

Special Matrices

- They have ~~same~~ same no. of rows and columns
i.e we can say that they are $n \times n$ matrices
 or we can say that they are square matrices.
- Matrices with a relatively high proportion of zero entries.
- Some special forms of square matrices are

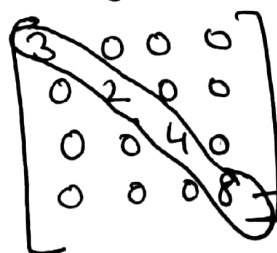
- Diagonal
- Tri diagonal
- Lower Triangular
- Upper Triangular
- Symmetric

Say name of the matrices is $A[i, j]$
 with the subscript i & j where i represent
 rows & j represent column.

Diagonal Matrices :-

$$A[i, j] = 0 \text{ for } i \neq j$$

It means all the position of the matrix will hold
 zero except diagonal because at diagonal $i=j$

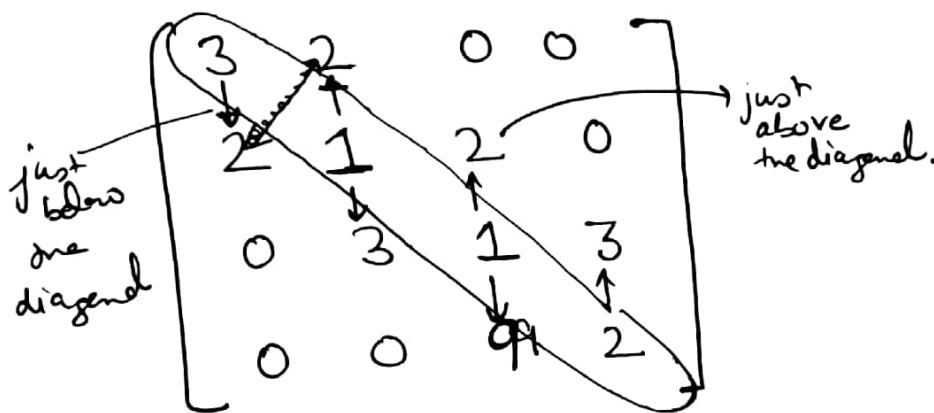


non-zero entries
at $i=j$

Tridiagonal :-

$$A[i, j] = 0 \text{ for } |i - j| > 1$$

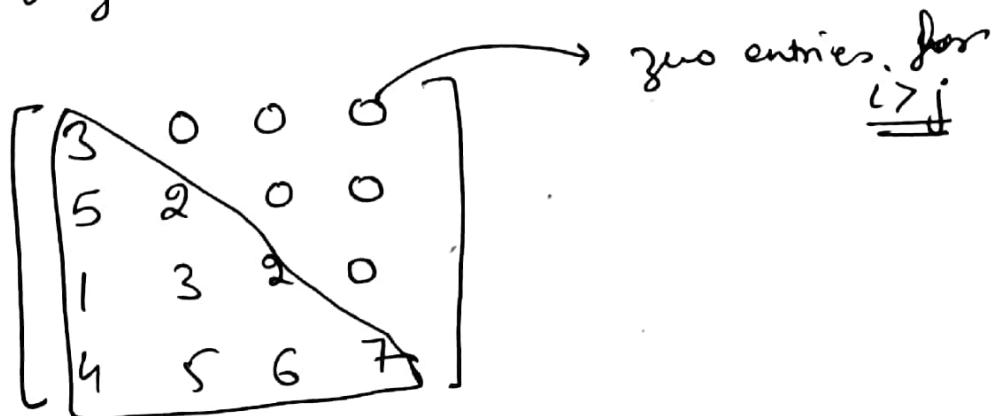
It means all the position of the matrices will hold zero whose absolute value of $|i - j| > 1$ & rest all will have non-zero entries. Therefore we can say that non-zero elements will be on diagonal, just below & just above the diagonal.



Lower Triangular :-

$$A[i, j] = 0 \text{ for } i > j$$

It means all the position of the matrices will hold zero for ~~where~~ $i > j$ whereas we will have non-zero entries at diagonal and below the diagonal.



Upper Triangular:-

$$A[i,j] = 0 \text{ for } i < j$$

It means all the position of the matrices will hold zero for $i < j$ where as we will have non-zero entries at diagonal & ~~below~~ ~~the~~ above the diagonal.

2	7	4	1
0	3	2	3
0	0	5	2
0	0	0	1

Symmetric square matrix :-

$$A[i,j] = A[j,i] \text{ for all } i \text{ and } j$$

It means all the position of the matrices will have same value for $A[i,j] = A[j,i]$

	a_{11}	a_{12}			
	2	4		6	0
a_{21}	4	1	a_{23}	9	5
	6	9		4	7
a_{31}	0	5	7	0	

As you can see

$$a_{12} = a_{21}$$

$$a_{23} = a_{32}$$

$$\underline{\underline{4 \text{ so on}}}$$

Our Motive :-

Natural method of representing matrices in memory as 2-D arrays may not be suitable for special matrices. That is, one may save space by storing only those entries which may be non-zero.

To locate the index no. of an element in Tridiagonal Matrices

Elements in tridiagonal are on the diagonal & just above & below the diagonal as shown below.
Here the dot represent the data.

	1	2	3	4	5	6
1	a_{11}	a_{12}				
2	a_{21}	a_{22}	a_{23}			
3		a_{32}	a_{33}	a_{34}		
4			a_{43}	a_{44}	a_{45}	
5				a_{54}	a_{55}	a_{56}
6					a_{65}	a_{66}

Total no. of elements in a tridiagonal matrix

As matrix is a square matrix
Therefore the diagonal will have n elements
Elements above the diagonal are $n-1$
Elements below the diagonal are $n-1$

$$\therefore \text{Total no. of elements} \Rightarrow n + n-1 + n-1$$

$$\Rightarrow 3n-2$$

So we have $3n-2$ non-zero elements in a tridiagonal matrices.

As the above matrix has name $A[i, j]$ with subscript i & j

i	J
for $i=1$	1, 2 (valid values for J)
for $i=2$	1, 2, 3 [ie you can say that data is sending at a_{21}, a_{22} & a_{23} when $i=2$]
$i=3$	2, 3, 4
$i=4$	3, 4, 5
$i=5$	4, 5, 6
$i=6$	5, 6

If I want to locate $A[i, j]$ as $i=4$ & $j=5$
 So when I am at i^{th} row that means I had
 crossed ~~the~~ $i-1$ rows ie row 1, 2 & 3 [∵ $i=4$]

So the main fact over here is that all the
 non-zero data will be mapped at linear array.

ie we have $3n-2 \Rightarrow$ so if $n=6$

∴ we have 16 data elements

All these 16 data is mapped to a linear array
 having location number ranging from 1 to 16
 [if I take array index start from 1]

a_{11}	a_{12}	a_{21}	a_{22}	a_{23}	a_{32}	a_{33}	a_{34}	a_{43}	a_{44}	a_{45}	a_{54}	...	a_{66}
1	2	3	4	5	6	7	8	9	10	11	12	...	16

So we can clearly see that $A[4,5]$ will reside at location 11

Now we can calculate the location :-

Observe ~~at~~ how many elements are above this i^{th} row ie 4^{th} row

If I want to come at this 4^{th} row, it means I have to cross 3 rows.

Out of these 3 rows \Rightarrow 2 rows are having 3 data
1 row is having 2 data

\therefore we can say that out $i-1$ rows

$i-2$ is having 3 data

& 1^{st} row is having 2 data.

\therefore how many elements (data) are before this i^{th} row

$$\Rightarrow \underbrace{3 \times (i-2)}_{\substack{\text{data in } i-2 \\ \text{rows}}} + \underbrace{2}_{\substack{\text{data in } 1^{\text{st}} \text{ row}}}$$

Now to reach at a particular location,

$$L = \left[3 \times (i-2) + 2 \right] + \begin{array}{c} \text{no. of elements present} \\ \text{in left to the} \\ \text{particular location} \end{array} + \begin{array}{c} \text{the exact} \\ \text{element at} \\ \text{that particular} \\ \text{location} \end{array}$$

$$\Downarrow \qquad \qquad \qquad \Downarrow$$

$$j+1-i \qquad + \qquad 1$$

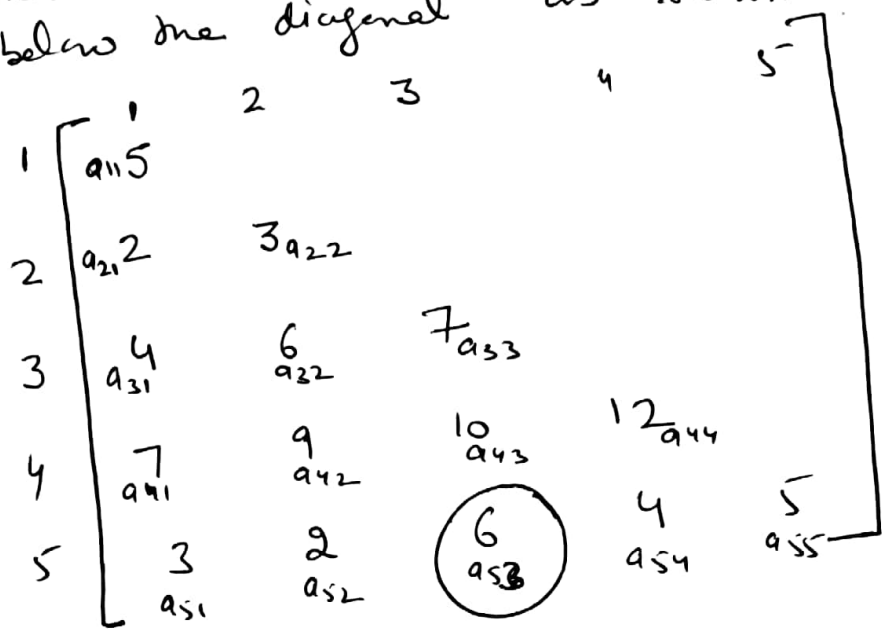
$$\boxed{L = 2i + j - 2} \Rightarrow \text{Final formula}$$

So by putting any value of i, j
We'll get the location.

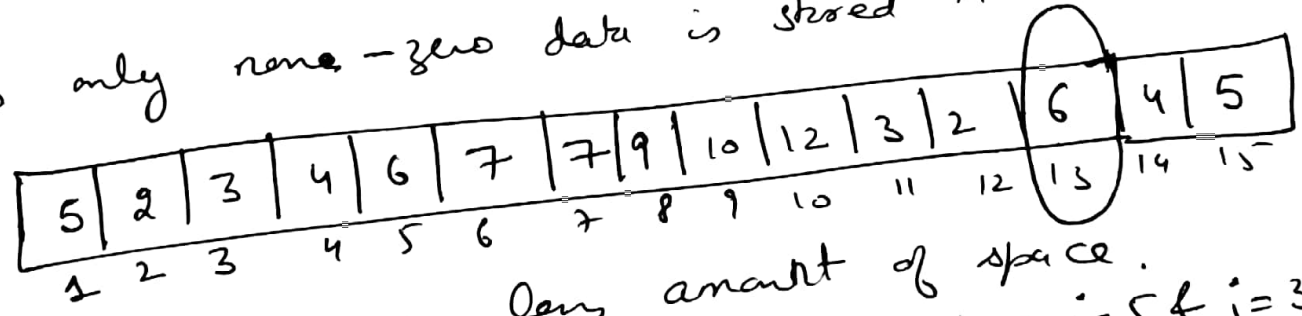
Lower Triangular

to locate the index no. of an element in lower triangular matrix

Elements in lower triangular are on one diagonal & below one diagonal as shown below.

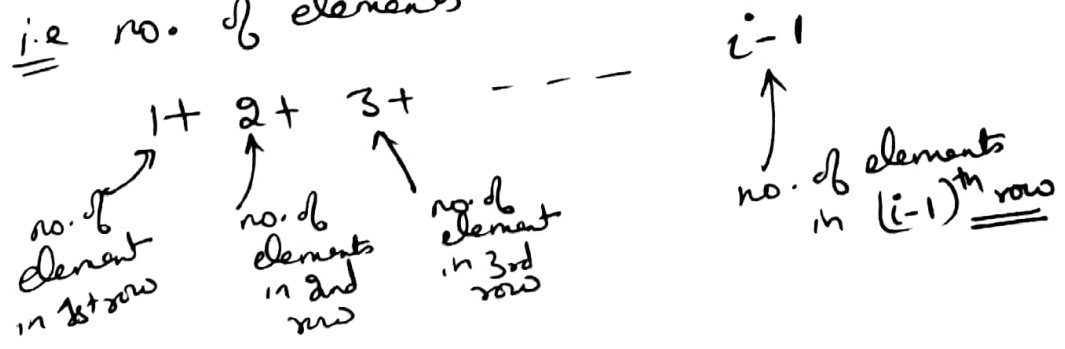


So only non-zero data is stored in 1-D array



Therefore we use less amount of space.
Say I want to find the location for $i=5$ & $j=3$
To reach at this row, I have to cross $i-1$ rows

i.e no. of elements in $i-1$ rows in $A[i, j]$ is



This series is natural no. sum in form of $(i-1)$

$$\Rightarrow \frac{i(i-1)}{2} \Rightarrow \text{There are elements above } i^{\text{th}} \text{ row}$$

There are j elements in the row i^{th} i.e including one element $A[i,j]$ also

$$\underline{\text{So}} \quad L = \frac{i(i-1)}{2} + j$$

$$\underline{\text{i.e}} \quad L = \frac{i(i-1)}{2} + j$$

Putting value of $i=5$ & $j=3$ we get

$$L \Rightarrow \frac{5(5-1)}{2} + 3 \Rightarrow \underline{\underline{13}}$$

Similarly you can compute the formula for Diagonal, upper triangular.

Sparse Matrix

- Matrix with many of elements are zero.
- This is not a dense matrix.
- Can be represented by
 - Array [Also known as Triplet]
 - Linklist

eg

$$\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 4 & 5 & 0 \end{bmatrix}$$

→ This is a sparse matrix & it do not follows any pattern for non-zero element. They can be anywhere ie at any position.

Array Representation :-

Why it is named as Triplet as

3 columns

- → one for no. of rows
- → one for no. of columns
- → one for no. of non-zero values.

Rows

- number of rows will be 1 more than no. of non-zero elements.

We can use row-major or column major
But Here, we are representing using row-major
only.

Triplet Representation

	0	1	2	3	4	5
0	2	0	0	3	0	5
1	0	1	4	0	0	0
2	0	0	0	6	0	0
3	0	0	0	0	0	0
4	9	0	0	0	0	0
5	0	0	2	0	0	0

⇒

Row	col	Value
6	6	8
0	0	2
0	3	3
0	5	5
1	1	1
1	2	4
2	3	6
4	0	9
5	2	2

In a Triplet

We have 3 columns

& no. of rows = 1 + no. of non-zero value
 as in the 0th row of triplet we are
 showing no. of rows, no. of column &
 no. of non-zero value. So we
 need 1 extra row ∴ 1 + no. of non-zero.

Operations on sparse Matrices

- Transpose
- Addition
- Multiplication.

Transpose :- Consider the given matrix &
 first represent it ~~to~~ in form of triplet as
 shown below :-

	0	1	2
0	10	0	0
1	0	0	2
2	0	5	0



row	col	value
3	3	3
0	0	10
1	2	2
2	1	5

Original triplet.

Now in transpose, we have to simply interchange the row and column values. So we exchange the row col values, & the resultant triplet we get is in column major form as shown below.

row	col	value
3	3	3
0	0	10
2	1	2
1	2	5

column major



Now we have to translate it to row-major

row	col	value
3	3	3
0	0	10
1	2	5
2	1	2

row major final resultant

To get the final resultant triplet in row major what we have to do is just arrange the rows in lexicographic order.

i.e value of (0,0); (1,2); (2,1)

or you can say that we arranged the row no. in increasing order.

	0	1	2
0	10	0	0
1	0	0	5
2	2	0	0

⇐ final resultant transpose matrix.

Addition operation :-

- We have to verify the same dimension of both the matrices before ~~proceed~~ proceeding.
- We have to look at elements that are lying or placed at same index position. so that we can add those elements.

$$\begin{array}{c|c|c|c}
 & 0 & 1 & 2 \\
 \hline
 0 & 8 & 0 & 0 \\
 1 & 0 & 0 & 3 \\
 2 & 6 & 0 & 0
 \end{array}
 +
 \begin{array}{c|c|c|c}
 & 0 & 1 & 2 \\
 \hline
 0 & 3 & 0 & 1 \\
 1 & 0 & 0 & 4 \\
 2 & 0 & 0 & 5
 \end{array}
 \Rightarrow
 \begin{array}{c|c|c|c}
 & 0 & 1 & 2 \\
 \hline
 0 & 11 & 0 & 1 \\
 1 & 0 & 0 & 7 \\
 2 & 6 & 0 & 5
 \end{array}$$

(A) (B) (C)

Triplet Representation of ~~the~~ both the matrices A & B

Triplet of A

row	col	value
3	3	3
0	0	8
1	2	3
2	0	6

Triplet of B

row	col	value
3	3	4
0	0	3
0	2	1
1	2	4
2	2	5

Resultant Triplet C

row	col	value
3	3	??
0	0	11
0	2	1
1	2	7
2	0	6
2	2	5

- Dimension of resultant matrix will remain same i.e. 3×3
- But we are not sure about ^{number of} non-zero value so we keep the value entry in the resultant matrix as empty.
- We ~~maintain~~ maintain a counter variable, initially count = 0 & we increase the value of counter when we encounter any non-zero value in the resultant triplet.
- At the end the value entry will be filled by ^{final} value of counter.
- Mention the lexicographic order for rows while doing addition.

Multiplication Operation:-

- To compute $A \times B$, first we have to take transpose of B in case of sparse Matrices.
- Multiply the corresponding elements & add them for each position in the resultant matrix.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 10 & 0 & 12 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 2 & 15 & 12 & 0 & 0 \end{bmatrix}$$

A

↓ taking A as it is.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 10 & 0 & 12 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 2 & 15 & 12 & 0 & 0 \end{bmatrix}$$

A

$$B = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 8 & 0 \\ 1 & 0 & 0 & 23 \\ 0 & 0 & 9 & 0 \\ 20 & 25 & 0 & 0 \end{bmatrix}$$

B.

↓ Taking transpose of B

$$B^T = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 8 & 0 \\ 1 & 0 & 0 & 23 \\ 0 & 0 & 9 & 0 \\ 20 & 25 & 0 & 0 \end{bmatrix}$$

B

We will multiply \Rightarrow first row of A with first row of B to get one first element.

\Rightarrow then first row of A with 2nd row of B to get one 2nd element.

\Rightarrow and so on.

$$A \times B^T = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 240 & 300 & 0 & 230 \\ 1 & 0 & 0 & 23 \\ 0 & 0 & 9 & 0 \\ 20 & 25 & 0 & 0 \end{bmatrix}$$

\Rightarrow

So in this way we can calculate for rest of the position.

Now to represent this multiplication we will use Triplet.

Triplet of Matrix A

Row	Col	Value
4	4	5
0	1	10
0	3	12
2	2	5
3	0	15
3	1	12

Triplet of B (Before Transpose)

Row	Col	Value
4	4	5
0	2	8
1	3	23
2	2	9
3	0	20
3	1	25

B (After Transpose)

Row	Col	Value
4	4	5
0	3	20
1	3	25
2	0	8
2	2	9
3	1	23

Transpose of B in row-major form

• Pick ~~row~~ 0th row of matrix A

ie

0,1
0,3
A

0,3
1,3
2,2
2,2
3,1
B

• We have to look for same column to do the multiplication

$\Rightarrow (0,1)$ ~~can~~ match with $(3,1)$
A B

\Rightarrow so when we multiply the elements at these position the ~~dimension~~ dimension of the resultant matrix will be

$(0,1) \times (3,1)$

$(0,3)$

ie we have to pick row no. of matrix B

for making column for the resultant.

⊙ Match for $(0,3)$ is with A

$$\begin{pmatrix} 0,3 \\ 1,3 \end{pmatrix} B$$

So the when we pick them for multiplication the final dimension will be.

$$\begin{pmatrix} 0,3 \\ A \end{pmatrix} \times \begin{pmatrix} 0,3 \\ B \end{pmatrix} \xrightarrow{\text{same column}} \begin{pmatrix} 0,0 \end{pmatrix} \Rightarrow \text{for resultant}$$

$$\begin{pmatrix} 0,3 \\ A \end{pmatrix} \times \begin{pmatrix} 1,3 \\ B \end{pmatrix} \xrightarrow{\text{same column}} \begin{pmatrix} 0,1 \end{pmatrix} \text{ for the resultant matrix}$$

$$\text{Value at } \begin{pmatrix} 0,3 \\ A \end{pmatrix} \times \text{value at } \begin{pmatrix} 0,3 \\ B \end{pmatrix} \xrightarrow{\text{same column}} 12 \times 20 \Rightarrow \underline{\underline{240}}$$

$$\begin{pmatrix} 0,3 \\ A \end{pmatrix} \times \begin{pmatrix} 1,3 \\ B \end{pmatrix} \xrightarrow{\text{same column}} 12 \times 25 \xrightarrow{\text{same column}} \underline{\underline{300}}$$

Now as you can see that for 0th row of A we have 3 possible dimension followed as $(0,0)$ $(0,1)$ $(0,3)$ for the resultant matrix. I had already written them in lexicographic order.

Now, in same way pick the next row of matrix A. ie row no. 2

$$\begin{pmatrix} 2,2 \\ A \end{pmatrix}$$

$$\begin{pmatrix} 0,3 \\ 1,3 \\ 2,0 \\ 2,2 \\ 3,1 \end{pmatrix} B$$

• Look for the same column for performing multiplication.

$$\therefore \begin{pmatrix} 2,2 \\ A \end{pmatrix} \times \begin{pmatrix} 2,2 \\ B \end{pmatrix} \xrightarrow{\text{same column}}$$

$(2,2)$ dimension for resultant matrix

In the same way, we can process for rest of the rows of A. Therefore, the resultant matrix triplet will look like

row	col	value
4	4	
0	0	240
0	1	300
0	3	230
2	2	45
3	2	120
3	3	276

Whenever same row no. appears arrange them according to their order.

same here.

Keep this empty & maintain a counter. After all possible combinations we will get to know the number of non-zero elements.

The corresponding sparse matrix will be

	0	1	2	3
0	240	300	0	230
1	0	0	0	0
2	0	0	45	0
3	0	0	120	276