Comparison of Trimmed Mean and Least Trimmed Squares based estimators for linear regression

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Problem Statement

Define trimmed mean in the simple linear regression model. Compare the performances between the LTS estimator and your proposed trimmed mean.

Trimmed Mean

- Trimmed Means are robust estimators of the central tendency
- Let $x_{(1)}, x_{(1)}, ..., x_{(n)}$ denote the ordered observations, then the trimmed mean is defined by,

$$T(x_1, x_2, ..., x_n; \alpha) = \frac{1}{n - 2[n\alpha]} \sum_{i=[n\alpha]+1}^{n-[n\alpha]} x_{(i)}$$

  where α is the trimming parameter

- For calculating trimmed mean in linear regression, we use regression quantiles

Regression Quantiles

- Consider the linear model

$$\vec{y} = \vec{\beta}X + \vec{Z}$$

where $\vec{y}^{\mathsf{T}} = [y_1, y_2, ..., y_n]$, X is a nxp matrix of known constants whose i^{th} row is $\vec{x_i}$, $\vec{\beta}^{\mathsf{T}} = [\beta_1, \beta_2, ..., \beta_n]$ is a vector of unknown parameters, and $\vec{Z}^{\mathsf{T}} = [Z_1, Z_2, ..., Z_n]$ is a vector of independent and identically distributed random variables.

- Koenker and Bassett(1978) extended the concept of quantiles to the linear model. Let $0 < \theta < 1$. Define

$$\psi_\theta(x) = \theta - \mathbb{I}(x < 0)$$

$$\rho_\theta = x\psi_\theta(x)$$

Then the θ^{th} regression Quantile is given as

$$\hat{\beta}(\theta) = \underset{\vec{b}}{\text{argmin}} \sum_{i=1}^{n} \rho_\theta(y_i - \vec{x_i}^{\mathsf{T}}\vec{b})$$

Regression Quantiles

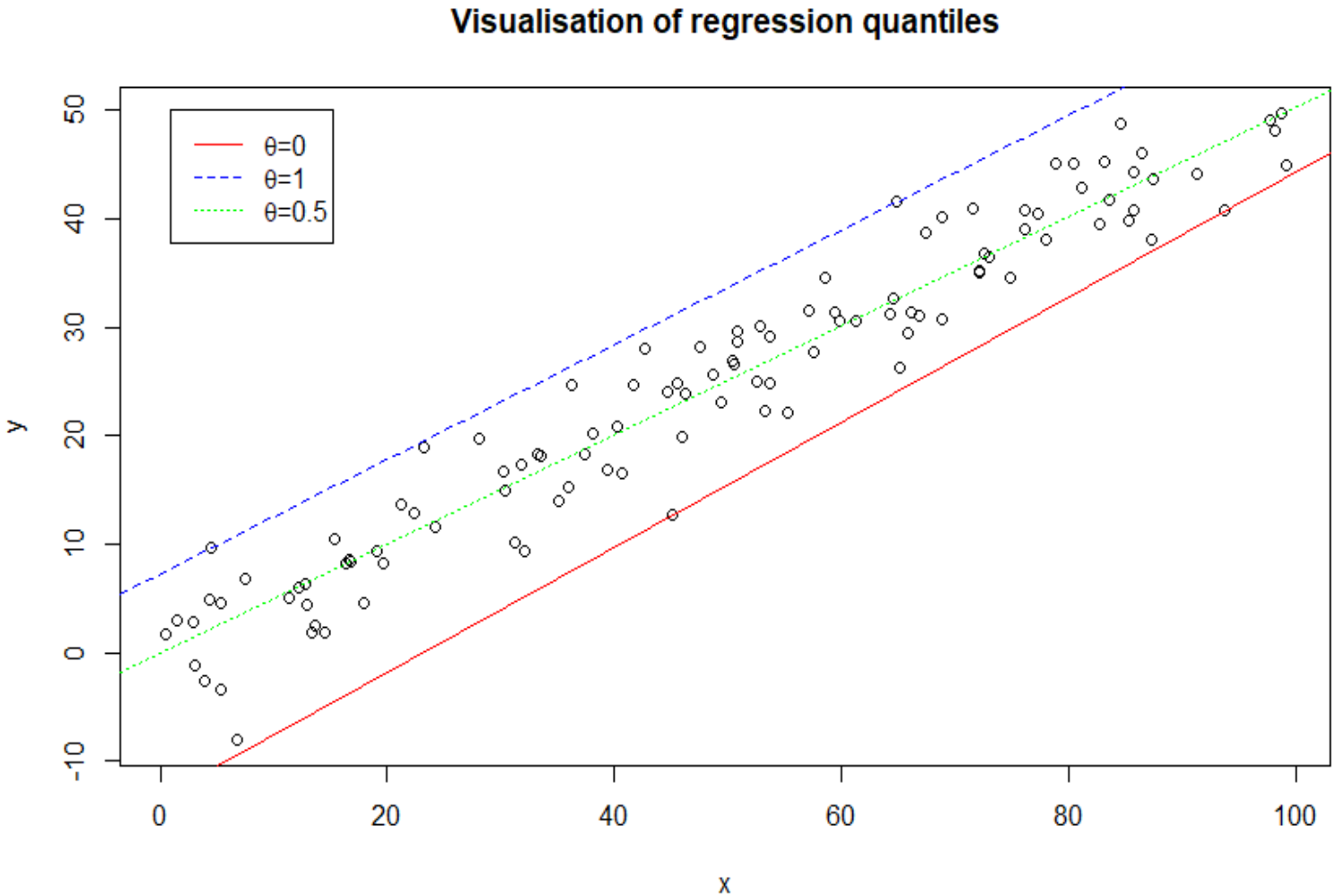
- Note that the above definition can be reduced to a form similar to the definition of multivariate quantiles we saw earlier.i.e

$$\hat{\beta}(\theta) = \underset{\vec{b}}{\text{argmin}} \sum_{i=1}^{n} [|y_i - \vec{x_i}^{\mathsf{T}} \vec{b}| + u(y_i - \vec{x_i}^{\mathsf{T}} \vec{b})]$$

  where $u = (2\theta - 1)$

- $\theta = 0$ and $\theta = 1$ covers the spread of our data
- For $\theta = 0.5$, $\hat{\beta}(\theta)$ gives the LAD regression estimator.

Visualisation of Regression Quantiles



Trimmed Mean for Linear Regression

- Koenker and Bassett proposed the following definition of α trimmed mean for regression and they call it $\hat{\beta}_{KB}$: Remove from the sample any observations whose residual from $\hat{\beta}(\alpha)$ is negative or whose residual from $\hat{\beta}(1-\alpha)$ is positive and calculate the least squares estimator using the remaining observations.

- We are using the above definition of trimmed mean for our purpose.

Computation of Trimmed Mean

- Consider the set of equations

$$y_i - \vec{x_i}^T \vec{\beta}(1-\alpha) \geq 0 \text{ or } y_i - \vec{x_i}^T \vec{\beta}(\alpha) \leq 0$$

- Let $b_i = 0$ or 1 according as i satisfies the above set of equations or not, and let B be the nXn diagonal matrix with $B_{ii} = b_i$. Then

$$\hat{\vec{\beta}}_{KB}(\alpha) = (X^T B X)^{-1}(X^T B \vec{y})$$

where $(X^T B X)^{-1}$ is the inverse of $(X^T B X)$

Least Trimmed Squares

- A method for robust regression that minimises the sum of squared residuals over a subset of data points.
- Let $(y_1, x_1), (y_2, x_2), ..., (y_n, x_n)$ denote the data points with the linear regression model defined as $y_i = \beta x_i + \epsilon_i$ for $i = 1, ..., n$
- Then, $\hat{\beta}^{(LTS)}$, the LTS estimator of β, is defined as

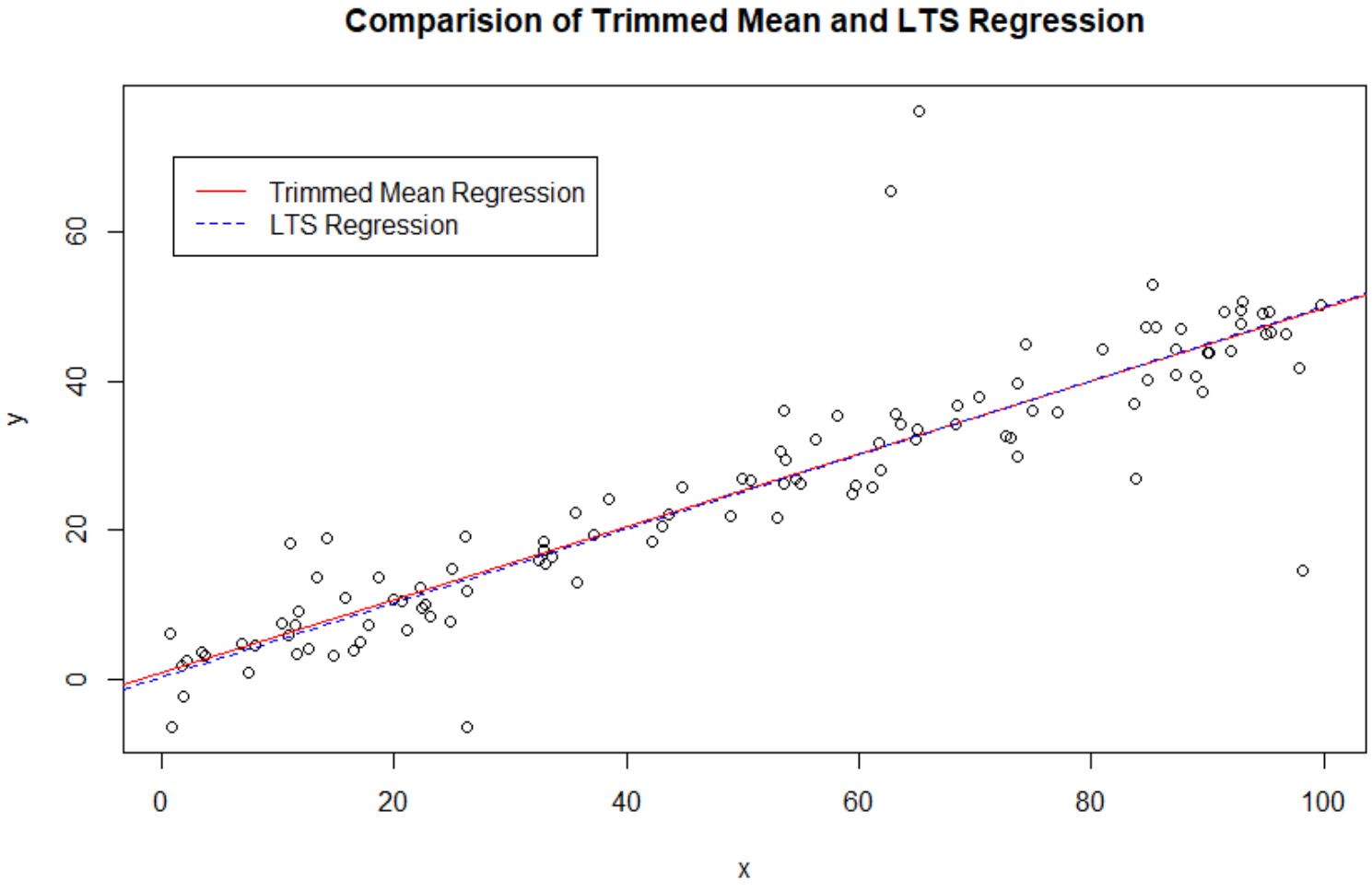
$$\hat{\beta}^{(LTS)} = \underset{\beta}{\text{argmin}} \sum_{i=1}^{h} \epsilon_i^2$$

where $n/2 \leq h \leq n$

Methodology for simulated data

- We chose the Noise distribution to be a mixture of a low(9) and a high variance(400) Gaussians. We added a small proportion of high variance Gaussians to get some outliers in our data.
- We generated the data from the from the process $y_i = 0.5x_i + \epsilon_i$, where x_i is generated from $U[0, 100]$ distribution.
- We fitted a Trimmed Mean and an LTS estimator to the data.
- The parameters of Trimmed Mean and LTS estimator are chosen such that the Mean Squared Error(MSE) is minimised(Best Possible Fit).
- We repeated this process 100 times and calculated the MSE for all the replications.

Results for Simulated Data



- For this particular replication
  MSE(Trimmed Mean) = 58.50234 MSE(LTS) = 58.72007

Results for Simulated Data

- For around 95% replications, the MSE for the Trimmed Mean is better than LTS.
- Although keep in mind that the MSEs for all the replications are very close to each other.
- From the above exercise, we may infer that Trimmed Mean based regression performs slightly better than LTS regression.

Methodology for real data

- **Data:** Our data contains the price series for the closing price of Dow Jones Industrial Average(DJIA) in the period 1980-1995.
- Tried to get the best fit Trimmed Mean and LTS estimators of $\vec{\beta} = [\beta_0, \beta_1]^{\mathsf{T}}$(Let it be $\hat{\vec{\beta}}$). (The one having the minimum MSE)
- Compared the MSEs of both the estimators.
- Also tried to compare their performance using bootstrapping. Generated 100 replications of the data.
- For each replication, we tried fitting both the trimmed mean as well as the LTS estimator. Lets denote them by $\vec{\beta}_i^{1*}$ and $\vec{\beta}_i^{2*}$ respectively.(where i denotes the replication index)

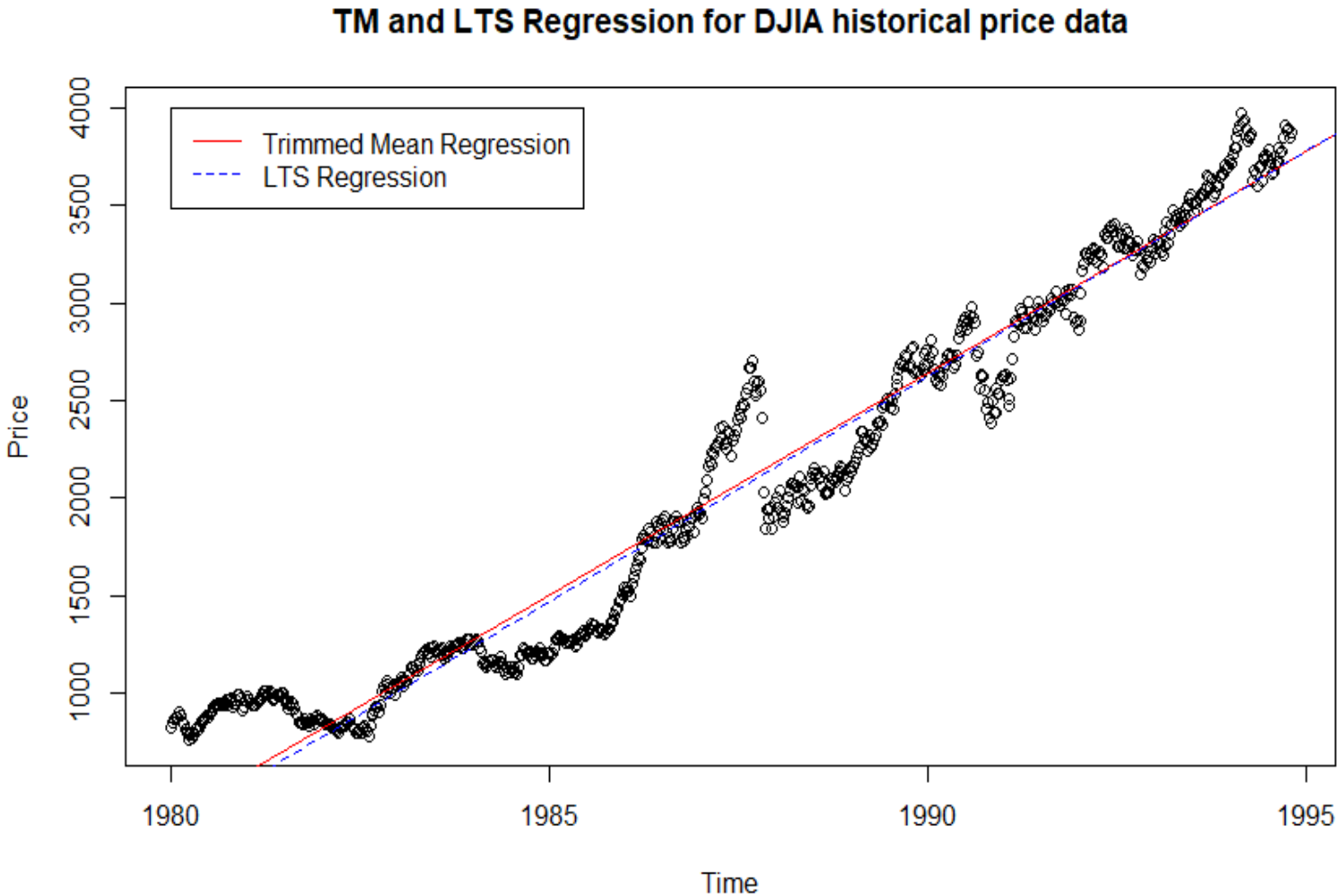
Methodology for real data

- **Method 1 :** We calculate the MSE for both the estimators in all the replications. In the end, we calculate the fraction of times Trimmed Mean MSE is better than the LTS MSE.
- **Method 2 :** We get the values of $\vec{\beta}_i^{1*}$ and $\vec{\beta}_i^{2*}$ for all the replications. Using these values, we calculate the Bootstrap Mean Squared Error.
- The Bootstrap Mean Squared Error is defined as :

$$E_j = \frac{1}{nboot} \sum_{i=1}^{nboot} \|(\hat{\vec{\beta}} - \vec{\beta}_i^{j*})\|^2$$

where nboot = number of replications in bootstrap and j =1,2 represents the trimmed mean and the LTS Bootstrap MSE respectively.

Results for real data



Results for real data

- MSE(Trimmed Mean) =44719.23
- MSE(LTS) = 44715.09
- $MSE(TrimmedMean) > MSE(LTS)$
- For around 70% of the replications, Trimmed Mean Regression had a better MSE than LTS Regression.
- Bootstrap MSE(Trimmed Mean) = 16247482
- Bootstrap MSE(LTS) = 33333897
- $BootstrapMSE(TrimmedMean) < BootstrapMSE(LTS)$
- We may conclude from the above that Trimmed Mean Estimator is better than LTS Regression.

References

- A.H. Welsh.The Trimmed Mean in the Linear Model(1987)
- David Ruppert and Raymond J. Carroll.Trimmed Least Squares Estimation in the Linear Model(1980)
- Roger Koenker, Gilbert Bassett, Jr. Regression Quantiles(1978)
- Peter J. Rousseeuw. Least Median of Squares Regression
- (Data source) https://vincentarelbundock.github.io/Rdatasets/datasets.html

Thank You!