# Solution to Problem Set 1: The Armington-Anderson Model

Armen Khederlarian ECO862

1. *Show that the CES utility collapses to CD, Leontief, and linear utilities.*
   Define the CES utility as $u(x,y) = (\alpha x^\rho + (1-\alpha)y^\rho)^{1/\rho}$, where $\sigma = \frac{1}{1-\rho}$.
   $\rho = 1$ then $u$ is linear
   $u(x,y) = \alpha x + (1-\alpha)y$
   $\rho = 0, \sigma \to \infty$ then $u$ is Cobb-Douglas
   First transform utility into logs: $u(x,y) = \frac{\ln(\alpha x^\rho + (1-\alpha)y^\rho}{\rho}$
   Then consider L'Hopital rule: $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$, where $f = \ln(\alpha x^\rho + (1-\alpha)y^\rho$ and $g = \rho, g' = 1$.

$$f' = \frac{d(\alpha x^\rho + (1-\alpha)y^\rho)}{d\rho} \Big/ (\alpha x^\rho + (1-\alpha)y^\rho) = \frac{\alpha x^\rho \ln(x) + (1-\alpha)y^\rho \ln(y)}{\alpha x^\rho + (1-\alpha)y^\rho}$$
$$\lim_{\rho\to 0} f' = \frac{\alpha \ln(x) + (1-\alpha)\ln(y)}{\alpha + 1 - \alpha} = \alpha \ln(x) + (1-\alpha)\ln(y)$$

   where the first equality in the first line uses $\frac{d \log f(x)}{dx} = \frac{f'(x)}{f(x)}$ and the second uses $\frac{da^x}{dx} = a^x \ln(a)$.
   $\rho \to -\infty, \sigma = 0$ then $u$ is Leontief
   Suppose WLOG $x < y$, the we need to show $\lim_{\rho\to -\infty}(\alpha x^\rho + (1-\alpha)y^\rho)^{1/\rho} = x$
   First, let $\rho < 0$, since $x, y \geq 0$, we have that

$$ax^\rho \leq \alpha x^\rho + (1-\alpha)y^\rho \Rightarrow (ax^\rho)^{1/\rho} \geq (\alpha x^\rho + (1-\alpha)y^\rho)^{1/\rho}$$

   Second, because $x < y$, $x^\rho \geq y^\rho$, so that

$$\alpha x^\rho + (1-\alpha)x^\rho \geq \alpha x^\rho + (1-\alpha)y^\rho \Rightarrow (\alpha x^\rho + (1-\alpha)x^\rho)^{1/\rho} \leq (\alpha x^\rho + (1-\alpha)y^\rho)^{1/\rho}$$

   By combining points 1 and 2 we obtain:

$$(\alpha x^\rho + (1-\alpha)x^\rho)^{1/\rho} \leq (\alpha x^\rho + (1-\alpha)y^\rho)^{1/\rho} \leq (ax^\rho)^{1/\rho}$$

   Finally, the limit of $\rho \to -\infty$ of the first and third expression is $x$:

$$\lim_{\rho\to-\infty}(\alpha x^\rho + (1-\alpha)x^\rho)^{1/\rho} = \lim_{\rho\to-\infty}(x^\rho)^{1/\rho} = x$$
$$\lim_{\rho\to-\infty}(\alpha x^\rho)^{1/\rho} = \lim_{\rho\to-\infty}\alpha^{1/\rho}x = x$$

2. *Derive the CES price index we saw in class.*
   The price index is the price of obtaining 1 unit of utility. To derive it we can use the expenditure

minimization problem:

$$\min_{\{q(\omega)\}} \int_{\omega\in\Omega} p(\omega)q(\omega)d\omega \text{ subject to } U = \Big(\int_{\omega\in\Omega} \nu(\omega)^{1/\sigma} q(\omega)^{\frac{\sigma-1}{\sigma}} d\omega\Big)^{\frac{\sigma}{\sigma-1}} \geq 1$$

After setting up the lagrangian and solving for the first order condition we obtain:

$$q(\omega)^{1/\sigma} = \frac{\lambda}{p(\omega}\nu(\omega)\Big(\int_{\omega\in\Omega} \nu(\omega)^{1/\sigma} q(\omega)^{\frac{1}{1-\sigma}} d\omega\Big)^{\frac{\sigma}{\sigma-1}}$$

We can then consider the ratio of relative demand for $\omega, \omega'$ to obtain:

$$\frac{q(\omega)}{q(\omega')} = \frac{\nu(\omega)}{\nu(\omega')}\Big(\frac{p(\omega)}{p(\omega')}\Big)^{-\sigma} \iff q(\omega) = \frac{\nu(\omega)}{\nu(\omega')}\Big(\frac{p(\omega)}{p(\omega')}\Big)^{-\sigma} q(\omega')$$

By plugging this expression into the constraint we obtain:

$$\begin{aligned}
&\int_{\omega\in\Omega} \nu(\omega)^{1/\sigma}\Big[\frac{\nu(\omega)}{\nu(\omega')}\Big(\frac{p(\omega)}{p(\omega')}\Big)^{-\sigma} q(\omega')\Big]^{\frac{\sigma-1}{\sigma}} d\omega = 1\\
&p(\omega')^{\sigma-1}\Big(\frac{q(\omega')}{\nu(\omega')}\Big)^{\frac{\sigma-1}{\sigma}}\Big(\int_{\omega\in\Omega} \nu(\omega)p(\omega)^{1-\sigma}d\omega\Big) = 1\\
\iff &q(\omega')^{\frac{\sigma-1}{\sigma}} = p(\omega')^{1-\sigma}\nu(\omega')^{\frac{\sigma-1}{\sigma}}\Big(\int_{\omega\in\Omega} \nu(\omega)p(\omega)^{1-\sigma}d\omega\Big)^{-1}\\
\iff &q(\omega') = p(\omega')^{-\sigma}\nu(\omega')\Big(\int_{\omega\in\Omega} \nu(\omega)p(\omega)^{1-\sigma}d\omega\Big)^{\frac{\sigma}{1-\sigma}}
\end{aligned}$$

Finally we can plug this expression (for $\omega$) into the objective function:

$$\begin{aligned}
P &= \int_{\omega\in\Omega} p(\omega)q(\omega)d\omega = \int_{\omega\in\Omega} p(\omega)^{1-\sigma}\nu(\omega)\Big(\int_{\omega\in\Omega} \nu(\omega)p(\omega)^{1-\sigma}d\omega\Big)^{\frac{\sigma}{1-\sigma}}\\
&= \Big(\int_{\omega\in\Omega} \nu(\omega)p(\omega)^{1-\sigma}d\omega\Big)^{1+\frac{\sigma}{1-\sigma}} = \Big(\int_{\omega\in\Omega} \nu(\omega)p(\omega)^{1-\sigma}d\omega\Big)^{\frac{1}{1-\sigma}}
\end{aligned}$$

3. *Show that welfare in the Armington(1969)-Anderson(1979) model is increasing trade openness and then show how trade openness depends on $\kappa$.*

   In class we derived the following expression of welfare in terms of domestic expenditure share:

$$U_j = \nu_{jj}^{\frac{1}{\sigma-1}} A_j \lambda_{jj}^{\frac{1}{1-\sigma}}$$

where $\lambda_{jj} = 1 - \sum_{i\neq j} \lambda_{ij}$ and $\sum_{i\neq j} \lambda_{ij}$ is imports over GDP. Given $\sigma > 1$, $dU_j/d\lambda_{jj} = \frac{1}{1-\sigma}\lambda_{jj}^{\frac{\sigma}{1-\sigma}}\nu_{jj}^{\frac{1}{\sigma-1}}A_j < 0$, so that welfare is decreasing in the domestic expenditure share and increasing in the import share. To show the relation between trade openness and $\kappa$ we can just consider the gravity equation:

$$X_{ij}(\omega) = \nu_{ij}\kappa_{ij}^{1-\sigma}\frac{Y_i}{\pi_i^{1-\sigma}}\frac{Y_j}{P_j^{1-\sigma}}$$

The elasticity of $X_{ij}$ to $\kappa$ is $1-\sigma$, so that the import share is decreasing in $\kappa$.

4. *Consider a version of the model without the demand shifter $\nu$ and then consider a move from autarky to a case with trade. Write down the change in welfare. How does the change in welfare depend on the trade elasticity? Why is it problematic to say that welfare changes are increasing or decreasing in the trade elasticity?*

   We think of autarky as $\kappa \to \infty$ and open trade as $1 < \kappa < \infty$. Considering the log change of utility from autarky to trade yields:

$$\log(U_j(\kappa_{ij} < \infty)/U_j(\kappa_{ij} = \infty)) = \frac{1}{1-\sigma}\log(\lambda_{jj(\kappa<\infty)}/\lambda_{jj}(\kappa = \infty)) = \frac{1}{1-\sigma}\log\lambda_{jj(\kappa<\infty)}$$

   The change in welfare decreasing in $\sigma$, i.e. when goods are less substitutable the gains from trade are larger. However, note if $\sigma$ is low, changing the import share, $1-\lambda$, requires larger trade cost changes. In other words, for the same change in trade costs, $\lambda_{jj(\kappa<\infty)}$ is going to be relatively large, thus undermining the gains from trade.

5. *Write a short summary of Anderson and van Wincoop (2003) main research question, methodological approach and key finding (maximum one page). You should be able to do this by reading just the introduction and skimming the paper.*

## References

Anderson, James E., and Eric van Wincoop (2003) 'Gravity with gravitas: A solution to the border puzzle.' *American Economic Review* 93(1), 170–192