INTRO TO DG 90min lecture @ SXS-Con 2024

Harald Pfeiffer
MPI for Grav. Physics

SPECTRAL METHODS

idea: approx functions by expansion in basis-fn.

$$u(x) \simeq u^{(N)}(x) = \sum_{k=0}^{N-1} \tilde{u}_k \phi_k(x)$$

Example: $u(x) = e^{-2(\cos x + 1)}$, $x \in [0, 2\pi]$. Calc derivative.

~~N points $x_j = \frac{2\pi j}{N}$, $j=0,\dots,N-1$~~

$$u^{(N)}(x) = \sum_{k=0}^{N/2} \tilde{u}_k e^{ikx}$$

$$x_j = \frac{2\pi j}{N} \quad j=0,..,N-1$$

Calc $\tilde{u}_k$ w/ orthogonality $\frac{1}{2\pi}\int e^{ikx} e^{-ijx} dx = \delta_{jk}$

$$\tilde{u}_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} u(x)\, dx \simeq \frac{1}{N}\sum_{j=0}^{N-1} u(x_j)\, e^{-ikx_j}$$

Now

$$\frac{du^{(N)}}{dx} = \sum_{k=0}^{N/2} ik\,\tilde{u}_k e^{ikx}$$

[SLIDES 1]

* expansion in basis-functions ϕ_k
* derivs easy (b/c $\phi_k(x)$ known)
* interpolation easy (b/c $\phi_k(x)$ are functions)
* astonishing accuracy (in <u>right</u> circumstances!)

Why / when does this work?

$$2\pi \tilde{u}_k = \int_0^{2\pi} u(x)\, e^{-ikx}\, dx$$

integrate by parts (IBP)

$$= \frac{i}{k} \underbrace{\left[u(x)\, e^{-ikx} \right]_0^{2\pi}}_{=0 \text{ if } u \text{ periodic}} + \frac{-i}{k} \int_0^{2\pi} u'(x)\, e^{-ikx}\, dx$$

$$\stackrel{IBP}{=} \frac{-i}{k} \frac{i}{k} \underbrace{\left[u'(x)\, e^{-ikx} \right]_0^{2\pi}}_{=0 \text{ if } u' \text{ periodic}} + \left(\frac{-i}{k}\right)^2 \int_0^{2\pi} u'' \, e^{-ikx}\, dx$$

$\vdots\ n \times$

$$= \qquad + \left(\frac{-i}{k}\right)^n \int_0^{2\pi} u^{(n)}\, e^{-ikx}\, dx$$

- if $u(x)$ is smooth & periodic ~~and~~:

  $\tilde{u}_k = O\left(\frac{1}{k^n}\right) \quad \forall n \in \mathbb{N}$

  $\Rightarrow \tilde{u}_k$ decay faster than any power of $\frac{1}{k}$ "Exponential", "spectral" conv.

- if $u^{(k)}$ not periodic $\Rightarrow \tilde{u}_k \sim 1/k^{n+1}$

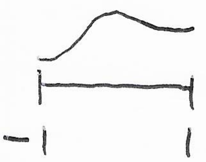
  if $u \in C^n$: IBP breaks down $\Rightarrow \tilde{u}_k \sim 1/k^{n+2}$

[SLIDES 2]

* Shape of $u(x)$ determines (exp) conv. rate

$\Longrightarrow$ choose domain-decomp s.t. solution looks "simple" in each element ~~el~~ PP

Open intervals (non-periodic)



Use eigenfunctions of singular Sturm-Liouville problem

$$\left(p(x)\,\phi_k'(x)\right)' + q(x)\,\phi_k(x) = \lambda\, w(x)\, \phi_k(x)$$

$p(\pm 1) = 0$

$\Longrightarrow$

* orthogonality $\displaystyle\int_{-1}^{1} \phi_j(x)\,\phi_k(x)\, w(x)\,dx = \delta_{ij}$

* completeness

* Gauß integration formula:

for $N \in \mathbb{N}$ $\exists$ w_i, x_i s.t. $\displaystyle\int_{-1}^{1} f(x)\,w(x)\,dx = \sum_{i=0}^{N-1} w_i\, f(x_i)$ exact for f = poly of degree $\leq 2N$

w_i, x_i: ~~either~~ Gauß (interior)

Gauß-Lobatto (incl. $x = \pm 1$)

* Recurrence relations

e.g. Chebyshev $\displaystyle\frac{T_{k+1}'}{k+1} = \frac{T_{k-1}'}{k-1} + 2T_k$

Legendre polynomials

have $w(x)=1$

orthogonality + Gauß integration w/o weight

$$\int \phi_j(x)\,\phi_k(x)\,dx = \delta_{jk}$$

$$\int f(x)\,dx = \sum_{i=0}^{N-1} w_i\, f(x_i) \quad \text{exact for poly of degree} \leq 2N$$

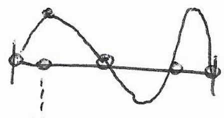
(specific for Legendre)

Cardinal basis

N fixed, $x_i =$ collocation pts of basis $\{\phi_k\}$, $i=0,N-1$

* consider Lagrange interpolation poly's

$\ell_j(x): \quad \ell_j(x_i) = \delta_{ij}$



$$\mathrm{span}\{\ell_j \; j=0,..,N-1\} = \mathrm{span}\{\phi_j \; j=0,..,N-1\} = \mathrm{poly}_{N-1}$$

* ~~by definition~~ $\Rightarrow$

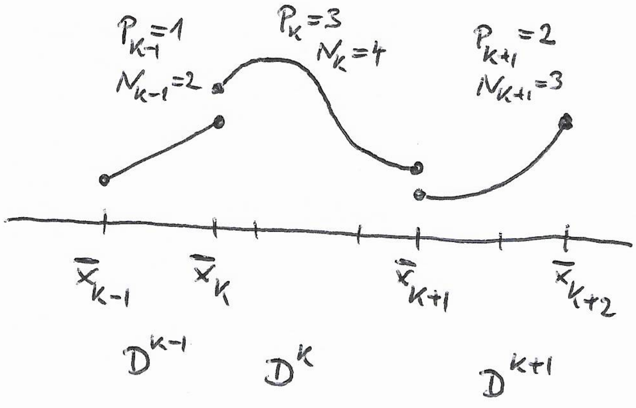
$$u^{(N)} = \sum_{k=0}^{N-1} \tilde{u}_k \phi_k(x) = \sum_{i=0}^{N-1} u(x_i)\, \ell_i(x)$$

both poly of degree $N-1$ that agree at N points x_i

- in Cardinal basis, grid-point values are expansion coefs.
- still exponentially convergent if $x_i =$ collocation pts and u smooth.

DG

many elements, each w/ spectral expansion



← poly order & N allowed to vary

← approx allowed to be discontinuous.

$$x \in D^K: \quad u_h(x,t) = \sum_{n=0}^{N_K-1} \tilde{u}_n^K(t)\,\phi_n^K(x)$$

coefs time-dependent for evolution problems

e.g. solve PDE in flux form

$$\frac{\partial u}{\partial t} + \frac{\partial f[u]}{\partial x} - g = 0$$

$f[u] = $ flux

e.g. $f[u] = au$ for advection ~~pde~~ eqn w/ speed a

Goal: find eqs for coefficients $\{\tilde{u}_n^K\}$.

Require residual $R[u_h] = \dot{u}_h + \frac{\partial f[u_h]}{\partial x} - g$.

to be orthogonal to basis fn

$$\int_D R[u_h]\,\phi_{n_0}^{K_0}\,dx = 0 \qquad \forall n_0, K_0$$

$$\Rightarrow \sum_{D^k} \underbrace{\tilde{\dot{u}}_n^k \int_{D^k} \phi_n^k \phi_{n_0}^{k_0}\, dx}_{\underline{\underline{\mathcal{M}}}^k \text{ mass-matrix}} + \underbrace{\int_{D^k} \frac{\partial f}{\partial x} \phi_{n_0}^{k_0}\, dx}_{} - \tilde{g}_n^k \underbrace{\int_{D^k} \phi_n^k \phi_{n_0}^{k_0}\, dx}_{\underline{\underline{\mathcal{M}}}^k} = 0$$

element-local b/c

ϕ_n^k has support only in D^k

on D^k

$$\overset{IBP}{=} -\int_{D^k} f \frac{\partial \phi_{n_0}^{k_0}}{\partial x}\, dx + \left[f \phi_{n_0}^{k_0} \right]_{\bar{x}_k}^{\bar{x}_{k+1}}$$

- on $\cancel{D_k}$ boundaries, f dual-valued.
- define unique flux $f^*(\bar{x}_k)$ at each bdry

$$\rightarrow -\int_{D^k} f \frac{\partial \phi_{n_0}^{k_0}}{\partial x}\, dx + \left[f^* \phi_{n_0}^{k_0} \right]_{\bar{x}_k}^{\bar{x}_{k+1}}$$

$$\overset{IBP^{-1}}{=} \underbrace{\int_{D^k} \frac{\partial f}{\partial x} \phi_{n_0}^{k_0}\, dx}_{\tilde{f}_n^k \underbrace{\int_{D_k} \frac{\partial \phi_n^k}{\partial x} \phi_{n_0}^{k_0}\, dx}_{\underline{\underline{S}}^k \text{ stiffness matrix}}} - \left[(f - f^*) \phi_{n_0}^{k_0} \right]_{\bar{x}_k}^{\bar{x}_{k+1}}$$

$$\Rightarrow \quad \underline{\underline{M}}\{\dot{u}_n^k\} + \underline{\underline{S}}\{f_n^k\} - \underline{\underline{M}}\{g_n^k\} = \text{Flux Corr}$$

$$\{\dot{u}_n^k\} = -\underline{\underline{M}}^{-1}\underline{\underline{S}}\{f_n^k\} + g_n^k + \underline{\underline{M}}^{-1}\,\text{Flux Corr}$$

Method of lines form!

Legendre polynomials + gridpoints ~~[illegible]~~

- exp conv.

- Gauß quadrature
- <u>Shape</u> of elements wrt. solution determines conv. rate.

Cardinal basis

- gridpoint values $\equiv$ expansion coeffs
- flux-terms local to bdry

DG

- $\underline{\underline{M}}$, $\underline{\underline{S}}$ element local

- Flux only needs bdry data communicated

  $\Rightarrow$ parallelisable

- h, p refinement flexibility