Lab Solutions 05

1. Assume that Z is a random variable that only takes values between 0 and 5. Any value in this range is equally likely to occur. Write down the formula for the density function f that corresponds to this random variable.

The function f should be constant between 0 and 5 and zero elsewhere. We know that the total integral has to be 1, so the height of the function must be $1/5$. This gives:

$$f(z) = \begin{cases} 0, & z < 0 \\ 1/5, & z \in [0, 5] \\ 0, & z > 5 \end{cases} \tag{5.1}$$

Some students may worry about the edges and equality signs at the points 0 and 5. In truth, it doesn't matter and is not important because the density function only has a meaning when integrated. Its value at a particular point is not even well-defined in terms of the probability function.

2. Without doing any calculations, what do you expect to be the expected value of Z in the previous question?

The random variable is equally likely to be between 0 and 5, so it seems reasonable that the expected value is right in the middle at 2.5.

3. Using the formulae in the notes, show that your guess matches the mathematical definition of $\mathbb{E}Z$.

We can see this mathematically as follows:

$$\begin{aligned}
\mathbb{E}Z &= \int_{-\infty}^{+\infty} z \cdot f(z) dz && (5.2) \\
&= \int_{-\infty}^{0} z \cdot f(z) dz + \int_{0}^{5} z \cdot f(z) dz + \int_{5}^{+\infty} z \cdot f(z) dz && (5.3) \\
&= 0 + \int_{0}^{5} \frac{1}{5} \cdot z dz + 0 && (5.4) \\
&= \frac{1}{5} \cdot \left[z^2/2 \right]_{z=0}^{z=5} && (5.5) \\
&= \frac{5^2}{2 \cdot 5} = 2.5. && (5.6)
\end{aligned}$$

Which matches our intuition.

4. Prove the equation given for $\mathbb{V}ar(AZ)$.

Using the definition of the variance we have:

$$\begin{aligned}
\mathbb{V}ar(A\epsilon) &= \mathbb{E}\left[(A\epsilon - \mathbb{E}A\epsilon)\cdot(A\epsilon - \mathbb{E}A\epsilon)^t\right] && (5.7)\\
&= \mathbb{E}\left[(A\epsilon - A\mathbb{E}\epsilon)\cdot(A\epsilon - A\mathbb{E}\epsilon)^t\right] && (5.8)\\
&= \mathbb{E}\left[A\left(\epsilon - \mathbb{E}\epsilon\right)\cdot\left(A\left(\epsilon - \mathbb{E}\epsilon\right)\right)^t\right] && (5.9)\\
&= A\mathbb{E}\left[(\epsilon - \mathbb{E}\epsilon)\cdot(\epsilon - \mathbb{E}\epsilon)^t A^t\right] && (5.10)\\
&= A\mathbb{E}\left[(\epsilon - \mathbb{E}\epsilon)\cdot(\epsilon - \mathbb{E}\epsilon)^t\right] A^t && (5.11)\\
&= A\mathbb{V}ar(\epsilon)A^t. && (5.12)
\end{aligned}$$

Notice that I needed to use the rule $(AB)^T = B^t A^t$ in the second to last step.

5. Assume that we have a random variable y defined in terms of a random variable ϵ such that:

$$y = Xb + \epsilon. \qquad (5.13)$$

Further assume that the expected value of the ϵ term is the zero vector (a vector with zeros in every component). Show that the expected value of the ordinary least squares equation for the estimate β is equal to b. This means that β is an *unbiased* estimator of b.

Note that we can think of the quantity $(X^tX)^{-1}X^t$ as a single matrix. This makes the computation fairly straightforward:

$$\begin{aligned}
\mathbb{E}\beta &= \mathbb{E}\left[(X^tX)^{-1}X^ty\right] && (5.14)\\
&= \mathbb{E}\left[(X^tX)^{-1}X^t(Xb+\epsilon)\right] && (5.15)\\
&= \mathbb{E}\left[(X^tX)^{-1}X^tXb\right] + \mathbb{E}\left[(X^tX)^{-1}X^t\epsilon\right] && (5.16)\\
&= (X^tX)^{-1}(X^tX)b + (X^tX)^{-1}X^t\mathbb{E}\epsilon && (5.17)\\
&= b + 0 = b. && (5.18)
\end{aligned}$$

6. Using the same set-up as above, further assume that:

$$\mathbb{V}ar(\epsilon) = \sigma^2 \cdot I_n. \qquad (5.19)$$

For some fixed value $\sigma^2 > 0$. Derive a formula for the variance of the ordinary

least squares estimate β.

Here we just use the variance rule for a sum:

$$\mathbb{V}ar(y) = \mathbb{V}ar(Xb + \epsilon) = \mathbb{V}ar(\epsilon) = \sigma^2 \cdot I_n. \tag{5.20}$$

7. Finally, derive a formula for the variance of the ordinary least squares estimate β.

Once again, considering $(X^tX)^{-1}X^t$ as a single matrix, and using the previous result, we have:

$$\begin{aligned}
\mathbb{V}ar(\beta) &= \mathbb{V}ar((X^tX)^{-1}X^ty) && (5.21)\\
&= (X^tX)^{-1}X^t \cdot \mathbb{V}ar(y) \cdot \left[(X^tX)^{-1}X^t\right]^2 && (5.22)\\
&= (X^tX)^{-1}X^t \cdot \sigma^2 \cdot I_n \cdot \left[(X^tX)^{-1}X^t\right]^2 && (5.23)\\
&= \sigma^2 \cdot (X^tX)^{-1}X^tX(X^tX)^{-1} && (5.24)\\
&= \sigma^2 \cdot (X^tX)^{-1}(X^tX)(X^tX)^{-1} && (5.25)\\
&= \sigma^2 \cdot (X^tX)^{-1} && (5.26)
\end{aligned}$$