rd\theta - \lambda \int_{-\theta_0}^{\theta_0} wRd\theta \tag{BA-3.45}$$

The boundary conditions for $v(\theta), w(\theta)$ are still given by $(\text{BA-3.3})_2$. Notice also that the external potential energy in view of its linear dependence on w remains the same as in the perfect case. For the pressurized imperfect arch problem an analytical solution exists and it will be derived in the sequel together with its asymptotic expansion according to the general theory in Section AC-4.

iia) *Exact Solution of Imperfect Structure*

The equilibrium equation of the imperfect structure is $\overline{\mathcal{E}}_{,u}\,\delta u = 0$. From (BA-3.44)-(BA-3.45) the corresponding pointwise (Euler-Lagrange) equilibrium equations of the imperfect structure are

$$\begin{aligned} \delta v \;&:\; \frac{d}{d\theta}(EA\overline{\epsilon}) = 0 \\ \delta w \;&:\; \frac{EI}{R^3}\frac{d^4w}{d\theta^4} - \frac{d}{d\theta}\Big(\frac{EA\overline{\epsilon}}{R}\Big(\frac{dw}{d\theta} + \frac{d\overline{w}}{d\theta}\Big)\Big) - EA\overline{\epsilon} - \lambda R = 0 \end{aligned} \tag{BA-3.46}$$

The boundary conditions for v and w are still given by (BA-3.3)$_2$. By using once again the convenient dimensionless quantities introduced in (BA-3.6) and noting that the axial force $N = EA\bar{\epsilon}$ is again a constant independent of θ, the linear in w equilibrium equation (BA-3.46)$_2$ has the following general solution

$$w(\theta) = C_1 + C_2\theta + C_3 \sin(\mu\theta) + C_4 \cos(\mu\theta) + \frac{R\theta^2}{2}(g-1) - \mu \int_0^\theta \sin[\mu(\theta - \phi)]\overline{w}(\phi)d\phi \tag{BA-3.47}$$

The above result has been obtained by noticing that (BA-3.46)$_2$ and (BA-3.5)$_2$ share the same linear operator in w, i.e. $(d^4w/d\theta^4 + \mu^2 d^2w/d\theta^2)$ but their right hand side terms differ by $-\mu^2 d^2\overline{w}/d\theta^2$. Consequently one has to add to (BA-3.7) the term $-\mu \int_0^\theta \sin[\mu(\theta - \phi)]\overline{w}(\phi)d\phi$ which is a particular solution to $d^4w/d\theta^4 + \mu^2 d^2w/d\theta^2 = \mu^2 d^2\overline{w}/d\theta^2$. The boundary conditions (BA-3.3)$_2$ applied to (BA-3.47) give the following system for the four constants C_i

$$\begin{aligned}
w(\pm\theta_0) \ : \quad & C_1 + C_4\cos(\mu\theta_0) + \frac{R\theta_0^2}{2}(g-1) - \mu\int_0^{\theta_0} \overline{w}_s(\phi)\sin[\mu(\theta_0-\phi)]d\phi = 0 \\
& C_2\theta_0 + C_3\sin(\mu\theta_0) - \mu\int_0^{\theta_0} \overline{w}_a(\phi)\sin[\mu(\theta_0-\phi)]d\phi = 0 \\
\frac{dw}{d\theta}(\pm\theta_0) \ : \quad & -C_4\mu\sin(\mu\theta_0) + R\theta_0(g-1) - \mu^2\int_0^{\theta_0} \overline{w}_s(\phi)\cos[\mu(\theta_0-\phi)]d\phi = 0 \\
& C_2 + C_3\mu\cos(\mu\theta_0) - \mu^2\int_0^{\theta_0} \overline{w}_a(\phi)\cos[\mu(\theta_0-\phi)]d\phi = 0
\end{aligned} \tag{BA-3.48}$$

where $\overline{w}_s(\theta) \equiv [w(\theta) + w(-\theta)]/2$ and $\overline{w}_a(\theta) \equiv [w(\theta) - w(-\theta)]/2$ denote the symmetric and antisymmetric parts of the imperfection $\overline{w}(\theta)$. The solution of (BA-3.48) for C_i gives, after recalling the definition for the displacement $\overset{0}{w}(\theta)$ of the perfect structure's principal branch

$$\begin{aligned}
w(\theta) = \ & \overset{0}{w}(\theta) - \mu\int_0^{\theta_0} \overline{w}(\phi)\sin[\mu(\theta-\phi)]d\phi + \frac{\mu}{\sin(\mu\theta_0)}\int_0^{\theta_0} \overline{w}_s(\phi)\{\cos(\mu\phi) - \cos(\mu\theta)\cos[\mu(\theta_0-\phi)]\}d\phi \ + \\
& \frac{\mu}{(\mu\theta_0)\cos(\mu\theta_0) - \sin(\mu\theta_0)}\int_0^{\theta_0} \overline{w}_a(\phi)\{-\mu\theta\sin(\mu\phi) + [\mu\theta_0\cos[\mu(\theta_0-\phi)] - \sin[\mu(\theta_0-\phi)]]\sin(\mu\theta)\}d\phi
\end{aligned} \tag{BA-3.49}$$

Notice from (BA-3.48) that for a symmetric imperfection $\overline{w}(\phi) = \overline{w}(-\phi)$, $\overline{w}_a(\phi) = 0$ and consequently $C_2 = C_3 = 0$, assuming of course $\mu\theta_0 \neq \tan(\mu\theta_0)$. We thus conclude that for a symmetric imperfection, a symmetric equilibrium solution will result which will bifurcate at the point $\mu\theta_0 = x_b$. For an arbitrary imperfection, i.e., when $\overline{w}_a(\phi) \neq 0$, the corresponding solution will always have a non-symmetric component even near zero load with no bifurcation point on it.

The axial displacement is found from the remaining equilibrium equation $N = EA\bar{\epsilon}$ (see (BA-3.48)$_1$), the definition of $\bar{\epsilon}$ (see (BA-3.44)$_1$) and the boundary condition $v(\pm\theta_0) = 0$ (see

(BA-3.3)$_2$)

$$v(\theta)=\int_{-\theta_0}^{\theta}\Big[w-\frac{1}{2R}\Big(\frac{dw}{d\theta}\Big)^2-\frac{1}{R}\frac{dw}{d\theta}\frac{d\overline{w}}{d\theta}-(\mu\theta_0)^2\frac{s}{k^2}R\theta_0^2\Big]d\theta \tag{BA-3.50}$$

Moreover, the relation between $\mu\theta_0$ and g (or equivalently between N and λ) for the imperfect structure is found from (BA-3.50) for $\theta=\theta_0$ with the help of (BA-3.49) following the same steps as in the derivation of the corresponding equation (BA-3.12) for the perfect structure. The resulting $\mu\theta_0-g$ relationship is cumbersome and hence will not be displayed here.

iib) *Asymptotic Solution of Imperfect Structure*

After presenting the analytical solution for the imperfect pressurized circular arch, we present the asymptotic solution to the same problem according to the general theory of Section AC-4.

The load drop $\Delta\lambda_s$, defined as the difference between the critical load of the perfect structure and the maximum load of the imperfect one is given, to the lowest order in the imperfection amplitude parameter and by (AC-4.31)$_2$, which requires the evaluation of $(\overline{\mathcal{E}}^c_{,uw}\overset{1}{u})\overline{w}$ for the present application. From (BA-3.44)-(BA-3.45), and after using also (BA-3.6) and (BA-3.25) follows

$$((\overline{\mathcal{E}}^c_{,uw})\overset{1}{u})\overline{w}=\int_{-\theta_0}^{\theta_0}\Big[N\frac{d\overset{1}{w}}{d\theta}\frac{d\overline{w}}{d\theta}/R^2+C\frac{d\overset{0}{w}}{d\theta}\frac{d\overset{1}{w}}{d\theta}/R^2\Big]Rd\theta=-\frac{\mu_b^2EI}{R^3}\int_{-\theta_0}^{\theta_0}\frac{d\overset{1}{w}}{d\theta}\frac{d\overline{w}}{d\theta}d\theta \tag{BA-3.51}$$

where the constant $C=0$ at bifurcation as discussed in the previous subsection on the bifurcation point. The remaining quantities required for the determination of $\Delta\lambda_s$, i.e. λ_2 and $((d\mathcal{E}_{,uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$ have already been evaluated in (BA-3.41) and (BA-3.39)$_3$ respectively. Consequently, by substituting (BA-3.39)$_3$, (BA-3.41) and (BA-3.51) into (AC-4.31)$_2$, we find that the asymptotic expansion of the load drop $\Delta\lambda_s$ near the first bifurcation of the pressurized circular arch is

$$\Delta\lambda_s=-\frac{3}{2}\Big(\frac{3}{\theta_0}\frac{EI}{(R\theta_0)^5}\frac{x_b^6/[1+x_b^2]}{5(g_b-1)+2}\Big)^{\frac{1}{3}}\Big[\frac{\mu^2\epsilon}{(d\mu^2/d\lambda)_b}\int_{-\theta_0}^{\theta_0}\frac{d\overset{1}{w}}{d\theta}\frac{d\overline{w}_a}{d\theta}d\theta\Big/\int_{-\theta_0}^{\theta_0}\Big(\frac{d\overset{1}{w}}{d\theta}\Big)^2d\theta\Big]^{2/3}+O(\epsilon) \tag{BA-3.52}$$

Notice in (BA-3.52) only the antisymmetric part $\overline{w}_a$ of the imperfection $\overline{w}$ contributes to $\Delta\lambda_s$. This conclusion is in agreement with the results of the previous subsection on the exact imperfect solution, according to which (see (BA-3.49)) an imperfect solution with no bifurcation point exists near $\overline{\lambda}_I$ only when $\overline{w}_a\neq 0$, while for $\overline{w}_a=0$ the structure bifurcates at $\mu\theta_0=x_b$ and hence $\Delta\lambda_s=0$.

It is obvious from (BA-3.52) that $\Delta\lambda_s$ is maximized when $\overline{w}=\overline{w}_a=\overset{1}{w}$ (recall that $\overline{w}_s$ does not contribute to the $O(\epsilon^{2/3})$ term of $\Delta\lambda_s$) in which case the maximum value $(\Delta\lambda_s)_{\max}$ is found, with the additional help of (BA-3.6) and (BA-3.40) to be

$$(\Delta\lambda_s)_{\max}/\lambda_c = -\frac{3}{4}\frac{\{[3\,x_b^4/2(1+x_b^2)][45(g_b-1)^2+48(g_b-1)+8]^2\}^{1/3}}{[R\theta_0^2]^{2/3}[5(g_b-1)+2]g_b}\epsilon^{2/3}+O(\epsilon) \tag{BA-3.53}$$

where g_b corresponds to the $\overline{\lambda}_I$ root in (BA-3.21) since we are interested in the first bifurcation point encountered as the pressure on the arch increases.

BB TWO-DIMENSIONAL STRUCTURES

Attention is now turned in applications involving two-dimensional problems in structural mechanics. The first application to be given is the stability of a flat plate subjected to in-plane loading. Two cases will be considered: a plate with a single eigenmode at the critical load and a plate with a double mode at the critical load.

BB-1 RECTANGULAR PLATE

The first application of the general theory presented in Section AC for two dimensional structures is the buckling of a simply supported rectangular plate which is uniformly compressed in its own plane. Depending on the geometry of the plate, the lowest critical eigenvalue can be either simple or double. Both cases will be studied.

i) *Von Karman Plate Model*

The simplest possible nonlinear plate model that can be successfully employed in the prediction of buckling of an axially compressed plate is due to T. Von Karman. This model is in essence a kinematically nonlinear (but constitutively linear) elastic model and takes into account the contributions to the strains of the squares of the rotations of the middle surface.

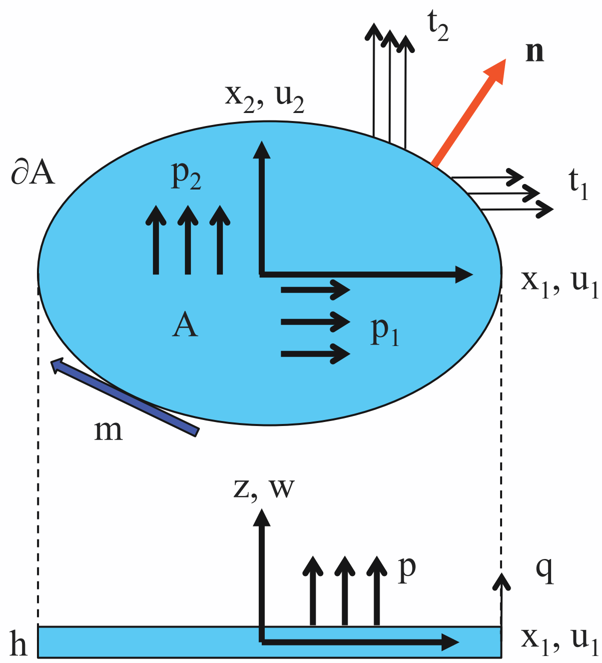

Figure BB-1.1: Von Karman plate.

Consider a thin rectangular plate whose middle surface is a region A in the x_1, x_2 plane and with thickness h, as shown in Fig. BB-1.2. Let $u_\alpha(x_1, x_2)$ $(\alpha = 1, 2)$ be the tangential displacements of an arbitrary point on the middle surface along x_α and $w(x_1, x_2)$ be the vertical displacement of the same point. The small strain - moderate rotation kinematical assumption leads to the following strain-displacement relations for the in-plane membrane strains $E_{\alpha\beta}$ and the curvature strains $K_{\alpha\beta}$

$$E_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + \frac{1}{2}w,_\alpha w,_\beta \,, \quad K_{\alpha\beta} = -w,_{\alpha\beta}, \tag{BB-1.1}$$

where $(\ )_{,\alpha} \equiv \partial(\ )/\partial x_\alpha$. Note also that from here and subsequently in this section Greek indexes range from 1 to 2 while the presence of repeated indexes in an expression implies summation with respect to those indexes from 1 to 2 (Einstein's summation convention).

The membrane resultants $N_{\alpha\beta}$ (N_{11}, N_{22} are the axial forces while $N_{12} = N_{21}$ is the shear force) and the moment resultants $M_{\alpha\beta}$ (M_{11}, M_{22} are the bending moments while $M_{12} = M_{21}$ is the twisting moment) are related to their work conjugate strains $E_{\alpha\beta}$ and $K_{\alpha\beta}$ by

$$N_{\alpha\beta} = hL_{\alpha\beta\gamma\delta}E_{\gamma\delta}, \quad M_{\alpha\beta} = \frac{h^3}{12}L_{\alpha\beta\gamma\delta}K_{\gamma\delta}, \tag{BB-1.2}$$

where the plane stress elastic moduli $L_{\alpha\beta\gamma\delta}$ of the material are given in terms of its Young's modulus E and its Poisson's ratio ν by

$$L_{\alpha\beta\gamma\delta} = \frac{E}{1-\nu^2}\left[\frac{1-\nu}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) + \nu\delta_{\alpha\beta}\delta_{\gamma\delta}\right]. \tag{BB-1.3}$$

The plate's potential energy $\mathcal{E}$, assuming that its midsurface is a region A in the (x_1, x_2) plane with boundary ∂A is

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}\int_A [L_{\alpha\beta\gamma\delta}(E_{\alpha\beta}E_{\gamma\delta} + \frac{h^2}{12}K_{\alpha\beta}K_{\gamma\delta})h]dA \\ &- \int_A [p_\alpha u_\alpha + pw]\, dA - \int_{\partial A} [t_\alpha u_\alpha + qw + m(-w_{,n})]\, ds,\end{aligned} \tag{BB-1.4}$$

where p_α, p are the distributed forces on the surface of the plate and t_α, q are the boundary tractions acting along the x_α, z directions respectively while m is the imposed bending moment at the plate's edge and $(-w_{,n})$ is its work conjugate edge rotation ($w_{,n}$ is the directional derivative of w along the outward normal $\mathbf{n}$ to the boundary ∂A).

ia) *Simply Supported Perfect Plate Under Compression – Simple Eigenvalue Case*

Consider the thin rectangular plate shown in Fig. BB-1.2 of dimensions $a_1 \times a_2$ and of thickness h. The plate is simply supported on all its four edges. Moreover the plate is axially compressed along the x_1 and x_2 directions with the help of four rigid bars which keep the plate's edges straight. The rigid bars are considered to be lubricated at their surface of contact with the thin plate and can transmit normal stresses but not shear ones. The normal forces F_α acting on the rigid bars are assumed to increase in proportion to a scalar parameter $\lambda \geq 0$. More specifically it is assumed that $F_1 = \lambda\overset{0}{\sigma}_{11}a_2h$, $F_2 = \lambda\overset{0}{\sigma}_{22}a_1h$ where $\overset{0}{\sigma}_{\alpha\beta}$ are constants whose physical meaning will be made apparent in the sequel.

From the absence of distributed forces $p_\alpha = p = 0$, and in view of the simple support at the boundary, the vertical displacement $w = 0$ and the edge bending moment $m = 0$. Consequently, the external loading part of the potential energy $\mathcal{E}$ of the plate takes the form

$$\mathcal{E}_{\text{ext}} = -\int_{\partial A} [t_\alpha u_\alpha]ds. \tag{BB-1.5}$$

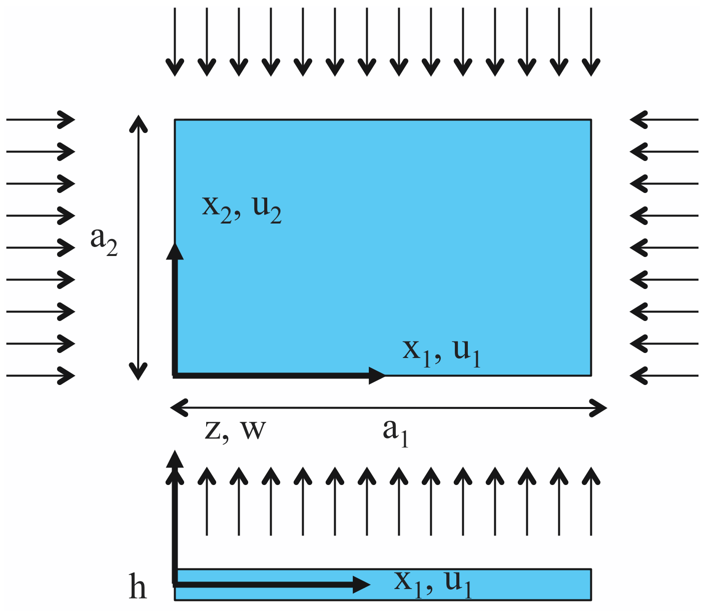

Figure BB-1.2: Rectangular Von Karman plate under in-plane loading.

The trivial principal solution to the plate problem is the one in which the forces applied on the rigid bars are uniformly distributed along the edges of the plate, in which case the traction $t_\alpha = \lambda \overset{0}{\sigma}_{\alpha\beta} n_\beta h$ ($\mathbf{n}$ the outer normal to ∂A) thus implying that $\lambda \overset{0}{\sigma}_{\alpha\beta}$ is the resulting constant stress field inside the plate. From the divergence theorem applied to the two dimensional domain A one has from (BB-1.5)

$$\mathcal{E}_{\text{ext}} = -\lambda \int_A [\overset{0}{\sigma}_{\alpha\beta} u_{\alpha,\beta}] h dA, \tag{BB-1.6}$$

and consequently from (BB-1.4) – (BB-1.6) the plate's potential energy can be written as

$$\begin{aligned}
&\mathcal{E} = \int_A [\frac{1}{2} L_{\alpha\beta\gamma\delta}(E_{\alpha\beta}E_{\gamma\delta} + \frac{h^2}{12} K_{\alpha\beta}K_{\gamma\delta}) - \lambda \overset{0}{\sigma}_{\alpha\beta} u_{\alpha,\beta}] hdA \\
&w(0, x_2) = \ w(a_1, x_2) = w(x_1, 0) = w(x_1, a_2) = 0 \\
&u_1(0, x_2) = \ u_{1,2}(a_1, x_2) = u_2(x_1, 0) = u_{2,1}(x_1, a_2) = 0,
\end{aligned} \tag{BB-1.7}$$

with the membrane ($E_{\alpha\beta}$) and curvature ($K_{\alpha\beta}$) strains given by (BB-1.1). The above essential boundary conditions given in (BB-1.7)$_2$ reflect the fact that the vertical displacement at the edges of the plate has to vanish and that the ends of the plate have to remain straight during the deformation. It is also tacitly assumed that the admissible displacements u_α, w have adequate smoothness as to ensure the finiteness of the potential energy integral in (BB-1.7)$_1$.

The equilibrium equations of the plate are given by extremizing $\mathcal{E}$ with respect to the displacement (see (AC-2.2)). Hence from (BB-1.7) and by recalling (BB-1.1) one obtains

$$\begin{aligned}
&\mathcal{E},_u \delta u = \int_A [L_{\alpha\beta\gamma\delta}(E_{\alpha\beta}\delta E_{\gamma\delta} + \frac{h^2}{12}K_{\alpha\beta}\delta K_{\gamma\delta}) - \lambda\overset{0}{\sigma}_{\alpha\beta}\delta u_{\alpha,\beta}]hdA = 0 \\
&\delta E_{\alpha\beta} = \frac{1}{2}(\delta u_{\alpha,\beta} + \delta u_{\beta,\alpha}) + \frac{1}{2}(w,_\alpha \delta w,_\beta + w,_\beta \delta w,_\alpha), \quad \delta K_{\alpha\beta} = -\delta w,_{\alpha\beta},
\end{aligned} \tag{BB-1.8}$$

where the admissible displacement δu_α, δw satisfy the essential boundary conditions (BB-1.7)$_2$.

It is not difficult to see that the principal solution to (BB-1.8) is the constant membrane force, zero moment solution, i.e.,

$$\begin{aligned}
&\overset{0}{N}_{\alpha\beta} = \lambda h\overset{0}{\sigma}_{\alpha\beta}, \quad (\overset{0}{\sigma}_{11}, \overset{0}{\sigma}_{22} \neq 0, \quad \overset{0}{\sigma}_{12} = \overset{0}{\sigma}_{21} = 0); \quad \overset{0}{M}_{\alpha\beta} = 0 \\
&\overset{0}{u}_1 = (\lambda x_1/E)(\overset{0}{\sigma}_{11} - \nu\overset{0}{\sigma}_{22}), \quad \overset{0}{u}_2 = (\lambda x_2/E)(\overset{0}{\sigma}_{22} - \nu\overset{0}{\sigma}_{11}); \quad \overset{0}{w} = 0,
\end{aligned} \tag{BB-1.9}$$

since a simple substitution of (BB-1.9) into (BB-1.8) proves that the equilibrium equations and boundary conditions are satisfied and since the solution vanishes for $\lambda = 0$.

Of interest is the lowest critical load λ_c for the above found principal solution. To this end one has to find the lowest load λ_c that satisfies (AC-2.7), namely $(\mathcal{E},_{uu}(\overset{0}{u}(\lambda_c), \lambda_c)\overset{1}{u})\delta u = 0$. Upon taking an additional derivative of (BB-1.8) with respect to u, and after noting also from the principal solution (BB-1.9) that $\overset{0}{N}_{\alpha\beta} = hL_{\alpha\beta\gamma\delta}\overset{0}{E}_{\alpha\beta} = \lambda h\overset{0}{\sigma}_{\alpha\beta}$, one obtains at the critical load λ_c

$$(\mathcal{E}^c,_{uu}\overset{1}{u})\delta u = \int_A [L_{\alpha\beta\gamma\delta}(\overset{1}{u}_{\alpha,\beta}\delta u_{\gamma,\delta} + \frac{h^2}{12}\overset{1}{w},_{\alpha\beta}\delta w,_{\gamma\delta}) + \lambda\overset{0}{\sigma}_{\alpha\beta}\overset{1}{w},_\alpha \delta w,_\beta]hdA = 0. \tag{BB-1.10}$$

Note that in the derivation of (BB-1.10) use was also made of the symmetries of the plane stress moduli $L_{\alpha\beta\gamma\delta}$ given in (BB-1.3). It is also understood that the eigenmode $\overset{1}{u} = (\overset{1}{u}_\alpha, \overset{1}{w})$ has to satisfy the essential boundary conditions (BB-1.7)$_2$.

Integration of (BB-1.10) by parts gives the following Euler-Lagrange form of the corresponding equations plus boundary conditions:

$$\begin{aligned}
\delta_{u_\alpha}: \quad & (L_{\alpha\beta\gamma\delta}\overset{1}{u}_{\gamma,\delta}),_\beta = 0 \text{ in } A, \\
& L_{12\gamma\delta}\overset{1}{u}_{\gamma,\delta} = 0 \text{ on } \partial A, \quad \overset{1}{u}_1(0,x_2) = \overset{1}{u}_{1,2}(a_1,x_2) = \overset{1}{u}_2(x_1,0) = \overset{1}{u}_{2,1}(x_1,a_2) = 0, \\
\delta w: \quad & (h^2/12)L_{\alpha\beta\gamma\delta}\overset{1}{w},_{\alpha\beta\gamma\delta} - \lambda\overset{0}{\sigma}_{\alpha\beta}\overset{1}{w},_{\alpha\beta} = 0 \text{ in } A, \\
& \overset{1}{w} = 0 \text{ on } \partial A, \quad \overset{1}{w},_{11}(0,x_2) = \overset{1}{w},_{11}(a_1,x_2) = \overset{1}{w},_{22}(x_1,0) = \overset{1}{w},_{22}(x_1,a_2) = 0.
\end{aligned} \tag{BB-1.11}$$

Observe that (BB-1.11)$_1$ is the equilibrium equation of plane stress linear elasticity with zero distributed loads on A and zero boundary conditions on ∂A. Consequently the unique solution for $\overset{1}{u}_\alpha$ is the zero one i.e.

$$\overset{1}{u}_\alpha = 0. \tag{BB-1.12}$$

On the other hand, one observes that (BB-1.11)$_2$ admits the following solutions $\overset{1}{w}$

$$\overset{1}{w} = h\sin(m\pi x_1/a_1)\sin(n\pi x_2/a_2), \tag{BB-1.13}$$

where m and n are arbitrary positive integers ($m, n \in \mathbb{N}$). A substitution of (BB-1.13) into (BB-1.11)$_2$ gives for the value of $\lambda = \lambda(m,n)$ corresponding to the $h\sin(m\pi x_1/a_1)\sin(n\pi x_2/a_2)$ eigenmode

$$\lambda(m,n) = -\frac{h^2}{12}\,\frac{E}{1-\nu^2}\,\frac{\left[(m\pi/a_1)^2 + (n\pi/a_2)^2\right]^2}{\overset{0}{\sigma}_{11}(m\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n\pi/a_2)^2}. \tag{BB-1.14}$$

Consequently there exists an infinity of critical points for the stability operator of the simply supported plate. Moreover all these points are bifurcation points since according to the general theory from (AC-2.27) $\mathcal{E}^c,_{u\lambda}\overset{1}{u} = 0$. Indeed from (BB-1.8) and (BB-1.12) we deduce

$$\mathcal{E}^c,_{u\lambda}\overset{1}{u} = -\int_A [\overset{0}{\sigma}_{\alpha\beta}\overset{1}{u}_{\alpha,\beta}] h dA = 0. \tag{BB-1.15}$$

Of all these bifurcation points of interest of course is the one corresponding to the lowest critical load. Assume for the time being that the minimum of $\lambda(m,n)$ is achieved for a unique pair of real integers (m_c, n_c)

$$\lambda_c = \lambda(m_c, n_c) = \min_{m,n\in\mathbb{N}}[\lambda(m,n)], \tag{BB-1.16}$$

then the $h\sin(m_c\pi x_1/a_1)\sin(n_c\pi x_2/a_2)$ is the unique eigenmode corresponding to λ_c at which point the principal equilibrium branch encounters a simple bifurcation.

One expects the principal branch of the solution to be stable for $0 \le \lambda < \lambda_c$. Indeed it will be shown that the stability functional of the principal solution $(\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u$ is positive definite in the aforementioned range of loads λ. From (BB-1.8) and (BB-1.9) we have

$$(\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u = \int_A [L_{\alpha\beta\gamma\delta}(\delta u_{\alpha,\beta}\delta u_{\gamma,\delta} + \frac{h^2}{12}\delta w,_{\alpha\beta}\delta w,_{\gamma\delta}) + \lambda\overset{0}{\sigma}_{\alpha\beta}\delta w,_\alpha \delta w,_\beta] h dA. \tag{BB-1.17}$$

A straightforward calculation using (BB-1.3) shows that for $1 > \nu > 0$, $L_{\alpha\beta\gamma\delta}\delta u_{\alpha,\beta}\delta u_{\gamma,\delta} > 0$ if $[\delta u_{\alpha,\beta}]_s \neq 0$ and hence one has to concentrate on the δw dependent part of the integrand in (BB-1.17). Noting that any admissible δw satisfying the essential boundary conditions $\delta w = 0$ or ∂A (see (BB-1.7)$_2$) can be put in the form

$$\delta w = h\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\delta W_{mn}\sin(m\pi x_1/a_1)\sin(n\pi x_2/a_2), \tag{BB-1.18}$$

where the δw dependent part of the quadratic in δu form in (BB-1.17) takes the form:

$$\begin{aligned}&\int_A [\frac{h^2}{12}L_{\alpha\beta\gamma\delta}\delta w,_{\alpha\beta}\delta w,_{\gamma\delta} + \lambda\overset{0}{\sigma}_{\alpha\beta}\delta w,_\alpha \delta w,_\beta] h dA = \\ &\frac{a_1 a_2 h^3}{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\delta W_{mn})^2\,[\lambda(m,n) - \lambda]\,[-\overset{0}{\sigma}_{11}(m\pi/a_1)^2 - \overset{0}{\sigma}_{22}(n\pi/a_2)^2].\end{aligned} \tag{BB-1.19}$$

Since $\overset{0}{\sigma}_{11}$, $\overset{0}{\sigma}_{22} < 0$ (the plate is compressed), it follows from (BB-1.7) and (BB-1.19) that for $0 \leq \lambda < \lambda_c$ (with λ_c given by (BB-1.16)) that the δw part of the stability functional (BB-1.17) is positive definite and hence, given that the δu part of the stability functional is always positive irrespective of the value of λ, the stability of the principal solution for loads up to the lowest bifurcation load λ_c follows from the just proved positive definiteness of (BB-1.17).

One should note at this point that with some additional effort, and after defining an inner product in the space of all admissible displacements, one could have shown the ellipticity of the stability operator in the sense of (AC-2.4), i.e., that a $\overset{0}{\beta}(\lambda) > 0$ exists $(\mathcal{E},_{uu}(\overset{0}{u}(\lambda), \lambda)\delta u)\delta u \geq \overset{0}{\beta}(\lambda)(\delta u, \delta u)$ for $0 \leq \lambda < \lambda_c$, exactly as shown for the relatively simpler case of the elastica (see (BA-1.5)).

Having found the lowest bifurcation load $\lambda_c = \lambda(m_c, n_c)$, attention is focused on the post-bifurcated expansion of the bifurcated equilibrium path. To this end notice that $((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$ can be found from (BB-1.7), (BB-1.9) and (BB-1.13)

$$((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = \int_A [\overset{0}{\sigma}_{\alpha\beta}\overset{1}{w},_\alpha \overset{1}{w},_\beta]hdA = \frac{h^3}{4}a_1 a_2[\overset{0}{\sigma}_{11}(m_c\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n_c\pi/a_2)^2]. \tag{BB-1.20}$$

Notice that since $\overset{0}{\sigma}_{\alpha\beta} < 0$, $((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} < 0$, as expected from the general theory in Section AC-3.

According to the general theory, the first term λ_1 in the asymptotic expansion of the bifurcated equilibrium path through λ_c is given by (AC-3.13). The denominator in that expression has just been calculated above in (BB-1.20) while the numerator, with the help of (BB-1.7), (BB-1.9) and (BB-1.12) yields

$$((\mathcal{E}^c,_{uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = \int_A 3[L_{\alpha\beta\gamma\delta}(\overset{1}{w},_\alpha \overset{1}{w},_\beta \overset{1}{u}_{\gamma,\delta})]hdA = 0, \tag{BB-1.21}$$

which implies from (AC-3.13) that $\lambda_1 = 0$ and hence that the structure will undergo a symmetric bifurcation.

The calculation of the next term λ_2 in the expansion of the load requires according to (AC-3.14) the determination of the next order term $v_{\xi\xi}$ in the expansion of the displacement which is found according to the general theory from (AC-3.8). With the help of (BB-1.7), (BB-1.9), one obtains from (AC-3.8) the following variational equation for $v_{\xi\xi} = (\overset{2}{u}_\alpha, \overset{2}{w})$

$$\begin{aligned} 0 = ((\mathcal{E}^c,_{uuu}\overset{1}{u})\overset{1}{u} + \mathcal{E}^c,_{uu}v_{\xi\xi})\ \delta v = \int_A [L_{\alpha\beta\gamma\delta}(\overset{1}{w},_\gamma \overset{1}{w},_\delta \delta u_{\alpha,\beta} + 2\overset{1}{u}_{\gamma,\delta}\overset{1}{w},_\alpha \delta w,_\beta)]hdA \\ + \int_A L_{\alpha\beta\gamma\delta}(\overset{2}{u}_{\gamma,\delta}\delta u_{\alpha,\beta} + \frac{h^2}{12}\ \overset{2}{w},_{\gamma\delta}\delta w,_{\alpha\beta}) + \lambda_c\overset{0}{\sigma}_{\alpha\beta}\overset{2}{w},_\alpha \delta w,_\beta]hdA. \end{aligned} \tag{BB-1.22}$$

Integration of (BB-1.22) by parts and recalling the admissibility conditions $(\text{BB-1.7})_2$

gives the following point-wise equations and boundary conditions for $\overset{2}{u}_\alpha, \overset{2}{w}$

$$\begin{aligned}
\delta u_\alpha : \quad & (L_{\alpha\beta\gamma\delta}(\overset{2}{u}_{\gamma,\delta} + \overset{1}{w}_{,\gamma}\overset{1}{w}_{,\delta}))_{,\beta} = 0 \text{ in } A; \\
& L_{12\gamma\delta}(\overset{2}{u}_{\gamma,\delta} + \overset{1}{w}_{,\gamma}\overset{1}{w}_{,\delta}) = 0 \text{ on } \partial A,\ \overset{2}{u}_1(0,x_2) = \overset{2}{u}_{1,2}(a_1,x_2) = \overset{2}{u}_2(x_1,0) = \overset{2}{u}_{2,1}(x_1,a_2) = 0 \\
\delta w : \quad & (h^2/12)L_{\alpha\beta\gamma\delta}\overset{2}{w}_{,\alpha\beta\gamma\delta} - \lambda_c \overset{0}{\sigma}_{\alpha\beta}\overset{2}{w}_{,\alpha\beta} = 0 \text{ in } A; \\
& \overset{2}{w} = 0 \text{ on } \partial A, \quad \overset{2}{w}_{,11}(0,x_2) = \overset{2}{w}_{,11}(a_1,x_2) = \overset{2}{w}_{,22}(x_1,0) = \overset{2}{w}_{,22}(x_1,a_2) = 0.
\end{aligned} \tag{BB-1.23}$$

Notice that the equation governing $\overset{2}{w}$ is identical to the one for $\overset{1}{w}$ (compare (BB-1.23)$_1$ and (BB-1.11)$_2$ for $\lambda = \lambda_c$) and hence $\overset{2}{w} = c\overset{1}{w}$. However, in view of the orthogonality $(v_{\xi\xi}, \overset{1}{u}) = 0$ (recall that $v_{\xi\xi} \in \mathcal{N}^\perp$ according to the general theory in subsection AC-3) and the fact that $\overset{1}{u}_\alpha = 0$ from (BB-1.12), one concludes $0 = (v_{\xi\xi}, \overset{1}{u}) = (\overset{2}{w}, \overset{1}{w}) = c(\overset{1}{w}, \overset{1}{w})$ which implies that the constant $c = 0$ and hence

$$\overset{2}{w} = 0. \tag{BB-1.24}$$

The determination of $\overset{2}{u}_\alpha$ is a somewhat more tricky task and is based on the observation that (BB-1.23)$_1$ are the equations of plane stress linear elasticity with a distributed body force $(L_{\alpha\beta\gamma\delta}\overset{1}{w}_{,\gamma}\overset{1}{w}_{,\delta})_{,\beta}$. Consequently, the Airy stress function technique of linear elasticity will be employed in this case. To this end, define the following quantities

$$s_{\alpha\beta} \equiv L_{\alpha\beta\gamma\delta}(\overset{2}{u}_{\gamma,\delta} + \overset{1}{w}_{,\gamma}\overset{1}{w}_{,\delta}) = L_{\alpha\beta\gamma\delta}\mathrm{e}_{\gamma\delta}, \quad e_{\alpha\beta} \equiv \frac{1}{2}(\overset{2}{u}_{\alpha,\beta} + \overset{2}{u}_{\beta,\alpha}) + \overset{1}{w}_{,\alpha}\overset{1}{w}_{,\beta}\,. \tag{BB-1.25}$$

Moreover define the Airy stress function $f(x_1, x_2)$ to be

$$s_{11} \equiv f_{,22}\ , \quad s_{22} \equiv f_{,11}\ , \quad s_{12} = s_{21} \equiv -f_{,12}\,. \tag{BB-1.26}$$

It is not difficult to see that with this definition the in-plane equilibrium equations in (BB-1.23)$_1$ take the form $s_{\alpha\beta,\beta} = 0$ i.e. they are identically satisfied. On the other hand, by inverting the relation between $s_{\alpha\beta}$ and $e_{\alpha\beta}$ from (BB-1.25) by using the elastic moduli expressions in (BB-1.3), one has

$$e_{11} = \frac{1}{E}(s_{11} - \nu s_{22}), \quad e_{22} = \frac{1}{E}(s_{22} - \nu s_{11}), \quad e_{12} = e_{21} = \frac{1+\nu}{E}s_{12}. \tag{BB-1.27}$$

In addition, by exploiting the relation between $e_{\alpha\beta}$, $\overset{2}{u}_\alpha$ and $\overset{1}{w}$ one obtains the compatibility condition

$$e_{11,22} + e_{22,11} - 2e_{12,12} = 2[(\overset{1}{w}_{,12})^2 - \overset{1}{w}_{,11}\overset{1}{w}_{,22}]. \tag{BB-1.28}$$

Upon substitution of (BB-1.27), (BB-1.26) and (BB-1.13) into (BB-1.28) one obtains the following governing equation for f

$$\frac{1}{E}\nabla^4 f = h^2(m_c\pi/a_1)^2(n_c\pi/a_2)^2[\cos(2m_c\pi x_1/a_1) + \cos(2n_c\pi x_2/a_2)], \tag{BB-1.29}$$

where ∇^4 denotes the biharmonic operator i.e. $\nabla^4 f \equiv f_{,1111} + 2f_{,1122} + f_{,2222}$.

The solution of (BB-1.29) is found to be

$$f = \frac{Eh^2}{16}\left[\frac{(n_c/a_2)^2}{(m_c/a_1)^2}\cos(2m_c\pi x_1/a_1) + \frac{(m_c/a_1)^2}{(n_c/a_2)^2}\cos(2n_c\pi x_2/a_2)\right], \tag{BB-1.30}$$

and thus from (BB-1.26), (BB-1.27) upon substitution of (BB-1.30) one has for $s_{\alpha\beta}$

$$s_{11} = -\frac{Eh^2}{4}(m_c\pi/a_1)^2\cos(2n_c\pi x_2/a_2), \quad s_{22} = -\frac{Eh^2}{4}(n_c\pi/a_2)^2\cos(2m_c\pi x_1/a_1), \quad s_{12} = s_{21} = 0, \tag{BB-1.31}$$

and for $e_{\alpha\beta}$

$$\begin{aligned} e_{11} = & -\frac{h^2}{4}[(m_c\pi/a_1)^2\cos(2n_c\pi x_2/a_2) - \nu(n_c\pi/a_2)^2\cos(2m_c\pi x_1/a_1)], \quad e_{12} = 0. \\ e_{22} = & -\frac{h^2}{4}[(n_c\pi/a_2)^2\cos(2m_c\pi x_1/a_1) - \nu(m_c\pi/a_1)^2\cos(2n_c\pi x_2/a_2)], \quad e_{12} = 0. \end{aligned} \tag{BB-1.32}$$

Finally, from (BB-1.25)$_2$ and (BB-1.32) one obtains for $\overset{2}{u}_\alpha$

$$\begin{aligned} \overset{2}{u}_1 = -\frac{h^2}{4}\{[(m_c\pi/a_1)^2x_1 + [\frac{(m_c\pi/a_1)^2 - \nu(n_c\pi/a_2)^2}{2(m_c\pi/a_1)} - \frac{1}{2}(m_c\pi/a_1)\cos(2n_c\pi x_2/a_2)]\sin(2m_c\pi x_1/a_1)\} \\ \overset{2}{u}_2 = -\frac{h^2}{4}\{[(n_c\pi/a_2)^2x_2 + [\frac{(n_c\pi/a_2)^2 - \nu(m_c\pi/a_1)^2}{2(n_c\pi/a_1)} - \frac{1}{2}(n_c\pi/a_2)\cos(2m_c\pi x_1/a_1)]\sin(2n_c\pi x_2/a_2)\} \end{aligned} \tag{BB-1.33}$$

It is not difficult to verify that the $\overset{2}{u}_\alpha$ found above, also verify all the boundary conditions in (BB-1.23)$_1$.

The last ingredient according to the general theory (see (AC-3.14)) required for the calculation of λ_2 is the numerator $(((\mathcal{E}^c{,_{uuuu}}\overset{1}{u})\overset{1}{u} + 3\,\mathcal{E}^c{,_{uuu}}\,v_{\xi\xi})\overset{1}{u})\overset{1}{u}$. To this end by using (BB-1.7), the wanted numerator in the expression for λ_2 according to (AC-3.14) becomes

$$\begin{aligned} (((\mathcal{E}^c{,_{uuuu}}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E}^c{,_{uuu}}\,v_{\xi\xi})\overset{1}{u})\overset{1}{u} &= 3\int_A [L_{\alpha\beta\gamma\delta}(\overset{2}{u}_{\gamma,\delta} + \overset{1}{w}_{,\gamma}\overset{1}{w}_{,\delta})\overset{1}{w}_{,\alpha}\overset{1}{w}_{,\beta}]hdA = \\ 3\int_A [s_{\alpha\beta}\overset{1}{w}_{,\alpha}\overset{1}{w}_{,\beta}]hdA &= \frac{3}{32}Eh^5a_1a_2[(m_c\pi/a_1)^4 + (n_c\pi/a_2)^4], \end{aligned} \tag{BB-1.34}$$

where in the derivation of (BB-1.34) use was also made of (BB-1.12), (BB-1.13), (BB-1.24), (BB-1.25), (BB-1.31). The denominator of λ_2 has already been found in (BB-1.20) and consequently from (AC-3.14) λ_2 is finally given by

$$\lambda_2 = -\frac{Eh^2}{8}\;\frac{(m_c\pi/a_1)^4 + (n_c\pi/a_2)^4}{\overset{0}{\sigma}_{11}(m_c\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n_c\pi/a_2)^2} = \frac{3}{2}\lambda_c(1-\nu^2)\frac{(m_c\pi/a_1)^4 + (n_c\pi/a_2)^4}{[(m_c\pi/a_1)^2 + (n_c\pi/a_2)^2]^2}. \tag{BB-1.35}$$

Recalling from the discussion of (BB-1.20) that $\overset{0}{\sigma}_{11}(m_c\pi/a_1)^2 + \overset{0}{\sigma}_{22}(n_c\pi/a_2)^2 < 0$ (or equivalently since $\lambda_c > 0$) we conclude that $\lambda_2 > 0$ which implies that the simple symmetric bifurcation of the simply supported plate gives a stable bifurcation equilibrium branch.

Example

As an application to the above analysis consider the case of a square plate $a_1 = a_2 = a$ under equibiaxial compression $\overset{0}{\sigma}_{11} = \overset{0}{\sigma}_{22} = -1$. Then from (BB-1.14), (BB-1.35)

$$\lambda_c = \frac{\pi^2}{6}\ \frac{E}{1-\nu^2}\left(\frac{h}{a}\right)^2, \quad m_c = n_c = 1; \quad \lambda_2 = \frac{3}{4}\ \frac{\lambda_c}{1-\nu^2}. \tag{BB-1.36}$$

ib) *Simply Supported Perfect Plate Under Compression – Double Eigenvalue Case*

The analysis in the previous subsection is valid when the lowest bifurcation load λ_c corresponds to a unique eigenmode i.e. when the pair of integers m_c, n_c that minimizes $\lambda(m, n)$ (see (BB-1.14)) is unique. Suppose now that $a_1 = \sqrt{2}a$, $a_2 = a$ and that the plate is compressed only along the x_1 direction, i.e. $\overset{0}{\sigma}_{11} = -1$, $\overset{0}{\sigma}_{22} = 0$. Under those conditions it is not difficult to see that the lowest bifurcation load λ_c corresponds to a double eigenmode since from (BB-1.16)

$$\lambda_c = \frac{3\pi^2}{8}\frac{E}{1-\nu^2}\left(\frac{h}{a}\right)^2, \quad (m_c, n_c) = (1,1) \text{ or } (2,1), \tag{BB-1.37}$$

while the corresponding bifurcation eigenmodes are (see (BB-1.13))

$$\overset{1}{w} = h\sin(\pi x_1/a_1)\sin(\pi x_2/a_2), \quad \overset{2}{w} = h\sin(2\pi x_1/a_1)\sin(\pi x_2/a_2); \quad a_1 = \sqrt{2}a_2. \tag{BB-1.38}$$

The corresponding in-plane displacements $\overset{i}{u}_\alpha$ to each mode with $\overset{i}{w}$ out-of-plane displacement are from (BB-1.12)

$$\overset{1}{u}_\alpha = 0, \quad \overset{2}{u}_\alpha = 0. \tag{BB-1.39}$$

The bifurcation is still a symmetric one i.e. $\mathcal{E}_{ijk} = 0$ since according to (AC-5.11) and with the help of (BB-1.7), (BB-1.9) and (BB-1.39) one obtains

$$((\mathcal{E}^c{,_{uuu}}\overset{i}{u})\overset{j}{u})\overset{k}{u} = \int_A [L_{\alpha\beta\gamma\delta}(\overset{i}{w}{,_\alpha}\overset{j}{w}{,_\beta}\overset{k}{u}_{\gamma,\delta} + \overset{j}{w}{,_\alpha}\overset{k}{w}{,_\beta}\overset{i}{u}_{\gamma,\delta} + \overset{k}{w}{,_\alpha}\overset{i}{w}{,_\beta}\overset{j}{u}_{\gamma,\delta})]hdA = 0 \tag{BB-1.40}$$

Following the general theory developed in Section AC-5, the calculation of the bifurcated equilibrium paths through λ_c requires first the evaluation of $v_{ij} = (\overset{ij}{u}_\alpha, \overset{ij}{w})$ satisfying (see (AC-5.9) for the corresponding definition)

$$\begin{aligned} 0 = \ & [(\mathcal{E}^c{,_{uuu}}\overset{i}{u})\overset{j}{u} + \mathcal{E}^c_{,uu}v_{ij})\delta v = \int_A [L_{\alpha\beta\gamma\delta}(\overset{i}{w}{,_\gamma}\overset{j}{w}{,_\delta}\delta u_{\alpha,\beta} + \overset{i}{u}_{\gamma,\delta}\overset{j}{w}{,_\alpha}\delta w{,_\beta} + \overset{j}{u}_{\gamma,\delta}\overset{i}{w}{,_\alpha}\delta w{,_\beta}]hdA \\ & + \int_A [L_{\alpha\beta\gamma\delta}(\overset{ij}{u}_{\gamma,\delta}\delta u_{\alpha,\beta} + \frac{h^2}{12}\overset{ij}{w}{,_{\gamma\delta}}\delta w{,_{\alpha\beta}}) + \lambda_c\overset{0}{\sigma}_{\alpha\beta}\overset{ij}{w}{,_\alpha}\delta w{,_\beta}]hdA \end{aligned} \tag{BB-1.41}$$

In the above derivations use was made of (BB-1.7), (BB-1.9). Integrating (BB-1.41) by parts and recalling once more the admissibility conditions $(\text{BB-1.7})_2$ as well as (BB-1.39) we obtain the following Euler - Lagrange equations for $\overset{ij}{u}$, $\overset{ij}{w}$

$$
\begin{aligned}
\delta u_\alpha \quad : \quad & L_{\alpha\beta\gamma\delta}(\overset{ij}{u}_{\gamma,\delta} + \overset{i}{w}_{,\gamma}\,\overset{j}{w}_{,\delta}\,)_{,\beta} = 0 \text{ in } A, \\
& L_{12\gamma\delta}(\overset{ij}{u}_{\gamma,\delta} + \overset{i}{w}_{,\gamma}\,\overset{j}{w}_{,\delta}\,) = 0 \text{ on } \partial A, \quad \overset{ij}{u}_1(0,x_2) = \overset{ij}{u}_{1,2}(a_1,x_2) = \overset{ij}{u}_2(x_1,0) = \overset{ij}{u}_{2,1}(x_1,a_2) = 0 \\
\delta w \quad : \quad & \frac{h^2}{12} L_{\alpha\beta\gamma\delta}\overset{ij}{w}_{,\,\alpha\beta\gamma\delta} - \lambda_c \overset{0}{\sigma}_{\alpha\beta}\overset{ij}{w}_{,\alpha\beta} = 0 \text{ in } A, \\
& \overset{ij}{w} = 0 \text{ on } \partial A, \quad \overset{ij}{w}_{,11}(0,x_2) = \overset{ij}{w}_{,11}(a_1,x_2) = \overset{ij}{w}_{,22}(x_1,0) = \overset{ij}{w}_{,22}(x_1,a_2) = 0.
\end{aligned}
\tag{BB-1.42}
$$

Notice that similarly to the simple eigenmode case, the equation governing $\overset{ij}{w}$ is identical to (BB-1.11)$_2$ governing each $\overset{k}{w}$ and thus $\overset{ij}{w} = c_{ij}\overset{1}{w} + d_{ij}\overset{2}{w}$ where c_{ij} and d_{ij} are constants. Since $v_{ij} = (\overset{ij}{u}_\alpha, \overset{ij}{w})$ is orthogonal to the eigenmodes $\overset{k}{u} = (0, \overset{k}{w}),\ k = 1,2$ at λ_c, we have $0 = (v_{ij}, \overset{k}{u}) = (\overset{ij}{w}, \overset{k}{w})$ which implies $c_{ij} = d_{ij} = 0$ and hence

$$\overset{ij}{w} = 0. \tag{BB-1.43}$$

The determination of $\overset{ij}{u}_\alpha$ is done in a similar way as for the simple buckling mode case. To this end define, in analogy to (BB-1.25) the stress $\overset{ij}{s}_{\alpha\beta}$ and strain $\overset{ij}{e}_{\alpha\beta}$ quantities

$$\overset{ij}{s}_{\alpha\beta} \equiv L_{\alpha\beta\gamma\delta}(\overset{ij}{u}_{\gamma,\delta} + \overset{i}{w}_{,\gamma}\,\overset{j}{w}_{,\delta}\,) = L_{\alpha\beta\gamma\delta}\overset{ij}{e}_{\gamma\delta}, \quad \overset{ij}{e}_{\alpha\beta} \equiv \frac{1}{2}(\overset{ij}{u}_{\alpha,\beta} + \overset{ij}{u}_{\beta,\alpha}) + \frac{1}{2}(\overset{i}{w}_{,\alpha}\,\overset{j}{w}_{,\beta} + \overset{j}{w}_{,\alpha}\,\overset{i}{w}_{,\beta}\,). \tag{BB-1.44}$$

In analogy to (BB-1.26) one also defines the Airy stress functions $\overset{ij}{f}$ to be

$$\overset{ij}{s}_{11} \equiv \overset{ij}{f}_{,22}\,, \quad \overset{ij}{s}_{22} \equiv \overset{ij}{f}_{,11}\,, \quad \overset{ij}{s}_{12} = \overset{ij}{s}_{21} \equiv -\overset{ij}{f}_{,12}\,, \tag{BB-1.45}$$

which satisfy automatically the equilibrium equations (BB-1.42)$_1$ in A, i.e. $\overset{ij}{s}_{\alpha\beta,\beta} = 0$. By inverting the relation (BB-1.44) between $\overset{ij}{s}_{\alpha\beta}$ and $\overset{ij}{e}_{\alpha\beta}$ one has, in analogy to (BB-1.27)

$$\overset{ij}{e}_{11} = \frac{1}{E}(\overset{ij}{s}_{11} - \nu\overset{ij}{s}_{22}), \quad \overset{ij}{e}_{22} = \frac{1}{E}(\overset{ij}{s}_{22} - \nu\overset{ij}{s}_{11}), \quad \overset{ij}{e}_{12} = \overset{ij}{e}_{21} = \frac{1+\nu}{E}\overset{ij}{s}_{12}. \tag{BB-1.46}$$

The compatibility condition, analogous to (BB-1.28) is found by exploiting the definition of $\overset{ij}{e}_{\alpha\beta}$ in terms of $\overset{ij}{u}_\alpha$ and $\overset{i}{w}$ in (BB-1.44) to be

$$\overset{ij}{e}_{11,22} + \overset{ij}{e}_{22,11} - 2\overset{ij}{e}_{12,12} = 2\overset{i}{w}_{,12}\,\overset{j}{w}_{,12} - \overset{i}{w}_{,11}\,\overset{j}{w}_{22} - \overset{j}{w}_{,11}\,\overset{i}{w}_{,22}\,. \tag{BB-1.47}$$

Upon substitution of (BB-1.38) and (BB-1.45), (BB-1.46) into (BB-1.47) one obtains the following equations for $\overset{ij}{f}$

$$\frac{1}{E}\nabla^4(\overset{ij}{f}) = 2\overset{i}{w}_{,12}\,\overset{j}{w}_{,12} - \overset{i}{w}_{,11}\,\overset{j}{w}_{22} - \overset{j}{w}_{,11}\,\overset{i}{w}_{,22} \tag{BB-1.48}$$

The solution of (BB-1.48) gives the following results for $\overset{ij}{f}$

$$\overset{11}{f} = \frac{Eh^2}{16}\left[\left(\frac{a_1}{a_2}\right)^2\cos(2\pi x_1/a_1) + \left(\frac{a_2}{a_1}\right)^2\cos(2\pi x_2/a_2)\right]$$

$$\overset{22}{f} = \frac{Eh^2}{16}\left[\left(\frac{a_1}{2a_2}\right)^2\cos(4\pi x_1/a_1) + \left(\frac{2a_2}{a_1}\right)^2\cos(2\pi x_2/a_2)\right]$$

$$\overset{12}{f} = \frac{Eh^2}{4}\left\{\frac{\cos(3\pi x_1/a_1)}{(3a_2/a_1)^2} - \frac{\cos(\pi x_1/a_1)}{(a_2/a_1)^2} + \left[\frac{9\cos(\pi x_1/a_1)}{[(a_2/a_1)+(4a_1/a_2)]^2} - \frac{\cos(3\pi x_1/a_1)}{[(9a_2/a_1)+(4a_1/a_2)]^2}\right]\cos(2\pi x_2/a_2)\right\} \tag{BB-1.49}$$

Consequently, using (BB-1.49) into (BB-1.45) one obtains for $\overset{ij}{s}_{\alpha\beta}$. Likewise one can also calculate $\overset{ij}{e}_{\alpha\beta}$ from (BB-1.46) and (BB-1.49) and subsequently from (BB-1.44) with the help of (BB-1.39) one can find $\overset{ij}{u}_{\alpha}$. It turns out only $\overset{ij}{s}_{\alpha\beta}$ are required in the subsequent calculations for λ_2.

As discussed in the general theory in Section AC-5, the coefficients $\mathcal{E}_{ijkl}$ and $\mathcal{E}_{ij\lambda}$ are required in order to calculate λ_2 and the initial tangents $\{\alpha_i^1\}$ to the different equilibrium paths through λ_c for the symmetric multiple bifurcation (see (AC-5.15) and the definitions for $\mathcal{E}_{ijkl}$ and $\mathcal{E}_{ij\lambda}$ in (AC-5.11). By using (BB-1.3), (BB-1.7) and (BB-1.44) one obtains for $\mathcal{E}_{ijkl}$

$$\begin{aligned}\mathcal{E}_{ijkl} &= \int_A [L_{\alpha\beta\gamma\delta}[\overset{i}{w}_{,\alpha}\overset{j}{w}_{,\beta}(\overset{kl}{u}_{\gamma,\delta} + \overset{k}{w}_{,\gamma}\overset{l}{w}_{,\delta}) + \overset{i}{w}_{,\alpha}\overset{l}{w}_{,\beta}(\overset{jk}{u}_{\gamma,\delta} + \overset{j}{w}_{,\gamma}\overset{k}{w}_{,\delta}) + \overset{i}{w}_{,\alpha}\overset{k}{w}_{,\beta}(\overset{jl}{u}_{\gamma,\delta} + \overset{j}{w}_{,\gamma}\overset{l}{w}_{,\delta})]hdA \\ &= \int_A [\overset{kl}{s}_{\alpha\beta}\overset{i}{w}_{,\alpha}\overset{j}{w}_{,\beta} + \overset{kj}{s}_{\alpha\beta}\overset{i}{w}_{,\alpha}\overset{l}{w}_{,\beta} + \overset{jl}{s}_{\alpha\beta}\overset{i}{w}_{,\alpha}\overset{k}{w}_{,\beta}]hdA\end{aligned} \tag{BB-1.50}$$

and for $\mathcal{E}_{ij\lambda}$

$$\mathcal{E}_{ij\lambda} = \int_A [\overset{0}{\sigma}_{\alpha\beta}\overset{i}{w}_{,\alpha}\overset{j}{w}_{,\beta}]hdA \tag{BB-1.51}$$

It is interesting to note that $\mathcal{E}_{ijkl}$ and $\mathcal{E}_{ij\lambda}$ are completely symmetric with respect to any permutation of indexes. By using (BB-1.49), as well as (BB-1.38), into (BB-1.50) one obtains for the nonzero coefficients $\mathcal{E}_{ijkl}$

$$\mathcal{E}_{1111} = \frac{15\sqrt{2}}{128}Eh^5\frac{\pi^4}{a^2}, \quad \mathcal{E}_{2222} = \frac{15\sqrt{2}}{32}Eh^5\frac{\pi^4}{a^2}, \quad \mathcal{E}_{1122} = \frac{14135\sqrt{2}}{83232}Eh^5\frac{\pi^4}{a^2} \tag{BB-1.52}$$

The remaining nonzero $\mathcal{E}_{ijkl}$ are given by all the possible index permutations in view of the symmetries discussed. The $\mathcal{E}_{ij\lambda}$ coefficients are found by substituting (BB-1.38) into (BB-1.51) (and recalling also that the only nonzero $\overset{0}{\sigma}_{\alpha\beta}$ is $\overset{0}{\sigma}_{11} = -1$. Hence one obtains for the nonzero coefficients $\mathcal{E}_{ij\lambda}$

$$\mathcal{E}_{11\lambda} = -\frac{\sqrt{2}}{8}h^3\pi^2, \quad \mathcal{E}_{22\lambda} = -\frac{\sqrt{2}}{2}h^3\pi^2 \tag{BB-1.53}$$

A more convenient way to proceed with the algebraic calculations is to rewrite the nonzero components of the coefficients $\mathcal{E}_{ijkl}$ and $\mathcal{E}_{ij\lambda}$ in terms of three positive constants $\alpha,\ \delta,\ \gamma$

as follows:

$$\mathcal{E}_{1111} = \alpha,\ \mathcal{E}_{2222} = 4\alpha,\ \mathcal{E}_{1122} = \mathcal{E}_{2211} = (4\alpha/3)(1+\delta); \quad \mathcal{E}_{11\lambda} = -\gamma,\ \mathcal{E}_{22\lambda} = -4\gamma,$$
$$\alpha \equiv (15\sqrt{2}/128)Eh^5\pi^4/a^2, \quad \gamma \equiv (\sqrt{2}/8)h^3\pi^2, \quad \delta \equiv 226/2601. \tag{BB-1.54}$$

Consequently, from (AC-5.15) of the general theory and by substituting in (BB-1.54), the initial direction α_i^1 of the bifurcated equilibrium solutions at λ_c satisfy the following system

$$\begin{aligned} \alpha(\alpha_1^1)\left[(\alpha_1^1)^2 + 4(1+\delta)(\alpha_2^1)^2 - 3\lambda_2(\gamma/\alpha)\right] &= 0 \\ 4\alpha(\alpha_2^1)\left[(1+\delta)(\alpha_1^1)^2 + (\alpha_2^1)^2 - 3\lambda_2(\gamma/\alpha)\right] &= 0 \\ (\alpha_1^1)^2 + (\alpha_2^1)^2 &= 1 \end{aligned} \tag{BB-1.55}$$

The above system admits four real solutions

$$\begin{aligned} &N1: \ \alpha_1^1 = 1, \quad \alpha_2^1 = 0; \quad \lambda_2 = \alpha/3\gamma \\ &N2: \ \alpha_1^1 = 0, \quad \alpha_2^1 = 1; \quad \lambda_2 = \alpha/3\gamma \\ &N3: \ \alpha_2^1/\alpha_1^1 = [\delta/(3+4\delta)]^{1/2}; \quad \lambda_2 = (\alpha/3\gamma)[1 + \delta(3+4\delta)/(3+5\delta)] \\ &N4: \ \alpha_2^1/\alpha_1^1 = -[\delta/(3+4\delta)]^{1/2}; \quad \lambda_2 = (\alpha/3\gamma)[1 + \delta(3+4\delta)/(3+5\delta)] \end{aligned} \tag{BB-1.56}$$

Notice that since all $\lambda_2 > 0$, all the bifurcated paths are supercritical at λ_c, i.e. occur under increasing load.

Next the stability of the above bifurcated branches is to be examined, at least in the neighborhood of λ_c. To this end one has according to the general theory presented in Section AC-5 to examine the positive definiteness of the matrix B_{ij} given by (AC-5.16)

$$B_{ij} = \begin{bmatrix} \mathcal{E}_{1111}(\alpha_1^1)^2 + \mathcal{E}_{1122}(\alpha_2^1)^2 + \lambda_2\mathcal{E}_{11\lambda} & 2\mathcal{E}_{1122}\alpha_1^1\alpha_2^1 \\ 2\mathcal{E}_{1122}\alpha_1^1\alpha_2^1 & \mathcal{E}_{2211}(\alpha_1^1)^2 + \mathcal{E}_{2222}(\alpha_2^1)^2 + \lambda_2\mathcal{E}_{22\lambda} \end{bmatrix} \tag{BB-1.57}$$

where in the derivation of (BB-1.57) from (AC-5.16) use was made of the fact that the only nonzero coefficients $\mathcal{E}_{ijkl}$ and $\mathcal{E}_{ij\lambda}$ are given by (BB-1.52), (BB-1.53). The evaluation of the stability matrix (BB-1.57) for the four bifurcated branches found in (BB-1.56) gives the following results

$$\begin{aligned} &N1: \quad B_{ij} = \frac{2\alpha}{3}\begin{bmatrix} 1 & 0 \\ 0 & 2\delta \end{bmatrix} \implies \text{Stable} \\ &N2: \quad B_{ij} = \frac{\alpha}{3}\begin{bmatrix} 3+4\delta & 0 \\ 0 & 8 \end{bmatrix} \implies \text{Stable} \\ &N3: \quad B_{ij} = \frac{2\alpha}{3(3+5\delta)}\begin{bmatrix} 3+4\delta & 4(1+\delta)\sqrt{\delta(3+4\delta)} \\ 4(1+\delta)\sqrt{\delta(3+4\delta)} & 4\delta \end{bmatrix} \implies \text{Unstable} \\ &N4: \quad B_{ij} = \frac{2\alpha}{3(3+5\delta)}\begin{bmatrix} 3+4\delta & -4(1+\delta)\sqrt{\delta(3+4\delta)} \\ -4(1+\delta)\sqrt{\delta(3+4\delta)} & 4\delta \end{bmatrix} \implies \text{Unstable} \end{aligned} \tag{BB-1.58}$$

It is interesting to note that unlike the simple mode case where the post-bifurcation equilibrium branch of the plate was stable, in the double eigenmode buckling case one finds that two of the four bifurcated equilibrium branches, N3 and N4, are unstable. However, since for all the post-bifurcated branches $\lambda_2 > 0$, no imperfection sensitivity is expected and the corresponding imperfect plate will not exhibit any snap-through instability at a load lower than λ_c.

BB-2 CIRCULAR CYLINDER

The second application of the general theory presented in Section AC for two dimensional structures is the buckling of a circular cylinder held between two frictionless parallel rigid plates under two different dead loads: a) a uniform lateral surface compression and b) axial compression. The boundary conditions are chosen to ensure constant stresses at the corresponding principal solutions, thus permitting analytic expressions for the asymptotic expansions of the bifurcated equilibrium paths.

The cylinder under lateral surface compression has a simple bifurcation at the lowest critical load while the cylinder under axial compression exhibits a number of simultaneous and nearly simultaneous modes near the lowest critical load. Due to the high number of interacting modes, whose number increases significantly with the decreasing thickness of the shell, the axially compressed cylinder is one of the most common examples of a structure with a highly imperfection sensitive buckling.

i) *Sanders Cylinder Model*

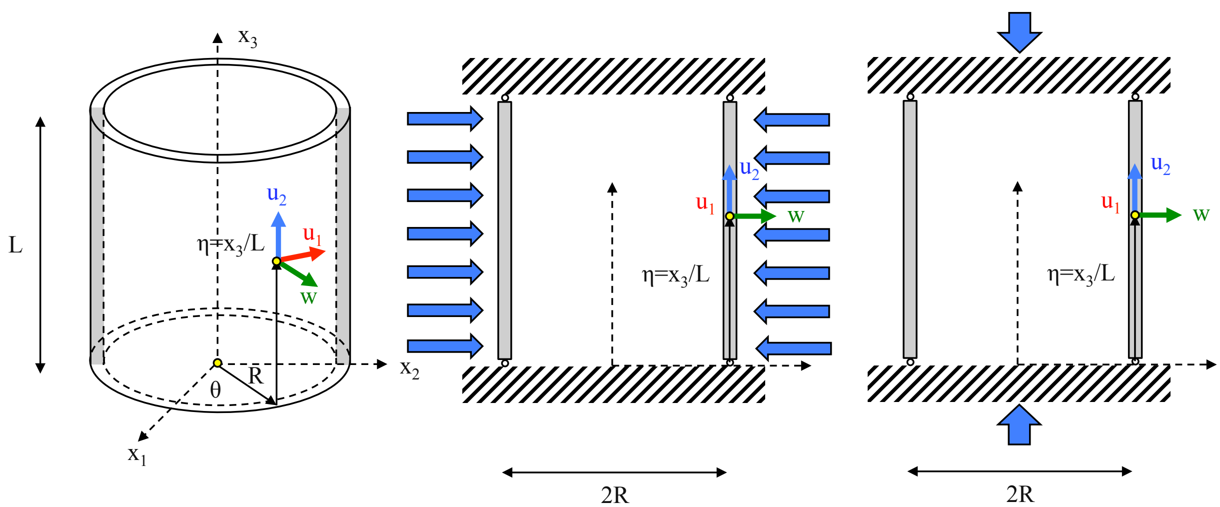

Figure BB-2.1: Circular cylinder under lateral pressure and axial compression.

The simplest possible nonlinear cylinder model that can be employed in the study of the two cylinder buckling problems of interest here, and which assumes a thin shell ($h/R \ll 1$) under small elastic strains and a linearly elastic response under plane stress conditions, is due to Sanders. This model is the cylinder generalization of the Von Karman model for the flat plate to which it reduces when the cylinder radius becomes infinite. The nonlinearity of the model comes from the kinematic assumptions according to which the squares of the rotations of the middle surface about its tangents contribute to the axial strain, exactly as in the Von Karman plate theory, while the rotation about the normal of the shell is negligible and does not contribute to the strains. Also, as in the Von Karman plate theory, the normality assumption for cross sections initially perpendicular to the middle surface is adopted, which leads to the absence of transverse shear strains in the theory.

Consider a thin circular cylindrical shell with radius R, initial length L, and initial thickness h held between two lubricated parallel rigid plates as shown in Fig. BB-2.1. The parameterization of the cylinder's middle surface requires two dimensionless coordinates namely θ (θ is the circumferential angle of longitude) and $\eta \equiv x_3/R$ (x_3 is the distance from the plane $x_1\,x_2$). Note that $0 \leq \theta \leq 2\pi,\ 0 \leq \eta \leq \ell \equiv L/R$.

The middle surface of the cylinder has displacements $Ru_1(\theta,\eta),\ Ru_2(\theta,\eta),\ Rw(\theta,\eta)$ along the circumferential, the axial and the normal direction respectively. The dimensionless membrane strains $E_{\alpha\beta}$ and bending strains $K_{\alpha\beta}$ are given in terms of the dimensionless displacements $u,\ v,\ w$ and rotations β_α about the tangents to the $\theta = const$ and $\eta = const$ lines on the surface

$$
\begin{aligned}
&E_{\alpha\beta} = e_{\alpha\beta} + \frac{1}{2}\beta_\alpha\beta_\beta, \quad K_{\alpha\beta} = \frac{1}{2}(\beta_{\alpha,\beta} + \beta_{\beta,\alpha}),\\
&e_{11} = u_{1,\theta} + w, \quad e_{22} = u_{2,\eta}, \quad e_{12} = e_{21} = \frac{1}{2}(u_{1,\eta} + u_{2,\theta}),\\
&\beta_1 = u_1 - w_{,\theta}\,, \quad \beta_2 = -w_{,\eta}\,.
\end{aligned}
\tag{BB-2.1}
$$

The membrane and bending resultants $N_{\alpha\beta}$ and $M_{\alpha\beta}$ are given in terms of the corresponding strain measures defined in ??:

$$
N_{\alpha\beta} = EhL_{\alpha\beta\gamma\delta}E_{\gamma\delta}, \quad M_{\alpha\beta} = E\frac{h^3}{12R}L_{\alpha\beta\gamma\delta}K_{\gamma\delta},
\tag{BB-2.2}
$$

where the dimensionless plane stress incremental moduli $L_{\alpha\beta\gamma\delta}$ are given in terms of the Poisson ratio ν by (compare to ??):

$$
L_{\alpha\beta\gamma\delta} = \frac{1}{1-\nu^2}\left[\frac{1-\nu}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) + \nu\delta_{\alpha\beta}\delta_{\gamma\delta}\right].
\tag{BB-2.3}
$$

The internal and external (corresponding to dead loads) potential energy of the cylinder are given by:

$$
\begin{aligned}
&\mathcal{E}'_{int} = \frac{1}{2}ER^2h\int_0^\ell\int_0^{2\pi}[L_{\alpha\beta\gamma\delta}E_{\alpha\beta}E_{\gamma\delta} + \frac{h^2}{12R^2}K_{\alpha\beta}K_{\gamma\delta}]d\theta d\eta,\\
&\mathcal{E}'_{ext} = -ER^3\int_0^\ell\int_0^{2\pi}[p_\alpha u_\alpha + pw]d\theta d\eta - ER^2h\int_0^{2\pi}\left[\left[t_\alpha u_\alpha + qw + m(-w_{,\eta})\right]_{\eta=0}^{\eta=\ell}\right]d\theta,
\end{aligned}
\tag{BB-2.4}
$$

where $p_\alpha,\ p$ are the dimensionless distributed dead loads (two tangent, one normal) on the surface of the cylinder, t_α and q are the dimensionless boundary (two in − plane tractions, one shear force) resultants and m is the dimensionless bending moment at the boundary with $-w_{,\eta}$ its work conjugate rotation.

For the two different loadings considered, and in view of the adopted loading ($p_\alpha = 0$) and boundary conditions ($t_1 = q = 0,\ u_{2,\theta} = w_{,\eta} = 0$), the principal solution has constant membrane resultants $\overset{0}{N}_{\alpha\beta} = \lambda Eh\overset{0}{\sigma}_{\alpha\beta}$ (the dimensionless constants $\overset{0}{\sigma}_{\alpha\beta}$ will be specified subsequently for each of the two different loadings) and no moments $\overset{0}{M}_{\alpha\beta} = 0$. From the principle of virtual work applied to the above mentioned principal solution, i.e., $\mathcal{E}'_u\,\delta u = 0$; $\mathcal{E}' \equiv \mathcal{E}'_{int} + \mathcal{E}'_{ext}$, one obtains from ?? that the external part of the potential energy can be rewritten as (compare to ??, ??):

$$\mathcal{E}'_{ext} = -ER^2h \int_0^\ell \int_0^{2\pi} [\lambda\overset{0}{\sigma}_{\alpha\beta} e_{\alpha\beta}] d\theta d\eta. \tag{BB-2.5}$$

Consequently, from ?? and ?? the cylinder's dimensionless potential energy $\mathcal{E} = \mathcal{E}'/ER^2h$ can be written as:

$$\mathcal{E} = \int_0^\ell \int_0^{2\pi} \Big[\frac{1}{2}L_{\alpha\beta\gamma\delta}(E_{\alpha\beta}E_{\gamma\delta} + \frac{h^2}{12R^2}K_{\alpha\beta}K_{\gamma\delta}) - \lambda\overset{0}{\sigma}_{\alpha\beta}e_{\alpha\beta}\Big] d\theta d\eta. \tag{BB-2.6}$$

The accompanying essential boundary conditions as well as $\overset{0}{\sigma}_{\alpha\beta}$ for the two types of loads considered will be given along with the corresponding bifurcation study.

ia) *Cylinder Under Uniform Lateral Compression*

For the circular cylinder held between two fixed, parallel rigid plates and subjected to uniform lateral external compression, the essential boundary conditions that correspond to the potential energy in ?? are:

$$u_2(\theta,0) = u_2(\theta,\ell) = w_{,\eta}(\theta,0) = w_{,\eta}(\theta,\ell) = 0, \tag{BB-2.7}$$

and the corresponding dimensionless constants $\overset{0}{\sigma}_{\alpha\beta}$ are:

$$\overset{0}{\sigma}_{11} = -R/h, \quad \overset{0}{\sigma}_{12} = \overset{0}{\sigma}_{21} = 0, \quad \overset{0}{\sigma}_{22} = -\nu R/h, \tag{BB-2.8}$$

where λ ($\lambda = -p$ according to the definition in ?? is the dimensionless external normal dead load applied. To exclude rigid body rotations, one must add the requirement $u_1(0,0) = 0$ to ??.

The equilibrium equations for the cylinder are obtained by extremizing $\mathcal{E}$ with respect to the displacement field. Hence from ?? one obtains with the help of the strain definitions in ??:

$$\begin{aligned}
&\mathcal{E}_{,u}\,\delta u = \int_0^\ell \int_0^{2\pi} [L_{\alpha\beta\gamma\delta}(E_{\alpha\beta}\delta E_{\gamma\delta} + \frac{h^2}{12R^2}K_{\alpha\beta}\delta K_{\gamma\delta}) - \lambda\overset{0}{\sigma}_{\alpha\beta}\delta e_{\alpha\beta}] d\theta d\eta = 0,\\
&\delta E_{\alpha\beta} = \delta e_{\alpha\beta} + \frac{1}{2}(\beta_\alpha\delta\beta_\beta + \beta_\beta\delta\beta_\alpha), \quad \delta K_{\alpha\beta} = \frac{1}{2}(\delta\beta_{\alpha,\beta} + \delta\beta_{\beta,\alpha}),\\
&\delta e_{11} = \delta u_{1,\theta} + \delta w, \quad \delta e_{22} = \delta u_{2,\eta}, \quad \delta e_{12} = \delta e_{21} = \frac{1}{2}(\delta u_{1,\eta} + \delta u_{2,\theta}),\\
&\delta\beta_1 = \delta u_1 - \delta w_{,\theta}\,, \quad \delta\beta_2 = -\delta w_{,\eta}\,,
\end{aligned} \tag{BB-2.9}$$

where the admissible displacements $\delta u_\alpha, \delta w$ satisfy ??. It is not difficult to see that the principal solution to ?? is the constant membrane force, zero moment solution, i.e.:

$$
\begin{aligned}
&\overset{0}{N}_{11} = -\lambda RE, \quad \overset{0}{N}_{12} = \overset{0}{N}_{21} = 0, \quad \overset{0}{N}_{22} = -\lambda\nu RE; \quad \overset{0}{M}_{\alpha\beta} = 0,\\
&\overset{0}{u}_1 = 0, \quad \overset{0}{u}_2 = 0, \quad \overset{0}{w} = -\lambda R(1-\nu^2)/h,
\end{aligned}
\tag{BB-2.10}
$$

because a straightforward substitution of ?? into ?? shows that the equilibrium equations and boundary conditions are satisfied and since the solution vanishes for $\lambda = 0$.

Of interest is the lowest critical load λ_c of the above found principal solution. To this end, one has to find the lowest dimensionless surface load λ_c that satisfies ??, namely $(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c), \lambda_c)\overset{1}{u})\delta u = 0$. Upon taking an additional derivative of ?? with respect to u, and after noting also from the principal solution ?? that $\overset{0}{N}_{\alpha\beta} = EhL_{\alpha\beta\gamma\delta}\overset{0}{E}_{\gamma\delta} = \lambda h\overset{0}{\sigma}_{\alpha\beta}$, one obtains λ_c and $\overset{1}{u}$ by solving:

$$
(\mathcal{E}^c_{,uu}\overset{1}{u})\delta u = \int_0^\ell \int_0^{2\pi} [L_{\alpha\beta\gamma\delta}(\overset{1}{e}_{\alpha\beta}\delta e_{\gamma\delta} + \frac{h^2}{12R^2}\overset{1}{K}_{\alpha\beta}\delta K_{\gamma\delta}) + \lambda_c\overset{0}{\sigma}_{\alpha\beta}\overset{1}{\beta}_\alpha\delta\beta_\beta]d\theta d\eta = 0. \tag{BB-2.11}
$$

It is understood that the eigenmode $\overset{1}{u} = (\overset{1}{u}_\alpha, \overset{1}{w})$ has to satisfy the essential boundary conditions in ??.

Integrating ?? by parts gives the following Euler – Lagrange equations plus boundary conditions:

$$
\begin{aligned}
\delta u_1 : \quad & (1+k)[\overset{1}{u}_{1,\theta\theta} + ((1-\nu)/2)\overset{1}{u}_{1,\eta\eta}] + \Lambda\overset{1}{u}_1 + [(1+\nu)/2]\overset{1}{u}_{2,\theta\eta} +\\
& [-k\nabla^2\overset{1}{w} + (1-\Lambda)\overset{1}{w}]_{,\theta} = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad 0 \le \eta \le \ell\\
& (1+k)\overset{1}{u}_{1,\eta} + \overset{1}{u}_{2,\theta} = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad \eta = 0, \ell\\
\delta u_2 : \quad & [(1+\nu)/2]\overset{1}{u}_{1,\eta\theta} + [(1-\nu)/2]\overset{1}{u}_{2,\theta\theta} + \overset{1}{u}_{2,\eta\eta} +\\
& \nu\overset{1}{w}_{,\eta} = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad 0 \le \eta \le \ell\\
& \overset{1}{u}_2 = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad \eta = 0, \ell\\
\delta w : \quad & [-k\nabla^2\overset{1}{u}_1 + (1-\Lambda)\overset{1}{u}_1]_{,\theta} + \nu\overset{1}{u}_{2,\eta} +\\
& k\nabla^4\overset{1}{w} + \Lambda(\overset{1}{w}_{,\theta\theta} + \nu\overset{1}{w}_{,\eta\eta}) + \overset{1}{w} = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad 0 \le \eta \le \ell\\
& \overset{1}{w}_{,\eta} = \overset{1}{u}_{1,\theta\eta} - \overset{1}{w}_{,\eta\eta\eta} = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad \eta = 0, \ell\\
& \text{where} : k \equiv h^2/12R^2, \quad \Lambda = \lambda(1-\nu^2)R/h.
\end{aligned}
\tag{BB-2.12}
$$

The dimensionless parameters k, Λ are introduced for convenience. Upon inspection, the system **??** admits the following solutions:

$$\left\{\begin{aligned} \overset{1}{u}_1(\theta,\eta) &= u_{mn}\cos(\mu\eta)\sin(n\theta) \\ \overset{1}{u}_2(\theta,\eta) &= v_{mn}\sin(\mu\eta)\cos(n\theta) \\ \overset{1}{w}(\theta,\eta) &= w_{mn}\cos(\mu\eta)\cos(n\theta) \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} \overset{1}{u}_1(\theta,\eta) &= u'_{mn}\cos(\mu\eta)\cos(n\theta) \\ \overset{1}{u}_2(\theta,\eta) &= v'_{mn}\sin(\mu\eta)\sin(n\theta) \\ \overset{1}{w}(\theta,\eta) &= w'_{mn}\cos(\mu\eta)\sin(n\theta) \end{aligned}\right\} \tag{BB-2.13}$$

$$\text{where}: \ \mu \equiv m\pi/\ell = m\pi R/L; \quad m,n \in \mathbb{N}_0,$$

with $\mathbb{N}_0$ the set of all integers including 0.

The presence of two modes for each pair of integers (m,n) had to be expected in view of the rotational invariance of the model, which in turn implies the same invariance for the equilibrium equations **??** and the eigenmode equations **??**. By appealing to the additional essential boundary condition $\overset{1}{u}_1(0,0)=0$ the $\overset{1}{u}'$ mode in **??** is excluded (note that the $\overset{1}{u}'$ mode is obtained by a $\pi/2$, $\theta-$ rotation with respect to the axis of the cylinder). By substituting $\overset{1}{u}$ in **??** into **??** we obtain the following linear system for $(u_{mn}, v_{mn}, w_{mn}) \equiv \mathbf{U}_{mn}$:

$$\mathbf{S}(n,\mu,\Lambda)\mathbf{U}_{mn} = 0$$

$$\mathbf{S} \equiv \begin{bmatrix} (1+k)[n^2+\mu^2(1-\nu)/2]-\Lambda, & \mu n(1+\nu)/2, & n[k(\mu^2+n^2)+1-\Lambda] \\ \mu n(1+\nu)/2, & \mu^2+n^2(1-\nu)/2 & \nu\mu \\ n[k(\mu^2+n^2)+1-\Lambda], & \nu\mu, & k(\mu^2+n^2)^2+1-\Lambda(n^2+\nu\mu^2) \end{bmatrix} \tag{BB-2.14}$$

A nontrivial solution to the system is possible when:

$$\mathrm{Det}\mathbf{S}(n,\mu,\Lambda,(n,\mu)) = 0, \tag{BB-2.15}$$

where $\Lambda(n,\mu)$ is a root of **??**. Thus there exists an infinity of critical points for the stability operator of the externally compressed cylinder (for each wavenumber pair (n,m) at most two critical surface loads are possible since **??** is quadratic in Λ). All these points are bifurcation points since according to the general theory from **??**, $\mathcal{E}^c_{,u\lambda}\overset{1}{u}=0$. Indeed from **??** and **??** we obtain:

$$\mathcal{E}^c_{,u\lambda}\overset{1}{u} = -\int_0^{\ell}\int_0^{2\pi}[\overset{0}{\sigma}_{\alpha\beta}\overset{1}{e}_{\alpha\beta}]d\theta d\eta = 0. \tag{BB-2.16}$$

Of all these bifurcation points, of interest is the one corresponding to the lowest critical load:

$$\lambda_c = \min_{m,n\in\mathbb{N}_0}\{[h/R(1-\nu^2)]\Lambda_I(n,m\pi/\ell)\} = [h/R(1-\nu^2)]\Lambda_I(n_c,m_c\pi/\ell), \tag{BB-2.17}$$

where $\Lambda_I(n,\mu)$, $\Lambda_{II}(n,\mu)$ are the minimum and maximum roots respectively of the quadratic in Λ **??**. The minimum value for Λ_I is achieved for $m=0$. Notice from **??** however that $\mathbf{S}(1,0,\Lambda)$ is singular for any value of Λ. This result has to be expected, since the corresponding eigenmode

$\overset{1}{u} = (-w_{01}\sin\theta, 0, w_{01}\cos\theta)$ represents a rigid body mode for which $\overset{1}{\beta}_\alpha = 0$ and $\overset{1}{e}_{\alpha\beta} = \overset{1}{K}_{\alpha\beta} = 0$ according to **??**$_2$. From Eqs. **??**, **??** one can easily deduce that for $n \neq 1$, $\Lambda_I(n, 0) = kn^2$ and hence in view of **??**:

$$\begin{aligned} &\lambda_c = h^3/[3(1-\nu^2)R^3], \quad m_c = 0, \quad n_c = 2, \\ &\overset{1}{u}_1 = -\frac{1}{2}\sin(2\theta), \qquad \overset{1}{u}_2 = 0, \quad \overset{1}{w} = \cos(2\theta), \end{aligned} \tag{BB-2.18}$$

which corresponds to a simple mode since the minimizing pair (m_c, n_c) of $\Lambda_I(n, m\pi/\ell)$ (recall that Λ_I is defined as the minimum root of **??** is unique).

We expect the principal branch of the equilibrium solution to be stable for $0 \leq \lambda < \lambda_c$. To prove this property we should show that the stability functional evaluated on the principal solution $(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda)\delta u)\delta u$ is positive definite in the aforementioned load range. From Eqs. **??**, **??** and **??** one has:

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda)\delta u)\delta u = \int_0^\ell \int_0^{2\pi} [L_{\alpha\beta\gamma\delta}(\delta e_{\alpha\beta}\delta e_{\gamma\delta} + \frac{h^2}{12R^2}\delta K_{\alpha\beta}\delta K_{\gamma\delta}) + \lambda\overset{0}{\sigma}_{\alpha\beta}\delta\beta_\alpha\delta\beta_\beta] d\theta d\eta. \tag{BB-2.19}$$

Noting that in view of the essential boundary conditions **??** and the additional requirement $\delta u_1(0,0) = 0$ (to exclude rigid body rotation about the cylinder axis), any admissible mode $\delta u = (\delta u_\alpha,\ \delta w)$ can be put in the form:

$$\begin{aligned} &\delta u_1 = \delta u_{mn}\cos(\mu\eta)\sin(n\theta), \\ &\delta u_2 = \delta v_{mn}\sin(\mu\eta)\cos(n\theta); \quad \mu = m\pi/\ell, \quad m, n \in \mathbb{N}_0, \\ &\delta w = \delta w_{mn}\cos(\mu\eta)\cos(n\theta). \end{aligned} \tag{BB-2.20}$$

Of the above admissible modes one must exclude $\delta u = (-\sin\theta, 0, \cos\theta)$ which is a rigid body mode since from **??**$_2$ we have $\delta E_{\alpha\beta} = \delta K_{\alpha\beta} = \delta\beta_\alpha = 0$.

By introducing **??** into **??** one obtains:

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda)\delta u)\delta u = \frac{\pi\ell}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(\delta\mathbf{U}_{mn})^{\mathrm{T}}\mathbf{S}(n, \mu, \Lambda)\delta\mathbf{U}_{mn}, \tag{BB-2.21}$$

where $\delta\mathbf{U}_{mn} = (\delta u_{mn}, \delta v_{mn}, \delta w_{mn})$ and where the $(n, m) = (1, 0)$ term is excluded from the summation in **??** since the corresponding admissible displacement corresponds to a rigid body mode as explained in **??**. It is not difficult to verify that for $0 \leq \lambda < \lambda_c$ (where λ_c is given by **??**) the matrices $\mathbf{S}(n, \mu, \Lambda)$ are positive definite (assuming of course $(n, \mu) \neq (1, 0)$) which from **??** implies the positivity of the stability functional. With some additional effort, and after defining an inner product in the space of all admissible functions, one can even show the ellipticity of the stability operator in the sense of **??**, i.e., that a $\overset{0}{\beta}(\lambda) > 0$ exists such that $(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda), \lambda)\delta u)\delta u \geq \overset{0}{\beta}(\lambda)(\delta u, \delta u)$ for $0 \leq \lambda < \lambda_c$, exactly as in the relatively simpler cases of the elastica (see **??**) and the thin walled beam (see **??**).

Having found the lowest bifurcation load λ_c, our attention is turned to the postbifurcation asymptotic expansion of the bifurcated equilibrium branch. At first we calculate $((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u}$ which from Eqs. ??, ?? and ?? is found to be:

$$((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} = \int_0^{\ell}\int_0^{2\pi}[\overset{0}{\sigma}_{\alpha\beta}\overset{1}{\beta}_{\alpha}\overset{1}{\beta}_{\beta}]d\theta d\eta = -\frac{9\pi}{4}\frac{R\ell}{h}. \tag{BB-2.22}$$

As expected from the general theory in Section AC-3, $((d\mathcal{E},_{uu}/d\lambda)_c\overset{1}{u})\overset{1}{u} < 0$.

The first term λ_1 in the asymptotic expansion of the bifurcated equilibrium branch through λ_c is given according to the general theory by ??. The denominator in that expression has just been calculated above in ?? while the numerator with the help of Eqs. ??, ?? and ?? yields:

$$((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = \int_0^{\ell}\int_0^{2\pi}[3L_{\alpha\beta\gamma\delta}\overset{1}{\beta}_{\alpha}\overset{1}{\beta}_{\beta}\overset{1}{e}_{\gamma\delta}]d\theta d\eta = 0, \tag{BB-2.23}$$

which from ?? implies that $\lambda_1 = 0$ and hence that the corresponding bifurcation at λ_c is symmetric.

The calculation of the next term λ_2 in the asymptotic expansion of the load requires, according to ??, the determination of the next order term $(v_{\xi\xi})$ in the expansion of the displacement which is determined from ?? of the general theory. With the help of Eqs. ??, ?? and ??, one obtains from ?? the following variational equation for $v_{\xi\xi} \equiv (\overset{2}{u}_{\alpha}, \overset{2}{w})$:

$$\begin{aligned} 0 &= ((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u} + \mathcal{E}^c_{,uu}v_{\xi\xi})\delta u = \int_0^{\ell}\int_0^{2\pi}[L_{\alpha\beta\gamma\delta}(\overset{1}{\beta}_{\gamma}\overset{1}{\beta}_{\delta}\delta e_{\alpha\beta} + 2\overset{1}{e}_{\gamma\delta}\overset{1}{\beta}_{\alpha}\delta\beta_{\beta})]d\theta d\eta\ + \\ &+ \int_0^{\ell}\int_0^{2\pi}[L_{\alpha\beta\gamma\delta}(\overset{2}{e}_{\gamma\delta}\delta e_{\alpha\beta} + \frac{h^2}{12R^2}\overset{2}{K}_{\gamma\delta}\delta K_{\alpha\beta}) + \lambda_c\overset{0}{\sigma}_{\alpha\beta}\overset{2}{\beta}_{\alpha}\delta\beta_{\beta}]d\theta d\eta, \end{aligned} \tag{BB-2.24}$$

where $\overset{2}{\beta}_{\alpha}$, $\overset{2}{e}_{\alpha\beta}$ and $\overset{2}{K}_{\alpha\beta}$ are given in terms of $(\overset{2}{u}_{\alpha}, \overset{2}{w})$ by ??$_2$ in which $(\overset{2}{u}_{\alpha}, \overset{2}{w})$ replace $(\delta u_{\alpha}, \delta w)$. Integration of ?? by parts, and recalling the eigenmode from ?? and the admissibility conditions from ?? gives the following Euler – Lagrange equations and boundary conditions for

$\overset{2}{u}_\alpha, \overset{2}{w}$:

$$
\begin{aligned}
\delta u_1 : \quad & (1+k)[\overset{2}{u}_{1,\theta\theta} + ((1-\nu)/2)\overset{2}{u}_{1,\eta\eta} + 4k\overset{2}{u}_1] + [(1+\nu)/2]\overset{2}{u}_{2,\theta\eta} + \\
& [-k\nabla^2\overset{2}{w} + (1-4k)\overset{2}{w}]_{,\theta} = -\frac{9}{2}\sin 4\theta \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad 0 \le \eta \le \ell \\
& (1+k)\overset{2}{u}_{1,\eta} + \overset{2}{u}_{2,\theta} = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad \eta = 0, \ell \\
\delta u_2 : \quad & [(1+\nu)/2]\overset{2}{u}_{1,\eta\theta} + [(1-\nu)/2]\overset{2}{u}_{2,\theta\theta} + \overset{2}{u}_{2,\eta\eta} + \\
& \nu\overset{2}{w}_{,\eta} = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad 0 \le \eta \le \ell \\
& \overset{2}{u}_2 = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad \eta = 0, \ell \\
\delta w : \quad & [-k\nabla^2\overset{2}{u}_1 + (1-4k)\overset{2}{u}_1]_{,\theta} + \nu\overset{2}{u}_{2,\eta} + \\
& k\nabla^4\overset{2}{w} + 4k(\overset{2}{w}_{,\theta\theta} + \nu\overset{2}{w}_{,\eta\eta}) + \overset{2}{w} = -(\frac{3}{2}\sin 2\theta)^2 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad \eta = 0, \ell \\
& \overset{2}{w}_{,\eta} = \overset{2}{u}_{1,\theta\eta} - \overset{2}{w}_{,\eta\eta\eta} = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad \eta = 0, \ell
\end{aligned}
\tag{BB-2.25}
$$

By inspection, the solution $(\overset{2}{u}_\alpha, \overset{2}{w})$ to **??** is found to be:

$$
\overset{2}{u}_1 = \frac{9}{8}\frac{180}{675-k-192k^2}\sin(4\theta), \quad \overset{2}{u}_2 = 0, \quad \overset{2}{w} = -\frac{9}{8}\Big[1 + \frac{45+k}{675-k-192k^2}\cos(4\theta)\Big] \tag{BB-2.26}
$$

As expected from the requirement of the general theory in Section AC-3, $v_{\xi\xi}$ is orthogonal to the eigenmode $\overset{1}{u}$. For thin walled cylinders $k \ll 1$ ($k = h^2/12R^2$ according to Eqs. **??**$_4$) and **??** simplifies to $\overset{2}{u}_1 = (3/10)\sin(4\theta)$, $\overset{2}{u}_2 = 0$, $\overset{2}{w} = -(3/40)[15 + \cos(4\theta)]$.

The last ingredient required for the calculation of the first nontrivial term λ_2 in the asymptotic expansion of the load according to the general theory from **??** is the quantity $(((\mathcal{E}^c_{,uuuu}\overset{1}{u})\overset{1}{u} + 3\mathcal{E}^c_{,uuu}v_{\xi\xi})\overset{1}{u})\overset{1}{u}$. To this end by using Eqs. **??**, **??** and **??** one obtains:

$$
\begin{aligned}
&(((\mathcal{E}^c_{,uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E}^c_{,uuu}v_{\xi\xi})\overset{1}{u})\overset{1}{u} = 3\int_0^\ell\int_0^{2\pi}[L_{\alpha\beta\gamma\delta}(\overset{2}{e}_{\gamma\delta} + \overset{1}{\beta}_\gamma\overset{1}{\beta}_\delta)\overset{1}{\beta}_\alpha\overset{1}{\beta}_\beta]d\theta d\eta \\
&= 3\int_0^\ell\int_0^{2\pi}[L_{1111}(\overset{2}{e}_{11} + \overset{1}{\beta}_1\overset{1}{\beta}_1)\overset{1}{\beta}_1\overset{1}{\beta}_1]d\theta d\eta = -\frac{3\ell\pi}{1-\nu^2}\left(\frac{9}{8}\right)^2\frac{192k^2}{675-k-192k^2}.
\end{aligned}
\tag{BB-2.27}
$$

Thus finally from **??** the wanted term λ_2 is:

$$
\lambda_2 = -\frac{3}{4(1-\nu^2)}\frac{(h/R)^5}{675-k-192k^2} = -\frac{27\lambda_c k}{675-k-192k^2}, \tag{BB-2.28}
$$

which for $k \ll 1$ simplifies to $\lambda_2/\lambda_c \simeq -k/25$. Since $\lambda_c < 0$ the postbifurcation equilibrium branch of the cylinder is unstable. It should be noticed at this point that the bifurcated equilibrium solution corresponds to a plane strain deformation of the cylinder since $\overset{1}{u}_{1,\eta} = \overset{1}{w}_{,\eta} = 0$, $\overset{1}{u}_2 = 0$.

ib) *Cylinder Under Axial Compression*

For the circular cylinder with no lateral surface loads and compressed between two parallel rigid plates, the essential boundary conditions that correspond to the potential energy in **??** are:

$$u_2(\theta, 0) = u_{2,\theta}(\theta, \ell) = w_{,\eta}(\theta, 0) = w_{,\eta}(\theta, \ell) = 0, \tag{BB-2.29}$$

and the corresponding dimensionless constants $\overset{0}{\sigma}_{\alpha\beta}$ are:

$$\overset{0}{\sigma}_{11} = \overset{0}{\sigma}_{12} = \overset{0}{\sigma}_{21} = 0, \quad \overset{0}{\sigma}_{22} = -1 \tag{BB-2.30}$$

Here the dimensionless load parameter λ is related to the applied axial force $N = -2\pi\lambda ERh$. For a force controlled experiment, the axial displacement of the top surface, $u_2(\theta, \ell)$ is a constant independent of θ which will be specified as part of the solution of the problem. Hence, in contrast to the lateral surface load problem (see **??**), for axial compression the corresponding essential boundary condition for u_2 is $u_{2,\theta}(\theta, \ell) = 0$ as seen in **??**. To exclude rigid body rotations one must add to **??** the requirement $u_1(0, 0) = 0$.

The equilibrium equations for the cylinder, which are obtained by extremizing its potential energy $\mathcal{E}$ with respect to the displacement field, are still given by **??**, but with the admissible displacements satisfying **??** and with $\overset{0}{\sigma}_{\alpha\beta}$ given by **??**. It is not difficult to see that under those conditions, the corresponding principal solution is the following constant membrane force, zero moment solution:

$$\begin{aligned} &\overset{0}{N}_{11} = \overset{0}{N}_{12} = \overset{0}{N}_{21} = 0, \quad \overset{0}{N}_{22} = -\lambda Eh; \quad \overset{0}{M}_{\alpha\beta} = 0, \\ &\overset{0}{u}_1 = 0, \quad \overset{0}{u}_2 = -\lambda\eta, \quad \overset{0}{w} = \nu\lambda. \end{aligned} \tag{BB-2.31}$$

Of interest again is the lowest critical load λ_c of the above principal solution. Proceeding exactly as for the lateral load case in the previous subsection, the eigenvalue problem determining λ_c is still given by **??**, with the understanding of course that δu and $\overset{1}{u}$ satisfy the admissibility conditions of **??** and that $\overset{0}{\sigma}_{\alpha\beta}$ is given by **??**. Integration of **??** by parts gives the following Euler-Lagrange equations plus boundary conditions:

$$\begin{aligned}
\delta u_1 : \quad & (1+k)[\overset{1}{u}_{1,\theta\theta} + [(1-\nu)/2]\overset{1}{u}_{1,\eta\eta}] + [(1+\nu)/2]\overset{1}{u}_{2,\theta\eta} + \\
& [-k\nabla^2\overset{1}{w} + \overset{1}{w}]_{,\theta} = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad 0 \le \eta \le \ell, \\
& (1+k)\overset{1}{u}_{1,\eta} + \overset{1}{u}_{2,\theta} = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad \eta = 0, \ell, \\
\delta u_2 : \quad & [(1+\nu)/2]\overset{1}{u}_{1,\eta\theta} + [(1-\nu)/2]\overset{1}{u}_{2,\theta\theta} + \overset{1}{u}_{2,\eta\eta} + \\
& \nu\overset{1}{w}_{,\eta} = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad 0 \le \eta \le \ell, \\
& \overset{1}{u}_2 = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \;\; \eta = 0, \qquad\qquad \text{(BB-2.32)} \\
& \overset{1}{u}_{2,\theta} = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \;\; \eta = \ell; \;\; \int_0^{2\pi} [\overset{1}{u}_{2,\eta} + \nu(\overset{1}{u}_{1,\theta} + \overset{1}{w})]d\theta = 0 \quad \text{for} \quad \eta = \ell, \\
\delta w : \quad & [-k\nabla^2\overset{1}{u}_1 + \overset{1}{u}_1]_{,\theta} + \nu\overset{1}{u}_{2,\eta} + \\
& k\nabla^4\overset{1}{w} + \lambda(1-\overset{2}{\nu})\overset{1}{w}_{,\eta\eta} + \overset{1}{w} = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad 0 \le \eta \le \ell, \\
& \overset{1}{w}_{,\eta} = \overset{1}{u}_{1,\eta\theta} - \overset{1}{w}_{,\eta\eta\eta} = 0 \quad \text{for} \quad 0 \le \theta \le 2\pi, \quad \eta = 0, \ell,
\end{aligned}$$

where the dimensionless parameter k for the axially loaded cylinder is again given by ??$_4$. Morover, upon inspection, the system of ?? still admits the eigenmodes in ??, where in view of the boundary condition $u_1(0,0) = 0$, the second set of modes $\overset{1}{u}'$ is excluded.

A direct substitution of the eigenmode $\overset{1}{u}$ in ?? into ?? gives the following linear system for $(u_{mn}, v_{mn}, w_{mn}) \equiv \mathbf{U}_{mn}$:

$$\mathbf{S}(n,\mu,\lambda)\mathbf{U}_{mn} \equiv 0$$

$$\mathbf{S} \equiv \begin{bmatrix} (1+k)[n^2+\mu^2(1-\nu)/2] & \mu n(1+\nu)/2 & n[k(\mu^2+n^2)+1] \\ \mu n(1+\nu)/2 & \mu^2+n^2(1-\nu)/2 & \nu\mu \\ n[k(\mu^2+n^2)+1] & \nu\mu & k(n^2+\mu^2)^2+1-\lambda(1-\nu^2)\mu^2 \end{bmatrix} \qquad \text{(BB-2.33)}$$

Once again a nontrivial solution to ?? is possible when $\mathbf{S}$ is singular, i.e., when it satisfies $\text{Det } \mathbf{S}(n,\mu,\lambda(n,\mu)) = 0$. (similarly to ?? but with $\mathbf{S}$ given by ??$_2$). The explicit expression for λ, which is a rational function of μ^2 and n^2 is cumbersome. A simplified expression for λ can be obtained by observing that for thin shells $k \equiv h^2/12R^2 \ll 1$, in which case for $\mu \neq 0$ one can write, under the additional assumption that $k(\mu^2+n^2) \ll 1$:

$$\lambda(n,\mu) = \left[\frac{k}{1-v^2}\frac{(\mu^2+n^2)^2}{\mu^2} + \frac{\mu^2}{(\mu^2+n^2)^2}\right](1+O[k(\mu^2+n^2)]). \qquad \text{(BB-2.34)}$$

As seen from ??, there exist an infinity of critical points for the stability of the axially compressed cylinder. All these critical points are bifurcation points since they satisfy according to the general theory (see **??**) $\mathcal{E}^c_{,u\lambda}\overset{1}{u} = 0$. As for the laterally compressed cylinder case, from Eqs. **??** and **??** we obtain once more **??** which proves that all the above critical points are bifurcation points.

Of interest is again the lowest value of the axial load λ. Hence from **??**, the critical load λ_c is the minimum $\lambda(n, m\pi/\ell)$ over all integers n and m. Unlike the laterally compressed cylinder (see **??**), where the minimum was an isolated point achieved for a given pair of integers, the minimum engenvalue for the axially compressed cylinder is always approached and sometimes even achieved by a number of different pairs of integers. Hence the axially compressed cylinder has simultaneous and nearly simultaneous eigenmodes, the number of which increases with the increasing thinness of the shell.

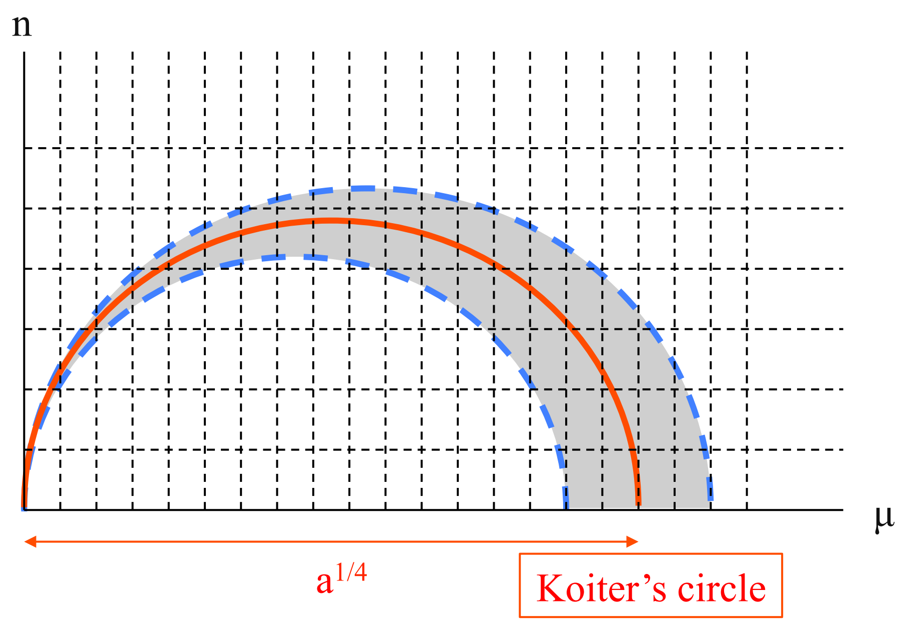

Figure BB-2.2: Koiter's circle indicating wavelengths (μ, n) of interacting eigenmodes.

An easy pictorial representation of the locus of integers in the n, m plane that correspond to the above mentioned nearly simultaneous eigenmodes can be found as follows: By allowing the variables n and μ in **??** to take any positive real value and by tacitly assuming that $k(\eta^2 + \mu^2) \ll 1$ (an assumption that will be verified a-posteriori), one can easily deduce that the minimum value of $\lambda(n, \mu)$ is $\lambda_c = 2a^{-1/2}$ and is achieved for all points on the circle $\mu^2 + n^2 = a^{1/4}\mu$ where $a \equiv (1-\nu^2)/k = 12(1-\nu^2)R^2/h^2$ (circle in solid line in Fig. BB-2.2). This circle is called the *Koiter's circle* and has a center at $(n, \mu) = (0, (1/2)a^{1/4})$ and a diameter $a^{1/4}$. For the most practical applications h/R varies between 10^{-1} to 10^{-3}, in which case one can easily verify that a large number of eigenmodes with (n, μ) near the Koiter circle give eigenvalues close to the minimum $\lambda_c = 2a^{-1/2}$. The eigenmodes whose eigenvalues λ satisfy $2a^{-1/2} \leq \lambda \leq 2a^{-1/2}(1+\zeta)$ where $0 < \zeta \ll 1$ have wavenumbers (n, μ) that lie in the shaded area in Fig. BB-2.2 between the two dotted line circles with origins on the μ axis and diameters $a^{1/4}\{1+\zeta\pm[(1+\zeta)^2-1]^{1/2}\}^{1/2}$.

The above analysis, that leads to Koiter's circle as the locus of (n,μ) pairs that minimize the critical load of the cylinder under axial compression, made use of the assumption $k(\mu^2+n^2)\ll 1$ which will now be verified. Indeed since at the minimum possible critical load $k/(1-\nu^2)=\mu^4/(\mu^2+n^2)^4$, one can easily show (since $\mu>0,\ n>0,\ 0<\nu<1$) that $k(\mu^2+n^2)^2<(1-\nu^2)$. Hence $k(\mu^2+n^2)$ is of $O(k^{1/2})$, thus justifying the assumption $k(\mu^2+n^2)\ll 1$.

Having discussed the bifurcation points on the principal branch, we turn our attention to its stability. Since the minimum possible value of $\lambda_c=2a^{-1/2}$, we expect the principal branch to be stable for $0\le\lambda<2a^{-1/2}$. The proof of this assertion is identical to the one used in the previous subsection on the laterally compressed cylinder. To show the positive definiteness of **??**, where $\overset{0}{\sigma}_{\alpha\beta}$ is this time given by **??**, one has to show the possitivity of **??** for all $\delta\mathbf{U}_{mn}$, where $\mathbf{S}(n,\mu,\lambda)$ is now given by **??**$_2$. The positive definiteness of all matrices $\mathbf{S}(n,\mu,\lambda)$ for $0\le\lambda 2a^{-1/2}$ is easy to establish. Some extra effort is needed to establish the ellipticity of the stability operator on the principal branch in the sense of **??**, i.e., the existence of a $\overset{0}{\beta}(\lambda)>0$ such that $(\mathcal{E},_{uu}(\overset{0}{u}(\lambda),\lambda)\delta u)\delta u\ge\overset{0}{\beta}(\lambda)(\delta u,\delta u)$ for $0\le\lambda<2a^{-1/2}$. The comments made about this proof for the laterally compressed cylinder are also applicable to the axially compressed one.

The presence of many nearly simultaneous modes in the neighborhood of the critical load implies mode interaction. Consequently, the general theory developed in Section AC-6 will be applied in this case. The small parameter ζ that determines the spacing of the nearly simultaneous modes is not explicitated here, i.e., no explicit expressions will be given for the critical load $\overset{i}{\hat{\lambda}}(\zeta)$ which corresponds to the eigenmode with wavenumbers (n_i,m_i). All the interacting modes will have a critical load satisfying $2a^{-1/2}\le\overset{i}{\hat{\lambda}}(\zeta)\le 2a^{-1/2}(1+\zeta)$, so that the corresponding (n_i,m_i) fall within the shaded zone of Fig. BB-2.2. According to Sections AC-6 and AC-7 , the asymptotic general study of the imperfection sensitivity and worst imperfection shape in imperfect structures with nearly simultaneous modes is based on the properties of the corresponding perfect structure. Hence all the required quantities for the study of the initial postcritical behavior of the imperfect axially compressed cylinder will be calculated from the perfect structure.

The first step in the asymptotic calculations for problems with mode interaction is the determination of the coefficients $\mathcal{E},_{ij\lambda}\equiv(d\mathcal{E},_{uu}/d\lambda)_c\overset{i}{u})\overset{j}{u}$ (see definition in **??**$_3$). From Eqs. **??**$_7$, **??** and **??**:

$$((d\mathcal{E},_{uu}/d\lambda)_c\overset{i}{u})\overset{j}{u}=-\int_0^{\ell}\int_0^{2\pi}[\overset{i}{w},_{\eta}\overset{j}{w},_{\eta}]d\theta d\eta. \tag{BB-2.35}$$

As expected from the general theory in Section AC-5, the matrix of coefficients $\mathcal{E}_{ij\lambda}$ is negative definite. The convenient mode normalization condition $\mathcal{E},_{ij\lambda}=-\delta_{ij}$ is added and hence from

Eqs. **??**, **??** and **??** the eigenmodes $\overset{i}{u} = (\overset{i}{u}_1, \overset{i}{u}_2, \overset{i}{w})$ are:

$$\begin{aligned}
\overset{i}{u}_1(\theta,\eta) = & \quad -\gamma_i \frac{n_i[(2+\nu)\mu_i^2 + n_i^2]}{\mu_i(\mu_i^2+n_i^2)^2} \cos(\mu_i\eta)\sin(n_i\theta) \\
\overset{i}{u}_2(\theta,\eta) = & \quad \gamma_i \frac{n_i^2 - \nu\mu_i^2}{(\mu_i^2+n_i^2)^2} \sin(\mu_i\eta)\cos(n_i\theta) \\
\overset{i}{w}(\theta,\eta) = & \quad \gamma_i \frac{1}{\mu_i} \cos(\mu_i\zeta)\cos(n_i\theta)
\end{aligned} \tag{BB-2.36}$$

$$\mu_i = m_i, \pi/\ell; \quad \gamma_i \quad \equiv (2/\pi\ell)^{1/2} \quad \text{for} \quad n_i \neq 0, \; \gamma_i \equiv (1/\pi\ell)^{1/2} \quad \text{for} \quad n_i = 0$$

A tacit assumption in the derivation of **??**, which is justified by the mode normalization condition **??**, is that $\mu_i \neq 0$, in addition to the already proved property of the modes that $k(n_i^2 + \mu_i^2) \ll 1$.

The next step in the asymptotic calculations for problems with mode interaction is the determination of the coefficients $\mathcal{E}_{ijk} \equiv ((\mathcal{E}^c_{,uuu}\overset{i}{u})\overset{j}{u})\overset{k}{u}$ (see definition in **??**$_2$). From Eqs. **??** and **??**:

$$((\mathcal{E}^c_{,uuu}\overset{i}{u})\overset{j}{u})\overset{k}{u} = \int_0^\ell \int_0^{2\pi} [L_{\alpha\beta\gamma\delta}(\overset{i}{e}_{\alpha\beta}\overset{j}{\beta}_\gamma\overset{k}{\beta}_\delta + \overset{j}{e}_{\alpha\beta}\overset{k}{\beta}_\gamma\overset{i}{\beta}_\delta + \overset{k}{e}_{\alpha\beta}\overset{i}{\beta}_\gamma\overset{j}{\beta}_\delta)] d\theta d\eta, \tag{BB-2.37}$$

where $\overset{i}{e}_{\alpha\beta}$, $\overset{j}{e}_{\alpha\beta}$, $\overset{k}{e}_{\alpha\beta}$ and $\overset{i}{\beta}_\alpha$, $\overset{j}{\beta}_\alpha$, $\overset{k}{\beta}_\alpha$ are linear functions of $\overset{i}{u}$, $\overset{j}{u}$, $\overset{k}{u}$ given by (BB 2.9) where $\delta u = (\delta u_1, \delta u_2, \delta w)$ is replaced respectively by $\overset{i}{u} = (\overset{i}{u}_1, \overset{i}{u}_2, \overset{i}{w})$, $\overset{j}{u} = (\overset{j}{u}_1, \overset{j}{u}_2, \overset{j}{w})$, $\overset{k}{u} = (\overset{k}{u}_1, \overset{k}{u}_2, \overset{k}{w})$.

Substitution of the eigenmodes **??** into **??** and recalling **??** yields after some straightforward but tedious manipulations:

$$\begin{aligned}
\mathcal{E}_{ijk} = \frac{\pi\ell}{8}\frac{\gamma_i\gamma_j\gamma_k}{\mu_i\mu_j\mu_k} \quad \Bigg\{ & \frac{\mu_i^2}{(\mu_i^2+n_i^2)^2}\Big[\mu_i^2\eta_j\eta_k\chi^i_{jk}(\eta)\chi_{ijk}(m) + n_i^2\mu_j\mu_k\chi_{ijk}(\eta)\chi^i_{jk}(m) + \\
& \qquad n_i\mu_i\Big(\eta_j\mu_k\chi^k_{ij}(\eta)\chi^j_{ik}(m) + \eta_k\mu_j\chi^j_{ik}(\eta)\chi^k_{ij}(m)\Big)\Big] + \\
& \frac{\mu_j^2}{(\mu_j^2+\eta_j^2)^2}\Big[\mu_j^2\eta_k n_i\chi^j_{ki}(\eta)\chi_{ijk}(m) + \eta_j^2\mu_k\mu_i\chi_{ijk}(\eta)\chi^j_{ki}(m) + \\
& \qquad \eta_j\mu_j\Big(\eta_k\mu_i\chi^i_{jk}(\eta)\chi^k_{ji}(m) + n_i\mu_k\chi^k_{ji}(\eta)\chi^i_{jk}(m)\Big)\Big] + \\
& \frac{\mu_k^2}{(\mu_k^2+\eta_k^2)^2}\Big[\mu_k^2\eta_k n_i\chi^k_{ij}(\eta)\chi_{ijk}(m) + \eta_k^2\mu_i\mu_j\chi_{ijk}(\eta)\chi^k_{ij}(m) + \\
& \qquad \eta_k\mu_k\Big(\eta_i\mu_j\chi^j_{ki}(\eta)\chi^i_{kj}(m) + \eta_j\mu_i\chi^i_{kj}(\eta)\chi^j_{ki}(m)\Big)\Big]\Bigg\}
\end{aligned} \tag{BB-2.38}$$

$$\chi_{ijk}(x) \quad \equiv \chi_i(x) + \chi_j(x) + \chi_k(x); \quad \chi^k_{ij} \equiv \chi_i(x) + \chi_j(x) - \chi_k(x)$$

$$\chi_i(x) \quad = \begin{cases} 1 & \text{if} \quad x_i + x_j + x_k = 2x_i \\ 0 & \text{if} \quad x_i + x_j + x_k \neq 2x_i \end{cases}$$

In the above derivations, we have also used the simplification $\eta^2+\mu^2 \gg 1$, which is in agreement with our assumption $k \ll 1$ (recall that $\mu^2+\eta^2$ is of $O(k^{-1/2})$ according to our discussion about Koiter's circle). It should be mentioned at this point that $\mathcal{E}_{ijk} \neq 0$ only when both the combinations $m_i \pm m_j \pm_k = 0$ and $n_i \pm n_j \pm n_k = 0$ are simultaneously satisfied.

It has long been established experimentally that axially loaded cylinders show extreme sensitivity to imperfections with collapse loads as far down as 20% of the critical load λ_c of the perfect cylinder. The explanation for this behavior has to be found in the post bifurcated equilibrium paths originating in the neighborhood of λ_c. It can also be shown from numerical calculations using the exact equilibrium equations of the cylinder (see Eqs. **??**, **??** and **??**) that in number of the above mentioned post-bifurcation equilibrium paths the total load λ and its corresponding work conjugate displacement $-u_2(\theta, \ell)$ both decrease rapidly with an increasing bifurcation amplitude parameter ξ. Moreover these paths are unstable in the neighborhood of the critical load. Consequently the axially loaded cylinder will experience a snap-through catastrophic buckling in either force or displacement controlled loading.

The existence of bifurcated equilibrium paths emerging near λ_c, in which both the dimensionless force acting on the cylinder $-N/EA = 2\pi\lambda$ and its work conjugate dimensionless displacement $-u_2(\theta, \ell)$ decrease as the bifurcation amplitude ξ $(\xi > 0)$ increases in the neighborhood of λ_c, is easy to show. For the applied axial force notice that $2\pi\lambda = 2\pi(\lambda_c + \lambda, \xi + O(\xi^2))$ while for its work conjugate displacement from Eqs. **??** and **??** notice that $-u_2(\theta, \ell; \xi) = -\overset{0}{u}_2(\theta, \ell; \lambda(\xi)) - \xi \overset{i}{u}_2(\theta, \ell) + O(\xi^2) = \ell(\lambda_c + \lambda, \xi) = O(\xi^2)$. When $\lambda_1 < 0$, always possible when a non-trivial solution to **??** exists, both the total force and the corresponding displacement applied on the cylinder decrease, thus proving our assertion.

A final comment is in order at this point about our analysis of the axially loaded cylinder problem. Recall that in the derivation of the bifurcation loads $\lambda(n, \mu)$ from **??** we assumed that $k(n^2 + \mu^2) \ll 1$. The same assumption was subsequently used in the derivation of the amplitudes of the eigenmodes $\overset{i}{u}$ in **??**. Had one assumed instead of the Sander's strain-displacement equation for the cylinder in **??**, the simpler Donnell's strain-displacement relations, according to which $\beta_1 = w,_\theta$, $\quad \beta_2 = -w,_\eta$, then **??**, **??** and consequently **??** would have been exact relations (and not accurate up to an $O(k(n^2 + \mu^2))$).

Example

An example will be given for the buckling of a cylinder which is axially compressed between two lubricated, rigid, flat plates. The cylinder is assumed to have the following geometric dimensions and material properties.

$$\pi/\ell = \pi R/L = 1, \quad \nu = 0.25, \quad a = 12(1-\nu^2)R^2/h^2 = 625 \tag{BB-2.39}$$

Assuming that we want all the interacting modes whose corresponding buckling load is up to 0.25% higher than the minimum critical load $\lambda_c = 2a^{-1/2} = 1/25$, i.e. for $\zeta = 2.5\ 10^{-3}$ the following four modes, which fall within the shaded area of Koiter's circle (see Fig. BB-2.2) can interact:

$$
\begin{aligned}
&\overset{1}{u}, \quad (m_1, n_1) = (1,2), \qquad \lambda = 1/25 \\
&\overset{2}{u}, \quad (m_2, n_2) = (4,2), \qquad \lambda = 1/25 \\
&\overset{3}{u}, \quad (m_3, n_3) = (5.0), \qquad \lambda = 1/25 \\
&\overset{4}{u}, \quad (m_4, n_4) = (5,1), \qquad \lambda = 1/25[(1+1/25)^2 + (1+1/25)^{-2}]
\end{aligned}
\tag{BB-2.40}
$$

Notice that $\overset{1}{u}, \overset{2}{u}, \overset{3}{u}$ are simultaneous modes with wavenumbers that lie on Koiter's circle while the wavenumbers of $\overset{4}{u}$ lie slightly off the Koiter circle. According to the results in ?? only modes $\overset{1}{u}, \overset{2}{u}, \overset{3}{u}$ interact since $m_1 + m_2 - m_3 = 0$ and $n_1 + n_2 \pm n_3 = 0$. Consequently the only non-zero coefficient $\mathcal{E}_{ijk}$ is found from ?? to be $\mathcal{E}_{123}$ (and all the other coefficient $\mathcal{E}_{ijk}$ resulting from all possible permutations of i, j, k).

$$
\mathcal{E}_{ijk} = \begin{cases} 1/5 \quad \text{if} \quad i \neq j \neq k \neq i \\ & 1 \leq i, k, k \leq 3 \\ 0 \quad \text{otherwise} \end{cases}
\tag{BB-2.41}
$$

Using ?? as well as the normalization condition $\mathcal{E}_{ij\lambda} = -\delta_{ij}$ into ?? one obtains for λ_1:

$$
\lambda_1 \alpha_i^1 = \frac{1}{5\pi} \alpha_j^1 \alpha_k^1 \quad \text{for} \quad i \neq j \neq k \neq i, \quad (\alpha_1^1)^2 + (\alpha_2^1)^2 + (\alpha_3^1)^2 = 1
\tag{BB-2.42}
$$

The solution of the above cyclic-symmetric second order system gives:

$$
(\alpha_1^1) = (\alpha_2^1) = (\alpha_3^1) = \pm 1/\sqrt{3}, \quad \lambda_1 = \pm 1/5\sqrt{3}\pi
\tag{BB-2.43}
$$

As discussed in Section AC-5, the stability of the above found bifurcated branch depends on the eigenvalues of the matrix of coefficients B_{ij} defined in ??. From Eqs. ?? and ?? one obtains:

$$
[B_{ij}] = \lambda_1 \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix},
\tag{BB-2.44}
$$

which has a maximum and minimum eigenvalue of opposite signs. Hence according to the discussion of ?? the equilibrium path in ?? is unstable in the neighborhood of λ_c.

BC TWO AND THREE DIMENSIONAL CONTINUA

For the material in this chapter it will be assumed that the reader is familiar with tensor calculus Euclidean spaces. However, for reasons of completeness, a brief introduction into the theory of finite deformations in nonlinear elastic media will be presented here.

BC-1 PRELIMINARIES ON FINITE ELASTICITY

i) *Kinematics and Strain Measure*

Consider a solid whose undeformed configuration occupies a volume V with boundary ∂V. The material points of the solid are identified by a set of coordinates θ^i called *convected coordinates*. The undeformed position vector $\mathbf{P}$ of a material point, say M, is given by:

$$\mathbf{P}(\theta^i) = \mathbf{e}_j X_j(\theta^i) \ , \tag{BC-1.1}$$

where $\mathbf{e}_j$ are the unit vectors of a fixed orthonormal basis of the space and X_j the corresponding (reference) Cartesian coordinates of the reference position vector $\mathbf{P}$. The reader is reminded at this point of the Einstein's summation convention over the repeated indexes of an expression; recall also that Greek indexes range from 1 to 2 while Latin ones range from 1 to 3.

Following the application of external loads and/or boundary displacements, the material point M assumes a new position, with the corresponding current position vector $\mathbf{p}$ given by:

$$\mathbf{p}(\theta^i) = \mathbf{P}(\theta^i) + \mathbf{u}(\theta^i) = \mathbf{e}_j x_j(\theta^i) \ , \tag{BC-1.2}$$

where $\mathbf{u}$ is the displacement vector of material point M and x_i the corresponding (current) Cartesian coordinates of the current position vector $\mathbf{p}$. It should be mentioned at this point that quantities associated with the reference configuration are denoted by capital letter symbols while their current configuration counterparts are denoted by small letter symbols.

A convenient basis for the case of employment of the convected coordinates θ^i is the covariant basis associated with the material point $M(\theta^i)$. The reference *covariant basis* $\{\mathbf{G}_i\}$ and current *covariant basis* $\{\mathbf{g}_i\}$ can be found from the corresponding position vectors as follows:

$$\mathbf{G_i} = \frac{\partial \mathbf{P}}{\partial \theta^i}, \quad \mathbf{g}_i = \frac{\partial \mathbf{p}}{\partial \theta^i} \ . \tag{BC-1.3}$$

Of use are also the associated dual bases (also termed contravariant bases) $\{\mathbf{G}^i\}$ and $\{\mathbf{g}^i\}$ which are related to the corresponding covariant bases $\{\mathbf{G}_i\}$ and $\{\mathbf{g}_i\}$ by:

$$\mathbf{G}_i \bullet \mathbf{G}^j = \delta_i^j, \quad \mathbf{g}_i \bullet \mathbf{g}^j = \delta_i^j \ . \tag{BC-1.4}$$

For future use the covariant and contravariant components of the reference and current metric tensor are also introduced:

$$\begin{aligned} &\mathbf{G}_{ij} \equiv \mathbf{G}_i \bullet \mathbf{G}_j, \quad G^{ij} \equiv \mathbf{G}^i \bullet \mathbf{G}_j \quad (\text{note} : G^{ij} = (G_{ij})^{-1}) \ , \\ &g_{ij} \equiv \mathbf{g}_i \bullet \mathbf{g}_j, \qquad g^{ij} \equiv \mathbf{g}^i \bullet \mathbf{g}^j \qquad (\text{note} : g^{ij} = (g_{ij})^{-1}) \ . \end{aligned} \tag{BC-1.5}$$

Since the curvature of the Euclidean space is zero, the corresponding metric tensor is the second rank identity tensor $\mathbf{I}$ which assumes the following useful forms in terms of the bases introduced in (BC-1.3) and (BC-1.4):

$$\begin{aligned}\mathbf{I} &= G_{ij}\mathbf{G}^i\mathbf{G}^j = \mathbf{G}^{ij}\mathbf{G}_j = \mathbf{G}_i\mathbf{G}^i = \mathbf{G}^i\mathbf{G}_i \ , \\ \mathbf{I} &= g_{ij}\mathbf{g}^i\mathbf{g}^j = g^{ij}\mathbf{g}_i\mathbf{g}_j = \mathbf{g}_i\mathbf{g}^i = \mathbf{g}^i\mathbf{g}_i \ .\end{aligned} \tag{BC-1.6}$$

The second rank identity tensor $\mathbf{I}$ has the property $\mathbf{I} \bullet \mathbf{A} = \mathbf{A} \bullet \mathbf{I} = \mathbf{A}$ for any tensor $\mathbf{A}$.

For certain applications a Cartesian description is preferable to a convected one, in which case material points are identified by their reference coordinates X_j (i.e., $x_i = x_i(X_j)$, $u_i = u_i(X_j)$) and the fixed orthonormal basis $\mathbf{e}_i$ is employed in both the reference and current configurations.

Of major importance for the analysis of deformation is the *deformation gradient* tensor $\mathbf{F}$ defined by:

$$\mathbf{F} \equiv \mathbf{p}\nabla = (\mathbf{P} + \mathbf{u})\nabla = \mathbf{I} + \mathbf{u}\nabla \ , \tag{BC-1.7}$$

where the *gradient operator* ∇ is with respect to the undeformed position $\mathbf{p}$, i.e., $\nabla \equiv \partial(\ )/\partial\mathbf{P}$. By definition, considering $\mathbf{p}$ as a function of $\mathbf{P}$, $d\mathbf{p} = (\mathbf{p}\nabla) \bullet d\mathbf{P}$ and where:

$$\nabla = \mathbf{G}^i \frac{\partial}{\partial \theta^i} \ \text{(convected)} \quad \nabla = \mathbf{e}_i \frac{\partial}{\partial X_i} \ \text{(Cartesian)} \ . \tag{BC-1.8}$$

Employing (BC-1.8) as well as (BC-1.3) in the deformation gradient definition (BC-1.7) one obtains:

$$\mathbf{F} = \mathbf{g}_i\mathbf{G}^i \ \text{(convected)} \quad \mathbf{F} = \frac{\partial \chi_i}{\partial X_j}\mathbf{e}_i\mathbf{e}_j \ \text{(Cartesian)} \ . \tag{BC-1.9}$$

The deformation gradient $\mathbf{F}$ which describes the deformation of the solid (from reference to current configuration) in the neighborhood of the material point M can be decomposed into a rigid body rotation superimposed on pure stretching. Two are the most meaningful ways to consider such a decomposition, namely:

$$\mathbf{F} = \mathbf{R} \bullet \mathbf{U}, \quad \mathbf{F} = \mathbf{V} \bullet \mathbf{R} \ , \tag{BC-1.10}$$

where $\mathbf{U}$ and $\mathbf{V}$ are two symmetric positive definite rank two tensors called *right* and *left stretch* tensors respectively. Both $\mathbf{U}$ and $\mathbf{V}$ share the same principal values λ_i (termed *principal stretches*) while their corresponding eigenvectors (termed reference and current principal directions respectively) are $\widehat{\mathbf{E}}_i$ and $\widehat{\mathbf{e}}_i$ and hence can be written as:

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \widehat{\mathbf{E}}, \widehat{\mathbf{E}}_i = \mathbf{U}^T, \quad \mathbf{V} = \sum_{i=1}^{3} \lambda_i \widehat{\mathbf{e}}, \widehat{\mathbf{e}}_i = \mathbf{V}^T \ . \tag{BC-1.11}$$

It should be mentioned at this point that in order for the deformation to have a physical meaning:

$$J \equiv \det(\mathbf{F}) > 0 \quad (J = dv/dV) \ , \tag{BC-1.12}$$

where from the definition of $\mathbf{F}$ follows that its determinant J is the ratio of the current (dv) to the reference (dV) volume element. For future use the following expressions for J are recorded:

$$J = [(\text{Det } g_{ij})/(\text{Det } G_{ij})]^{1/2} \text{ (convected)}, \quad J = \text{Det}(\partial \chi_i / \partial X_i) \text{ (Cartesian)} . \tag{BC-1.13}$$

The squares of $\mathbf{U}$ and $\mathbf{V}$, namely $\mathbf{C} = \mathbf{U} \bullet \mathbf{U}$ and $\mathbf{B} = \mathbf{V} \bullet \mathbf{V}$, play an important role in the analysis of deformation and are termed *right* and *left Cauchy-Green* tensors and are defined as follows (see also (BC-1.10)):

$$\mathbf{C} \equiv \mathbf{F}^T \bullet \mathbf{F} = \mathbf{U}^2 = \sum_{i=1}^{3} (\lambda_i)^2 \widehat{\mathbf{E}}_i \widehat{\mathbf{E}}_i, \quad \mathbf{B} \equiv \mathbf{F} \bullet \mathbf{F}^T = \mathbf{V}^2 = \sum_{i=1}^{3} (\lambda_i)^2 \widehat{\mathbf{e}}_i \widehat{\mathbf{e}}_i \, . \tag{BC-1.14}$$

From their respective definitions one can easily see that both $\mathbf{C}$ and $\mathbf{B}$ are symmetric while their positive definiteness follows from the fact that $d\mathbf{p} \bullet d\mathbf{p} = d\mathbf{P} \bullet \mathbf{C} \bullet d\mathbf{P} > 0$ for $d\mathbf{p} \neq 0$ and $d\mathbf{P} \bullet d\mathbf{P} = d\mathbf{p} \bullet \mathbf{B}^{-1} \bullet d\mathbf{p} > 0$ for $d\mathbf{p} \neq 0$.

In view of (BC-1.10), (BC-1.11) the following physical interpretation of the deformation at the neighborhood of point M can be given. The solids first deforms by pure stretching in the three mutually perpendicular directions $\widehat{\mathbf{E}}_i$ by corresponding stretch of λ_i followed by a rigid body rotation bringing $\widehat{\mathbf{E}}_i$ to its new position $\widehat{\mathbf{e}}_i$. In other words, a unit cube with edges along $\widehat{\mathbf{E}}_i$ deforms into an orthogonal parallelepiped of dimensions λ_i with edges along $\widehat{\mathbf{e}}_i$.

Of the many strain tensors that one can define the most useful for here is the Lagrangian strain tensor $\mathbf{E}$ defined as:

$$\mathbf{E} \equiv \frac{1}{2}(\mathbf{C} - \mathbf{I}) \, . \tag{BC-1.15}$$

By employing (BC-1.9), (BC-1.14) as well as (BC-1.6) one finds the following useful expressions for $\mathbf{E}$:

$$\mathbf{E} = \frac{1}{2}(g_{ij} - G_{ij}) \mathbf{G}^i \mathbf{G}^j \text{ (convected)}, \quad \mathbf{E} = \frac{1}{2}(\frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} - \delta_{ij}) \mathbf{e}_i \mathbf{e}_j \text{ (Cartesian)} . \tag{BC-1.16}$$

It is not difficult to check that the above defined tensor $\mathbf{E}$ satisfies all the required properties for a strain measure namely that $\mathbf{E}$ should vanish for zero displacements, $\mathbf{E}$ should be independent of the rigid body rotation part of the deformation $\mathbf{R}$ and that the linearized version of $\mathbf{E}$ should coincide with the small strain tensor $\mathbf{e} \equiv \frac{1}{2}(\mathbf{u}\nabla + \nabla\mathbf{u})$ all easily verifiable properties.

ii) *Stress Measures and Equilibrium Equations*

The most frequently used stress measure in continuum mechanics is the second rank tensor $\boldsymbol{\sigma}$ called *Cauchy stress* which is related to the *current traction* vector $\mathbf{t}$ acting on a point on the surface with normal $\mathbf{n}$ by:

$$\mathbf{t} = \mathbf{n} \bullet \boldsymbol{\sigma} \quad (\mathbf{t} \equiv d\mathbf{f}/da) \, . \tag{BC-1.17}$$

The above relation, which is easily proved using the familiar Cauchy tetrahedron argument of continuum mechanics, relates the current traction $\mathbf{t}$, i.e., the current force $d\mathbf{f}$ acting on a surface area da in the current configuration, with the normal $\mathbf{n}$ to the aforementioned area da.

For solids undergoing large deformations and strains one can define alternative measures for surface traction. One such possibility is the *nominal traction* vector $\mathbf{T}$ defined as the current force $d\mathbf{f}$ acting on the reference area dA. The rank two stess tensor that relates the normal $\mathbf{N}$ to the element of surface dA in the reference configuration to the nominal traction $\mathbf{T}$ is called the *first Piola-Kirchhoff stress* $\mathbf{\Pi}$ and satisfies:

$$\mathbf{T} = \mathbf{N} \bullet \mathbf{\Pi} \quad (\mathbf{T} \equiv d\mathbf{f}/dA) \ . \tag{BC-1.18}$$

Another possibility for a surface traction measure is the *pseudotraction* vector $\mathbf{T}^*$ which is defined to be the pseudoforce $d\mathbf{f}^*$ per unit reference area $d\mathcal{A}$, where $d\mathbf{f}^*$ is related to $d\mathbf{f}$ via the deformation gradient $\mathbf{F}$. The rank two stress tensor relating the reference normal $\mathbf{N}$ to the pseudotraction $\mathbf{T}^*$ is called the *second Piola-Kirchoff stress* $\mathbf{S}$ and satisfies:

$$\mathbf{T}^* = \mathbf{N} \bullet \mathbf{S} \quad (\mathbf{T}^* \equiv d\mathbf{f}^*/dA, \ d\mathbf{f} = \mathbf{F} \bullet d\mathbf{f}^*) \ . \tag{BC-1.19}$$

From the definitions (BC-1.17) to (BC-1.19) as well as from the relation between the current and reference surface areas da and dA, namely $J\mathbf{N}dA = \mathbf{F}^T \bullet \mathbf{n}da$, one can deduce the following relations among the afore-introduced stress measures:

$$\mathbf{\Pi} = J\mathbf{F}^{-1} \bullet \boldsymbol{\sigma}, \quad \mathbf{S} = J\mathbf{F}^{-1} \bullet \boldsymbol{\sigma} \bullet (\mathbf{F}^{-1})^T, \quad \mathbf{\Pi} = \mathbf{S} \bullet \mathbf{F}^T \ . \tag{BC-1.20}$$

Since from the principle of angular momentum conservation the Cauchy stress tensor is symmetric, i.e., $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, one can easily deduce from (BC-1.20) that the second Piola-Kirchhoff stress $\mathbf{S}$ is also symmetric, i.e., $\mathbf{S} = \mathbf{S}^T$ while the first Piola-Kirchhoff stress $\mathbf{\Pi}$ is not, i.e., $\mathbf{\Pi} \neq \mathbf{\Pi}^T$.

The principle of virtual work formulation (or weak formulation) of the problem gives in one step both the equilibrium equations as well as the boundary and interface (if any) conditions. Two different versions of the principle of virtual work are considered according to whether the reference or the current configuration is employed. More specifically, the reference configuration version of the principle of virtual work is:

$$\begin{aligned} &\int_V \mathbf{\Pi} \bullet\bullet \delta\mathbf{F} dV = \int_V \rho\mathbf{b} \bullet \delta\mathbf{u} dV + \int_{\partial V} \mathbf{T} \bullet \delta\mathbf{u} dA \ , \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{or equivalently} \\ &\int_V \mathbf{S} \bullet\bullet \delta\mathbf{E} dV = \int_V \rho\mathbf{b} \bullet \delta\mathbf{u} dV + \int_{\partial V} \mathbf{T} \bullet \delta\mathbf{u} dA \ . \end{aligned} \tag{BC-1.21}$$

where the solid occupies a volume V in the reference configuration with boundary ∂V and is subjected to body forces $\mathbf{b}$ per unit mass and nominal traction $\mathbf{T}$ on one part of the surface say ∂V_t while on the remaining part of the surface ∂V_u ($\partial V = \partial V_u \cup \partial V_t$) the displacements are prescribes. Moreover ρ is the reference density of the material (mass per unit reference volume) while δu is any kinematically admissible displacement field (recall that admissibility implies $\delta \mathbf{u} = 0$ on ∂V_u).

Assuming adequate continuity in the stress and displacement fields involved in (BC-1.21) one derives with the help of (BC-1.7), (BC-1.14), (BC-1.15) and the divergence theorem the following equilibrium equations:

$$\nabla \bullet \mathbf{\Pi} + \rho \mathbf{b} = 0, \text{ or equivalently } \nabla \bullet (\mathbf{S} \bullet \mathbf{F}^T) + \rho \mathbf{b} = 0 \ , \tag{BC-1.22}$$

while the current configuration is employed instead of the reference one the principle of virtual work assumes the form:

$$\int_v \boldsymbol{\sigma} \bullet \bullet \delta \mathbf{e} dv = \int_v \rho \mathbf{b} \bullet \delta \mathbf{u} dv + \int_v \mathbf{t} \bullet \delta \mathbf{u} da \quad (\mathbf{e} \equiv \frac{1}{2}(\mathbf{u}\nabla + \nabla \mathbf{u})) \ , \tag{BC-1.23}$$

where $\mathbf{b}$ is again the body force, ρ is the current density/mass per unit current volume) and $\mathbf{e}$ is the small strain tensor. Notice that in this case the gradient ∇ is considered with respect to the current position vector $\mathbf{p}$, i.e., $\nabla \equiv \partial(\ )/\partial \mathbf{p}$ since all field quantities are now functions of $\mathbf{p}$. Using once more the divergence theorem one obtains the following differential equation for equilibrium with respect to the current configuration (counterpart of (BC-1.22)).

$$\nabla \bullet \boldsymbol{\sigma} + \rho \mathbf{b} = 0 \ , \tag{BC-1.24}$$

while the corresponding traction boundary conditions are still given by (BC-1.17).

iii) *Constitutive Equations*

All the previous results concerning kinematics and equilibrium are valid for arbitrary solids undergoing finite deformations. Of interest here are only rubber elastic materials. Their behavior can be idealized as time independent and their final stresses depend only on their final strains (which by the way can be considerably large) of the O(1). Moreover these materials posess a *stored elastic energy* function W per unit reference volume which is termed strain energy density and which depends on the total strain $\mathbf{E}$ at the point in question, i.e., $W = W(\mathbf{E})$.

Materials that are characterized by such an energy density function are also called hyperelastic materials and their second Piola-Kirchhoff stress is the derivative of the energy density with respect to the Lagrangian strain, i.e.,

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} \ . \tag{BC-1.25}$$

The above constitutive equation holds true for arbitreary, i.e. anisotropic, compressible hyperelastic materials by exploiting the relations (BC 1.20) between the different stress measures one can find expressions equivalent to (BC-1.25) relating the remaining stress measures to derivatives of the energy with respect to corresponding deformation measures. For the case of incompressible, but still anisotropic, hyperelastic materials the constitutive equation (BC-1.25) takes the form:

$$\mathbf{S} = \frac{\partial W}{\partial E} + p\mathbf{C}^{-1}, \quad (\mathbf{C} = \mathbf{F}^T \bullet \mathbf{F},\ \det\mathbf{C} = 1)\ , \tag{BC-1.26}$$

where p is the solid's hydrostatic pressure, specifiable through the boundary conditions of the problem.

The equivalent expressions for the first Piola-Kirchhoff stress is found, with the help of (BC-1.20) for the compressible case:

$$\mathbf{\Pi} = \left(\frac{\partial W}{\partial \mathbf{F}}\right)^T\ , \tag{BC-1.27}$$

while for the incompressible case:

$$\mathbf{\Pi} = \left(\frac{\partial W}{\partial \mathbf{F}}\right)^T + p\mathbf{F}^{-1}\ . \tag{BC-1.28}$$

A very interesting subclass of hyperelastic materials is the class of isotropic hyperelastic materials, i.e., materials whose properties are the same with respect to any set of mutually perpendicular material axes. For a compressible, isotropic hyperelastic material, the energy density W is a function of the invariants of the strain $\mathbf{E}$, or equivalently, in view of (BC-1.15), W is a function of the invariants of the right Cauchy-Green tensor $\mathbf{C}$ or also equivalently in view of (BC-1.14) W is a function of the stretch ratios, namely:

$$W = W(I_1, I_2, I_3), \quad \text{or equivalently} \quad W = W(\lambda_1, \lambda_2, \lambda_3)\ , \tag{BC-1.29}$$

where the invariants of $\mathbf{C}$ are defined by:

$$I_1 \equiv \text{tr}\ \mathbf{C}, \quad I_2 \equiv \frac{1}{2}[(\text{tr}\ \mathbf{C})^2 - \text{tr}\ \mathbf{C}^2], \quad I_3 \equiv \text{Det}\ \mathbf{C}\ , \tag{BC-1.30}$$

with the trace of a second rank tensor $\mathbf{C}$ defined as the scalar inner product of $\mathbf{C}$ by the identity rank two tensor $\mathbf{I}$, i.e., tr $\mathbf{C} \equiv \mathbf{I} \bullet\bullet \mathbf{C}$. It should be noted here that when W is expressed as a function of the principal stretches λ_i the corresponding functional dependence is completely symmetric with respect to each of the variables, i.e., $W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_2, \lambda_1, \lambda_3) = W(\lambda_1, \lambda_3, \lambda_2)$.

For the case of an incompressible isotropic hyperelastic material the energy density takes the form:

$$W = W(I_1, I_2),\ I_3 = 1 \text{ or equivalently } W = W(\lambda_1, \lambda_2, \lambda_3),\ \lambda_1, \lambda_2, \lambda_3 = 1\ . \tag{BC-1.31}$$

Using (BC-1.29), (BC-1.30) into (BC-1.25) and recalling (BC-1.14), the stress-strain relation for the compressible isotropic hyperelastic material takes the form:

$$\mathbf{S} = 2\left[\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\mathbf{I} - \frac{\partial W}{\partial I_2}\mathbf{C} + \frac{\partial W}{\partial I_3}I_3\mathbf{C}^{-1}\right]\ . \tag{BC-1.32}$$

An equivalent expression to (BC-1.32) in terms of the Cauchy stress $\boldsymbol{\sigma}$ is found with the help of (BC-1.20) to be (see also (BC-1.12)):

$$\boldsymbol{\sigma} = \frac{2}{J}\left[\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\mathbf{B} - \frac{\partial W}{\partial I_2}\mathbf{B}^2 + \frac{\partial W}{\partial I_3}I_3\mathbf{I}\right] . \tag{BC-1.33}$$

From the definition of the invariants I_i of $\mathbf{C}$ in (BC-1.30) and from (BC-1.14) one obtains the following relations between I_i and the principal stretches λ_i:

$$I_1 = (\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2, \quad I_2 = (\lambda_1\lambda_2)^2 + (\lambda_2\lambda_3)^2 + (\lambda_3\lambda_1)^2, \quad I_3 = (\lambda_1\lambda_2\lambda_3)^2 . \tag{BC-1.34}$$

Employing the above result into (BC-1.33)) and taking also into account (BC-1.14)) one obtains that the principal axes of the Cauchy stress $\boldsymbol{\sigma}$ are the same with the current principal stretch directions $\widehat{\mathbf{e}}_i$ while the corresponding principal stresses σ_i are related to the corresponding principal stretches λ_i by:

$$\sigma_1 = \frac{1}{\lambda_2\lambda_3}\frac{\partial W}{\partial \lambda_1}, \quad \sigma_2 = \frac{1}{\lambda_3\lambda_1}\frac{\partial W}{\partial \lambda_2}, \quad \sigma_3 = \frac{1}{\lambda_1\lambda_2}\frac{\partial W}{\partial \lambda_3} . \tag{BC-1.35}$$

An analogous result holds for the second Piola-Kirchhoff stress whose principal directions are as expected the reference principal stretch directions $\widehat{\mathbf{E}}_i$.

For the case of an incompressible isotropic hyperelastic material from (BC-1.26) the stress-strain relations corresponding to (BC-1.32) and (BC-1.33) take the form:

$$\mathbf{S} = 2\left[\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\mathbf{I} - \frac{\partial W}{\partial I_2}\mathbf{C}\ ,\right] + p\mathbf{C}^{-1} \quad (I_3 = 1) \tag{BC-1.36}$$

while the corresponding to (BC-1.35) $\sigma_i - \lambda_i$ relationship becomes:

$$\sigma_1 = \lambda_1\frac{\partial w}{\partial \lambda_1} + p, \quad \sigma_2 = \lambda_2\frac{\partial W}{\partial \lambda_2} + p, \quad \sigma_3 = \lambda_3\frac{\partial W}{\partial \lambda_3} + p, \quad (\lambda_1\lambda_2\lambda_3 = 1) . \tag{BC-1.37}$$

iv) *Incremental Moduli*

The stability analyses that follow, require the evaluation of the solid's inrcremental moduli on the principal solution. Of the several choices, which relate the rate of a particular stress measure to its work-conjugate strain measure, the most useful moduli are the ones relating the rate of the first Piola-Kirchhoff stress $\dot{\mathbf{\Pi}}$ to the rate of the deformation gradient $\dot{\mathbf{F}}$ for a complessible, hyperelastic material.

Indeed for a hyperelastic solid, its potential energy $\mathcal{E}$ is:

$$\mathcal{E} = \int_V W(\mathbf{F})\, dA - \int_V \rho\mathbf{b} \bullet \mathbf{u} dV - \int_{\partial V} \mathbf{T} \bullet \mathbf{u} dA\ , \tag{BC-1.38}$$

where $\mathbf{T}$ and $\mathbf{b}$ are the nominal traction and body force applied on the solid (see (BC-1.21)).

Equilibrium is obtained by setting the first functional derivative of the energy to zero i.e. $\mathcal{E}_u\delta u = 0$, which in view of the constitutive law in (BC-1.27), leads to (BC-1.21).

To investigate the solid's stability, one must evaluate the second functional derivative of the potential energy on the principal equilibrium branch:

$$(\mathcal{E}_{,uu}\,\Delta u)\delta u = \int_V \left[\Delta\mathbf{F} \bullet\bullet \frac{\partial^2 W}{\partial\mathbf{F}\partial\mathbf{F}} \bullet\bullet \delta\mathbf{F}\right] dA; \ . \tag{BC-1.39}$$

The naturally appearing in (BC-1.39) second derivative of the energy density W with respect to the deformation gradient $\mathbf{F}$ is the sought incremental moduli tensor relating the rate of the first Piola-Kirchhoff stress $\dot{\mathbf{\Pi}}$ to the rate of the deformation gradient $\dot{\mathbf{F}}$ for a complessible, hyperelastic material.:

$$\dot{\mathbf{\Pi}} = \mathbf{L} \bullet\bullet \dot{\mathbf{F}}^T; \quad \mathbf{L} \equiv \frac{\partial^2 W}{\partial\mathbf{F}\partial\mathbf{F}} \quad \left(L_{ijkl} \equiv \frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}\right) , \tag{BC-1.40}$$

where the component form of the incremental moduli are given in parentheses in (BC-1.39).

For an isotropic, compressible, hyperelastic material, one obtains from the definitions of the invariants in (BC-1.30) and the chain rule of differentiation:

$$\begin{aligned}
L_{ijkl} \;=\;& 4\frac{\partial^2 W}{\partial I_1\partial I_1}F_{ij}F_{kl} + 4\frac{\partial^2 W}{\partial I_1\partial I_2}\left[F_{ij}F_{kq}(I_1\delta_{ql} - C_{ql}) + F_{ip}(I_1\delta_{pj} - C_{pj})F_{kl}\right] \\
&+ 4\frac{\partial^2 W}{\partial I_1\partial I_3}I_3\left[F_{ij}F^{-1}_{lk} + F^{-1}_{ji}F_{kl}\right] + 4\frac{\partial^2 W}{\partial I_2\partial I_2}F_{ip}(I_1\delta_{pj} - C_{pj})F_{kq}(I_1\delta_{ql} - C_{ql}) \\
&+ 4\frac{\partial^2 W}{\partial I_2\partial I_3}I_3\left[F^{-1}_{ji}F_{kq}(I_1\delta_{ql} - C_{ql}) + F_{ip}(I_1\delta_{pj} - C_{pj})F^{-1}_{lk}\right] + 4\frac{\partial^2 W}{\partial I_3\partial I_3}(I_3)^2 F^{-1}_{ji}F^{-1}_{lk} \\
&+ 2\frac{\partial W}{\partial I_1}\delta_{ik}\delta_{jl} + 2\frac{\partial W}{\partial I_2}\left[2F_{ij}F_{kl} - B_{ik}\delta_{jl} - \delta_{ik}C_{jl} - F_{il}F_{kj} + I_1\delta_{ik}\delta_{jl}\right] \\
&+ 2\frac{\partial W}{\partial I_3}I_3\left[2F^{-1}_{ji}F^{-1}_{lk} - F^{-1}_{li}F^{-1}_{jk}\right] .
\end{aligned} \tag{BC-1.41}$$

In several applications, the principal axes of deformation in the fundamental solution are also the most convenient cartesian axes to use for the bifurcation problem. In this case, where $\mathbf{F} = \mathrm{diag}[\lambda_i]$, a further simplification of (BC-1.41) is possible:

$$L_{ijkl} \;=\; \begin{cases} \dfrac{\partial^2 W}{\partial\lambda_i\partial\lambda_k}; & i = j,\ k = l\ , \\[2ex] \dfrac{\lambda_i\dfrac{\partial W}{\partial\lambda_i} - \lambda_j\dfrac{\partial W}{\partial\lambda_j}}{(\lambda_i)^2 - (\lambda_j)^2}; & i = k,\ j = l,\ (\lambda_i \neq \lambda_j)\ , \\[2ex] \dfrac{\lambda_i\dfrac{\partial W}{\partial\lambda_j} - \lambda_j\dfrac{\partial W}{\partial\lambda_i}}{(\lambda_j)^2 - (\lambda_i)^2}; & i = l,\ j = k,\ (\lambda_i \neq \lambda_j)\ , \end{cases} \tag{BC-1.42}$$

BC-2 GENERAL CONSIDERATIONS; RANK ONE CONVEXITY

For the material in this section....

BC-3 LATERALLY STRESSED HALF-SPACES IN 2D AND 3D

i) *Plane Strain Case – 2D*

We start by investigating the stability of a laterally stressed half-space, as shown in Fig. BC-3.1. The half-space is loaded under plane strain conditions by lateral straining (load parameter λ is the axial strain $1-\lambda_1 > 0$, taken positive in the case of compression) along its X_1 direction, while the face $X_2 = 0$ remains traction free. The stress-free configuration is

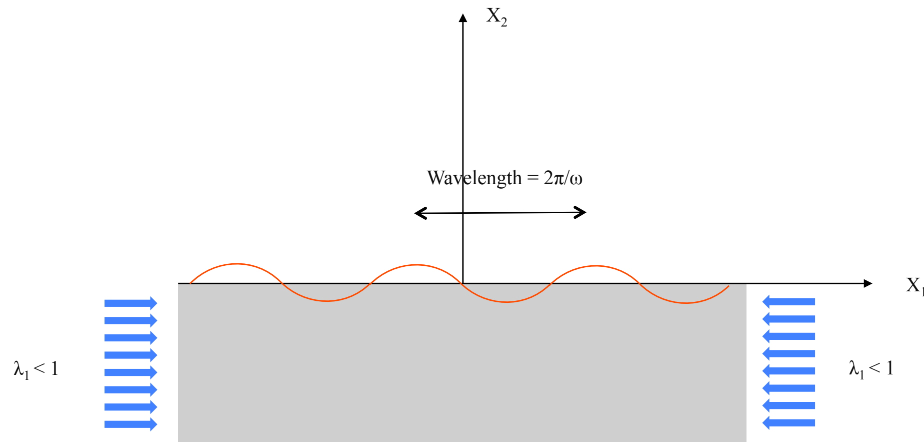

Figure BC-3.1: Reference configuration and surface buckling mode of a half-space loaded under plane strain conditions by lateral compression along its X_1 direction.

The principal solution is the uniform strain solution with constant deformation gradient $\overset{0}{\mathbf{F}} = \mathrm{diag}[\lambda_1, \lambda_2]$, where $\lambda = 1 - \lambda_1$ and $\lambda_2(\lambda)$ is found from the requirement of a zero surface traction. For an isotropic, compressible solid, one has from (BC-1.35):

$$\Pi_2 = \frac{\partial W}{\partial \lambda_2} = 0\,, \quad \lambda_1 = 1 - \lambda\,, \tag{BC-3.1}$$

A displacement control test is considered which is monotonically increasing with deformation. For small values of the load parameter, the homogeneous principal solution is stable, i.e. it is also a minimizer of the potential energy satisfying $(\mathcal{E}_{uu}\delta u)\delta u > 0$ for all nonzero admissible displacement fields δu. As λ increases, it reaches a value λ_c where the principal solution $\overset{0}{u}(\lambda_c)$ is no longer a minimizer of the potential energy, but where the energy vanishes along a particular direction $\overset{1}{u}$, called the "*critical mode*" which satisfies (BC-3.2). At this load, termed the "*critical load*", for the problem at hand a bifurcated equilibrium branch emerges with tangent $\overset{1}{u}$ from the principal solution. As shown in Section AC-3, the presence of a bifurcated branch at the critical point is guaranteed by:

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c), \lambda_c)\overset{1}{u})\delta u = 0\,. \tag{BC-3.2}$$

Noticing that the cartesian coordinate axes are also the principal axes of deformation, and hence the special form of the incremental moduli in (BC-1.42), the Euler-Lagrange differential equations of this eigenvalue problem, which result by a standard integration by parts of the variational equation (BC-3.2), are:

$$\begin{aligned} &L^c_{1111}\overset{1}{u}_{1,11} + L^c_{1122}\overset{1}{u}_{2,21} + L^c_{1212}\overset{1}{u}_{1,22} + L^c_{1221}\overset{1}{u}_{2,12} = 0 \;, \\ &L^c_{2112}\overset{1}{u}_{1,21} + L^c_{2121}\overset{1}{u}_{2,11} + L^c_{2211}\overset{1}{u}_{1,12} + L^c_{2222}\overset{1}{u}_{2,22} = 0 \;, \end{aligned} \tag{BC-3.3}$$

for any point in the reference domain ($-\infty < X_1 < \infty, -\infty < X_2 < 0$).

The corresponding natural boundary conditions at the traction-free surface $X_2 = 0$ are:

$$L^c_{1212}\overset{1}{u}_{1,2} + L^c_{1221}\overset{1}{u}_{2,1} = 0 \;, \quad L^c_{2211}\overset{1}{u}_{1,1} + L^c_{2222}\overset{1}{u}_{2,2} = 0 \;, \tag{BC-3.4}$$

where the first equation corresponds to the absence of shear and the second the absence of normal traction increment.

The nonzero components of the incremental moduli evaluated on the principal branch according to (BC-1.41), are found to be:

$$\begin{aligned} &L_{1111} = 4(\lambda_1)^2[W_{,11} + 2(\lambda_2)^2 W_{,12} + (\lambda_2)^4 W_{,22}] + 2[W_{,1} + (\lambda_2)^2 W_{,2}] \;, \\ &L_{2222} = 4(\lambda_2)^2[W_{,11} + 2(\lambda_1)^2 W_{,12} + (\lambda_1)^4 W_{,22}] + 2[W_{,1} + (\lambda_1)^2 W_{,2}] \;, \\ &L_{1122} = L^c_{2211} = 4\lambda_1\lambda_2[W_{,11} + ((\lambda_1)^2 + (\lambda_2)^2) W_{,12} + (\lambda_1\lambda_2)^2 W_{,22} + W_{,2}] \;, \\ &L_{1212} = L^c_{2121} = 2W_{,1} \;, \\ &L_{1221} = L_{2112} = -2\lambda_1\lambda_2 W_{,2} \;, \end{aligned} \tag{BC-3.5}$$

where use is made of the notation $W_{,i} \equiv \partial W/\partial I_i$ and $W_{,ij} \equiv \partial^2 W/\partial I_i \partial I_j$.

The above system of differential equations with constant coefficients admits the following solution:

$$\overset{1}{u}_i(X_1, X_2) = U_i \exp[i\omega(zX_2 + X_1)] \;, \tag{BC-3.6}$$

where ω is the wavenumber of the eigenmode and $z \in \mathbb{C}$, since the material is operating in the elliptic domain. Moreover, $\Im(z) < 0$ since the eigenmode should decay away from the surface as $X_2 \longrightarrow \infty$.

Substituting the eigenmode (BC-3.6) into the governing equations (BC-3.3) one obtains the following system for z and the amplitude U_i:

$$\begin{bmatrix} L_{1111} + z^2 L_{1212} & z(L_{1122} + L_{1221}) \\ z(L_{2112} + L_{2211}) & L_{2121} + z^2 L_{2222} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \;, \tag{BC-3.7}$$

where z is the root of the bi-quadratic:

$$(L_{1111} + z^2 L_{1212})(L_{2121} + z^2 L_{2222}) - z^2 (L_{1122} + L_{1221})^2 = 0 \;. \tag{BC-3.8}$$

Denote the two roots of the bi-quadratic (BC-3.8) with negative imaginary part $(\Im(z) < 0)$ by z_I the corresponding amplitudes by $\overset{I}{U}_i$. From the surface boundary conditions (BC-3.4) and using also (BC-3.7) to relate $\overset{I}{U}_1$to $\overset{I}{U}_2$, one obtains the following system relating, at the onset of bifurcation, the amplitudes $\overset{I}{U}_1$ of the two eigenmodes that decay exponentially away from the free surface:

$$\sum_{J=1}^{2} S_{IJ}\overset{J}{U}_1 = 0;\ \ I = 1, 2\ ,$$
$$S_{1J} \equiv z_J L_{1212} - L_{1221}\frac{L_{1111} + (z_J)^2 L_{1212}}{z_J(L_{1122} + L_{1221})},\ \ S_{2J} \equiv L_{2211} - L_{2222}\frac{L_{1111} + (z_J)^2 L_{1212}}{L_{1122} + L_{1221}};\ \ J = 1, 2\ . \tag{BC-3.9}$$

A nontrivial solution of the above system exists when:

$$\det[S_{IJ}(\lambda_c)] = 0\ , \tag{BC-3.10}$$

where the critical load λ_c is the lowest positive root of (BC-3.10).

BC-4 PLANE STRAIN TENSION AND COMPRESSION OF A RECTANGULAR BLOCK

This section pertains to the application of the general asymptotic theory of Lyapunov–Schmidt–Koiter for the single eigenmode case, developed in Section AC-3, to the determination of the initial post–bifurcation response of a hyperelastic rectangular block subjected to plane–strain (tensile or compressive) loading. The first part presents the principal solution and the general asymptotic expansions along the bifurcated equilibrium path of the load and the displacement field about the critical point. The last two parts pertain to the determination of the first and second order coefficients in the bifurcated equilibrium path's asymptotic expansions.

i) *Principal Solution and General Asymptotic Expansions for Bifurcated Paths.*

The reference configuration of the solid under investigation is its stress–free configuration and the corresponding $2L_1 \times 2L_2$ rectangular block is shown in Fig. BC-4.1 The block which has an aspect ratio $r \equiv 2L_1/2L_2$ is loaded in the X_2 direction between two rigid, parallel, flat plates that cannot transmit shear, while the faces $X_1 = \pm L_1$ remain traction free.

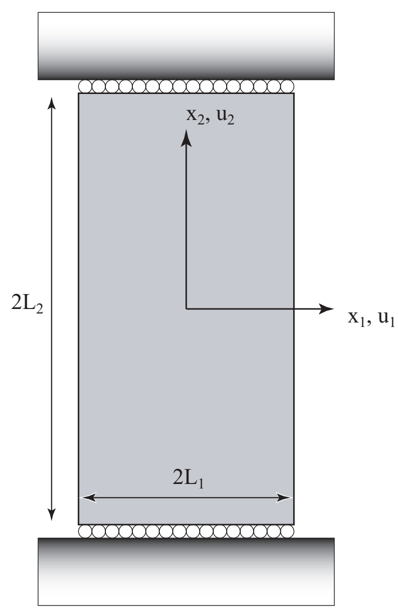

Figure BC-4.1: Rectangular bloc under plane strain axial compression or tension along its X_2 direction.

The rectangular block is made of a hyperelastic, compressible material with a two–dimensional strain energy $W(I_1, I_2)$ (usually obtained from its three–dimensional counterpart under plane strain assumptions) where I_1 and I_2 are the two invariants of the right Cauchy–Green deformation tensor $\mathbf{C}$, namely:

$$I_1 = C_{ii} \ , \quad I_2 = \det C_{ij} \ ; \quad C_{ij} \equiv F_{ki}F_{kj} \ , \quad F_{ij} \equiv \delta_{ij} + u_{i,j} \ , \qquad \text{(BC-4.1)}$$

with $u_i(X_1, X_2)$ the displacement components of a material point with reference configuration Cartesian coordinates X_i. In (BC-4.1) are also recorded for completeness the definitions of right Cauchy–Green deformation tensor $\mathbf{C}$ and the deformation gradient tensor $\mathbf{F}$. Again we adopt Einstein's summation convention over repeated Latin indexes, which range from 1 to 2, as well as the use of a comma followed by an index to denote partial differentiation with respect to the corresponding Cartesian coordinate, i.e. $f_i \equiv \partial f / \partial X_i$.

A displacement control test is considered, both in tension and in compression. The choice of a stiff loading device is dictated by the presence of a maximum force in the plane–strain tension test for some materials. Hence, the monotonically increasing with deformation "*load parameter*" λ ($\lambda \geq 0$) is chosen to be the absolute value of the block's engineering strain (relative displacement of the two rigid end plates divided by their initial distance).

The potential energy $\mathcal{E}$ of the solid is a λ–dependent functional of the field $u(\mathbf{X})$:

$$\mathcal{E}(u, \lambda) = \int_A W(I_1, I_2)\, dA \,, \tag{BC-4.2}$$

with A denoting the reference domain of the solid. In addition to the smoothness conditions required for the potential energy $\mathcal{E}$, an admissible displacement field must also satisfy the essential boundary conditions:

$$u_{2,1}(X_1, -L_2) = u_{2,1}(X_1, L_2) = 0 \; ; \; u_1(0,0) = u_2(0,0) = 0 \; ; \; |u_2(X_1, L_2) - u_2(X_1, -L_2)| = 2L_2\lambda \,, \tag{BC-4.3}$$

where the second set of constraints, i.e. fixing the displacement of the block's center, eliminate rigid body translations along X_i.

For each load parameter λ, the equilibrium displacement field is found by extremizing the solid's potential energy given in (BC-4.2), i.e. by setting to zero the first functional derivative of the potential energy with respect to the admissible displacement field u:

$$\mathcal{E}_{,u}(u, \lambda)\delta u = 0 \,. \tag{BC-4.4}$$

One obvious solution to (BC-4.4) is the "*principal solution*", denoted by $\overset{0}{u}(\lambda)$, which corresponds to a constant strain equilibrium of the rectangular block and which is given in terms of the principal stretch ratios $\lambda_i(\lambda)$ by:

$$\overset{0}{u}_1(\lambda) = [\lambda_1(\lambda) - 1]X_1 \,, \quad \overset{0}{u}_2(\lambda) = [\lambda_2(\lambda) - 1]X_2 \,. \tag{BC-4.5}$$

The principal stretch ratios $\lambda_i(\lambda)$ depend on the solid's constitutive response and the load parameter λ and are determined by:

$$\Pi_1 = \frac{\partial W}{\partial \lambda_1} = 0 \,, \quad \lambda_2 = 1 \pm \lambda \,, \tag{BC-4.6}$$

where the first equation in (BC-4.6) expresses the vanishing of the lateral first Piola–Kirchhoff stress Π_1 and thus gives λ_1 in terms of λ_2, while the second equation gives λ_2 in terms of the load parameter λ (with sign + for tension and sign – for compression). To find the principal solution $\lambda_i(\lambda)$, the invariants I_i of W in (BC-4.6) must be expressed in terms of the principal stretch ratios:

$$I_1 = (\lambda_1)^2 + (\lambda_2)^2 \ , \quad I_2 = (\lambda_1 \lambda_2)^2 \ . \tag{BC-4.7}$$

Notice that in the absence of loading, the displacement field vanishes $\overset{0}{u}(0) = 0$. For small values of the load parameter, the homogeneous principal solution is stable, i.e. it is also a minimizer of the potential energy satisfying $(\mathcal{E}_{uu}\delta u)\delta u > 0$ for all nonzero admissible displacement fields δu. As λ increases, it reaches a value λ_c where the principal solution $\overset{0}{u}(\lambda_c)$ is no longer a minimizer of the potential energy, but where the energy vanishes along a particular direction $\overset{1}{u}$, called the "*critical mode*" which satisfies the first equation in (BC-4.8). At this load, termed the "*critical load*", for the problem at hand a bifurcated equilibrium branch emerges with tangent $\overset{1}{u}$ from the principal solution. As shown in Section AC-3, the presence of a bifurcated branch at the critical point is guaranteed by the second equation in (BC-4.8):

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c), \lambda_c)\overset{1}{u})\delta u = 0 \ , \quad (\mathcal{E}_{,u}(\overset{0}{u}(\lambda_c), \lambda_c)\overset{1}{u}) = 0 \ . \tag{BC-4.8}$$

For the axially loaded rectangular block problem at hand, the eigenmode $\overset{1}{u}$ of the bilinear stability operator $\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c), \lambda_c)$ is unique (up to a multiplicative scalar) and hence leads to a simple bifurcation at λ_c.

Of interest here is the determination of the bifurcated equilibrium path emerging at λ_c. Proving the existence of such a global bifurcated solution is a difficult and highly technical mathematical problem, which is beyond the scope of this work. It is therefore tacitly assumed that at the neighborhood of the critical point all the necessary conditions are met which allow for the existence of a bifurcated path. The displacement field of this path is found analytically using the general LSK asymptotic expansion with respect to the bifurcation amplitude parameter ξ, according to Section AC-3:

$$\xi \equiv < u - \overset{0}{u}, \overset{1}{u} > \ , \tag{BC-4.9}$$

where by $< u, v >$ is denoted the inner product of two admissible displacement fields u and v. The choice presently adopted for the inner product is:

$$< u, v > \equiv \frac{1}{|A|} \int_A u_i v_i \, dA \ , \tag{BC-4.10}$$

where $|A|$ denotes the area of the reference domain A (and equals $4L_1L_2$).

According to the general theory presented in Section AC-3, the LSK asymptotic expansion for the bifurcated equilibrium path about the critical point λ_c is given by:

$$u = \overset{0}{u}(\lambda) + \xi \overset{1}{u} + \frac{\xi^2}{2}\overset{2}{u} + O(\xi^3) \ , \quad \lambda = \lambda_c + \frac{\xi^2}{2}\lambda_2 + O(\xi^4) \ . \tag{BC-4.11}$$

Notice that the asymptotic expansion of λ is in terms of even powers of ξ, due to the symmetry of the problem (it can be shown with the help of the eigenmode expressions (BC-4.17) below that for all cases analyzed here $((\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\overset{1}{u} = 0)$. The coefficients for the second order terms in the expansion of the displacement and the load parameter are:

$$(\mathcal{E}^c_{,uu}\overset{2}{u} + (\mathcal{E}^c_{,uuu}\overset{1}{u})\overset{1}{u})\delta v = 0, \quad < \delta v, \overset{1}{u} >= 0,$$
$$\lambda_2 = -\frac{1}{3}\frac{(((\mathcal{E}^c_{,uuuu}\overset{1}{u})\overset{1}{u})\overset{1}{u})\overset{1}{u} + 3((\mathcal{E}^c_{,uuu}\overset{2}{u})\overset{1}{u})\overset{1}{u}}{((d\mathcal{E}_{,uu}/d\lambda)^c\overset{1}{u})\overset{1}{u}}, \tag{BC-4.12}$$

where in the above equations the superscript $(\ )^c$ denotes evaluation of the operator in question at the critical point $(\overset{0}{u}(\lambda_c), \lambda_c)$.

According to Section AC-3, the stability of the bifurcated equilibrium path of the perfect block in the neighborhood of λ_c depends on the sign of λ_2; if $\lambda_2 > 0$ the bifurcated path is stable, i.e. it minimizes the block's potential energy $\mathcal{E}$ in a neighborhood of the critical point, while for $\lambda_2 < 0$ it is unstable near the critical point. Consequently according to the general theory in Section AC-4, an imperfect block whose perfect counterpart has $\lambda_2 < 0$ will exhibit a maximum average strain (recall that a displacement control is considered in the present analysis) lower than λ_c and thus lead to a snap–through type instability. On the other hand, a $\lambda_2 > 0$ for the perfect block, guarantees a stable equilibrium path of its imperfect counterpart near λ_c and hence allows average strain values exceeding λ_c in a quasistatic loading process.

ii) *Critical Load* λ_c *and Mode* $\overset{1}{u}$

Using the energy definition in (BC-4.2), the rectangular block's eigenvalue problem in $(\text{BC-4.8})_1$ can be rewritten as:

$$(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda_c), \lambda_c)\overset{1}{u})\delta u = \int_A [L^c_{ijk\ell}\overset{1}{u}_{i,j}\delta u_{k,\ell}]dA = 0\ ; \quad L^c_{ijk\ell} \equiv \frac{\partial^2 W(\overset{0}{u}(\lambda_c))}{\partial F_{ij}\partial F_{k\ell}}\ . \tag{BC-4.13}$$

The Euler-Lagrange differential equations of the eigenvalue problem, which result by a standard integration by parts of the variational equation (BC-4.13), are given by:

$$L^c_{1111}\overset{1}{u}_{1,11} + L^c_{1122}\overset{1}{u}_{2,21} + L^c_{1212}\overset{1}{u}_{1,22} + L^c_{1221}\overset{1}{u}_{2,12} = 0\ ,$$
$$L^c_{2112}\overset{1}{u}_{1,21} + L^c_{2121}\overset{1}{u}_{2,11} + L^c_{2211}\overset{1}{u}_{1,12} + L^c_{2222}\overset{1}{u}_{2,22} = 0\ , \tag{BC-4.14}$$

for any point in the reference domain $(-L_1 \leq X_1 \leq L_1, -L_2 \leq X_2 \leq L_2)$.

The corresponding boundary conditions are:

$$L^c_{1111}\overset{1}{u}_{1,1} + L^c_{1122}\overset{1}{u}_{2,2} = 0\ , \quad L^c_{2112}\overset{1}{u}_{1,2} + L^c_{2121}\overset{1}{u}_{2,1} = 0\ , \quad (X_1 = \pm L_1)\ ,$$
$$\overset{1}{u}_{1,2} = 0\ , \qquad \overset{1}{u}_{2,1} = 0\ , \qquad (X_2 = \pm L_2)\ , \tag{BC-4.15}$$

where the first three equations are natural boundary conditions, while the last results from the kinematic admissibility condition (BC-4.3).

The orthotropy of the incremental moduli tensor $\mathbf{L}$ with respect to coordinate axes is used in the derivation of the above equations. The nonzero components of the incremental moduli evaluated on the principal branch according to (BC-4.13)$_2$, are found with the help of (BC-4.1) and (BC-4.7) to be:

$$\begin{aligned}
L^c_{1111} &= 4(\lambda_1)^2[W_{,11} + 2(\lambda_2)^2 W_{,12} + (\lambda_2)^4 W_{,22}] + 2[W_{,1} + (\lambda_2)^2 W_{,2}] \ , \\
L^c_{2222} &= 4(\lambda_2)^2[W_{,11} + 2(\lambda_1)^2 W_{,12} + (\lambda_1)^4 W_{,22}] + 2[W_{,1} + (\lambda_1)^2 W_{,2}] \ , \\
L^c_{1122} &= L^c_{2211} = 4\lambda_1\lambda_2[W_{,11} + ((\lambda_1)^2 + (\lambda_2)^2)W_{,12} + (\lambda_1\lambda_2)^2 W_{,22} + W_{,2}] \ , \\
L^c_{1212} &= L^c_{2121} = 2W_{,1} \ , \\
L^c_{1221} &= L^c_{2112} = -2\lambda_1\lambda_2 W_{,2} \ ,
\end{aligned} \tag{BC-4.16}$$

where use is made of the notation $W_{,i} \equiv \partial W/\partial I_i$ and $W_{,ij} \equiv \partial^2 W/\partial I_i \partial I_j$.

The above system of differential equations with constant coefficients has separable (in X_1 and X_2) solutions for the critical mode $\overset{1}{u}$, leading to the following expressions:

$$\mathcal{S}^1 : \left\{ \begin{array}{l} \overset{1}{u}_1 = v_1(X_1)\cos(p_2X_2) - v_1(0) \\ \\ \overset{1}{u}_2 = -v_2(X_1)\sin(p_2X_2) \\ \\ p_2 = n\pi/L_2 \end{array} \right\} , \quad \mathcal{A}^1 : \left\{ \begin{array}{l} \overset{1}{u}_1 = v_1(X_1)\sin(p_2X_2) \\ \\ \overset{1}{u}_2 = v_2(X_1)\cos(p_2X_2) - v_2(0) \\ \\ p_2 = (n - \dfrac{1}{2})\pi/L_2 \end{array} \right\} , \tag{BC-4.17}$$

where the symbols $\mathcal{S}^1$ and $\mathcal{A}^1$ designate the symmetric and antisymmetric, with respect to X_1, modes. Here n is an arbitrary integer (to be determined subsequently).

The mode's X_1 dependence is recorded below by giving the expressions for $v_i(X_1)$. Two cases $\mathcal{S}^2$ and $\mathcal{A}^2$ are distinguished, according to the symmetry or antisymmetry of the mode with respect to the X_2 axis:

$$\mathcal{S}^2 : \left\{ \begin{array}{rcl} v_1(X_1) &=& A_\alpha \sinh(\alpha p_2 X_1) + A_\beta \sinh(\beta p_2 X_1) \\ \\ v_2(X_1) &=& A_\alpha K(\alpha)\cosh(\alpha p_2 X_1) + A_\beta K(\beta)\cosh(\beta p_2 X_1) \\ \\ A_\beta &=& -\dfrac{(L^c_{1111}\alpha - L^c_{1122}K(\alpha))\cosh(\alpha p_2 L_1)}{(L^c_{1111}\beta - L^c_{1122}K(\beta))\cosh(\beta p_2 L_1)} A_\alpha \end{array} \right\} ,$$

$$\mathcal{A}^2 : \left\{ \begin{array}{rcl} v_1(X_1) &=& B_\alpha \cosh(\alpha p_2 X_1) + B_\beta \cosh(\beta p_2 X_1) \\ \\ v_2(X_1) &=& B_\alpha K(\alpha)\sinh(\alpha p_2 X_1) + B_\beta K(\beta)\sinh(\beta p_2 X_1) \\ \\ B_\beta &=& -\dfrac{(L^c_{1111}\alpha - L^c_{1122}K(\alpha))\sinh(\alpha p_2 L_1)}{(L^c_{1111}\beta - L^c_{1122}K(\beta))\sinh(\beta p_2 L_1)} B_\alpha \end{array} \right\} , \tag{BC-4.18}$$

$$K(x) \equiv [L^c_{1111}x^2 - L^c_{1212}]/[(L^c_{1122} + L^c_{1221})x] \ , \quad (x = \alpha,\ \beta) \ .$$

The constants α and β entering (BC-4.18) are related to the roots y of the (biquadratic) characteristic equation of the system of differential equations (BC-4.14), namely:

$$\begin{gathered} ay^4 + 2by^2 + c = 0 \; ; \quad y = \pm i\alpha \; , \; \pm i\beta \; , \\ a \equiv L^c_{1111} L^c_{2121} \; , \quad 2b \equiv L^c_{1111} L^c_{2222} + L^c_{1212} L^c_{2121} - (L^c_{1122} + L^c_{2112})^2 \; , \quad c \equiv L^c_{2222} L^c_{1212} \; . \end{gathered} \tag{BC-4.19}$$

Since the bifurcated solutions of interest are in the elliptic regime of the material response, the characteristic equation (BC-4.19) evaluated at the critical load cannot have real solutions and hence $a > 0, \; c > 0, \; b > -\sqrt{ac}$. The solution has $\alpha, \beta \in \mathbb{R}$ if $b > \sqrt{ac}$, in which case the material is said to be in the *EI regime* (elliptic-imaginary case since the roots of (BC-4.19) are purely imaginary $\pm i\alpha, \pm i\beta$). When $|b| < \sqrt{ac}$, the material is said to be in the *EC regime* (elliptic-complex case since the roots of (BC-4.19) are complex conjugate $\delta \pm i\gamma$ – i.e. $(\alpha, \beta) = (\gamma + i\delta, \gamma - i\delta)$). It should also be noted that the eigenmode components $v_i(X_1)$ are always real, since for the EC regime $\beta = \overline{\alpha}$ and $A_\beta = \overline{A_\alpha}, \; B_\beta = \overline{B_\alpha}$ (remark is obvious for the EI regime since $\alpha, \beta \in \mathbb{R}$).

Substituting (BC-4.18) into the boundary conditions (BC-4.15) and using (BC-4.14), (BC-4.19)), one finds that one of the following equations must be satisfied at the critical load:

$$\begin{aligned} &\left[\frac{\tanh(\pi\alpha\eta)}{\tanh(\pi\beta\eta)}\right]^{\varepsilon_2} = \frac{\alpha}{\beta}\left[\frac{\beta^2 - \chi}{\alpha^2 - \chi}\right] \; , \qquad \text{EI case} \; , \; (\alpha, \beta) \; = [(b \pm (b^2 - ac)^{1/2})/a]^{1/2} \; , \\ &\frac{\sinh(2\pi\gamma\eta)}{\sin(2\pi\delta\eta)} = \varepsilon_2 \frac{\gamma}{\delta}\left[\frac{\chi - \gamma^2 - \delta^2}{\chi + \gamma^2 + \delta^2}\right] \; , \quad \text{EC case} \; , \; (\gamma, \delta) \; = \; [((ac)^{1/2} \pm b)/2a]^{1/2} \; , \\ &\eta \equiv p_2 L_1/\pi = nr \text{ or } (n - 1/2)r \; , \quad \chi \equiv \frac{[L^c_{2222}(L^c_{1212}L^c_{2121} - (L^c_{1221})^2)]}{[L^c_{2121}(L^c_{1111}L^c_{2222} - (L^c_{1122})^2)]} \; , \end{aligned} \tag{BC-4.20}$$

where $\varepsilon_2 = +1$ for the $\mathcal{S}^2$ type modes and $\varepsilon_2 = -1$ for the $\mathcal{A}^2$ type modes in (BC-4.18). The critical load λ_c for a given block, i.e. for a given energy density W and aspect ratio r, is the minimum nontrivial root of the two equations in (BC-4.20), where the minimum is taken over all integers n and for $\varepsilon_2 = \pm 1$.

iii) *Second Order Terms λ_2 and $\overset{2}{u}$*

The calculation of λ_2, required to determine the stability of the bifurcated equilibrium solution for the hyperelastic block, necessitates as an intermediate step the determination of the auxiliary displacement field $\overset{2}{u}$. To this end, from the general variational statement defining $\overset{2}{u}$ in (BC-4.12)$_1$, one obtains:

$$\int\limits_A [(L^c_{ijk\ell} \overset{2}{u}_{k,\ell} + M^c_{ijk\ell mn} \; \overset{1}{u}_{k,\ell} \overset{1}{u}_{m,n}) \delta v_{i,j}] dA = 0 \; , \quad < \overset{1}{u}, \delta v >= 0 \; , \quad M^c_{ijk\ell mn} \equiv \frac{\partial^3 W(\overset{\circ}{u}\,(\Lambda_c))}{\partial F_{ij} \partial F_{k\ell} \partial F_{mn}} \; . \tag{BC-4.21}$$

Due to the symmetries of the problem and making use of (BC-4.17), it will be shown that the solution of (BC-4.21) results in the following expression for $\overset{2}{u}$:

$$\overset{2}{u}_1 = w_1(X_1)\cos(2p_2X_2) + \widetilde{w}_1(X_1) \; , \quad \overset{2}{u}_2 = w_2(X_1)\sin(2p_2X_2) \; , \tag{BC-4.22}$$

where the expressions for $w_i(X_1)$ and $\widetilde{w}_1(X_1)$ will be detailed in the solution procedure that follows.

The Euler-Lagrange differential equations for $\overset{2}{u}$, found from the variational equation (BC-4.21) using integration by parts, are:

$$
\begin{aligned}
& L^c_{1111}\, \overset{2}{u}_{1,11} + (L^c_{1122} + L^c_{1221})\, \overset{2}{u}_{2,12} + L^c_{1212}\, \overset{2}{u}_{1,22} = \\
& -[(M^c_{111111}(\overset{1}{u}_{1,1})^2 + 2M^c_{111122}\, \overset{1}{u}_{1,1}\overset{1}{u}_{2,2} + M^c_{112222}(\overset{1}{u}_{2,2})^2 \\
& + M^c_{111212}(\overset{1}{u}_{1,2})^2 + M^c_{112121}(\overset{1}{u}_{2,1})^2 + 2M^c_{112112}\, \overset{1}{u}_{1,2}\overset{1}{u}_{2,1})_{,1} \\
& + 2(M^c_{121112}\, \overset{1}{u}_{1,1}\overset{1}{u}_{1,2} + M^c_{121121}\, \overset{1}{u}_{1,1}\overset{1}{u}_{2,1} + M^c_{122212}\, \overset{1}{u}_{2,2}\overset{1}{u}_{1,2} + M^c_{122221}\, \overset{1}{u}_{2,2}\overset{1}{u}_{2,1})_{,2}\,]\ , \\
& L^c_{2121}\, \overset{2}{u}_{2,11} + (L^c_{2112} + L^c_{2211})\, \overset{2}{u}_{1,12} + L^c_{2222}\, \overset{2}{u}_{2,22} = \\
& -[(M^c_{222222}(\overset{1}{u}_{2,2})^2 + 2M^c_{221122}\, \overset{1}{u}_{1,1}\overset{1}{u}_{2,2} + M^c_{221111}(\overset{1}{u}_{1,1})^2 \\
& + M^c_{221212}(\overset{1}{u}_{1,2})^2 + M^c_{222121}(\overset{1}{u}_{2,1})^2 + 2M^c_{221221}\, \overset{1}{u}_{1,2}\overset{1}{u}_{2,1})_{,2} \\
& + 2(M^c_{211112}\, \overset{1}{u}_{1,1}\overset{1}{u}_{1,2} + M^c_{211121}\, \overset{1}{u}_{1,1}\overset{1}{u}_{2,1} + M^c_{212221}\, \overset{1}{u}_{2,2}\overset{1}{u}_{2,1} + M^c_{212212}\, \overset{1}{u}_{2,2}\overset{1}{u}_{1,2})_{,1}\,]\ ,
\end{aligned}
\tag{BC-4.23}
$$

for any point in the reference domain $(-L_1 \le X_1 \le L_1, -L_2 \le X_2 \le L_2)$.

The corresponding boundary conditions are:

$$
\left.
\begin{aligned}
& L^c_{1111}\, \overset{2}{u}_{1,1} + L^c_{1122}\, \overset{2}{u}_{2,2} = -[M^c_{111111}(\overset{1}{u}_{1,1})^2 + 2M^c_{111122}\, \overset{1}{u}_{1,1}\overset{1}{u}_{2,2} \\
& + M^c_{112222}(\overset{1}{u}_{2,2})^2 + M^c_{111212}(\overset{1}{u}_{1,2})^2 + 2M^c_{111221}\, \overset{1}{u}_{1,2}\overset{1}{u}_{2,1} + M^c_{112121}(\overset{1}{u}_{2,1})^2] \\
& L^c_{2112}\, \overset{2}{u}_{1,2} + L_{2121}\, \overset{2}{u}_{2,1} = -2[M^c_{211112}\, \overset{1}{u}_{1,1}\overset{1}{u}_{1,2} + M^c_{211121}\, \overset{1}{u}_{1,1}\overset{1}{u}_{2,1} \\
& + M^c_{212221}\, \overset{1}{u}_{2,2}\overset{1}{u}_{2,1} + M^c_{212212}\, \overset{1}{u}_{2,2}\overset{1}{u}_{1,2}]
\end{aligned}
\right\}\ ;\ (X_1 = \pm L_1)\ ,
\tag{BC-4.24}
$$

$$
\overset{2}{u}_{2,1} = 0\ ,\quad \overset{2}{u}_{1,2} = 0\ ; \qquad (X_2 = \pm L_2)\ ,
$$

where the first three equations are natural boundary conditions while the last results from the kinematic admissibility condition (BC-4.3). The components of rank six tensor $\mathbf{M}$, defined in (BC-4.21)$_3$, are calculated in a straightforward way with the help of (BC-4.1) and (BC-4.7), but due to the resulting cumbersome expressions will not be recorded here. The orthotropy of the principal solution implies that the only nonzero components of $\mathbf{M}$ are those containing an even number of similar indexes, just like the components of $\mathbf{L}$. This property has been used in the derivations for $\overset{2}{u}$ presented here.

Substituting the eigenmode expressions of (BC-4.17) into the system of (BC-4.23) and (BC-4.24) for the boundary value problem in $\overset{2}{u}$, one finds the X_2 dependence of $\overset{2}{u}$ recorded in

(BC-4.22). The X_1 dependence of $\overset{2}{u}$, i.e. the functions $w_1(X_1)$, $\widetilde{w}_1(X_1)$ and $w_2(X_1)$, are obtained by introducing the expressions for $\overset{2}{u}$ in (BC-4.22) into the ordinary differential equations (BC-4.23)

$$
\begin{aligned}
L^c_{1111}w_1''(X_1) + 2p_2(L^c_{1122} + L^c_{1221})w_2'(X_1) - (2p_2)^2 L^c_{1212}w_1(X_1) &= -\varepsilon_1 E_1(X_1) \; , \\
L^c_{1111}\widetilde{w}_1''(X_1) &= -\widetilde{E}_1'(X_1) \; , \\
L^c_{2121}w_2''(X_1) - 2p_2(L^c_{2211} + L^c_{2112})w_1'(X_1) - (2p_2)^2 L^c_{2222}w_2(X_1) &= -\varepsilon_1 E_2(X_1) \; ,
\end{aligned} \qquad \text{(BC-4.25)}
$$

$$
\begin{aligned}
E_1(X_1) \;\; \equiv \;\; & \frac{1}{2}[M^c_{111111}(v_1')^2 - 2M^c_{111122}p_2v_1'v_2 + M^c_{112222}(p_2v_2)^2 \\
- \;\; & M^c_{111212}(p_2v_1)^2 - 2M^c_{111221}p_2v_1v_2' - M^c_{112121}(v_2')^2]' \\
+ \;\; & 2p_2[-M^c_{121112}p_2v_1v_1' - M^c_{121121}v_1'v_2' + M^c_{122212}(p_2)^2v_2v_1 + M^c_{122221}p_2v_2v_2'] \; ,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{E}_1(X_1) \;\; \equiv \;\; & \frac{1}{2}[M^c_{111111}(v_1')^2 - 2M^c_{111122}p_2v_1'v_2 + M^c_{112222}(p_2v_2)^2 \\
+ \;\; & M^c_{111212}(p_2v_1)^2 + 2M^c_{111221}p_2v_1v_2' + M^c_{112121}(v_2')^2] \; ,
\end{aligned}
$$

$$
\begin{aligned}
E_2(X_1) \;\; \equiv \;\; & [-M^c_{211112}p_2v_1'v_1 - M^c_{211121}v_1'v_2' + M^c_{212212}(p_2)^2v_1v_2 + M^c_{212221}p_2v_2v_2']' \\
+ \;\; & p_2[-M^c_{221111}(v_1')^2 + 2M^c_{221122}p_2v_1'v_2 - M^c_{222222}(p_2v_2)^2 \\
+ \;\; & M^c_{221212}(p_2v_1)^2 + 2M^c_{221221}p_2v_1v_2' + M^c_{222121}(v_2')^2] \; ,
\end{aligned}
$$

with $\varepsilon_1 = +1$ for an $\mathcal{S}^1$ mode and $\varepsilon_1 = -1$ for an $\mathcal{A}^1$ mode, according to (BC-4.17) and where the functions $v_i(X_1)$ are given by (BC-4.18). Note that a symbol followed by a prime denotes ordinary differentiation with respect to X_1.

Substitution of the eigenmode $\overset{2}{u}$ in (BC-4.22) into the boundary conditions (BC-4.24) yields:

$$
\left.
\begin{aligned}
L^c_{1111}w_1' + 2p_2L^c_{1122}w_2 &= -\varepsilon_1 F_1 \\
L^c_{1111}\widetilde{w}_1' &= -\widetilde{E}_1 \\
L^c_{2121}w_2' - 2p_2L^c_{2112}w_1 &= -\varepsilon_1 F_2
\end{aligned}
\right\} ; \; (X_1 = \pm L_1) \; , \qquad \text{(BC-4.26)}
$$

$$
\begin{aligned}
F_1 \;\; \equiv \;\; & \frac{1}{2}[M^c_{111111}(v_1')^2 - 2M^c_{111122}p_2v_1'v_2 + M^c_{112222}(p_2v_2)^2 \\
& -M^c_{111212}(p_2v_1)^2 - 2M^c_{111221}p_2v_1v_2' - M^c_{112121}(v_2')^2] \; ,
\end{aligned}
$$

$$
F_2 \;\; \equiv \;\; -M^c_{211112}p_2v_1v_1' - M^c_{211121}v_1'v_2' + M^c_{212212}(p_2)^2v_2v_1 + M^c_{212221}p_2v_2v_2' \; ,
$$

where the definitions of ε_1 and $v_i(X_1)$ are the same as for (BC-4.25).

The determination of $\widetilde{w}_1(X_1)$ from the differential equation (BC-4.25) and boundary condition (BC-4.26) is straightforward:

$$\widetilde{w}_1(X_1) = -\frac{1}{L^c_{1111}} \int_0^{X_1} \widetilde{E}_1(y)\, dy \,, \tag{BC-4.27}$$

where the constant of integration is fixed from the constraint against rigid body displacement in $(\text{BC-4.3})_2$.

Finding $w_i(X_1)$ is a considerably more complicated task which proceeds as follows: Notice that the system of differential equations for $w_i(X_1)$ can be decoupled to yield:

$$a w_i''''(X_1) - 2b(2p_2)^2 w_i''(X_1) + c(2p_2)^4 w_i(X_1) \;=\; \varepsilon_1 \widehat{E}_i(X_1) \,, \quad (i = 1, 2) \,, \tag{BC-4.28}$$

$$\widehat{E}_1(X_1) \;\equiv\; -E_1'' L^c_{2121} + E_2'(2p_2)(L^c_{1122} + L^c_{1221}) + E_1(2p_2)^2 L^c_{2222} \,,$$

$$\widehat{E}_2(X_1) \;\equiv\; -E_2'' L^c_{1111} - E_1'(2p_2)(L^c_{2211} + L^c_{2112}) + E_2(2p_2)^2 L^c_{1212} \,,$$

where the definition of the constants $a,\ b,\ c$ is given in (BC-4.19). The solution to the above fourth order ordinary differential equations with constant coefficients is found, with the help of (BC-4.25), to be:

$$\begin{aligned}
w_1(X_1) \;&=\; \widehat{A}_\alpha \sinh(2\alpha p_2 X_1) + \widehat{A}_\beta \sinh(2\beta p_2 X_1) + \varepsilon_1 \int_0^{X_1} G(X_1 - y)\widehat{E}_1(y)\, dy \,, \\
w_2(X_1) \;&=\; -[\, (\widehat{A}_\alpha K(\alpha) + \varepsilon_1 J(\alpha)) \cosh(2\alpha p_2 X_1) + (\widehat{A}_\beta K(\beta) - \varepsilon_1 J(\beta)) \cosh(2\beta p_2 X_1) \,] \\
&+\; \varepsilon_1 \int_0^{X_1} G(X_1 - y)\widehat{E}_2(y)\, dy \,,
\end{aligned} \tag{BC-4.29}$$

$$G(X_1) \;\equiv\; \frac{1}{a(2p_2)^3(\alpha^2 - \beta^2)} \left[\frac{\sinh(2\alpha p_2 X_1)}{\alpha} - \frac{\sinh(2\beta p_2 X_1)}{\beta} \right] \,,$$

$$J(x) \;\equiv\; \frac{1}{(2p_2)^3 L^c_{1111}(\alpha^2 - \beta^2)} \left[\frac{K(x)}{x} E_1'(0) + \frac{2p_2}{x^2} E_2(0) \right] \,, \quad (x = \alpha, \beta) \,,$$

where the expression for $K(x)$ is defined in (BC-4.18). It should be mentioned here that in deriving the above results with the help of(BC-4.18), one notices that E_1, $\widehat{E}_1$, F_2 and E_2, $\widehat{E}_2$, F_1 defined in (BC-4.25), (BC-4.26) are odd and even functions respectively of their argument X_1. These properties imply from (BC-4.25), (BC-4.26) that w_1 and w_2 are in their turn odd and even functions respectively of X_1.

Using the above results for $w_i(X_1)$ into the corresponding boundary conditions in (BC-4.26), one obtains a linear system for the two unknown constants $\widehat{A}_\alpha$, $\widehat{A}_\beta$, appearing in (BC-4.29) :

$$\left[\begin{array}{c} P_c(\alpha)\widehat{A}_\alpha + P_c(\beta)\widehat{A}_\beta = \varepsilon_1 Q_c \\ P_s(\alpha)\widehat{A}_\alpha + P_s(\beta)\widehat{A}_\beta = \varepsilon_1 Q_s \end{array} \right] \,, \tag{BC-4.30}$$

$$P_c(x) \equiv [L_{1111}x - L_{1122}K(x)]\cosh(2xp_2L_1)\ , \quad (x = \alpha, \beta)\ ,$$

$$P_s(x) \equiv [L_{2121}xK(x) + L_{2112}]\sinh(2xp_2L_1)\ , \quad (x = \alpha, \beta)\ ,$$

$$\begin{aligned} Q_c &\equiv L_{1122}[\, J(\alpha)\cosh(2\alpha p_2L_1) - J(\beta)\cosh(2\beta p_2L_1)\,] - \frac{F_1(L_1)}{2p_2} \\ &- L_{1122}\int_0^{L_1} G(L_1 - y)\widehat{E}_2(y)\,dy - \frac{L_{1111}}{2p_2}\int_0^{L_1} G'(L_1 - y)\widehat{E}_1(y)\,dy\ , \end{aligned}$$

$$\begin{aligned} Q_s &\equiv -L_{2121}[\,\alpha J(\alpha)\sinh(2\alpha p_2L_1) - \beta J(\beta)\sinh(2\beta p_2L_1)\,] + \frac{F_2(L_1)}{2p_2} \\ &- L_{2112}\int_0^{L_1} G(L_1 - y)\widehat{E}_1(y)\,dy + \frac{L_{2121}}{2p_2}\int_0^{L_1} G'(L_1 - y)\widehat{E}_2(y)\,dy\ . \end{aligned}$$

Solution of the above system for $\widehat{A}_\alpha$, $\widehat{A}_\beta$ completes the determination of $w_i(X_1)$ and hence of $\overset{2}{u}$ according to (BC-4.22). It can be shown that $w_i(X_1)$, $\widetilde{w}_1(X_1)$ and consequently $\overset{2}{u}$ are always real functions (obvious for the EI regime and following in the EC regime from the fact that $\overline{\alpha} = \beta$, which implies $\overline{\widehat{A}_\alpha} = \widehat{A}_\beta$).

The determination of Λ_2 follows by substituting the above found functions $\overset{1}{u}$ and $\overset{2}{u}$ into (BC-4.12)$_3$. All the required calculations for arbitrary values of the aspect ratio r, are based on a FORTRAN code written for this purpose.

iv) *Material Selection*

Three different isotropic hyperelastic material models are used in the calculations presented here. The first model is a compressible *neo–Hookean model*, which has the following energy density for plane–strain deformations:

$$W(I_1, I_2) = \mu\left[\frac{1}{2}(I_1 - 2 - \ln I_2) + \frac{\nu}{1-\nu}(I_2^{1/2} - 1)^2\right]\ , \tag{BC-4.31}$$

where μ and ν are respectively the solid's shear modulus and plane–strain Poisson ratio at zero stress. As $\nu \to 1$ the material becomes incompressible, i.e. $I_2 \to 1$.

The second model, termed *Gent model*, is an experimentally based model for natural rubbers, with the following energy density for plane–strain deformations:

$$W(I_1, I_2) = \mu\left[-\frac{J_m}{2}\ln\left(1 - \frac{I_1 - 2}{J_m}\right) - \frac{1}{2}\ln I_2 + \left(\frac{\nu}{1-\nu} - \frac{1}{J_m}\right)(I_2^{1/2} - 1)^2\right]\ , \tag{BC-4.32}$$

where, in addition to the shear modulus μ and plane–strain Poisson ratio ν, the material requires a third constant J_m which determines its locking strain in a uniaxial tension experiment. When $J_m \to \infty$, a simple calculation shows that the Gent model in (BC-4.32) reduces to the neo–Hookean solid in (BC-4.31).

The third model, termed *Blatz-Ko model*, is based on experiments in compressible foam rubbers and has the following energy density in plane–strain:

$$W(I_1, I_2) = \mu \left[\frac{1}{2} \frac{I_1}{I_2} + I_2^{1/2} - 2 \right] , \tag{BC-4.33}$$

v) *Results*

The general theory is hereby applied to the three specific materials presented above. The critical loads and modes of the corresponding rectangular blocks are presented first followed by the initial (i.e. at the critical load) curvature of the bifurcated equilibrium paths as function of the rectangular block's slenderness ratio.

va) *Principal Solution*

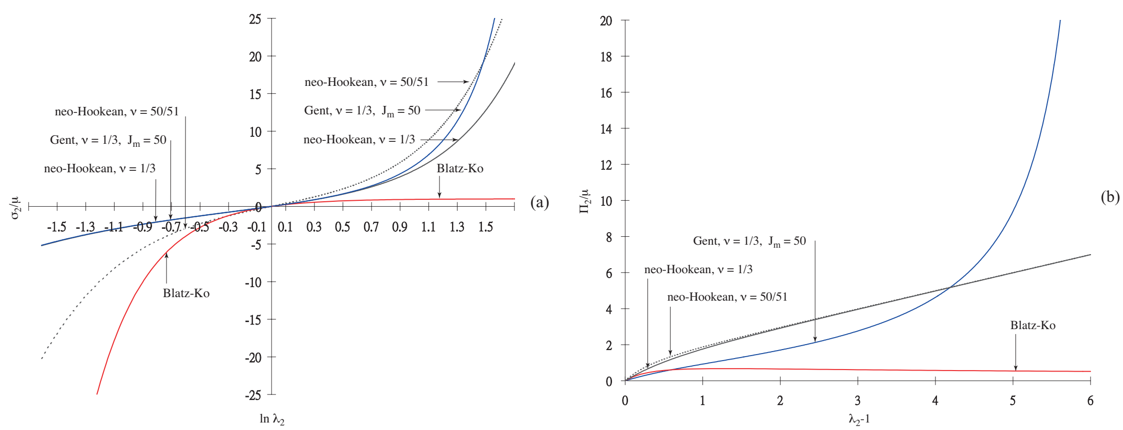

Figure BC-4.2: In (a) is plotted the dimensionless Cauchy Stress σ_2/μ versus logarithmic strain $\epsilon_2 = \ln \lambda_2$ of a rectangular block in tension and compression, while in (b) is plotted the dimensionless Kirchhoff stress Π_2/μ versus engineering strain $\lambda_2 - 1$ of the same block in tension. All four constitutive laws used have the same initial shear modulus μ, while in addition the Gent, the neo–Hookean for $\nu = 1/3$ and the Blatz–Ko materials share the same initial Poisson ratio $\nu = 1/3$.

The principal solution, i.e. the uniaxial plane–strain response of the rectangular block for the above introduced three different constitutive laws (BC-4.31), (BC-4.33) and (BC-4.32), is calculated with the help (BC-4.6), (BC-4.7); the results are depicted in Fig. BC-4.2. More specifically the dimensionless Cauchy stress (σ_2/μ) versus logarithmic strain ($\ln \lambda_2$) under tension and compression is plotted in Fig. BC-4.2a. The dimensionless first Piola–Kirchhoff stress (Π_2/μ, which is the dimensionless force per unit width applied on the block) versus engineering strain ($\lambda_2 - 1$) under tension is plotted in Fig. BC-4.2b. The response of the compressible neo–Hookean, Gent (for $J_m = 50$) and Blatz–Ko solids with the same initial shear modulus (μ) and plane–strain Poisson ratio ($\nu = 1/3$) is plotted in solid lines, while the response of an almost incompressible neo–Hookean solid ($\nu = 50/51$) is plotted in dotted line. The choice of the specific value for the plane–strain Poisson ratio of the Gent and neo–Hookean solid ($\nu = 1/3$) is dictated by the value of this constant for the Blatz–Ko material (recall $\nu = -(d\lambda_1/d\lambda_2)_{\lambda_2=1}$).

Although near zero strain the response of the three compressible materials is the same, their finite strain behavior differs significantly. Notice from Fig. BC-4.2a that the Blatz–Ko solid has the stiffest response under compression, while the response of the compressible neo–Hookean and Gent solids (both with $\nu = 1/3$) is almost indistinguishable in the compressive range. In tension, the Gent solid has ultimately the stiffest response, while the Blatz–Ko material is the softest and its Cauchy stress reaches an asymptote. The difference in the uniaxial response of the block under tension is better appreciated from Fig. BC-4.2b, which shows that for large strains, the force applied on the block behaves almost linearly for the neo–Hookean solid, shows strain locking (i.e. reaches an asymptote at a finite strain) for the Gent solid and has a maximum for the Blatz–Ko material at $\lambda_2 = 3^{0.75} \approx 2.28$.

A more fundamental difference between the above constitutive laws, is that while the neo–Hookean and Gent solids are rank one convex at all strains, the Blatz–Ko solid looses its rank one convexity at finite strains. In other words, the incremental equilibrium equations of the neo–Hookean and Gent are always strongly elliptic, while for the Blatz–Ko solid they exhibit real characteristics at finite strains. However, at the neighborhood of the critical load λ_c all solids investigated here are shown to satisfy the strong ellipticity condition as well as the strong complementing boundary condition (i.e. no surface bifurcations are possible in the vicinity of the critical load).

vb) *Critical Loads and Modes*

The determination of the critical load as a function of the block's geometry for the above three constitutive laws, is presented in Fig. BC-4.3 to Fig. BC-4.6. More specifically, in Fig. BC-4.3 and Fig. BC-4.4 are plotted, for compression and tension respectively, the lowest in absolute value bifurcation strains $(\lambda_2)_c - 1$ (i.e. the solutions of the bifurcation load equations (BC-4.20)), as functions of the geometric parameter η for the antisymmetric ($\mathcal{A}^2$) and symmetric ($\mathcal{S}^2$) cases. Based on these results, Fig. BC-4.5 and Fig. BC-4.6 record the critical loads λ_c of the same blocks as functions of the slenderness ratios r. This two step approach is necessary to explain the discontinuities, due to mode changes, in some of the λ_c vs r curves appearing in Fig. BC-4.5 and in Fig. BC-4.6.

For all constitutive laws, the dependence of bifurcation strain $(\lambda_2)_c - 1$ on η in blocks under compression is presented in Fig. BC-4.3. Observe that for each constitutive law, the absolute value of the bifurcation strain increases (decreases) for the antisymmetric (symmetric) mode

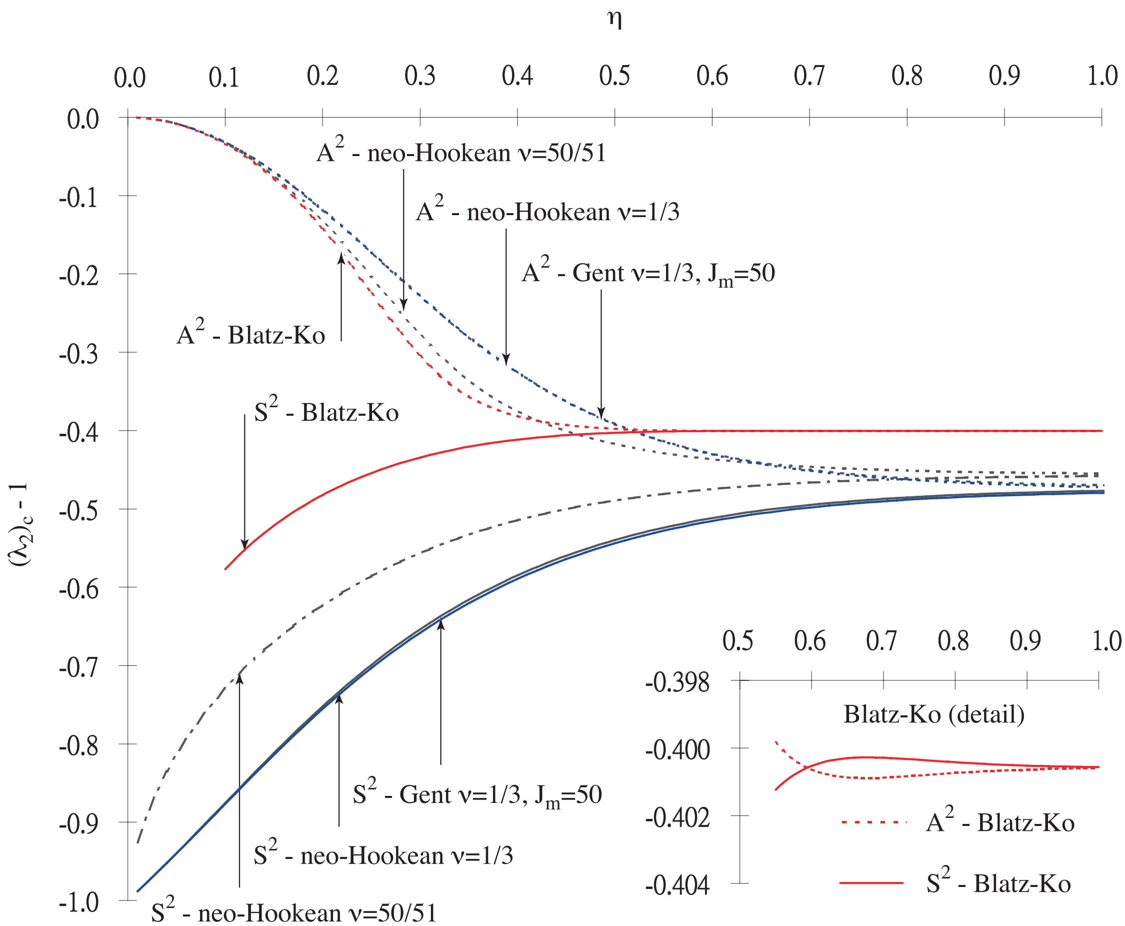

Figure BC-4.3: Lowest bifurcation strain $(\lambda_2)_c - 1$ as a function of the dimensionless wave number η at the onset of bifurcation in a rectangular block under plane–strain compression for asymmetric and symmetric bifurcation modes for four different constitutive laws.

with increasing η, until they both reach the same asymptotic, material–dependent limit. At this limit, the bifurcation strain corresponds to a high wavenumber surface bifurcation of the half–space under compression, which is the only mode of instability possible for the stubby block. At the opposite end, the limit of a slender block ($\eta \to 0$) all antisymmetric mode curves go through zero, as expected from the vanishing buckling strain of a thin beam.

The bifurcation strains for the neo–Hookean and Gent solids are always monotonic functions of η. Moreover, there is a near coincidence between the bifurcation strains of the neo–Hookean and Gent solids that have the same plane–strain Poisson ratio $\nu = 1/3$, as expected from their almost identical response in compression, according to Fig. BC-4.2. In contrast to the neo–Hookean and Gent blocks, the antisymmetric and symmetric mode bifurcation strains for the Blatz–Ko block are no longer monotonic functions of η and cross each other, for the first time as η increases from zero, at $\eta \approx 0.6$. The monotonicity (nonmonotonicity) of the $(\lambda_2)_c - 1$ vs η curves is the reason for the monotonic (nonmonotonic) critical load λ_c versus slenderness r curves of the different material blocks, as it will be subsequently discussed.

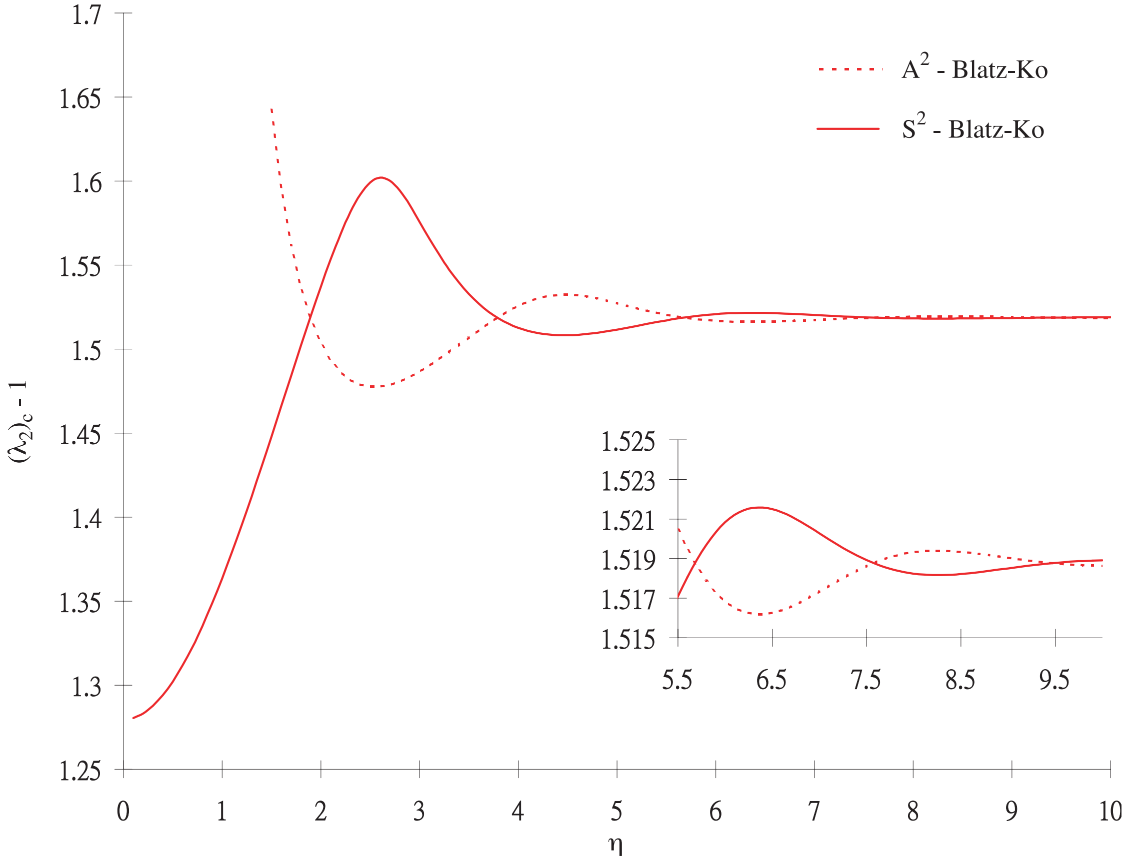

Figure BC-4.4: Lowest bifurcation strain $(\lambda_2)_c - 1$ as a function of the dimensionless wave number η at the onset of bifurcation in a rectangular block under plane–strain tension for asymmetric and symmetric bifurcation modes for a Blatz–Ko material.

The dependence of the bifurcation strain $(\lambda_2)_c - 1$ on η for Blatz–Ko blocks under tension is presented in Fig. BC-4.4. As for the compression case, the bifurcation strain curves corresponding to antisymmetric (symmetric) modes reach the same asymptote, which corresponds to a surface instability of the Blatz–Ko half–space under tension. At the opposite end, the limit of a slender block ($\eta \to 0$) the symmetric mode curve goes through $(\lambda_2)_c = 3^{0.75}$, since the instability of a thin rod occurs at maximum force (see discussion of Fig. BC-4.2b). Again, as for the compressive case, the nonmonotonicity of the $(\lambda_2)_c - 1$ η curve is the reason for the nonmonotonic critical load λ_c versus slenderness r curve of the Blatz–Ko block in tension.

From the results in Fig. BC-4.3 and Fig. BC-4.4 one can find the critical load λ_c versus slenderness r relations of the corresponding blocks by evaluating the bifurcation strain $(\lambda_2)_c - 1$ for $\eta = r/2,\ r,\ 3r/2,\ r,\ \ldots$ and then selecting the critical strain as the lowest absolute value $|(\lambda_2)_c - 1| = \lambda_c$. If the minimum occurs for an η that is an integral (fractional) multiple of the block slenderness r, the corresponding critical mode is symmetric (antisymmetric) with respect to X_1, according to (BC-4.17).

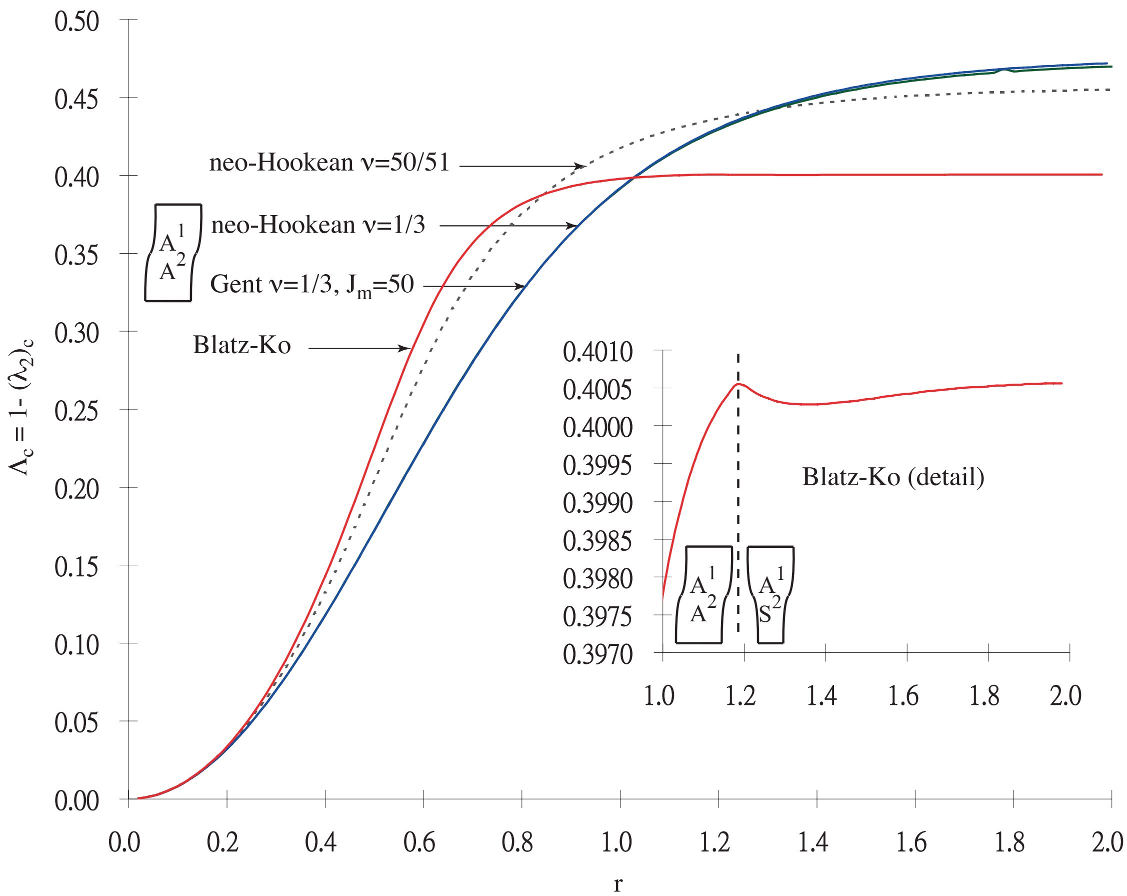

Figure BC-4.5: Critical load λ_c as a function of the block's aspect ratio r at the onset of bifurcation in a rectangular block under plane–strain compression for four different constitutive laws.

For the neo–Hookean and Gent blocks under compression, according to Fig. BC-4.3 the lowest strains always occur for the antisymmetric $\mathcal{A}^2$ mode, whose monotonic dependence on η results in a minimum strain occurring at $\eta = r/2$ and hence in a monotonic λ_c vs r dependence as seen in Fig. BC-4.5. The resulting critical mode has $n = 1$ and is antisymmetric with respect to both axes ($\mathcal{A}^1,\ \mathcal{A}^2$). For the Blatz–Ko solid under compression, the critical mode for $r < 1.2$ is also antisymmetric with respect to both axes ($\mathcal{A}^1,\ \mathcal{A}^2$), while for $r > 1.2$ it becomes symmetric with respect to X_2 axis ($\mathcal{A}^1,\ \mathcal{S}^2$). The minimum strain always occurs at $\eta = r/2$ and hence the critical mode has $n = 1$. The change of mode, expected from the crossing of antisymmetric and symmetric bifurcation strain curves at $\eta \approx 0.6$ in Fig. BC-4.3, is reflected by the discontinuity of the critical load curve in Fig. BC-4.5 at $r \approx 1.2$.

For the Blatz–Ko blocks under tension according to Fig. BC-4.4, the minimum bifurcation strain occurs alternatively for symmetric or antisymmetric in X_2 modes but always for $\eta = r/2$, i.e. for the antisymmetric in X_1 mode. Consequently as seen in Fig. BC-4.6, the critical load λ_c versus slenderness r curve is discontinuous and changes from an ($\mathcal{A}^1,\ \mathcal{S}^2$) to an ($\mathcal{A}^1,\ \mathcal{A}^2$) mode and back each time a discontinuity point is encountered as η increases from zero in

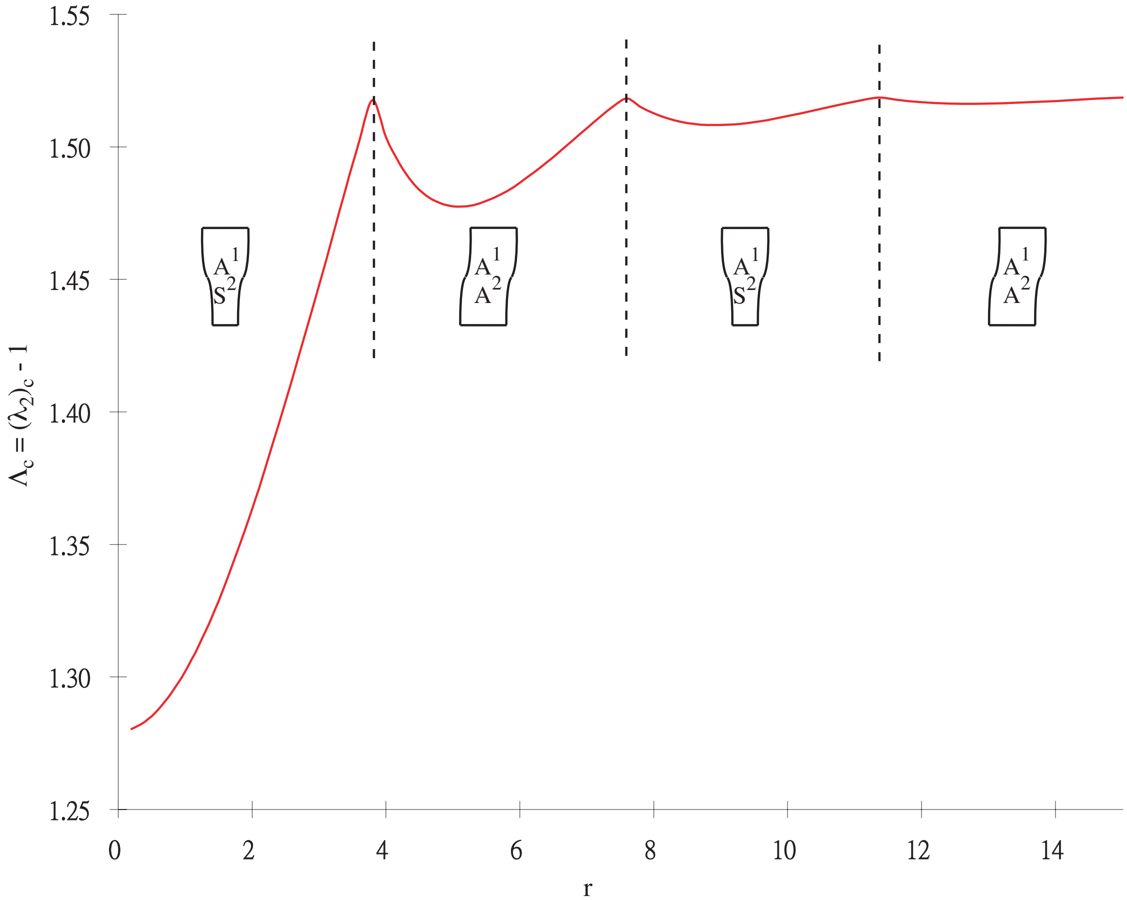

Figure BC-4.6: Critical load λ_c as a function of the block's aspect ratio r at the onset of bifurcation in a rectangular block under plane–strain tension for a Blatz–Ko material.

Fig. BC-4.4.

The results in Fig. BC-4.5 and Fig. BC-4.6 give for a rectangular block with known material properties and aspect ratio, the critical load and corresponding eigenmode type, information required to calculate the curvature of the bifurcated equilibrium path at the critical point, as discussed next.

vc) *Curvature of the Bifurcated Equilibrium Path*

Having determined the critical load and corresponding eigenmode for a given block, the stage is set to calculate the curvature λ_2 of the bifurcated equilibrium path according to (BC-4.12)$_3$.

To avoid numerical difficulties associated with slender blocks (i.e. when $r \ll 1$), all calculations reported here use as block dimensions $L_1 = 1$ and $L_2 = 1/r$. Evaluation of the curvature λ_2 requires numerical integrations in the interval $[0, L_1]$. To this end the interval $[0, L_1]$ is divided into 10^4 equal subintervals. All integrations required in the X_1 direction use a simple trapezoidal rule based on this grid. The eigenmode $\overset{1}{u}$ and the second term in the post–bifurcation equilibrium displacement expansion $\overset{2}{u}$, which in turn requires the calculation of the auxiliary functions $w_i(X_1)$ defined in (BC-4.29), are all evaluated on the same grid. Numerical experiments with denser grids on the interval $[0, L_1]$ gave almost identical results (errors of less than 10^{-4}). It is noteworthy that since all functions involved in the calculations of λ_2 are separable – i.e. there are products of an X_1 function by a trigonometric X_2 function – the integrations in the $[0, L_2]$ interval are done analytically.

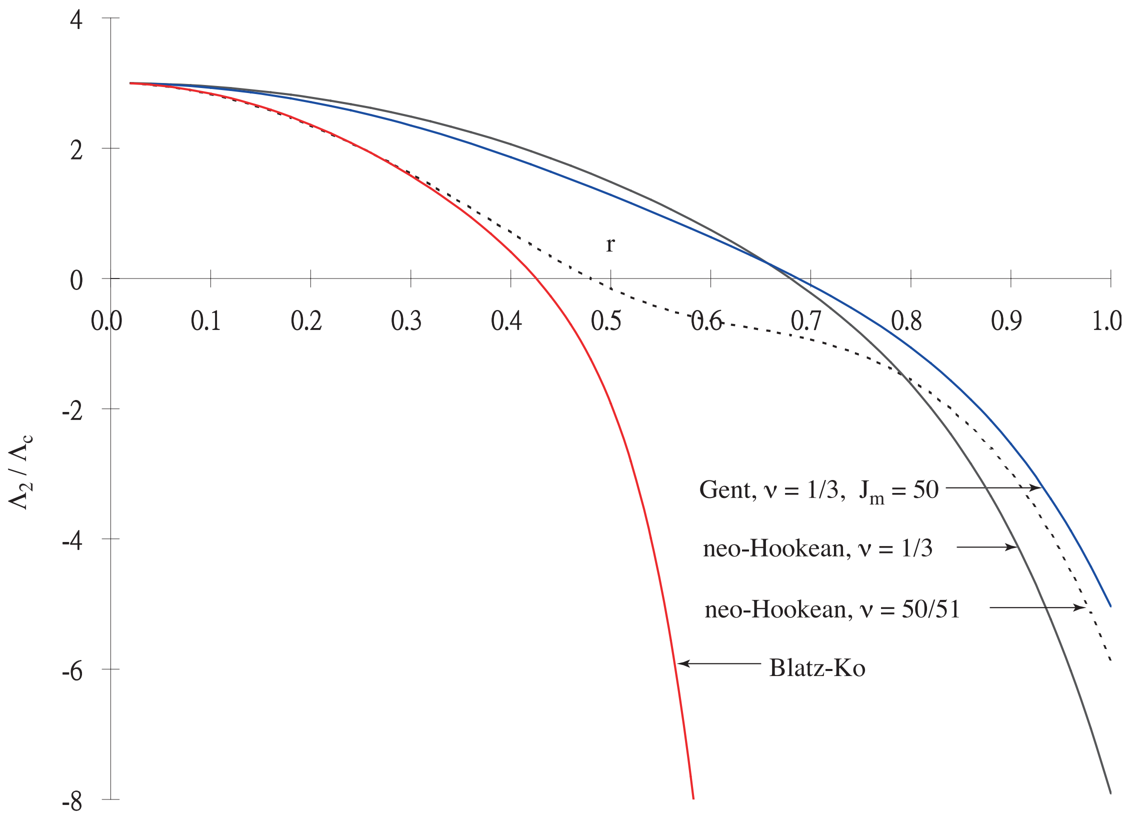

Figure BC-4.7: Dimensionless curvature at critical load of the bifurcated equilibrium path λ_2/λ_c as a function of the block's aspect ratio r in a rectangular block under plane–strain compression for four different constitutive laws.

For the case of compression, the bifurcated equilibrium path's curvature at the critical load λ_2/λ_c versus the block's slenderness r is given in Fig. BC-4.7. Notice that, irrespective of the material properties, $\lambda_2/\lambda_c \to 3$ as $r \to 0$. From simple nonlinear beam models — such as Euler's elastica — it is expected that at the thin beam limit, the post–bifurcated behavior should be stable (i.e. $\lambda_2 > 0$). What is rather surprising is that the curvature of the bifurcated equilibrium path at criticality is found to be independent of the constitutive law. A small–strain, moderate rotation structural beam model that accounts for axial compressibility does indeed show that $\lambda_2/\lambda_c \to 3$ as $r \to 0$ when a displacement–based norm corresponding to (BC-4.10) is adopted.

A rather surprising feature of the λ_2/λ_c vs r curves in Fig. BC-4.7 is that the post–

bifurcated equilibrium solutions become unstable (i.e. $\lambda_2 < 0$) for moderately stubby beams (at $r \approx 0.43$ for the Blatz–Ko block). The reason for this behavior is to be found in the moment–curvature relation for stubby beams. The post–buckling behavior of the thin beam under compression is stable since the increase in vertical displacement produces higher curvature at the end–sections which are compatible with the higher moments at these sections. As the beam becomes stubbier, the overall shortening of the beam in the bifurcated equilibrium solution, which results in a curvature increase at the end–sections is no longer sustainable because the moment, as a function of curvature, reaches a maximum. The curvature at which a stubby beam reaches a maximum moment is material–dependent, thus explaining the different points on the r axis that the λ_2/λ_c curves cross for the different constitutive laws. Given that the Gent solid has the stiffest response, it is not surprising that it is the corresponding block that looses bifurcated path stability under compression at the highest aspect ratio ($r \approx 0.69$).

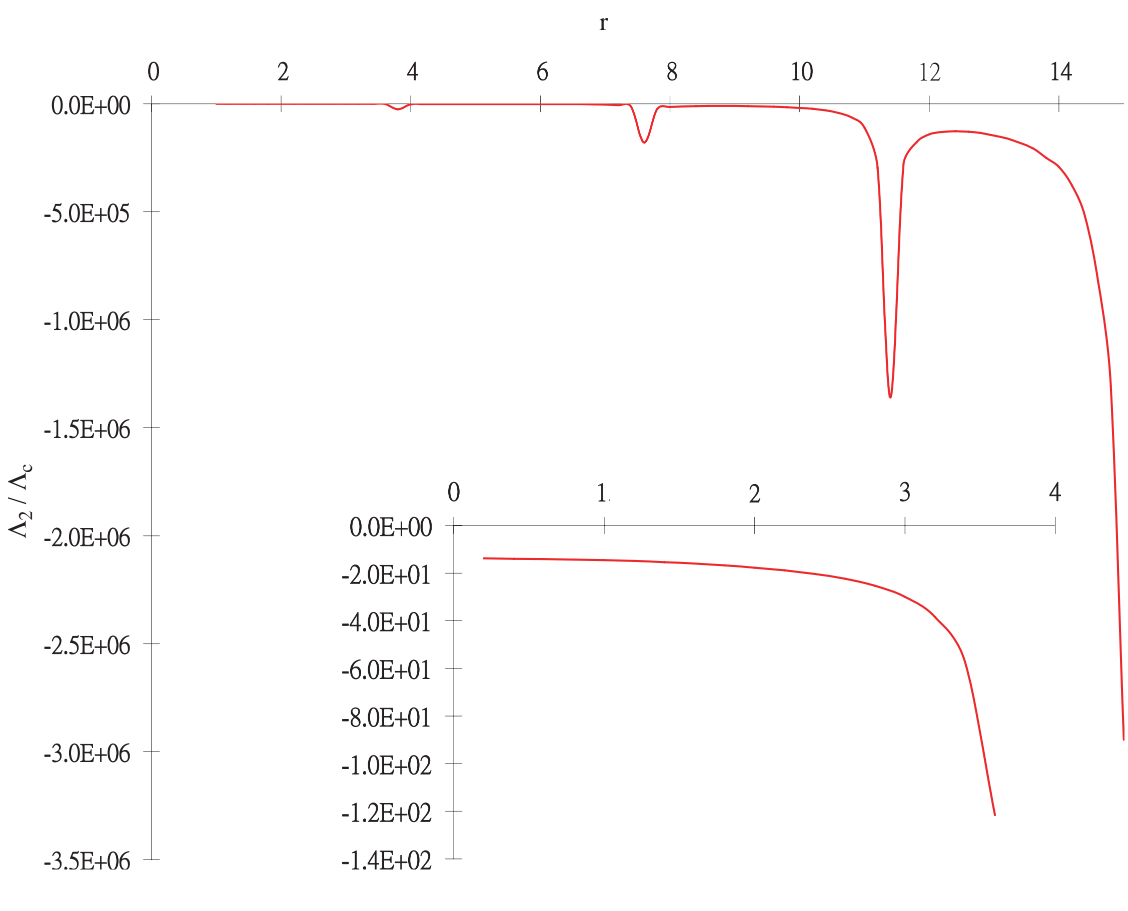

Figure BC-4.8: Dimensionless curvature at critical load of the bifurcated equilibrium path λ_2/λ_c as a function of the block's aspect ratio r in a rectangular block under plane–strain tension for a Blatz–Ko material.

For the case of a Blatz–Ko block in tension, the bifurcated equilibrium path's curvature at the critical load λ_2/λ_c versus the block's slenderness r is given in Fig. BC-4.8. Notice that

$\lambda_2/\lambda_c \to -27/2(3^{1/2} - 3^{-1/4})$ for the thin rod limit $r \to 0$. The unstable $(\lambda_2 < 0)$ bifurcated equilibrium path for the thin block under tension is expected since the block, having reached a maximum force at criticality, snaps back by developing a highly strained neck zone at one end while strains are lowered in the rest of its length. The unstable bifurcated path behavior near the critical load persists for any aspect ratio. The λ_2/λ_c vs r curve shows bumps near the regions where the Blatz–Ko block under tension changes eigenmodes at criticality, as seen by comparing Fig. BC-4.8 to Fig. BC-4.6.

The complex calculations leading to the evaluation of $\lambda_2(r)/\lambda_c(r)$ for arbitrary r need independent verification. To this end, and in addition to the asymptotic calculations done with MATHEMATICA for $r \to 0$, finite element (FEM) calculations for the rectangular elastic blocks are also performed with ABAQUS using slightly imperfect initial geometries. For the compressive case, blocks with a vertical mid–line at $X_1 = \zeta L_1 \sin(\pi X_2/L_2)$ (instead at $X_1 = 0$ for the perfect block) are considered, where the imperfection amplitude parameter $\zeta = 10^{-3}$, thus generating slightly asymmetric blocks in the shape of the relevant – near $r = 0$ – critical eigenmode in compression $(\mathcal{A}^1,\ \mathcal{A}^2)$. Meshes of 20×40 bilinear quadrilateral elements (of equal sides $L_1/10$ and $L_2/20$ along X_1 and X_2) are used to calculate, with a help of a continuation method, the vertical displacement V_m versus the horizontal displacement H_m of the top mid–node. By changing the block's aspect ratio, one finds the value of the slenderness parameter r above which the V_m vs H_m curve reaches a maximum in V_m, thus indicating crossing of the r axis in the λ_2/λ_c curve in Fig. BC-4.7. This way it is possible to independently verify, with an 1% accuracy, the results of the corresponding calculations for the compressive case. Using the same mesh but a different initial geometry, i.e. a straight vertical axis but a slightly varying block thickness $2L_1 = 2L_1(1+\zeta\sin(\pi X_2/L_2)$, imperfect block geometries are produced in the shape of the relevant – near $r = 0$ – critical eigenmode in tension $(\mathcal{A}^1,\ \mathcal{S}^2)$. These FEM calculations using imperfect blocks, show unstable equilibrium paths past a maximum force, thus independently checking the validity of the stability results for the corresponding perfect block's bifurcated equilibrium paths.

vi) *Asymptotic Behavior of Thin Blocks $r \to 0$*

The asymptotic calculations for the thin block limiting case $(r \to 0)$ are based on the above presented analysis and use the symbolic calculations program MATHEMATICA. Recall from the discussion of the initial post–bifurcation curvature results in Section AC-3 that the dimensions of the block are: $L_1 = 1$ and $L_2 = 1/r$ and that the eigenmode $\overset{1}{u}$ is normalized using the inner product definition in (BC-4.10).

via) *Compression (Critical Mode $\mathcal{A}^1$, $\mathcal{A}^2$; Critical Wavenumber $n_c = 1$)*

To calculate λ_2, one starts with the asymptotic expressions of the stretch ratios λ_i and

$v_i(X_1)$ (the X_1–dependent part of the eigenmode $\overset{1}{u}$) at the critical point:

$$(\lambda_1)_c = 1 + \frac{\nu}{12}(\pi r)^2 + O(r^4)\ , \quad (\lambda_2)_c = 1 - \frac{1}{12}(\pi r)^2 + O(r^4)\ ,$$
$$v_1(X_1) = \sqrt{2} + O(r^2)\ , \qquad v_2(X_1) = -\frac{X_1}{\sqrt{2}}\pi r + O(r^3)\ . \tag{BC-4.34}$$

The next piece of information needed are the asymptotic expansions for $w_i(X_1)$, $\widetilde{w}_1(X_1)$ (the X_1–dependent parts of $\overset{2}{u}$) given below to their leading order in r:

$$w_1(X_1) = -\frac{X_1}{4}(\pi r)^2\ , \quad \widetilde{w}_1(X_1) = -\frac{(1+\nu)X_1}{4}(\pi r)^2\ , \quad w_2(X_1) = -\frac{1}{4}\pi r\ . \tag{BC-4.35}$$

Introducing these expressions into (BC-4.40), (BC-4.41), (BC-4.42) and (BC-4.12), one obtains to the leading order in r:

$$((\mathcal{E}^c_{,uuuu}\ \overset{1}{u})\ \overset{1}{u})\ \overset{1}{u})\ \overset{1}{u} = \frac{9(1+\nu)}{2(1-\nu)}\pi^4 r^3\ , \quad ((\mathcal{E}^c_{,uuu}\ \overset{2}{u})\ \overset{1}{u})\ \overset{1}{u} = -\frac{2+3\nu+\nu^2}{2(1-\nu)}\pi^4 r^3\ ,$$
$$((d\mathcal{E}_{,uu}/d\lambda)^c\ \overset{1}{u})\ \overset{1}{u} = -2(1+\nu)\pi^2 r\ , \quad \lambda_2 = \frac{1}{4}(\pi r)^2\ , \tag{BC-4.36}$$

thus establishing that the initial curvature of the critical load, i.e. the limit $\lambda_2(r)/\lambda_c(r) \to 3$ as $r \to 0$ is independent of the constitutive law.

vib *Tension (Critical Mode $\mathcal{A}^1$, $\mathcal{S}^2$; Critical Wavenumber $n_c = 1$)*

The asymptotic expressions given below correspond to the Blatz–Ko material under tension. These expressions are given up to the lowest nontrivial order in r required for the calculation of λ_2, the post–bifurcation curvature at the critical point.

To calculate λ_2, one starts again with the asymptotic expressions of the stretch ratios λ_i and $v_i(X_1)$ (the X_1–dependent part of the eigenmode $\overset{1}{u}$) at the critical point:

$$(\lambda_1)_c = 3^{-1/4}\left[1 - \frac{1}{2916}(\pi r)^2\right] + O(r^4)\ , \quad (\lambda_2)_c = 3^{3/4}\left[1 + \frac{1}{972}(\pi r)^2\right] + O(r^4)\ ,$$
$$v_1(X_1) = \frac{X_1}{9\sqrt{2}}\pi r + O(r^3)\ . \qquad v_2(X_1) = \sqrt{2} + O(r^2)\ . \tag{BC-4.37}$$

The asymptotic expansions for $w_i(X_1)$, $\widetilde{w}_1(X_1)$ (the X_1–dependent parts of $\overset{2}{u}$) are recorded next. For the calculations of λ_2 in tension, the first two terms in the r–expansions of $w_i(X_1)$ are required:

$$w_1(X_1) = 3^{1/4}\left[3X_1 - \frac{(X_1)^3}{486}(\pi r)^2\right],\ \widetilde{w}_1(X_1) = 3^{1/4}\frac{X_1}{81}(\pi r)^2,\ w_2(X_1) = 3^{1/4}\left[-\frac{27}{\pi r} + \frac{(X_1)^2}{2}\pi r\right]. \tag{BC-4.38}$$

Upon introduction of the above expressions into (BC-4.40), (BC-4.41), (BC-4.42) and (BC-4.12), one obtains to the leading order in r:

$$((\mathcal{E}^c_{,uuuu}\ \overset{1}{u})\ \overset{1}{u})\ \overset{1}{u})\ \overset{1}{u} = \frac{25}{81\sqrt{3}}\pi^4 r^3\ , \quad ((\mathcal{E}^c_{,uuu}\ \overset{2}{u})\ \overset{1}{u})\ \overset{1}{u} = -\frac{4}{\sqrt{3}}\pi^2 r\ ,$$
$$((d\mathcal{E}_{,uu}/d\lambda)^c\ \overset{1}{u})\ \overset{1}{u} = -\frac{3^{1/4}8}{81}\pi^2 r\ , \quad \lambda_2 = \frac{3^{3/4}27}{2}\ , \tag{BC-4.39}$$

thus establishing with a different method that the limit $\lambda_2(r)/\lambda_c(r) \to -27/2(3^{1/2} - 3^{-1/4})$ as $r \to 0$ coincides with the one found in the numerical calculations reported in Fig. BC-4.8.

Finally, the determination of λ_2 from the general theory according to (BC-4.12)$_3$ requires the calculation of the following quantities: The first term in the numerator of λ_2 is found with the help of the energy definition in (BC-4.2) to be:

$$((\mathcal{E}^c_{,uuuu} \overset{1}{u}) \overset{1}{u}) \overset{1}{u}) \overset{1}{u} = \int_A [N^c_{ijk\ell mnpq} \overset{1}{u}_{i,j} \overset{1}{u}_{k,\ell} \overset{1}{u}_{m,n} \overset{1}{u}_{p,q}] dA \,, \quad N^c_{ijk\ell mn} \equiv \frac{\partial^4 W(\overset{0}{u}(\lambda_c))}{\partial F_{ij} \partial F_{k\ell} \partial F_{mn} \partial F_{pq}} \,. \qquad \text{(BC-4.40)}$$

The second term in the numerator of λ_2 is found with the help of (BC-4.2), and also from (BC-4.12)$_1$ by substituting $\delta v = \overset{2}{u}$, to have the following two equivalent expressions:

$$((\mathcal{E}^c_{,uuu} \overset{2}{u}) \overset{1}{u}) \overset{1}{u} = \int_A [M^c_{ijk\ell mn} \overset{2}{u}_{i,j} \overset{1}{u}_{k,\ell} \overset{1}{u}_{m,n}] dA = -\int_A [L^c_{ijk\ell} \overset{2}{u}_{i,j} \overset{2}{u}_{k,\ell}] dA \,. \qquad \text{(BC-4.41)}$$

Finally, the denominator of λ_2 is also found with the help of the energy definition in (BC-4.2) and the definition of tensor $\mathbf{M}$ in (BC-4.21)$_3$ to be:

$$((d\mathcal{E}_{,uu}/d\lambda)^c \overset{1}{u}) \overset{1}{u} = \int_A [(dL_{ijk\ell}/d\lambda)^c \overset{1}{u}_{i,j} \overset{1}{u}_{k,\ell}] dA \,, \quad (dL_{ijk\ell}/d\lambda)^c = M^c_{ijk\ell mn} (d \overset{0}{u}_{m,n} / d\lambda)^c \,. \qquad \text{(BC-4.42)}$$

BC-5 PLANE STRAIN PURE BENDING OF A RECTANGULAR PLATE

For the material in this section....

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