Optimization problem

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May 27, 2023

Summary

Generality

Definition

Résolution avec l'ADMM:

- Équation : $\min_{x,z}\{f(x)+g(z)\}$
- Contrainte : $Ax+Bz=c$

Lagrangien augmenté :

$$\begin{aligned} L(x,z,u) &= f(x)+g(z)+u^T(Ax+Bz-c)+\frac{\rho}{2}\|Ax+Bz-c\|_2^2 \\ &= f(x)+g(z)+\frac{\rho}{2}\|Ax+Bz-c+\frac{u}{\rho}\|_2^2-\frac{\|u\|_2^2}{2\rho} \end{aligned}$$

Tikhonov : $R(x) = \|x\|_2^2$

$$
\begin{aligned}
&\hat{x} = \arg\min_x \{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|x\|_2^2\} \\
&\frac{d}{dx}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|x\|_2^2\} = 0 \\
&\iff -H^T(y - Hx) + 2\lambda x = 0 \\
&\iff -H^T y + H^T H x + 2\lambda x = 0 \\
&\iff -H^T y + (H^T H + 2\lambda I)x = 0 \\
&\iff x = (H^T H + 2\lambda I)^{-1} H^T y
\end{aligned}
$$

ADMM

Definition

Résolution avec l'ADMM:

- Équation : $\min_{x,z}\{f(x)+g(z)\}$
- Contrainte : $Ax+Bz=c$

Lagrangien augmenté :

$$\begin{aligned} L(x,z,u) &= f(x)+g(z)+u^T(Ax+Bz-c)+\frac{\rho}{2}\|Ax+Bz-c\|_2^2 \\ &= f(x)+g(z)+\frac{\rho}{2}\|Ax+Bz-c+\frac{u}{\rho}\|_2^2-\frac{\|u\|_2^2}{2\rho} \end{aligned}$$

Algorithm

Algorithm 1 ADMM

procedure ADMM
Input: y, λ, ρ, $nb\_iterations$
Output: $x_{nb\_iterations}$
  for $k \in 0 ... nb\_iterations$ **do**
    $x_{k+1} = \arg\min_x L(x_k, z_k, u_k)$
    $z_{k+1} = \arg\min_z L(x_{k+1}, z, u_k)$
    $u_{k+1} = u_k + \rho(z_{k+1} - x_{k+1})$
  end for
end procedure

$L_1 : R(x) = \|x\|_1$

Résolution avec l'ADMM:

- Équation : $\min\limits_{x,z}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1\}$
- Contrainte : $x = z$

Lagrangien augmenté :

$$\begin{aligned} L(x, z, u) &= \frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + u^T(x - z) + \frac{\rho}{2}\|x - z\|_2^2 \\ &= \frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho} \end{aligned}$$

$L_1 : R(x) = \|x\|_1$

$$\|y - Hx\|_2^2 = \| \begin{bmatrix} y_0 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} H_{0,0} & \dots & H_{0,N} \\ \vdots & \ddots & \vdots \\ H_{N,0} & \dots & H_{N,N} \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_N \end{bmatrix} \|_2^2$$

$$= \| \begin{bmatrix} y_0 \\ \vdots \\ y_N \end{bmatrix} - \begin{bmatrix} H_{0,0}x_0 + \cdots + H_{0,N}x_N \\ \vdots \\ H_{N,0}x_0 + \cdots + H_{N,N}x_N \end{bmatrix} \|_2^2$$

$$= \| \begin{bmatrix} y_0 - (H_{0,0}x_0 + \cdots + H_{0,N}x_N) \\ \vdots \\ y_N - (H_{N,0}x_0 + \cdots + H_{N,N}x_N) \end{bmatrix} \|_2^2$$

$\arg\min_x L(x, z, u)$

$$
\begin{aligned}
&\frac{d}{dx}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho}\} = 0 \\
&\iff -H^T(y - Hx) + \rho(x - z + \frac{u}{\rho}) = 0 \\
&\iff -H^T y + H^T Hx + \rho x - \rho z + u = 0 \\
&\iff -H^T y + (H^T H + \rho I)x - \rho z + u = 0 \\
&\iff (H^T H + \rho I)x = H^T y + \rho z - u \\
&\iff x = (H^T H + \rho I)^{-1}(H^T y + \rho z - u)
\end{aligned}
$$

$\arg\min_z L(x, z, u)$

$$\frac{d}{dz}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho}\} = 0$$

$$\iff \lambda sign(z) + \rho(x - z + \frac{u}{\rho}) = 0$$

$$\iff \lambda sign(z) + \rho x - \rho z + u = 0$$

$$\iff -\lambda sign(z) + \rho z = \rho x + u$$

$$\iff z - \frac{\lambda}{\rho} sign(z) = x + \frac{u}{\rho}$$

$$\iff z = sign(x + \frac{u}{\rho}) \times max(|x + \frac{u}{\rho}| - \frac{\lambda}{\rho}, 0)$$

$$\iff z = Soft_{\frac{\lambda}{\rho}}(x + \frac{u}{\rho})$$

Algorithm

Algorithm 2 ADMM

procedure ADMM

Input: y, λ, ρ, $nb\_iterations$

Output: $x_{nb\_iterations}$

  for $k \in 0 ... nb\_iterations$ **do**

    $x_{k+1} = (H^T H + \rho I)^{-1}(H^T y_k + \rho z_k - u_k)$

    $z_{k+1} = Soft_{\frac{\lambda}{\rho}}(x_{k+1} + \frac{u_k}{\rho})$

    $u_{k+1} = u_k + \rho(z_{k+1} - x_{k+1})$

  end for

end procedure

Split Bregman

Remarques sur la notation ∇ (gradient)

$$\nabla = \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \end{bmatrix} = \begin{bmatrix} \nabla_x \\ \nabla_y \end{bmatrix}$$

$$\nabla^T = \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \end{bmatrix}^T = \begin{bmatrix} \frac{d}{dx} & \frac{d}{dy} \end{bmatrix}$$

$$= \begin{bmatrix} \nabla_x \\ \nabla_y \end{bmatrix}^T = \begin{bmatrix} \nabla_x^T & \nabla_y^T \end{bmatrix} \text{(Transpose of block matrix)}$$

$$\nabla f = \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \end{bmatrix} f = \begin{bmatrix} \frac{df}{dx} \\ \frac{df}{dy} \end{bmatrix} = \begin{bmatrix} \nabla_x f \\ \nabla_y f \end{bmatrix}$$

Remarques sur la notation Δ (Laplacien)

$$\begin{aligned}\nabla^T\nabla &= \begin{bmatrix}\frac{d}{dx}\\ \frac{d}{dy}\end{bmatrix}^T \begin{bmatrix}\frac{d}{dx}\\ \frac{d}{dy}\end{bmatrix} = \begin{bmatrix}\frac{d}{dx} & \frac{d}{dy}\end{bmatrix} \begin{bmatrix}\frac{d}{dx}\\ \frac{d}{dy}\end{bmatrix} = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} = \Delta\\ &= \begin{bmatrix}\nabla_x\\ \nabla_y\end{bmatrix}^T \begin{bmatrix}\nabla_x\\ \nabla_y\end{bmatrix} = \begin{bmatrix}\nabla_x^T & \nabla_y^T\end{bmatrix} \begin{bmatrix}\nabla_x\\ \nabla_y\end{bmatrix} = \nabla_x^T\nabla_x + \nabla_y^T\nabla_y = \Delta\end{aligned}$$

$R(x) = \|\nabla x\|_1$

Résolution avec l'ADMM:

- Équation : $\min\limits_{x,z}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1\}$
- Contrainte : $\nabla x = (d_x, d_y) = z \iff (d_x = z_x)$ and $(d_y = z_y)$

Lagrangien augmenté :

$$
\begin{aligned}
L(x,z,u) &= \frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + u^T(\nabla x - z) + \frac{\rho}{2}\|\nabla x - z\|_2^2 \\
&= \frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|\nabla x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho} \\
&= \frac{1}{2}\|y - Hx\|_2^2 + \lambda\|(d_x, d_y)\|_1 + \frac{\rho}{2}\|d_x - z_x + \frac{u_x}{\rho}\|_2^2 - \frac{\|u_x\|_2^2}{2\rho} + \frac{\rho}{2}\|d_y - z_y + \frac{u_y}{\rho}\|_2^2 - \frac{\|u_y\|_2^2}{2\rho}
\end{aligned}
$$

$\arg\min_x L(x, z, u)$

$$\frac{d}{dx}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|\nabla x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho}\} = 0$$

$$\iff -H^T(y - Hx) + \rho\nabla^T(\nabla x - z + \frac{u}{\rho}) = 0$$

$$\iff -H^T(y - Hx) + \rho\nabla^T\nabla x - \rho\nabla^T z + \rho\nabla^T u = 0$$

$$\iff -H^T y + H^T Hx + \rho\nabla^T\nabla x - \rho\nabla^T z + \rho\nabla^T u = 0$$

$$\iff -H^T y + (H^T H + \rho\nabla^T\nabla)x - \rho\nabla^T z + \rho\nabla^T u = 0$$

$$\iff (H^T H + \rho\nabla^T\nabla)x = H^T y + \rho\nabla^T z - \rho\nabla^T u$$

$$\iff x = (H^T H + \rho\nabla^T\nabla)^{-1}(H^T y + \rho\nabla^T z - \rho\nabla^T u)$$

$\arg\min_z L(x, z, u)$

$$\frac{d}{dz}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho}\} = 0$$

$$\iff \lambda sign(z) + \rho(x - z + \frac{u}{\rho}) = 0$$

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$$\iff z - \frac{\lambda}{\rho} sign(z) = x + \frac{u}{\rho}$$

$$\iff z = sign(x + \frac{u}{\rho}) \times max(|x + \frac{u}{\rho}| - \frac{\lambda}{\rho}, 0)$$

$$\iff z = Soft_{\frac{\lambda}{\rho}}(x + \frac{u}{\rho})$$

Algorithm

Algorithm 3 ADMM

procedure ADMM
  Input: y, λ, ρ, $nb\_iterations$
Output: $x_{nb\_iterations}$
  for $k \in 0...nb\_iterations$ **do**
    $x_{k+1} = (H^T H + \rho I)^{-1}(H^T y_k + \rho z_k - u_k)$
    $z_{k+1} = Soft_{\frac{\lambda}{\rho}}(x_{k+1} + \frac{u_k}{\rho})$
    $u_{k+1} = u_k + \rho(z_{k+1} - x_{k+1})$
  end for
end procedure

Chambolle Pock

Remarques sur la notation ∇ (gradient)

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$$\nabla f = \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \end{bmatrix} f = \begin{bmatrix} \frac{df}{dx} \\ \frac{df}{dy} \end{bmatrix} = \begin{bmatrix} \nabla_x f \\ \nabla_y f \end{bmatrix}$$

Remarques sur la notation Δ (Laplacien)

$$\nabla^T\nabla = \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \end{bmatrix}^T \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \end{bmatrix} = \begin{bmatrix} \frac{d}{dx} & \frac{d}{dy} \end{bmatrix} \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \end{bmatrix} = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} = \Delta$$

$$= \begin{bmatrix} \nabla_x \\ \nabla_y \end{bmatrix}^T \begin{bmatrix} \nabla_x \\ \nabla_y \end{bmatrix} = \begin{bmatrix} \nabla_x^T & \nabla_y^T \end{bmatrix} \begin{bmatrix} \nabla_x \\ \nabla_y \end{bmatrix} = \nabla_x^T\nabla_x + \nabla_y^T\nabla_y = \Delta$$

$R(x) = \|\nabla x\|_1$

Résolution avec l'ADMM:

- Équation : $\min_{x,z}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1\}$
- Contrainte : $\nabla x = (d_x, d_y) = z \iff (d_x = z_x)$ and $(d_y = z_y)$

Lagrangien augmenté :

$$\begin{aligned} L(x,z,u) &= \frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + u^T(\nabla x - z) + \frac{\rho}{2}\|\nabla x - z\|_2^2 \\ &= \frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|\nabla x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho} \\ &= \frac{1}{2}\|y - Hx\|_2^2 + \lambda\|(d_x, d_y)\|_1 + \frac{\rho}{2}\|d_x - z_x + \frac{u_x}{\rho}\|_2^2 - \frac{\|u_x\|_2^2}{2\rho} + \frac{\rho}{2}\|d_y - z_y + \frac{u_y}{\rho}\|_2^2 - \frac{\|u_y\|_2^2}{2\rho} \end{aligned}$$

$\arg\min_x L(x, z, u)$

$$
\begin{aligned}
&\frac{d}{dx}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|\nabla x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho}\} = 0 \\
\iff\ & -H^T(y - Hx) + \rho\nabla^T(\nabla x - z + \frac{u}{\rho}) = 0 \\
\iff\ & -H^T(y - Hx) + \rho\nabla^T\nabla x - \rho\nabla^T z + \rho\nabla^T u = 0 \\
\iff\ & -H^T y + H^T H x + \rho\nabla^T\nabla x - \rho\nabla^T z + \rho\nabla^T u = 0 \\
\iff\ & -H^T y + (H^T H + \rho\nabla^T\nabla)x - \rho\nabla^T z + \rho\nabla^T u = 0 \\
\iff\ & (H^T H + \rho\nabla^T\nabla)x = H^T y + \rho\nabla^T z - \rho\nabla^T u \\
\iff\ & x = (H^T H + \rho\nabla^T\nabla)^{-1}(H^T y + \rho\nabla^T z - \rho\nabla^T u)
\end{aligned}
$$

$\arg\min_z L(x, z, u)$

$$
\begin{aligned}
&\frac{d}{dz}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho}\} = 0 \\
&\iff \lambda sign(z) + \rho(x - z + \frac{u}{\rho}) = 0 \\
&\iff \lambda sign(z) + \rho x - \rho z + u = 0 \\
&\iff -\lambda sign(z) + \rho z = \rho x + u \\
&\iff z - \frac{\lambda}{\rho} sign(z) = x + \frac{u}{\rho} \\
&\iff z = sign(x + \frac{u}{\rho}) \times max(|x + \frac{u}{\rho}| - \frac{\lambda}{\rho}, 0) \\
&\iff z = Soft_{\frac{\lambda}{\rho}}(x + \frac{u}{\rho})
\end{aligned}
$$

Algorithm

Algorithm 4 ADMM

procedure ADMM

Input: y, λ, ρ, $nb\_iterations$

Output: $x_{nb\_iterations}$

for $k \in 0...nb\_iterations$ **do**

$x_{k+1} = (H^T H + \rho I)^{-1}(H^T y_k + \rho z_k - u_k)$

$z_{k+1} = Soft_{\frac{\lambda}{\rho}}(x_{k+1} + \frac{u_k}{\rho})$

$u_{k+1} = u_k + \rho(z_{k+1} - x_{k+1})$

end for

end procedure

The End