Assignment 5 (Dimensionality Reduction)

1. Finding CUR of a small matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}$$

Answer: Picking column 0, 1 and row 1, 2 we get W matrix as:

$W = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}$ and we know that matrix W can be decomposed as, $W = X\Sigma Y^T$ [SVD decomposition] where X and Y are orthonormal matrices ($XX^T = I$ and $YY^T = I$) and Σ is diagonal matrix.

$$W^TW = (X\Sigma Y^T)^T \times (X\Sigma Y^T)$$
$$W^TW = Y\Sigma^T X^T X\Sigma Y^T$$
$$W^TW = Y\Sigma^T \Sigma Y^T \dots\dots\dots\dots (1)$$
$$WY = X\Sigma \dots\dots\dots\dots\dots\dots . (2)$$

Using equation (1) and (2) we can find SVD decomposition of matrix W and hence can find matrix U

$$Where,\ U = W^+ = Y\,\Sigma^+ X^T$$

In equation to $Y^T = Y^{-1}$ since the matrix is orthonormal and hence taken to LHS of the equation.

$$W^TW = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \times \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 18 & 24 \\ 24 & 32 \end{bmatrix}$$

$$Det(W^TW - I) = \begin{vmatrix} 18 - \lambda & 24 \\ 24 & 32 - \lambda \end{vmatrix} = 0$$
$$\lambda^2 - 50\lambda = 0$$
$$\lambda = 0, 50$$
$$\Rightarrow (W^TW - 0 \times I)Y_1 = 0\ [Finding\ Eigen\ Vectors]$$
$$\begin{bmatrix} 18 & 24 \\ 24 & 32 \end{bmatrix} Y_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}\ and\ similarly, Y_2 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

$$\Rightarrow Y = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}\ and\ \Sigma = \begin{bmatrix} 0 & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}\ [Using\ Equation\ (1)]$$

$$Also, \quad \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix} = X\Sigma\ [Using\ equ\ (2)]$$

$$Or, \quad \begin{bmatrix} 0 & 5 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}\ [\ Hence, SVD\ decomposition\ obtained]$$

$$\Sigma^+ = \begin{bmatrix} 0 & 0 \\ 0 & 1/5\sqrt{2} \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 1/5\sqrt{2} \end{bmatrix}\ [Pseudo\ Inverse\ of\ \Sigma]$$

$$U = W^+ = Y\Sigma^+ X^T$$

$$U = \begin{bmatrix} -{}^{4}/_{5} & {}^{3}/_{5} \\ {}^{3}/_{5} & {}^{4}/_{5} \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ 0 & {}^{1}/_{5\sqrt{2}} \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ {}^{1}/_{\sqrt{2}} & {}^{1}/_{\sqrt{2}} \end{bmatrix}$$

$$U = \begin{bmatrix} {}^{3}/_{50} & {}^{3}/_{50} \\ {}^{2}/_{25} & {}^{2}/_{25} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \\ 3 & 3 \\ 4 & 4 \\ 5 & 5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \; and \; R = \begin{bmatrix} 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \end{bmatrix}$$

$$CUR = \begin{bmatrix} 1 & 1 \\ 3 & 3 \\ 4 & 4 \\ 5 & 5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} {}^{3}/_{50} & {}^{3}/_{50} \\ {}^{2}/_{25} & {}^{2}/_{25} \end{bmatrix} \times \begin{bmatrix} 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \end{bmatrix}$$