Lect – 9 to 12

Special Matrices

- They have ~~same~~ same no. of rows and columns i.e we can say that they are $n \times n$ matrices or we can say that they are square matrices.
- Matrices with a relatively high proportion of zero entries.
- Some special forms of square matrices are

→ Diagonal

→ Tri diagonal

→ Lower Triangular

→ Upper Triangular

→ Symmetric

Say name of the matrices is $A[i, j]$ with the subscript i & j where i represent rows & j represent column.

Diagonal Matrices :–

$$A[i,j] = 0 \text{ for } i \neq j$$

It means all the position of the matrix will hold zero except diagonal because at diagonal $\boxed{i=j}$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \rightarrow \text{non-zero entries at } i=j$$

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Tridiagonal :-

$$A[i,j] = 0 \quad \text{for } |i-j| > 1$$

It means all the position of the matrices will hold zero whose absolute value of $|i-j| > 1$ & rest all will have non-zero entries. Therefore we can say that non-zero elements will be on diagonal, just below & just above the diagonal.

$$\begin{bmatrix} 3 & 2 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

just below the diagonal

just above the diagonal.

Lower Triangular :-

$$A[i,j] = 0 \quad \text{for } i > j$$

It means all the position of the matrices will hold zero for ~~where~~ $i > j$ whereas we will have non-zero entries at diagonal and below the diagonal.

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

zero entries for $i > j$

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Upper Triangular :-

$$A[i,j] = 0 \quad \text{for} \quad i<j$$

It means all the position of the matrices will hold zero for $i<j$ where as we will have non-zero entries at diagonal & ~~below~~ ~~the to~~ above the diagonal.

$$\begin{bmatrix} 2 & 7 & 4 & 1 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Symmetric square matrix :-

$$A[i,j] = A[j,i] \quad \text{for all } i \text{ and } j$$

It means all the position of the matrices will have same value for $A[i,j] = A[j,i]$

$$\begin{bmatrix} 2 & 4 & 6 & 0 \\ 4 & 1 & 9 & 5 \\ 6 & 9 & 4 & 7 \\ 0 & 5 & 7 & 0 \end{bmatrix}$$

As you can see
$a_{12} = a_{21}$
$a_{23} = a_{32}$
& so on

Our Motive :-

Natural method of representing matrices in memory as 2-D arrays may not be suitable for special matrices. That is, one may save space by storing only those entries which may be non-zero.

To locate the index no. of an element in Tridiagonal Matrices

Elements in tridiagonal are on the diagonal & just above & below the diagonal as shown below.
Here the dot represent the data.

$$
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & a_{11} & a_{12} & & & & \\
2 & a_{21} & a_{22} & a_{23} & & & \\
3 & & a_{32} & a_{33} & a_{34} & & \\
4 & & & a_{43} & a_{44} & a_{45} & \\
5 & & & & a_{54} & a_{55} & a_{56} \\
6 & & & & & a_{65} & a_{66}
\end{array}
$$

Total no. of elements in a tridiagonal matrix

As matrix is a square matrix
Therefore the diagonal will have n elements
Elements above the diagonal are $n-1$
Elements below the diagonal are $n-1$

$\therefore$ Total no. of elements $\Rightarrow n + n-1 + n-1$

$\Rightarrow 3n-2$

So we have $3n-2$ non-zero elements in a triadiagonal matrices.

As the above matrix has name $A[i, j]$ with subscipt i & j

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i	J
for i=1	1, 2 (valid values for J)
for i=2	1, 2, 3 [ie you can say that data is descending at a_{21}, a_{22} & a_{23} when i=2
i=3	2, 3, 4
i=4	3, 4, 5
i=5	4, 5, 6
i=6	5, 6.

If I want to locate A[i, j] as i=4 & j=5

So when I am at i^{th} row that means I had crossed i-1 rows ie row 1, 2 & 3 [∵ i=4]

So the main fact over here is that all the non-zero data will be mapped at linear array

ie we have 3n-2 ⇒ so if our n=6

∴ we have 16 data elements

All these 16 data is mapped to a linear array having location number ranging from 1 to 16

[if I take array index start from 1]

a_{11}	a_{12}	a_{21}	a_{22}	a_{23}	a_{32}	a_{33}	a_{34}	a_{43}	a_{44}	a_{45}	a_{54}	....	a_{66}
1	2	3	4	5	6	7	8	9	10	11	12	----	16

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So we can clearly see that $A[4,5]$ will reside at location 11

<u>How we can calculate the location :-</u>

Observe ~~at~~ how many elements are above this i^{th} row <u>i.e.</u> 4^{th} row

If I want to come at this 4^{th} row, it means I have to cross 3 rows.

Out of these 3 rows $\Rightarrow$ 2 rows are having 3 data
1 row is having 2 data

$\therefore$ we can say that out $i-1$ rows
$i-2$ is having 3 data
& 1st row is having 2 data.

$\therefore$ how many elements (data) are before this i^{th} row

$$\Rightarrow \underbrace{3 * (i-2)}_{\text{data in } i-2 \text{ rows}} + \underbrace{2}_{\text{data in 1st row}}$$

<u>Now</u> to reach at a particular location.

$$L = [3 * (i-2) + 2] + \underbrace{\text{no. of elements present in left to the particular location}}_{j+1-i} + \underbrace{\text{the exact element at that particular location}}_{1}$$

$$\boxed{L = 2i + j - 2.}$$

$\Rightarrow$ Final formula, so by putting any value of i & j we get the location.

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Lower Triangular

- To locate the index no. of an element in lower triangular matrix

Elements in lower triangular are on the diagonal & below the diagonal as shown below.

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & a_{11}\,5 & & & & \\
2 & a_{21}\,2 & 3\,a_{22} & & & \\
3 & 4\,a_{31} & 6\,a_{32} & 7\,a_{33} & & \\
4 & 7\,a_{41} & 9\,a_{42} & 10\,a_{43} & 12\,a_{44} & \\
5 & 3\,a_{51} & 2\,a_{52} & 6\,a_{53} & 4\,a_{54} & 5\,a_{55}
\end{array}
$$

So only none-zero data is stored in 1-D array

5	2	3	4	6	7	7	9	10	12	3	2	6	4	5
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

- Therefore we use less amount of space.

Say I want to find the location for $i=5$ & $j=3$. I have to cross $i-1$ rows. To reach at this row

no. of elements in $i-1$ rows in $A[i, j]$ is

i.e no. of elements in $i-1$ rows

$$1 + 2 + 3 + \text{---} \; i-1$$

no. of element in 1st row

no. of elements in 2nd row

no. of element in 3rd row

no. of elements in $(i-1)^{th}$ row

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This series is natural no. sum in form of $(i-1)$

$$\Rightarrow \frac{i(i-1)}{2} \Big] \Rightarrow$$ These are elements above i^{th} row

There are j elements in the row i^{th} i.e including the element $A[i,j]$ also

So $$L = \frac{i(i-1)}{2} + j$$

i.e $$L = \frac{i(i-1)}{2} + j.$$

Putting value of $i = 5$ & $j = 3$ we get

$$L \Rightarrow \frac{5(5-1)}{2} + 3 \Rightarrow 13$$

Similarly you can compute the formula for Diagonal, upper triangular.

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Sparse Matrix

- Matrix with many of elements are zero.
- This is not a dense matrix.
- Can be represented by
  → Array [Also known as Triplet]
  → Linklist

eg

$$\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 4 & 5 & 0 \end{bmatrix}$$

→ This is a sparse matrix 4 it do not follow any patter for non-zero element. They can be anywhere ie at any position.

Array Representation :-

Why it is named as Triplet as

3 columns

- → one for no. of rows
- → one for no. of columns
- → one for no. of non-zero values.

Rows

- number of rows will be 1 more than no. of non-zero elements.

We can use row-major or column major
But Here, we are representing using row-major only.

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Triplet Representation

	0	1	2	3	4	5
0	2	0	0	3	0	5
1	0	1	4	0	0	0
2	0	0	0	6	0	0
3	0	0	0	0	0	0
4	9	0	0	0	0	0
5	0	0	2	0	0	0

$\Rightarrow$

Row	Col	Value
6	6	8
0	0	2
0	3	3
0	5	5
1	1	1
1	2	4
2	3	6
4	0	9
5	2	2

In a Triplet

We have 3 columns
& no. of rows = 1 + no. of non-zero value
as in the 0th row of triplet we are showing no. of rows, no. of column & no. of non-zero value. So we need 1 extra row ∴ 1 + no. of non-zero.

- Operations on Sparse Matrices

- Transpose
- Addition
- Multiplication.

Transpose :- Consider the given matrix & first represent it ~~the~~ in form of triplet as shown below :-

	0	1	2
0	10	0	0
1	0	0	2
2	0	5	0

$\Rightarrow$

row	col	Value
3	3	3
0	0	10
1	2	2
2	1	5

Original triplet.

Now In transpose, we have to simply interchange the row and column values. So we exchange the row col values, & the resultant triplet we get is in column major form as shown below.

row	col	value
3	3	3
0	0	10
2	1	2
1	2	5

column major

$\Rightarrow$ Now we have to translate it to row-major

row	col	value
3	3	3
0	0	10
1	2	5
2	1	2

row major
final resultant

To get the final resultant triplet in row major what we have to do is just ~~at~~ arrange the rows in lexicographic order.

i.e value of (0,0); (1,2); (2,1)

or you can say that we arranged the row no. in increasing order.

	0	1	2
0	10	0	0
1	0	0	5
2	2	0	0.

$\Leftarrow$ final resultant transpose matrix.

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Addition operation :-

- We have to verify the same dimension of both the matrices before ~~proceedd~~ proceeding.
- We have to look at elements that are lying or placed at same index position. so that we can add those elements.

eg

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 8 & 0 & 0 \\ 1 & 0 & 0 & 3 \\ 2 & 6 & 0 & 0 \end{array} + \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 3 & 0 & 1 \\ 1 & 0 & 0 & 4 \\ 2 & 0 & 0 & 5 \end{array} \Rightarrow \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 11 & 0 & 1 \\ 1 & 0 & 0 & 7 \\ 2 & 6 & 0 & 5. \end{array}$$

(A) (B) (C).

Triplet Representation of ~~ate~~ both the matrices A & B

Triplet of A

row	col	Value
3	3	3
0	0	8
1	2	3
2	0	6

+

Triplet of B

row	col	value
3	3	4
0	0	3
0	2	1
1	2	4
2	2	5

⇒

Resultant Triple C

Row	col	Value
3	3	?
0	0	11
0	2	1
1	2	7
2	0	6
2	2	5

??

- Dimension of resultant matric will remain same ie 3x3
- But we are not sure about number of non-zero value so we Keep the value entry in the resultan matrix as empty.
- We ~~mainta~~ mentain a counter variable, intially counter=0 & we increase the value of counter when we encounter any non-zero value in the resultant triplet.
- At the end the value entry will be filled by final value of counter
- Mentain the lexicographic order for rows while doing addition.

Multiplication Operation:-

- To compute $A * B$, first we have to take transpose of B in case of sparse Matrices.
- Multiply the corresponding elements & add them for each position in the resultant matrix.

$$
\begin{array}{c|c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 10 & 0 & 12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 15 & 12 & 0 & 0 \end{bmatrix}
\end{array}
\times
\begin{array}{c|c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 23 \\ 0 & 0 & 9 & 0 \\ 20 & 25 & 0 & 0 \end{bmatrix}
\end{array}
$$

A   B.

$\Downarrow$ taking A as it is. $\Downarrow$ Taking transpose of B

$$
\begin{array}{c|c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 10 & 0 & 12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 15 & 12 & 0 & 0 \end{bmatrix}
\end{array}
\times
\begin{array}{c|c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 25 \\ 8 & 0 & 9 & 0 \\ 0 & 23 & 0 & 0 \end{bmatrix}
\end{array}
$$

A   B

We will multiply as $\Rightarrow$ First row of A with first row of B to get the first element.

$\Rightarrow$ then first row of A with 2nd row of B to get the 2nd element.

$\Rightarrow$ and so on.

$$
\begin{array}{c|c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 240 & 300 & 0 & 230 \\ & & & \\ & & & \\ & & & \end{bmatrix}
\end{array}
$$

$\Rightarrow$ So in this way we can calculate for rest of the position.

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Now to represent this multiplication we will use Triplet.

Triplet of Matrix A

Row	Col	Value
4	4	5
0	1	10
0	3	12
2	2	5
3	0	15
3	1	12

Triplet of B (Before Transpose)

Row	Col	Value
4	4	5
0	2	8
1	3	23
2	2	9
3	0	20
3	1	25

B (After Transpose)

row	col	value
4	4	5
0	3	20
1	3	25
2	0	8
2	2	9
3	1	23

Transpose of B in row-major form

- Pick ~~row 0~~ 0^{th} row of matrix A

ie

A: 0,1; 0,3

B: 0,3; 1,3; 2,0; 2,2; 3,1

- We have to look for same column to do the multiplication

$\Rightarrow$ (0, 1) match with (3, 1)
A B

$\Rightarrow$ So when we multiply the elements at these position The dimension of the resultant matrix will be

$(0,1) \times (3,1)$

$(0,3)$ ie we have to pick row no. of matrix B for making column for the resultant.

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- Match for $\underset{A}{(0,3)}$ is with $\underset{B}{\begin{matrix}0,3\\1,3\end{matrix}}$

So the when we pick them for multiplication the final dimension will be.

$\underset{A}{(0,3)} \times \underset{B}{(0,3)}$ — same column

$\Downarrow$

$(0,0) \Rightarrow$ for resultant

$\underset{A}{(0,3)} \times \underset{B}{(1,3)}$ — same column

$\Downarrow$

$(0,1)$

for the resultant matrix

Value at $\underset{A}{(0,3)}$ $\times$ Value at $\underset{B}{(0,3)}$

$\Downarrow$

$12 \times 20 \Rightarrow \underline{240}$

$\Downarrow$

$(0,3) \times (1,3)$

12×25

$\Downarrow$

300.

Now as you can see that for 0^{th} row of A followed as we have 3 possible dimension for the resultant matrix

$(0,0) \quad (0,1) \quad (0,3)$

I had already written them in lexicographic order.

Now, in same way pick the next row of matrix A. ie row no. 2

$\underset{A}{(2,2)}$

$\underset{B}{\begin{matrix}0,3\\1,3\\2,0\\2,2\\3,1\end{matrix}}$

- Look for the same column for performing multiplication.

$\therefore \underset{A}{(2,2)} \times \underset{B}{(2,2)}$

$\Downarrow$

$(2,2)$ dimension for resultant matrix

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In the same way, we can process for rest of the rows of A. Therefore, the resultant matrix triplet will look like

row	col	Value
4	4	
0	0	240
0	1	300
0	3	230
2	2	45
3	2	120
3	3	276

?? keep this empty & maintain a counter. After all possible combination we will get to know the number ~~value~~ of non-zero elements.

Whenever same row no. appears arrange them according to their order.

same here.

The corresponding sparse matrix will be

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 240 & 300 & 0 & 230 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 45 & 0 \\ 3 & 0 & 0 & 120 & 276 \end{array}$$

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