Coupled Heave-Pitch Motions and Froude Krylov Excitation Forces

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1. Coupled Equation of Motion in Heave and Pitch

Once we have set up the simple equation of motion for a vessel in heave it is natural to extend this discussion to the coupled heave-pitch equations of motion. Considering a ship floating on the free surface in waves. This ship will naturally heave and pitch due to the incident waves. It is not guaranteed that these two motions will be independent, however. Thus it becomes necessary to consider the motions together.

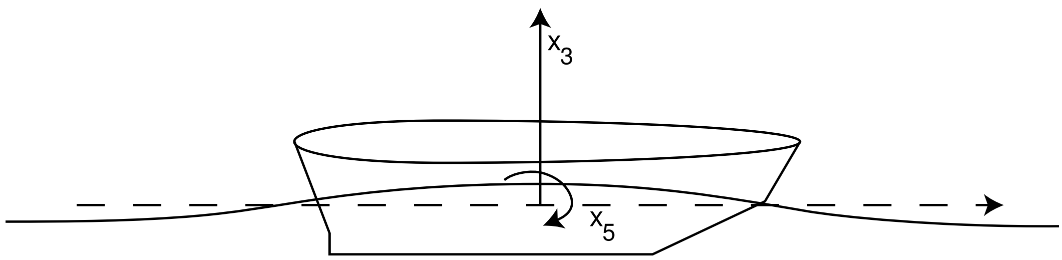


Figure 1. Ship in coupled heave (x_3) and pitch (x_5) motions.

The body boundary condition must be properly specified, thus we need to know both the linear and angular velocities of the vessel:

$$\vec{V}_B = \frac{dx_3}{dt}\,\widehat{k} + \vec{r} \times \left(\frac{dx_5}{dt}\widehat{j}\right) = (z\,\dot{x}_5,\;\; \dot{x}_3 - x\,\dot{x}_5) \tag{1.1}$$

It follows that the velocity normal to the vessel hull is

$$\vec{V}_B \cdot \widehat{n} = n_x\,z\,\dot{x}_5 + n_z(\dot{x}_3 - x\,\dot{x}_5) = n_z\,\dot{x}_3 + (n_x\,z - n_z\,x)\dot{x}_5 \tag{1.2}$$

where the position vector, $\vec{r}$, crossed with the unit outward normal, $\widehat{n}$, is

$$\begin{pmatrix} n_4 \\ n_5 \\ n_6 \end{pmatrix} = \vec{r} \times \widehat{n} \tag{1.3}$$

In general the components of the wave total potential function do not change significantly from our previous cases. It now is necessary, however, to account for the pitching motion in the radiation component of the total potential,

$$\phi(x,z,t) = \phi_I(x,z,t) + \phi_D(x,z,t) + \phi_R(x,z,t) \tag{1.4}$$

$$= Re\left\{a\left[\widehat{\phi}_I(x,z) + \widehat{\phi}_D(x,z)\right]e^{i\omega t} + \widehat{x}_3\widehat{\phi}_3(x,z)e^{i\omega t} + \widehat{x}_5\widehat{\phi}_5(x,z)e^{i\omega t}\right\}, \tag{1.5}$$

where the motions in the heave (3) and pitch (5) directions are

$$x_3(t) = Re\left\{\widehat{x}_3 e^{i\omega t}\right\} \tag{1.6}$$

$$x_5(t) = Re\left\{\widehat{x}_5 e^{i\omega t}\right\}. \tag{1.7}$$

The boundary conditions for the potentials, $\widehat{\phi}_I$, $\widehat{\phi}_D$, and $\widehat{\phi}_3$ are the same as in the pure heave case. We must now, however, consider also the boundary conditions for the radiation potential due to pitching motions, $\widehat{\phi}_5$. Following the same formula from the previous reading, the equation of motion and boundary conditions at the free surface, sea floor and on the body are:

(1) $\nabla^2\widehat{\phi}_5 = 0$

(2) $-\omega^2\widehat{\phi}_5 + g\frac{\partial\widehat{\phi}_5}{\partial z} = 0$

(3) $\frac{\partial\widehat{\phi}_5}{\partial z} = 0$ on $z = -H$

(4) $\frac{\partial\widehat{\phi}_5}{\partial n} = n_5$; on the body

In addition to the force in the vertical direction, we also need to formulate the pitching moment (about the y-axis) acting around the center of gravity. First we can re-write the vertical force dependent on the two motions as:

$$
\begin{aligned}
(1.8) \qquad F_3(t) &= \int\int_{\overline{S}} \rho\, n_z \frac{\partial \phi_T}{\partial t} ds \\
(1.9) \qquad &= Re\left\{ \int\int_{\overline{S}} i\, \rho\, \omega\, e^{i\omega t} n_z \left(a \left[\widehat{\phi}_I + \widehat{\phi}_D \right] + \widehat{x}_3 \widehat{\phi}_3 + \widehat{x}_5 \widehat{\phi}_5 \right) ds \right\} \\
(1.10) \qquad &= Re\left\{ a\, e^{i\omega t} \left[\widehat{F}_{I3} + \widehat{F}_{D3} \right] + \widehat{x}_3 e^{i\omega t} \left(\widehat{F}_{33} \right) + \widehat{x}_5 e^{i\omega t} \left(\widehat{F}_{35} \right) \right\} \\
(1.11) \qquad &= Re\left\{ a\, e^{i\omega t} \left[\widehat{F}_{I3} + \widehat{F}_{D3} \right] + \widehat{x}_3 e^{i\omega t} \left(\omega^2 A_{33} - i\omega B_{33} \right) + \widehat{x}_5 e^{i\omega t} \left(\omega^2 A_{35} - i\omega B_{35} \right) \right\}
\end{aligned}
$$

Next, we can find F_5, the moment on the body due to pitching and heaving motion.

$$
M_2 = F_5 = \int\int_{\overline{S}} \rho \frac{\partial \phi}{\partial t} (\vec{r} \times \widehat{n}) dS \tag{1.12}
$$

where

$$
\begin{aligned}
(1.13) \qquad F_5(t) &= \int\int_{\overline{S}} \rho \frac{\partial \phi_T}{\partial t} n_5\, dS \\
(1.14) \qquad &= Re\left\{ a\, e^{i\omega t} \left[\widehat{F}_{I5} + \widehat{F}_{D5} \right] + \widehat{x}_3 e^{i\omega t} \left(\widehat{F}_{53} \right) + \widehat{x}_5 e^{i\omega t} \left(\widehat{F}_{55} \right) \right\} \\
(1.15) \qquad &= Re\left\{ a\, e^{i\omega t} \left[\widehat{F}_{I5} + \widehat{F}_{D5} \right] + \widehat{x}_3 e^{i\omega t} \left(\omega^2 A_{53} - i\omega B_{53} \right) + \widehat{x}_5 e^{i\omega t} \left(\omega^2 A_{55} - i\omega B_{55} \right) \right\}
\end{aligned}
$$

The hydrostatic forces in the two directions are

$$
\begin{aligned}
(1.16) \qquad F_{3h} &= -C_{33}\, x_3 - C_{35}\, x_5 \\
(1.17) \qquad F_{5h} &= -C_{53}\, x_3 - C_{33}\, x_5
\end{aligned}
$$

where the coefficients, C_{ij}, are based on the vessel geometry.

Finally, taking the pitch motion about the center of gravity (usually the c.g. is near midships), where $x_{cg} = 0$, we can write the complete, coupled equations of motions in heave and pitch

$$m\,\frac{d^2 x_3}{dt^2} = F_3 + F_{3h} \tag{1.18}$$

$$I\,\frac{d^2 x_5}{dt^2} = F_5 + F_{5h} \tag{1.19}$$

Substituting in for F_3 and F_5 with equations (1.11) and (1.15) respectively, and the appropriate hydrostatic forcing, the system of equations can now be written in matrix form assuming a harmonic input and an LTI system:

$$\begin{bmatrix} m + A_{33} & A_{35} \\ A_{53} & I + A_{55} \end{bmatrix} \begin{bmatrix} \ddot{x}_3 \\ \ddot{x}_5 \end{bmatrix} + \begin{bmatrix} B_{33} & B_{35} \\ B_{53} & B_{55} \end{bmatrix} \begin{bmatrix} \dot{x}_3 \\ \dot{x}_5 \end{bmatrix} + \begin{bmatrix} C_{33} & C_{35} \\ C_{53} & C_{55} \end{bmatrix} \begin{bmatrix} x_3 \\ x_5 \end{bmatrix} = a \begin{bmatrix} \widehat{F}_{I3} + \widehat{F}_{D3} \\ \widehat{F}_{I5} + \widehat{F}_{D5} \end{bmatrix} e^{i\omega t}$$

Simply expressed in matrix notation, the system of equations for the coupled pitch and heave motions for a freely floating body can be rewritten as

$$[M + A]\,\underline{\ddot{x}} + [B]\,\underline{\dot{x}} + [C]\,\underline{x} = a\,\underline{\widehat{F}}\,e^{i\omega t} \tag{1.20}$$

where the vector motion is comprised of the linear heave motion, x_3, and the angular pitch motion, x_5.

$$\underline{x} = Re \left\{ \begin{pmatrix} \widehat{x}_3 \\ \widehat{x}_5 \end{pmatrix} e^{i\omega t} \right\} \tag{1.21}$$

Writing the equation in matrix form allows us to better determine the amplitude response function (transfer function) between the forcing and the ship motions:

$$\underline{\widehat{x}} = \frac{a}{\{-\omega^2\,[M + A] + i\omega\,[B] + [C]\}}\;\underline{\widehat{F}} \tag{1.22}$$

Note! In the problem given for homework, the buoy is upright and surge and pitch are coupled motions. Following the discussion above you should be able to derive the coupled surge-pitch equations of motion. It can be assumed that the heave motion for the spar buoy is uncoupled, for the most part, from the other two motions.

2. Determination of Excitation Forces

2.1. **Approximation of Diffraction Forces.** Recalling from the last reading, the total potential is simply a linear superposition of the incident, diffraction, and radiation potentials:

$$\phi = (\phi_I + \phi_D + \phi_R)\ e^{i\omega t}. \tag{2.1}$$

The **radiation** potential is comprised of six components due to the motions in the six directions, ϕ_j where $j = 1, 2, 3, 4, 5, 6$. Each function ϕ_j is the potential resulting from a unit motion in j^{th} direction for a body floating in a quiescent fluid. The resulting body boundary condition follows for both linear (eq. 2.2) and angular motions (eq. 2.3):

$$\frac{\partial \phi_j}{\partial n} = i\omega n_j;\ (j = 1, 2, 3) \tag{2.2}$$

$$\frac{\partial \phi_j}{\partial n} = i\omega(\vec{r} \times \widehat{n})_{j-3};\ (j = 4, 5, 6) \tag{2.3}$$

where the position vector away from the center of rotation (usually the origin $(x, y, z) = (0, 0, 0)$) is $\vec{r} = (x, y, z)$ and the outward surface normal vector is $\widehat{n} = n_j$ for $(j = 1, 2, 3) = (n_x, n_y, n_z)$. In order to meet all the boundary conditions these must radiate away from the body. Thus

$$\phi_j \propto e^{\mp ikx};\ \text{as } x \ \rightarrow\ \pm\infty \tag{2.4}$$

For the **diffraction** problem we know that the derivative of the *total* potential in the absence of any motion causing wave radiation, (i.e. take the incident potential plus the diffraction potential without consideration of the radiation potential), normal to the body surface must be zero on the body: $\frac{\partial \phi_T}{\partial n} = 0$ on S_B, where $\phi_T = \phi_I + \phi_D$. Therefore

$$\frac{\partial \phi_I}{\partial n} = -\frac{\partial \phi_D}{\partial n};\ \text{on } S_B \tag{2.5}$$

We have so far talked primarily about the incident potential. The formulation of the incident potential is straight forward from the boundary value problem (BVP) setup in lecture 15. There exist several viable forms of this incident potential function, each are essentially a phase shifted version of another. The diffraction potential can also be found in the same fashion using the BVP for the diffraction potential with the appropriate boundary condition on the body. This potential can be approximated for a *long wave* condition. This long wave approximation assumes that the incident wavelength is very long compared to the body diameter and thus the induced velocity field from the incident waves on the structure can be assumed constant over the body and approximated by the following potential function:

$$\phi_D \simeq \frac{i}{\omega}\left[\frac{\partial \phi_I}{\partial x}\ \phi_1\ +\ \frac{\partial \phi_I}{\partial y}\ \phi_2\ +\ \frac{\partial \phi_I}{\partial z}\ \phi_3\right] \tag{2.6}$$

The details of this are beyond the scope of this course, but it is useful to have a general concept of how to approximate this diffraction potential. Further explanation of this approximation can be found in Newman (p. 301).

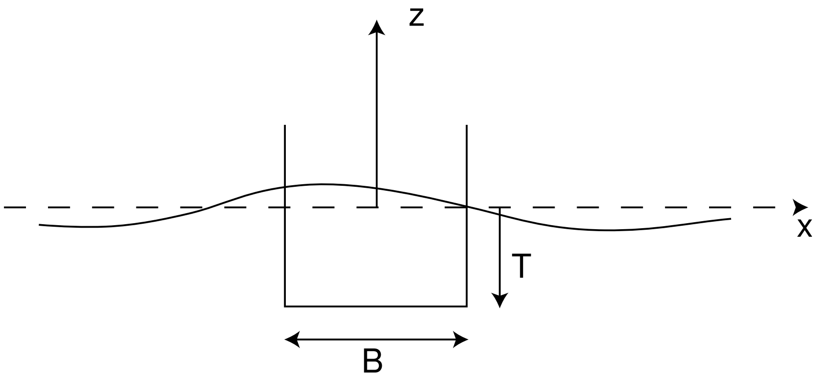
Ultimately, if we assume the body to be sufficiently small as not to affect the pressure field due to an incident wave, then the diffraction effects can be completely ignored. This assumption comes from the *Froude-Krylov hypothesis* and assures us that the resulting excitation force equivalent to the Froude-Krylov force:

$$F^{FK}(t) = -\rho \int\int \frac{\partial \phi_I}{\partial t}\widehat{n}\ dS \tag{2.7}$$

2.2. **Froude Krylov Forces.**

2.2.1. *Vertical Froude-Krylov Force on a Single Hull Vessel.* Given the familiar form of the deep water incident wave potential:

$$\phi_I = \frac{a\omega}{k}\ e^{kz}\ Re\left\{i\ e^{i(\omega t - kx)}\right\} \tag{2.8}$$



We can find the force in the vertical direction by integrating eq. 2.7 around the bottom of the vessel. Here the normal in the z-direction, n_z, is negative, thus $n_z = -1$, and the force per unit length in the z-direction is

$$F_z^{FK} = Re\left\{\int_{-B/2}^{B/2} -\rho\ i\ \omega\ \frac{ia\omega}{k}\ e^{-kT}\ e^{i(\omega t - kx)}\ dx\right\} \tag{2.9}$$

$$= Re\left\{\frac{\rho o \omega^2}{k^2}\ a\ e^{-kT}\ e^{i\omega t}\ \left(e^{-ikB/2} - e^{ikB/2}\right)\right\} \tag{2.10}$$

$$= Re\left\{2\rho\ \frac{\omega^2}{k^2}\ a\ e^{-kT}\ e^{i\omega t}\ \sin(kB/2)\right\} \tag{2.11}$$

Recall that $sin\ z = \frac{e^{iz} - e^{-iz}}{2i}$.

Using the vertical velocity for the fluid under the incident wave, we can rewrite the force in terms of the velocity or acceleration at the vessel/buoy centerline at the free surface:

$$w(t) = Re\left\{a\omega\ e^{kz}\ i\ e^{i(\omega t - kx)}\right\} \tag{2.12}$$

$$\dot{w}(t) = Re\left\{-a\omega^2\ e^{kz}\ e^{i(\omega t - kx)}\right\} \tag{2.13}$$

$$\dot{w}(x = 0, z = 0, t) = Re\left\{a\ \omega^2\ e^{i\omega t}\right\}. \tag{2.14}$$

Such that the force per unit length, in the vertical direction, as a function of the vertical (heave) acceleration,

$$F_z^{FK} = Re\left\{\frac{2\rho}{k^2}\ e^{-kT}\ sin(kB/2)\ \dot{w}(0,0,t)\right\}. \tag{2.15}$$

For the case where $\omega \to 0$ (i.e. low frequency and long wavelength waves) the wavenumber, $k = \omega^2/g$, also goes to zero and the following simplifications can be made:

$$e^{kT} \simeq 1 - kT \tag{2.16}$$

$$\sin(kB/2) \simeq kB/2 \tag{2.17}$$

to yield a simplified heave force. The vertical Froude Krylov force (per unit length) on the body using the long wavelength approximation is then

$$F_z^{FK} \simeq Re\left\{2\rho\,\frac{\omega^2}{k^2}\,a\,(1-kT)\,(kB/2)\,e^{i\omega t}\right\} \tag{2.18}$$

$$\simeq Re\left\{\rho\,g\,aB\,\left(1-\frac{\omega^2}{g}T\right)\,e^{i\omega t}\right\} \tag{2.19}$$

Now considering the heave restoring coefficient (per unit length), $C_{33} = \rho\,g\,B$, and the free surface elevation, $\eta(x,t) = Re\left\{a\,e^{i(\omega t - kx)}\right\}$ we can rewrite this force (per unit length), for long wavelength waves, as

$$F_z^{FK} \simeq Re\left\{C_{33}\,\eta(x=0,t)\right\} \tag{2.20}$$

2.2.2. *Horizontal Froude-Krylov Force on a Single Hull Vessel.* The horizontal force on the vessel above can be found in a similar fashion to the vertical force.

$$F_x = \int\!\!\int_{S_B} \rho\frac{\partial \phi_I}{\partial t}\,n_x\,dS \tag{2.21}$$

$$= Re\left\{\rho\,i\,\omega\,\frac{i\omega a}{k}\int_{-T}^{0} e^{kz}\,dz\left[e^{i(\omega t - kB/2)} - e^{i(\omega t + kB/2)}\right]\right\} \tag{2.22}$$

$$= Re\left\{i\rho\frac{a\omega^2}{k}\left(1-e^{-kT}\right)\,e^{i\omega t}\,2\,\sin(kB/2)\right\} \tag{2.23}$$

For low frequency waves (long wavelength) similar simplifications can be made like above for the vertical force:

$$F_x(t) \simeq Re\left\{i\rho\frac{a\omega^2}{k}\ (KT)\ e^{i\omega t}\ 2\ kB/2\right\} \tag{2.24}$$

$$u(t) = Re\left\{a\omega\ e^{kz}\ e^{i(\omega t-kx)}\right\} \tag{2.25}$$

$$\dot{u}(t) = Re\left\{i\ a\ \omega^2\ e^{kz}\ e^{i(\omega t-kx)}\right\} \tag{2.26}$$

$$F_x(t) \simeq Re\left\{\rho TB\ \dot{u}(x=0, z=0, t)\right\} \tag{2.27}$$

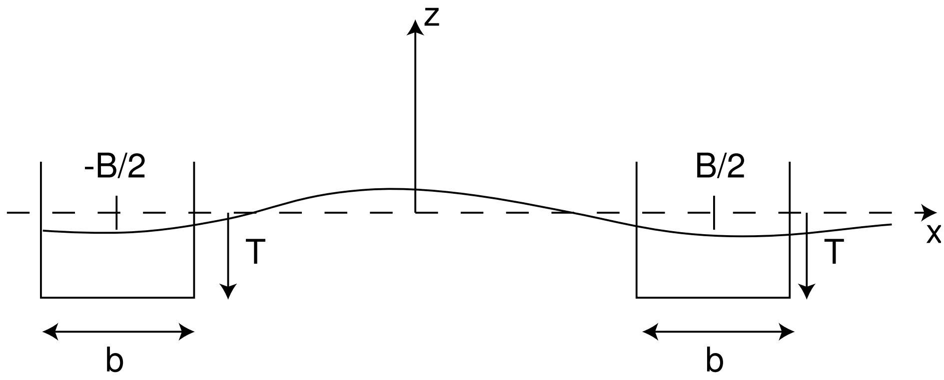
Where $\rho TB = \rho\forall$, and $\forall$ is the vessel volume per unit length such that we are left with the surge force (per unit length)

$$F_x \simeq \rho\forall\ \dot{u} \tag{2.28}$$

From the last section the horizontal force is

$$F_z \simeq C_{33}\eta + \rho\forall\dot{w} \tag{2.29}$$

yielding two components of force acting on the buoy.



2.2.3. *Multi Hulled Vessel.* One important configuration is a multi-hulled vessel or platform. Here the multiple pontoons may have some effect on each other, especially if diffraction is an issue. In order to simplify the problem, and to determine at the force acting on the vessel in the x-direction, a few basic assumptions about the waves and the structure are made: $(b/\lambda << 1)$, $(B/\lambda \sim 1)$, $(a < b)$, and $(b \sim T)$. So the Froude-Krylov force in the surge direction, for an incident wave train with free surface elevation $\eta(x,t) = a\cos(\omega t - kx)$ is:

$$F_x^{FK} \simeq \rho\ bT\ \dot{u}(x = -B/2, z = 0, t) + \rho\ bT\ \dot{u}(x = B/2, z = 0, t) \tag{2.30}$$

where

$$\dot{u}(x,z,t) = -a\ \omega^2\ e^{kz}\ \sin(\omega t - kx). \tag{2.31}$$

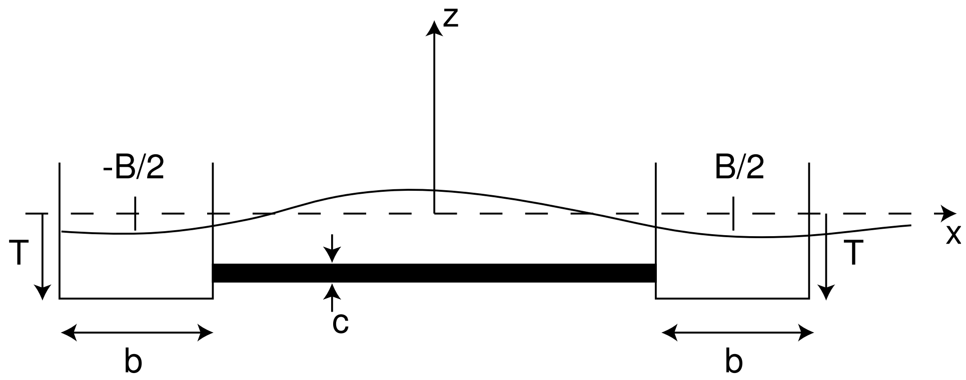
This is the contribution to the x-force from both pontoons. Simplifying the equation using trigonometry identities we get the force per unit length:

$$F_x^{FK} \simeq \rho bT\ (-a\ \omega^2)\ \{\sin(\omega t + kB/2)\ +\ \sin(\omega t - kB/2)\} \tag{2.32}$$

$$\simeq -2\rho\ b\ T\ (a\ \omega^2)\ \cos(kB/2)\ \sin(\omega t) \tag{2.33}$$

Note that when $kB/2 = \pi/2$ (or $B = \lambda/2$) then $F_x^{FK}(t) = 0$. This indicates that an optimal design exists that can be used to minimize the surge force on the vessel. This is *very* useful to keep in mind when designing a structure.

2.2.4. *Multi Hulled Vessel with additional pontoon.* Using the same assumptions from above we can find the x-force acting on the twin-hulled vessel adjusted for the additional pontoon between the two hulls. Since the pontoon is fully submerged it does not affect the hydrostatic restoring force but does effect the force in the



surge and heave directions. In surge the force per unit length becomes:

$$\begin{aligned} F_x^{FK} \quad &\simeq \quad -2\ \rho\ b\ T\ (a\ \omega^2)\ \cos(kB/2)\ \sin(\omega t) \\ &+c\ p(x=-B/2+b/2, z=0,t) \\ &-c\ p(x=B/2-b/2, z=0,t) \end{aligned}$$

The last two terms are the adjustment to the force for the addition of the pontoon, $\delta F_x^{FK}(t)$. Pressure is found from the incident potential: $p(x,z,t) = \rho g\ a\ e^{kz}\ \cos(\omega t - kx)$, such that the incremental force due to the presence of the pontoon is

(2.34)
$$\delta F_x^{FK} = -2\ c\ \rho\ g\ a\ \sin(\omega t)\ \sin\left(\frac{k}{2}(B-b)\right)$$

For deep water waves with long wavelength incident on a multi-hulled structure with a connecting pontoon where $B >> b$, we get a force per unit length acting in the x-direction equivalent to:

(2.35)
$$F_x^{FK}(t) \simeq -2\ \rho\ a\ \omega^2\ \sin(\omega t)\,\{bT\cos(kB/2) + \delta/2\ \sin(kB/2)\}$$

This analysis can be similarly extended to the heave and pitch motions.