Optimization problem

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June 3, 2023

Summary

Generality

Definition

Résolution avec l'ADMM:

- Équation : $\min_{x,z}\{f(x) + g(z)\}$
- Contrainte : $Ax + Bz = c$

Lagrangien augmenté :

$$\begin{aligned} L(x, z, u) &= f(x) + g(z) + u^T(Ax + Bz - c) + \frac{\rho}{2}\|Ax + Bz - c\|_2^2 \\ &= f(x) + g(z) + \frac{\rho}{2}\|Ax + Bz - c + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho} \end{aligned}$$

Tikhonov : $R(x) = \|x\|_2^2$

$$\hat{x} = \arg\min_x \{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|x\|_2^2\}$$

$$\frac{d}{dx}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|x\|_2^2\} = 0$$

$$\iff -H^T(y - Hx) + 2\lambda x = 0$$

$$\iff -H^Ty + H^THx + 2\lambda x = 0$$

$$\iff -H^Ty + (H^TH + 2\lambda I)x = 0$$

$$\iff x = (H^TH + 2\lambda I)^{-1}H^Ty$$

ADMM

Definition

Résolution avec l'ADMM:

- Équation : $\min_{x,z}\{f(x)+g(z)\}$
- Contrainte : $Ax+Bz=c$

Lagrangien augmenté :

$$
\begin{aligned}
L(x,z,u) &= f(x)+g(z)+u^T(Ax+Bz-c)+\frac{\rho}{2}\|Ax+Bz-c\|_2^2 \\
&= f(x)+g(z)+\frac{\rho}{2}\|Ax+Bz-c+\frac{u}{\rho}\|_2^2-\frac{\|u\|_2^2}{2\rho}
\end{aligned}
$$

Algorithm

Algorithm 1 ADMM

```
procedure ADMM
    Input: y, λ, ρ, nb_iterations
Output: x_{nb_iterations}
    for k ∈ 0...nb_iterations do
        x_{k+1} = argmin_x L(x_k, z_k, u_k)
        z_{k+1} = argmin_z L(x_{k+1}, z, u_k)
        u_{k+1} = u_k + ρ(z_{k+1} − x_{k+1})
    end for
end procedure
```

procedure ADMM

Input: y, λ, ρ, $nb\_iterations$

Output: $x_{nb\_iterations}$

for $k \in 0 \ldots nb\_iterations$ **do**

$$x_{k+1} = \arg\min_{x} L(x_k, z_k, u_k)$$

$$z_{k+1} = \arg\min_{z} L(x_{k+1}, z, u_k)$$

$$u_{k+1} = u_k + \rho(z_{k+1} - x_{k+1})$$

end for

end procedure

$L_1 : R(x) = \|x\|_1$

Résolution avec l'ADMM:

- Équation : $\min\limits_{x,z}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1\}$
- Contrainte : $x = z$

Lagrangien augmenté :

$$\begin{aligned} L(x,z,u) &= \frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + u^T(x - z) + \frac{\rho}{2}\|x - z\|_2^2 \\ &= \frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho} \end{aligned}$$

$L_1 : R(x) = \|x\|_1$

$$\frac{d}{dx}\{\|y - Hx\|_2^2\} = -2H^T(y - Hx) = -2H^Ty + 2H^THx$$
$$\frac{d}{dx}\{\|Hx - y\|_2^2\} = 2H^T(Hx - y) = 2H^THx - 2H^Ty$$

$\arg\min_x L(x, z, u)$

$$
\begin{aligned}
&\frac{d}{dx}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho}\} = 0 \\
&\iff -H^T(y - Hx) + \rho(x - z + \frac{u}{\rho}) = 0 \\
&\iff -H^Ty + H^THx + \rho x - \rho z + u = 0 \\
&\iff -H^Ty + (H^TH + \rho I)x - \rho z + u = 0 \\
&\iff (H^TH + \rho I)x = H^Ty + \rho z - u \\
&\iff x = (H^TH + \rho I)^{-1}(H^Ty + \rho z - u)
\end{aligned}
$$

$\arg\min_z L(x, z, u)$

$$
\begin{aligned}
&\frac{d}{dz}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho}\} = 0 \\
&\iff \lambda sign(z) - \rho(x - z + \frac{u}{\rho}) = 0 \\
&\iff \lambda sign(z) - \rho x + \rho z - u = 0 \\
&\iff \lambda sign(z) + \rho z = \rho x + u \\
&\iff z + \frac{\lambda}{\rho} sign(z) = x + \frac{u}{\rho} \\
&\iff z = sign(x + \frac{u}{\rho}) \times max(|x + \frac{u}{\rho}| - \frac{\lambda}{\rho}, 0) \\
&\iff z = Soft_{\frac{\lambda}{\rho}}(x + \frac{u}{\rho})
\end{aligned}
$$

Algorithm

Algorithm 2 ADMM-L_1

procedure ADMM
  Input: y, λ, ρ, $nb\_iterations$
Output: $x_{nb\_iterations}$
  for $k \in 0...nb\_iterations$ **do**
    $x_{k+1} = (H^T H + \rho I)^{-1}(H^T y_k + \rho z_k - u_k)$
    $z_{k+1} = Soft_{\frac{\lambda}{\rho}}(x_{k+1} + \frac{u_k}{\rho})$
    $u_{k+1} = u_k + \rho(z_{k+1} - x_{k+1})$
  end for
end procedure

Total Variation : $R(x) = \|\nabla x\|_1$

Résolution avec l'ADMM:

- Équation : $\min_{x,z}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1\}$
- Contrainte : $\nabla x = z$

Lagrangien augmenté :

$$\begin{aligned} L(x,z,u) &= \frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + u^T(\nabla x - z) + \frac{\rho}{2}\|\nabla x - z\|_2^2 \\ &= \frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|\nabla x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho} \end{aligned}$$

$\arg\min_x L(x, z, u)$

$$\frac{d}{dx}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|\nabla x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho}\} = 0$$

$$\iff -H^T(y - Hx) + \rho\nabla^T(\nabla x - z + \frac{u}{\rho}) = 0$$

$$\iff -H^Ty + H^THx + \rho\nabla^T\nabla x - \rho\nabla^Tz + \nabla^Tu = 0$$

$$\iff -H^Ty + (H^TH + \rho\nabla^T\nabla)x - \rho\nabla^Tz + \nabla^Tu = 0$$

$$\iff (H^TH + \rho\nabla^T\nabla)x = \rho\nabla^Tz - \nabla^Tu + H^Ty$$

$$\iff (H^TH + \rho\nabla^T\nabla)x = \rho\nabla^T(z - \frac{u}{\rho}) + H^Ty$$

$$\iff x = (H^TH + \rho\nabla^T\nabla)^{-1}[\rho\nabla^T(z - \frac{u}{\rho}) + H^Ty]$$

$\arg\min_{z} L(x, z, u)$

$$\frac{d}{dz}\{\frac{1}{2}\|y - Hx\|_2^2 + \lambda\|z\|_1 + \frac{\rho}{2}\|\nabla x - z + \frac{u}{\rho}\|_2^2 - \frac{\|u\|_2^2}{2\rho}\} = 0$$

$$\iff \lambda sign(z) - \rho(\nabla x - z + \frac{u}{\rho}) = 0$$

$$\iff \lambda sign(z) - \rho\nabla x + \rho z - u = 0$$

$$\iff \lambda sign(z) + \rho z = \rho\nabla x + u$$

$$\iff z + \frac{\lambda}{\rho} sign(z) = \nabla x + \frac{u}{\rho}$$

$$\iff z = sign(\nabla x + \frac{u}{\rho}) \times max(|\nabla x + \frac{u}{\rho}| - \frac{\lambda}{\rho}, 0)$$

$$\iff z = Soft_{\frac{\lambda}{\rho}}(\nabla x + \frac{u}{\rho})$$

Algorithm

Algorithm 3 ADMM-L_1

procedure ADMM
  Input: y, λ, ρ, $nb\_iterations$
Output: $x_{nb\_iterations}$
  for $k \in 0 ... nb\_iterations$ **do**
    $x_{k+1} = (H^T H + \rho \nabla^T \nabla)^{-1} [\rho \nabla^T (z_k - \frac{u_k}{\rho}) + H^T y]$
    $z_{k+1} = Soft_{\frac{\lambda}{\rho}} (\nabla x_{k+1} + \frac{u_k}{\rho})$
    $u_{k+1} = u_k + \rho (z_{k+1} - x_{k+1})$
  end for
end procedure

Split Bregman

Definition

Problème de minimisation

$$\hat{f} = \arg\min_{f} \{ \frac{1}{2} \|g - Hf\|_2^2 + \lambda \|\nabla f\|_1 \}$$

Réécriture de $\|\nabla f\|_1$

En posant,

$$\begin{cases} d_x = \nabla_x f = \dfrac{df}{dx} \\ d_y = \nabla_y f = \dfrac{df}{dy} \\ \|\nabla f\|_1 = \|(d_x, d_y)\|_1 = \displaystyle\sum_i \sqrt{(d_x)_i^2 + (d_y)_i^2} \end{cases}$$

nous avons:

$$\hat{f} = \arg\min_f \{\frac{1}{2}\|g - Hf\|_2^2 + \lambda\|(d_x, d_y)\|_1\}$$

Réécriture de $\|\nabla f\|_1$ (Relaxation)

nous avons,

$$(f, d_x, d_y) = \arg \min_{f, d_x, d_y} \{\frac{1}{2}\|g - Hf\|_2^2 + \lambda\|(d_x, d_y)\|_1 + \frac{\sigma}{2}\|d_x - \nabla_x f\|_2^2 + \frac{\sigma}{2}\|d_y - \nabla_y f\|_2^2\}$$

Introduction du processus itératif

$$(f^{k+1}, d_x^{k+1}, d_y^{k+1}) = \arg\min_{f,d_x,d_y} \{\frac{1}{2}\|g - Hf^k\|_2^2 \lambda \|(d_x^k, d_y^k)\|_1 + \frac{\sigma}{2}\|d_x^k - \nabla_x f^k - b_x^k\|_2^2 + \frac{\sigma}{2}\|d_y^k - \nabla_y f^k - b_y^k\|_2^2\}$$

Avec,

$$\begin{cases} b_x^{k+1} = b_x^k + (\nabla_x f^{k+1} - d_x^{k+1}) \\ b_y^{k+1} = b_y^k + (\nabla_y f^{k+1} - d_y^{k+1}) \end{cases}$$

Division en deux sous-problème

1. Calcul de f
2. Calcul de (d_x, d_y)

Calcul de f

$$
\begin{aligned}
f^{k+1} &= \arg\min_{f}\{\frac{1}{2}\|g - Hf^k\|_2^2 + \lambda\|(d_x^k, d_y^k)\|_1 + \frac{\sigma}{2}\|d_x^k - \nabla_x f^k - b_x^k\|_2^2 + \frac{\sigma}{2}\|d_y^k - \nabla_y f^k - b_y^k\|_2^2\} \\
&= \arg\min_{f}\{\frac{1}{2}\|g - SHf^k\|_2^2 + \frac{\sigma}{2}\|d_x^k - \nabla_x f^k - b_x^k\|_2^2 + \frac{\sigma}{2}\|d_y^k - \nabla_y f^k - b_y^k\|_2^2\}
\end{aligned}
$$

Calcul de f

$$\frac{d}{df}\{\frac{1}{2}\|g - Hf\|_2^2 + \frac{\sigma}{2}\|d_x - \nabla_x f - b_x\|_2^2 + \frac{\sigma}{2}\|d_y - \nabla_y f - b_y\|_2^2\} = 0$$

$$\Longrightarrow \; -H^T(g - Hf) - \sigma\nabla_x^T(d_x - \nabla_x f - b_x) - \sigma\nabla_y^T(d_y - \nabla_y f - b_y) = 0$$

$$\Longleftrightarrow \; -H^Tg + H^THf - \sigma\nabla_x^Td_x + \sigma\nabla_x^T\nabla_x f + \sigma\nabla_x^Tb_x - \sigma\nabla_y^Td_y + \sigma\nabla_y^T\nabla_y f + \sigma\nabla_y^Tb_y = 0$$

$$\Longleftrightarrow \; H^THf + \sigma\nabla_x^T\nabla_x f + \sigma\nabla_y^T\nabla_y f = \sigma\nabla_x^Td_x - \sigma\nabla_x^Tb_x + \sigma\nabla_y^Td_y - \sigma\nabla_y^Tb_y + H^Tg$$

$$\begin{aligned} &H^THf + \sigma\nabla_x^T\nabla_x f + \sigma\nabla_y^T\nabla_y f \\ &= [H^TH + \sigma(\nabla_x^T\nabla_x + \nabla_y^T\nabla_y)]f \\ &= [H^TH + \sigma\Delta]f \end{aligned}$$

$$\begin{aligned} &\sigma\nabla_x^Td_x - \sigma\nabla_x^Tb_x + \sigma\nabla_y^Td_y - \sigma\nabla_y^Tb_y + H^Tg \\ &= \sigma\nabla_x^T(d_x - b_x) + \sigma\nabla_y^T(d_y - b_y) + H^Tg \\ &= \sigma[\nabla_x^T(d_x - b_x) + \nabla_y^T(d_y - b_y)] + H^Tg \end{aligned}$$

Calcul de f

Ainsi, dans le domaine spatial, nous avons :

$$f = [H^T H + \sigma\Delta]^{-1}[\sigma(\nabla_x^T(d_x - b_x) + \nabla_y^T(d_y - b_y)) + H^T g]$$

Calcul de (d_x, d_y)

$$
\begin{aligned}
(d_x^{k+1}, d_y^{k+1}) &= \arg\min_{d_x, d_y}\{\frac{1}{2}\|g - Hf^k\|_2^2 + \lambda\|(d_x^k, d_y^k)\|_1 + \frac{\sigma}{2}\|d_x^k - \nabla_x f^k - b_x^k\|_2^2 + \frac{\sigma}{2}\|d_y^k - \nabla_y f^k - b_y^k\|_2^2\} \\
&= \arg\min_{d_x, d_y}\{\lambda\|(d_x^k, d_y^k)\|_1 + \frac{\sigma}{2}\|d_x^k - \nabla_x f^k - b_x^k\|_2^2 + \frac{\sigma}{2}\|d_y^k - \nabla_y f^k - b_y^k\|_2^2\} \\
&= \arg\min_{d_x, d_y}\{\lambda\|(d_x^k, d_y^k)\|_1 + \frac{\sigma}{2}\|d_x^k - (\nabla_x f^k + b_x^k)\|_2^2 + \frac{\sigma}{2}\|d_y^k - (\nabla_y f^k + b_y^k)\|_2^2\}
\end{aligned}
$$

$$
\begin{cases}
d_x^{k+1} = \max(s^k - \dfrac{\lambda}{\sigma}, 0)\dfrac{s_x^k}{s^k} \\
d_y^{k+1} = \max(s^k - \dfrac{\lambda}{\sigma}, 0)\dfrac{s_y^k}{s^k}
\end{cases}
$$

Avec,

$$
\begin{cases}
s_x^k = \nabla_x f^k + b_x^k \\
s_y^k = \nabla_y f^k + b_y^k \\
s^k = \sqrt{(s_x^k)^2 + (s_y^k)^2}
\end{cases}
$$

The End