1^{st} Topic

Laplace Transforms

- **Introduction,**
- **Applications of Laplace Transform,**
- **Definitions of Laplace Transform,**
- **Laplace Transformation Operator,**
- **Existence of Laplace Transforms,**
- **Laplace Transforms of some Elementary Functions**

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Introduction:

A **transformation** is a **mathematical device**, which converts one function into another.

For example, when a differential operator $D\left(\equiv \frac{d}{dx}\right)$ operates on $f(x) = \sin x$, it gives a new function $g(x) = D[f(x)] = \cos x$.

Laplace transform or Laplace transformation is widely used by scientists and engineers. It is particularly effective in **solving linear differential equations**- ordinary as well as partial. It reduces an ordinary differential equation into an algebraic equation.

Laplace transform directly gives the solution of differential equations with given initial conditions without the necessity of first finding the general solution and then evaluating the arbitrary constants.

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Applications of Laplace Transform:

Laplace transform is very useful in obtaining solution of linear differential equations, both ordinary and partial, solution of system of simultaneous differential equations, solution of integral equations, solution of linear difference equations and in the evaluation of definite integrals.

Advantages of Laplace Transform:

1. With the application of Laplace transform, particular solution of **differential equation** is obtained directly without the necessity of first determining general solution and then obtaining the particular solution (by substitution of initial conditions).
2. Laplace transform solves **non-homogeneous differential equation** without the necessity of first solving the corresponding homogeneous differential equation.
3. Laplace transform is applicable not only to continuous function but also the **piecewise continuous** functions, complicated **periodic** functions, **step** function and **impulse** functions.
4. Laplace transform of various functions are readily available (in tabulated form).

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LAPLACE TRANSFORM:

DEFINITION:

Let f(t) be a function of t defined for all positive values of t, i.e. $0 \le t < \infty$.

Then, the Laplace transforms of f(t), denoted by $L\{f(t)\}$ is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt,$$

for all 's', such that this integral converges.

Thus, the operation of multiplying $f(t)$ by e^{-st} and integrating from 0 to ∞ is called **Laplace transformation.**

The parameter 's' is a real or complex number. In general, the parameter 's' is taken to be real positive number.

Now, since $L\{f(t)\}$ is a function of 's', then it can be briefly written as $\bar{f}(s)$,

i.e. $L\{f(t)\} = \bar{f}(s)$.

Sometimes, we use symbol p for the parameter s.

Thus, $L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\,dt = \bar{f}(p)$.

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Laplace transformation operator::

The symbol L, which transforms f(t) into $\bar{f}(s)$, is called the **Laplace transformation operator**.

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Notation:

In various textbooks, authors follow two types of notations:

(i). **Functions** are denoted by lower case letter $f(t), g(t), h(t), \ldots\ldots\ldots$.

and their **Laplace transforms** are denoted by

$\bar{f}(s), \bar{g}(s), \bar{h}(s), \ldots\ldots\ldots$ respectively or by $\bar{f}(p), \bar{g}(p), \bar{h}(p), \ldots\ldots\ldots$.

(ii). **Functions** are denoted by capital letters $F(t), G(t), H(t), \ldots\ldots\ldots$

and their **Laplace transforms** are denoted by corresponding lower case letters

$f(s), g(s), h(s), \ldots\ldots\ldots$ respectively or by $f(p), g(p), h(p), \ldots\ldots\ldots$.

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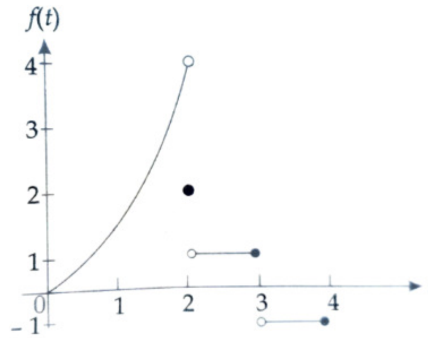
Existence of Laplace transforms:

The Laplace transforms is said to be exist, if the integral $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ is **convergent** for some values of 's'.

Otherwise, we may use the following theorem:

Sufficient conditions for the existence of Laplace Transform of f(t):



Let f(t) be a piecewise continuous on $0 \le t < \infty$.

This means that f is continuous on any finite interval [a, b] except perhaps at finitely many points in [a, b] at which the function f(t) has finite jumps.

For example, let $f(t) = \begin{cases} t^2, & 0 \le t < 2 \\ 2, & t = 2 \\ 1, & 2 < t \le 3 \\ -1, & 3 < t \le 4 \end{cases}$

Here f is continuous on the interval [0, 4] except at 2 and 3, where f has finite discontinuities. A graph of this function is shown above.

If function f(t) is piecewise continuous on $[0, \tau]$, then $e^{-st} f(t)$ is piecewise continuous and hence the integral $\int_0^\tau e^{-st} f(t)\,dt$ exists.

But the existence of integral $\int_0^\tau e^{-st} f(t)\,dt$ for every $\tau > 0$ does not necessarily ensure the existence of the integral $\int_0^\tau e^{-st} f(t)\,dt$ as $\tau \to \infty$.

Example: $f(t) = e^{t^2}$ is continuous on every interval [0, k] for some finite k, but $\int_0^\infty e^{-st} e^{t^2}\,dt$ diverges for every real value of s.

Thus for convergence of integral $\int_0^\infty e^{-st} f(t)\,dt$, we need some further condition on f(t).

The form of integral suggests that, if f(t) is of exponential order α, i.e. if there exist some constants α and M >0 such that $|f(t)| \le Me^{\alpha t}$

which implies $e^{-st}|f(t)| \le Me^{(\alpha-s)t}$ for $t \ge 0$.

Further, since the integral $\int_0^\infty Me^{(\alpha-s)t}dt$ converges to $\frac{M}{s-\alpha}$, then by comparison test, the integral $\int_0^\infty e^{-st}|f(t)|dt$ converges, and hence, the integral $\int_0^\infty e^{-st}f(t)dt$ converges for $s > \alpha$.

This ensures the existence of the Laplace transform for f(x).

Thus we have the following result:

If f(t) is a piecewise continuous function on the interval $[0,\infty)$ and is of exponential order α for $t \ge 0$, then Laplace transform of f(t) exists for $s > \alpha$.

Geometrically, the condition of exponential order α means that the graph of f(t), $t \ge 0$ does not grow faster than the graph of the exponential function $g(t) = Me^{\alpha t}, t \ge 0$.

Many functions, e.g., sin at, cos at, e^{at}, etc satisfy this condition.

Also, we note that the conditions stated above are sufficient but not necessary for a function to have a Laplace transform.

Remarks:

It should be noted that, the above conditions are **sufficient** rather than **necessary**.

For example, $L\left(\frac{1}{\sqrt{t}}\right)$ exists, though $\frac{1}{\sqrt{t}}$ is infinite at t = 0 and hence is not piecewise continuous on interval $[0,\infty)$. The Laplace transform of f(t) is

$$L\left(\frac{1}{\sqrt{t}}\right) = \int_0^\infty e^{-st}\frac{1}{\sqrt{t}}dt = 2\int_0^\infty e^{-sx^2}dx, \quad \left(\text{Put } t = x^2\right)$$

$$= \frac{2}{\sqrt{s}}\int_0^\infty e^{-u^2}du, \quad \left(\text{Put } u = x\sqrt{s}\right)$$

$$= \frac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}, \quad s > 0, \quad \text{since } \Gamma(1/2) = \sqrt{\pi}$$

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INVERSE LAPLACE TRANSFORM:

$L\{f(t)\} = \bar{f}(s)$ can also be written as $f(t) = L^{-1}\{\bar{f}(s)\}$.

Then f(t) is called the **inverse Laplace transform** of $\bar{f}(s)$.

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LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS:

The direct application of the definition gives the following formulae:

(1) $L(1) = \dfrac{1}{s}$ $[s > 0]$

(2) $L(t^n) = \dfrac{n!}{s^{n+1}}$, when n =0, 1, 2, 3, ... $\left[\text{Otherwise } \dfrac{\Gamma(n+1)}{s^{n+1}}\right]$

(3) $L(e^{at}) = \dfrac{1}{s-a}$ $[s > a]$

(4) $L(\sin at) = \dfrac{a}{s^2 + a^2}$ $[s > 0]$

(5) $L(\cos at) = \dfrac{s}{s^2 + a^2}$ $[s > 0]$

(6) $L(\sinh at) = \dfrac{a}{s^2 - a^2}$ $[s > |a|]$

(7) $L(\cosh at) = \dfrac{s}{s^2 - a^2}$ $[s > |a|]$

Proofs:

(1) To show: $L(1) = \dfrac{1}{s}$, $[s > 0]$.

Proof: $L(1) = \displaystyle\int_0^\infty e^{-st}.1dt = \left| -\frac{e^{-st}}{s} \right|_0^\infty = \frac{1}{s}$, if $s > 0$.

(2) To show: $L(t^n) = \dfrac{n!}{s^{n+1}}$, when n =0, 1, 2, 3, ... **.**

Proof: $L(t^n) = \displaystyle\int_0^\infty e^{-st}.t^n dt = \int_0^\infty e^{-p}.\left(\frac{p}{s}\right)^n \frac{dp}{s}$, (on putting st = p, dt = dp/s)

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-p}.p^n dp = \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } n > -1 \text{ and } s > 0.$$

In particular, $L\left(t^{-1/2}\right) = \dfrac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \sqrt{\dfrac{\pi}{s}}$; $L\left(t^{1/2}\right) = \dfrac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} = \dfrac{\sqrt{\pi}}{2.s^{3/2}}$.

If n be a positive integer, $\Gamma(n+1) = n!$.

Therefore, $L\left(t^n\right) = \dfrac{n!}{s^{n+1}}$.

(3) To show: $L\left(e^{at}\right) = \dfrac{1}{s-a}$, $[s > a]$.

Proof: $L\left(e^{at}\right) = \displaystyle\int_0^\infty e^{-st}.e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \left|\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty = \frac{1}{s-a}$, if $s > a$.

(4) To show: $L(\sin at) = \dfrac{a}{s^2 + a^2}$, $[s > 0]$.

Proof: $L(\sin at) = \displaystyle\int_0^\infty e^{-st} \sin at dt = \left|\frac{e^{-st}}{s^2 + a^2}(-s \sin at - a \cos at)\right|_0^\infty = \frac{a}{s^2 + a^2}$.

(5) To show: $L(\cos at) = \dfrac{s}{s^2 + a^2}$, $[s > 0]$.

Proof: $L(\cos at) = \displaystyle\int_0^\infty e^{-st} \cos at dt = \left|\frac{e^{-st}}{s^2 + a^2}(-s \cos at - a \sin at)\right|_0^\infty = \frac{s}{s^2 + a^2}$.

(6) To show: $L(\sinh at) = \dfrac{a}{s^2 - a^2}$, $[s > |a|]$.

Proof: $L(\sinh at) = \displaystyle\int_0^\infty e^{-st} \sinh at.dt = \int_0^\infty e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right) dt$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t} dt - \int_0^\infty e^{-(s+a)t} dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2 - a^2}, \text{ for } s > |a|.$$

(7) To show: $L(\cosh at) = \dfrac{s}{s^2 - a^2}$, $[s > |a|]$.

Proof: $L(\cosh at) = \int_0^\infty e^{-st} \cosh at.dt = \int_0^\infty e^{-st}\left(\dfrac{e^{at} + e^{-at}}{2}\right)dt$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}dt + \int_0^\infty e^{-(s+a)t}dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2 - a^2}, \text{ for } s > |a|.$$

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2^{nd} Topic

Laplace Transforms

General Properties of Laplace Transform and their Applications

- Linearity Property
- First Shifting Property
- Change of Scale Property

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General Properties of Laplace Transform:

Since $\bar{f}(s)$, the Laplace transform of f(t) is obtained from the definition of Laplace transform, but in practice, Laplace transforms are obtained by the application of some of the following important properties of Laplace transform:

1. Linearity Property:

This property states that Laplace transform of a linear combination (sum) is the linear combination (sum) of Laplace transforms.

2. First Shift Theorem:

This theorem proves that multiplication of f(t) by e^{at} amounts to replacement s by $s - a$ in $\bar{f}(s)$.

3. In Change of Scale:

Here the argument t of f is multiplied by constant a, s is replaced by $\frac{s}{a}$ in $\bar{f}(s)$ and then multiplied by $\frac{1}{a}$.

4. Laplace Transform of a Derivative:

This amounts to multiplication of $\bar{f}(s)$ by $s - f(0)$.

i.e. $L\{f'(t)\} = s\bar{f}(s) - f(0)$, provided $\lim_{t\to\infty}\left[e^{-st} f(t)\right] = 0$.

5. Laplace Transform of an Integral:

This amounts to amounts to division of $\bar{f}(s)$ by s.

i.e. $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)$.

6. Laplace Transform of f(t) multiplied by t^n :

This amounts to differentiation of $\bar{f}(s)$ n times w.r.t. s with $(-1)^n$ as sign.

i.e. $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[\bar{f}(s)\right]$, where n = 1, 2, 3,...........

7. Laplace Transform of f(t) divided by t:

This amounts to integration of $\bar{f}(s)$ between the limits s to ∞.

i.e. $L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty \bar{f}(s)\,ds$.

8. Second Shift Theorem:

This proves that Laplace transform of shifted function $f(t-a)u(t-a)$ is obtained by multiplying $\bar{f}(s)$ by e^{-at}.

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PROPERTIES OF LAPLACE TRANSFORMATIONS:

(1). Linearity Property:

If a, b, c be any constants and f, g, h any functions of t, then

$$L[a\,f(t) + b\,g(t) - c\,h(t)] = a\,L\{f(t)\} + b\,L\{g(t)\} - c\,L\{h(t)\}.$$

or

Linearity property states that Laplace transform of a linear combination (sum) is the linear combination (sum) of Laplace transforms.

Proof: L.H.S. = $L[a\,f(t) + b\,g(t) - c\,h(t)]$

$$= \int_0^\infty e^{-st}\{af(t)+bg(t)-ch(t)\}dt \qquad \text{(by definitions)}$$

$$= a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt - c\int_0^\infty e^{-st}h(t)dt$$

$$= aL\{f(t)\}+bL\{g(t)\}-cL\{h(t)\}.$$

Thus, linear transform is a linear operator, additive, subtractive and homogeneous.

This result can easily be generalized to more than three functions.

Because of the above property of L, it is called a **linear operator**.

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(2). First Shifting or First Translation Property or s-Shift Theorem:

(Replacement of s by s-a in transform)

If $L\{f(t)\}=\bar{f}(s)$, then $L\{e^{at}f(t)\}=\bar{f}(s-a)$.

or

First shift theorem proves that multiplication of f(t) by e^{at} ***amounts to replacement s by*** $s-a$ ***in*** $\bar{f}(s)$.

Proof: Given $L\{f(t)\}=\bar{f}(s)$.

To show: $L\{e^{at}f(t)\}=\bar{f}(s-a)$.

Since $L\{e^{at}f(t)\}=\int_0^\infty e^{-st}\left(e^{at}f(t)\right)dt = \int_0^\infty e^{-(s-a)t}f(t)dt$ (by definition)

$$= \int_0^\infty e^{-rt}f(t)dt \qquad \text{(where } r=s-a\text{)}$$

$$= \bar{f}(r)=\bar{f}(s-a).$$

Thus, if we know the transform $\bar{f}(s)$ of f(t), then we can write the transform of $e^{at}f(t)$ simply replacing s by $s-a$ to get $\bar{f}(s-a)$.

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Application of this property leads to the following useful results:

(1) $L(e^{at}) = \dfrac{1}{s-a}$ $\left[\because L(1) = \dfrac{1}{s}\right]$

(2) $L(e^{at}t^n) = \dfrac{n!}{(s-a)^{n+1}}$ $\left[\because L(t^n) = \dfrac{n!}{s^{n+1}}\right]$

(3) $L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$ $\left[\because L(\sin bt) = \dfrac{b}{s^2+b^2}\right]$

(4) $L(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2+b^2}$ $\left[\because L(\cos bt) = \dfrac{s}{s^2+b^2}\right]$

(5) $L(e^{at}\sinh bt) = \dfrac{b}{(s-a)^2-b^2}$ $\left[\because L(\sinh bt) = \dfrac{b}{s^2-b^2}\right]$

(6) $L(e^{at}\cosh bt) = \dfrac{s-a}{(s-a)^2-b^2}$ $\left[\because L(\cosh bt) = \dfrac{s}{s^2-b^2}\right]$

where in each case $s > a$.

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(3). Change of Scale Property:

$$\text{If } L\{f(t)\} = \bar{f}(s), \text{ then } L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$$

or

In change of scale, where the argument t of f is multiplied by constant a, s is replaced by $\dfrac{s}{a}$ ***in*** $\bar{f}(s)$ ***and then multiplied by*** $\dfrac{1}{a}$.

Proof: $L\{f(at)\} = \displaystyle\int_0^\infty e^{-st} f(at)\,dt = \int_0^\infty e^{-su/a} f(u).\frac{du}{a}$ $\left[\text{Put } at = u \Rightarrow dt = \dfrac{du}{a}\right]$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} f(u)\,du = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$$

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REMEMBER:

(1). Linearity Property:

If a, b, c be any constants and f, g, h any functions of t, then

$L[a\,f(t)+b\,g(t)-c\,h(t)]=a\,L\{f(t)\}+b\,L\{g(t)\}-c\,L\{h(t)\}$.

Q.No.1.: Find the Laplace transforms of

(i) $\sin 2t\sin 3t$, **(ii)** $\cos^2 2t$, **(iii)** $\sin^3 2t$.

Sol. (i). Since $\sin 2t\sin 3t=\frac{1}{2}(\cos t-\cos 5t)$.

$\therefore L(\sin 2t\sin 3t)=\frac{1}{2}[L(\cos t)-L(\cos 5t)]$ (by linearity property)

$$=\frac{1}{2}\left[\frac{s}{s^2+1^2}-\frac{s}{s^2+5^2}\right]=\frac{12s}{(s^2+1)(s^2+25)}.\ \textbf{Ans.}$$

(ii) Since $\cos^2 2t=\frac{1}{2}(1+\cos 4t)$.

$\therefore L(\cos^2 2t)=\frac{1}{2}\{L(1)+L(\cos 4t)\}$ (by linearity property)

$$=\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+16}\right).\ \textbf{Ans.}$$

(iii) Since $\sin 6t=3\sin 2t-4\sin^3 2t\Rightarrow \sin^3 2t=\frac{3}{4}\sin 2t-\frac{1}{4}\sin 6t$.

$\therefore L(\sin^3 2t)=L\left(\frac{3}{4}\sin 2t-\frac{1}{4}\sin 6t\right)=\frac{3}{4}L(\sin 2t)-\frac{1}{4}L(\sin 6t)$ (by linearity property)

$$=\frac{3}{4}.\frac{2}{s^2+2^2}-\frac{1}{4}.\frac{6}{s^2+6^2}=\frac{48}{(s^2+4)(s^2+36)}.\ \textbf{Ans.}$$

Q.No.2.: Find the Laplace transform of $e^{2t}+4t^3-2\sin 3t+3\cos 3t$.

Sol.: $L(e^{2t}+4t^3-2\sin 3t+3\cos 3t)=L(e^{2t})+4L(t^3)-2L(\sin 3t)+3L(\cos 3t)$

(by linearity property)

$$=\frac{1}{s-2}+\frac{4.3!}{s^4}-\frac{2.3}{s^2+9}+\frac{3s}{s^2+9}$$

$$=\frac{1}{s-2}+\frac{24}{s^4}-\frac{6}{s^2+9}+\frac{3s}{s^2+9}$$

$$=\frac{1}{s-2}+\frac{24}{s^4}+\frac{3(s-2)}{s^2+9}.\ \text{Ans.}$$

Q.No.3.: Find the Laplace transform of $1+2\sqrt{t}+\frac{3}{\sqrt{t}}$.

Sol.: $L\left(1+2\sqrt{t}+\frac{3}{\sqrt{t}}\right)=L(1)+2L\left(\sqrt{t}\right)+3L\left(\frac{1}{\sqrt{t}}\right)$ (by linearity property)

$$=\frac{1}{s}+\frac{2\Gamma\left(\frac{1}{2}+1\right)}{s^{3/2}}+\frac{3\Gamma\left(-\frac{1}{2}+1\right)}{s^{1/2}}=\frac{1}{s}+\frac{2.\frac{1}{2}\sqrt{\pi}}{s^{3/2}}+\frac{3\sqrt{\pi}}{s^{1/2}}$$

$$=\frac{1}{s}+\frac{\sqrt{\pi}}{s^{3/2}}+3\sqrt{\left(\frac{\pi}{s}\right)}.\ \text{Ans.}$$

Q.No.4.: Find the Laplace transform of $\cosh at-\cos at$.

Sol.: $L(\cosh at-\cos at)=L(\cosh at)-L(\cos at)$ (by linearity property)

$$=\frac{s}{s^2-a^2}-\frac{s}{s^2+a^2}=\frac{2a^2s}{s^4-a^4}.\ \text{Ans.}$$

Q.No.5.: Find the Laplace transform of $\cos(at+b)$.

Sol.: $L[\cos(at+b)]=L(\cos at\cos b-\sin at\sin b)$ (by linearity property)

$$=\cos bL(\cos at)-\sin bL(\sin at)=\cos b\frac{s}{s^2+a^2}-\sin b\frac{a}{s^2+a^2}$$

$$=\frac{s\cos b-a\sin b}{s^2+a^2}.\ \text{Ans.}$$

Q.No.6.: Find the Laplace transform of $(\sin t-\cos t)^2$.

Sol.: Since $(\sin t-\cos t)^2=\sin^2 t+\cos^2 t-2\sin t\cos t=1-2\sin t\cos t=1-\sin 2t$.

$\therefore L(1-\sin 2t)=L(1)-L(\sin 2t)$ (by linearity property)

$$=\frac{1}{s}-\frac{2}{s^2+4}=\frac{s^2+4-2s}{s\left(s^2+4\right)}=\frac{s^2-2s+4}{s\left(s^2+4\right)}.\ \text{Ans.}$$

Q.No.7.: Find the Laplace transform of $\sin 2t\cos 3t$.

Sol.: Since $\sin 2t\cos 3t = \frac{1}{2}(\sin 5t - \sin t)$.

$\therefore L\left(\frac{\sin 5t - \sin t}{2}\right) = \frac{1}{2}[L(\sin 5t) - L(\sin t)]$ (by linearity property)

$$= \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right] = \frac{1}{2}\cdot\frac{4s^2-20}{(s^2+1)(s^2+25)} = \frac{2(s^2-5)}{(s^2+1)(s^2+25)}.\text{ Ans.}$$

Q.No.8.: Find the Laplace transform of $\sin at\sin bt$.

Sol.: $L(\sin at\sin bt) = \frac{1}{2}L[\cos(a-b)t - \cos(a+b)t]$

$= \frac{1}{2}[L(\cos(a-b)t) - L(\cos(a+b)t)]$ (by linearity property)

$$= \frac{1}{2}\left[\frac{s}{s^2+(a-b)^2} - \frac{s}{s^2+(a+b)^2}\right] = \frac{2abs}{\left[s^2+(a+b)^2\right]\left[s^2+(a-b)^2\right]}.\text{ Ans.}$$

Q.No.9.: Find the Laplace transform of $\sin^2 3t$.

Sol.: Since $\sin^2 3t = \frac{1}{2}(1-\cos 6t)$.

$\therefore L(\sin^2 3t) = \frac{1}{2}L(1-\cos 6t) = \frac{1}{2}(L(1) - L(\cos 6t))$ (by linearity property)

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right] = \frac{18}{s(s^2+36)}.\text{ Ans.}$$

Q.No.10.: Find the Laplace transform of $\cos^3 2t$.

Sol.: Since $\cos^3 2t = \frac{1}{4}[\cos 6t + 3\cos 2t]$.

$\therefore L(\cos^3 2t) = \frac{1}{4}L[\cos 6t + 3\cos 2t] = \frac{1}{4}[L(\cos 6t) + L(3\cos 2t)]$ (by linearity property)

$$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{3s}{s^2+4}\right] = \frac{1}{4}\left[\frac{s^3+4s+3s^3+108s}{(s^2+36)(s^2+4)}\right] = \frac{1}{4}\left[\frac{4s^3+112s}{(s^2+36)(s^2+4)}\right]$$

$$= \frac{s(s^2+28)}{(s^2+36)(s^2+4)}.\text{ Ans.}$$

Q.No.11.: Find the Laplace transform of $t - \sinh 2t$.

Sol.: $L(t-\sinh 2t) = L(t) - L(\sinh 2t)$ (by linearity property)

$$=\frac{1}{s^2}-\frac{2}{s^2-4}=\frac{4+s^2}{s^2\left(4-s^2\right)}.\ \text{Ans.}$$

Q.No.12.: Find the Laplace transforms of $\cosh^3 2t$.

Sol.: Since $\cosh 6t = 4\cosh^3 2t - 3\cosh 2t \Rightarrow \cosh^3 2t = \frac{3}{4}\cosh 2t + \frac{1}{4}\cosh 6t$.

$$\therefore L\left(\cosh^3 2t\right)=L\left(\frac{3}{4}\cosh 2t+\frac{1}{4}\cosh 6t\right)=\frac{3}{4}L(\cosh 2t)+\frac{1}{4}L(\cosh 6t) \quad \text{(by linearity property)}$$

$$=\frac{3}{4}\cdot\frac{s}{s^2-2^2}+\frac{1}{4}\cdot\frac{s}{s^2-6^2}=\frac{s\left(s^2-28\right)}{\left(s^2-4\right)\left(s^2-36\right)}.$$

Q.No.13.: Find the Laplace transforms of $e^{at}-e^{bt}$.

Sol.: $L\left(e^{at}-e^{bt}\right)=L\left(e^{at}\right)-L\left(e^{bt}\right)$ (by linearity property)

$$=\frac{1}{s-a}-\frac{1}{s-b}=\frac{a-b}{(s-a)(s-b)}.\ \text{Ans.}$$

Q.No.14.: Find the Laplace transforms of $\cos^2 kt$.

Sol.: $L\left\{\cos^2 kt\right\}=L\left\{\frac{1+\cos 2kt}{2}\right\}=\frac{1}{2}L\{1\}+\frac{1}{2}L\{\cos 2kt\}$ (by linearity property)

$$=\frac{1}{2}\frac{1}{s}+\frac{1}{2}\frac{s}{s^2+4k^2}.\ \text{Ans.}$$

Q.No.15.: Find the Laplace transforms of $\left\{5e^{2t}-3\right\}^2$.

Sol.: $L\left\{\left(5e^{2t}-3\right)^2\right\}=L\left\{25e^{4t}-30e^{2t}+9\right\}$

$$=25L\left\{e^{4t}\right\}-30L\left\{e^{2t}\right\}+9L\{1\} \quad \text{(by linearity property)}$$

$$=25.\frac{1}{s-4}-30\frac{1}{s-2}+9.\frac{1}{s}.\ \text{Ans.}$$

Q.No.16.: Find the Laplace transforms of $3t^4-2t^3+4e^{-3t}-2\sin 5t+3\cos 2t$.

Sol.: $L\left\{3t^4-2t^3+4e^{-3t}-2\sin 5t+3\cos 2t\right\}$

$$=3L\left\{t^4\right\}-2L\left\{t^3\right\}+4L\left\{e^{-3t}\right\}-2L\{\sin 5t\}+3L\{\cos 2t\} \quad \text{(by linearity property)}$$

$$=3.\frac{4!}{s^5}-2.\frac{3!}{s^4}+4\frac{1}{s+3}-2.\frac{5}{s^2+5^2}+3.\frac{s}{s^2+2^2}.\ \text{Ans.}$$

Q.No.17.: Find the Laplace transforms of $\cos\sqrt{t}$.

Sol.: Expanding in series $\cos\sqrt{t} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(t^{\frac{1}{2}}\right)^{2n}}{(2n)!} = \sum \frac{(-1)^n}{(2n)!} t^n$.

$$\therefore L\{\cos\sqrt{t}\} = L\left[\sum \frac{(-1)^n}{(2n)!} t^n\right] = \sum L\left[\frac{(-1)^n}{(2n)!} t^n\right] \qquad \text{(by linearity property)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} L\left(t^n\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{n!}{s^{n+1}}. \text{ Ans.}$$

Q.No.18.: Find the Laplace transform of $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3$.

Sol.: Since $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3 = \left(t^{3/2}\right) - 3\left(t^{1/2}\right) + 3\left(t^{-1/2}\right) - \left(t^{-3/2}\right)$

$$\therefore L\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3 = L\left(t^{3/2}\right) - 3L\left(t^{1/2}\right) + 3L\left(t^{-1/2}\right) - L\left(t^{-3/2}\right) \qquad \text{(by linearity property)}$$

$$= \frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{3/2+1}} - 3\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{1/2+1}} + 3\frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-1/2+1}} - \frac{\Gamma\left(-\frac{3}{2}+1\right)}{s^{-3/2+1}}$$

$$= \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{5/2}} - 3\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} + 3\frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} - \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-1/2}}$$

$$= \frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}} - \frac{3}{2}\frac{\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}} + 2\sqrt{\pi}\sqrt{s} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},\ \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}\right]$$

$$= \frac{\sqrt{\pi}}{4}\left(\frac{3}{s^{5/2}} - \frac{6}{s^{3/2}} + \frac{12}{s^{1/2}} + \frac{8}{s^{-1/2}}\right). \text{ Ans.}$$

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REMEMBER:

(2). First Shifting or First Translation Property or s-Shift Theorem:

(Replacement of s by s-a in transform)

If $L\{f(t)\}=\bar{f}(s)$, then $L\{e^{at}f(t)\}=\bar{f}(s-a)$.

Q.No.1.: Show that **(i)** $L(t\sin at)=\dfrac{2as}{\left(s^2+a^2\right)^2}$, **(ii)** $L(t\cos at)=\dfrac{s^2-a^2}{\left(s^2+a^2\right)^2}$.

Sol.: Since $L(t)=\dfrac{1}{s^2}$.

$$\therefore L\left(e^{iat}t\right)=\frac{1}{(s-ia)^2}=\frac{(s+ia)^2}{[(s-ia)(s+ia)]^2}.$$ (by shifting property)

$$\Rightarrow L[t(\cos at+i\sin at)]=\frac{\left(s^2-a^2\right)+i(2as)}{\left(s^2+a^2\right)^2}.$$

Equating the real and imaginary parts from both sides, we get

$$L(t\sin at)=\frac{2as}{\left(s^2+a^2\right)^2} \text{ and } L(t\cos at)=\frac{s^2-a^2}{\left(s^2+a^2\right)^2}.$$

Q.No.2.: Find the Laplace transform of $e^{-at}\sinh bt$.

Sol.: $L\left(e^{-at}\sinh bt\right)=\dfrac{b}{(s+a)^2-b^2}$. Ans. (by shifting property)

Q.No.3.: Find the Laplace transform of $t^3.e^{-3t}$.

Sol.: $L\left(t^3.e^{-3t}\right)=\dfrac{\Gamma(4)}{(s+3)^4}=\dfrac{3!}{(s+3)^4}=\dfrac{6}{(s+3)^4}$. Ans. (by shifting property)

Q.No.4.: Find the Laplace transform of $e^{-2t}\sin 4t$.

Sol.: $L\left(e^{-2t}\sin 4t\right)=\dfrac{4}{(s+2)^2+16}=\dfrac{4}{s^2+4s+20}$. Ans. (by shifting property)

Q.No.5.: Find the Laplace transforms of $t^2\sin at$.

Sol.: Since $L\left(t^2\right)=\dfrac{2!}{s^3}=\dfrac{2}{s^3}$.

$$\therefore L\left\{t^2 e^{iat}\right\} = \frac{2}{(s-ia)^3} = \frac{2(s+ia)^3}{[(s-ia)(s+ia)]^3}$$ (by shifting property)

$$\Rightarrow L\left\{t^2(\cos at + i\sin at)\right\} = \frac{2\left[\left(s^3 - 3a^2 s\right) + i\left(3as^2 - a^3\right)\right]}{\left(s^2 + a^2\right)^3}.$$

Equating the imaginary parts on both sides, we get

$$L\left\{t^2 \sin at\right\} = \frac{2a\left(3s^2 - a^2\right)}{\left(s^2 + a^2\right)^3}.$$ Ans.

Q.No.6.: Find the Laplace transforms of $te^{-4t}\sin 3t$.

Sol.: Since $L\{t\} = \frac{1}{t^2}$.

$$\therefore L\left\{te^{3it}\right\} = \frac{1}{(s-3i)^2} = \frac{(s+3i)^2}{[(s-3i)(s+3i)]^2}$$ (by shifting property)

$$\Rightarrow L\{t(\cos 3t + i\sin 3t)\} = \frac{\left(s^2 - 9\right) + 6is}{\left(s^2 + 9\right)^2}.$$

Equating the imaginary parts on both sides, we get

$$L[t\sin 3t] = \frac{6s}{\left(s^2 + 9\right)^2}.$$

Again, applying the first shifting theorem, we have

$$L\left\{e^{-4t}.t\sin 3t\right\} = \frac{6(s+4)}{\left[(s+4)^2 + 9\right]^2} = \frac{6(s+4)}{\left(s^2 + 8s + 25\right)^2}.$$ Ans.

Q.No.7.: Find the Laplace transforms of $f(t) = t^{\frac{7}{2}} e^{3t}$.

Sol.: $$L\left\{t^{\frac{7}{2}}\right\} = \frac{\Gamma\left(\frac{7}{2}+1\right)}{s^{\frac{7}{2}+1}} = \frac{\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{9}{2}}} = \frac{105\sqrt{\pi}}{16s^{\frac{9}{2}}}.$$

$$\therefore L\left\{e^{3t}.t^{\frac{7}{2}}\right\} = \left.\frac{105\sqrt{\pi}}{16s^{\frac{9}{2}}}\right|_{\text{at } s=s-3}$$ (by shifting property)

$$=\frac{105\sqrt{\pi}}{16(s-3)^{\frac{9}{2}}}.$$

Q.No.8.: Find the Laplace transforms of

(i) $e^{-3t}(2\cos 5t-3\sin 5t)$, (ii) $e^{3t}\sin^2 t$, (iii) $e^{4t}\sin 2t\cos t$.

Sol.: (i) $L\left\{e^{-3t}(2\cos 5t-3\sin 5t)\right\}=2L\left(e^{-3t}\cos 5t\right)-3L\left(e^{-3t}\sin 5t\right)$ (by linearity property)

$$=2.\frac{s+3}{(s+3)^2+5^2}-3.\frac{5}{(s+3)^2+5^2} \text{ (by shifting property)}$$

$$=\frac{2s-9}{s^2+6s+34}. \text{ Ans.}$$

(ii) Since $L\left(\sin^2 t\right)=\frac{1}{2}L(1-\cos 2t)=\frac{1}{2}(L(1)-L(\cos 2t))$ (by linearity property)

$$=\frac{1}{2}\left\{\frac{1}{s}-\frac{s}{s^2+4}\right\}.$$

$$\therefore\quad L\left(e^{3t}\sin^2 t\right)=\frac{1}{2}\left\{\frac{1}{s-3}-\frac{s-3}{(s-3)^2+4}\right\}. \text{ Ans.} \quad \text{(by shifting property)}$$

(iii) Since $L(\sin 2t\cos t)=\frac{1}{2}L(\sin 3t+\sin t)=\frac{1}{2}(L(\sin 3t)+L(\sin t))$ (by linearity property)

$$=\frac{1}{2}\left\{\frac{3}{s^2+3^2}+\frac{1}{s^2+1^2}\right\}.$$

$$\therefore\quad L\left(e^{4t}\sin 2t\cos t\right)=\frac{1}{2}\left\{\frac{3}{(s-4)^2+9}+\frac{1}{(s-4)^2+1}\right\}. \text{ Ans.} \quad \text{(by shifting property)}$$

Q. No.9.: If $L\{f(t)\}=\bar{f}(s)$, show that

$$L[(\sinh at)f(t)]=\frac{1}{2}\left[\bar{f}(s-a)-\bar{f}(s+a)\right]$$

$$L[(\cosh at)f(t)]=\frac{1}{2}\left[\bar{f}(s-a)+\bar{f}(s+a)\right]$$

Hence evaluate **(i)** $L(\sinh 2t\sin 3t)$ **(ii)** $L(\cosh 3t\cos 2t)$.

Sol. We have $L\{(\sinh at)f(t)\}=L\left[\frac{1}{2}\left(e^{at}-e^{-at}\right)f(t)\right]$

$$=\frac{1}{2}\left[L\left\{e^{at}f(t)\right\}-L\left\{e^{-at}f(t)\right\}\right] \qquad \text{(by linearity property)}$$

$$=\frac{1}{2}\left[\bar{f}(s-a)-\bar{f}(s+a)\right]. \qquad \text{(by shifting property)}$$

Similarly, $L\{(\cosh at)f(t)\}=L\left[\frac{1}{2}\left(e^{at}+e^{-at}\right)f(t)\right]$

$$=\frac{1}{2}\left[L\left\{e^{at}f(t)\right\}+L\left\{e^{-at}f(t)\right\}\right] \qquad \text{(by linearity property)}$$

$$=\frac{1}{2}\left[\bar{f}(s-a)+\bar{f}(s+a)\right]. \qquad \text{(by shifting property)}$$

(i) Since $L(\sin 3t)=\frac{3}{s^2+3^2}$, the first result gives

$$L(\sinh 2t\sin 3t)=\frac{1}{2}\left\{\frac{3}{(s-2)^2+3^2}-\frac{3}{(s+2)^2+3^2}\right\}=\frac{12s}{s^4+10s^2+169}.\ \text{Ans.}$$

(ii) Since $L(\cos 2t)=\frac{s}{s^2+2^2}$, the second result gives

$$L(\cosh 3t\cos 2t)=\frac{1}{2}\left\{\frac{s-3}{(s-3)^2+2^2}+\frac{s+3}{(s+3)^2+2^2}\right\}=\frac{2s\left(s^2-5\right)}{s^4-10s^2+169}.\ \text{Ans.}$$

Q.No.10.: Find the Laplace transform of $(t+2)^2.e^t$.

Sol.: $L\left[\left(t^2+4t+4\right)e^t\right]=L\left(e^t t^2\right)+L\left(e^t 4t\right)+L\left(e^t 4\right)$ (by linearity property)

$$=\frac{2!}{(s-1)^3}+\frac{4}{(s-1)^2}+\frac{4}{(s-1)} \qquad \text{(by shifting property)}$$

$$=\frac{2}{(s-1)^3}+\frac{4}{(s-1)^2}+\frac{4}{(s-1)}$$

$$=\frac{2+4(s-1)+4(s-1)^2}{(s-1)^3}=\frac{2\left(2s^2-2s+1\right)}{(s-1)^3}.\ \text{Ans.}$$

Q.No.11.: Find the Laplace transform of $e^{-t}\sin^2 t$.

Sol.: $L\left(e^{-t}\sin^2 t\right)=\frac{1}{2}L\left\{e^{-t}(1-\cos 2t)\right\}$

$$=\frac{1}{2}\left\{L\left(e^{-t}1\right)-L\left(e^{-t}\cos 2t\right)\right\} \qquad \text{(by linearity property)}$$

$$=\frac{1}{2}\left[\frac{1}{s+1}-\frac{(s+1)}{(s+1)^2+4}\right] \qquad \text{(by shifting property)}$$

$$=\frac{2}{(s+1)\left(s^2+2s+5\right)}. \text{ Ans.}$$

Q.No.12.: Find the Laplace transform of $\cosh at \sin at$.

Sol.: $L(\cosh at \sin at)=\frac{1}{2}L\left[\left(e^{at}+e^{-at}\right)\sin at\right]$

$$=\frac{1}{2}\left\{L\left(e^{at}\sin at\right)+L\left(e^{-at}\sin at\right)\right\} \qquad \text{(by linearity property)}$$

$$=\frac{1}{2}\left[\frac{a}{(s-a)^2+a^2}+\frac{a}{(s+a)^2+a^2}\right] \qquad \text{(by shifting property)}$$

$$=\frac{a\left(s^2+2a^2\right)}{s^4+4a^4}. \text{ Ans.}$$

Q.No.13.: Find the Laplace transform of $\sinh 3t \cos^2 t$.

Sol.: Since $\sinh 3t \cos^2 t=\left[\frac{e^{3t}-e^{-3t}}{2}\right]\left[\frac{1+\cos 2t}{2}\right]$.

$$\therefore L\left(\sinh 3t\cos^2 t\right)=\frac{1}{4}L\left[e^{3t}+e^{3t}\cos 2t-e^{-3t}-e^{-3t}\cos 2t\right]$$

$$=\frac{1}{4}\left[L\left(e^{3t}\right)+L\left(e^{3t}\cos 2t\right)-L\left(e^{-3t}\right)-L\left(e^{-3t}\cos 2t\right)\right]$$

(by linearity property)

$$=\frac{1}{4}\left[\frac{1}{s-3}-\frac{1}{s+3}+\frac{s-3}{(s-3)^2+4}-\frac{s+3}{(s+3)^2+4}\right] \qquad \text{(by shifting property)}$$

$$=\frac{1}{4}\left[\frac{6}{s^2-9}+\frac{6\left(s^2-1\right)}{s^4-10s^2+169}\right]=\frac{3}{2}\left[\frac{1}{s^2-9}+\frac{s^2-1}{s^4-10s^2+169}\right]. \text{ Ans.}$$

Q.No.14.: Find the Laplace transforms of $e^{-3t}(\cos 4t+3\sin 4t)$.

Sol.: $L\left\{e^{-3t}(\cos 4t+3\sin 4t)\right\}=L\left\{e^{-3t}\cos 4t\right\}+3L\left\{e^{-3t}\sin 4t\right\}$ (by linearity property)

$$=\frac{s+3}{(s+3)^2+4^2}+3.\frac{4}{(s+3)^2+4^2} \qquad \text{(by shifting property)}$$

$$=\frac{s+15}{s^2+6s+25}\text{. Ans.}$$

Q.No.15.: Find the Laplace transforms of $\left\{3t^5-2t^4+4e^{-5t}-3\sin 6t+4\cos 4t\right\}e^{2t}$.

Sol.: $L\left\{3t^5-2t^4+4e^{-5t}-3\sin 6t+4\cos 4t\right\}$

$$=3L\left\{t^5\right\}-2L\left(t^4\right)+4L\left\{e^{-5t}\right\}-3L(\sin 6t)+4L\{\cos 4t\} \qquad \text{(by linearity property)}$$

$$=3\frac{5!}{s^6}-2\frac{4!}{s^5}+4\frac{1}{s+5}-3\frac{6}{s^2+36}+4\frac{s}{s^2+16}.$$

$\therefore L\left\{3t^5-2t^4+4e^{-5t}-3\sin 6t+4\cos 4t\right\}e^{2t}$

$$=\left.\frac{360}{s^6}-\frac{48}{s^5}+\frac{4}{s+5}-\frac{18}{s^2+36}+\frac{4s}{s^2+16}\right|_{\text{with s replaced by s - 2}}. \qquad \text{(by shifting property)}$$

$$=\frac{360}{(s-2)^6}-\frac{48}{(s-2)^5}+\frac{4}{s+3}-\frac{18}{(s-2)^2+36}+\frac{4(s-2)}{(s-2)^2+16}\text{. Ans.}$$

Q.No.16.: Find the Laplace transforms of $f(t)=\cosh at.\cos bt$.

Sol.: $L\{f(t)\}=L\{\cosh at.\cos bt\}=L\left\{\frac{1}{2}\left(e^{at}+e^{-at}\right)\cos bt\right\}$

$$=\frac{1}{2}L\left\{e^{at}\cos bt\right\}+\frac{1}{2}L\left\{e^{-at}\cos bt\right\} \qquad \text{(by linearity property)}$$

$$=\left.\frac{1}{2}\frac{s}{s^2+b^2}\right|_{s=s-a}+\left.\frac{1}{2}\frac{s}{s^2+b^2}\right|_{s=s+a} \qquad \text{(by shifting property)}$$

$$=\frac{1}{2}\frac{s-a}{(s-a)^2+b^2}+\frac{1}{2}\frac{s+a}{(s+a)^2+b^2}\text{. Ans.}$$

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REMEMBER:

(3). Change of Scale Property:

If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$.

Q.No.1.: Find $L\left\{\frac{\sin at}{t}\right\}$, given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right)$.

Sol.: Since given $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right)$.

$$L\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\tan^{-1}\left\{\frac{1}{s/a}\right\} \qquad \text{(by change of scale property)}$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{a}{s}\right)$$

Thus, $L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left(\frac{a}{s}\right)$. Ans.

Q.No.2.: If $L\{f(t)\} = \frac{e^{-\frac{1}{s}}}{s}$, find $L\{e^{-t}f(3t)\}$.

Sol.: Given $L\{f(t)\} = \frac{e^{-\frac{1}{s}}}{s}$.

$$\Rightarrow L\{f(3t)\} = \frac{1}{3}\frac{e^{-\frac{3}{s}}}{\frac{s}{3}} = \frac{e^{-\frac{3}{s}}}{s}. \qquad \text{(by change of scale property) } \left(a = 3, \text{ replace } s \text{ by } \frac{s}{3}\right).$$

$$\therefore \quad L\{e^{-t}f(3t)\} = \frac{e^{-\frac{3}{(s+1)}}}{s+1}. \text{ Ans.} \qquad \text{(by shifting property)}$$

Q.No.3.: Find the Laplace transform of $\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t$.

Sol.: $L\left(\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t\right) = L\left[\left(\frac{e^{1/2} - e^{-1/2}}{2}\right)\sin\frac{\sqrt{3}}{2}t\right]$

$$=\frac{1}{2}\left[L\left(e^{1/2}\sin\frac{\sqrt{3}}{2}t\right)\right]-\frac{1}{2}\left[L\left(e^{-1/2}\sin\frac{\sqrt{3}}{2}t\right)\right]$$

$$=\frac{1}{2}\left[\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right]-\frac{1}{2}\cdot\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} \qquad \left[\because L\left(\sin\frac{\sqrt{3}}{2}t\right)=\frac{\frac{\sqrt{3}}{2}}{s^2+\frac{3}{4}}\right]$$

$$=\frac{\sqrt{3}}{4}\cdot\frac{\left(s^2+s+1\right)-\left(s^2-s+1\right)}{\left(s^2+1-s\right)\left(s^2+1+ys\right)}$$

$$=\frac{\sqrt{3}}{4}\cdot\frac{2s}{\left(s^2+1\right)^2-s^2}=\frac{\sqrt{3}.s}{2\left(s^4+1+2s^2-s^2\right)}=\frac{\frac{\sqrt{3}}{2}s}{s^4+1+s^2}.\text{Ans.}$$

Q.No.4.: If $L(f(t))=\bar{f}(s)$, then show that $L\left(f\left(\frac{t}{a}\right)\right)=a\bar{f}(as)$.

Sol.: $L\left(f\left(\frac{t}{a}\right)\right)=\int_0^\infty e^{-st}.f(t/a)dt$ $\qquad \left[\begin{array}{l}\text{Put } \frac{t}{a}=z, t=az, dt=adz\\ \text{when } t\to\infty, z\to\infty,\ t\to 0,\ z\to 0\end{array}\right]$

$$=\int_0^\infty e^{-saz}f(z)adz=a\int_0^\infty e^{-(as)z}f(z)dz$$

$$=a\bar{f}(as).\text{ Ans.} \qquad \left[L(f(t))=\int_0^\infty e^{-st}f(t)dt=\bar{f}(s)\right]$$

Q.No.5.: Show that $L(\sin kt\sinh kt)=\frac{2k^2s}{s^2+4k^4}$.

Sol.: $L(\sin kt\sinh kt)=L\left[\frac{e^{kt}-e^{-kt}}{2}\sin kt\right]=\frac{1}{2}L\left(e^{kt}\sin kt\right)-\frac{1}{2}L\left(e^{-kt}\sin kt\right)\left[\sinh\theta=\frac{e^{\theta}-e^{-\theta}}{2}\right]$

$$=\frac{1}{2}\left[\frac{k}{(s-k)^2+k^2}-\frac{k}{(s+k)^2+k^2}\right]$$

$$=\frac{k}{2}\left(\frac{(s+k)^2+k^2-(s-k)^2-k^2}{\left(s^2+2k^2-2ks\right)\left(s^2+2k^2+2ks\right)}\right)$$

$$=\frac{k}{2}\frac{\left(s^2+2k^2+2ks-s^2-2k^2+2ks\right)}{\left(s^2+2k^2\right)^2-4k^2s^2}=\frac{k}{2}\left(\frac{4ks}{s^4+4k^4}\right)=\frac{2k^2s}{s^4+4k^4}.\ \text{Ans.}$$

Q.No.6.: Find the Laplace transforms of $f(t)$ defined as

$$f(t)=\frac{t}{\tau},\ \text{when } 0<t<\tau$$

$$=1,\ \text{when } t>\tau$$

Sol.: $$L\{f(t)\}=\int_0^\tau e^{-st}.\frac{t}{\tau}dt+\int_\tau^\infty e^{-st}.1dt=\frac{1}{\tau}\left[\left|t.\frac{e^{-st}}{-s}\right|_0^\tau-\int_0^\tau 1.\frac{e^{-st}}{-s}dt\right]+\left|\frac{e^{-st}}{-s}\right|_\tau^\infty$$

$$=\frac{1}{\tau}\left[\frac{\tau e^{-s\tau}-0}{-s}-\left|\frac{e^{-s\tau}}{s^2}\right|_0^\tau\right]+\frac{0-e^{-s\tau}}{-s}$$

$$=\frac{-e^{-s\tau}}{s}-\frac{e^{-s\tau}-1}{\tau s^2}+\frac{e^{-s\tau}}{s}=\frac{1-e^{-s\tau}}{\tau s^2}.\ \text{Ans.}$$

Q.No.7.: Find the Laplace transform of $f(t)=\begin{cases}e^t, & 0<t<1\\ 0, & t>1\end{cases}$.

Sol.: $$L[f(t)]=\int_0^1 e^{-st}e^t dt+\int_1^\infty 0.dt=\int_0^1 e^{-st}e^t dt$$

$$=\left[\frac{e^{(1-s)}}{1-s}-\frac{1}{1-s}\right]=\frac{1}{1-s}\left[e^{1-s}-1\right].\ \text{Ans.}$$

Q.No.8.: Find the Laplace transform of $f(t)=\begin{cases}\sin t, & 0<t<\pi\\ 0, & t>\pi\end{cases}$.

Sol.: $$L[f(t)]=\int_0^\pi e^{-st}\sin t dt+\int_\pi^\infty 0.dt=\int_0^\pi e^{-st}\sin t dt.$$

Let $$I=\int e^{-st}\sin t dt=\sin t\frac{e^{-st}}{-s}-\int \cos t\frac{e^{-st}}{-s}dt=-\sin t\frac{e^{-st}}{s}+\frac{1}{s}\int \cos t e^{-st}dt$$

$$=-\sin t\frac{e^{-st}}{-s}+\frac{1}{s}\left[\cos t\frac{e^{-st}}{-s}-\int -\sin t\frac{e^{-st}}{-s}dt\right]$$

$$= \frac{-\sin te^{-st}}{s} - \frac{\cos te^{-st}}{s^2} - \frac{1}{s^2}\int e^{-st}\sin tdt$$

$$= -\frac{\sin te^{-st}}{s} - \frac{\cos te^{-st}}{s^2} - \frac{I}{s^2}$$

$$\Rightarrow I\left[1+\frac{1}{s^2}\right] = \left[\frac{-\sin te^{-st}}{s} - \frac{\cos te^{-st}}{s^2}\right] = \frac{1}{s^2}\left[-s.\sin te^{-st} - \cos te^{-st}\right]$$

.

$$\therefore L[f(t)] = \frac{1}{s^2+1}\left[-s.\sin te^{-st} - \cos te^{-st}\right]_0^\pi$$

$$= \frac{1}{s^2+1}\left[e^{-\pi s} + e^0\right] = \frac{e^{-\pi s}+1}{s^2+1}. \text{ Ans.}$$

Q.No.9.: Find the Laplace transform of $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ t-1, & 2 < t < 3 \\ 7, & t > 3 \end{cases}$.

Sol.: $L[f(t)] = \int_0^2 t^2e^{-st}dt + \int_2^3 (t-1)e^{-st}dt + \int_3^\infty 7e^{-st}dt$

$$= \left[\frac{t^2e^{-st}}{-s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{-s^3}\right]_0^2 + \left[\frac{(t-1)e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_2^3 + \left[\frac{7e^{-st}}{-s}\right]_3^\infty$$

$$= \left[\left(\frac{4e^{-2s}}{-s} - \frac{4e^{-2s}}{s^2} + \frac{2e^{-2s}}{s^3}\right) + \left(\frac{-2}{s^3}\right)\right] + \left[\left(\frac{2e^{-3s}}{-s} - \frac{e^{-3s}}{s^2}\right) - \left(\frac{e^{-2s}}{-s} - \frac{e^{-2s}}{s^2}\right)\right] + 7\left[0 + \frac{e^{-3s}}{s}\right]$$

$$= \left[-\frac{4e^{-2s}}{s} - \frac{2e^{-2s}}{s^2} + \frac{4e^{-2s}}{s^3} - \frac{2}{s^3}\right] + \left[-\frac{2e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2}\right] + \frac{7e^{-3s}}{s}$$

$$= -\frac{e^{-2s}}{s^3}\left[4s^2 + 4s - 2 - s^2 - s\right] - \frac{2}{s^3} + \frac{e^{-3s}}{s^2}\left[-2s - 1 + 7s\right]$$

$$= -\frac{2}{s^3} - \frac{e^{-2s}}{s^3}\left(3s^2 + 3s - 2\right) + \frac{e^{-3s}}{s^2}(5s-1). \text{ Ans.}$$

Q.No.10.: Find the Laplace transform of $f(t) = |t-1| + |t+1| \qquad t \geq 0$.

Sol.: Given $f(t) = |t-1| + |t+1|$.

$$\Rightarrow f(t)=\begin{cases}-(t-1)+(t+1), & t<1\\ 2, & t=1\\ 2t, & t>1\end{cases}$$

$$\therefore L[f(t)]=\int_0^1 e^{-st}.2dt+\int_1^\infty e^{-st}.2tdt=2\left[\frac{e^{-st}}{-s}\right]_0^1+2\left[\frac{te^{-st}}{-s}-\frac{e^{-st}}{s^2}\right]_1^\infty$$

$$=-\frac{2}{s}\left[e^{-s}-1\right]+2\left[-\frac{e^{-s}}{-s}+\frac{e^{-s}}{s^2}\right]=-\frac{2}{s}\left[\frac{1-e^{s}}{e^{s}}\right]+2\left[\frac{se^{-s}+e^{-s}}{s^2}\right]$$

$$=\frac{2\left(e^{s}-1\right)}{se^{s}}+\frac{2e^{-s}(s+1)}{s^2}=\frac{2}{s}+\frac{2e^{-s}}{s^2}=\frac{2}{s}\left(1+\frac{e^{-s}}{s}\right).\ \text{Ans.}$$

Q.No.11.: Find the Laplace transform of $f(t)=\begin{cases}\cos\left(t-\frac{2\pi}{3}\right), & t>\frac{2\pi}{3}\\ 0, & t<\frac{2\pi}{3}\end{cases}$.

Sol.: $L[f(t)]=\int e^{-st}\cos\left(t-\frac{2\pi}{3}\right)dt$.

Let $I=\int e^{-st}\cos\left(t-\frac{2\pi}{3}\right)dt=-\cos\left(t-\frac{2\pi}{3}\right)\frac{e^{st}}{s}-\frac{1}{s}\int\sin\left(t-\frac{2\pi}{3}\right)e^{-st}dt$

$$=\frac{-\cos\left(t-\frac{2\pi}{3}\right)e^{-st}}{s}-\frac{1}{s}\left[\sin\left(t-\frac{2\pi}{3}\right)\frac{e^{-st}}{-s}-\int\cos\left(t-\frac{2\pi}{3}\right)\frac{e^{-st}}{-s}dt\right]$$

$$=\frac{-\cos\left(t-\frac{2\pi}{3}\right)e^{-st}}{s}+\frac{1}{s^2}\sin\left(t-\frac{2\pi}{3}\right)e^{-st}-\frac{I}{s^2}.$$

$$\Rightarrow I\left[1+\frac{1}{s^2}\right]=\left[\frac{-\cos\left(t-\frac{2\pi}{3}\right)e^{-st}}{s}+\frac{\sin\left(t-\frac{2\pi}{3}\right)e^{-st}}{s^2}\right]$$

$$\Rightarrow L[f(t)]=\frac{s^2}{s^2+1}\left[\frac{-\cos\left(t-\frac{2\pi}{3}\right)e^{-st}}{s}+\frac{\sin\left(t-\frac{2\pi}{3}\right)e^{-st}}{s^2}\right]_{2\pi/3}^{\infty}$$

$$= \frac{s^2}{s^2+1}\left[0 + \frac{e^{(-2\pi/3)s}}{s} + 0\right] = \frac{s^2}{s^2+1}.\frac{e^{-(2\pi/3)s}}{s} = e^{-2\pi s/3}.\frac{s}{s^2+1}. \text{ Ans.}$$

Q.No.12.: Find the Laplace Transform of f(t) defined by

$$f(t) = \begin{cases} t, & 0 < t < 4 \\ 5, & t > 4 \end{cases}.$$

Sol.: Here $f(t) = \begin{cases} t, & 0 < t < 4 \\ 5, & t > 4 \end{cases}$.

Consider $L(f(t)) = \int_0^{\infty} e^{-st} f(t)dt = \int_0^4 \underset{\text{II}}{e^{-st}} . \underset{\text{I}}{t}\, dt + \int_4^{\infty} e^{-st}.5.dt$.

Integrating by parts, we get

$$= (t)\left|\frac{e^{-st}}{-s}\right|_0^4 - \left|(1)\left(\frac{e^{-st}}{-s^2}\right)\right|_0^4 + 5.\int_4^{\infty} e^{-st}dt = -\frac{1}{s}\left(4e^{-4s}\right) - \frac{1}{s^2}\left(e^{-4s} - 1\right) + 5\left|\frac{e^{-st}}{-s}\right|_4^{\infty}$$

$$= e^{-4s}\left(\frac{-4}{s} - \frac{1}{s^2} + \frac{5}{s}\right) - \frac{5}{s} + \frac{1}{s^2} = e^{-4s}\left(\frac{1}{s} - \frac{1}{s^2}\right) + \frac{1}{s^2} - \frac{5}{s}. \text{ Ans.}$$

Q.No.13.: Find the Laplace Transform of f(t) defined by

$$f(t) = \begin{cases} \cos t, & 0 < t < 2\pi \\ 0, & t > 2\pi \end{cases}.$$

Sol.: Here $f(t) = \begin{cases} \cos t, & 0 < t < 2\pi \\ 0, & t > 2\pi \end{cases}$.

Consider $L(f(t)) = \int_0^{\infty} e^{-st} f(t)dt = \int_0^{2\pi} e^{-st}.\cos t dt + \int_{2\pi}^{\infty} 0 dt$

$$= \left|\frac{e^{-st}}{s^2+1}\left((-s)\cos t + \sin t\right)\right|_0^{2\pi} \qquad \left[\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx - b\sin bx)\right]$$

$$= \frac{e^{-2\pi s}}{s^2+1}(-s) - \frac{1}{s^2+1}(-s) = s\left(\frac{1-e^{-2\pi s}}{s^2+1}\right).\text{Ans.} \qquad [\text{Here } a = -s, b = 1]$$

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Home Assignments

Find the Laplace Transforms of the following functions:

Q.No.	Function	Answer
1.	$\cosh at \cos at$	$\frac{a(s^2+2a^2)}{s^4+4a^4}$
2.	$t^2 e^{-2t}$	$\frac{2}{(s+2)^3}$
3.	$(1+te^{-t})^3$	$\frac{1}{s}+\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}+\frac{6}{(s+3)^4}$
4.	$\sin^2 kt$	$\frac{2k^2}{s(s^2+4k^2)}$
5.	$4e^{5t}+6t^3-3\sin 4t+2\cos 2t$	$\frac{4}{s-5}+\frac{36}{s^4}-\frac{12}{s^2+16}+\frac{2s}{s^2+4}$
6.	$\cos^3 at$	$\frac{s(s^2+7a^2)}{(s^2+a^2)(s^2+9a^2)}$
7.	$\cos 3t.\cos 2t \cos t$	$\frac{1}{4}\left(\frac{s}{s^2+36}+\frac{s}{s^2+16}+\frac{s}{s^2+4}+\frac{1}{s}\right)$
8.	$\sin\sqrt{t}$	**Hint:** Use $\Gamma\left(n+\frac{1}{2}\right)=\frac{1.3.5....(2n-1)}{2^n}\sqrt{\pi}$ for n positive integer. $\frac{\sqrt{\pi}}{2s\sqrt{s}}\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(-\frac{1}{4s}\right)^{n-1}=\frac{\sqrt{\pi}}{2s\sqrt{s}}e^{-\left(\frac{1}{4s}\right)}$. Ans
9.	If $L\{f(t)\}=\frac{20-4s}{s^2-4s+20}$. Find $L\{f(3t)\}$	$\frac{4(15-s)}{(s^2-12s+180)}$
10.	If $L\{\sin t\}=\frac{1}{s^2+1}$.	$\frac{3}{s^2+9}$

	Find $L\{\sin 3t\}$	
11.	$(t+2)^2 e^t$	$\dfrac{(4s^2-4s+2)}{(s-1)^3}$
12.	$e^{-4t}\cosh 2t$	$\dfrac{(s+4)}{(s^2+8s+12)}$
13.	$\sinh at \sin at$	$\dfrac{2a^2 s}{(s^2+4a^4)}$
14.	$e^{2t}(3\sin 4t-4\cos 4t)$	$\dfrac{(20-4s)}{(s^2-4s+20)}$
15.	$\dfrac{t^{n-1}}{1-e^{-t}}$	$\sum\limits_{m=0}^{\infty}\dfrac{\Gamma(n)}{(s+m)^n}$
16.	$e^{-2t}\sin^3 t$	$\dfrac{3}{4}\dfrac{1}{s^2+4s+5}-\dfrac{3}{4}\dfrac{1}{s^2+4s+13}$
17.	$\sin^4 t.e^{2t}$	$\dfrac{1}{8}\left[\dfrac{3}{s-2}-\dfrac{4(s-2)}{(s-2)^2+4}+\dfrac{s-2}{(s-2)^2+16}\right]$

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3rd Topic

Laplace Transforms

General Properties of Laplace Transform and their Applications

- LT of Derivatives of f(t)
- LT of Integrals
- LT of f(t) Multiplied by t^n
- LT of f(t) Divided by t
- Evaluation of Integrals by LT

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General properties of Laplace transform:

4. Laplace Transforms of Derivatives:

If $f'(t)$ be continuous and $L\{f(t)\} = \bar{f}(s)$,

then $L\{f'(t)\} = s\bar{f}(s) - f(0)$, provided $\lim_{t\to\infty}\left[e^{-st}f(t)\right] = 0$.

Proof: $L\{f'(t)\} = \int_0^\infty e^{-st}f'(t)dt$

$$= \left|e^{-st}f(t)\right|_0^\infty - \int_0^\infty (-s)e^{-st}.f(t)dt \qquad \text{(Integrating by parts)}$$

$$= \underset{t\to\infty}{\mathrm{Lt}}\ e^{-st}f(t) - f(0) + s\int_0^\infty e^{-st}f(t)dt$$

$= 0 - f(0) + sL\{f(t)\} = s\bar{f}(s) - f(0)$. This completes the proof.

Note: If f(t) to be such that $\underset{t\to\infty}{Lt}\ e^{-st}f(t)=0$, then f(t) is said to be of exponential order s.

Remarks: The above theorem can be generalized.

Theorem: If $f'(t)$ and its first $(n-1)$ derivatives be continuous, then

$$L\{f^n(t)\}=s^n\bar{f}(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots\ldots\ldots\ldots-f^{n-1}(0).$$

Proof: Using the general rule of integration by parts, we get

$$L\{f^n(t)\}=\int_0^\infty e^{-st}f^n(t)dt$$

$$=\Big|e^{-st}f^{n-1}(t)-(-s)e^{-st}f^{n-2}(t)+(-s)^2e^{-st}f^{n-3}(t)\ldots\ldots\ldots\ldots$$

$$+(-1)^{n-1}(-s)^{n-1}e^{-st}.f(t)\Big|_0^\infty+(-1)^n(-s)^n\int_0^\infty e^{-st}f(t)dt$$

$$=-f^{n-1}(0)-sf^{n-2}(0)-s^2f^{n-3}(0)-\ldots\ldots\ldots-s^{n-1}f(0)+s^n\int_0^\infty e^{-st}f(t)dt$$

Assuming that $\underset{t\to\infty}{Lim}\,e^{-st}f^m(t)=0$ for m = 0, 1, 2, 3, ..., $n-1$.

$$L\{f^n(t)\}=s^n\bar{f}(s)-s^{n-1}f(0)-s^{n-2}f(0)-\ldots\ldots\ldots\ldots-f^{n-1}(0).$$

This proves the required result.

Remarks: In particular, when n = 0, 1, 2, 3, ..., we have

$$L\{f''(t)\}=s^2\bar{f}(s)-sf(0)-f'(0)$$

$$L\{f'''(t)\}=s^3\bar{f}(s)-s^2f(0)-sf(0)-f''(0)$$

$$L\{f^{iv}(t)\}=s^4\bar{f}(s)-s^3f(0)-s^2f(0)-sf(0)-f'''(0)$$ and so on.

These results will be used in the solution of differential equations.

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5. Laplace Transforms of an Integrals:

If $L\{f(t)\}=\bar{f}(s)$, then $L\left\{\int_0^t f(u)du\right\}=\frac{1}{s}\bar{f}(s)$.

or

Laplace transform of an integral f amounts to division of $\bar{f}(s)$ by s.

Proof: Let $\phi(t) = \int_0^t f(u)du$, then $\phi'(t) = f(t)$ and $\phi(0) = 0$.

$\therefore L\{\phi'(t)\} = s\bar{\phi}(s) - \phi(0) \Rightarrow L\{f(t)\} = s\bar{\phi}(s) \Rightarrow \bar{f}(s) = sL\{\phi(t)\} \Rightarrow L\{\phi(t)\} = \frac{1}{s}\bar{f}(s)$.

$$\Rightarrow L\left(\int_0^t f(u)du\right) = \frac{1}{s}\bar{f}(s).$$

This completes the proof.

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6. Laplace transform of f(t) multiplied by t^n :

If $L\{f(t)\} = \bar{f}(s)$, then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[\bar{f}(s)\right]$, where n = 1, 2, 3,............

or

Laplace transform of f(t) multiplied by t^n amounts to differentiation of $\bar{f}(s)$, n times w.r.t. s with $(-1)^n$ as sign.

Proof: We prove this theorem by induction.

Since given $L\{f(t)\} = \bar{f}(s) \Rightarrow \int_0^\infty e^{-st} f(t)dt = \bar{f}(s)$.

Differentiating both sides w.r.t. s (using Leibnitz's rule for differentiation under the integral sign), we get

$$\frac{d}{ds}\left\{\int_0^\infty e^{-st} f(t)dt\right\} = \frac{d}{ds}\{\bar{f}(s)\} \Rightarrow \int_0^\infty \frac{d}{ds}\left(e^{-st}\right)f(t)dt = \frac{d}{ds}\{\bar{f}(s)\}$$

$$\Rightarrow \int_0^\infty \{-te^{-st}.f(t)\}dt = \frac{d}{ds}\{\bar{f}(s)\} \Rightarrow \int_0^\infty e^{-st}[tf(t)]dt = -\frac{d}{ds}\left[\bar{f}(s)\right]$$

$$\Rightarrow L\{tf(t)\} = -\frac{d}{ds}\left[\bar{f}(s)\right],$$

which proves the theorem for n = 1.

Now assume the theorem to be true for n = m (say), so that

$$L\{t^m f(t)\} = (-1)^m \frac{d^m}{ds^m}[\bar{f}(s)] \Rightarrow \int_0^\infty e^{-st}[t^m f(t)]dt = (-1)^m \frac{d^m}{ds^m}[\bar{f}(s)].$$

Differentiating both sides w.r.t. s, we get

$$\frac{d}{ds}\left[\int_0^\infty e^{-st}t^m f(t)dt\right] = (-1)^m \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)] \Rightarrow \left[\int_0^\infty \frac{d}{ds}e^{-st}t^m f(t)dt\right] = (-1)^m \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)]$$

$$\Rightarrow \int_0^\infty (-te^{-st})t^m f(t)dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)] \Rightarrow \int_0^\infty e^{-st}[t^{m+1}f(t)]dt = (-1)^{m+1}\frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)].$$

$$\Rightarrow L\{t^{m+1}f(t)\} = (-1)^{m+1}\frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)]$$

This shows that, the theorem is true for n = m + 1.

Hence, by mathematical induction, the theorem is true for all positive integral values of n.

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7. Laplace transform of f(t) divided by t:

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty \bar{f}(s)ds$, provided the integral exists.

or

Laplace transform of f(t) divided by t amounts to integration of $\bar{f}(s)$ between the limits s to ∞.

Proof: We have $\bar{f}(s) = \int_0^\infty e^{-st}f(t)dt$.

Integrating both sides w. r. t. s from s to ∞, we get

$$\int_s^\infty \bar{f}(s)ds = \int_s^\infty\left[\int_0^\infty e^{-st}f(t)dt\right]ds.$$

Since s and t are independent, so changing the order of integration on the right hand side, we have

$$\int_s^\infty \bar{f}(s)ds = \int_0^\infty \left(\int_s^\infty f(t)e^{-st}ds\right)dt$$

$$= \int_0^\infty f(t)\left(\int_s^\infty e^{-st}ds\right)dt = \int_0^\infty f(t)\left|\frac{e^{-st}}{-t}\right|_s^\infty dt = \int_0^\infty e^{-st}.\frac{f(t)}{t}dt = L\left\{\frac{1}{t}f(t)\right\}.$$

Thus $L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty \bar{f}(s)ds$.

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Now let us understand the present concepts by the evaluation of Laplace transforms of the various types of functions:

Problems on the topic "Laplace transforms of an Integrals":

REMEMBER

5. Laplace Transforms of an Integrals:

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\bar{f}(s)$.

Q. No.1.: Find $L\left(\int_0^t e^{-t}\cos t dt\right)$.

Sol.: $L\left(\int_0^t e^{-t}\cos t dt\right)$.

Since $L(\cos t) = \frac{s}{s^2+1}$.

$$\therefore L\left(e^{-t}\cos t\right) = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{\left(s^2+2s+1\right)+1} = \frac{s+1}{s^2+2s+2}.$$

$$\therefore L\left(\int_0^t e^{-t}\cos t dt\right) = \frac{s+1}{s\left(s^2+2s+2\right)}.$$ Ans.

Q.No.2.: Evaluate **(i)** $L\left(\int_0^t te^{-t}\sin 4t dt\right)$, **(ii)** $L\left(\int_0^t \frac{\cos at - \cos bt}{t}dt\right)$.

Sol.: (i). $L\left(\int\limits_0^t te^{-t}\sin 4t dt\right)$

$$L(t\sin 4t) = -\frac{d}{ds}\left(\frac{4}{s^2+16}\right) = \frac{8s}{\left(s^2+16\right)}.$$

$$\therefore L\left(e^{-t}t\sin 4t\right) = \frac{8(s+1)}{\left[(s+1)^2+16\right]^2} = \frac{8(s+1)}{\left(s^2+2s+17\right)^2}$$

$$L\left(\int\limits_0^t te^{-t}\sin 4t dt\right) = \frac{8(s+1)}{s\left(s^2+2s+17\right)^2}. \text{ Ans.}$$

(ii). $L\left(\int\limits_0^t \frac{\cos at - \cos bt}{t} dt\right)$.

$$L(\cos at - \cos bt) = \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right).$$

$$L\left(\frac{\cos at - \cos bt}{t}\right) = \int\limits_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds = \frac{1}{2}\left[\log\left(s^2+a^2\right) - \log\left(s^2+b^2\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty = \frac{1}{2}\left[\log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]_s^\infty = -\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}.$$

$$\therefore L\left(\int\limits_0^t \frac{\cos at - \cos bt}{t} dt\right) = -\frac{1}{2s}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) = \frac{1}{2s}\log\left(\frac{s^2+b^2}{s^2+a^2}\right). \text{ Ans.}$$

Q.No.3.: Find the Laplace transform of $L\left\{\int_0^t \frac{1-e^{-u}}{u} du\right\}$.

Sol.: Here the integrand is $f(t) = \left(1-e^{-t}\right)/t$.

We know that $L\{f(t)\} = L\left\{\frac{1-e^{-t}}{t}\right\} = F(s) = \log\left(1+\frac{1}{s}\right)$.

Using the theorem of Laplace transform of integrals

$$L\left\{\int_0^t \frac{1-e^{-u}}{u}\,du\right\} = \frac{1}{s}\log\left(1+\frac{1}{s}\right)$$

Q.No.4.: Find the Laplace' transform of $L\left\{\int_0^t \int_0^t \int_0^t \cos au\,du\,du\,du\right\}$.

Sol.: Here integrand is $f(t) = \cos at$.

$$L\{f(t)\} = L\{\cos at\} = \frac{s}{s^2+a^2}.$$

Using the theorem of LT of integrals

$$L\left\{\int_0^t \cos au\,du\right\} = \frac{1}{s}\cdot\frac{s}{s^2+a^2} = \frac{1}{s^2+a^2}.$$

Applying repeatedly

$$L\left\{\int_0^t \int_0^t \cos au\,du\,du\right\} = \frac{1}{s}\cdot\frac{1}{s^2+a^2}.$$

$$L\left\{\int_0^t \int_0^t \int_0^t \cos au\,du\,du\,du\right\} = \frac{1}{s^2}\cdot\frac{1}{s^2+a^2}.$$

Q.No.5.: Find the Laplace' transform of $L\left\{\sinh ct\int_0^t e^{au}\sinh bu\,du\right\}$**.**

Sol.: $L\{\sinh bt\} = \dfrac{b}{s^2-b^2}$**.**

$$L\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2-b^2}.$$

$$L\left\{\int_0^t e^{au}\sinh bu\,du\right\} = \frac{1}{s}\cdot\frac{b}{(s-a)^2-b^2}. \qquad \text{(i)}$$

Now consider

$$L\left\{\sinh ct\int_0^t e^{au}\sinh bu\,du\right\} = L\left\{\left(\frac{e^{ct}-e^{-ct}}{2}\right)\int_0^t e^{au}\sinh budu\right\}$$

$$= \frac{1}{2}L\left\{e^{ct}\int_0^t e^{au}\sinh budu\right\} + \frac{1}{2}L\left\{e^{-ct}\int_0^t e^{au}\sinh budu\right\}$$

Using (i)

$$=\frac{1}{2}\frac{b}{s\left[(s-a)^2-b^2\right]}\Bigg|_{s=s-c}+\frac{1}{2}\frac{b}{s\left[(s-a)^2-b^2\right]}\Bigg|_{s=s+c}$$

$$=\frac{1}{2}\frac{b}{(s-c)\left[(s-c-a)^2-b^2\right]}+\frac{1}{2}\frac{b}{(s+c)\left[(s+c-a)^2-b^2\right]}.$$

Q.No.6.: Find the Laplace' transform of $\int_0^\infty \frac{\cos 6t-\cos 4t}{t}dt$.

Sol.: This integral can be looked upon as

$$\int_0^\infty e^{-0.t}.\left(\frac{\cos 6t-\cos 4t}{t}\right)dt$$

So that the given integral is Laplace transform of $\frac{\cos 6t-\cos 4t}{t}$ with s = 0.

$$L\left(\frac{\cos 6t-\cos 4t}{t}\right)_{\text{with } s=0}$$

$$=\frac{1}{2}\text{In}\left(\frac{s^2+b^2}{s^2+a^2}\right)\Bigg|_{\substack{s=0\\a=6\\b=4}}=\text{In}\frac{2}{3}.$$

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Problems on the topic "Multiplication by t^n":

REMEMBER

6. Laplace transform of f(t) multiplied by t^n :

If $L\{f(t)\}=\bar{f}(s)$, then $L\{t^n f(t)\}=(-1)^n\frac{d^n}{ds^n}\left[\bar{f}(s)\right]$, where n = 1, 2, 3,............

Q.No.1.: Find the Laplace transforms of

(i) $t\cos at$, **(ii)** $t^2\sin at$, **(iii)** t^3e^{-3t}, **(iv)** $te^{-t}\sin 3t$.

Sol.: (i) Since $L(\cos at)=\frac{s}{\left(s^2+a^2\right)}$.

$$\therefore L(t\cos at)=-\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)=-\frac{s^2+a^2-s.2s}{\left(s^2+a^2\right)^2}=\frac{s^2-a^2}{\left(s^2+a^2\right)^2}. \text{ Ans.}$$

(ii) Since $L(\sin at) = \dfrac{a}{s^2 + a^2}$.

$$\therefore L\left(t^2 \sin at\right) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2 + a^2}\right) = \frac{d}{ds}\left\{\frac{-2as}{\left(s^2 + a^2\right)^2}\right\} = \frac{2a\left(3s^2 - a^2\right)}{\left(s^2 + a^2\right)^3}. \text{ Ans.}$$

(iii) Since $L\left(e^{-3t}\right) = \dfrac{1}{(s+3)}$.

$$\therefore L\left(t^3 e^{-3t}\right) = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s+3}\right) = -\frac{(-1)^3 .3!}{(s+3)^{3+1}} = \frac{6}{(s+3)^4}. \text{ Ans.}$$

(iv) Since $L(\sin 3t) = \dfrac{3}{s^2 + 3^2}$, therefore $L(t \sin 3t) = -\dfrac{d}{ds}\left(\dfrac{3}{s^2 + 3^2}\right) = \dfrac{6s}{\left(s^2 + 9\right)^2}$.

By using the **shifting property**, we get

$$L\left(e^{-t} t \sin 3t\right) = \frac{6(s+1)}{\left[(s+1)^2 + 9\right]^2} = \frac{6(s+1)}{\left(s^2 + 2s + 10\right)^2}. \text{ Ans.}$$

Q.No.2.: Find the Laplace transforms of the function $t \sin^2 t$.

Sol.: Since $t \sin^2 t = \dfrac{t}{2}(1 - \cos 2t)$.

$$\therefore L\left(t \sin^2 t\right) = L\left(\frac{t}{2}\right) - L\left(\frac{t}{2}\cos 2t\right) = \frac{1}{2}L(t) - \frac{1}{2}L(t \cos 2t) \qquad \text{(by linear property)}$$

$$= \frac{1}{2}.\frac{1}{s^2} - \frac{1}{2}\left[-\frac{d}{ds}\left(\frac{s}{s^2 + 4}\right)\right] = \frac{1}{2s^2} + \frac{1}{2}\left[\frac{s^2 + 4 - s.2s}{\left(s^2 + 4\right)^2}\right]$$

$$= \frac{1}{2}\frac{s^4 + 16 + 8s^2 - s^4 + 4s^2}{s^2\left(s^2 + 4\right)^2} = \frac{1}{2}\left[\frac{16 + 12s^2}{s^2\left(s^2 + 4\right)^2}\right]$$

$$= \frac{2\left(3s^2 + 4\right)}{s^2\left(s^2 + 4\right)^2}. \text{ Ans.}$$

Q.No.3.: Find the Laplace transforms of the function $t \sin 3t.\cos 2t$.

Sol.: Since $\sin 3t.\cos 2t = \dfrac{1}{2}(\sin 5t + \sin t)$.

$$\therefore L(t \sin 3t.\cos 2t) = \frac{1}{2}\left[L(t \sin 5t) + L(t \sin t)\right] = \frac{1}{2}\left[-\frac{d}{ds}\left(\frac{5}{s^2 + 25}\right) - \frac{d}{ds}\left(\frac{1}{s^2 + 1}\right)\right]$$

$$=\frac{1}{2}\left[\frac{-(-5.2s)}{\left(s^2+25\right)^2}-\frac{(-2s)}{\left(s^2+1\right)^2}\right]=\frac{1}{2}\left[\frac{10s}{\left(s^2+25\right)^2}+\frac{2s}{\left(s^2+1\right)^2}\right]$$

$$=\frac{5s}{\left(s^2+25\right)^2}+\frac{s}{\left(s^2+1\right)^2}. \text{ Ans.}$$

Q.No.4.: Find the Laplace transforms of the function $t^2\cos at$.

Sol.: Since we know that $L(\cos at)=\frac{s}{s^2+a^2}$.

$$\therefore L\left(t^2\cos at\right)=\frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)=\frac{d}{ds}\left[\frac{s^2+a^2-2s.s}{\left(s^2+a^2\right)^2}\right]=\frac{d}{ds}\left[\frac{1}{\left(s^2+a^2\right)}-\frac{2s^2}{\left(s^2+a^2\right)^2}\right]$$

$$=\frac{-2s}{\left(s^2+a^2\right)^2}-\left[\frac{\left(s^2+a^2\right)^2(4s)-2s^2.2\left(s^2+a^2\right)2s}{\left(s^2+a^2\right)^4}\right]$$

$$=\frac{-2s\left(s^2+a^2\right)-4s\left[\left(s^2+a^2\right)-2s^2\right]}{\left(s^2+a^2\right)^3}=\frac{-2s^3-2s.a^2-4s^3-4sa^2+8s^3}{\left(s^2+a^2\right)^3}$$

$$=\frac{2s^3-6sa^2}{\left(s^2+a^2\right)^3}. \text{ Ans.}$$

Q.No.5.: Find the Laplace transforms of the function $t\sinh at$.

Sol.: Since we know that $L(\sinh at)=\frac{a}{s^2-a^2}$.

$$\therefore L(t\sinh at)=\frac{-d}{ds}\left(\frac{a}{s^2-a^2}\right)=-a\left[\frac{-2s}{\left(s^2-a^2\right)^2}\right]=\frac{2as}{\left(s^2-a^2\right)^2}. \text{ Ans.}$$

Q.No.6.: Find the Laplace transforms of the function $te^{-2t}\sin 2t$.

Sol.: Since $L(\sin 2t)=\frac{2}{s^2+4}$.

$$\therefore L(t\sin 2t)=\frac{-d}{ds}\left(\frac{2}{s^2+4}\right)=-2\left[\frac{-2s}{\left(s^2+4\right)^2}\right]=\frac{4s}{\left(s^2+4\right)^2}.$$

$$\therefore L\left(te^{-2t}\sin 2t\right)=\frac{4(s+2)}{\left[(s+2)^2+4\right]^2}=\frac{4(s+2)}{\left(s^2+4s+4+4\right)^2}=\frac{4(s+2)}{\left(s^2+4s+8\right)^2}. \text{ Ans.}$$

Q.No.7.: Find the Laplace transforms of the function $t\,e^{-t}\cosh t$.

Sol.: Since we know that $L(\cosh at)=\dfrac{s}{s^2-a^2}$.

$$\therefore L(t\cosh at)=\frac{-d}{ds}\left(\frac{s}{s^2-a^2}\right)=-\left[\frac{-2s^2}{\left(s^2-a^2\right)^2}+\frac{1}{s^2-a^2}\right]$$

$$=\frac{-s^2+a^2+2s^2}{\left(s^2-a^2\right)^2}=\frac{s^2+a^2}{\left(s^2-a^2\right)^2}$$

$$\therefore L\left(te^{-t}\cosh at\right)=\frac{(s+1)^2+a^2}{\left[(s+1)^2-a^2\right]^2}.$$

Substituting a = 1, we get

$$\therefore L\left(te^{-t}\cosh t\right)=\frac{s^2+2s+2}{\left(s^2+2s\right)^2}.\ \text{Ans.}$$

Q.No.8.: Find the Laplace transforms of the function $t^2e^{-3t}\sin 2t$

Sol.: Since $L(\sin 2t)=\dfrac{2}{s^2+4}$.

$$L(t\sin 2t)=\frac{-d}{ds}\left(\frac{2}{s^2+4}\right)=\frac{4s}{\left(s^2+4\right)^2}.$$

$$\therefore L\left(t^2\sin 2t\right)=-\frac{d}{ds}\frac{4s}{\left(s^2+4\right)^2}=-4\left[\frac{1}{\left(s^2+4\right)^2}-\frac{(2s)^2}{\left(s^2+4\right)^3}\right]=4\left[\frac{4s^2}{\left(s^2+4\right)^3}-\frac{1}{\left(s^2+4\right)^2}\right]$$

$$=4\left[\frac{4s^2-s^2-4}{\left(s^2+4\right)^3}\right]=4\left[\frac{3s^2-4}{\left(s^2+4\right)^3}\right].$$

$$\therefore L\left(e^{-3t}t^2\sin 2t\right)=4\left[\frac{3(s+3)^2-4}{\left\{(s+3)^2+4\right\}^3}\right]=\frac{4\left[3\left(s^2+6s+9\right)-4\right]}{\left(s^2+6s+13\right)^3}=\frac{4\left(3s^2+18s+23\right)}{\left(s^2+6s+13\right)^3}.\ \text{Ans.}$$

Q.No.9.: Find the Laplace' transform of $f(t)=t^2$.

Sol.: $f(0)=0$, $f'(0)=0$, $f''(t)=2$.

By theorem

$L\{f''\} = s^2 L\{f\} - sf(0) - f'(0)$

$L(2) = s^2 L\{t^2\} - s.0 - 0$.

$\therefore L\{t^2\} = \frac{1}{s^2} L\{2\} = \frac{2}{s^2} L\{1\} = \frac{2}{s^2} \cdot \frac{1}{s} = \frac{2}{s^3}$.

Q.No.10.: Given $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$, prove that $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \left(\frac{\pi}{s}\right)^{\frac{1}{2}} e^{-\frac{1}{4s}}$.

Sol.: Take $f(t) = \frac{\cos\sqrt{t}}{\sqrt{t}}$.

Since $\frac{\cos\sqrt{t}}{\sqrt{t}}$ is the derivative of $\sin\sqrt{t}$, choose $g(t) = \sin\sqrt{t}$, then $g'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$,

$g(0) = 0$

Using theorem $L\{g'\} = sL\{g\} - g(0)$, we have

$L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s.L\{\sin\sqrt{t}\} - 0$.

$L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s.\sqrt{\pi}.\frac{e^{-\frac{1}{4s}}}{2s^{3/2}}$.

$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{s^{1/2}}.e^{-\frac{1}{4s}}$.

Q.No.11.: Show that $L\{t\sin at\} = \frac{2as}{\left(s^2 + a^2\right)^2}$.

Sol.: $L\{\sin at\} = \frac{a}{s^2 + a^2}$

$\therefore L\{t.\sin at\} = -\frac{d}{ds}\left\{\frac{a}{s^2 + a^2}\right\} = \frac{2as}{\left(s^2 + a^2\right)^2}$.

Q.No.12.: Find the Laplace' transform of $L\{t.e^{-2t}\sin t\}$.

Sol.: $L(\sin t) = \frac{1}{s^2 + 1}$.

$$L\left\{e^{-2t}\sin t\right\}=\frac{1}{(s+2)^2+1}=\frac{1}{s^2+4s+5}.$$

$$L\left\{te^{-2t}\sin t\right\}=-\frac{d}{ds}\left(\frac{1}{s^2+4s+5}\right)=\frac{2s+4}{\left(s^2+4s+5\right)^2}. \text{ Ans.}$$

Q.No.13.: Find the Laplace' transform of $L\left\{\left(t^2-3t+2\right)\sin 3t\right\}$.

Sol.: $L\{\sin 3t\}=\dfrac{3}{s^2+3^2}=\dfrac{3}{s^2+9}$

$$L\left\{\left(t^2-3t+2\right)\sin 3t\right\}=L\left\{t^2.\sin 3t\right\}-3L\{t.\sin 3t\}+2L\{\sin 3t\}.$$

Using multiplication by t

$$=(-1)^2\frac{d^2}{ds^2}\left\{\frac{3}{s^2+9}\right\}-3.(-1)\frac{d}{ds}\left\{\frac{3}{s^2+9}\right\}+2.\frac{3}{s^2+9}$$

$$=\frac{\left(18s^2-54\right)}{\left(s^2+9\right)^3}+3.\frac{(-6s)}{\left(s^2+9\right)^2}+\frac{6}{s^2+9}$$

$$=\frac{6s^4-18s^3+126s^2-162s+432}{\left(s^2+9\right)^3}.$$

Q.No.14.: Find the Laplace transform of $\sin at-at\cos at$.

Sol.: $L[\sin at-at\cos at]=L(\sin at)-aL(t\cos at)$

$$=\frac{a^2}{s^2+a^2}-a(-1)^1\frac{d}{ds}L(\cos at)$$

$$=\frac{a}{s^2+a^2}+a\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)=\frac{a}{s^2+a^2}+a\frac{\left(s^2+a^2\right).1-s.2s}{\left(s^2+a^2\right)^2}$$

$$=\frac{a}{s^2+a^2}+\frac{a\left(a^2-s^2\right)}{\left(s^2+a^2\right)^2}=\frac{a}{s^2+a^2}+\frac{a\left(a^2+s^2\right)+a\left(a^2-s^2\right)}{\left(s^2+a^2\right)^2}=\frac{2a^3}{\left(s^2+a^2\right)^2}. \text{ Ans.}$$

Q.No.15.: Find the Laplace transform of $e^{-t}(\sin 2t-2t\cos 2t)$.

Sol.: $L(\sin 2t-2t\cos 2t)=L(\sin 2t)-L(2t\cos 2t)$

$$=\frac{2}{s^2+4}-2L(t\cos 2t)=\frac{2}{s^2+4}-2(-1)^1\frac{d}{ds}L(\cos 2t)$$

$$=\frac{2}{s^2+4}+2\frac{d}{ds}\frac{s}{s^2+4}=\frac{2}{s^2+4}+2.\frac{\left(s^2+4\right)-s.2s}{\left(s^2+4\right)^2}$$

$$=\frac{2}{s^2+4}+\frac{2\left(4-s^2\right)}{s^2+4}=\frac{2\left(s^2+4\right)+8-2s^2}{\left(s^2+4\right)^2}=\frac{16}{\left(s^2+4\right)^2}=f(s)\text{ ,(say).}$$

Therefore, $L\left[e^{-t}(\sin 2t-2t\cos 2t)\right]=\frac{16}{\left[(s+1)^2+4\right]^2}=\frac{16}{\left(s^2+2s+5\right)^2}$. Ans.

Q.No.16.: Find the Laplace transform of $te^{at}\sin at$.

Sol.: Since we know $L(\sin at)=\frac{a}{s^2+a^2}$

Now $L(t\sin at)=(-1)^1\frac{d}{ds}(L(\sin at))=-\frac{d}{ds}\frac{a}{s^2+a^2}=(-a)\frac{-1.2s}{\left(s^2+a^2\right)^2}=\frac{2as}{\left(s^2+a^2\right)^2}=f(s)$

$\therefore\ L\left(e^{at}t\sin at\right)=\frac{2a(s-a)}{\left((s-a)^2+a^2\right)^2}=\frac{2a(s-a)}{\left(s^2+a^2-2as+a^2\right)^2}=\frac{2a(s-a)}{\left(s^2-2as+2a^2\right)^2}$.Ans.

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Problems on the topic "Division by t":

REMEMBER

7. Laplace transform of f(t) divided by t:

If $L\{f(t)\}=\bar{f}(s)$, then $L\left\{\frac{1}{t}f(t)\right\}=\int_s^\infty \bar{f}(s)ds$, provided the integral exists.

Q.No.1.: Find the Laplace transform of

(i) $\frac{\left(1-e^t\right)}{t}$, (ii) $\frac{\cos at-\cos bt}{t}$.

Sol.: (i) Since $L\left(1-e^t\right)=L(1)-L\left(e^t\right)=\frac{1}{s}-\frac{1}{s-1}$.

$\therefore L\left(\frac{1-e^t}{t}\right)=\int_s^\infty\left(\frac{1}{s}-\frac{1}{s-1}\right)ds=\left|\log s-\log(s-1)\right|_s^\infty$

$$=\left|\log\left(\frac{s}{s-1}\right)\right|_s^\infty=-\log\left[\frac{1}{1-\frac{1}{s}}\right]=\log\left(\frac{s-1}{s}\right). \text{ Ans.}$$

(ii) Since $L(\cos at-\cos bt)=\dfrac{s}{s^2+a^2}-\dfrac{s}{s^2+b^2}$

$$\therefore L\left(\frac{\cos at-\cos bt}{t}\right)=\int_s^\infty\left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)ds=\left|\frac{1}{2}\log\left(s^2+a^2\right)-\frac{1}{2}\log\left(s^2+b^2\right)\right|_s^\infty$$

$$=\frac{1}{2}\underset{s\to\infty}{\text{Lt}}\log\frac{s^2+a^2}{s^2+b^2}-\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$=\frac{1}{2}\log\left(\frac{1+0}{1+0}\right)-\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)=\log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2}. \text{ Ans.} [\because \log 1=0]$$

Q.No.2.: Find the Laplace transforms of the function $\dfrac{\left(e^{-at}-e^{-bt}\right)}{t}$.

Sol.: Since $L\left(e^{-at}-e^{-bt}\right)=\left[\dfrac{1}{s+a}-\dfrac{1}{s+b}\right]$.

$$\therefore L\left(\frac{e^{-at}-e^{-bt}}{t}\right)=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds=\left|\log\left(\frac{s+a}{s+b}\right)\right|_s^\infty$$

$$=-\log\left(\frac{s+a}{s+b}\right)=\log\left(\frac{s+b}{s+a}\right). \text{ Ans.}$$

Q.No.3.: Find the Laplace transforms of the function $\dfrac{(\sin at)}{t}$.

Sol.: Since $L(\sin at)=\dfrac{a}{s^2+a^2}$.

$$\therefore L\left(\frac{\sin at}{t}\right)=\int_s^\infty\frac{a}{s^2+a^2}ds=a.\left|\frac{1}{a}\tan^{-1}\frac{s}{a}\right|_s^\infty=\left|\tan^{-1}\frac{s}{a}\right|_s^\infty$$

$$=\tan^{-1}\infty-\tan^{-1}\frac{s}{a}=\frac{\pi}{2}-\tan^{-1}\frac{s}{a}=\cot^{-1}\frac{s}{a}. \text{ Ans.}$$

Q.No.4.: Find the Laplace transforms of the function $\dfrac{(\cos 2t-\cos 3t)}{t}$.

Sol.: Since $L(\cos 2t - \cos 3t) = \frac{s}{s^2+4} - \frac{s}{s^2+9}$.

$$\therefore L\left(\frac{\cos 2t - \cos 3t}{t}\right) = \int_s^\infty \left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right)ds = \frac{1}{2}\left[\log(s^2+4) - \log(s^2+9)\right]_s^\infty$$

$$= \frac{1}{2}\left|\log\left(\frac{s^2+4}{s^2+9}\right)\right|_s^\infty = -\frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+9}{s^2+4}\right). \text{ Ans.}$$

Q.No.5.: Find the Laplace transforms of the function $\frac{\left(e^{at} - \cosh bt\right)}{t}$.

Sol.: Since $L\left(e^{at} - \cosh bt\right) = \frac{1}{s-a} - \frac{s}{s^2-b^2}$.

$$\therefore L\left(\frac{e^{at} - \cosh bt}{t}\right) = \int_s^\infty \left(\frac{1}{s-a} - \frac{s}{s^2-b^2}\right)ds = \left|\log(s-a) - \frac{1}{2}\log(s^2-b^2)\right|_s^\infty$$

$$= \left|\log\frac{(s-a)}{\sqrt{s^2-b^2}}\right|_s^\infty = -\log\frac{(s-a)}{\sqrt{s^2-b^2}} = \log\left[\frac{s^2-b^2}{(s-a)^2}\right]^{1/2}$$

$$= \frac{1}{2}\log\left[\frac{s^2-b^2}{(s-a)^2}\right]. \text{ Ans.}$$

Q.No.6.: Find the Laplace transforms of the function $\frac{\left(e^{-t}\sin t\right)}{t}$.

Sol.: Since $L(\sin t) = \frac{1}{s^2+1}$.

$$\therefore L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1}ds = \left|\tan^{-1}s\right|_s^\infty = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s.$$

$$\therefore L\left(\frac{e^{-t}\sin t}{t}\right) = \cot^{-1}(s+1). \text{ Ans.}$$

Q.No.7.: Find the Laplace transforms of the function $\frac{(1-\cos 2t)}{t}$.

Sol.: Since $L(1-\cos 2t)=\frac{1}{s}-\frac{s}{s^2+4}$.

$$\therefore L\left(\frac{1-\cos 2t}{t}\right)=\int_s^\infty\left(\frac{1}{s}-\frac{s}{s^2+4}\right)ds=\left|\log s-\frac{1}{2}\log\left(s^2+4\right)\right|_s^\infty=\left|\log\frac{s}{\left(s^2+4\right)^{1/2}}\right|_s^\infty$$

$$=-\log\left(\frac{s^2}{s^2+4}\right)^{1/2}=\frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right).\text{ Ans.}$$

Q.No.8.: Find the Laplace transforms of the function $\frac{(1-\cos t)}{t^2}$**.**

Sol.: Since $L(1-\cos t)=\frac{1}{s}-\frac{s}{s^2+1}$.

$$\therefore L\left(\frac{1-\cos t}{t}\right)=\int_s^\infty\left(\frac{1}{s}-\frac{s}{s^2+1}\right)ds=\left|\log s-\frac{1}{2}\log\left(s^2+1\right)\right|_s^\infty=\left|\log\left(\frac{s}{\sqrt{s^2+1}}\right)\right|_s^\infty$$

$$=-\log\left(\frac{s^2}{s^2+1}\right)^{1/2}=\frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right).$$

$$\therefore L\left(\frac{1-\cos t}{t^2}\right)=\int_s^\infty\frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)ds=\int_s^\infty\log\frac{\left(s^2+1\right)^{1/2}}{s}ds$$

$$=\int_s^\infty\frac{1}{2}\log\left(s^2+1\right)ds-\int_s^\infty\log s\,ds$$

$$=\left|\frac{s}{2}\log\left(s^2+1\right)\right|_s^\infty-\int_s^\infty\frac{2s}{s^2+1}.\frac{s}{2}ds-\left|s(\log s-1)\right|_s^\infty$$

$$=\left|\frac{s}{2}\log\left(s^2+1\right)\right|_s^\infty-|s|_s^\infty+\left|\tan^{-1}s\right|_s^\infty-|s\log s|_s^\infty+|s|_s^\infty$$

$$=-\frac{1}{2}s\log\left(1+s^{-2}\right)+\cot^{-1}s=\cot^{-1}s-\frac{1}{2}s\log\left(1+s^{-2}\right).\text{ Ans.}$$

Q.No.9.: Find the Laplace' transform of $\frac{\sin^2 t}{t}$.

Sol.: Let $\sin^2 t=L\left(\frac{1-\cos 2t}{2}\right)=L\left(\frac{1}{2}\right)-\frac{1}{2}L(\cos 2t)$

$$=\frac{1}{2}\frac{1}{s}-\frac{1}{2}\frac{s}{s^2+2^2}.$$

Using division by t

$$L\left\{\frac{1}{t}.\sin^2 t\right\}=\frac{1}{2}\int_s^\infty\left(\frac{ds}{s}-\frac{sds}{s^2+4}\right)=\left[\frac{1}{2}\text{In } s-\frac{1}{4}\text{In}\left(s^2+4\right)\right]_s^\infty$$

$$=\text{In}\left(\frac{\sqrt{s}}{\left(s^2+4\right)^{1/4}}\right)\Bigg|_s^\infty=\frac{1}{4}\text{In}\left(\frac{s^2+4}{s^2}\right).$$

Q.No.10.: Evaluate $L\left(\int_0^t\frac{\sin t}{t}dt\right)$.

Sol.: $L\left(\int_0^t\frac{\sin t}{t}dt\right)$.

$$L(\sin t)=\frac{1}{s^2+1}.$$

$$\therefore L\left(\frac{\sin t}{t}\right)=\int_s^\infty\frac{1}{s^2+1}ds=\left|\tan^{-1}s\right|_s^\infty=\frac{\pi}{2}-\tan^{-1}s=\cot^{-1}s.$$

$$\therefore L\left(\int_0^t\frac{\sin t}{t}dt\right)=\frac{1}{s}\cot^{-1}s.\text{ Ans.}$$

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Evaluation of Integrals by Laplace Transforms:

Q.No.1.: Evaluate **(i)** $\int_0^\infty te^{-2t}\sin tdt$, **(ii)** $\int_0^\infty\frac{\sin mt}{t}dt$ and **(iii)** $L\left\{\int_0^t\frac{e^t\sin t}{t}dt\right\}$.

Sol.: (i) $\int_0^\infty te^{-2t}\sin tdt=\int_0^\infty e^{-st}(t\sin t)dt$, where $s=2$

$$=L(t\sin t),\text{ (by definition)}$$

$$=(-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right)=\frac{2s}{\left(s^2+1\right)^2}=\frac{2\times2}{\left(2^2+1\right)^2}=\frac{4}{25}.\text{ Ans.}$$

(ii) Since $L(\sin mt) = \frac{m}{(s^2 + m^2)} = f(s)$, say.

$$\therefore L\left(\frac{\sin mt}{t}\right) = \int_s^\infty f(s) ds = \int_s^\infty \frac{m ds}{s^2 + m^2} = \left|\tan^{-1}\frac{s}{m}\right|_s^\infty .$$

By definition, $\int_0^\infty e^{-st}\frac{\sin mt}{t} dt = \frac{\pi}{2} - \tan^{-1}\frac{s}{m}$.

Now $\underset{s \to 0}{\mathrm{Lt}} \tan^{-1}\left(\frac{s}{m}\right) = 0$, if $m > 0$ or π if $m < 0$.

Thus, taking limits as $s \to 0$, we get

$$\int_0^\infty \frac{\sin mt}{t} dt = \frac{\pi}{2} \text{ if } m > 0 \text{ or } -\frac{\pi}{2}, \text{ if } m < 0. \text{ Ans.}$$

(iii) Since $L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2 + 1} = \tan^{-1} s \,\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$.

By using the **shifting property**, we get

$$L\left\{e^t\left(\frac{\sin t}{t}\right)\right\} = \cot^{-1}(s-1) .$$

Using the "Laplace transforms of integrals" i.e.

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t f(u) du\right\} = \frac{1}{s}\bar{f}(s)$.

We get

$$L\left[\int_0^t \left\{e^t\left(\frac{\sin t}{t}\right)\right\} dt\right] = \frac{1}{s}\cot^{-1}(s-1). \text{ Ans.}$$

Q.No.2.: Evaluate **(i)** $\int_0^\infty te^{-3t} \sin t dt$, **(ii)** $\int_0^\infty te^{-2t} \cos t dt$.

Sol.: (i) To find: $\int_0^\infty te^{-3t} \sin t dt$.

$$L(\sin t) = \frac{1}{s^2 + 1} \qquad \therefore L(t \sin t) = -\frac{d}{ds}\left(\frac{1}{s^2 + 1}\right) = \frac{2s}{(s^2 + 1)^2}.$$

$\therefore$ By definition of Laplace transform $L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt$

$$\int_0^\infty e^{-st}(t\sin t)\,dt=\frac{2s}{(s^2+1)^2}.$$

Put s = 3

$$\int_0^\infty te^{-3t}\sin t\,dt=\frac{2.3}{(3^2+1)^2}=\frac{6}{100}=\frac{3}{50}.\text{ Ans.}$$

(ii) To find: $\int_0^\infty te^{-2t}\cos t\,dt$

$$L(t\cos t)=-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)=\frac{s(2s)-(s^2+1)}{(s^2+1)^2}=\frac{s^2-1}{(s^2+1)^2}$$

$\therefore$ By definition of Laplace transform $L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt$

$$\int_0^\infty e^{-st}(t\cos t)dt=\frac{s^2-1}{(s^2+1)^2}.$$

Put s = 2, we get

$$\int_0^\infty e^{-2t}(t\cos t)dt=\frac{4-1}{(4+1)^2}=\frac{3}{25}.\text{ Ans.}$$

Q.No.3.: Prove that **(i)** $\int_0^\infty \left(\frac{e^{-t}-e^{-3t}}{t}\right)dt=\log 3$, **(ii)** $\int_0^\infty \frac{e^{-t}\sin^2 t}{t}dt=\frac{1}{4}\log 5$.

Sol.: $L\left(e^{-t}-e^{-3t}\right)=\frac{1}{s+1}-\frac{1}{s+3}$.

$$\therefore L\left(\frac{e^{-t}-e^{-3t}}{t}\right)=\int_s^\infty\left(\frac{1}{s+1}-\frac{1}{s+3}\right)ds=\left[\log(s+1)-\log(s+3)\right]_s^\infty$$

$$=\left[\log\frac{s+1}{s+3}\right]_s^\infty=\left[\log\frac{1+\frac{1}{s}}{1+\frac{3}{s}}\right]_s^\infty=0-\log\frac{s+1}{s+3}.$$

$\therefore$ By definition

$$\int_0^\infty e^{-st}.\frac{e^{-t}-e^{-3t}}{t}dt = -\log\frac{s+1}{s+3}$$

Put s = 0, we get

$$\int_0^\infty \left(\frac{e^{-t}-e^{-3t}}{t}\right)dt = -\log\left(\frac{1}{3}\right) = \log 3$$

This completes the proof.

(ii). $\int_0^\infty \frac{e^{-t}\sin^2 t}{t}dt$.

$$\int_0^\infty e^{-t}\left(\frac{\sin^2 t}{t}\right)dt = \int_0^\infty e^{-st}\left(\frac{\sin^2 t}{t}\right)dt = L\left(\frac{\sin^2 t}{t}\right)dt = L\left(\frac{1-\cos 2t}{2t}\right)dt$$

$$= \frac{1}{2}L\left(\frac{1-\cos 2t}{t}\right) = \frac{1}{2}\int_s^\infty \left(\frac{1}{s}-\frac{s}{s^2+4}\right)ds = \frac{1}{2}\left|\log s - \frac{1}{2}\log\left(s^2+4\right)\right|_s^\infty$$

$$= \frac{1}{2}\left|\log\frac{s}{\left(s^2+4\right)^{1/2}}\right|_s^\infty = \frac{1}{2}.\frac{1}{2}\left|\log\left(\frac{s^2}{s^2+4}\right)\right|_s^\infty = \frac{1}{4}\left[-\log\left(\frac{s^2}{s^2+4}\right)\right]$$

$$= \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right) = \frac{1}{4}\log\left(\frac{1+4}{1}\right) = \frac{1}{4}\log 5.$$

This completes the proof.

Q.No.4.: Show that $\int_0^\infty t^2 e^{-4t}.\sin 2t dt = \frac{11}{500}$.

Sol.: LHS is written as

$$\int_0^\infty e^{-4t}.\left(t^2\sin 2t\right)dt.$$

i.e. it is L.T. of $t^2\sin 2t$ with s = 4. We know that

$$L\{\sin 2t\} = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}.$$

Then $L\left(t^2\sin 2t\right) = (-1)^2\frac{d^2}{ds^2}\left\{\frac{2}{s^2+4}\right\} = \frac{d}{ds}\left\{\frac{-4s}{\left(s^2+4\right)^2}\right\} = -4.\left.\frac{\left(4-3s^2\right)}{\left(s^2+4\right)^3}\right|_{s=4} = \frac{11}{500}.$

Now let us solve some more problems:

Q.No.1.: Find $L(\cos at)$ and deduce from it $L(\sin at)$.

Sol.: $L(\cos at)=\int_0^{\infty} e^{-st}\cos at.dt=\dfrac{s}{s^2+a^2}$.

Now $(\sin at)=\int_0^{t} a\cos at.dt$.

$\therefore L(\sin at)=L\left(\int_0^{t} a\cos atdt\right)=a.\dfrac{1}{s}.\dfrac{s}{s^2+a^2}=\dfrac{a}{s^2+a^2}$. Ans. $\left[\because L\left\{\int_0^{t} f(t)dt\right\}=\dfrac{1}{s}.\bar{f}(s)\right]$

Q.No.2.: Given $L\left\{2\sqrt{\left(\dfrac{t}{\pi}\right)}\right\}=\dfrac{1}{s^{3/2}}$, show that $L\left\{\dfrac{1}{\sqrt{(\pi t)}}\right\}=\dfrac{1}{\sqrt{s}}$.

Sol.: $L\left\{2\sqrt{\left(\dfrac{t}{\pi}\right)}\right\}=\dfrac{1}{s^{3/2}}=\bar{f}(s)$.

Let $f(t)=2\sqrt{\dfrac{t}{\pi}}$, $\therefore f'(t)=\dfrac{2}{\sqrt{\pi}}.\dfrac{1}{2}.\dfrac{1}{\sqrt{t}}=\dfrac{1}{\sqrt{\pi t}}$.

$\therefore L\left\{\dfrac{1}{\sqrt{\pi t}}\right\}=L\{f'(t)\}=s\bar{f}(s)-f(0)=s.\dfrac{1}{s^{3/2}}-0=\dfrac{1}{\sqrt{s}}$.

This completes the proof.

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Home Assignments

Find the Laplace transform of the following functions:

Q. No.	Function	Answer
1.	$\sin^2 t$	$\dfrac{2}{s(s^2+4)}$

2.	$\cos at$	$\frac{s}{s^2+a^2}$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$t\cosh at$	$\frac{s^2+a^2}{(s^2-a^2)^2}$
5.	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
6.	$t^2\sin t$	$\frac{6s^2-2}{(s^2+1)^3}$
7.	$t.\sinh 2t$	$\frac{4s}{(s^2-4)^2}$
8.	$t(3\sin 2t-2\cos 2t)$	$\frac{8+12s-2s^2}{(s^2+4)^2}$. $t.e^{at}.\sin at$.
9.	$te^{2t}\sin 3t$	$\frac{6(s-2)}{(s^2-4s+13)^2}$
10	$\sin 2t-2t\cos 2t$	$\frac{16}{(s^2+4)^2}$
11	$\frac{1-e^{2t}}{t}$	$\text{In}\frac{s-2}{s}$
12	$\frac{\sinh t}{t}$	$\frac{1}{2}\text{In}\left(\frac{s+1}{s-1}\right)$
13	$\frac{1-\cos at}{t}$	$\frac{1}{2}\text{In}\frac{s^2+a^2}{s}$
14	$\frac{\sin 3t\cos t}{t}$	$\frac{1}{2}\left[\pi-\tan^{-1}\left(\frac{s}{4}\right)-\tan^{-1}\left(\frac{s}{2}\right)\right]$
15	$\int_0^t e^u.\frac{\sin u}{u}du$	$\frac{1}{s}\cot^{-1}(s-1)$

16	$\int_0^t \frac{e^{-4u}\sin 3u}{u}du$	$\frac{1}{s}\cot^{-1}\left(\frac{s+4}{3}\right)$
17	$\cosh t\int_0^t e^u \cosh u du$	$\frac{1}{2}\left[\frac{s-2}{(s-1)^2(s-3)}+\frac{s}{(s+1)^2(s-1)}\right]$
18	$\int_0^\infty te^{-2t}\sin 3t dt$	$\frac{12}{169}$
19	$\left\{\int_0^t e^t.\frac{\sin t}{t}dt\right\}$	$\frac{1}{s}\cot^{-1}(s-1)$
20	Prove that $\int_0^t e^{-3t}t\cos t dt=\frac{2}{25}$	
21	Prove that $\int_0^\infty t^3e^{-t}.\sin t dt=0$.	
22	Prove that $\int_0^\infty t.e^{-3t}.\sin t dt=\frac{3}{50}$.	
23	Prove that $\int_0^\infty e^{-2t}\sin^3 t dt=\frac{6}{65}$.	
24	Show that $\int_0^\infty \frac{e^{-3t}-e^{6t}}{t}dt=\text{In } 2$.	
25	Show that $\int_0^\infty \frac{\sin^2 t}{t^2}dt=\frac{\pi}{2}$.	
26	Show that $\int_0^\infty .e^{-t}.\frac{\sin^2 t}{t}dt=\frac{1}{2}\text{In } 5$.	
27	Show that $\int_0^t e^{-2t}\frac{(2\sin t-3\sinh t)}{t}dt=2\cot^{-1}2+\frac{3}{2}\log\left(\frac{1}{3}\right)$	
28	Show that $\int_{t=0}^\infty \int_{u=0}^t \frac{e^{-t}\sin u}{u}du dt=\frac{\pi}{4}$	
29	Show that $\int_0^\infty \frac{e^{-at}--e^{-bt}}{t}dt=\log\frac{b}{a}$	

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4th Topic

Laplace Transforms

Inverse Laplace Transforms

Methods of finding the Inverse Laplace Transform

- **Method of Partial Fractions**
- **Use of some Important Properties**
- **Convolution Theorem**

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Inverse Laplace Transform:

If $L\{f(t)\} = \bar{f}(s)$, then f(t) is called the **inverse Laplace transform or inverse transform or simply inverse of** $\bar{f}(s)$, and is denoted by $L^{-1}\{\bar{f}(s)\}$.

Thus $f(t) = L^{-1}\{\bar{f}(s)\}$.

L^{-1} is known as the inverse Laplace transform operator and is such that

$$LL^{-1} = L^{-1}L = 1.$$

In the inverse problem, $\bar{f}(s)$ is given (known) and f(t) is to be determine.

Remarks:

Inverse Laplace transform of $\bar{f}(s)$ need not exists for all $\bar{f}(s)$.

Some methods for finding inverse Laplace transform are:

- **Use of partial fractions (Method of Partial Fractions)**
- **Use of some important properties**
- **Convolution Theorem**
- **Use of Laplace transform tables**

Here we will discuss "Method of Partial Fractions" only.

METHOD OF PARTIAL FRACTIONS:

To find the inverse transforms, we first express the given function of s into partial fraction, which will, then, be recognizable as one of the following standard forms:

(1) $L^{-1}\left(\frac{1}{s}\right) = 1$

(2) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

(3) $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$, $n = 1, 2, 3, \ldots\ldots$

(4) $L^{-1}\left[\frac{1}{(s-a)^n}\right] = \frac{e^{at}t^{n-1}}{(n-1)\,!}$

(5) $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

(6) $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

(7) $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$

(8) $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

(9) $L^{-1}\left(\frac{1}{(s-a)^2+b^2}\right) = \frac{1}{b}e^{at}\sin bt$

(10) $L^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos bt$

(11) $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{1}{2a}t\sin at$

(12) $L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \frac{1}{2a^3}(\sin at - at\cos at)$

Remarks on Partial Fraction:

A fraction of the form $\frac{a_0s^m + a_1s^{m-1} + \ldots\ldots\ldots. + a_m}{b_0s^n + b_1s^{n-1} + \ldots\ldots\ldots. + b_n}$,

in which m and n are positive integers, is called a rational algebraic function. When numerator is of lower degree than the denominator, it is called proper fraction.

To resolve a given fraction into partial fraction, we first factorise the denominator into real factor. These will be either linear or quadratic, and some factor s repeated. We know from algebra that a proper fraction can be resolved into a sum of partial fractions such that:

(i) **To a non-repeated linear factor:**

If $s-a$ occurring once in the denominator, then partial fraction is of the form $\frac{A}{(s-a)}$.

e.g. $\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}=\frac{A}{(s-1)}+\frac{B}{(s-2)}+\frac{C}{(s-3)}$

(ii) **To a repeated linear factor:**

If $(s-a)^r$ occurring r times in the denominator, then the sum of r partial fractions is of the form $\frac{A_1}{s-a}+\frac{A_2}{(s-a)^2}+\frac{A_3}{(s-a)^3}+\dots\dots\dots+\frac{A_r}{(s-a)^r}$.

e.g. $\frac{4s+5}{(s-1)^2}=\frac{A}{s-1}+\frac{B}{(s-1)^2}$.

(iii) **To a non repeated quadratic factor:**

If (s^2+as+b) occurring once in the denominator, then a partial fraction of the form $\frac{As+B}{s^2+as+b}$.

e.g.. $\frac{s}{s^4+4a^4}=\frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)}=\frac{As+B}{(s^2+2as+2a^2)}+\frac{Cs+D}{(s^2-2as+2a^2)}$.

(iv) to a **repeated quadratic factor** $(s^2+as+b)^r$ (occurring r times) in the denominator, corresponds the sum of r partial fractions of the form $\frac{A_1s+B_1}{s^2+as+b}+\frac{A_2s+B_2}{(s^2+as+b)^2}+\dots\dots\dots+\frac{A_rs+B_r}{(s^2+as+b)^r}$.

Then, we have to determine the unknown constant $A,\ A_1,\ B_1$, …..etc.

Method to obtain Partial Fraction:

- **Non-repeated linear factor:** To obtain the partial fraction corresponding to the non-repeated linear factor $s-a$ in the denominator, put s = a everywhere in the given fraction except in the factor $s-a$ itself.

e.g. $\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}=\frac{A}{(s-1)}+\frac{B}{(s-2)}+\frac{C}{(s-3)}$

Then $$A=\left[\frac{2.1^2-6.1+5}{(1-2)(1-3)}\right]=\frac{1}{2},\ B=\left[\frac{2.2^2-6.2+5}{(2-1)(2-3)}\right]=-1,\text{and } C=\left[\frac{2.3^2-6.3+5}{(3-1)(3-2)}\right]=\frac{5}{2}.$$

- In all other cases, equate the given fraction to the sum of suitable partial fractions in accordance with (i) to (iv) above, having found the partial fraction corresponding to the non-repeated linear factors by the above rule.

  e.g. $\dfrac{4s+5}{(s-1)^2}=\dfrac{A}{s-1}+\dfrac{B}{(s-1)^2}$.

$$\frac{s}{s^4+4a^4}=\frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)}=\frac{As+B}{(s^2+2as+2a^2)}+\frac{Cs+D}{(s^2-2as+2a^2)}.$$

$$\frac{5s+3}{(s-1)(s^2+2s+5)}=\frac{A}{(s-1)}+\frac{Bs+C}{(s^2+2s+5)}.$$

$$\Rightarrow\frac{5s+3}{(s-1)(s^2+2s+5)}=\left(\frac{5(1)+3}{(1^2+2.1+5)}\right)\frac{1}{(s-1)}+\frac{Bs+C}{(s^2+2s+5)}$$

$$\Rightarrow\frac{5s+3}{(s-1)(s^2+2s+5)}=\frac{1}{(s-1)}+\frac{Bs+C}{(s^2+2s+5)}$$

- Then multiply both sides by the denominator of the given fraction and equate the coefficients of the powers of the s or substitute the convenient numerical values of s on both sides.
- Finally, solve the simplest of the resulting equations to find the unknown constant.

Now using the concept of partial fractions, let us find inverse Laplace transforms of some functions:

Q.No.1.: Find the inverse Laplace transforms of **(i)** $\dfrac{s^2-3s+4}{s^3}$,

(ii) $\dfrac{s+2}{s^2-4s+13}$.

Sol.: (i) $$L^{-1}\left(\frac{s^2-3s+4}{s^3}\right)=L^{-1}\left(\frac{1}{s}\right)-3L^{-1}\left(\frac{1}{s^2}\right)+4L^{-1}\left(\frac{1}{s^3}\right)$$

$$=1-3t+4.\frac{t^2}{2!}=1-3t+2t^2.$$

(ii) $L^{-1}\left(\frac{s+2}{s^2-4s+13}\right)=L^{-1}\left(\frac{s+2}{(s-2)^2+9}\right)=L^{-1}\left(\frac{s-2+4}{(s-2)^2+3^2}\right)$

$$=L^{-1}\left(\frac{s-2}{(s-2)^2+3^2}\right)+4L^{-1}\left(\frac{1}{(s-2)^2+3^2}\right)$$

$$=e^{2t}\cos 3t+\frac{4}{3}e^{2t}\sin 3t\,.\ \text{Ans.}$$

Q.No.2.: Find the inverse Laplace transforms of **(i)** $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$,

(ii) $\frac{4s+5}{(s-1)^2(s+2)}$.

Sol.: (i) $\overline{f}(s)=\frac{2s^2-6s+5}{s^3-6s^2+11s-6}=\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$.

Use the Method of Partial Fractions:

Here, we can write $\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}=\frac{A}{(s-1)}+\frac{B}{(s-2)}+\frac{C}{(s-3)}$.

To find the values of A, B and C:

Then $A=\left[\frac{2.1^2-6.1+5}{(1-2)(1-3)}\right]=\frac{1}{2}$,

$B=\left[\frac{2.2^2-6.2+5}{(2-1)(2-3)}\right]=-1$,

and $C=\left[\frac{2.3^2-6.3+5}{(3-1)(3-2)}\right]=\frac{5}{2}$.

To find Inverse Laplace Transform:

$$\therefore L^{-1}\left(\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right)=\frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right)-L^{-1}\left(\frac{1}{s-2}\right)+\frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right)$$

$$=\frac{1}{2}e^{t}-e^{2t}+\frac{5}{2}e^{3t}.\ \text{Ans.}$$

(ii). Given $\overline{f}(s) = \frac{4s+5}{(s-1)^2(s+2)}$

Use the Method of Partial Fractions:

Here, we can write $\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{4(-2)+5}{(-2-1)^2(s+2)}$.

To find the values of A and B:

Multiplying both sides by $(s-1)^2(s+2)$, we get

$$4s+5 = A(s-1)(s+2) + B(s+2) - \frac{1}{3}(s-1)^2.$$

- Equating the coefficient of s^2 from both sides, we get

  $0 = A - \frac{1}{3} \Rightarrow A = \frac{1}{3}$.

- By Putting $s = 1$, we get $9 = 3B \Rightarrow B = 3$.

To find Inverse Laplace Transform:

$$\therefore L^{-1}\left(\frac{4s+5}{(s-1)^2(s+2)}\right) = \frac{1}{3}L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left(\frac{1}{(s-1)^2}\right) - \frac{1}{3}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{1}{3}e^t + 3te^t - \frac{1}{3}e^{-2t}. \text{ Ans.}$$

Q.No.3.: Find the inverse Laplace transforms of **(i)** $\frac{5s+3}{(s-1)(s^2+2s+5)}$,

(ii) $\frac{s}{s^4+4a^4}$.

Sol.: (i) Given $\overline{f}(s) = \frac{5s+3}{(s-1)(s^2+2s+5)}$.

Use the Method of Partial Fractions:

Here, we can write $\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+2s+5)}$.

$$\Rightarrow \frac{5s+3}{(s-1)(s^2+2s+5)} = \left(\frac{5(1)+3}{(1^2+2.1+5)}\right)\frac{1}{(s-1)} + \frac{Bs+C}{(s^2+2s+5)}$$

$$\Rightarrow \frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{(s-1)} + \frac{Bs+C}{(s^2+2s+5)}$$

To find the values of B and C:

Multiplying both sides by $(s-1)(s^2+2s+5)$, we get

$5s+3 = 1.(s^2+2s+5)+(Bs+C)(s-1)$.

- Equating the coefficient of s^2 from both sides, we get
  $0 = 1+B \Rightarrow B = -1$.
- Putting $s=0$, we get $3 = 5-C \Rightarrow C = 2$.

To find Inverse Laplace Transform:

$$\therefore L^{-1}\left(\frac{5s+3}{(s-1)(s^2+2s+5)}\right) = L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{-s+2}{s^2+2s+5}\right)$$

$$= L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{-(s+1)+3}{(s+1)^2+4}\right)$$

$$= L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] + 3L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= e^t - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t. \text{ Ans.}$$

(ii) Given $\bar{f}(s) = \frac{s}{s^4+4a^4} = \frac{s}{(s^2+2a^2)^2-(2as)^2} = \frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)}$

Use the Method of Partial Fractions:

Here, we can write $\frac{s}{s^4+4a^4} = \frac{As+B}{(s^2+2as+2a^2)} + \frac{Cs+D}{(s^2-2as+2a^2)}$.

To find the values of A, B, C and D:

Multiplying both sides by s^4+4a^4, we get

$s = (As+B)(s^2-2as+2a^2)+(Cs+D)(s^2+2as+2a^2)$.

Equating the coefficient of s^3, $0 = A+C$ (i)

Equating the coefficient of s^2, $0 = -2aA+B+2aC+D$ (ii)

Equating the coefficient of s, $1 = 2a^2A-2aB+2a^2C+2aD$ (iii)

Putting $s = 0$, $\quad 0 = 2a^2B + 2a^2D$. (iv)

From (iv), $\quad B + D = 0$. (v)

$\therefore$ (ii) becomes $\quad -A + C = 0$, and by (i), we get $A = C = 0$.

Then (iii) reduce to $D - B = \dfrac{1}{2a}$ and by (v), $B = -\dfrac{1}{4a}$, $D = \dfrac{1}{4a}$.

To find Inverse Laplace Transform:

$$\therefore L^{-1}\left(\frac{s}{s^4 + 4a^4}\right) = -\frac{1}{4a}L^{-1}\left(\frac{1}{s^2 + 2as + 2a^2}\right) + \frac{1}{4a}L^{-1}\left(\frac{1}{s^2 - 2as + 2a^2}\right)$$

$$= -\frac{1}{4a}L^{-1}\left(\frac{1}{(s+a)^2 + a^2}\right) + \frac{1}{4a}L^{-1}\left(\frac{1}{(s-a)^2 + a^2}\right)$$

$$= -\frac{1}{4a}.\frac{1}{a}e^{-at}\sin at + \frac{1}{4a}.\frac{1}{a}e^{at}\sin at = \frac{1}{2a^2}\sin at\left(\frac{e^{at} - e^{-at}}{2}\right)$$

$$= \frac{1}{2a^2}\sin at \sinh at \text{. Ans.}$$

Q.No.4.: Find the inverse Laplace transform of $\dfrac{3\left(s^2 - 2\right)^2}{2s^5}$.

Sol.: Here $\bar{f}(s) = \dfrac{3\left(s^4 + 4 - 4s^2\right)}{2s^5} = \dfrac{3s^4 + 12 - 12s^2}{2s^5} = \dfrac{3}{2s} + \dfrac{6}{s^5} - \dfrac{6}{s^3}$.

To find Inverse Laplace Transform:

Taking the inverse Laplace transform, we get

$$L^{-1}\left(\frac{3}{2s} + \frac{6}{s^5} - \frac{6}{s^3}\right) = L^{-1}\left(\frac{3}{2s}\right) + L^{-1}\left(\frac{6}{s^5}\right) - L^{-1}\left(\frac{6}{s^3}\right)$$

$$= \frac{3}{2} + \frac{6t^4}{4!} - \frac{6t^2}{2!} = \frac{3}{2} + \frac{t^4}{4} - 3t^2 = \frac{3}{2} - 3t^2 + \frac{1}{4}t^4 \text{. Ans.}$$

Q.No.5.: Find the inverse Laplace transform of $\dfrac{2s - 5}{4s^2 + 25} + \dfrac{4s - 18}{9 - s^2}$.

Sol.: Given $\bar{f}(s) = \dfrac{2s - 5}{4s^2 + 25} + \dfrac{4s - 18}{9 - s^2}$.

Now $\dfrac{2s - 5}{4s^2 + 25} = \dfrac{2s}{4s^2 + 25} - \dfrac{5}{4s^2 + 25}$ and $\dfrac{4s - 18}{9 - s^2} = \dfrac{4s}{9 - s^2} - \dfrac{18}{9 - s^2}$

Then

$$\bar{f}(s)=\left(\frac{2s}{4s^2+25}-\frac{5}{4s^2+25}\right)+\left(\frac{4s}{9-s^2}-\frac{18}{9-s^2}\right)$$

$$=\left(\frac{2s}{4\left(s^2+\frac{25}{4}\right)}-\frac{5}{4\left(s^2+\frac{25}{4}\right)}\right)+\left(\frac{4s}{9-s^2}-\frac{18}{9-s^2}\right)$$

$$=\frac{2}{4}\left(\frac{s}{s^2+\frac{25}{4}}\right)-\frac{5}{4}\left(\frac{1}{s^2+\frac{25}{4}}\right)+4\left(\frac{s}{9-s^2}\right)-18\left(\frac{1}{9-s^2}\right)$$

$$=\frac{2}{4}\left(\frac{s}{s^2+\left(\frac{5}{2}\right)^2}\right)-\frac{5}{4}\left(\frac{1}{s^2+\left(\frac{5}{2}\right)^2}\right)-4\left(\frac{s}{s^2-3^2}\right)+18\left(\frac{1}{s^2-3^2}\right)$$

To find Inverse Laplace Transform:

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{2s-5}{4s^2+25}+\frac{4s-18}{9-s^2}\right]=\frac{2}{4}L^{-1}\left(\frac{s}{s^2+\left(\frac{5}{2}\right)^2}\right)-\frac{5}{4}L^{-1}\left(\frac{1}{s^2+\left(\frac{5}{2}\right)^2}\right)-4L^{-1}\left(\frac{s}{s^2-3^2}\right)+18L^{-1}\left(\frac{1}{s^2-3^2}\right)$$

$$=\frac{2}{4}\cos\frac{5}{2}t-\frac{5}{4}.\frac{2}{5}.\sin\frac{5}{2}t-4\cosh 3t+\frac{18}{3}\sinh 3t$$

$$=\frac{1}{2}\left(\cos\frac{5}{2}t-\sin\frac{5}{2}t\right)-4\cosh 3t+6\sinh 3t\text{ . Ans.}$$

Q.No.6.: Find the inverse Laplace transform of $\frac{s}{(s+3)^2+4}$.

Sol.: Here $\frac{s+3-3}{(s+3)^2+4}=\frac{s+3}{(s+3)^2+4}-\frac{3}{(s+3)^2+4}$.

Taking the inverse Laplace transform, we get

$$L^{-1}\left(\frac{s+3}{(s+3)^2+(2)^2}\right)-3L^{-1}\left(\frac{1}{(s+3)^2+(2)^2}\right)=e^{-3t}\cos 2t-\frac{3.e^{-3t}}{2}\sin 2t$$

$$= e^{-3t}\left(\cos 2t - \frac{3}{2}\sin 2t\right). \text{ Ans.}$$

Q.No.7.: Find the inverse Laplace transform of $\dfrac{3s}{s^2+2s-8}$.

Sol.: Here $\dfrac{3s}{s^2+2s+1-9} = \dfrac{3s}{(s+1)^2-(3)^2} = \dfrac{3(s+1-1)}{(s+1)^2-(3)^2} = \dfrac{3(s+1)}{(s+1)^2-3^2} - \dfrac{3}{(s+1)^2-3^2}$.

Taking the inverse Laplace transform, we get

$$L^{-1}\left(\frac{3s}{s^2+2s-8}\right) = 3e^{-t}\cosh 3t - e^{-t}\sinh 3t = e^{-t}(3\cosh 3t - \sinh 3t)$$

$$= e^{-t}\left[3\left(\frac{e^{-3t}+e^{3t}}{2}\right) - \left(\frac{e^{3t}-e^{-3t}}{2}\right)\right] = e^{-t}\left[\frac{3}{2}e^{-3t} + \frac{3}{2}e^{3t} - \frac{e^{3t}}{2} + \frac{e^{-3t}}{2}\right]$$

$$= e^{-t}\left[2e^{-3t} + e^{3t}\right] = 2e^{-4t} + e^{2t}. \text{ Ans.}$$

Q.No.8.: Find the inverse Laplace transform of $\dfrac{3s+7}{s^2-2s-3}$.

Sol.: Here $\dfrac{3s+7}{s^2-2s-3} = \dfrac{3s+7}{(s-1)^2-(2)^2} = \dfrac{3s}{(s-1)^2-(2)^2} + \dfrac{7}{(s-1)^2-(2)^2}$

$$= \frac{3(s+1-1)}{(s-1)^2-(2)^2} + \frac{7}{(s-1)^2-(2)^2}$$

$$= \frac{3(s-1)}{(s-1)^2-(2)^2} + \frac{7}{(s-1)^2-(2)^2} + \frac{3}{(s-1)^2-(2)^2}$$

$$= \frac{3(s-1)}{(s-1)^2-(2)^2} + \frac{10}{(s-1)^2-(2)^2}.$$

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right] = 3e^t\cosh 2t + \frac{10}{2}e^t\sinh 2t = 3e^t\cosh 2t + 5e^t\sinh 2t$$

$$= 3e^t\left[\frac{e^{-2t}+e^{2t}}{2}\right] + 5e^t\left[\frac{e^{2t}-e^{-2t}}{2}\right] = 3\left[\frac{e^{-t}+e^{3t}}{2}\right] + 5\left[\frac{e^{3t}-e^{-t}}{2}\right]$$

$$= \frac{3}{2}e^{-t} + \frac{3}{2}e^{3t} + \frac{5}{2}e^{3t} - \frac{5}{2}e^{-t} = \left[-e^{-t} + 4e^{3t}\right]. \text{ Ans.}$$

Q.No.9.: Find the inverse Laplace transform of $\dfrac{s^2+s-2}{s(s+3)(s-2)}$.

Sol.: Here $\dfrac{s^2+s-2}{s(s+3)(s-2)}=\dfrac{A}{s}+\dfrac{B}{s+3}+\dfrac{C}{s-2}$.

Then $A=\dfrac{[0+0-2]}{(3)(-2)}=\dfrac{-2}{-6}=\dfrac{1}{3}$, $B=\dfrac{\left[(-3)^2+(-3)-2\right]}{(-3)(-5)}=\dfrac{9-3-2}{15}=\dfrac{4}{15}$,

$C=\dfrac{\left[(2)^2+(2)-2\right]}{2.5}=\dfrac{4}{10}=\dfrac{2}{5}$

$\Rightarrow \dfrac{A}{s}+\dfrac{B}{s+3}+\dfrac{C}{s-2}=\dfrac{1}{3s}+\dfrac{4}{15(s+3)}+\dfrac{2}{5(s-2)}$.

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{s^2+s-2}{s(s+3)(s-2)}\right]=L^{-1}\left[\frac{1}{3s}+\frac{4}{15(s+3)}+\frac{2}{5(s-2)}\right]=\frac{1}{3}+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}. \text{ Ans.}$$

Q.No.10.: Find the inverse Laplace transform of $\dfrac{s^2-10s+13}{(s-7)\left(s^2-5s+6\right)}$.

Sol.: Here $\dfrac{s^2-10s+13}{(s-7)(s-3)(s-2)}=\dfrac{A}{s-7}+\dfrac{B}{s-3}+\dfrac{C}{s-2}$.

$$\Rightarrow s^2-10s+13=A(s-3)(s-2)+B(s-7)(s-2)+C(s-7)(s-3)$$

$$=A\left(s^2-5s+6\right)+B\left(s^2-9s+14\right)+C\left(s^2-10s+21\right)$$

$$=(A+B+C)s^2+(-5A-9B-10C)s+6A+14B+21C.$$

$A+B+C=1$,

$-5A-9B-10C=-10$,

$6A+14B+21C=13$.

Solving, we get

$A=-\dfrac{2}{5}$, $B=2$, $C=-\dfrac{3}{5}$

$$\left[\frac{s^2+10s+13}{(s-7)(s-3)(s-2)}\right]=\frac{-2}{5(s-7)}+\frac{2}{s-3}+\left(\frac{-3}{5}\right)\frac{1}{s-2}.$$

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{s^2-10s+13}{(s-7)(s^2-5s+6)}\right]=L^{-1}\left[\frac{-2}{5(s-7)}+\frac{2}{s-3}+\left(\frac{-3}{5}\right)\frac{1}{s-2}\right]$$

$$=-\frac{2}{7}e^{7t}+2e^{3t}-\frac{3}{5}e^{2t}\text{. Ans.}$$

Q.No.11.: Find the inverse Laplace transform of $\dfrac{s}{(s^2-1)^2}$.

Sol.: Here $\dfrac{s}{(s+1)^2(s-1)^2}=\dfrac{A}{(s-1)}+\dfrac{B}{(s-1)^2}+\dfrac{C}{(s+1)}+\dfrac{D}{(s+1)^2}$.

Equating the coefficient of A, B, C and D, we get

$A+C=0$

$A+B-C+D=0$

$-A+2B-C-2D=1$

$-A+B+C+D=0$

Solving these equations, we get

$B=\dfrac{1}{4},\quad D=-\dfrac{1}{4},\quad A=0,\quad C=0$

$$\therefore \frac{s}{(s^2-1)^2}=\frac{1}{4(s-1)^2}-\frac{1}{4(s+1)^2}.$$

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{s}{(s^2-1)^2}\right]=L^{-1}\left[\frac{1}{4(s-1)^2}\right]-L^{-1}\left[\frac{1}{4(s+1)^2}\right]$$

$$=\frac{te^{t}}{4}-\frac{te^{-t}}{4}=\frac{t}{2}\left[\frac{e^{t}-e^{-t}}{2}\right]=\frac{t}{2}\sinh t\text{. Ans.}$$

Q.No.12.: Find the inverse Laplace transform of $\dfrac{(1+2s)}{(s+2)^2(s-1)^2}$.

Sol.: Here $\dfrac{(1+2s)}{(s+2)^2(s-1)^2}=\dfrac{A}{(s+2)}+\dfrac{B}{(s+2)^2}+\dfrac{C}{(s-1)}+\dfrac{D}{(s-1)^2}$.

$$1+2s=A(s+2)(s-1)^2+B(s-1)^2+C(s+2)^2(s-1)+D(s+2)^2$$

$$1+2s=A(s+2)(s^2+1-2s)+B(s^2+1-2s)+C(s^2+4+4s)(s-1)+D(s^2+4s+4)$$

$$1+2s = A\left(s^3+s-2s^2+2s^2+2-4s\right)+B\left(s^2+1-2s\right)+C\left(s^3+4s+4s^2-s^2-4-4s\right)$$
$$+D\left(s^2+4+4s\right)$$

Equating the coefficient of A, B, C and D, we get

$A+C=0$

$B+3C+D=0$

$-3A-2B+4D=2$

$2A+B-4C+4D=1$

Solving these equations, we get

$$A=0,\quad C=0,\quad B=-\frac{1}{3},\quad D=\frac{1}{3}$$

$$\therefore \frac{1+2s}{(s+2)^2(s-1)^2}=\frac{-1}{3(s+2)^2}+\frac{1}{3(s-1)^2}.$$

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right]=L^{-1}\left[\frac{-1}{3(s+2)^2}\right]+L^{-1}\left[\frac{1}{3(s-1)^2}\right]=-\frac{1}{3}e^{-2t}t+\frac{1}{3}e^{t}t=\frac{t}{3}\left[e^{t}-e^{-2t}\right].\text{ Ans.}$$

Q.No.13.: Find the inverse Laplace transform of $\dfrac{s}{(s-3)\left(s^2+4\right)}$.

Sol.: Here $\dfrac{s}{(s-3)\left(s^2+4\right)}=\dfrac{A}{s-3}+\dfrac{Bs+C}{s^2+4}$.

$$s=A\left(s^2+4\right)+(Bs+C)(s-3)$$

$$s=As^2+4A+Bs^2-3Bs+Cs-3C$$

Comparing the coefficient of A, B, C, we get

$$A=\frac{3}{13},\quad B=-\frac{3}{13},\quad C=\frac{4}{13}$$

$$\therefore \frac{s}{(s-3)\left(s^2+4\right)}=\frac{3}{13(s-3)}-\frac{3s}{13\left(s^2+4\right)}+\frac{4}{2.13}\frac{2}{\left(s^2+4\right)}.$$

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{s}{(s-3)\left(s^2+4\right)}\right]=L^{-1}\left[\frac{3}{13(s-3)}\right]-L^{-1}\left[\frac{3s}{13\left(s^2+4\right)}\right]+L^{-1}\left[\frac{4}{2.13}\frac{2}{\left(s^2+4\right)}\right]$$

$$= \frac{3}{13}e^{3t} - \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t = \frac{1}{13}\left[3e^{3t} - 3\cos 2t + 2\sin 2t\right]. \text{ Ans.}$$

Q.No.14.: Find the inverse Laplace transform of $\frac{s}{(s+1)^2(s^2+1)}$.

Sol.: Here $\frac{s}{(s+1)^2(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$.

$$s = A(s+1)(s^2+1) + B(s^2+1) + Cs(s+1)^2 + D(s+1)^2$$

$$s = A(s^3+s+s^2+1) + B(s^2+1) + Cs(s^2+2s+1) + D(s^2+2s+1)$$

Equating the coefficient of A, B, C and D, we get

$$A = 0,\ B = -\frac{1}{2},\ D = \frac{1}{2},\ C = 0.$$

$$\therefore \frac{s}{(s+1)^2(s^2+1)} = -\frac{1}{2(s+1)^2} + \frac{1}{2(s^2+1)}.$$

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{s}{(s+1)^2(s^2+1)}\right] = L^{-1}\left[-\frac{1}{2(s+1)^2}\right] + L^{-1}\left[\frac{1}{2(s^2+1)}\right]$$

$$= -\frac{1}{2}te^{-t} + \frac{1}{2}\sin t = \frac{1}{2}\left(\sin t - te^{-t}\right). \text{ Ans.}$$

Q.No.15.: Find the inverse Laplace transform of $\frac{s^3}{s^4-a^4}$.

Sol.: Here $\frac{s^3}{(s-a)(s+a)(s^2+a^2)} = \frac{A}{(s-a)} + \frac{B}{(s+a)} + \frac{Cs+D}{s^2+a^2}$.

$$A = \frac{a^3}{(2a)(2a^2)} = \frac{a^3}{4a^3} = \frac{1}{4}$$

$$B = \frac{(-a^3)}{(-2a)(2a^2)} = \frac{-a^3}{-4a^3} = \frac{1}{4}$$

$$\therefore \frac{s^3}{(s^4-a^4)} = \frac{1}{4(s-a)} + \frac{1}{4(s+a)} + \frac{Cs+D}{s^2+a^2}$$

$$= \frac{1}{4(s-a)} + \frac{1}{4(s+a)} + \frac{1}{2}\cdot\frac{s}{(s^2+a^2)}. \qquad \left[\because C = \frac{1}{2},\ D = 0\right]$$

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{s^3}{s^4-a^4}\right]=L^{-1}\left[\frac{1}{4(s-a)}+\frac{1}{4(s+a)}+\frac{1}{2}\cdot\frac{s}{\left(s^2+a^2\right)}\right]=\frac{1}{4}e^{at}+\frac{1}{4}e^{-at}+\frac{1}{2}\cos at$$

$$=\frac{1}{2}\left[\frac{e^{at}+e^{-at}}{2}\right]+\frac{1}{2}\cos at=\frac{1}{2}\left[\cos at+\cosh at\right].\text{ Ans.}$$

Q.No.16.: Find the inverse Laplace transform of $\frac{1}{s^3-a^3}$.

Sol.: Here $\frac{1}{(s-a)\left(s^2+a^2+as\right)}=\frac{A}{s-a}+\frac{Bs+C}{\left(s^2+a^2+as\right)}$.

Solving for A

$$A=\frac{1}{a^2+a^2+a^2}=\frac{1}{3a^2}.$$

$$\frac{1}{(s-a)\left(s^2+a^2+as\right)}=\frac{1}{3a^2(s-a)}+\frac{Bs+C}{\left(s^2+a^2+as\right)}$$

$$3a^2=s^2+a^2+as+(Bs+C)\left(3a^2s-3a^3\right)$$

$$3a^2=s^2+a^2+as+3a^2Bs^2-3a^3Bs+3a^2Cs-3a^3C$$

Equating the coefficient, we get

$$C=\frac{-2}{3a},\quad B=-\frac{1}{3a^2}$$

$$\frac{1}{s^3-a^3}=\frac{1}{3a^2(s-a)}-\frac{s}{3a^2\left(s^2+as+a^2\right)}-\frac{2}{3a\left(s^2+as+a^2\right)}$$

$$=\frac{1}{3a^2(s-a)}-\frac{1}{3a^2}\frac{\left(s+\frac{a}{2}-\frac{a}{2}\right)}{\left(s+\frac{a}{2}\right)^2+\frac{3a^2}{4}}-\frac{1}{3a}\frac{2}{\left(s+\frac{a}{2}\right)^2+\frac{3a^2}{4}}$$

$$=\frac{1}{3a^2(s-a)}-\frac{1}{3a^2}\frac{\left(s+\frac{a}{2}\right)}{\left(s+\frac{a}{2}\right)^2+\frac{3a^2}{4}}+\frac{1}{\left(s+\frac{a}{2}\right)^2+\frac{3a^2}{4}}\left(\frac{1}{6a}-\frac{2}{3a}\right).$$

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{1}{s^3-a^3}\right]=\frac{1}{3a^2}e^{at}-\frac{1}{3a^2}e^{-at/2}\cos\frac{\sqrt{3}}{2}at-\frac{1}{2a}\cdot\frac{1}{\frac{\sqrt{3}\,a}{2}}e^{-at/2}\sin\frac{\sqrt{3}\,at}{2}$$

$$=\frac{1}{3a^2}e^{at}-\frac{1}{3a^2}e^{-at/2}\cos\frac{\sqrt{3}\,at}{2}-\frac{1}{\sqrt{3}\,a^2}e^{-at/2}\sin\frac{\sqrt{3}\,at}{2}$$

$$=\frac{1}{3a^2}\left[e^{at}-e^{-at/2}\left\{\cos\frac{\sqrt{3}\,at}{2}+\sqrt{3}\sin\frac{\sqrt{3}\,at}{2}\right\}\right]. \text{ Ans.}$$

Q.No.17.: Find the inverse Laplace transform of $\frac{s^2+6}{(s^2+1)(s^2+4)}$.

Sol.: Here $\frac{s^2+6}{(s^2+1)(s^2+4)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4}$.

$$s^2+6=(As+B)(s^2+4)+(Cs+D)(s^2+1)$$

$$s^2+6=As^3+4As+Bs^2+4B+Cs^3+Cs+Ds^2+D.$$

Solving, we get

$$A=0,\ B=\frac{5}{3},\ C=0,\ D=-\frac{2}{3}$$

$$\frac{s^2+6}{(s^2+1)(s^2+4)}=\frac{5}{3(s^2+1)}-\frac{2}{3(s^2+4)}.$$

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{s^2+6}{(s^2+1)(s^2+4)}\right]=\frac{5}{3}L^{-1}\left[\frac{1}{(s^2+1)}\right]-\frac{2}{3}L^{-1}\left[\frac{1}{(s^2+4)}\right]$$

$$=\frac{5}{3}\sin t-\frac{1}{3}\sin 2t=\frac{1}{3}[5\sin t-\sin 2t]. \text{ Ans.}$$

Q.No.18.: Find the inverse Laplace transform of $\frac{2s-3}{(s^2+4s+13)}$.

Sol.: Here $\frac{2s-3}{(s^2+4s+4+9)}=\frac{2s-3}{(s+2)^2+(3)^2}=\frac{2(s+2-2)}{(s+2)^2+(3)^2}-\frac{3}{(s+2)^2+(3)^2}$

$$=\frac{2(s+2)}{(s+2)^2+(3)^2}-\frac{7}{(s+2)^2+3^2}$$

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{2s-3}{\left(s^2+4s+13\right)}\right]=2e^{-2t}\cos 3t-\frac{7}{3}e^{-2t}\sin 3t=\frac{1}{3}e^{-2t}[6\cos 3t-7\sin 3t]. \text{ Ans.}$$

Q.No.19.: Find the inverse Laplace transform of $\frac{s^2+s}{\left(s^2+1\right)\left(s^2+2s+2\right)}$.

Sol.: Here $\frac{s^2+s}{\left(s^2+1\right)\left(s^2+2s+2\right)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+2s+2}$.

$$s^2+s=(As+B)\left(s^2+2s+2\right)+(Cs+D)\left(s^2+1\right)$$

$$=(A+C)s^3+(2A+B+D)s^2+(2A+2B+C)s+2B+D$$

Solving and equating the coefficients, we get

$A+C=0$

$2A+B+D=1$

$2A+2B+C=1$

$2B+D=0$

$D=-\frac{2}{5},\ C=-\frac{3}{5},\ B=\frac{1}{5},\ A=\frac{3}{5}$

$$\therefore \frac{s^2+s}{\left(s^2+1\right)\left(s^2+2s+2\right)}=\frac{3}{5}\left(\frac{s}{s^2+1}\right)+\frac{1}{5}\left(\frac{1}{s^2+1}\right)-\frac{3}{5}\left(\frac{s+1-1}{(s+1)^2+(1)^2}\right)-\frac{2}{5}\left(\frac{1}{(s+1)^2+(1)^2}\right).$$

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{s^2+s}{\left(s^2+1\right)\left(s^2+2s+2\right)}\right]=\frac{3}{5}L^{-1}\left(\frac{s}{s^2+1}\right)+\frac{1}{5}L^{-1}\left(\frac{1}{s^2+1}\right)-\frac{3}{5}L^{-1}\left(\frac{s+1-1}{(s+1)^2+(1)^2}\right)$$

$$-\frac{2}{5}L^{-1}\left(\frac{1}{(s+1)^2+1^2}\right)$$

$$=\frac{3}{5}\cos t+\frac{1}{5}\sin t-\frac{3}{5}e^{-t}\cos t+\frac{1}{5}e^{-t}\sin t$$

$$=\frac{1}{5}\left(1+e^{-t}\right)\sin t+\frac{3}{5}\left(1-e^{-t}\right)\cos t. \text{ Ans.}$$

Q.No.20.: Find the inverse Laplace transform of $\frac{s+2}{\left(s^2+4s+5\right)^2}$.

Sol.: Here $\frac{s+2}{\left(s^2+4s+5\right)^2}$.

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{s+2}{\left(s^2+4s+5\right)^2}\right]=L^{-1}\left[\frac{s+2}{(s+2)^2+1^2}\right].$$

Since $L^{-1}\left[\frac{s}{\left(s^2+b^2\right)^2}\right]=\frac{1}{2b}t\sin bt$ and $L\left(e^{-at}t\sin t\right)=\frac{2b(s+a)}{\left[(s+a)^2+b^2\right]^2}$.

$$L^{-1}\left[\frac{s+2}{\left(s^2+4s+5\right)^2}\right]=\frac{1}{2}e^{-2t}t\sin t\text{ . Ans.}$$

Q.No.21.: Find the inverse Laplace transform of $\frac{s+3}{\left(s^2+6s+13\right)^2}$.

Sol.: Here $\frac{s+3}{\left(s^2+6s+13\right)^2}=\frac{s+3}{\left[(s+3)^2+2^2\right]^2}$.

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{s+3}{\left[(s+3)^2+2^2\right]^2}\right]=\frac{1}{2.2}e^{-3t}-t\sin 2t=\frac{1}{4}e^{-3t}-\sin 2t\text{ . Ans.}$$

Q.No.22.: Find the inverse Laplace transform of $\frac{s}{\left(s^4+s^2+1\right)}$.

Sol.: Here $\frac{s}{\left(s^4+s^2+1\right)}$.

Since $\left(s^4+s^2+1\right)=\left(s^2+s+1\right)\left(s^2-s+1\right)$.

$$\therefore \frac{s}{\left(s^4+s^2+1\right)}=\frac{1}{2}\frac{2s}{\left(s^2+s+1\right)\left(s^2-s+1\right)}.$$

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{s}{\left(s^4+s^2+1\right)}\right]=\frac{1}{2}L^{-1}\left(\frac{1}{s^2-s+1}\right)-\frac{1}{2}L^{-1}\left(\frac{1}{s^2+s+1}\right)$$

$$=\frac{1}{2}L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\}-\frac{1}{2}L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$=\frac{1}{2.\frac{\sqrt{3}}{2}}e^{t/2}\sin\frac{\sqrt{3}}{2}t-\frac{1}{2.\frac{\sqrt{3}}{2}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t$$

$$=\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}\,t}{2}\left(\frac{e^{t/2}-e^{-t/2}}{2}\right)=\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}\,t}{2}\sinh\frac{t}{2}.\text{ Ans.}$$

Q.No.23.: Find the inverse Laplace transform of $\frac{a\left(s^2-2a^2\right)}{s^4+4a^4}$.

Sol.: Here $\frac{a\left(s^2-2a^2\right)}{s^4+4a^4}$.

Since $s^4+4a^4=\left(s^2+2as+2a^2\right)\left(s^2-2as+2a^2\right)$

$$\Rightarrow\frac{a\left(s^2-2a^2\right)}{s^4+4a^4}=\frac{As+B}{s^2+2as+2a^2}+\frac{Cs+D}{s^2-2as+2a^2}.$$

$$as^2-2a^3=(As+B)\left(s^2-2as+2a^2\right)+(Cs+D)\left(s^2+2as+2a^2\right)$$

Equating the coefficient of s^3; $A+C=0$

Equating the coefficient of s^2; $a=-2Aa+B+2aC+D$

Equating the coefficient of s; $0=2a^2A-2aB+2a^2C+2aD$

Putting $s=0$, we get

$$-2a^3=2a^2B+2a^2D\Rightarrow B+D=-a$$

$$\therefore a=-2aA+2aC-a$$

$$2a=-2aA+2aC$$

$$C-A=1$$

$$C+A=0.$$

Solving these equations, we get

$$2C=1\Rightarrow C=\frac{1}{2},\quad A=-\frac{1}{2},\quad D=B=-\frac{a}{2}$$

$$\Rightarrow \frac{-\frac{s}{2}-\frac{a}{2}}{(s+a)^2+a^2}+\frac{\frac{1}{2}s-\frac{a}{2}}{(s-a)^2+a^2} \Rightarrow -\frac{1}{2}\cdot\frac{s+a}{(s+a)^2+a^2}+\frac{1}{2}\cdot\frac{s-a}{(s-a)^2+a^2}$$

Taking the inverse Laplace transform, we get

$$L^{-1}\left[\frac{a\left(s^2-2a^2\right)}{s^4+4a^4}\right]=-\frac{1}{2}L^{-1}\left[\frac{s+a}{(s+a)^2+a^2}\right]+\frac{1}{2}L^{-1}\left[\frac{s-a}{(s-a)^2+a^2}\right]$$

$$=-\frac{1}{2}e^{-at}\cos at+\frac{1}{2}e^{at}\cos at=\cos at\left[\frac{e^{at}-e^{-at}}{2}\right]$$

$$=\cos at.\sinh at \text{ . Ans.}$$

Q.No.24.: Find the inverse Laplace transform of $\dfrac{s^3+6s^2+14s}{(s+2)^4}$.

Sol.: Let $\dfrac{s^3+6s^2+14s}{(s+2)^4}=\dfrac{A}{(s+2)^4}+\dfrac{B}{(s+2)^3}+\dfrac{C}{(s+2)^2}+\dfrac{D}{(s+2)}$

s^3+6s^2+14s

$=A+B(s+2)+C(s+2)^2+D(s+2)^3$

$=Ds^3+(6D+C)s^2+(12D+4C+B)s+(8D+4C+2B+A)$

Equating the coefficient of s

$A=-12, \quad B=2, \quad C=0, \quad D=1.$

$$L^{-1}\left\{\frac{s^3+6s^2+14s}{(s+2)^4}\right\}=-12L^{-1}\left\{\frac{1}{(s+2)^4}\right\}+2L^{-1}\left\{\frac{1}{(s+2)^3}\right\}+L^{-1}\left\{\frac{1}{(s+2)}\right\}$$

$$=-2e^{-2t}t^3+e^{-2t}t^2+e^{-2t}=e^{-2t}\left\{1+t^2-2t^3\right\}.$$

Q.No.25.: Find the inverse Laplace transform of $\dfrac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)}$

Sol.: $\dfrac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)}=\dfrac{As+B}{s^2+2s+2}+\dfrac{Cs+D}{s^2+2s+5}$

s^2+2s+3

$=(As+B)\left(s^2+2s+5\right)+(Cs+D)\left(s^2+2s+2\right)$

$$= (A+C)s^3 + (2A+B+2C+D)s^2 + (5A+2B+2C+2D)s + 5B + 2D$$

Comparing coefficient of s on either side

$A + C = 0,\ 2A + B + 2C + D = 1,\ 5A + 2B + 2C + 2D = 2,\ 5B + 2D = 3.$

By solving these equations, we get

$A = 0,\ B = \frac{1}{3},\ C = 0,\ \ D = \frac{2}{3}.$

$$L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$$

$$= L^{-1}\left\{\frac{\frac{1}{3}}{s^2+2s+2}\right\} + L^{-1}\left\{\frac{\frac{2}{3}}{s^2+2s+5}\right\}$$

$$= \frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$

$$= \frac{1}{3}e^{-t}\sin t + \frac{2}{3}.\frac{1}{2}.e^{-t}\sin 2t = \frac{1}{3}2^{-t}(\sin t + \sin 2t).$$

Q.No.26.: Find the inverse Laplace transform of $\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$.

Sol.: Let $\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2} = \dfrac{As+B}{(s^2-2s+2)^2} + \dfrac{Cs+D}{(s^2-2s+2)}$

$$s^3 - 3s^2 + 6s - 4$$
$$= As + B + (Cs+D)(s^2-2s+2) = Cs^3 + (D-2C)s^2 + (A+2C-2D)s + B + 2D$$

Equating & solving

$A = 2,\ B = -2,\ C = 1,\ \ D = -1$ and rewriting $s^2 - 2s + 2 = (s-1)^2 + 1$

$$L^{-1}\left\{\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}\right\} = L^{-1}\left\{\frac{2s-2}{\{(s-1)^2+1\}^2}\right\} + L^{-1}\left\{\frac{s-1}{\{(s-1)^2+1\}}\right\}$$

$$= e^tL^{-1}\left\{\frac{2s}{(s^2+1)^2} + \frac{s}{(s^2+1)}\right\} = 2e^tL^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} + e^t.\cos t$$

$$= 2e^{t}.\frac{t}{2}.\sin t + e^{t}\cos t = e^{t}\{t\sin t + \cos t\}. \text{ Ans.}$$

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Home Assignments

Use of partial fractions:

Q.No.1.: Find the inverse Laplace transform of $\frac{s-2}{s^2+5s+6}$.

Ans.: $-4e^{-2t}+5e^{-3t}$.

Q.No.2.: Find the inverse Laplace transform of $\frac{2s^2-4}{(s+1)(s-2)(s-3)}$.

Ans.: $\frac{-1}{6}e^{-t}-\frac{4}{3}e^{2t}+\frac{7}{2}e^{3t}$.

Q.No.3.: Find the inverse Laplace transform of $\frac{s^2-7s+24}{s^3-7s^2+14s-8}$.

Ans.: $6e^{t}-7e^{2t}+2e^{4t}$.

Q.No.4.: Find the inverse Laplace transform of $\frac{s+17}{(s-1)(s+3)}$.

Ans.: $\frac{9}{2}e^{t}-\frac{7}{2}e^{-3t}$.

Q.No.5.: Find the inverse Laplace transform of $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$.

Ans.: $\frac{1}{2}e^{t}-e^{2t}+\frac{5}{2}e^{3t}$.

Q.No.6.: Find the inverse Laplace transform of $\frac{5s}{s^2+4s+4}$.

Ans.: $5e^{-2t}(1-2t)$.

Q.No.7.: Find the inverse Laplace transform of $\dfrac{5}{(s-2)^4}$.

Ans.: $\dfrac{5t^3e^{2t}}{6}$.

Q.No.8.: Find the inverse Laplace transform of $\dfrac{7}{(2s+1)^3}$.

Ans.: $\dfrac{7}{16}t^2e^{\frac{-t}{2}}$.

Q.No.9.: Find the inverse Laplace transform of $\dfrac{s+2}{s^2+4s+7}$.

Ans.: $e^{-2t}.\cos\sqrt{3t}$.

Q.No.10.: Find the inverse Laplace transform of $\dfrac{2s+12}{s^2+6s+13}$.

Ans.: $e^{-3t}(2\cos 2t+3\sin 2t)$.

Q.No.11.: Find the inverse Laplace transform of $\dfrac{s^2+9s-9}{s^3-9s}$.

Ans.: $1+3\sinh 3t$.

Q.No.12.: Find the inverse Laplace transform of $\dfrac{s}{(s^2-2s+2)(s^2+2s+2)}$.

Ans.: $\dfrac{1}{2}\sin t.\sinh t$.

Q.No.13.: Find the inverse Laplace transform of $\dfrac{3s^3-3s^2-40s+36}{(s^2-4)^2}$.

Ans.: $(5t+3)e^{-2t}-2te^{2t}$.

Q.No.14.: Find the inverse Laplace transform of $\dfrac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}$.

Ans.: $\dfrac{1}{2}\sin t+\dfrac{1}{2}t\cos t-te^{-t}$.

Q.No.15.: Find the inverse Laplace transform of $\frac{5s^2-7s+17}{(s-1)(s^2+4)}$.

Ans.: $3e^t+2\cos 2t-\frac{5}{2}\sin 2t$.

Q.No.16.: Find the inverse Laplace transform of $\frac{2s^2+15s+7}{(s+1)^2(s-2)}$.

Ans.: $(2t-3)e^{-t}+5e^{2t}$.

Q.No.17.: Find the inverse Laplace transform of $\frac{s+1}{(s^2+1)(s^2+4)}$.

Ans.: $\frac{1}{6}(2\cos t-2\cos 2t+2\sin t-\sin 2t)$.

Q.No.18.: Find the inverse Laplace transform of $\frac{10}{s(s^2-2s+5)}$.

Ans.: $2-e^t(2\cos 2t-\sin 2t)$

Q.No.19.: Find the inverse Laplace transform of $\frac{1}{s(s+1)^2}$.

Ans.: $1-e^{-t}-te^{-t}$.

Q.No.20.: Find the inverse Laplace transform of $\frac{s^2+8s+27}{(s+1)(s^2+4s+13)}$.

Ans.: $2e^{-t}+e^{-2t}(\sin 3t-\cos 3t)$.

Q.No.21.: Find the inverse Laplace transform of $\frac{s}{s^4+s^2+1}$.

Ans.: $\frac{1}{\sqrt{3}}\left[e^{\frac{t}{2}}\frac{\sin\sqrt{3}}{2}t-e^{-\frac{t}{2}}\frac{\sin\sqrt{3}}{2}t\right]$

Q.No.22.: Find the inverse Laplace transform of $\frac{s^2}{s^4+4a^4}$.

Ans.: $\frac{1}{2a}[\sinh at\cos at+\cosh at.\sin at]$.

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5^{th} Topic

Laplace Transforms

General Properties of Inverse Laplace Transforms

- Linearity Property
- Shifting Property (Translation Theorem)
- Change of Scale Property
- Multiplication by 's'
- Division by Powers of 's'
- Inverse Laplace Transform of Derivatives
- Inverse Laplace Transform of Integrals

Convolution Theorem

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As we have mentioned in last topic that for finding inverse Laplace transform, the following methods have been used, which are given below::

- **Use of partial fractions (Method of Partial Fractions)√**
- **Use of some important properties**
- **Convolution Theorem**
- **Use of Laplace transform tables**

So in this topic, we will discuss General properties of Inverse Laplace Transforms.

General properties of Inverse Laplace Transforms

- Linearity Property
- Shifting Property (Translation theorem)
- Change of Scale Property
- Multiplication by s
- Division by Powers of s
- Inverse Laplace Transform of Derivatives
- Inverse Laplace Transform of Integrals

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Linearity Property:

If $L\{f(t)\} = \bar{f}(s)$ and $L\{g(t)\} = \bar{g}(s)$, the

$$L^{-1}\left[c_1\bar{f}(s) + c_2\bar{g}(s)\right] = c_1L^{-1}\bar{f}(s) + c_2L^{-1}\bar{g}(s) = c_1f(t) + c_2g(t),$$

where c_1 and c_2 are any two constants.

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Shifting Property for Inverse Laplace Transforms:

(First Shift or Translation Theorem)

If $L^{-1}\left[\bar{f}(s)\right] = f(t)$, then $L^{-1}\left[\bar{f}(s-a)\right] = e^{at}f(t) = e^{at}L^{-1}\left[\bar{f}(s)\right]$.

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Change of Scale Property:

$$L^{-1}\left[\bar{f}(ks)\right] = \frac{1}{k}f\left(\frac{t}{k}\right).$$

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Multiplication by s:

If $L^{-1}\left[\bar{f}(s)\right] = f(t)$, and $f(0) = 0$, then $L^{-1}\left\{s\bar{f}(s)\right\} = \frac{d}{dt}\{f(t)\}$.

In general, $L^{-1}\left\{s^n\bar{f}(s)\right\} = \frac{d^n}{dt^n}\{f(t)\}$, provided $f(0) = f'(0) = \dots\dots\dots = f^{n-1}(0) = 0$.

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Division by powers of s:

$$L^{-1}\left[\bar{f}(s)\right]=f(t),\text{ then } L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\}=\int_0^t f(t)dt.$$

Also $$L^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\}=\int_0^t\left\{\int_0^t f(t)dt\right\}dt.$$

$$L^{-1}\left\{\frac{\bar{f}(s)}{s^3}\right\}=\int_0^t\left\{\int_0^t\left(\int_0^t f(t)dt\right)dt\right\}dt \text{ and so on}.$$

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Inverse Laplace Transform of Derivatives:

If $L^{-1}\left[\bar{f}(s)\right]=f(t)$, then $tf(t)=L^{-1}\left\{-\frac{d}{ds}\left[\bar{f}(s)\right]\right\}$

The above result follows from $L\{tf(t)\}=-\frac{d}{ds}\{\bar{f}(s)\}$.

In general, $L^{-1}\left\{\bar{f}^{\,n}(s)\right\}=(-1)^n t^n f(t)$, $n = 1, 2, 3, \ldots\ldots.$

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Inverse Laplace Transform of Integrals:

If $L^{-1}\{\bar{f}(s)\}=f(t)$, then $L^{-1}\left\{\int_s^\infty \bar{f}(s)ds\right\}=\frac{f(t)}{t}$.

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CONVOLUTION THEOREM:

Statement: If $L^{-1}\{\bar{f}(s)\}=f(t)$ and $L^{-1}\{\bar{g}(s)\}=g(t)$,

then $$L^{-1}\{\bar{f}(s)\bar{g}(s)\}=\int_0^t f(u)g(t-u)du = F*G.$$

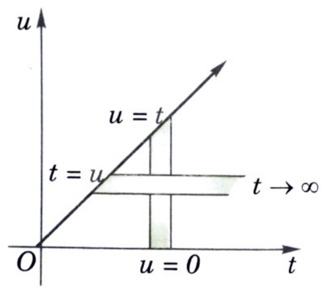
[F*G is called the convolution or falting of F and G.]

Proof: Let $\phi(t)=\int_0^t f(u)g(t-u)du$

Then by definition of Laplace Transform, we get

$$L\{\phi(t)\}=\int_0^\infty e^{-st}\left\{\int_0^t f(u)g(t-u)du\right\}dt=\int_0^\infty\int_0^t e^{-st}f(u)g(t-u)du.dt \qquad \text{(i)}$$

The domain of integration for this double integral is the entire area lying between the lines u = 0 and u = t.



On changing the order of integration, we get

$$L\{\phi(t)\}=\int_0^\infty\int_u^\infty e^{-st}f(u)g(t-u)dt.du=\int_0^\infty e^{-su}f(u)\left\{\int_u^\infty e^{-s(t-u)}g(t-u)dt\right\}du$$

$$=\int_0^\infty e^{-su}f(u)\left\{\int_0^\infty e^{-sv}g(v)dv\right\}du \text{ , on putting } t-u=v$$

$$=\int_0^\infty e^{-su}f(u)\bar{g}(s)du=\int_0^\infty e^{-su}f(u)du.\bar{g}(s)=\bar{f}(s).\bar{g}(s).$$

$$\Rightarrow\{\bar{f}(s)\bar{g}(s)\}=L\{\phi(t)\}=L\left[\int_0^t f(u)g(t-u)du\right]$$

Thus $L^{-1}\{\bar{f}(s)\bar{g}(s)\}=\int_0^t f(u)g(t-u)du=F*G$.

This completes the proof.

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Now let us find the Inverse Laplace Transform:

Always Remember: Linearity Property:

If $L\{f(t)\}=\bar{f}(s)$ and $L\{g(t)\}=\bar{g}(s)$, the

$$L^{-1}\left[c_1\bar{f}(s)+c_2\bar{g}(s)\right]=c_1L^{-1}\bar{f}(s)+c_2L^{-1}\bar{g}(s)=c_1f(t)+c_2g(t),$$

where c_1 and c_2 are any two constants.

Q.No.1.: Find the Inverse Laplace's transform of : $\dfrac{2s+1}{s^2-4}$.

Sol.: Given $\bar{f}(s)=\dfrac{2s+1}{s^2-4}$.

To find Inverse Laplace Transform:

Now $L^{-1}\left(\bar{f}(s)\right)=L^{-1}\left\{\dfrac{2s+1}{s^2-4}\right\}=L^{-1}\left\{\dfrac{2s}{s^2-4}\right\}+L^{-1}\left\{\dfrac{1}{s^2-4}\right\}$

$$=2.\cosh 2t+\frac{1}{2}\sinh 2t.$$

Q.No.2.: Find the Inverse Laplace's transform of : $\dfrac{1}{s}e^{-\frac{1}{\sqrt{s}}}$.

Sol.: Given $\bar{f}(s)=\dfrac{1}{s}e^{-\frac{1}{\sqrt{s}}}$.

Since we know that $e^{-\frac{1}{\sqrt{s}}}=\displaystyle\sum_{n=0}^{\infty}\left(-\frac{1}{\sqrt{s}}\right)^n\frac{1}{n!}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{s^{\frac{n}{2}}}$.

$$\Rightarrow \frac{1}{s}e^{-\frac{1}{\sqrt{s}}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{s^{\frac{n}{2}+1}}.$$

To find Inverse Laplace Transform:

Now $L^{-1}\left\{\dfrac{1}{s}e^{-\frac{1}{\sqrt{s}}}\right\}=L^{-1}\left\{\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{s^{\frac{n}{2}+1}}\right\}$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}L^{-1}\left\{\frac{1}{s^{\frac{n}{2}+1}}\right\}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{t^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} . \quad \left[\because L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{\Gamma(n)}, n = 0, 1, 2, 3, \ldots\ldots \right]$$

Q.No.3.: Show that **(i)** $L^{-1}\left(\frac{1}{s}\sin\frac{1}{s}\right) = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \frac{t^7}{(7!)^2} + \ldots\ldots\ldots,$

(ii) $L^{-1}\left(\frac{1}{s}\cos\frac{1}{s}\right) = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \ldots\ldots\ldots$**.**

Sol.: (i) $L^{-1}\left(\frac{1}{s}\sin\frac{1}{s}\right) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{s} - \frac{1}{s^3(3!)} + \frac{1}{s^5(5!)} - \frac{1}{s^7(7!)} + \ldots\ldots\ldots\right\}\right]$

$$= L^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^4(3!)} + \frac{1}{s^6(5!)} - \frac{1}{s^8(7!)} + \ldots\ldots\ldots\right\}$$

$$= L^{-1}\left(\frac{1}{s^2}\right) - L^{-1}\left(\frac{1}{s^4(3!)}\right) + L^{-1}\left(\frac{1}{s^6(5!)}\right) - L^{-1}\left(\frac{1}{s^8(7!)}\right) + \ldots\ldots\ldots$$

$$= t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \frac{t^7}{(7!)^2} \ldots\ldots\ldots\ldots$$

Thus $L^{-1}\left(\frac{1}{s}\sin\frac{1}{s}\right) = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \frac{t^7}{(7!)^2} \ldots\ldots\ldots\ldots$**.Ans.**

(ii) $L^{-1}\left(\frac{1}{s}\cos\frac{1}{s}\right) = L^{-1}\left[\frac{1}{s}\left\{1 - \frac{1}{s^2(2!)} + \frac{1}{s^4(4!)} - \frac{1}{s^6(6!)} + \ldots\ldots\ldots\right\}\right]$

$$= L^{-1}\left\{\frac{1}{s} - \frac{1}{s^3(2!)} + \frac{1}{s^5(4!)} - \frac{1}{s^7(6!)} + \ldots\ldots\ldots\right\}$$

$$= L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s^3(2!)}\right) + L^{-1}\left(\frac{1}{s^5(4!)}\right) - L^{-1}\left(\frac{1}{s^7(6!)}\right) + \ldots\ldots\ldots$$

$$= 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \ldots\ldots\ldots\ldots$$

$L^{-1}\left(\frac{1}{s}\cos\frac{1}{s}\right) = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \ldots\ldots\ldots\ldots$ **Ans.**

Always Remember: Shifting Property for Inverse Laplace Transforms:

(First Shift or Translation Theorem)

If $L^{-1}\left[\bar{f}(s)\right] = f(t)$, then $L^{-1}\left[\bar{f}(s-a)\right] = e^{at}f(t) = e^{at}L^{-1}\left[\bar{f}(s)\right]$.

Q.No.1.: Find the Inverse Laplace's transform of $\frac{1}{(s+a)^{n+1}}$, n: non-negative integer.

Sol.: Given $\bar{f}(s) = \frac{1}{(s+a)^{n+1}}$.

To find Inverse Laplace Transform:

$$L^{-1}\left\{\frac{1}{(s+a)^{n+1}}\right\} = e^{-at}L^{-1}\left\{\frac{1}{s^{n+1}}\right\}. \qquad \text{(by Shifting Property)}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(s+a)^{n+1}}\right\} = e^{-at}\frac{t^n}{n!}. \qquad \left[\because L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}, n = 1, 2, 3, \ldots\ldots\right]$$

Q.No.2.: Find the Inverse Laplace's transform of $\frac{s}{(s+a)^2+b^2}$.

Sol.: Given $\bar{f}(s) = \frac{s}{(s+a)^2+b^2}$.

To find Inverse Laplace Transform:

$$L^{-1}\left[\frac{s}{(s+a)^2+b^2}\right] = e^{-at}L^{-1}\left[\frac{s-a}{s^2+b^2}\right] \qquad \text{(by Shifting Property)}$$

$$\Rightarrow L^{-1}\left[\frac{s}{(s+a)^2+b^2}\right] = e^{-at}\left\{L^{-1}\left[\frac{s}{s^2+b^2}\right] - \frac{a}{b}L^{-1}\left[\frac{b}{s^2+b^2}\right]\right\} \quad \text{(by Linearity Property)}$$

$$= e^{-at}\left\{\cos bt - \frac{a}{b}.\sin bt\right\}. \quad \left[\because L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at \text{ and } L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at\right]$$

Q.No.3.: Find the Inverse Laplace's transform of $\frac{3s}{s^2-25}$.

Sol.: $L^{-1}\left\{\frac{3s}{s^2-25}\right\} = 3L^{-1}\left\{\frac{s}{s^2-5^2}\right\} = 3\cosh 5t$.

Q.No.4.: Find the Inverse Laplace's transform of $\dfrac{1}{s^{\frac{3}{2}}}$**.**

Sol.: $L^{-1}\left\{\dfrac{1}{s^{\frac{3}{2}}}\right\}=\dfrac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}=\dfrac{t^{\frac{1}{2}}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}=2\sqrt{\dfrac{t}{\pi}}$.

Q.No.5.: Find the Inverse Laplace's transform of $\dfrac{3s+1}{(s+1)^4}$

Sol.: $L^{-1}\left(\dfrac{3s+1}{(s+1)^4}\right)=e^{-t}L^{-1}\left(\dfrac{3(s-1)+1}{s^4}\right)=e^{-t}\left\{3L^{-1}\left(\dfrac{1}{s^3}\right)-2L^{-1}\left(\dfrac{1}{s^4}\right)\right\}$

$$=e^{-t}\left\{3.\frac{t^2}{2!}-2.\frac{t^3}{3!}\right\}=e^{-t}\left\{\frac{3}{2}t^2-\frac{1}{3}t^3\right\}.$$

Q.No.6.: Find the Inverse Laplace's transform of $\dfrac{s+1}{s^2+6s+25}$.

Sol.: $L^{-1}\left\{\dfrac{s+1}{s^2+6s+25}\right\}=L^{-1}\left\{\dfrac{s+1}{(s+3)^2+16}\right\}$

Applying shift theorem, we get

$$=e^{-3t}L^{-1}\left\{\frac{(s-3)+1}{s^2+16}\right\}=e^{-3t}\left\{L^{-1}\left[\frac{s-2}{s^2+16}\right]\right\}=e^{-3t}.\left\{\cos 4t-\frac{1}{2}\sin 4t\right\}.$$

Q.No.7.: Find the Inverse Laplace's transform of $F(as+b)$.

Sol.: $F(as+b)=\int_0^\infty e^{-(as+b)t}f(t)\,dt=\int_0^\infty e^{-ast}e^{-bt}f(t)dt$

Put $at=u,\ dt=\dfrac{du}{a}$

$$F(as+b)=\int_0^\infty e^{-su}.e^{\frac{b}{a}u}.f\left(\frac{u}{a}\right).\frac{1}{a}.du=\frac{1}{a}\int_0^\infty e^{-su}\left\{.e^{-\frac{b}{a}u}.f\left(\frac{u}{a}\right)\right\}du$$

$$=\frac{1}{a}L\left\{.e^{-\frac{b}{a}u}.f\left(\frac{u}{a}\right)\right\}=L\left\{\frac{1}{a}e^{-bt}.f(t)\right\}$$

$$\therefore L^{-1}\{F(as+b)\} = \frac{1}{a}e^{-bt}.f(t).$$

Q.No.8.: Find the Inverse Laplace's transform of $\dfrac{1}{\sqrt{2s+3}}$.

Sol.: $L^{-1}\left\{\dfrac{1}{\sqrt{2s+3}}\right\} = \dfrac{1}{\sqrt{2}}L^{-1}\left\{\dfrac{1}{\left(s+\dfrac{3}{2}\right)^{1/2}}\right\}$.

Applying shift theorem, we get

$$= \frac{1}{\sqrt{2}}e^{\frac{-3t}{2}}.L^{-1}\left\{\frac{1}{s^{\frac{1}{2}}}\right\} = \frac{1}{\sqrt{2}}e^{\frac{-3t}{2}}.\frac{t^{\frac{-1}{2}}}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{2}}\frac{t^{\frac{-1}{2}}.e^{\frac{-3t}{2}}}{\sqrt{\pi}}.$$

Q.No.9.: Find the inverse Laplace transforms of the following: **(i)** $\dfrac{s^2}{(s-2)^3}$,

(ii) $\dfrac{s+3}{s^2-4s+13}$.

Sol.: **(i)** Since $s^2 = (s-2)^2 + 4(s-2) + 4$

$$\therefore \frac{s^2}{(s-2)^3} = \frac{1}{s-2} + \frac{4}{(s-2)^2} + \frac{4}{(s-2)^3}$$

$$\therefore L^{-1}\left\{\frac{s^2}{(s-2)^3}\right\} = L^{-1}\left(\frac{1}{s-2}\right) + 4L^{-1}\left(\frac{1}{(s-2)^2}\right) + 4L^{-1}\left(\frac{1}{(s-2)^3}\right) = e^{2t} + 4e^{2t}t + 2e^{2t}t^2.$$

(ii) $\dfrac{s+3}{s^2-4s+13} = \dfrac{s-2}{(s-2)^2+3^2} + \dfrac{5}{(s-2)^2+3^2}$

$$\therefore L^{-1}\frac{s+3}{s^2-4s+13} = L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + \frac{5}{3}L^{-1}\left\{\frac{3}{(s-2)^2+3^2}\right\} = e^{2t}\cos 3t + \frac{5}{3}e^{2t}\sin 3t.$$

Q.No.10.: Find the Inverse Laplace's transform of $\dfrac{s}{(s+a)^2}$.

Sol.: $L^{-1}\left[\dfrac{s}{(s+a)^2}\right]$

Resolving $\frac{s}{(s+a)^2}$ into partial fractions

$$= L^{-1}\left[\frac{1}{s+a} - \frac{a}{(s+a)^2}\right] = L^{-1}\left(\frac{1}{s+a}\right) - aL^{-1}\left(\frac{1}{(s+a)^2}\right) = e^{-at} - ae^{-at}.t$$

Thus $L^{-1}\left[\frac{s}{(s+a)^2}\right] = e^{-at}[1-at]$. Ans.

Q.No.11.: Find the Inverse Laplace's transform of $\frac{s^2}{(s+a)^3}$.

Sol.: $L^{-1}\left[\frac{s^2}{(s+a)^3}\right]$

Now, $\frac{s^2}{(s+a)^3} = \frac{A}{s+a} + \frac{B}{(s+a)^2} + \frac{C}{(s+a)^3}$

$s^2 = A(s+a)^2 + B(s+a) + C$

$s^2 = As^2 + (2aA + B)s + a^2A + aB + C$ Equating the coefficient of s^2, we get

$A = 1$

Equating the coefficient of s, we get

$2aA + B = 0 \Rightarrow B = -2a$

Similarly equating the coefficient of constant term, we get

$a^2A + aB + C = 0 \Rightarrow a^2 - 2a^2 + C = 0 \Rightarrow C = a^2$

$$L^{-1}\left[\frac{s^2}{(s+a)^3}\right] = L^{-1}\left[\frac{1}{s+a} - \frac{2a}{(s+a)^2} + \frac{a^2}{(s+a)^3}\right]$$

$$= e^{-at} - 2ae^{-at}.t + a^2e^{-at}.\frac{t^2}{2!} = \frac{1}{2}e^{-at}\left[2 - 4at + a^2t^2\right]$$

$L^{-1}\left[\frac{s^2}{(s+a)^3}\right] = \frac{1}{2}e^{-at}\left[a^2t^2 - 4at + 2\right]$. Ans.

Always Remember: Change of Scale Property: $L^{-1}\left[\bar{f}(ks)\right]=\frac{1}{k}f\left(\frac{t}{k}\right)$.

Q.No.1.: Find $L^{-1}\left\{\frac{64}{81s^4-256}\right\}$.

Sol.: Given $\bar{f}(s)=\frac{64}{81s^4-256}=\frac{4^3}{(3s)^4-4^4}$.

To find Inverse Laplace Transform:

Since $L^{-1}\left\{\frac{a^3}{s^4-a^4}\right\}=\frac{1}{2}L^{-1}\left\{\frac{a}{s^2-a^2}-\frac{a}{s^2+a^2}\right\}=\frac{1}{2}(\sinh at-\sin at)=f(t)$

$$L^{-1}\left\{\frac{64}{81s^4-256}\right\}=L^{-1}\left\{\frac{4^3}{(3s)^4-4^4}\right\}=L^{-1}\{F(3s)\},\text{ with } a=4$$

Here $F(3s)=\frac{a^3}{(3s)^4-a^4}$ and $f(t)=\frac{1}{2}(\sinh at-\sin at)$.

Thus, applying change of scale property (with k = 3), we get

$$L^{-1}\{F(3s)\}=\frac{1}{3}f\left(\frac{t}{3}\right).$$

$$=\frac{1}{3}.\frac{1}{2}\left[\sinh\frac{4t}{3}-\sin\frac{4t}{3}\right]$$

$$=\frac{1}{6}\left[\sinh\frac{4t}{3}-\sin\frac{4t}{3}\right]$$

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Always Remember: Multiplication by s:

If $L^{-1}\left[\bar{f}(s)\right]=f(t)$, and $f(0)=0$, then $L^{-1}\left\{s\bar{f}(s)\right\}=\frac{d}{dt}\{f(t)\}$.

In general, $L^{-1}\left\{s^n\bar{f}(s)\right\}=\frac{d^n}{dt^n}\{f(t)\}$, provided $f(0)=f'(0)=\ldots\ldots\ldots=f^{n-1}(0)=0$.

Q.No.1.: Evaluate the following $L^{-1}\left\{\frac{s}{s^2-a^2}\right\}$.

Sol.: Given $\bar{f}(s) = \dfrac{s}{s^2 - a^2}$

To find Inverse Laplace Transform:

Since we have $L^{-1}\left\{\dfrac{1}{s^2 - a^2}\right\} = \dfrac{\sinh at}{a}$.

$$\therefore L^{-1}\left\{s.\frac{1}{s^2 - a^2}\right\} = \frac{d}{dt}\frac{\sinh at}{a} = \frac{a\cosh at}{a}$$

$$\Rightarrow L^{-1}\left\{\frac{s}{s^2 - a^2}\right\} = \cosh at.$$

Always Remember: Division by powers of s:

$$L^{-1}\left[\bar{f}(s)\right] = f(t), \text{ then } L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)dt$$

Also
$$L^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \left\{\int_0^t f(t)dt\right\}dt$$

$$L^{-1}\left\{\frac{\bar{f}(s)}{s^3}\right\} = \int_0^t \left\{\int_0^t \left(\int_0^t f(t)dt\right)dt\right\}dt \text{ and so on.}$$

Q.No.1.: Find the Inverse Laplace's transform of $\dfrac{s+2}{s^2(s+3)}$.

Sol.: Given $\bar{f}(s) = \dfrac{s+2}{s^2(s+3)}$

To find Inverse Laplace Transform:

$$L^{-1}\left\{\frac{s+2}{s^2(s+3)}\right\} = L^{-1}\left\{\frac{1}{s(s+3)}\right\} + L^{-1}\left\{\frac{2}{s^2(s+3)}\right\} = R_1 + R_2.$$

We know that $L^{-1}\left\{\dfrac{1}{s+3}\right\} = e^{-3t}.L^{-1}\left\{\dfrac{1}{s}\right\} = e^{-3t}$.

$$R_1 = L^{-1}\left\{\frac{1}{s}.\frac{1}{s+3}\right\} = \int_0^t e^{-3u}du \text{, (by division by powers of s property)}$$

$$= \left.\frac{e^{-3u}}{-3}\right|_0^t = \frac{1}{3}\left[1 - e^{-3t}\right].$$

$$R_2 = 2L^{-1}\left\{\frac{1}{s^2}\frac{1}{s+3}\right\} = 2L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s(s+3)}\right\}$$

$$= 2\int_0^t \frac{1}{3}\left(1 - e^{-3u}\right)du \qquad \text{(by } R_1\text{)}$$

$$= \frac{2}{3}\left[u - \frac{e^{-3u}}{-3}\right]_0^t = \frac{2}{3}\left[t + \frac{e^{-3t}}{3} - \frac{1}{3}\right].$$

$$\therefore L^{-1}\left\{\frac{s+2}{s^2(s+3)}\right\} = \frac{1}{3}\left[1 - e^{-3t}\right] + \frac{2}{3}\left[t + \frac{e^{-3t}}{3} - \frac{1}{3}\right]$$

$$= \frac{1}{3} - \frac{1}{3}e^{-3t} + \frac{2}{3}t + \frac{2}{9}e^{-3t} - \frac{2}{9}$$

$$= \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t}.$$

Q.No.2.: Find the Inverse Laplace's transform of $\frac{1}{s^3(s+1)}$.

Sol.: Given $\bar{f}(s) = \frac{1}{s^3(s+1)}$

To find Inverse Laplace Transform:

Since we have $L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}L^{-1}\left\{\frac{1}{s}\right\} = e^{-t}$.

Applying division by powers of s property, we get

$$L^{-1}\left\{\frac{1}{s}\cdot\left(\frac{1}{s+1}\right)\right\} = \int_0^t e^{-u}du = \left.\frac{e^{-u}}{-1}\right|_0^t = 1 - e^{-t}.$$

$$L^{-1}\left\{\frac{1}{s}\cdot\left(\frac{1}{s(s+1)}\right)\right\} = \int_0^t \left(1 - e^{-u}\right)du = \left[u - \frac{e^{-u}}{-1}\right]_0^t = t + e^{-t} - 1.$$

Finally, $L^{-1}\left\{\frac{1}{s}\cdot\left(\frac{1}{s^2(s+1)}\right)\right\} = \int_0^t \left(u + e^{-u} - 1\right)du$

$$=\left[\frac{u^2}{2}+\frac{e^{-u}}{-1}-u\right]_0^t=\frac{t^2}{2}-e^{-t}+1-t=1-t+\frac{1}{2}t^2-e^{-t}.$$

Q.No.3.: Find the Inverse Laplace's transform of $\frac{1}{\left(s^2+a^2\right)^2}$.

Sol.: Rewrite $\frac{1}{\left(s^2+a^2\right)^2}=\frac{1}{s}.\frac{s}{\left(s^2+a^2\right)^2}$.

We know that $L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}=-t\frac{\sin at}{2a}$.

(Using inverse Laplace transform for derivatives).

Now $L^{-1}\left\{\frac{1}{\left(s^2+a^2\right)^2}\right\}=L^{-1}\left\{\frac{1}{s}.\frac{s}{\left(s^2+a^2\right)^2}\right\}$.

Using inverse Laplace transform for integral, we get

$$=\int_0^t -u.\frac{\sin au}{2a}du.$$

Integrating by parts, we get

$$=-\frac{1}{2a^2}u.\cos au\Big|_0^t+\frac{1}{2a^2}.\frac{\sin au}{a}\Big|_0^t$$

$$=\frac{1}{2a^3}\{\sin at-at.\cos at\}.$$

Q.No.4.: Find the Inverse Laplace's transform of $\frac{s^2}{\left(s^2+a^2\right)^2}$.

Sol.: Rewriting $\frac{s^2}{\left(s^2+a^2\right)^2}=\frac{s^2+a^2-a^2}{\left(s^2+a^2\right)^2}=\frac{1}{\left(s^2+a^2\right)}-\frac{a^2}{\left(s^2+a^2\right)^2}$

$$L^{-1}\left\{\frac{s^2}{\left(s^2+a^2\right)^2}\right\}=L^{-1}\left\{\frac{1}{s^2+a^2}\right\}-a^2L^{-1}\left\{\frac{1}{\left(s^2+a^2\right)^2}\right\}$$

$$=\frac{\sin at}{a}-a^2.\frac{1}{2a^3}\{\sin at-at\cos at\}$$

$$= \frac{1}{2a}(\sin at + at\cos at).$$

Q.No.5.: Find the Inverse Laplace's transform of $\frac{1}{s^2(s^2+a^2)}$.

Sol.: $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]$

We know that

$$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1}{a}\int_0^t \sin at dt = -\frac{1}{a^2}[\cos at]_0^t = -\frac{1}{a^2}[\cos at - 1] = \frac{1}{a^2}[1-\cos at]$$

$$\therefore L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \frac{1}{a^2}\int_0^t (1-\cos at)dt = \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right]_0^t$$

$$= \frac{1}{a^2}\left(t - \frac{\sin at}{a} - 0 + \frac{\sin a(0)}{a}\right)$$

$$\Rightarrow L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \frac{1}{a^3}(at - \sin at). \text{ Ans.}$$

Q.No.6.: Find the Inverse Laplace's transform of $\frac{1}{s^3(s^2+1)}$.

Sol.: $L^{-1}\left[\frac{1}{s^3(s^2+1)}\right]$

We know that

$$L^{-1}\left(\frac{1}{s^2+1^2}\right) = \sin t$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2+1)}\right] = \int_0^t \sin t dt = -[\cos t]_0^t = -[\cos t - 1] = 1 - \cos t$$

$$\therefore L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = \int_0^t (1-\cos t)dt = [t - \sin t]_0^t = t - \sin t$$

$$\therefore L^{-1}\left[\frac{1}{s^3\left(s^2+1\right)}\right]=\int_0^t (t-\sin t)dt=\left[\frac{t^2}{2}+\cos t\right]_0^t$$

$$\Rightarrow L^{-1}\left[\frac{1}{s^3\left(s^2+1\right)}\right]=\frac{t^2}{2}+\cos t-1. \text{ Ans.}$$

Q.No.7.: Find the inverse Laplace transforms of **(i)** $\frac{1}{s\left(s^2+a^2\right)}$,

(ii) $\frac{1}{s(s+a)^3}$.

Sol.: (i) Since $L^{-1}\frac{1}{\left(s^2+a^2\right)}=\frac{1}{a}\sin at$

$$\therefore L^{-1}\left\{\frac{1}{s\left(s^2+a^2\right)}\right\}=\int_0^t \frac{1}{a}\sin at\,dt=\frac{1}{a^2}\left[-\cos at\right]_0^t=\frac{(1-\cos at)}{a^2}. \text{ Ans.}$$

(ii) $L^{-1}\left\{\frac{1}{s(s+a)^3}\right\}=L^{-1}\left\{\frac{1}{[(s+a)-a](s+a)^3}\right\}=e^{-at}L^{-1}\left\{\frac{1}{(s-a)s^3}\right\}$.

Now $L^{-1}\left\{\frac{1}{s-a}\right\}=e^{at}$ $\therefore L^{-1}\left\{\frac{1}{(s-a)s}\right\}=\int_0^t e^{at}dt=\frac{e^{at}}{a}-\frac{1}{a}$.

$$L^{-1}\left\{\frac{1}{s^2(s-a)}\right\}=\frac{1}{a}\int_0^t \left(e^{at}-1\right)dt=\frac{1}{a^2}\left(e^{at}-at-1\right)$$

$$L^{-1}\left\{\frac{1}{s^3(s-a)}\right\}=\frac{1}{a^2}\int_0^t \left(e^{at}-at-1\right)dt=\frac{1}{a^3}\left(e^{at}-\frac{a^2}{2}t^2-at-1\right).$$

Hence $L^{-1}\left\{\frac{1}{s(s+a)^3}\right\}=e^{-at}.\frac{1}{a^3}\left(e^{at}-\frac{a^2t^2}{2}-at-1\right)$

$$=\frac{1}{a^3}\left(1-e^{-at}-ate^{-at}-\frac{a^2}{2}t^2e^{-at}\right). \text{ Ans.}$$

Q.No.8.: Find the inverse Laplace transforms of $\frac{s+2}{s^2(s+1)(s-2)}$.

Sol.: $L^{-1}\left\{\frac{s+2}{(s+1)(s-2)}\right\}=\frac{4}{3}L^{-1}\left(\frac{1}{s-2}\right)-\frac{1}{3}L^{-1}\left(\frac{1}{s+1}\right)=\frac{4}{3}e^{2t}-\frac{1}{3}e^{-t}$

Then by formula $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)dt$,

$$L^{-1}\left\{\frac{s+2}{s(s+1)(s-2)}\right\} = \int_0^t L^{-1}\left(\frac{s+2}{(s+1)(s-2)}\right).dt = \int_0^t \left(\frac{4}{3}e^{2t} - \frac{1}{3}e^{-t}\right)dt = \frac{2}{3}e^{2t} + \frac{1}{3}e^{-t} - 1$$

Again by formula $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)dt$,

$$L^{-1}\left\{\frac{s+2}{s^2(s+1)(s-2)}\right\} = \int_0^t L^{-1}\left\{\frac{s+2}{s(s+1)(s-2)}\right\}dt$$

$$= \int_0^t \left(\frac{2}{3}e^{2t} + \frac{1}{3}e^{-t} - 1\right)dt = \frac{1}{3}\left(e^{2t} - e^{-t} - t\right).$$ **Ans.**

Q.No.9.: Find the Inverse Laplace's transform of $\frac{1}{s(s+2)}$.

Sol.: $L^{-1}\left[\frac{1}{s(s+2)}\right]$

We know that, $L^{-1}\left[\frac{\bar{f}(s)}{s}\right] = \int_0^t f(t)dt$

Let $\bar{f}(s) = \frac{1}{s+2} \Rightarrow f(t) = L^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}$

$$\therefore L^{-1}\left[\frac{1}{s(s+2)}\right] = \int_0^t e^{-2t}dt = -\frac{1}{2}\left|e^{-2t}\right|_0^t = -\frac{1}{2}\left[e^{-2t} - 1\right] = \frac{1}{2}\left[1 - e^{-2t}\right].$$ **Ans.**

Q.No.10.: Find the Inverse Laplace's transform of $\frac{1}{s^2(s+2)}$.

Sol.: Since $L^{-1}\left[\frac{1}{s(s+2)}\right] = \frac{1}{2}\left[1 - e^{-2t}\right]$.

Hence $L^{-1}\left[\frac{1}{s^2(s+2)}\right] = \frac{1}{2}\int_0^t \left[1 - e^{-2t}\right]dt = \frac{1}{2}\left[t - \frac{e^{-2t}}{-2}\right]_0^t = \frac{1}{2}\left[t + \frac{e^{-2t}}{2} - \frac{1}{2}\right]$

$$\Rightarrow L^{-1}\left[\frac{1}{s^2(s+2)}\right] = \frac{1}{4}\left[2t + e^{-2t} - 1\right].$$ **Ans.**

Q.No.11.: Find the Inverse Laplace's transform of $\frac{1}{s(s+2)^3}$.

Sol.: $L^{-1}\left[\frac{1}{s(s+2)^3}\right] = L^{-1}\left[\frac{1}{\{(s+2)-2\}(s+2)^3}\right] = e^{-2t}L^{-1}\left[\frac{1}{(s+2)s^3}\right]$

Since $L^{-1}\left[\frac{1}{s^2(s+2)}\right] = \frac{1}{4}\left[2t + e^{-2t} - 1\right]$

$$L^{-1}\left[\frac{1}{s^3(s+2)}\right] = \frac{1}{4}\int_0^t \left[e^{-2t} + 2t - 1\right]dt = \frac{1}{4}\left[\frac{-e^{-2t}}{2} + t^2 - t\right]_0^t$$

$$= \frac{1}{4}\left[\frac{-e^{-2t}}{2} + t^2 - t + \frac{1}{2}\right]_0^t = \frac{1}{8}\left[-e^{-2t} + 2t^2 - 2t + 1\right]$$

$$= \frac{-1}{8}\left[e^{-2t} - 2t^2 + 2t - 1\right]$$

$$L^{-1}\left[\frac{1}{s(s+2)^3}\right] = \frac{-e^{-2t}}{8}\left[e^{-2t} - 2t^2 + 2t - 1\right]. \text{ Ans.}$$

Always Remember: Inverse Laplace transform of derivatives:

If $L^{-1}\left[\bar{f}(s)\right] = f(t)$, then $tf(t) = L^{-1}\left\{-\frac{d}{ds}\left[\bar{f}(s)\right]\right\}$

The above result follows from $L\{tf(t)\} = -\frac{d}{ds}\{\bar{f}(s)\}$.

In general, $L^{-1}\left\{\bar{f}^{\,n}(s)\right\} = (-1)^n t^n f(t)$, n = 1, 2, 3, …….

Q.No.1.: Find the Inverse Laplace's transform of $\frac{s+1}{\left(s^2+2s+2\right)^2}$.

Sol.: Given $\bar{f}(s) = \frac{s+1}{\left(s^2+2s+2\right)^2}$

To find Inverse Laplace Transform:

Taking Inverse Laplace's transform on both sides, we get

$$L^{-1}\left\{\bar{f}(s)\right\}=L^{-1}\left\{\frac{s+1}{\left(s^2+2s+2\right)^2}\right\}=L^{-1}\left\{\frac{s+1}{\left\{(s+1)^2+1\right\}^2}\right\}=e^{-t}L^{-1}\left\{\frac{s}{\left(s^2+1\right)^2}\right\}$$

We know that $\frac{d}{ds}\left\{\frac{1}{s^2+1}\right\}=\frac{-2s}{\left(s^2+1\right)^2}$

So that $\frac{s}{\left(s^2+1\right)^2}=-\frac{1}{2}\frac{d}{ds}\left\{\frac{1}{s^2+1}\right\}$

Using inverse Laplace's transform of derivatives [(with n = 1) one differentiation]

$$L^{-1}\left\{\frac{s+1}{\left(s^2+2s+2\right)^2}\right\}=L^{-1}\left\{\frac{s+1}{\left\{(s+1)^2+1\right\}^2}\right\}=e^{-t}L^{-1}\left\{\frac{s}{\left(s^2+1\right)^2}\right\}$$

$$=e^{-t}.\frac{-1}{2}L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{\left(s^2+1\right)}\right)\right\}$$

$$=-\frac{1}{2}e^{-t}.(-1)^1.t^1.L^{-1}\left\{\frac{1}{s^2+1}\right\}=\frac{1}{2}e^{-t}.t.\sin t.\ \left[\because L^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t\right]$$

Q.No.2.: Find the Inverse Laplace's transform of $\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$.

Sol.: Here $\bar{f}(s)=\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$.

$$\Rightarrow \bar{f}(s)=\frac{1}{2}\left[\log\left(s^2+b^2\right)-\log\left(s^2+a^2\right)\right].$$

$$\therefore \bar{f}'(s)=\frac{1}{2}\frac{2s}{s^2+b^2}-\frac{1}{2}\frac{2s}{s^2+a^2}=\frac{s}{s^2+b^2}-\frac{s}{s^2+a^2}.$$

To find Inverse Laplace Transform:

Taking Inverse Laplace's transform on both sides, we get

$$L^{-1}\left\{\bar{f}'(s)\right\}=L^{-1}\left\{\frac{s}{s^2+b^2}-\frac{s}{s^2+a^2}\right\}=L^{-1}\left\{\frac{s}{s^2+b^2}\right\}-L^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos bt-\cos at.$$

Using inverse Laplace's transform of derivatives, we get

$$-tf(t)=L^{-1}\left\{\bar{f}'(s)\right\}=\cos bt-\cos at.$$

Thus, $f(t) = \dfrac{\cos at - \cos bt}{t}$.

Q.No.3.: Find the Inverse Laplace's transform of $\dfrac{1}{(s-a)^3}$.

Sol.: We know that $L^{-1}\dfrac{1}{s-a} = e^{at}$

Let $F(s) = \dfrac{1}{(s-a)}$ so that

$f(t) = L^{-1}\{F(s)\} = e^{at}$

Then

$$F'(s) = \frac{-1}{(s-a)^2}.F''(s) = \frac{(-1)(-2)}{(s-a)^3} = \frac{2}{(s-a)^3}$$

By inverse Laplace's transform of derivatives with n = 2, we get

$$L^{-1}\left\{\frac{1}{(s-a)^3}\right\} = \frac{1}{2}L^{-1}\{F''(s)\} = \frac{(-1)^2}{2}t^2.f(t) = 1.\frac{t^2.e^{at}}{2} = \frac{t^2.e^{at}}{2}.$$

Q.No.4.: Find the Inverse Laplace's transform of $\cot^{-1}\left(\dfrac{s+a}{b}\right)$.

Sol.: $F(s) = \cot^{-1}\left(\dfrac{s+a}{b}\right)$, $F'(s) = -\dfrac{1}{1+\left(\dfrac{s+a}{b}\right)^2}.\dfrac{1}{b}$

$$F'(s) = \frac{1}{b}\frac{-b^2}{(s+a)^2+b^2}, \text{ so}$$

$$L^{-1}\{F'(s)\} = -L^{-1}\left\{\frac{b}{(s+a)^2+b^2}\right\} = -e^{-at}\sin bt.$$

Since $-tf(t) = L^{-1}\{F'(s)\} = -e^{-at}\sin bt$.

Thus $f(t) = \dfrac{e^{-at}}{t}\sin bt$.

Q.No.5.: Find the Inverse Laplace's transform of $s\log\dfrac{s-1}{s+1}$.

Sol.: $L^{-1}\left[s\log\frac{s-1}{s+1}\right]$

If $f(t)=L^{-1}\left[s\log\frac{s-1}{s+1}\right]$

$$tf(t)=-L^{-1}\left[\frac{d}{ds}\left\{s\log\frac{s-1}{s+1}\right\}\right]=-L^{-1}\left[\left[s\left\{\frac{s+1}{s-1}.\frac{(s+1)-(s-1)}{(s+1)^2}\right\}\right]+\log\frac{s-1}{s+1}\right]$$

$$=-L^{-1}\left[\frac{2s}{(s-1)(s+1)}+\log\frac{s-1}{s+1}\right]=-L^{-1}\left[\frac{2s}{s^2-1}\right]+L^{-1}\log\frac{s-1}{s+1}$$

$$=-L^{-1}\left(\frac{2s}{s^2-1^2}\right)-\left(-\frac{1}{t}\right)L^{-1}\left[\frac{d}{ds}\log\frac{s-1}{s+1}\right]=-2\cosh t+\frac{1}{t}L^{-1}\left[\frac{s+1}{s-1}.\frac{(s+1)-(s-1)}{(s+1)^2}\right]$$

$$=-2\cosh t+\frac{1}{t}L^{-1}\left[\frac{2}{(s-1)(s+1)}\right]=-2\cosh t+\frac{2}{t}\sinh t$$

$$f(t)=-\frac{2}{t}\cosh t+\frac{2}{t^2}\sinh t$$

Thus $L^{-1}\left[s\log\frac{s-1}{s+1}\right]=-\frac{2}{t}\cosh t+\frac{2}{t^2}\sinh t=\frac{2(\sinh t-t\cosh t)}{t^2}$. Ans.

Q.No.6.: Find the Inverse Laplace's transform of $\tan^{-1}\left(\frac{2}{s}\right)$.

Sol.: $L^{-1}\left[\tan^{-1}\left(\frac{2}{s}\right)\right]$

If $f(t)=L^{-1}\left[\tan^{-1}\left(\frac{2}{s}\right)\right]$

$$tf(t)=L^{-1}\left[-\frac{d}{ds}\tan^{-1}\left(\frac{2}{s}\right)\right]=-L^{-1}\left[-\frac{2}{s^2+4}\right]=L^{-1}\left[\frac{2}{s^2+4}\right]=\sin 2t$$

$$f(t)=\frac{\sin 2t}{t}.$$

Thus $L^{-1}\left[\tan^{-1}\left(\frac{2}{s}\right)\right]=\frac{\sin 2t}{t}$.

Q.No.7.: Find the inverse Laplace transforms of

(i) $\log\frac{s+1}{s-1}$, **(ii)** $\log\frac{s^2+1}{s(s+1)}$, **(iii)** $\cot^{-1}\left(\frac{s}{2}\right)$, **(iv)** $\tan^{-1}\left(\frac{2}{s^2}\right)$.

Sol.: If $f(t)=L^{-1}\log\frac{s+1}{s-1}$

Then by formula, if $L^{-1}\left[\bar{f}(s)\right]=f(t)$, then $tf(t)=L^{-1}\left\{-\frac{d}{ds}\left[\bar{f}(s)\right]\right\}$

$$tf(t)=L^{-1}\left\{-\frac{d}{ds}\log\left(\frac{s+1}{s-1}\right)\right\}=-L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\}+L^{-1}\left\{\frac{d}{ds}\log(s-1)\right\}$$

$$=-L^{-1}\left(\frac{1}{s+1}\right)+L^{-1}\left(\frac{1}{s-1}\right)=-e^{-t}+e^{t}=2\sinh t$$

$\therefore f(t)=\frac{(2\sinh t)}{t}$. Ans.

(ii) If $f(t)=L^{-1}\log\frac{s^2+1}{s(s+1)}$,

Then by formula, if $L^{-1}\left[\bar{f}(s)\right]=f(t)$, then $tf(t)=L^{-1}\left\{-\frac{d}{ds}\left[\bar{f}(s)\right]\right\}$

$$t.f(t)=L^{-1}\left\{-\frac{d}{ds}\log\left(\frac{s^2+1}{s(s+1)}\right)\right\}=-L^{-1}\left\{\frac{d}{ds}\log(s^2+1)\right\}+L^{-1}\left\{\frac{d}{ds}\log s\right\}+L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\}$$

$$=-L^{-1}\left(\frac{2s}{s^2+1}\right)+L^{-1}\left(\frac{1}{s}\right)+L^{-1}\left(\frac{1}{s+1}\right)=-2\cos t+1+e^{-t}$$

Thus $f(t)=\frac{1}{t}\left(1+e^{-t}-2\cos t\right)$. Ans.

(iii) If $f(t)=L^{-1}\cot^{-1}\left(\frac{s}{2}\right)$.

Then by formula, if $L^{-1}\left[\bar{f}(s)\right]=f(t)$, then $tf(t)=L^{-1}\left\{-\frac{d}{ds}\left[\bar{f}(s)\right]\right\}$

$$tf(t)=L^{-1}\left\{-\frac{d}{ds}\cot^{-1}\left(\frac{s}{2}\right)\right\}=L^{-1}\left\{\frac{2}{s^2+2^2}\right\}=\sin 2t$$

Thus $f(t)=\frac{(\sin t)}{t}$. Ans.

(iv) If $f(t)=L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$

Then by formula, if $L^{-1}\left[\bar{f}(s)\right]=f(t)$, then $tf(t)=L^{-1}\left\{-\frac{d}{ds}\left[\bar{f}(s)\right]\right\}$.

$$t.f(t)=L^{-1}\left\{-\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)\right\}=L^{-1}\left\{\frac{4s}{s^4+4}\right\}$$

$$=L^{-1}\left\{\frac{4s}{\left(s^2+2\right)^2-(2s)^2}\right\}=L^{-1}\left\{\frac{4s}{\left(s^2+2+2s\right)\left(s^2+2-2s\right)}\right\}$$

$$=L^{-1}\left\{\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right\}=L^{-1}\left\{\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right\}$$

$$=e^t\sin t-e^{-t}\sin t=2\sinh t\sin t\text{. Ans.}$$

Q.No.8.: Find the inverse Laplace transforms of $\frac{s+2}{\left(s^2+4s+5\right)^2}$.

Sol.: $L^{-1}\left\{\frac{1}{\left(s^2+4s+5\right)}\right\}=L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}=e^{-2t}\sin t$

By formula if $L^{-1}\left[\bar{f}(s)\right]=f(t)$, then $L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\}=-t\{f(t)\}$.

$$L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{\left(s^2+4s+5\right)}\right)\right\}=(-1)^1 t.e^{-2t}\sin t$$

$$\Rightarrow L^{-1}\left\{\frac{-(2s+4)}{\left(s^2+4s+5\right)^2}\right\}=-t.e^{-2t}\sin t$$

$$\Rightarrow L^{-1}\left\{\frac{s+2}{\left(s^2+4s+5\right)^2}\right\}=\frac{1}{2}t.e^{-2t}\sin t\text{. Ans.}$$

Q.No.9.: Find the Inverse Laplace's transform of $\log\left(\frac{1+s}{s}\right)$.

Sol.: $\log\left(\frac{1+s}{s}\right)$

If $L^{-1}\left[\log\left(\frac{1+s}{s}\right)\right] = f(t)$

$$tf(t) = L^{-1}\left[\frac{-d}{ds}\log\left(\frac{1+s}{s}\right)\right] = -L^{-1}\left[\frac{s}{1+s}.\frac{s-(1+s)}{s^2}\right]$$

$$tf(t) = L^{-1}\left[\frac{1}{s(1+s)}\right]$$

Resolving $\left[\frac{1}{s(1+s)}\right]$ into partial, we get

$$tf(t) = L^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = 1 - e^{-t} \Rightarrow f(t) = \frac{1-e^{-t}}{t}$$

Thus $L^{-1}\left[\log\left(\frac{1+s}{s}\right)\right] = \frac{1-e^{-t}}{t}$

Q.No.10.: Find the Inverse Laplace's transform of $\log\left(\frac{s+a}{s+b}\right)$.

Sol.: $\log\left(\frac{s+a}{s+b}\right)$

If $f(t) = L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right]$

$$tf(t) = L^{-1}\left[\frac{-d}{ds}\log\left(\frac{s+a}{s+b}\right)\right] = -L^{-1}\left[\frac{s+b}{s+a}.\frac{(s+b)-(s+a)}{(s+b)^2}\right] = -L^{-1}\left[\frac{s+b}{s+a}.\frac{b-a}{(s+b)^2}\right]$$

$$= L^{-1}\left[\frac{a-b}{(s+a)(s+b)}\right]$$

Resolving $\frac{a-b}{(s+a)(s+b)}$ into partial fraction, we get

$$= L^{-1}\left[\frac{1}{(s+b)} - \frac{1}{(s+a)}\right]$$

$$= L^{-1}\left(\frac{1}{s+b}\right) - L^{-1}\left(\frac{1}{s+a}\right)$$

$$tf(t) = e^{-bt} - e^{-at}$$

$$f(t)=\frac{e^{-bt}-e^{-at}}{t}$$

Thus $L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right]=\frac{1}{t}\left[e^{-bt}-e^{-at}\right]$

Q.No.11.: Find the Inverse Laplace's transform of $L^{-1}\log\left\{\frac{s+1}{(s+2)(s+3)}\right\}$.

Sol.: $L^{-1}\log\left\{\frac{s+1}{(s+2)(s+3)}\right\}$

If $L^{-1}\log\left\{\frac{s+1}{(s+2)(s+3)}\right\}=f(t)$

Then $tf(t)=L^{-1}\left[\frac{-d}{ds}\log\frac{s+1}{(s+2)(s+3)}\right]=-L^{-1}\left[\frac{(s+2)(s+3)}{(s+1)}.\frac{(s+2)(s+3)-(s+1)(2s+5)}{(s+2)^2(s+3)^2}\right]$

$$=-L^{-1}\left[\frac{(s+2)(s+3)-(s+1)(2s+5)}{(s+1)(s+2)(s+3)}\right]=L^{-1}\left[\frac{s^2+2s-1}{(s+1)(s+2)(s+3)}\right]$$

$$=L^{-1}\left[\frac{1-2-1}{(s+1)(1)(2)}+\frac{4-4-1}{(-1)(s+2)(1)}+\frac{9-6-1}{(-2)(-1)(s+3)}\right]$$

$$=L^{-1}\left[-\frac{1}{s+1}+\frac{1}{s+2}+\frac{1}{s+3}\right]=-\left(e^{-t}-e^{-2t}-e^{-3t}\right)$$

Thus $L^{-1}\log\left\{\frac{s+1}{(s+2)(s+3)}\right\}=-\frac{1}{t}\left(e^{-t}-e^{-2t}-e^{-3t}\right)$. Ans.

Q.No.12.: Find the Inverse Laplace's transform of $\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$.

Sol.: $\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

If $f(t)=L^{-1}\left[\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right]$

Then $tf(t)=L^{-1}\left[-\frac{1}{2}\frac{d}{ds}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right]$

$$tf(t) = -\frac{1}{2}L^{-1}\left[\frac{s^2+a^2}{s^2+b^2}\cdot\frac{(s^2+a^2)(2s)-2s(s^2+b^2)}{(s^2+a^2)^2}\right] = L^{-1}\left\{-\frac{1}{2}\left[\frac{2(s^3+a^2s-s^3-b^2s)}{(s^2+a^2)(s^2+b^2)}\right]\right\}$$

$$= -L^{-1}\left[\frac{(a^2-b^2)s}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[-s\left\{\frac{1}{s^2+b^2}-\frac{1}{s^2+a^2}\right\}\right]$$

$$= -L^{-1}\left[\frac{s}{s^2+b^2}-\frac{s}{s^2+a^2}\right] = -[\cos bt - \cos at].$$

$$\Rightarrow f(t) = \frac{\cos at - \cos bt}{t}$$

$$L^{-1}\left[\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right] = \frac{\cos at - \cos bt}{t}. \text{ Ans.}$$

Q.No.13.: Find the Inverse Laplace's transform of $\log\left(1-\frac{a^2}{s^2}\right)$.

Sol.: $L^{-1}\log\left(1-\frac{a^2}{s^2}\right)$

$$f(t) = L^{-1}\log\left(1-\frac{a^2}{s^2}\right)$$

$$tf(t) = L^{-1}\left[-\frac{d}{ds}\log\left(1-\frac{a^2}{s^2}\right)\right] = -L^{-1}\left[\frac{s^2}{s^2-a^2}.(-a^2)\left(-\frac{2}{s^3}\right)\right]$$

$$= -L^{-1}\left[\frac{2a^2}{s(s^2-a^2)}\right] = -2aL^{-1}\left[\frac{a}{s(s^2-a^2)}\right].$$

Now, since $L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$.

$$\therefore L^{-1}\left(\frac{a}{s(s^2-a^2)}\right) = \int_o^t \sinh at.dt = \int_0^t \frac{e^{at}-e^{-at}}{2}dt = \frac{1}{2}\left|\frac{e^{at}}{a}+\frac{e^{-at}}{a}\right|_0^t = \frac{1}{2a}\left|e^{at}+e^{-at}-2\right|$$

$$\therefore tf(t) = 2-e^{at}-e^{-at}$$

$$f(t) = \frac{2-e^{at}-e^{-at}}{t}$$

Thus $L^{-1}\log\left(1-\frac{a^2}{s^2}\right)=\frac{2-e^{at}-e^{-at}}{t}=\frac{2}{t}(1-\cosh at)$. Ans**.**

Q.No.14.: Find the Inverse Laplace's transform of $\log\left(\frac{s(s+1)}{s^2+4}\right)$.

Sol.: $L^{-1}\log\left(\frac{s(s+1)}{s^2+4}\right)$

If $f(t)=L^{-1}\left[\log\left(\frac{s(s+1)}{s^2+4}\right)\right]$

$$\therefore tf(t)=L^{-1}\left[-\frac{d}{ds}\left(\log\frac{s(s+1)}{s^2+4}\right)\right]=-L^{-1}\left[\frac{s^2+4}{s(s+1)}\cdot\frac{(s^2+4)(2s+1)-s(s+1)(2s)}{(s^2+4)^2}\right]$$

$$=-L^{-1}\left[\frac{-s^2+8s+4}{s(s+1)(s^2+4)}\right]=L^{-1}\left[\frac{s^2-8s-4}{s(s+1)(s^2+4)}\right]$$

$$\left[\frac{s^2-8s-4}{s(s+1)(s^2+4)}\right]=\frac{-4}{s(4)}-\frac{5}{5(s+1)}+\frac{As+B}{s^2+4}$$

$$s^2-8s-4=-(s+1)(s^2+4)-s(s^2+4)+(As+B)s(s+1)$$

$$s^2-8s-4=(-1-1+A)s^3+(-1+B+A)s^2+(-4-4+B)s-4$$

Equating the coefficient of s^3 on both sides, we get

$-1-1+A=0\Rightarrow A=2$

Equating the coefficient of s^2 on both sides, we get

$-1+B+A=1\Rightarrow B=2-A\Rightarrow B=0$

$$\Rightarrow tf(t)=L^{-1}\left[-\frac{1}{s}-\frac{1}{s+1}+\frac{2s}{s^2+4}\right]=L^{-1}\left(-\frac{1}{s}\right)+L^{-1}\left(-\frac{1}{s+1}\right)+2L^{-1}\left(\frac{s}{s^2+2^2}\right)$$

$$=-1-e^{-t}+2\cos 2t$$

Thus $f(t)=\frac{2\cos 2t-e^{-t}-1}{t}$

$$L^{-1}\log\left(\frac{s(s+1)}{s^2+4}\right)=\frac{2\cos 2t-e^{-t}-1}{t}$$

Q.No.15.: Find the Inverse Laplace's transform of $\cot^{-1}(s+1)$.

Sol.: $L^{-1}\left[\cot^{-1}(s+1)\right]$

If $f(t)=L^{-1}\left[\cot^{-1}(s+1)\right]$

$$t.f(t)=L^{-1}\left[-\frac{d}{ds}\cot^{-1}(s+1)\right]=L^{-1}\left[-\frac{(-1)}{(s+1)^2+1}\right]=L^{-1}\left[\frac{1}{(s+1)^2+1}\right]=e^{-t}\sin t$$

$$f(t)=\frac{e^{-t}\sin t}{t}$$

Thus $L^{-1}\left[\cot^{-1}(s+1)\right]=\frac{e^{-t}\sin t}{t}$. Ans.

Always Remember: Inverse Laplace transform of integrals:

$$\text{If } L^{-1}\{\bar{f}(s)\}=f(t)\text{, then } L^{-1}\left\{\int_s^{\infty}\bar{f}(s)ds\right\}=\frac{f(t)}{t}.$$

Q.No.1.: Evaluate the following $L^{-1}\left\{\frac{1}{2}\log\left(\frac{s+1}{s-1}\right)\right\}$.

Sol.: Given $\bar{f}(s)=\frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$.

Since $\frac{1}{2}\log\left(\frac{s+1}{s-1}\right)=\frac{1}{2}\left[\log(s+1)-\log(s-1)\right]=\frac{1}{2}\int_s^{\infty}\left(\frac{-du}{u+1}+\frac{du}{u-1}\right)$.

To find Inverse Laplace Transform:

Taking Inverse Laplace's transform on both sides, we get

$$L^{-1}\left\{\frac{1}{2}\log\left(\frac{s+1}{s-1}\right)\right\}=L^{-1}\left\{\int_s^{\infty}\frac{1}{2}\left(\frac{du}{u-1}-\frac{du}{u+1}\right)\right\}$$

(by applying inverse linear transform of integrals)

$$=\frac{f(t)}{t}=\frac{\sinh t}{t}. \qquad \left[\because \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}-\frac{1}{s+1}\right\}=\frac{e^{t}-e^{-t}}{2}=\sinh t\right]$$

Q.No.2.: Evaluate the following $L^{-1}\left\{\int_s^{\infty}\left(\frac{u}{u^2+a^2}-\frac{u}{u^2+b^2}\right)du\right\}$.

Sol.: Consider $\bar{f}(s)=\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}$.

To find Inverse Laplace Transform:

Taking Inverse Laplace's transform on both sides, we get

$$f(t) = L^{-1}\left\{\bar{f}(s)\right\} = L^{-1}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) = \cos at - \cos bt\,.$$

Thus applying inverse linear transform of integrals, we get

$$L^{-t}\left\{\int_s^\infty \bar{f}(u)du\right\} = L^{-1}\left\{\int_s^\infty \left(\frac{u}{u^2+a^2} - \frac{u}{u^2+b^2}\right)du\right\} = \frac{f(t)}{t} = \frac{\cos at - \cos bt}{t}.$$

Always Remember: CONVOLUTION THEOREM:

Statement: If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $L^{-1}\{\bar{g}(s)\} = g(t)$,

then $L^{-1}\{\bar{f}(s)\bar{g}(s)\} = \int_0^t f(u)g(t-u)du = F * G$

[F*G is called the convolution or falting of F and G.]

Q.No.1.: Apply Convolution theorem to evaluate

$$\textbf{(i)}\ L^{-1}\frac{s}{\left(s^2+a^2\right)^2},\ \textbf{(ii)}\ L^{-1}\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}.$$

Sol.:(i) Since $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$ and $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$.

Thus $L^{-1}\{\bar{f}(s)\} = f(t)$ and $L^{-1}\{\bar{g}(s)\} = g(t)$.

Then $L^{-1}\{\bar{f}(s)\bar{g}(s)\} = \int_0^t f(u)g(t-u)du = F * G$, (by using Convolution theorem)

we obtain

$$L^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right) = \int_0^t \cos au \frac{\sin a(t-u)}{a}du \left[\because f(u) = \cos au,\ g(t-u) = \frac{1}{a}\sin a(t-u)\right]$$

$$= \frac{1}{2a}\int_0^t \left[\sin at + \sin\left(2au - at\right)\right]du = \frac{1}{2a}\left|u\sin at + \frac{1}{2a}\cos\left(2au - at\right)\right|_0^t$$

Hence, $L^{-1}\left(\frac{s}{\left(s^2+a^2\right)^2}\right) = \frac{1}{2a}t\sin at$. Ans.

(ii) Since $L^{-1}\left(\frac{s}{s^2+a^2}\right)=\cos at$ and $L^{-1}\left(\frac{s}{s^2+b^2}\right)=\cos bt$.

Thus $L^{-1}\{\bar{f}(s)\}=f(t)$ and $L^{-1}\{\bar{g}(s)\}=g(t)$.

Then $L^{-1}\{\bar{f}(s)\bar{g}(s)\}=\int_0^t f(u)g(t-u)du = F*G$, (by using Convolution theorem)

we obtain

$$L^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right)=\int_0^t \cos au.\cos b(t-u)\,du \quad [\because f(u)=\cos au,\ g(t-u)=\cos b(t-u)]$$

$$=\frac{1}{2}\int_0^t \{\cos[(a-b)u+bt]+\cos[(a+b)u-bt]\}du$$

$$[\because 2\cos A\cos B=\cos(A+B)+\cos(A-B)]$$

$$=\frac{1}{2}\left\{\frac{\sin[(a-b)u+bt]}{a-b}+\frac{\sin[(a+b)u-bt]}{a+b}\right\}_0^t$$

$$=\frac{1}{2}\left\{\frac{\sin at-\sin bt}{a-b}+\frac{\sin at+\sin bt}{a+b}\right\}$$

$$=\frac{a\sin at-b\sin bt}{a^2-b^2}\text{. Ans.}$$

Q.No.2.: Apply Convolution theorem to evaluate inverse Laplace transform of

$$\frac{2as}{\left(s^2+a^2\right)^2}.$$

Sol.: Here $L^{-1}\left[\frac{2as}{\left(s^2+a^2\right)^2}\right]=2aL^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]$.

Also $L^{-1}\left[\frac{s}{\left(s^2+a^2\right)}\right]=\cos at$, $L^{-1}\left[\frac{1}{\left(s^2+a^2\right)}\right]=\frac{1}{a}\sin at$.

Using Convolution theorem, we obtain

$$2aL^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right)=2a\int_0^t \cos a(t-u).\frac{\sin au}{a}du$$

$$=\int_0^t 2\cos a(t-u).\sin au.du=\int_0^t [\sin at+\sin(2au-at)]du$$

$$= \left| u \sin at - \frac{1}{2a} \cos(2au - at) \right|_0^t = \left| t \sin at - \frac{1}{2a} \cos at + \frac{1}{2a} \cos at \right|$$

$$L^{-1}\left[\frac{2as}{\left(s^2 + a^2\right)^2} \right] = t \sin at \text{. Ans.}$$

Q.No.3.: Using Convolution theorem to evaluate $L^{-1}\left\{ \frac{1}{(s+a)(s+b)} \right\}$.

Sol.: Since $L^{-1}\left(\frac{1}{s+a} \right) = e^{-at}$ and $L^{-1}\left(\frac{1}{s+b} \right) = e^{-bt}$.

Using Convolution theorem, we obtain

$$L^{-1}\left(\frac{1}{(s+a)(s+b)} \right) = \int_0^t e^{-a(t-u)} e^{-bu} du = \int_0^t e^{-at+(a-b)u} du = \frac{1}{a-b}\left[e^{-at+(a-b)u} \right]_0^t$$

$$= \frac{1}{a-b}\left[e^{-at+(a-b)t} - e^{-at} \right] = \frac{1}{a-b}\left[e^{-bt} - e^{-at} \right]$$

Thus $L^{-1}\left(\frac{1}{(s+a)(s+b)} \right) = \frac{1}{a-b}\left[e^{-bt} - e^{-at} \right]$. Ans.

Q.No.4.: Using Convolution theorem to evaluate $L^{-1}\left\{ \frac{1}{s\left(s^2+4\right)} \right\}$.

Sol.: Since $L^{-1}\left(\frac{1}{s} \right) = 1$ and $L^{-1}\left(\frac{1}{s^2+4} \right) = \frac{1}{2} \sin 2t$.

Using Convolution theorem, we obtain

$$L^{-1}\left\{ \frac{1}{s\left(s^2+4\right)} \right\} = \int_0^t 1.\frac{1}{2} \sin 2u du = \frac{1}{2}\left| \frac{-\cos 2u}{2} \right|_0^t = -\frac{1}{4}\left| \cos 2t - 1 \right|$$

$$L^{-1}\left\{ \frac{1}{s\left(s^2+4\right)} \right\} = \frac{1}{4}\left[1 - \cos 2t \right]. \text{ Ans.}$$

Q.No.5.: Using Convolution theorem to evaluate $L^{-1}\left\{ \frac{1}{\left(s^2+a^2\right)^2} \right\}$.

Sol.: $L^{-1}\left\{ \frac{1}{\left(s^2+a^2\right)^2} \right\}$

Since $L^{-1}\left\{\frac{1}{\left(s^2+a^2\right)}\right\}=\frac{1}{a}\sin at$.

Using Convolution theorem, we obtain

$$L^{-1}\left\{\frac{1}{\left(s^2+a^2\right)}\cdot\frac{1}{s^2+a^2}\right\}=\int_0^t \frac{1}{a}\sin a(t-u).\frac{1}{a}\sin au.du=\frac{1}{a^2}\int_0^t \sin a(t-u)\sin au.du$$

$$=\frac{1}{2a^2}\int_0^t 2\sin a(t-u)\sin au.du$$

$$=-\frac{1}{2a^2}\int_0^t -2\sin a(t-u)\sin au.du$$

$$=-\frac{1}{2a^2}\int_0^t [\cos at-\cos a(t-2u)]du$$

$$=-\frac{1}{2a^2}\left[u\cos at-\frac{\sin a(t-2u)}{-2a}\right]_0^t$$

$$=-\frac{1}{2a^2}\left[t\cos at-\frac{1}{2a}\sin at-\frac{1}{2a}\sin at\right]$$

$$=\frac{1}{2a^3}[\sin at-at\cos at].$$

Thus $L^{-1}\left\{\frac{1}{\left(s^2+a^2\right)^2}\right\}=\frac{1}{2a^3}[\sin at-at\cos at]$. Ans.

Q.No.6.: Using Convolution theorem evaluate $L^{-1}\left\{\frac{1}{s^2\left(s^2+a^2\right)}\right\}$.

Sol.: $L^{-1}\left\{\frac{1}{s^2\left(s^2+a^2\right)}\right\}$

Since $L^{-1}\left(\frac{1}{s^2}\right)=t$ and $L^{-1}\left\{\frac{1}{\left(s^2+a^2\right)}\right\}=\frac{1}{a}\sin at$

Using Convolution theorem, we obtain

$$L^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{\left(s^2+a^2\right)}\right\}=\int_0^t (t-u)\frac{1}{a}\sin au.du=\frac{1}{a}\int_0^t t\sin au.du-\frac{1}{a}\int_0^t u\sin au.du$$

$$=\frac{t}{a}\left|-\frac{\cos au}{a}\right|_0^t-\frac{1}{a}\left[u\frac{\cos au}{-a}-\left(-\frac{1}{a}\right)\int_0^t \cos au.du\right]_0^t$$

$$=-\frac{t}{a^2}\left|\cos at-1\right|+\frac{1}{a^2}\left|u\cos au-\int_0^t \cos au.du\right|_0^t$$

$$=-\frac{t}{a^2}(\cos at-1)+\frac{1}{a^2}\left|u\cos au-\frac{\sin au}{a}\right|_0^t$$

$$=\frac{t}{a^2}(1-\cos at)+\frac{1}{a^2}\left|t\cos at-\frac{\sin at}{a}\right|$$

$$=\frac{t}{a^2}-\frac{t}{a^2}\cos at+\frac{t}{a^2}\cos at-\frac{\sin at}{a^3}=\frac{1}{a^3}(at-\sin at)$$

Thus $L^{-1}\left\{\frac{1}{s^2\left(s^2+a^2\right)}\right\}=\frac{1}{a^3}(at-\sin at)$. **Ans.**

Q.No.7.: Using Convolution theorem evaluate $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$.

Sol.: $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$

Since $L^{-1}\left(\frac{1}{s^2}\right)=t$ and $L^{-1}\left(\frac{1}{(s+1)^2}\right)=e^{-t}t$

Using Convolution theorem, we obtain

$$L^{-1}\left(\frac{1}{s^2}\cdot\frac{1}{(s+1)^2}\right)=\int_0^t (t-u).e^{-u}u.du=\int_0^t tue^{-u}du-\int_0^t u^2e^{-u}du$$

$$=t\left[\frac{ue^{-u}}{-1}-\int\frac{e^{-u}}{-1}du\right]_0^t-\left[\frac{u^2e^{-u}}{-1}-\int\frac{2ue^{-u}}{-1}du\right]_0^t$$

$$=t\left[-ue^{-u}+\frac{e^{-u}}{-1}\right]_0^t-\left[-u^2e^{-u}+2\left\{\frac{ue^{-u}}{-1}+\int e^{-u}du\right\}\right]_0^t$$

$$=t\left[-ue^{-u}-e^{-u}\right]_0^t-\left[-u^2e^{-u}+2\left\{-ue^{-u}-e^{-u}\right\}\right]_0^t$$

$$=t\left[-te^{-t}-e^{-t}+1\right]-\left[-u^2e^{-u}-2ue^{-u}-2e^{-u}\right]_0^t$$

$$= t\left[-te^{-t} - e^{-t} + 1\right] + \left[t^2e^{-t} + 2te^{-t} + 2e^{-t} - 2\right]$$

$$= -t^2e^{-t} - te^{-t} + t + t^2e^{-t} + 2te^{-t} + 2e^{-t} - 2$$

$$= te^{-t} + 2e^{-t} + t - 2$$

Thus $L^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\} = te^{-1} + 2e^{-t} + t - 2$. Ans.

Q.No.8.: Using Convolution theorem to evaluate $L^{-1}\left\{\dfrac{1}{(s-2)(s+2)^2}\right\}$.

Sol.: $L^{-1}\left\{\dfrac{1}{(s-2)(s+2)^2}\right\}$

Since $L^{-1}\left\{\dfrac{1}{(s-2)}\right\} = e^{2t}$ and $L^{-1}\left\{\dfrac{1}{(s+2)^2}\right\} = e^{-2t}t$.

Using Convolution theorem, we obtain

$$L^{-1}\left\{\frac{1}{(s-2)}\cdot\frac{1}{(s+2)^2}\right\} = \int_0^t e^{2(t-u)}.e^{-2u}u.du = \int_0^t ue^{2t-4u}du.$$

Integrating by parts, we get

$$= \left[u\frac{e^{2t-4u}}{-4} - \int \frac{e^{2t-4u}}{-4}du\right]_0^t = \left[-\frac{u}{4}e^{2t-4u} + \frac{1}{4}\left(\frac{e^{2t-4u}}{-4}\right)\right]_0^t$$

$$= -\left[\frac{u}{4}e^{2t-4u} + \frac{1}{16}e^{2t-4u}\right]_0^t = -\left[\frac{t}{4}e^{-2t} + \frac{1}{16}e^{-2t} - \frac{1}{16}e^{2t}\right]$$

Thus $L^{-1}\left\{\dfrac{1}{(s-2)(s+2)^2}\right\} = \dfrac{1}{16}\left[e^{2t} - e^{-2t} - 4te^{-2t}\right]$. Ans.

Q.No.9.: Using Convolution theorem to evaluate $L^{-1}\left\{\dfrac{1}{(s+1)(s+9)^2}\right\}$.

Sol.: $L^{-1}\left\{\dfrac{1}{(s+1)(s+9)^2}\right\}$

Since $L^{-1}\dfrac{1}{(s+1)} = e^{-t}$ and $L^{-1}\dfrac{1}{(s+9)} = e^{-9t}.t$

Using Convolution theorem, we obtain

$$L^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{(s+9)^2}\right\}=\int_0^t e^{-(t-u)}.ue^{-9u}du=\int_0^t e^{-8u-t}udu$$

Integrating by parts, we get

$$=\left[u\frac{e^{-8u-t}}{-8}+\frac{1}{8}\int e^{-8u-t}dt\right]_0^t=\left[-\frac{u}{8}e^{-8u-t}-\frac{1}{64}e^{-8u-t}\right]_0^t$$

$$=\left[-\frac{t}{8}e^{-9t}-\frac{1}{64}e^{-9t}+\frac{1}{64}e^{-t}\right]$$

Thus $L^{-1}\left\{\frac{1}{(s+1)(s+9)^2}\right\}=\left[-\frac{t}{8}e^{-9t}-\frac{1}{64}e^{-9t}+\frac{1}{64}e^{-t}\right]=\frac{e^{-t}}{64}\left(1-e^{-8t}(1+8t)\right)$. Ans.

Q.No.10.: Using Convolution theorem to evaluate $L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}$.

Sol.: $L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}$

Since $L^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t$ and $L^{-1}\left\{\frac{s}{s^2+4}\right\}=\cos 2t$.

Using Convolution theorem, we obtain

$$L^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{s}{s^2+4}\right\}=\int_0^t \sin(t-u)\cos 2u.du=\frac{1}{2}\int_0^t [\sin(t+u)+\sin(t-3u)]du$$

$$=\frac{1}{2}\left|-\cos(t+u)\right|_0^t+\frac{1}{2}\left|\frac{-\cos(t-3u)}{-3}\right|_0^t$$

$$=-\frac{1}{2}|\cos 2t-\cos t|+\frac{1}{6}|\cos 2t-\cos t|$$

$$=\cos 2t\left(\frac{1}{6}-\frac{1}{2}\right)+\cos t\left(\frac{1}{2}-\frac{1}{6}\right)=-\frac{1}{3}\cos 2t+\frac{1}{3}\cos t$$

$$L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}=\frac{1}{3}(\cos t-\cos 2t).\ \text{Ans.}$$

Some more problems

Q.No.1.: Find the inverse Laplace transforms of

$$\textbf{(i)}\ \frac{s}{\left(s^2+a^2\right)^2},\quad \textbf{(ii)}\ \frac{s^2}{\left(s^2+a^2\right)^2},\quad \textbf{(iii)}\ \frac{1}{\left(s^2+a^2\right)^2}.$$

Sol.: If $f(t)=L^{-1}\dfrac{s}{\left(s^2+a^2\right)^2}$, then by formula $L\left\{\dfrac{f(t)}{t}\right\}=\displaystyle\int_s^{\infty}\bar{f}(s)ds$

$$L\left\{\frac{f(t)}{t}\right\}=\int_s^{\infty}\frac{s}{\left(s^2+a^2\right)^2}ds=\frac{1}{2}\int_s^{\infty}\frac{2s}{\left(s^2+a^2\right)^2}ds=-\frac{1}{2}\left(\frac{1}{s^2+a^2}\right)_s^{\infty}=\frac{1}{2}\cdot\frac{1}{s^2+a^2}$$

$$\therefore \frac{f(t)}{t}=\frac{1}{2}L^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{\sin at}{2a}.$$

Hence, $f(t)=\dfrac{1}{2a}t\sin at$. Ans.

(ii) In (i), we have proved that

$$L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}=\frac{1}{2a}t\sin at=f(t),\ \text{(say)}$$

Since $f(0)=0$, we get from formula $L^{-1}\{s\bar{f}(s)\}=\dfrac{d}{dt}\{f(t)\}$, that

$$L^{-1}\left\{\frac{s^2}{\left(s^2+a^2\right)^2}\right\}=L^{-1}\left\{s.\frac{s}{\left(s^2+a^2\right)^2}\right\}=\frac{d}{dt}\{f(t)\}$$

$$=\frac{d}{dt}\left(\frac{1}{2a}t\sin at\right)=\frac{1}{2a}(\sin at+at\cos at).\ \text{Ans.}$$

(iii) In (i), we have shown that

$$L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}=\frac{1}{2a}(t\sin at)=f(t),\ \text{say}$$

Then by formula $L^{-1}\left\{\dfrac{\bar{f}(s)}{s}\right\}=\displaystyle\int_0^t f(t)dt$, we get

$$L^{-1}\left\{\frac{1}{\left(s^2+a^2\right)^2}\right\}=L^{-1}\left\{\frac{1}{s}\cdot\frac{s}{\left(s^2+a^2\right)^2}\right\}=\int_0^t f(t)dt=\int_0^t\frac{t\sin at}{2a}dt$$

$$= \frac{1}{2a}\left\{\left|t.\frac{-\cos at}{a}\right|_0^t - \int_0^t 1.\left(\frac{-\cos at}{a}\right)dt\right\}$$

$$= \frac{1}{2a}\left\{\frac{-t\cos at}{a} + \frac{\sin at}{a^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)\text{. Ans.}$$

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Home Assignments

Linearity property:

Q.No.1.: Find the inverse Laplace transform of $\frac{3}{s+4}$.

Ans.: $3e^{-4t}$.

Q.No.2.: Find the inverse Laplace transform of $\frac{8s}{s^2+16}$.

Ans.: $8.\cos 4t$.

Q.No.3.: Find the inverse Laplace transform of $\frac{1}{2s-5}$.

Ans.: $\frac{1}{2}e^{\frac{5t}{2}}$.

Q.No.4.: Find the inverse Laplace transform of $\frac{6}{s^2+4}$.

Ans.: $3\sin 2t$.

Q.No.5.: Find the inverse Laplace transform of $\frac{3s-12}{s^2+8}$.

Ans.: $3\cos 2\sqrt{2}\,t - 3\sqrt{2}\sin 2\sqrt{2}\,t$.

Q.No.6.: Find the inverse Laplace transform of $\frac{(2s-5)}{(s^2-9)}$.

Ans.: $2\cosh 3t - \frac{5}{3}\sinh 3t$.

Q.No.7.: Find the inverse Laplace transform of $s^{-\frac{7}{2}}$.

Ans.: $\frac{8t^{\frac{5}{2}}}{15\sqrt{\pi}}$.

Q.No.8.: Find the inverse Laplace transform of $\frac{s+1}{s^{\frac{4}{3}}}$.

Ans.: $\frac{\left(t^{-\frac{2}{3}} + 3t^{\frac{1}{3}}\right)}{\Gamma\left(\frac{1}{3}\right)}$.

Q.No.9.: Find the inverse Laplace transform of $\left(\frac{\sqrt{s}-1}{s}\right)^2$.

Ans.: $1 + t - \frac{4t^{\frac{1}{2}}}{\sqrt{\pi}}$.

Q.No.10.: Find the inverse Laplace transform of $\frac{3s-8}{4s^2+25}$.

Ans.: $\frac{3}{4}\cos\frac{5t}{2} - \frac{4}{5}\sin\frac{5t}{2}$.

Q.No.11.: Find the inverse Laplace transform of $\frac{5s+10}{4s^2+25}$.

Ans.: $\frac{5}{9}\cosh\frac{4t}{3} + \frac{5}{6}\sinh\frac{4t}{3}$.

Q.No.12.: Find the inverse Laplace transform of $\frac{3\left(s^2-1\right)^2}{2s^5} + \frac{4s-18}{9-s^2} + \frac{(s+1)\left(2-s^{\frac{1}{2}}\right)}{s^{\frac{5}{2}}}$.

Ans.: $\frac{1}{2}-t-\frac{3}{2}t^2+\frac{1}{16}t^4+4\frac{t^{\frac{1}{2}}}{\sqrt{\pi}}+\frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}}-4\cosh 3t+6\sinh 3t$.

Q.No.13.: Find the inverse Laplace transform of $\frac{1}{s}\sin\left(\frac{1}{s}\right)$.

Ans.: $\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{[(2n-1)!]^2}t^{2n-1}$.

Q.No.14.: Find the inverse Laplace transform of $\frac{1}{s}e^{-\frac{1}{s}}$.

Ans.: $\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{[(n!)]^2}t^{2n-1}$.

First shift theorem:

Q.No.1.: Find the inverse Laplace transform of $\frac{5}{(s+2)^5}$.

Ans.: $\frac{5}{24}t^4e^{2t}$.

Q.No.2.: Find the inverse Laplace transform of $\frac{4s+12}{s^2+8s+16}$.

Ans.: $4e^{-4t}(1-t)$.

Q.No.3.: Find the inverse Laplace transform of $\frac{s}{(s+1)^5}$.

Ans.: $\frac{e^{-t}}{24}\left(4t^3-t^4\right)$.

Q.No.4.: Find the inverse Laplace transform of $\frac{s}{(s+1)^{\frac{5}{2}}}$.

Ans.: $\frac{2t^{\frac{1}{2}}(3-2t)}{3\sqrt{\pi}}$.

Q.No.5.: Find the inverse Laplace transform of $\frac{1}{\sqrt[3]{8s-27}}$.

Ans.: $\dfrac{t^{\frac{-2}{3}}e^{\frac{27t}{8}}}{2\Gamma\left(\frac{1}{3}\right)}$.

Q.No.6.: Find the inverse Laplace transform of $\dfrac{3s-14}{s^2-4s+8}$.

Ans.: $e^{2t}(3\cos 2t-4\sin 2t)$.

Q.No.7.: Find the inverse Laplace transform of $\dfrac{5s-2}{s^2+4s+8}$.

Ans.: $\dfrac{e^{\frac{-2t}{3}}}{15}\left\{25\cos\dfrac{2\sqrt{5}\,t}{3}-24\sqrt{5}\sin\dfrac{2\sqrt{5}\,t}{3}\right\}$

Q.No.8.: Find the inverse Laplace transform of $\dfrac{3s+2}{4s^2+12s+9}$.

Ans.: $\dfrac{3}{4}e^{\frac{-3t}{2}}-\dfrac{5}{8}te^{\frac{-3t}{2}}$.

Q.No.9.: Find the inverse Laplace transform of $\dfrac{8s+20}{s^2-12s+32}$.

Ans.: $2e^{6t}(4\cosh 2t+17\sinh 2t)$.

Q.No.10.: Find the inverse Laplace transform of $\dfrac{1}{\left(s^2+2s+5\right)^2}$.

Ans.: $\dfrac{e^{-t}}{16}(\sin 2t-2t\cos 2t)$.

Change of scale property:

Q.No.1.: If $L^{-1}\left\{\dfrac{e^{-\frac{1}{s}}}{\sqrt{s}}\right\}=\dfrac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$, show that $L^{-1}\left\{\dfrac{e^{-\frac{a}{s}}}{\sqrt{s}}\right\}=\dfrac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$ when $a>0$.

Q.No.2.: If $L^{-1}\{F(s)\}=f(t)$, prove that $L^{-1}\{F(as+b)\}=\dfrac{1}{a}e^{-\left(\frac{b}{a}\right)t}f\left(\dfrac{t}{a}\right)$, where $a>0$.

Q.No.3.: If $L^{-1}\left\{\dfrac{s}{\left(s^2+1\right)^2}\right\}=\dfrac{t\sin t}{2}$, show that $L^{-1}\left\{\dfrac{8s}{\left(4s^2+1\right)^2}\right\}=\dfrac{t}{2}\sin\dfrac{t}{2}$.

Q.No.4.: If $L^{-1}\left\{\dfrac{s^2-1}{\left(s^2+1\right)^2}\right\}=t\cos t$, show that $L^{-1}\left\{\dfrac{8s^2-1}{\left(9s^2+1\right)^2}\right\}=\dfrac{t}{9}\sin\dfrac{t}{3}$.

Multiplication by s:

Q.No.1.: Evaluate the inverse Laplace transform of $\dfrac{s^2}{s^2+1}$.

Ans.: $\cos t$.

Q.No.2.: Evaluate the inverse Laplace transform of $\dfrac{s}{\left(s^2+1\right)^2}$.

Ans.: $\dfrac{1}{2}t\sin t$.

Division by powers of s:

Q.No.1.: Evaluate the inverse Laplace transform of $\dfrac{1}{s^3\left(s^2+1\right)}$.

Ans.: $\dfrac{t^2}{2}+\cos t-1$.

Q.No.2.: Evaluate the inverse Laplace transform of $\dfrac{s}{(s-2)^5(s+1)}$.

Ans.: $e^{2t}\left(\dfrac{t^4}{36}+\dfrac{t^3}{54}-\dfrac{t^2}{54}+\dfrac{t}{81}-\dfrac{1}{243}\right)+\dfrac{e^t}{243}$.

Q.No.3.: Evaluate the inverse Laplace transform of $\dfrac{s^2}{\left(s^2-4s+5\right)^2}$.

Ans.: $te^{2t}(\cos t-\sin t)$.

Q.No.4.: Evaluate the inverse Laplace transform of $\dfrac{1}{s}\left(\dfrac{s-a}{s+a}\right)$.

Ans.: $2e^{-at}-1$.

Q.No.5.: Evaluate the inverse Laplace transform of $\frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right)$.

Ans.: $1+t-\cos t-\sin t$.

Q.No.6.: Evaluate the inverse Laplace transform of $\frac{1}{s^4-2s^3}$.

Ans.: $\frac{\left(e^{26}-1-2t-2t^2\right)}{8}$.

Q.No.7.: Evaluate the inverse Laplace transform of $\frac{1}{s\left(s^2+a^2\right)}$.

Ans.: $\frac{(1-\cos at)}{a^2}$.

Q.No.8.: Evaluate the inverse Laplace transform of $\frac{1}{s(s+a)^3}$.

Ans.: $-\frac{t^2e^{-at}}{2a}-\frac{te^{-at}}{a^2}-\frac{1}{a^3}\left(e^{-at}-1\right)$.

Q.No.9.: Evaluate the inverse Laplace transform of $\frac{1}{s^2\left(s^2+a^2\right)}$.

Ans.: $\frac{1}{a^2}\left(t-\frac{\sin at}{a}\right)$.

Inverse Laplace transform of derivatives:

Q.No.1.: Find the inverse Laplace transform of $\frac{1}{(s-a)^n}$, n = 1, 2, 3,..

Ans.: $\frac{t^{n-1}.e^{at}}{(n+)!}$.

Q.No.2.: Find the inverse Laplace transform of $\frac{s}{\left(s^2+a^2\right)^2}$.

Ans.: $\frac{t\sin at}{2a}$.

Q.No.3.: Find the inverse Laplace transform of $\frac{1}{s^2+4s+5}$.

Ans.: $\dfrac{te^{-2t}\sin t}{2}$.

Q.No.4.: Find the inverse Laplace transform of $\log\left(\dfrac{s+a}{s+b}\right)$.

Ans.: $\dfrac{e^{-at}-e^{-bt}}{-t}$.

Q.No.5.: Find the inverse Laplace transform of $\log\dfrac{s(s+1)}{\left(s^2+4\right)}$.

Ans.: $\dfrac{2\cos 2t - e^{-t} - 1}{t}$.

Q.No.6.: Find the inverse Laplace transform of $\log\dfrac{s+1}{s-1}$.

Ans.: $\dfrac{2\sinh t}{t}$.

Q.No.7.: Find the inverse Laplace transform of $\cot^{-1}\left(\dfrac{s+3}{2}\right)$.

Ans.: $\dfrac{2}{t}e^{-3t}\sin 2t$.

Inverse Laplace transform of integrals:

Q.No.1.: Evaluate the following $L^{-1}\left\{\int_s^\infty \left(\dfrac{1}{u}-\dfrac{1}{u+1}\right)dx\right\}$.

Ans.: $\dfrac{1-e^{-t}}{t}$.

Q.No.2.: Evaluate the following $L^{-1}\left\{\int_s^\infty \operatorname{In}\left(\dfrac{u+2}{u+1}\right)du\right\}$

Ans.: $\dfrac{e^{-t}-e^{-2t}}{t^2}$.

Q.No.3.: Evaluate the following $L^{-1}\left\{\int_s^\infty \tan^{-1}\left(\dfrac{2}{u^2}\right)du\right\}$

Ans.: $\dfrac{2\sin t \sinh t}{t^3}$.

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6^{th} Topic

Laplace Transforms

Applications to Differential Equations

(Solution of Differential Equation by Laplace Transforms)

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Applications to Differential Equations:

The Laplace transform can be used to solve ordinary as well as partial differential equations. We shall apply this method to solve ordinary linear differential equations with constant co-efficients. The advantage of this method is that it yields particular solution directly without the necessity of first finding the general solution and then evaluating the arbitrary constants.

This method is, in general, shorter than our earlier methods.

Working procedure:

1. Take the Laplace transforms of both sides of the given differential equation, using initial conditions.
   This gives an algebraic equation.
2. Solve the algebraic equation to get $\bar{y}$ in terms of s.
3. Take inverse Laplace transform of both sides.
   This gives y as a function of t which is the desired solution satisfying the given conditions.

Always Remember:

If $f'(t)$ be continuous and $L\{f(t)\} = \bar{f}(s)$, then $L\{f'(t)\} = s\bar{f}(s) - f(0)$, provided

$\lim\limits_{t\to\infty}\left[e^{-st}f(t)\right] = 0$.

The above result can be generalized.

If $f'(t)$ and its first $(n-1)$ derivatives be continuous, then

$L\{f^n(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots\ldots\ldots\ldots - f^{n-1}(0)$.

Now let us solve the differential equations by the method of transforms

Q.No.1.: Solve by the method of transforms, the equation

$$y''' + 2y'' - y' - 2y = 0 \text{ given } y(0) = y'(0) = 0 \text{ and } y''(0) = 6.$$

Sol.: Given: Differential equation: $y''' + 2y'' - y' - 2y = 0$.

Initial Conditions: $y(0) = y'(0) = 0$ and $y''(0) = 6$.

Step No. 01: Find the Laplace Transform of the given equation:

Taking the Laplace transform of both sides, we get

$$\left[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)\right] + 2\left[s^2\bar{y} - sy(0) - y'(0)\right] - \left[s\bar{y} - y(0)\right] - 2\bar{y} = 0.$$

Step No. 02: Find the expression for $\bar{y} = \bar{f}(s)$ using initial conditions:

Using the initial conditions, above equation reduces to

$$\left(s^3 + 2s^2 - s - 2\right)\bar{y} = 6$$

$$\therefore \bar{y} = \frac{6}{(s-1)(s+1)(s+2)} = \frac{6}{(s-1)6} + \frac{6}{(-2)(s+1)} + \frac{6}{3(s+2)}.$$

$$= \frac{1}{(s-1)} - \frac{3}{(s+1)} + \frac{2}{(s+2)}$$

Step No. 03: Find the Inverse Laplace Transform for evaluating y:

Taking the Inverse Laplace transform of both sides, we get

$$y = L^{-1}\left(\frac{1}{s-1}\right) - 3L^{-1}\left(\frac{1}{s+2}\right) + 2L^{-1}\left(\frac{1}{s+2}\right)$$

$\Rightarrow y = e^{t} - 3e^{-t} + 2e^{-2t}$, $\left[\because L^{-1}\left(\dfrac{1}{s-a}\right) = e^{at}\right]$

which is the required solution.

Q.No.2.: Use transform method to solve

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^{t} \text{ with } x = 2,\ \frac{dx}{dt} = -1 \text{ at } t = 0.$$

Sol.: Given: Differential equation: $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + x = e^{t}$.

Initial Conditions: $x = 2$, $\dfrac{dx}{dt} = -1$ at $t = 0$.

Step No. 01: Find the Laplace Transform of the given equation:

Taking the Laplace transform of both sides, we get

$$\left[s^2\bar{x} - sx(0) - x'(0)\right] - 2[\{s\bar{x} - x(0)\}] + \bar{x} = \frac{1}{s-1}.$$

Step No. 02: Find the expression for $\bar{x} = \bar{f}(s)$ using initial conditions:

Using the initial conditions, above equation reduces to

$$\left(s^2 - 2s + 1\right)\bar{x} = \frac{1}{s-1} + 2s - 5 = \frac{2s^2 - 7s + 6}{s-1}$$

$$\therefore \bar{x} = \frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}.$$

To find the values of A, B and C:

Multiplying both sides by $(s-1)^3$, we get

$$2s^2 - 7s + 6 = A(s-1)^2 + B(s-1) + C.$$

- Equating the coefficient of s^2 from both sides, we get
  $A = 2$.
- Equating the coefficient of s from both sides, we get
  $-2A + B = -7 \Rightarrow B = -3$.
- By Putting s = 1, we get $C = 1$.

$$\therefore \bar{x} = \frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$

Step No. 03: Find the Inverse Laplace Transform for evaluating x:

Taking the Inverse Laplace transform of both sides, we get

$$x = 2L^{-1}\left(\frac{1}{s-1}\right) - 3L^{-1}\left(\frac{1}{(s-1)^2}\right) + L^{-1}\left(\frac{1}{(s-1)^3}\right)$$

$$\Rightarrow x = 2e^t - \frac{3e^t.t}{1!} + \frac{3e^t.t^2}{2!} = 2e^t - 3te^t + \frac{1}{2}t^2e^2, \qquad \left[\because L^{-1}\left[\frac{1}{(s-a)^n}\right] = \frac{e^{at}t^{n-1}}{(n-1)\ !}\right]$$

which is the required solution.

Q.No.3.: Solve $(D^2 + n^2)x = a\sin(nt + \alpha)$, $x = Dx = 0$, at $t = 0$.

Sol.: Given: Differential equation: $(D^2 + n^2)x = a\sin(nt + \alpha)$.

Initial Conditions: $x = Dx = 0$, at $t = 0$.

Step No. 01: Find the Laplace Transform of the given equation:

Taking the Laplace transform of both sides, we get

$$\left[s^2\bar{x} - sx(0) - x'(0)\right] + n^2\bar{x} = aL[\sin nt.\cos\alpha + \cos nt.\sin\alpha]$$

$$= a\cos\alpha.\frac{n}{s^2+n^2} + a\sin\alpha.\frac{s}{s^2+n^2}.$$

$$\left[\because L(\sin at) = \frac{a}{s^2+a^2} \quad \text{and} \quad L(\cos at) = \frac{s}{s^2+a^2}\right]$$

Step No. 02: Find the expression for $\bar{x} = \bar{f}(s)$ using initial conditions:

Using the initial conditions, above equation reduces to

$$(s^2+n^2)\bar{x} = a\cos\alpha.\frac{n}{s^2+n^2} + a\sin\alpha.\frac{s}{s^2+n^2}.$$

$$\therefore \bar{x} = an\cos\alpha.\frac{1}{(s^2+n^2)^2} + a\sin\alpha.\frac{s}{(s^2+n^2)^2}.$$

Step No. 03: Find the Inverse Laplace Transform for evaluating x:

Taking the Inverse Laplace transform of both sides, we get

$$x = an\cos\alpha.\frac{1}{2n^3}(\sin nt - nt\cos nt) + a\sin\alpha.\frac{t}{2n}\sin nt$$

$$\left[\because L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \frac{1}{2a^3}(\sin at - at\cos at) \ \text{ and } L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{1}{2a}t\sin at\right]$$

$$\Rightarrow x = \frac{1}{2n^2}\left[a\cos\alpha.\left(\sin nt - nt\cos nt\right) + nt.a\sin\alpha.\sin nt\right]$$

$$\Rightarrow x = \frac{a}{2n^2}\left[\cos\alpha.\sin nt - nt(\cos\alpha.\cos nt - \sin\alpha.\sin nt)\right]$$

$$\Rightarrow x = \frac{a\{\sin nt\cos\alpha - nt\cot(nt+\alpha)\}}{2n^2},$$

which is the required solution. Ans.

Q.No.4.: Solve $\left(D^3 - 3D^2 + 3D - 1\right)y = t^2e^t$,

given that y (0) =1, $y' = (0) = 0,\ y''(0) = -2$.

Sol.: Taking the Laplace transforms of both sides, we get

$$\left[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)\right] - 3\left[s^2\bar{y} - sy(0) - y'(0)\right] + 3\left[s\bar{y} - y(0)\right] - \bar{y} = \frac{2}{(s-1)^3}.$$

Using the conditions, it reduce to

$$\bar{y} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6} = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}.$$

On inversion, we get, $y = 2L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{(s-1)^2}\right) - L^{-1}\left(\frac{1}{(s-1)^3}\right) + 2L^{-1}\frac{1}{(s-1)^6}$

$\Rightarrow y = e^t\left(1 - t - \frac{1}{2}t^2 + \frac{1}{60}t^5\right)$, which is the required solution. Ans.

Q.No.5.: Solve $\frac{d^2x}{dt^2} + 9x = \cos 2t$, if $x(0) = 1,\ x\left(\frac{\pi}{2}\right) = -1$.

Sol.: Since $x'(0)$ is not given, we assume that $x'(0) = a$.

Taking the Laplace transforms of both sides, we get

$$L(x'') + 9L(x) = L(\cos 2t)\ \text{ i. e. } \left[s^2\bar{x} - sx(0) - sx(0)\right] + 9\bar{x} = \frac{s}{s^2+4}$$

$$\left(s^2+9\right)\bar{x} = s + a + \frac{s}{s^2+4} \Rightarrow \bar{x} = \frac{s+a}{s^2+9} + \frac{s}{\left(s^2+4\right)\left(s^2+9\right)}$$

$$\Rightarrow \bar{x} = \frac{a}{s^2+9} + \frac{1}{5}.\frac{s}{s^2+4} + \frac{4}{5}.\frac{s}{s^2+9}.$$

On inversion, we get $x = \frac{a}{3}\sin 3t + \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t$

When $t = \frac{\pi}{2}$, $\quad -1 = -\frac{a}{3} - \frac{1}{5} \Rightarrow \frac{a}{3} = \frac{4}{5}$. $\quad \left[\because x\left(\frac{\pi}{2}\right) = -1\right]$

Hence the solution is $x = \frac{1}{5}(\cos 2t + 4\sin 3t + 4\cos 3t)$. Ans.

or

Sol.: Given differential equation is $\frac{d^2x}{dt^2} + 9x = \cos 2t$.

Taking Laplace transform of both sides, we get

$$L\left[\frac{d^2x}{dt^2} + 9x\right] = L(\cos 2t) \Rightarrow L\left[\frac{d^2x}{dt^2}\right] + 9L(x) = L(\cos 2t)$$

$$\Rightarrow s^2L(x) - sx(0) - x'(0) + 9L(x) = \frac{s}{s^2+4}$$

$$\Rightarrow (s^2+9)\bar{x} - s - c = \frac{s}{s^2+4}, \qquad \bar{x} = L(x) \quad [\text{Take } x'(0) = c\,]$$

$$\Rightarrow (s^2+9)\bar{x} = \frac{s}{s^2+4} + s + c \quad \bar{x} = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{c}{s^2+9}. \qquad \text{(i)}$$

Consider $L^{-1}\left(\frac{s}{(s^2+4)(s^2+9)}\right) = L^{-1}\left(\frac{s}{s^2+9} + \frac{c}{s^2+9}\right)$

$$= \int_0^t \cos 2u . \frac{1}{3}\sin 3(t-u)du \quad [\text{Using convolution theorem}]$$

$$= \frac{1}{3}.\frac{1}{2}\int_0^t 2\cos 2u \sin 3(t-u)du \quad [2\cos A \sin B] = \text{Sin}(A+B) - \sin(A-B)$$

$$= \frac{1}{6}\int_0^t [\sin(2u+3t-3u) - \sin(2u-3t-3u)]du = \frac{1}{6}\int_0^t [\sin(3t-u) - \sin(5u-3t)]du$$

$$= \frac{1}{6}\left| -\frac{\cos(3t-u)}{(-1)} + \frac{\cos(5u-3t)}{5}\right|_0^t = \frac{1}{6}\left[\cos 2t + \frac{1}{5}\cos 2t - \left(\cos 3t + \frac{1}{5}\cos 3t\right)\right]$$

$$=\frac{1}{6}\left[\frac{5\cos 2t+\cos 2t-5\cos 3t-\cos 3t}{5}\right]=\frac{6}{6}\left(\frac{\cos 2t-\cos 3t}{5}\right)=\frac{1}{5}(\cos 2t-\cos 3t)$$

$$\Rightarrow L^{-1}\left(\frac{s}{(s^2+4)(s^2+9)}\right)=\frac{1}{5}(\cos 2t-\cos 3t). \qquad (A)$$

Therefore from (i), taking inverse Laplace transform of both sides of (i), we get

$$x=L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right]+L^{-1}\left[\frac{s}{s^2+9}\right]+cL^{-1}\left[\frac{1}{s^2+9}\right]$$

$$\Rightarrow x(t)=x=\frac{1}{5}(\cos 2t-\cos 3t)+\cos 3t+\frac{c}{3}\sin 3t \qquad (B)$$

To find c: Given $x\left(\frac{\pi}{2}\right)=-1$

$$-1=\frac{1}{5}\left(\cos\pi-\cos\frac{3\pi}{2}\right)+\cos\frac{3\pi}{2}+\frac{c}{3}\sin\frac{3\pi}{2}$$

$$-1=\frac{1}{5}(-1)+\frac{c}{3}.\sin\left(\pi+\frac{\pi}{2}\right) \qquad \left[\cos(n+1)\frac{\pi}{2}=0, n\in Z\right]$$

$$-1=-\frac{1}{5}-\frac{c}{3} \qquad [\sin(180+\theta)=-\sin\theta]$$

$$\Rightarrow -\frac{c}{3}=-1+\frac{1}{5}=\frac{-4}{5}\Rightarrow c=\frac{12}{5}\ . \qquad [\cos\pi=-1, \sin\pi=0]$$

Therefore, from (B),

$$x=\frac{1}{5}(\cos 2t-\cos 3t)+\cos 3t+\frac{4}{5}\sin 3t$$

$$\Rightarrow x=\frac{1}{5}(\cos 2t+4\cos 3t+4\sin 3t),$$ is the required solution.

Q.No.6.: Solve $ty''+2y'+ty=\cos t$ given than y (0) =1.

Sol.: Taking the Laplace transforms of both sides, we get

$L[t(f(t))]=-\frac{d}{ds}[L\{f(t)\}]$, we get

$$-\frac{d}{ds}\left[s^2\bar{y}-sy(0)-y'(0)\right]-\frac{d}{ds}(\bar{y})=\frac{s}{s^2+1}$$

$$\Rightarrow -\left(s^2\frac{d\bar{y}}{ds}+2s\bar{y}\right)+y(0)+0+2s\bar{y}-2y(0)-\frac{d}{ds}(\bar{y})=\frac{s}{s^2+1}$$

$$\Rightarrow \left(s^2+1\right)\frac{d\bar{y}}{ds}+1=-\frac{s}{s^2+1}\Rightarrow \frac{d\bar{y}}{ds}=-\frac{1}{s^2+1}-\frac{s}{\left(s^2+1\right)^2}.$$

On inversion and noting that $L^{-1}\left[\bar{f}'(s)\right]=-tf(t)$, we get

$$-ty=-\sin t-\frac{1}{2}t\sin t\text{, or }\ y=\frac{1}{2}\left(1+\frac{2}{t}\right)\sin t,$$

which is the desired solution**.**

Q.No.7.: Solve the equation $\frac{d^3y}{dt^3}+2\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=0$;

where $y = 1$, $\frac{dy}{dt}=1$, $\frac{d^2y}{dt^2}=2$ at $t = 0$.

Sol.: The given equation is $y'''+2y''-y'-2y=0$.

Taking the Laplace transform of both sides, we get

$$\left[s^3\bar{y}-s^2y(0)-sy'(0)-y''(0)\right]+2\left[s^2\bar{y}-sy(0)-y'(0)\right]-\left[s\bar{y}-y(0)\right]-2\bar{y}=0 \qquad \text{(i)}$$

Using the given conditions $y(0) = 1$, $y'(0)=2$, $y''(0)=2$, equation (i) reduces to

$$\left(s^3+2s^2-s-2\right)\bar{y}=s^2+2s+2+2s+4-1$$

$$\Rightarrow \left(s^3+2s^2-s-2\right)\bar{y}=s^2+4s+5$$

$$\therefore \bar{y}=\frac{s^2+4s+5}{s^3+2s^2-s-2}=\frac{s^2+4s+5}{(s-1)(s+1)(s+2)}=\frac{5}{3(s-1)}-\frac{1}{s+1}+\frac{1}{3(s+2)}.$$

Taking the inverse Laplace transforms of both sides, we get

$$y=\frac{5}{3}L^{-1}\left\{\frac{1}{s-1}\right\}-L^{-1}\left\{\frac{1}{s-1}\right\}+\frac{1}{3}L^{-1}\left\{\frac{1}{s-2}\right\}=\frac{5}{3}e^t-e^{-t}+\frac{1}{3}e^{-2t}$$

$$\Rightarrow y=\frac{1}{3}\left(5e^t+e^{-2t}\right)-e^{-t}$$, which is the required solution.

Q.No.8.: Solve the equation $y''-3y'+2y=4t+e^{3t}$, when $y(0) = 1$ and $y'(0)=-1$.

Sol.: Taking the Laplace transform of both sides, we get

$$\left[s^2\bar{y}-sy(0)-y'(0)\right]-3\left[s\bar{y}-y(0)\right]+2\bar{y}=\frac{4}{s^2}+\frac{1}{s-3}.$$

Using the given conditions, it reduces to

$$\left(s^2-3s+2\right)\overline{y}-s+1+3=\frac{4}{s^2}+\frac{1}{s-3} \qquad \Rightarrow \left(s^2-3s+2\right)\overline{y}=\frac{4}{s^2}+\frac{1}{s-3}s-4$$

$$\therefore \overline{y}=\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}=\frac{3}{s}+\frac{2}{s^2}-\frac{1}{2(s-1)}-\frac{2}{s-2}+\frac{1}{2(s-3)}. \qquad \text{[Partial fraction]}$$

Taking the inverse Laplace transforms of both sides, we get

$$y=3L^{-1}\left\{\frac{1}{x}\right\}+2L^{-1}\left(\frac{1}{s^2}\right)-\frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\}-2L^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$=3+2t-\frac{1}{2}e^t-2e^{2t}+\frac{1}{2}e^{3t}.$$

$\Rightarrow y=3+2t+\frac{1}{2}\left(e^{3t}-e^t\right)-2e^{2t}$, which is the required solution.

Q.No.9.: Solve the equation $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=e^{-t}\sin t$, $\quad x(0)=0,\ x'(0)=1$.

Sol.: The given equation is $x''+2x'+5x=e^t\sin t$.

Taking the Laplace transform of both sides, we get

$$\left[s^2\overline{x}-sx(0)-x'(0)\right]+2\left[s\overline{x}-x(0)\right]+5\overline{x}=\frac{1}{(s+1)^2+1}.$$

Using the given conditions, it reduces to

$$\left(s^2+2s+5\right)\overline{x}-1=\frac{1}{s^2+2s+2}$$

$$\Rightarrow \overline{x}=\frac{1}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)}+\frac{1}{s^2+2s+5}=\left[\frac{1}{\left(s^2+2s+2\right)}-\frac{1}{\left(s^2+2s+5\right)}\right]+\frac{1}{s^2+2s+5}$$

$$=\frac{1}{3}\left[\frac{1}{s^2+2s+2}+\frac{2}{s^2+2s+5}\right]=\frac{1}{3}\left[\frac{1}{(s+1)^2+1}+\frac{2}{(s+1)^2+2^2}\right]$$

Taking the inverse Laplace transform of both sides, we get

$$x=\frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1}+2.\frac{1}{(s+1)^2+2^2}\right\}=\frac{1}{3}\left[e^{-t}\sin t+2.\frac{1}{2}e^{-t}\sin 2t\right].$$

$\Rightarrow x=\frac{1}{3}e^{-t}(\sin t+\sin 2t)$, which is the required solution.

Q.No.10.: A voltage Ee^{-at} is applied at t = 0 to a circuit of inductance L and resistance

R. Show that the current at time t is $\frac{E}{R-aL}\left(e^{at}-e^{-Rt/L}\right)$.

Sol.: Let I be the current in the circuit at any time t, then by Kirchoff's law, we have

$$L\frac{dI}{dt+RI=Ee^{-at}}, \text{ where } I(0)=0.$$

Taking the Laplace transforms of both sides

$$L\left[\left(s\bar{I}-I(0)\right)\right]+R\bar{I}=\frac{E}{s+a}.$$

$$\Rightarrow \bar{I}=\frac{E}{(s-a)(Ls+R)}=\frac{E}{R-aL}\left(\frac{1}{s+a}-\frac{L}{Ls+R}\right)=\frac{E}{R-aL}\left(\frac{1}{s+a}-\frac{1}{s+\frac{R}{L}}\right).$$

Taking the inverse Laplace transforms of both sides, we get

$$I=\frac{E}{R-aL}L^{-1}\left(\frac{1}{s+a}-\frac{1}{s+\frac{R}{L}}\right)=\frac{E}{R-aL}\left[e^{-at}-e^{-Rt/L}\right].$$

Q.No.11.: Solve the simultaneous equations:

$$\frac{dx}{dt}-y=e^t, \quad \frac{dy}{dt}+x=\sin t; \text{ given } x(0)=1, \; y(0)=0.$$

Sol.: Taking the Laplace transforms of the given equations, we get

$$\left[s\bar{x}-x(0)\right]-\bar{y}=\frac{1}{s-1}\Rightarrow s\bar{x}-\bar{y}=\frac{1}{s-1} \qquad \left[\because x(0)=1\right] \qquad \text{(i)}$$

and $$\left[s\bar{y}-y(0)\right]+\bar{x}=\frac{1}{s^2+1}\Rightarrow \bar{x}+s\bar{y}=\frac{1}{s^2+1} \quad \left[\because y(0)=0\right] \qquad \text{(ii)}$$

Solving (i) and (ii) for $\bar{x}$ and $\bar{y}$, we have

$$\bar{x}=\frac{s^2}{(s-1)\left(s^2+1\right)}+\frac{1}{\left(s^2+1\right)^2}=\frac{1}{2}\left[\frac{1}{s-1}+\frac{s}{s^2+1}+\frac{1}{s^2+1}\right]+\frac{1}{\left(s^2+1\right)^2}$$

and $$\bar{y}=\frac{s}{\left(s^2+1\right)^2}+\frac{s}{(s-1)\left(s^2+1\right)}=\frac{s}{\left(s^2+1\right)^2}-\frac{1}{2}\left[\frac{1}{s-1}-1\frac{s}{s^2+1}+\frac{1}{s^2+1}\right].$$

Taking inverse Laplace transform of both sides, we get

$$x=\frac{1}{2}L^{-1}\left[\frac{1}{s-1}+\frac{s}{s^2+1}+\frac{1}{s^2+1}\right]+L^{-1}\left[\frac{1}{\left(s^2+1\right)^2}\right]=\frac{1}{2}\left[e^t+\cos t+\sin t\right]+\frac{1}{2}(\sin t-t\cos t)$$

$$\left[\because L^{-1}\left[\frac{1}{\left(s^2+a^2\right)^2}\right]=\frac{1}{2a^2}\sin at-at\cos at\right]$$

$$=\frac{1}{2}\left[e^t+\cos t+2\sin t-t\cos t\right].$$

$$y=L^{-1}\left[\frac{s}{\left(s^2+1\right)^2}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{s-1}-\frac{s}{s^2+1}+\frac{1}{s^2+1}\right]\qquad\left[\because L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=\frac{1}{2a}t\sin at\right]$$

$$=\frac{1}{2}\left(t\sin t-e^t+\cos t+\sin t\right).$$

Hence $x=\frac{1}{2}\left[e^t+\cos t+2\sin t-t\cos t\right]$ and $y=\frac{1}{2}\left(t\sin t-e^t+\cos t+\sin t\right)$.

Q.No.12.: Solve the simultaneous equations:

$$\left(D^2-3\right)x-4y=0,\ x+\left(D^2+1\right)y=0 \text{ for } t>0,$$

given that $x=y=\frac{dy}{dt}$ and $\frac{dx}{dt}=2$ at $t=0$.

Sol.: Taking the Laplace transforms of the given equations, we get

$$s^2\bar{x}-sx(0)-x'(0)-3\bar{x}-4\bar{y}=0$$

$$\Rightarrow\left(s^2-3\right)\bar{x}-4\bar{y}=2\qquad\left[\because x(0)=0,\quad x'(0)=2\right]\qquad\text{(i)}$$

and $\bar{x}+s^2\bar{y}-sy(0)-y'(0)+\bar{y}=0$

$$\Rightarrow\bar{x}+\left(s^2-3\right)\bar{y}=0\qquad\left[\because y(0)=0,\quad y'(0)=0\right]\qquad\text{(ii)}$$

Solving (i) and (ii) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x}=\frac{2\left(s^2+1\right)}{\left(s^2-1\right)^2}=\frac{1}{(s-1)^2}+\frac{1}{(s+1)^2}$$

and $\bar{y}=-\frac{2}{\left(s^2-1\right)^2}=-\frac{1}{2}\left[\frac{1}{s+1}-\frac{1}{s-1}-\frac{1}{(s+1)^2}+\frac{1}{(s-1)^2}\right].$

Taking the inverse Laplace transforms of both sides, we get

$$x = L^{-1}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}\right] = te^t + te^{-t} = 2t\left(\frac{e^t + e^{-t}}{2}\right) = 2t\cosh t$$

and $y = -\frac{1}{2}L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1} - \frac{1}{(s+1)^2} + \frac{1}{(s+1)^2}\right]$

$$= -\frac{1}{2}\left(e^{-t} - e^t - te^{-t} + te^t\right) = \frac{e^t - e^{-t}}{2} - t\left(\frac{e^t - e^{-t}}{2}\right) = (1-t)\sinh t.$$

Hence $x = 2t\cosh t$ and $y = (1-t)\sinh t$.

Q.No.13.: Solve the equations by Laplace transform method:

$$\left(D^2 - 2D + 2\right)x = 0; \quad x = Dx = 1 \quad \text{at } t = 0.$$

Sol.: Given differential equation is $\left(D^2 - 2D + 2\right)x = 0$.

$$\Rightarrow D^2x - 2Dx + 2x = 0 \quad \Rightarrow \frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 0.$$

Taking Laplace transform both sides, we get $L\left[\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x\right] = L(0)$

$$\Rightarrow L\left(\frac{d^2x}{dt^2}\right) - 2L\left(\frac{dx}{dt}\right) + 2L(x) = 0$$

$$\Rightarrow s^2L(x) - sx(0) - x'(0) - 2(sL(x) - x(0)) + 2L(x) = 0$$

$$\Rightarrow \left(s^2 - 2s + 2\right)\bar{x} - s + 2 - 1 = 0, \quad \bar{x} = L(x) \qquad [\text{Given } x(0) = 1,\ x'(0) = 1]$$

$$\Rightarrow \bar{x} = \frac{s-1}{s^2 - 2s + 2} = \frac{s-1}{s^2 - 2s + 1 + 2 - 1} \qquad \left[\text{add and subtract}\left(\frac{1}{2}\text{coefficient of } s\right)^2\right]$$

$$\Rightarrow \bar{x} = \frac{s-1}{(s-1)^2 + 1}.$$

Taking inverse Laplace transform both sides, we get

$$\Rightarrow L^{-1}\left(\bar{x}\right) = L^{-1}\left(\frac{s-1}{(s-1)^2 + 1}\right) \Rightarrow x = e^tL^{-1}\left(\frac{s}{s^2 + 1}\right) = e^t\cos t, \text{ is the required solution.}$$

Q.No.14.: Solve the equations by Laplace transform method:

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t \text{ with } x = 2,\ \frac{dx}{dt} = -1 \quad \text{at } t = 0.$$

Sol.: Taking Laplace transform both sides, we get

$$L\left[\frac{d^2x}{dt^2}-2\frac{dx}{dt}+x\right]=L\left(e^t\right)\Rightarrow L\left(\frac{d^2x}{dt^2}\right)-2L\left(\frac{dx}{dt}\right)+L(x)=L(e^t)$$

$$\Rightarrow s^2L(x)-sx(0)-x'(0)-2(sL(x)-x(0))+L(x)=\frac{1}{(s+1)}\left[\text{Given }\ x(0)=2,\ x'(0)=-1\right]$$

$$\Rightarrow\left(s^2-2s+1\right)\bar{x}-2s+1+2.2=\frac{1}{s-1},\quad \bar{x}=L(x)$$

$$(s-1)^2\bar{x}=\frac{1}{s-1}+2s-5\Rightarrow\bar{x}=\frac{1}{(s-1)^3}+\frac{2s-5}{(s-1)^2}.$$

Consider $\dfrac{2s-5}{(s-1)^2}=\dfrac{A}{s-1}+\dfrac{B}{(s-1)^2}$.

Multiplying by $(s-1)^2$, we get

$$2s-5=A(s-1)+B \qquad \text{(i)}$$

From (i), equating the coefficient of s, we get

$A=2$.

Equating the coefficient of constant terms

$-5=-A+B\ \Rightarrow B=-5+2=-3$

$$\bar{x}=\frac{1}{(s-1)^3}+\frac{2}{s-1}-\frac{3}{(s-1)^2}$$

Taking the inverse Laplace of both sides, we get

$$L^{-1}\left(\bar{x}\right)=L^{-1}\left(\frac{1}{(s-1)^3}\right)+2L^{-1}\left(\frac{1}{s-1}\right)-3L^{-1}\left(\frac{1}{(s-1)^2}\right)$$

$$=e^{-t}L^{-1}\left(\frac{1}{s^3}\right)+2e^t-3e^tL^{-1}\left(\frac{1}{s^2}\right)$$

$$=e^t\frac{t^2}{2!}+2e^t-3e^t.\frac{t}{1!} \qquad \left[L^{-1}\left(\frac{1}{s^{n+1}}\right)=\frac{t^n}{n!},n\in Z^+\right]$$

$$=\frac{1}{2}e^tt^2-3te^t=e^t\left\{\frac{t^2}{2}-3t+2\right\}$$

$\Rightarrow x = e^{t}\left(\frac{t^2}{2} - 3t + 2\right)$ is the required solution.

Q.No.15.: Solve the differential equation by Laplace Transform Method

$$x'' - 3x' + 2x = 1 - e^{2t}, \quad x(0) = 1, \quad x'(0) = 0.$$

Sol.: Given differential equation is $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 1 - e^{2t}$.

Taking Laplace transform on both sides, we get

$$L\left[\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x\right] = L\left(1 - e^{2t}\right) \Rightarrow L\left(\frac{d^2x}{dt^2}\right) - 3L\left(\frac{dx}{dt}\right) + 2L(x) = L\left(1 - e^{2t}\right)$$

$$\Rightarrow s^2L(x) - sx(0) - x'(0) - 3(sL(x) - x(0)) + 2L(x) = \frac{1}{s} - \frac{1}{s-2}$$

$$\Rightarrow \left(s^2 - 3s + 2\right)\bar{x} - s = 3 = \frac{1}{s} - \frac{1}{s-2}$$

$$\Rightarrow \left(s^2 - 3s + 2\right)\bar{x} = \frac{1}{s} - \frac{1}{s-2} + s - 3$$

$$= \frac{s - 2 + s + (s-3)s(s-2)}{s(s-2)} = \frac{s^3 - 5s^2 + 6s - 2}{s(s-2)}$$

$$\Rightarrow \bar{x} = \frac{s^3 - 5s^2 + 6s - 2}{s(s-2)(s-1)(s-2)} = \frac{s^3 - 5s^2 + 6s - 2}{s(s-1)(s-2)^2} \qquad \text{(i)}$$

Now consider $\frac{s^3 - 5s^2 + 6s - 2}{s(s-1)(s-2)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$ (*)

Multiplying both sides by $s(s-1)(s-2)^2$, we get

$$s^3 - 5s^2 + 6s - 2 = A(s-1)(s-2)^2 + Ds(s-2)^2 + Cs(s-1)(s-2) + Ds(s-1) \qquad \text{(ii)}$$

Take $s = 0$ in (ii); $-2 = -4A \Rightarrow A = \frac{1}{2}$

Take $s = 1$ in (ii); $0 = B \Rightarrow B = 0$

Take $s = 2$ in (ii); $-2 = 2D \Rightarrow D = -1$

Equating the co-efficient of s^3 on both sides

$1 = A + B + C \Rightarrow C = 1 - \frac{1}{2} - 0 = \frac{1}{2}$.

Therefore, (*), we have $\frac{s^3-5s^2+6s-2}{s(s-1)(s-2)^2}=\frac{1}{2}\frac{1}{s}+\frac{1}{2}\frac{1}{s-2}-\frac{1}{(s-2)^2}$.

Hence, (i) gives, $\bar{x}=\frac{1}{2}.\frac{1}{s}+\frac{1}{2}.\frac{1}{s-2}-\frac{1}{(s-2)^2}$.

Taking inverse Laplace transform

$$L^{-1}(\bar{x})=\frac{1}{2}L^{-1}\left(\frac{1}{s}\right)+\frac{1}{2}L^{-1}\left(\frac{1}{s-2}\right)-L^{-1}\left(\frac{1}{(s-2)^2}\right)$$

$$\Rightarrow x=\frac{1}{2}+\frac{1}{2}e^{2t}-e^{2t}.L^{-1}\left(\frac{1}{s^2}\right)$$

$$x=\frac{1}{2}+\frac{1}{2}e^{2t}-te^{2t}. \qquad \left[L^{-1}\left(\frac{1}{s^{n+1}}\right)=\frac{t^n}{n!}, n\in Z^+\right]$$

Q.No.16.: Solve the differential equation by Laplace Transform Method

$$y''+y'-2y=t, \quad y(0)=1, y'(0)=0.$$

Sol.: Given differential equation is $\frac{d^2y}{dt^2}+\frac{dy}{dt}-2y=t$.

Taking Laplace transform on both sides, we get

$$\Rightarrow L\left[\frac{d^2y}{dt^2}+\frac{dy}{dt}-2y\right]=L(t)\Rightarrow L\left(\frac{d^2y}{dt^2}\right)+4L\left(\frac{dy}{dt}\right)+3L(y)=\frac{1}{s^2}.$$

$$\Rightarrow s^2L(y)-sy(0)-y'(0)+(sL(y)-y(0))-2L(y)=\frac{1}{s^2}\Rightarrow (s^2+4s-)L(y)-s-1=\frac{1}{s^2},$$

$$\Rightarrow (s^2+4s-)\bar{y}=\frac{1}{s^2}+s+1 \quad \bar{y}=L(y)$$

$$\Rightarrow \bar{y}=\frac{1+s^3+s^2+6}{s^2(s-1)(s+2)}. \qquad (*)$$

Consider $\frac{1+s^3+s^2+6}{s^2(s-1)(s+2)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}+\frac{D}{s+3}$ (i)

Multiplying by $s^2(s-1)(s-2)$, we get

$$s^3+s^2+1=As(s-1)(s+2)+B(s-1)(s+2)+Cs^2(s+2)+Ds^2(s-1) \qquad (ii)$$

Put $s=0$ in (ii), $1=B(-2)\Rightarrow B=-\frac{1}{2}$

Put $s=1$ in (ii), $3=3C \Rightarrow C=1$

Put $s=-2$ in (ii), $-3=12D \Rightarrow D=\frac{1}{4}$

Equating the co-efficient of s^3, we get

$1=A+C+D \Rightarrow A=1-1-\frac{1}{4}=-\frac{1}{4}$.

Therefore from (*), $\frac{1+s^2+s^3}{s^2(s-1)(s+2)}=-\frac{1}{4}.\frac{1}{s}-\frac{1}{2}.\frac{1}{s^2}+\frac{1}{s-1}+\frac{1}{4}\frac{1}{s+2}$

Hence, from (i), $\bar{y}=\frac{-1}{4}.L^{-1}\left(\frac{1}{s}\right)-\frac{1}{2}L^{-1}\left(\frac{1}{s^2}\right)+L^{-1}\left(\frac{1}{s-1}\right)+\frac{1}{4}L^{-1}\left(\frac{1}{s+2}\right)$ $\quad \left[L(y)=\bar{y} \Rightarrow L^{-1}(\bar{y})\right]$

$\Rightarrow y=-\frac{1}{4}-\frac{1}{2}t+e^t+\frac{1}{4}e^{-2t} \Rightarrow y=e^t+\frac{1}{4}\left(e^{-2t}-1\right)-\frac{1}{2}t$ is the required solution.

Q.No.17.: Solve the differential equation by Laplace Transform Method

$$\left(D^2+5D+6\right)x=5e^t,\ x(0)=2,\ x'(0)=1$$

Sol.: Given differential equation is $\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=5e^t \qquad x(0)=2,\ x'(0)=1$

Taking Laplace transform both sides, we get

$$L\left(\frac{d^2x}{dt^2}\right)+5L\left(\frac{dx}{dt}\right)+6L(x)=5L\left(e^t\right)$$

$$\Rightarrow \left[s^2L(x)-sx(0)-x'(0)\right]+5(sL(x)-x(0))+6L(x)=\frac{5}{s-1}$$

$$\Rightarrow \left(s^2+5s+6\right)L(x)-2s-1-10=\frac{5}{s-1}$$

$$\Rightarrow \left(s^2+5s+6\right)\bar{x}=\frac{5}{s-1}+2s+11 \qquad \bar{x}=L(x)$$

$$\Rightarrow \bar{x}=\frac{5}{(s-1)(s+3)(s+2)}+\frac{2s+11}{(s+3)(s+2)}=\frac{5+(2s+11).(s-1)}{(s-1)(s+3)(s+2)}$$

$$\Rightarrow \bar{x}=\frac{2s^2-2s+11s-11+5}{(s-1)(s+3)(s+2)} \Rightarrow \bar{x}=\frac{2s^2+9s-6}{(s-1)(s+3)(s+2)}. \qquad \text{(i)}$$

Now consider $\frac{2s^2+9s-6}{(s-1)(s+3)(s+2)}=\frac{A}{s-1}+\frac{B}{s+3}+\frac{C}{s+2}$ (ii)

Multiplying both by $(s-1)(s+3)(s+2)$, we get

$$2s^2+9s-6=A(s+3)(s+2)+B(s-1)(s+2)+C(s-1)(s+3). \qquad \text{(iii)}$$

Put s = 1 in (iii), we get $5=12A \Rightarrow A=\frac{5}{12}$

Put $s=-3$ in (iii), we get $18-27-6=(-4)(-1)B \Rightarrow 4B=-15 \Rightarrow B=\frac{-15}{4}$

Put $s=-2$ in (iii), we get $8-18-6=C(-3)(1) \Rightarrow C=\frac{16}{3}$.

Therefore, from (ii) $\frac{2s^2+9s-6}{(s-1)(s+3)(s+2)}=\frac{5}{12}.\frac{1}{s-1}-\frac{15}{4}.\frac{1}{s+3}+\frac{16}{3}\frac{1}{s+2}$.

Hence, from (i) $\bar{x}=L(x)=\frac{5}{12}.\frac{1}{s-1}-\frac{15}{4}.\frac{1}{s+3}+\frac{16}{3}\frac{1}{s+2}$

Taking inverse Laplace transform both sides,

$$x=\frac{5}{12}L^{-1}\left(\frac{1}{s-1}\right)-\frac{15}{4}L^{-1}\left(\frac{1}{s+3}\right)+\frac{16}{3}L^{-1}\left(\frac{1}{s+2}\right) \quad \left[L(x)=\bar{x}\Rightarrow x=L^{-1}(\bar{x})\right]$$

$=\frac{5}{12}e^t-\frac{15}{4}e^{-3t}+\frac{16}{3}e^{2t}$ is the required solution.

Q.No.18.: Solve the differential equation by Laplace Transform Method

$$y''+4y'+3y=e^{-t}, \quad y(0)=y'(0)=1.$$

Sol.: Given differential equation is $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+3y=e^{-t}$.

Taking Laplace transform on both sides, we get

$$L\left[\frac{d^2y}{dt^2}+4\frac{dy}{dt}+3y\right]=L\left(e^{-t}\right) \Rightarrow L\left(\frac{d^2y}{dt^2}\right)+4L\left(\frac{dy}{dt}\right)+3L(y)=\frac{1}{s+1}.$$

$$\Rightarrow s^2L(y)-sy(0)-y'(0)+4(sL(y)-y(0))+3L(y)=\frac{1}{s+1}$$

$$\Rightarrow \left(s^2+4s+3\right)\bar{y}-s-1-4=\frac{1}{s+1}, \quad \bar{y}=L(y).$$

$$\Rightarrow \bar{y}=\frac{1}{s^2+4s+3}\left(\frac{1}{s+1}+s+5\right)=\frac{1+(s+5)(s+1)}{(s+1)(s+3)(s+1)}=\frac{s^2+6s+6}{(s+1)^2(s+3)}. \qquad (*)$$

Consider $\frac{s^2+6s+6}{(s+1)^2(s+3)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{s+3}$. (i)

Multiplying by $(s+1)^2(s+3)$, we get

$$s^2+6s+6=A(s+1)(s+3)+B(s+3)+C(s+1)^2 \tag{ii}$$

Put $s=-1$ in (ii), $1 = B(2) \Rightarrow B=\frac{1}{2}$.

Put $s=-3$ in (ii), $-3=4C \Rightarrow C=-\frac{3}{4}$.

Equating the co-efficient of s^2, we get

$$1=A+C \Rightarrow A=1+\frac{3}{4}=\frac{7}{4}.$$

Therefore, from (i), $\frac{s^2+6s+6}{(s+1)^2(s+3)}=\frac{7}{4}.\frac{1}{s+1}+\frac{1}{2}.\frac{1}{(s+1)^2}-\frac{3}{4}.\frac{1}{s+3}$.

Hence, from (*)

$$\bar{y}=\frac{7}{4}\frac{1}{s+1}+\frac{1}{2(s+1)^2}-\frac{3}{4}\frac{1}{s+3}.$$

Taking inverse Laplace Transform both sides,

$$L^{-1}(\bar{y})=\frac{7}{4}L^{-1}\left(\frac{1}{s+1}\right)+\frac{1}{2}L^{-1}\left(\frac{1}{(s+1)^2}\right)-\frac{3}{4}L^{-1}\left(\frac{1}{s+3}\right)$$

$$=\frac{7}{4}e^{-t}+\frac{1}{2}e^{-t}L^{-1}\left(\frac{1}{s^2}\right)-\frac{3}{7}e^{-3t}=\frac{7}{4}e^{-t}+\frac{e^{-t}}{2}.\frac{t}{1!}-\frac{3}{4}e^{-3t}$$

$$\Rightarrow y=\frac{7}{4}e^{-t}+\frac{te^{-t}}{2}-\frac{3}{4}e^{-3t}.$$

Q.No.19.: Solve the differential equation by Laplace Transform Method

$$\frac{dx}{dt}+x=\sin wt\ ,\ \ x(0)=2.$$

Sol.: Taking Laplace transform both sides of the given equation

$$L\left[\frac{dx}{dt}+x\right]=L(\sin wt)\Rightarrow L\left(\frac{dx}{dt}\right)+L(x)=\frac{w}{s^2+w^2}$$

$$\Rightarrow sL(x)-L(0)+L(x)=\frac{w}{s^2+w^2}\Rightarrow (s+1)\bar{x}-2=\frac{w}{s^2+w^2},\quad \overline{x=L(x)}$$

$$\Rightarrow (s+1)\bar{x}=\frac{w}{s^2+w^2}+2\Rightarrow \bar{x}=\frac{w}{(s+1)(s^2+w^2)}+\frac{2}{(s+1)}. \tag{i}$$

Now consider $\frac{w}{(s+1)(s^2+w^2)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+w^2}$ (ii)

Multiplying both sides by, $(s+1)(s^2+w^2)$, we get

$w=A(s^2+w^2)+(Bs+C)(s+1)$. (iii)

Put $s=-1$ in (iii), $w=A(1+w^2)\Rightarrow A=\frac{w}{w^2+1}$.

Equating the coefficient of s^2 in (iii), we get

$0=A+B\Rightarrow B=-\frac{w}{w^2+1}$.

Equating the coefficient of s in (iii), we get $0=B+C\Rightarrow C=\frac{w}{w^2+1}$.

Therefore, from (ii), we have $\frac{w}{(s+1)(s^2+w^2)}=\frac{w}{w^2+1}.\frac{1}{s+1}-\frac{w}{w^2+1}.\frac{s-1}{s^2+w^2}$.

Hence, (i) gives $\bar{x}=\frac{w}{w^2+1}.\frac{1}{s+1}-\frac{w}{w^2+1}.\left(\frac{s-1}{s^2+w^2}\right)+\frac{2}{s+1}$.

Taking inverse Laplace transform on both sides, we get

$$L^{-}(\bar{x})=\frac{w}{w^2+1}L^{-1}\left(\frac{1}{s+1}\right)-\frac{w}{w^2+1}L^{-1}.\left(\frac{s}{s^2+w^2}\right)+\frac{w}{w^2+1}L^{-1}.\left(\frac{1}{s^2+w^2}\right)+2L^{-1}\left(\frac{1}{s+1}\right).$$

$\Rightarrow x=\frac{w}{w^2+1}e^{-t}-\frac{w}{w^2+1}\cos wt+\frac{w}{w^2+1}\sin wt+2e^{-t}$, is the required solution.

Q.No.20.: Solve the differential equation by Laplace Transform Method

$$(D^2-1)x=a\cosh t, \qquad x(0)=x'(0)=0.$$

Sol.: The given differential equation is $\frac{d^2x}{dt^2}-x=a\cosh t$.

Taking Laplace transform both sides, we get $L\left[\frac{d^2x}{dt^2}\right]-L(x)=aL(\cosh t)$

$\Rightarrow s^2L(x)-sx(0)-x'(0)-L(x)=a\frac{s}{s^2-1}$

$\Rightarrow (s^2-1)L(x)=\frac{as}{s^2-1}\Rightarrow \bar{x}=\frac{as}{(s^2-1)^2}$, $\bar{x}=L(x)$ [Using $x(0)=x'(0)=0$]

Taking Inverse Laplace Transform, we get

$$x = L^{-1}\left[\frac{as}{\left(s^2-1\right)^2}\right] = aL^{-1}\left[\frac{s}{\left(s^2-1\right)^2}\right] = a\frac{1}{2}t\sinh t \qquad \left[L^{-1}\left(\frac{s}{\left(s^2-a^2\right)^2}\right) = \frac{1}{2a}t\sinh at\right]$$

$$= \frac{a}{2}t\sinh t.$$

Q.No.21.: Solve the differential equation by using Laplace transform

$$\left(D^2 - D - 2\right)x = 20\sin 2t, \quad x(0) = -1, \quad x'(0) = 2.$$

Sol.: Given differential equation is $\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 20\sin 2t$.

Taking Laplace transform both sides, we get

$$L\left[\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x\right] = L(20\sin 2t) \Rightarrow L\left(\frac{d^2x}{dt^2}\right) - L\left(\frac{dx}{dt}\right) - 2L(x) = 20L(\sin 2t)$$

$$\Rightarrow s^2L(x) - sx(0) - x'(0) - (sL(x) - x(0)) - 2L(x) = 20.\frac{2}{s^2+4}$$

$$\Rightarrow \left(s^2 - s - 2\right)\bar{x} + s - 2 - 1 = \frac{40}{s^2+4}, \qquad \bar{x} = L(x)$$

$$\Rightarrow \left(s^2 - s - 2\right)\bar{x} = \frac{40}{s^2+4} - s + 3$$

$$\Rightarrow \bar{x} = \frac{40 - s^3 - 4s + 3s^2 + 12}{\left(s^2+4\right)(s+1)(s-2)} = \frac{3s^2 - s^3 - 4s + 52}{\left(s^2+4\right)(s+1)(s-2)} \qquad \text{(i)}$$

Now consider $\frac{3s^2 - s^3 - 4s + 52}{\left(s^2+4\right)(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{CS+D}{s^2+4}$ (*)

Multiplying both sides by $\left(s^2+4\right)(s+1)(s-2)$, we get

$$3s^2 - s^3 - 4s + 52 = A(s-2)(s^2+4) + B(s+1)(s^2+4) + \left(C_s + D\right)((s+1)(s-2)) \qquad \text{(ii)}$$

Put $s = -1$ in (ii), $60 = -15A \Rightarrow A = -4$

Put $s = 2$ in (ii), $48 = 24B \Rightarrow B = 2$.

Comparing the co-efficient of s^3 of equation (ii), we get

$-1 = A + B + C \Rightarrow C = -1 + 4 - 2 = 1$.

Comparing the co-efficient of s^2 of equation (ii), we get

$3 = -2A + B - C \Rightarrow D = 3 - 8 - 2 + 1 \Rightarrow D = -6$.

Therefore from (*), we get $\frac{3s^2 - s^3 - 4s + 52}{(s^2+4)(s+1)(s-2)} = -4.\frac{1}{s+1} + 2\frac{1}{s-2} + \frac{s-6}{s^2+4}$.

Hence using (i), we get $\bar{x} = -4.\frac{1}{s+1} + 2\frac{1}{s-2} + \frac{s-6}{s^2+4}$.

Taking inverse Laplace transform of both sides,

$$x = -4L^{-1}\left[\frac{1}{s+1}\right] + 2L^{-1}\left[\frac{1}{s-2}\right] + L^{-1}\left[\frac{s}{s^2+4}\right] - 6L^{-1}\left[\frac{s}{s^2+4}\right]$$

$$= -4e^{-t} + 2e^{2t} + \cos 2t - 6.\frac{1}{2}\sin 2t$$

$x = 2e^{2t} - 4e^{-t} + \cos 2t - 3\sin 2t$, is the required solution.

Q.No.22.: Solve the differential equation by using Laplace transform

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \frac{17}{2}\sin 2t \text{ given } y = 2 \text{ and } \frac{dy}{dt} = -4, \text{ when } t = 0.$$

Sol.: The given differential equation is $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \frac{17}{2}\sin 2t$.

Taking Laplace transform, we get $L\left[\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y\right] = \frac{17}{2}L(\sin 2t)$

$$\Rightarrow (s^2L(y) - sy(0) - y'(0)) + 2(sL(y) - y(0)) + 5L(y) = \frac{17}{2}.\frac{2}{(s^2+4)}$$

$$\Rightarrow (s^2 + 2s + 5)L(y) - 2s + 4 - 4 = \frac{17}{s^2+4}$$

$$\Rightarrow L(y) = \frac{17}{(s^2+4)(s^2+2s+5)} + \frac{2s}{(s^2+2s+5)}. \qquad \text{(i)}$$

Now consider $\frac{17}{(s^2+4)(s^2+2s+5)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+2s+5}$. (ii)

Multiplying both sides by $(s^2+4)(s^2+2s+5)$, we get

$$17 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+4)$$

$$= As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B + Cs^3 + 4Cs + Ds^2 + 4D'$$

$$= (A+C)s^3 + (2A+B+D)s^2 + (5A+2B+4C)s + 5B + 4D \qquad \text{(iii)}$$

Equating the coefficient of s^3 in (iii), we get $0 = A + C$ (A)

Equating the coefficient of s^2 in (iii), we get $0 = 2A + B + D$ (B)

Equating the coefficient of s in (iii), we get $0 = 5A + 2B + 4C$ (C)

Equating the constant terms in (iii), we get $17 = 5B + 4D$ (D)

Eliminating D from (iv) and (ii), for this

Operating (iv) –4(ii), we get $B - 8A = 17$ (E)

From (A), $C = -A$ and from (C), $A + 2B = 0 \Rightarrow A = -2B$

(E) gives $B + 16B = 17 \Rightarrow B = 1 \Rightarrow A = -2$

From (A) $C = -A = 2$.

From (B), $2A + B + D = 0 \Rightarrow D = -2A - B = 4 - 1 = 3 \Rightarrow D = 3$.

Hence from (ii), $\dfrac{17}{(s^2+4)(s^2+2s+5)} = \dfrac{-2s+1}{s^2+4} + \dfrac{2s+3}{s^2+2s+5}$.

Therefore, from (i)

$$L(y) = \frac{-2s+1}{s^2+4} + \frac{2s+3}{s^2+2s+5} + \frac{2s}{s^2+2s+5} = \frac{-2s+1}{s^2+4} + \frac{4s+3}{s^2+2s+5}$$

$$= -2\frac{s}{s^2+4} + \frac{1}{s^2+4} + \frac{4(s+1)-1}{(s+1)^2+4}$$

Taking Inverse Laplace Transform, we get

$$y = -2L^{-1}\left(\frac{s}{s^2+4}\right) + L^{-1}\left(\frac{1}{s^2+4}\right) + L^{-1}\left[\frac{4(s+1)-1}{(s+1)^2+4}\right]$$

$$= -2\cos 2t + \frac{1}{2}\sin 2t + e^{-t}L^{-1}\left(\frac{4s-1}{s^2+4}\right)$$

$$= -2\cos 2t + \frac{1}{2}\sin 2t + e^{-t}\left[L^{-1}\left(\frac{4s}{s^2+4}\right) - L^{-1}\left(\frac{1}{s^2+4}\right)\right]$$

$$= -2\cos 2t + \frac{1}{2}\sin 2t + e^{-t}\left(4\cos 2t - \frac{1}{2}\sin 2t\right)$$

$$= e^{-t}\left(4\cos 2t - \frac{1}{2}\sin 2t\right) - 2\cos 2t + \frac{1}{2}\sin 2t.$$

Q.No.23.: Solve the differential equation by using Laplace transform

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}-2y=3\cos 3t-11\sin 3t \text{ given } y(0)=0\,,\quad y'(0)=6\,.$$

Sol.: Taking Laplace Transform of both sides of the given differential equation,

$$L\left[\frac{d^2y}{dt^2}+\frac{dy}{dt}-2y\right]=L[3\cos 3t-11\sin 3t]$$

$$\Rightarrow L\left[\frac{d^2y}{dt^2}\right]+L\left[\frac{dy}{dt}\right]-2L(y)=3L(\cos 3t-11\sin 3t)$$

$$\Rightarrow s^2L(y)-sy(0)-y'(0)+sL(y)-y(0)-2L(y)=\frac{3s}{s^2+9}-11\frac{3}{s^2+9}=\frac{3s-33}{\left(s^2+9\right)}$$

$$\Rightarrow \left(s^2+s-2\right)\bar{y}-0-6-0=\frac{3s-33}{s^2+9}\Rightarrow \left(s^2+s-2\right)\bar{y}=\frac{3s-33}{s^2+9}+6$$

$$\Rightarrow \bar{y}=\frac{3s-33}{\left(s^2+9\right)(s-1)(s+2)}+\frac{6}{(s-1)(s+2)} \qquad \text{(i)}$$

Consider $\dfrac{3s-33}{(s-1)(s+2)\left(s^2+9\right)}=\dfrac{A}{s-1}+\dfrac{B}{s+2}+\dfrac{Cs+D}{s^2+9}$. (*)

Multiplying both sides by $(s-1)(s+2)\left(s^2+9\right)$, we get

$$3s-33=A(s+2)(s^2+9)+B(s-1)(s^2+9)+(Cs+D)(s-1)(s+2) \qquad \text{(ii)}$$

Put s = 1 in (ii), $-30=30A\Rightarrow A=-1$

Put $s=-2$ in (ii), $-39=-39B\Rightarrow B=1$.

Comparing the co-efficient of s^3, we get

$0=A+B+C\Rightarrow C=0$.

Comparing the co-efficient of s^2, we get

$0=2A-B+D+C\Rightarrow D=3$.

Therefore, from (*), we get

$$\frac{3s-33}{(s-1)(s+2)\left(s^2+9\right)}=\frac{1}{s-1}+\frac{1}{s+2}+\frac{3}{s^2+9}.$$

Consider $\dfrac{6}{(s-1)(s+2)}=\dfrac{A}{s-1}+\dfrac{B}{s+2}$

$\Rightarrow 6=A(s+2)+B(s-1)$

$0=A+B$

$6 = 2A - B \Rightarrow A = 2, \quad B = -2$.

Hence, from (i), we get

$$\bar{y} = -1.\frac{1}{s+1} + \frac{1}{s+2} + \frac{3}{s^2+9} + \frac{6}{(s-1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2} + \frac{3}{s^2+9}$$

Taking Inverse Laplace Transform, we get

$$y = L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right) + 3L^{-1}\left(\frac{1}{s^2+9}\right) \qquad \left[L(y) = \bar{y} \Rightarrow y = L^{-1}(\bar{y})\right]$$

$= e^t - e^{-2t} + 3.\frac{1}{3}\sin 3t = e^t - e^{-2t} + \sin 3t$, is the required solution.

Q.No.24.: Solve the differential equation by using Laplace transform

$$\frac{d^2x}{dt^2} + x = t\cos 2t \text{ given } x(0) = 0, \quad x'(0) = 0.$$

Sol.: Taking Laplace Transform of both sides of the given differential equation,

$$L\left[\frac{d^2x}{dt^2} + x\right] = L[t\cos 2t] \text{ take } f(t) = \cos t$$

$$\Rightarrow L\left[\frac{d^2x}{dt^2}\right] + L[x] = \frac{d}{dt}\bar{f}(s) \text{ where } \bar{f}(s) = L(\cos 2t) = \frac{s}{s^2+4}$$

$$= \frac{2s^2}{(s^2+4)^2}$$

$$\Rightarrow s^2L(x) - sx(0) - x'(0) + L(x) = \frac{2s^2}{(s^2+4)^2}$$

$$\Rightarrow (s^2+1)\bar{x} - 0 - 0 = \frac{2s^2}{(s^2+4)^2}, \quad \bar{x} = L(x)$$

$$\Rightarrow \bar{x} = \frac{2s^2}{(s^2+4)^2(s^2+1)}$$

Taking inverse Laplace transform, we get $x = 2L^{-1}\left[\frac{s}{(s^2+4)^2}.\frac{s}{(s^2+1)}\right]$

Here $\bar{f}(s) = \frac{s}{(s^2+4)^2}$, $f(t) = L^{-1}(\bar{f}(s)) = \frac{1}{2.2t\sin 2t}.t\sin 2t = \frac{1}{4}$

Also $\bar{g}(s)=\dfrac{s}{s^2+1}$, $g(t)=L^{-1}(\bar{g}(s))=\cos t$

$\left[L(x)=\bar{x}\Rightarrow x=L^{-1}(\bar{x})\right]$.

By convolution Theorem, we get

$$x=2\int_0^t \frac{1}{4}u\sin 2u\cos(t-u)du \qquad \left[L^{-1}(f(s)g(s)=\int_0^t f(u)g(t-u)du\right]$$

$$=\frac{1}{4}\int_0^t 2u\sin 2u\cos(t-u)du$$

$$=\frac{1}{4}\int_0^t u\left[\sin(u+t)+\sin(3u-t)\right]du \qquad \left[a\sin A\cos B=\sin(A+B)+\sin(A-B)\right]$$

Integrating by parts, we get

$$=\frac{1}{4}\left[(u)\left(-\cos(u+t)-\frac{\cos(3u-t)}{3}\right)-(1)\left(-\sin(u-t)-\frac{\cos(3u-t)}{9}\right)\right]_0^t$$

$$=\frac{1}{4}\left[\left(-t\cos 2t-\frac{t}{3}\cos 2t+\sin 2t+\frac{\sin 2t}{9}\right)-\left(\sin t-\frac{\sin t}{9}\right)\right]$$

$$=\frac{1}{4}\left[-\frac{4}{3}t\cos 2t+\frac{10}{9}\sin 2t-\frac{8}{9}\sin t\right]=\frac{1}{9}\left[-3t\cos 2t+\frac{5}{2}\sin 2t-2\sin t\right],$$ is the required solution.

Q.No.25.: Solve the differential equation by using Laplace transform

$$\left(D^3+D\right)x=2,\ x=3,\ Dx=1,\ D^2x=-2,\ \text{at } t=0.$$

Sol.: The given differential equation is $\dfrac{d^3x}{dt^3}+\dfrac{dx}{dt}=2$.

Taking Laplace transform, we get $L\left[\dfrac{d^3x}{dt^3}+\dfrac{dx}{dt}\right]=L(2)\Rightarrow L\left(\dfrac{d^3x}{dt^3}\right)+L\left[\dfrac{dx}{dt}\right]=\dfrac{2}{s}$

$$\Rightarrow s^3L(x)-s^2x(0)-sx'(0)-x''(0)+sL(x)-x(0)=\frac{2}{s}$$

$$\Rightarrow \left(s^3+s\right)\bar{x}-3s^2-s+2-3=\frac{2}{s} \quad \Rightarrow \left(s^3+s\right)\bar{x}=\frac{2}{s}+3s^2+s+1$$

$$\Rightarrow \bar{x} = \frac{2}{s^2(s^2+1)} + \frac{3s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} \qquad \text{(i)}$$

Now consider $\frac{1}{s^2(s^2+1)} = \frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$ (Put s^2 =1, before integration)

$\Rightarrow 1 = A(t+1) + Bt$

A = 1 Put t = 0

$B = -1$ Put $t = -1$

$$\therefore \frac{1}{s^2(s^2+1)} = \frac{1}{t} - \frac{1}{t+1} = \frac{1}{s^2} - \frac{1}{s^2+1} \qquad \text{(i)}$$

$$\text{Also } \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \qquad \text{(iii)}$$

$\Rightarrow 1 = A(s^2+1) + (Bs+C)s$ [On multiplication by $s(s^2+1)$]

Comparing the coefficient of s^2 ; 0 = A + B

Comparing the coefficient of s ; 0 = C

Comparing the constant terms ; 1 = A, Hence $B = -1$

$$\therefore \text{From (iii), } \frac{1}{s(s^2+1)} = \frac{1}{s} + \frac{s}{s^2+1} \qquad \text{(iv)}$$

Therefore using (ii) and (iv), we get

$$\bar{x} = 2\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) + \frac{3s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s} - \frac{s}{s^2+1} = \frac{2}{s^2} - \frac{2}{s^2+1} + \frac{1}{s^2+1} + \frac{3s}{s^2+1} - \frac{s}{s^2+1} + \frac{1}{s}$$

$$= \frac{2}{s^2} - \frac{1}{s^2+1} + \frac{2s}{s^2+1} + \frac{1}{s}. \qquad \text{(v)}$$

Taking inverse Laplace Transform of both sides of (v), we get

$$x = 2L^{-1}\left(\frac{1}{s^2}\right) - L^{-1}\left(\frac{1}{s^2+1}\right) + 2L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s}\right) \qquad \begin{bmatrix} L(x) = \bar{x} \\ x = L^{-1}(\bar{x}) \end{bmatrix}$$

$= 2t - \sin t + 2\cos t + 1$, is the required solution.

Q.No.26.: Solve the differential equation by using Laplace transform

$$(D^3 + D^2)x = 6t^2 + 4, \ x = D^2x, \ \ Dx = 2, \ \ \text{at } t = 0.$$

Sol.: The given differential equation is $\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} = 6t^2 + 4$.

Taking Laplace transform, we get

$$\Rightarrow L\left(\frac{d^3x}{dt^3}\right)+L\left[\frac{d^2x}{dt^2}\right]=L\left(6t^2+4\right)$$

$$\Rightarrow s^3L(x)-s^2x(0)-sx'(0)-x''(0)+s^2L(x)-sx(0)-x'(0)=6L\left(t^2\right)+4L(1)$$

$$\Rightarrow \left(s^3+s^2\right)L(x)-2s-2=6\frac{2}{s^3}+\frac{4}{s}$$

$$\Rightarrow \left(s^3+s^2\right)\bar{x}=\frac{12}{s^3}+\frac{4}{s}+2s+2,\quad \bar{x}=L(x)$$

$$\Rightarrow \bar{x}=\frac{12}{s^5(s+1)}+\frac{4}{s^3(s+1)}+\frac{2}{s^2}\frac{(s+1)}{(s+1)}=\frac{12}{s^5(s+1)}+\frac{4}{s^3(s+1)}+\frac{2}{s^2}. \qquad \text{(i)}$$

Taking inverse Laplace Transform of both sides of (i), we get

$$x=12L^{-1}\left(\frac{1}{s^5(s+1)}\right)+4L^{-1}\left(\frac{1}{s^3(s+1)}\right)+2L^{-1}\left(\frac{1}{s^2}\right) \qquad \text{(ii)}$$

$$\left[\begin{array}{l} L(x)=\bar{x} \\ x=L^{-1}(\bar{x}) \end{array}\right]$$

Now consider $L^{-1}\left(\frac{1}{s^5(s+1)}\right)=L^{-1}\left(\frac{1}{s^5}.\frac{1}{s+1}\right)$

$$\left[L^{-1}\left(\frac{1}{s^{n+1}}\right)=\frac{t^n}{n!}\right]$$

Take $\bar{f}(s)=\frac{1}{s^5}$, $\quad \bar{g}(s)=\frac{1}{s+1}$

$$L^{-1}(\bar{f}(s))=L^{-1}\left(\frac{1}{s^5}\right)=\frac{t^4}{4!}=\frac{t^4}{24}=f(t).$$

Also $L^{-1}\left(\bar{g}(g)\right)=L^{-1}\left(\frac{1}{s+1}\right)=e^{-t}=g(t)$.

$$\therefore L^{-1}\left(\frac{1}{s^5(s+1)}\right)=\int_0^t f(u).g(t-u)du=\int_0^t \frac{u^4}{24}.e^{-(t-u)}du.$$

$$=\frac{e^{-t}}{24}\int_0^t \underset{\text{I}}{u^4}\,\underset{\text{II}}{e^u}\,du \qquad \text{(By convolution theorem)}$$

$$=\frac{e^{-t}}{24}\left[\left(u^4\right)\left(e^u\right)-\left(4u^3\right)\left(e^u\right)+\left(12u^2\right)e^u-(24u)e^u+(24)e^u\right]_0^t$$

$$\Rightarrow \left(L^{-1}\right)\left[\frac{1}{s^5(s+1)}\right]=\frac{e^{-t}}{24}\left[t^4e^t-4t^3e^t+12t^2e^t-24te^t+24e^t-24\right]. \qquad \text{(A)}$$

Also $\left(L^{-1}\right)\left[\dfrac{1}{s^3(s+1)}\right]=L^{-1}\left[\dfrac{1}{s^3}\cdot\dfrac{1}{s+1}\right]=\int\limits_0^t f(\mu).g(t-u)d\mu$

By convolution Theorem

$$\bar{f}(s)=\frac{1}{s^2},\quad \bar{g}(s)=\frac{1}{s+1}$$

$$L^{-1}\left(\bar{f}(s)\right)L^{-1}\left(\frac{1}{s^3}\right)=\frac{t^2}{2!}=\frac{t^2}{2}=f(t)$$

$$L^{-1}\left(\bar{g}(s)\right)L^{-1}\left(\frac{1}{s+1}\right)=e^{-t}=g(t)$$

$$=\int\limits_0^t \frac{\mu^2}{2}.e^{-(t-\mu)}d\mu$$

$$=\frac{e^{-t}}{2}\int\limits_0^t \underset{I}{\mu^2}\,\underset{II}{e^{\mu}}\,d\mu=\frac{e^{-t}}{2}\left[\mu^2e^{\mu}-(2\mu)e^{\mu}+2e^{\mu}\right]_0^t$$

$$\Rightarrow L^{-1}\left[\frac{1}{s^3(s+1)}\right]=\frac{e^{-t}}{2}\left[t^2e^t-2te^t+2e^t-2\right].\qquad\text{(B)}$$

Therefore, from (ii) and using (A) and (B), we have

$$x=12\frac{e^{-t}}{24}\left[e^t\left(t^4-4t^3+12t^2-24t\right)+24\left(e^t-1\right)\right]+4\frac{e^{-t}}{2}\left[e^t\left(t^2-2t\right)+2\left(e^t-1\right)\right]+2t$$

$$=\frac{1}{2}t^4-2t^3+6t^2-12t+12e^{-t}\left(e^t-1\right)+2\left(t^2-2t\right)+2.2e^{-t}\left(e^{-t}-1\right)+2t$$

$x=\dfrac{t^4}{2}-2t^3+8t^2-14t+16\left(1-e^{-t}\right)$, is the required solution.

Q.No.27.: Find the solution of the differential equation by using Laplace Transform

$$\left(D^3-D\right)x=2\cos t,\quad x=3,\ Dx=2,\ \ D^2x=1\ \text{ at } t=0.$$

Sol.: Given the differential equation is $\dfrac{d^3x}{dt^3}-\dfrac{dx}{dt}=2\cos t$.

Taking Laplace Transform of both sides, we get

$$L\left[\frac{d^3x}{dt^3}\right]-L\left[\frac{dx}{dt}\right]=2L(\cos t)$$

$$\Rightarrow s^3L(x) - s^2x(0) - sx'(0) - x''(0) - (sL(x) - x(0)) = \frac{2s}{s^2+1}$$

$$\Rightarrow (s^3 - s)\bar{x} - 3s^2 - 2s - 1 + 3 = \frac{2s}{s^2+1}, \quad \bar{x} = L(x)$$

$$\Rightarrow (s^3 - s)\bar{x} = \frac{2s}{s^2+1} + 3s^2 + 2s - 2$$

$$\bar{x} = \frac{2}{(s^2-1)(s^2+1)} + \frac{3s}{s^2-1} + \frac{2}{s(s+1)}. \qquad \text{(i)}$$

Taking inverse Laplace transform of both sides of (i)

$$x = 2L^{-1}\left[\frac{1}{(s-1)(s+1)(s^2+1)}\right] + 3L^{-1}\left[\frac{s}{s^2-1}\right] + 2L^{-1}\left[\frac{1}{s(s+1)}\right]. \qquad \text{(ii)}$$

Consider $\frac{1}{(s-1)(s+1)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$. (*)

Multiplying both sides by $(s-1)(s+1)(s^2+1)$, we get

$$\Rightarrow 1 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s^2-1). \qquad \text{(iii)}$$

Put s = 1 in (iii), we get $1 = 4A \Rightarrow A = \frac{1}{4}$.

Put $s = -1$ in (iii). we get $1 = -4B \Rightarrow B = -\frac{1}{4}$.

Comparing the co-efficient of s^3 in (iii), we get $0 = A + B + C \Rightarrow C = 0$.

Comparing the co-efficient of s^2 in (iii), we get $0 = A - B + D \Rightarrow D = -\frac{1}{2}$.

Therefore, from (*) $\frac{1}{(s-1)(s+1)(s^2+1)} = \frac{1}{4}.\frac{1}{s-1} - \frac{1}{4}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s^2+1}$. (iv)

Also $\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$.

Multiplying by $s(s+1)$, we get $1 = A(s+1) + Bs$.

Equating the coefficient of s, A + B = 0.

Equating constant term, $1 = A \Rightarrow B = -1$.

$$\therefore \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}. \qquad \text{(v)}$$

Therefore, from (i), and using (iv) and (v), we get

$$x = 2L^{-1}\left[\frac{1}{4}.\frac{1}{s-1} - \frac{1}{4}.\frac{1}{s+1} - \frac{1}{2}.\frac{1}{s^2+1}\right] + 3L^{-1}\left(\frac{s}{s^2+1}\right) + 2L^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$= 2\left[\frac{1}{4}e^t - \frac{1}{4}e^{-t} - \frac{1}{2}\sin t\right] + 3\cosh t + 2\left[1 - e^{-t}\right]$$

$$= \frac{1}{2}\left(e^t - e^{-t}\right) - \sin t + 3\cosh t + 2\left(1 - e^{-t}\right) \qquad \left[\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}\right]$$

$= \sinh t - \sin t + 3\cosh t + 2\left(1 - e^{-t}\right)$, is the required solution.

Q.No.28.: Constant voltage E is applied at t = 0 to a circuit with an inductance L, Capacitance C and Resistance R. Find the current I at time t, if the initial current and charge are zero.

Sol.: By Kirchoff's law, the differential equation satisfy the given condition is

$$L\frac{dI}{dt} + RI = E \text{ where } I(0) = 0$$

Taking Laplace Transform both sides, we get

$$L[sL(I) - I(0)] + RL(I) + L(E) = \frac{E}{s} \Rightarrow (Ls + R)\bar{I} = \frac{E}{s}, \quad \bar{I} = L(I)$$

$$\bar{I} = \frac{E}{s(Ls + R)}.$$

Consider $\dfrac{E}{s(Ls+R)} = \dfrac{A}{s} + \dfrac{B}{Ls+R}$. (i)

Multiplying by s(Ls + R), on both sides, we get $E = A(Ls + R) + Bs$.

Comparing the coefficient of s, $0 = AL + B$.

Comparing the constant terms, $E = AR \Rightarrow A = \dfrac{E}{R}$.

And $B = -AL = -\dfrac{LE}{R}$.

$\therefore$ From (i), $\bar{I} = \dfrac{E}{R}.\dfrac{I}{s} + \left(\dfrac{-LE}{R}\right)\dfrac{1}{Ls+R} = E\left[\dfrac{1}{R}.\dfrac{1}{s} - \dfrac{L}{R}.\dfrac{1}{Ls+R}\right]$.

Taking inverse Laplace transform, we get

$$I=\frac{E}{R}L^{-1}\left(\frac{1}{s}\right)-\frac{EL}{R}L^{-1}\left(\frac{1}{Ls+R}\right) \qquad \left[L(I)=\bar{I}\Rightarrow I=L^{-1}(\bar{I})\right]$$

$$\Rightarrow I=\frac{E}{R}-\frac{E}{R}L^{-1}\left[\frac{1}{s+\frac{R}{L}}\right]=\frac{E}{R}-\frac{E}{R}e^{-\frac{R}{L}t}=\frac{E}{R}\left[1-e^{-\frac{R}{L}t}\right]$$, is the required solution.

Q.No.29.: Find the solution of simultaneous differential equation by using Laplace Transform $\frac{dx}{dt}-2x+3y=0$, $\frac{dy}{dt}+2x-y=0$, given that x(0) = 8, y(0) = 3.

Sol.: Given $\frac{dx}{dt}-2x+3y=0$, (i)

$$\frac{dy}{dt}+2x-y=0. \qquad \text{(ii)}$$

Taking Laplace transform of (i) and (ii), we get

$s\bar{x}-x(0)-2\bar{x}+3\bar{y}=0$, $\bar{x}=L(x)$ and $s\bar{y}-y(0)+2\bar{x}-\bar{y}=0$, $\bar{y}=L(y)$.

$$\Rightarrow (s-2)\bar{x}+3\bar{y}=8 \qquad \text{(iii)}$$

and $(s-2)\bar{y}+2\bar{x}=3$ (iv)

To find $\bar{x}$ and $\bar{y}$, multiply (iii) by $(s-1)$ and (iv) by 3 and subtracting, we get

$[(s-1)(s-1)-6]\bar{x}=8(s-1)-9$.

Consider $\frac{8s-17}{(s+1)(s-4)}=\frac{A}{s+1}+\frac{B}{s-4}$.

Multiplying by $(s+1)(s-4)$, we get $8s-17=A(s-4)+B(s+1)$.

Equating the coefficient of s, 8 = A + B.

Equating the constant term, $-17=-4A+B$.

Solving, A = 5, B = 3

$$\therefore \bar{x}=\frac{5}{s+1}+\frac{3}{s-4}. \qquad \text{(v)}$$

To find $\bar{y}$, multiply (iii) by 2 and (iv) by $(s-2)$ and subtracting, we get

$[6-(s-2)(s-1)]\bar{y}=16-3(s-2)\Rightarrow \bar{y}=\frac{3s-22}{(s^2-3s-4)}=\frac{3s-22}{(s+1)(s-4)}$.

Consider $\dfrac{3s-22}{(s+1)(s-4)}=\dfrac{A}{s+1}+\dfrac{B}{s-4}$.

Multiplying by $(s+1)(s-4)$, we get $3s-22=A(s-4)+B(s+1)$.

Equating the coefficient of s and constant terms, we get

$3=A+B$ and $-22=-4A+B$. Solving, we get $A=5,\ \ B=-2$.

$$\therefore \bar{y}=\frac{5}{s+1}-\frac{2}{s-4}. \qquad \text{(vi)}$$

From (v) and (vI, taking Inverse Laplace transform, we get

$$x=L^{-1}\left[\frac{5}{s+1}+\frac{3}{s-4}\right]=5.L^{-1}\left(\frac{1}{s+1}\right)+3.L^{-1}\left(\frac{1}{s-4}\right)$$

$x=5e^{-t}+3e^{4t}$.

Also, $y=L^{-1}\left[\dfrac{5}{s+1}-\dfrac{2}{s-4}\right]=5e^{-t}-2e^{4t}$.

Hence $x=5e^{-t}+3e^{4t}$

$y=5e^{-t}-2e^{4t}$, is the required solution.

Q.No.30.: Find the solution of simultaneous differential equation by using Laplace transform $(D^2-D)y+z=0$, $(D-1)y+Dz=0$, given that for t = 0, y = 0, z = 1 and $Dy=0$.

Sol.: Given $(D^2-D)y+z=0$, (i)

$(D-1)y+Dz=0$. (ii)

Taking Laplace transform of (i) and (ii), we get

$s^2\bar{y}-sy(0)-y'(0)-(s\bar{y}-y(0))+\bar{z}=0$

and $s\bar{y}-y(0)-\bar{y}+s\bar{z}-z(0)=0$, $\quad \bar{y}=L(y),\ \ \bar{z}=L(z)$

$$\Rightarrow (s^2-s)\bar{y}+\bar{z}=3 \qquad \text{(iii)}$$

$$\text{and}\quad (s-1)\bar{y}+s\bar{x}=1. \qquad \text{(iv)}$$

Solving (iii) and (iv) for $\bar{y}$ and $\bar{z}$ we get, on multiplying (iii) by $(s-1)$ and (iv) by (s^2-s).

$(s-1)\bar{z}-s(s^2-s)\bar{z}=-s(s-1)$

$$\bar{z}\left(s-1-s^2(s-1)\right)=-s(s-1)$$

$$\bar{z}=-\frac{s(s-1)}{(s-1)\left(1-s^2\right)}=\frac{s}{s^2+1}. \qquad \text{(v)}$$

To find $\bar{y}$ multiply (iii) by s and subtract from (4), we get

$$\left[s\left(s^2-s\right)-(s-1)\right]\bar{y}=-1$$

$$\bar{y}=\frac{-1}{(s-1)\left(s^2-1\right)}=\frac{-1}{(s-1)^2(s+1)}. \qquad \text{(vi)}$$

Now consider $\frac{-1}{(s-1)^2(s+1)}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s+1}$. (*)

Multiplying (*) by $(s-1)^2(s+1)$, we get

$$\Rightarrow -1=A(s-1)(s+1)+B(s+1)+C(s-1)^2. \qquad \text{(vii)}$$

Put s = 1 in (vii), $-1=2B\Rightarrow B=-\frac{1}{2}$.

Put $s=-1$ in (vii), $-1=4C\Rightarrow C=-\frac{1}{4}$.

Comparing the coefficient of s^2, in (vii)

$$0=A+C\Rightarrow A=\frac{1}{4}.$$

Therefore, from (*), $\frac{-1}{(s-1)^2(s+1)}=\frac{1}{4}.\frac{1}{s-1}-\frac{1}{2}.\frac{1}{(s-1)^2}-\frac{1}{4}.\frac{1}{s+1}$.

and hence from (vi), $\bar{y}=\frac{1}{4}.\frac{1}{s-1}-\frac{1}{2}.\frac{1}{(s-1)^2}-\frac{1}{4}.\frac{1}{s+1}$. (viii)

Now, taking inverse Laplace Transform, of both sides, of equations (v) and (viii), we get

$$z=L^{-1}\left(\frac{s}{s^2-1}\right)=\cosh t \qquad \text{[from (v)]}$$

$$y=L^{-1}\left[\frac{1}{4}.\frac{1}{s-1}-\frac{1}{2}.\frac{1}{(s-1)^2}-\frac{1}{4}.\frac{1}{s+1}\right] \qquad \text{[from (viii)]}$$

$$\Rightarrow y=\frac{1}{4}L^{-1}\left(\frac{1}{s-1}\right)-\frac{1}{2}L^{-1}\frac{1}{(s-1)^2}-\frac{1}{4}L^{-1}\left(\frac{1}{s+1}\right)$$

$$\Rightarrow y=\frac{1}{4}e^{t}-\frac{1}{2}e^{-t}t-\frac{1}{4}e^{-t}=\frac{1}{4}\left(e^{t}-e^{-t}\right)-\frac{1}{2}te^{t}=\frac{1}{2}\left[\sinh t-te^{t}\right] \qquad \left[\sinh\theta=\frac{e^{\theta}-e^{-\theta}}{2}\right]$$

is the required solution.

Q.No.31.: Find the solution of the simultaneous differential equation by Laplace Transform $D^2x+y=-5\cos 2t$, $D^2y+x=5\cos 2t$, where $x(0)=x'(0)=1$, $y'(0)=-1$

Sol.: Given $D^2x+y=-5\cos 2t$ (i)

$D^2y+x=5\cos 2t$. (ii)

Taking Laplace Transform of (i) and (ii) of both sides, we get

$$s^2\bar{x}-sx(0)-x'(0)+\bar{y}=-\frac{5s}{s^2+4}, \qquad \bar{x}=L(x)$$

and $s^2\bar{y}-sy(0)-y'(0)+\bar{x}=\dfrac{5s}{s^2+4}$, $\qquad \bar{y}=L(y)$.

$$\Rightarrow s^2\bar{x}+\bar{y}=-\frac{5s}{s^2+4}+s+1 \tag{iii}$$

and $s^2\bar{y}+\bar{x}=\dfrac{5s}{s^2+4}-s+1$. (iv)

To find $\bar{x}$, multiply (iii) by s^2 and subtracting (ii), we get

$$\left(s^4-1\right)\bar{x}=-\frac{5s^3}{s^2+4}+s^3+s^2-\frac{5s}{s^2+4}+s-1=\frac{-5s\left(s^2+1\right)}{s^2+4}+s^3+s^2+s-1$$

$$\Rightarrow \bar{x}=\frac{-5s}{s^2+4\left(s^2+1\right)}+\frac{s\left(s^2+1\right)}{s^4-1}+\frac{s^2-1}{s^4-1}=\frac{-5s}{(s-1)(s+1)\left(s^2+4\right)}+\frac{s}{s^2-1}+\frac{1}{s^2-1}. \tag{v}$$

Consider $\dfrac{-5s}{(s-1)(s+1)\left(s^2+4\right)}=\dfrac{A}{s-1}+\dfrac{B}{s+1}+\dfrac{Cs+D}{s^2+4}$. (vi)

Multiplying (vi) by $(s-1)(s+1)\left(s^2+4\right)$, we get

$$-5s=A(s+1)(s^2=4)+B(s-1)(s^2+4)+(Cs+D)(s^2-1). \tag{vii}$$

Put s = 1, in (vii), $\quad -5=10A\Rightarrow A=-\dfrac{1}{2}$.

Put $s=-1$, in (vii), $5=-10B \Rightarrow B=-\frac{1}{2}$.

Comparing the coefficient of s^3, $0=A+B+C \Rightarrow C=1$.

Comparing the coefficient of s^2, $0=A+(-B)+D \Rightarrow D=0$.

Therefore from (v) and (vi), we get

$$\bar{x}=\frac{-1}{2}.\frac{1}{s-1}-\frac{1}{2}.\frac{1}{s+1}+\frac{s}{s^2+4}+\frac{s}{s^2-1}+\frac{1}{s^2+1}. \qquad \text{(vii-a)}$$

Operating $s^2 \times (iv)-(iii)$, we get

$$\left(s^4-1\right)\bar{y}=\frac{5s^3}{s^2+4}-s^3+s^2+\frac{5s}{s^2+4}-s-1=\frac{5s\left(s^2+1\right)}{s^2+4}-s^3+s^2-s-1$$

$$\bar{y}=\frac{5s(s^2+1)}{(s^2+4)(s^4-1)}-\frac{s\left(s^2+1\right)}{s^4-1}+\frac{s^2-1}{s^4-1}=\frac{5s}{(s^2+4)(s^2-1)}-\frac{s}{s^2-1}+\frac{1}{s^2+1}$$

$$=-\left[\frac{-5s}{(s^2+4)(s^2-1)}\right]-\frac{s}{s^2-1}+\frac{1}{s^2+1}=-\left[\frac{-1}{2}.\frac{1}{s-1}-\frac{1}{2}.\frac{1}{s+1}+\frac{s}{s^2+4}\right]-\frac{s}{s^2-1}+\frac{1}{s^2+1}$$

$$=\frac{1}{2}\frac{s+1+s-1}{(s-1)(s+1)}-\frac{s}{s^2+4}-\frac{s}{s^2-1}+\frac{1}{s^2+1} \quad \text{[Using (vi)]}$$

$$=\frac{1}{2}\frac{2s}{s^2-1}-\frac{s}{s^2+4}-\frac{s}{s^2-1}+\frac{1}{s^2+1}=\frac{-s}{s^2+4}+\frac{1}{s^2+1}. \qquad \text{(viii)}$$

Taking inverse Laplace transform of both sides of (vii-a) and (viii), we get

$$x=\frac{-1}{2}\left(e^t-e^{-t}\right)+\cos 2t+\cosh t+\sin t=\cos 2t+\sin t,$$

$$y=-\cos 2t+\sin t,$$

is the required solution.

Q.No.32.: Find the solution of the simultaneous differential equation by Laplace Transform

$$\frac{dx}{dt}-\frac{dy}{dt}-2x+2y=1-2t, \quad \frac{d^2y}{dt^2}+2\frac{dy}{dt}+x=0$$

subject to the conditions $x=0$, $y=0$, $\frac{dx}{dt}=0$, when $t=0$.

Sol.: Given $\frac{dx}{dt}-\frac{dy}{dt}-2x+2y=1-2t$ (i)

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+x=0. \qquad \text{(ii)}$$

Taking Laplace Transform of (i) and (ii), both sides, we get

$$\left[s\bar{x}-x(0)\right]-\left[s\bar{y}-y(0)\right]-2\bar{x}+2\bar{y}=\frac{1}{s}-\frac{2}{s^2}$$

and $\left[s^2\bar{x}-sx(0)-x'(0)\right]+\left[2s\bar{y}-y(0)\right]+\bar{x}=0$, $\qquad \bar{x}=L(x), \bar{y}=L(y)$

$$\Rightarrow (s-2)\bar{x}-(s-2)\bar{y}=\frac{1}{s}-\frac{2}{s^2} \qquad \text{(iii)}$$

and $\left(s^2+1\right)\bar{x}-2s\bar{y}=0$. (iv)

Multiply (iii) by 2s and (iv) by $(s-2)$ and adding, we get

$$\left(2s^2-4s+s^3-2s^2+s-2\right)\bar{x}=2-\frac{4}{s}$$

$$\Rightarrow \left(s^3-3s-2\right)\bar{x}=2-\frac{4}{s} \qquad [\text{s=2 satisfies, } s^3-3s-2=0]$$

$$\Rightarrow (s-2)\left(s^2+1+2s\right)\bar{x}=2-\frac{4}{s} \Rightarrow (s-2)(s+1)^2\bar{x}=2-\frac{4}{s}$$

$$\Rightarrow \bar{x}=\frac{2}{(s-2)(s+1)^2}-\frac{4}{s(s-2)(s+1)^2}=\frac{2s-4}{s(s-2)(s+1)^2}-\frac{2}{s(s+1)^2}. \qquad \text{(v)}$$

Consider $\dfrac{2}{s((s+1)^2}=\dfrac{A}{2}+\dfrac{B}{s+1}+\dfrac{C}{(s+1)^2}$.

Multiplying both sides, by $s(s+1)^2$, we get

$$\Rightarrow 2=A(s+1)^2+Bs(s+1)+Cs. \qquad \text{(vi)}$$

Put $s=0$ in (vi), $2=A \Rightarrow A=2$.

Put $s=-1$ in (vi), $2=-C \Rightarrow C=-2$.

Comparing the coefficient of s^2 in (vi), we get $0=A+B \Rightarrow B=-2$.

Hence $\dfrac{2}{s(s+1)^2}=\dfrac{2}{s}-\dfrac{2}{s+1}-\dfrac{2}{(s+1)^2}$

Also, from (v) $\bar{x}=\dfrac{2}{s}-\dfrac{2}{s+1}-\dfrac{2}{(s+1)^2}$. (vii)

Again multiplying (iii) by $\left(s^2+1\right)$ and (iv) by $(s-2)$ and subtracting, we get

$$\left[-(s-2)(s^2+1)-2s(s-2)\right]\bar{y}=\frac{s^2+1}{s}-\frac{2(s^2+1)}{s^2}=\frac{s(s^2+1)-2(s^2+1)}{s^2}$$

$$\Rightarrow\left[-(s-2)(s^2+1+2s)\right]\bar{y}=\frac{(s^2+1)(s-2)}{s^2}\Rightarrow\left[-(s-2)(s+1)^2\right]\bar{y}=\frac{(s^2+1)(s-2)}{s^2}$$

$$\Rightarrow\bar{y}=-\frac{s^2+1}{s^2(s+1)^2}. \tag{vii-a}$$

Consider $-\dfrac{s^2+1}{s^2(s+1)^2}=\dfrac{A}{s}+\dfrac{B}{s^2}+\dfrac{C}{s+1}+\dfrac{D}{(s+1)^2}$. (*)

Multiplying both sides, by $s(s+1)^2$, we get

$$\Rightarrow -(s^2+1)=As(s+1)^2+B(s+1)^2+Cs^2(s+1)+Ds^2. \tag{vii-b}$$

Put s = 0 in (vii-b), $-1=B\Rightarrow B=-1$.

Put $s=-1$ in (vii-b), $-2=D\Rightarrow D=-2$.

Comparing the coefficient of s^2, in (vii-b), $-1=B+D+2A+C$.

Comparing the coefficient of s, in (vii-b), $\Rightarrow A=2$.

0 = A + 2B

$C=-1-B-D-2A=-1+1+2-4=-2\Rightarrow c=-2$.

Hence, from (*) $-\dfrac{s^2+1}{s^2(s+1)^2}=\dfrac{2}{s}-\dfrac{1}{s^2}-\dfrac{2}{s+1}-\dfrac{2}{(s+1)^2}$. (viii)

From (vii) and (vii) taking inverse Laplace, both sides

From (vii),

$$\bar{x}=L^{-1}\left[\frac{2}{s}-\frac{2}{s+1}-\frac{2}{(s+1)^2}\right]=2L^{-1}\left(\frac{1}{s}\right)-L^{-1}\left(\frac{1}{s+1}\right)-2L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$=2t-2e^{-t}-2e^{-t}L^{-1}\left(\frac{1}{s^2}\right)=2t-2e^{-t}-2e^{-t}.\frac{t}{1!}=2\left(t-e^{-t}-te^{-t}\right)$$

From (viii),

$$\bar{x}=L^{-1}\left[\frac{2}{s}-\frac{1}{s^2}-\frac{2}{s+1}-\frac{2}{(s+1)^2}\right]=2L^{-1}\left(\frac{1}{s}\right)-L^{-1}\left(\frac{1}{s^2}\right)-2L^{-1}\left(\frac{1}{s+1}\right)-2L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$=2-\frac{t}{1!}-2e^{-t}-2e^{-t}L^{-1}\left(\frac{1}{s^2}\right)=2-t-2e^{-t}-2e^{-t}.t.$$

Q.No.33.: The co-ordinates (x, y) of a particle moving along a plane curve, at any time t are given by $\frac{dy}{dt} + 2x = \sin 2t$, $\frac{dx}{dt} - 2y = \cos 2t$, $t > 0$. If at t = 0, x = 1 and $y = 0$. Show by using transform, that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.

Sol.: Given $\frac{dy}{dt} + 2x = \sin 2t$, (i)

$$\frac{dx}{dt} - 2y = \cos 2t . \quad \text{(ii)}$$

Taking Laplace Transform of both sides of (i) and (ii), we get

$$L\left(\frac{dy}{dt}\right) + 2L(x) = L(\sin 2t) \text{ and } L\left(\frac{dx}{dt}\right) - 2L(y) = L(\cos 2t)$$

$$\Rightarrow s\bar{y} - y(0) + 2\bar{x} = \frac{2}{s^2+4} \text{ and } s\bar{y} - x(0) - 2\bar{y} = \frac{2}{s^2+4}, \text{ where, } \bar{y} = L(y), \bar{x} = L(x).$$

$$\Rightarrow s\bar{y} + 2\bar{x} = \frac{2}{s^2+4} \quad \text{(iii)}$$

$$\text{and } s\bar{x} - 2\bar{y} = \frac{s}{s^2+4} + 1. \quad \text{(iv)}$$

Multiplying (iii) by s and (iv) by 2 and subtracting, we get

$$\left(s^2+4\right)\bar{y} = \frac{2s}{s^2+4} - \frac{2s}{s^2+4} - 2 = -2 \Rightarrow \bar{y} = \frac{-2}{s^2+4}. \quad \text{(v)}$$

Again multiply (iii) by (ii) and (iv) by s and adding, we get

$$\left(4+s^2\right)\bar{x} = \frac{4}{s^2+4} + \frac{s^2}{s^2+4} + s = \frac{4+s^2}{s^2+4} + s = 1 + s \Rightarrow \bar{x} = \frac{1+s}{s^2+4} = \frac{1}{s^2+4} + \frac{s}{s^2+4}. \quad \text{(vi)}$$

From (v) and (vi), taking inverse Laplace transform of both sides, we get

$$y = L^{-1}\left(-\frac{2}{s^2+4}\right) = -2L^{-1}\left(\frac{1}{s^2+4}\right) = -2.\frac{1}{2}\sin 2t = -\sin 2t .$$

$$\text{Also } x = L^{-1}\left(\frac{1}{s^2+4} + \frac{s}{s^2+4}\right) = L^{-1}\left(\frac{1}{s^2+4}\right) + L^{-1}\left(\frac{s}{s^2+4}\right) = \frac{1}{2}\sin 2t + \cos 2t .$$

Now $4x^2 + 4xy + 5y^2 = 4\left[\frac{1}{4}\sin^2 2t + \cos^2 2t + \sin 2t \cos 2t\right] + 4\left[\frac{1}{2}\sin 2t + \cos 2t\right](-\sin 2t) + 5\sin^2 2t$

$$= \sin^2 2t + 4\cos^2 2t + 4\sin 2t\cos 2t + \left(-2\sin^2 2t\right) - 4\sin 2t\cos 2t + 5\sin^2 2t = 4.$$

Q.No.34.: The currents i_1 and i_2 in mesh are given by differential equation $\frac{di_1}{dt} - wi_2 = a\cos pt$, $\frac{di_2}{dt} - wi_1 = a\sin pt$. Find the currents i_1 and i_2 by Laplace Transform if $i_1 = i_2 = 0$ at $t = 0$.

Sol.: Given $\frac{di_1}{dt} - wi_2 = a\cos pt$, (i)

$$\frac{di_2}{dt} - wi_1 = a\sin pt. \quad \text{(ii)}$$

Taking Laplace Transform of (i) and (ii), we get

$$s\overline{i_1} - i_1(0) - w\overline{i_2} = \frac{as}{s^2+p^2} \Rightarrow s\overline{i_1} - w\overline{i_2} = \frac{as}{s^2+p^2} \quad \text{(iii)}$$

and $s\overline{i_2} - i_2(0) + w\overline{i_1} = \frac{ap}{s^2+p^2} \Rightarrow w\overline{i_1} + s\overline{i_2} = \frac{ap}{s^2+p^2}$. (iv)

Operating $(iii)\times s + (ii)\times s$, we get $\left(s^2+w^2\right)\overline{i_1} = \frac{as^2}{s^2+p^2} + \frac{awp}{s^2+p^2}$

$$\Rightarrow \overline{i_1} = \frac{as^2}{\left(s^2+p^2\right)\left(s^2+w^2\right)} + \frac{awp}{\left(s^2+p^2\right)\left(s^2+w^2\right)}. \quad \text{(v)}$$

Operating $(iii)\times w - (ii)\times s$, we get

$$\left(-w^2-s^2\right)\overline{i_2} = \frac{asw}{s^2+p^2} - \frac{aps}{s^2+p^2} \Rightarrow \overline{i_2} = \frac{as(p-w)}{\left(s^2+p^2\right)\left(s^2+w^2\right)}. \quad \text{(vi)}$$

Taking Inverse Laplace Transform of (v) and (vi), we get

$$\Rightarrow i_1 = aL^{-1}\left[\frac{s^2}{\left(s^2+p^2\right)\left(s^2+w^2\right)}\right] + awpL^{-1}\left[\frac{1}{\left(s^2+p^2\right)\left(s^2+w^2\right)}\right]. \quad \text{(A)}$$

Consider $L^{-1}\left(\frac{s^2}{\left(s^2+p^2\right)\left(s^2+w^2\right)}\right) = L^{-1}\left(\frac{s}{s^2+p^2}\cdot\frac{s}{s^2+w^2}\right) = L^{-1}\left(\overline{f}(s).\overline{g}(s)\right)$, (*)

where $\overline{f}(s) = \frac{s}{s^2+p^2}$, $\overline{g}(s) = \frac{s}{s^2+w^2}$.

Now $L^{-1}(\overline{f}(s)) = L^{-1}\left(\frac{s}{s^2+p^2}\right) = \cos pt = f(t)$.

$$L^{-1}(\bar{g}(s)) = L^{-1}\left(\frac{s}{s^2+w^2}\right) = \cos wt = g(t).$$

Therefore, from (*), using Convolution Theorem, we get

$$L^{-1}\left(\frac{s^2}{(s^2+p^2)(s^2+w^2)}\right) = \int_0^t \cos pu.\cos w(t-u)du = \frac{1}{2}\int_0^t 2\cos pu \cos w(t-u)du$$

$$= \frac{1}{2}\int_0^t [\cos(pu+wt-wu)+\cos(pu-wt+wu)]du$$

$$= \frac{1}{2}\left[\frac{\sin(pu+wt-wu)}{p-w}+\frac{\sin(pu-wt+wu)}{p+w}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin pt}{p-w}+\frac{\sin pt}{p+w}\right]-\frac{1}{2}\left[\frac{\sin pt}{p-w}+\frac{\sin(-wt)}{p+w}\right] = \frac{\sin pt}{2}\left(\frac{p+w+p-w}{p^2-w^2}\right)-\frac{1}{2}\sin wt\left(\frac{p+w-p+w}{p^2-w^2}\right)$$

$$= \frac{p\sin pt - w\sin wt}{p^2-w^2}. \tag{B}$$

Further $\left(\frac{1}{(s^2+p^2)(s^2+w^2)}\right) = L^{-1}\left(\frac{1}{(s^2+p^2)}\cdot\frac{1}{(s^2+w^2)}\right) = L^{-1}(\bar{f}(s).\bar{g}(s))$, (**)

where $\bar{f}(s) = \frac{1}{s^2+p^2}$, $\bar{g}(s) = \frac{1}{s^2+w^2}$.

Also $L^{-1}(\bar{f}(s)) = L^{-1}\left(\frac{1}{s^2+p^2}\right) = \frac{1}{p}\sin pt = f(t)$

$$L^{-1}(\bar{g}(s)) = L^{-1}\left(\frac{1}{s^2+w^2}\right) = \frac{1}{w}\sin wt = g(t).$$

Therefore, from (**), using Convolution Theorem, we get

$$L^{-1}\left(\frac{1}{(s^2+p^2)(s^2+w^2)}\right) = \int_0^t \frac{1}{p}\sin pu.\frac{1}{w}\sin w(t-u)du = \frac{1}{2pw}\int_0^t 2\sin pu \sin w(t-u)du$$

$$= \frac{1}{2pw}\int_0^t [\cos(pu-wt+wu)-\cos(pu+wt-wu)]du$$

$$= \frac{1}{2pw}\left[\frac{\sin(pu-wt+wu)}{p+w}-\frac{\sin(pu+wt-pu)}{p-w}\right]_0^t$$

$$=\frac{1}{2pw}\left[\frac{\sin pt}{p+w}-\frac{\sin pt}{p-w}\right]-\frac{1}{2pw}\left[\frac{\sin(-wt)}{p+w}-\frac{\sin(wt)}{p-w}\right]$$

$$=\frac{1}{2pw}\sin pt\left(\frac{p-w-p-w}{p^2-w^2}\right)-\frac{1}{2pw}\sin wt\left(\frac{p-w+p+w}{p^2-w^2}\right)$$

$$=\frac{\sin pt}{p(w^2-p^2)}+\frac{\sin wt}{p(w^2-p^2)}. \tag{C}$$

Using (B) and (C) in (A), we get

$$\Rightarrow i_1=a\left(\frac{p\sin pt-w\sin wt}{p^2-w^2}\right)+\frac{aw}{w^2-p^2}\sin pt+\frac{ap}{p^2-w^2}\sin wt$$

$$\Rightarrow i_1=\frac{a\sin pt}{w^2-p^2}(w-p)+\frac{a}{p^2-w^2}(p-w)\sin wt=\frac{a}{w+p}(\sin pt+\sin wt).$$

To find i_2 from (vi), taking inverse Laplace Transform, we have

$$i_2=a(p-w)L^{-1}\left[\frac{s}{(s^2+p^2)(s^2+w^2)}\right]. \tag{D}$$

Consider $\frac{s}{(s^2+p^2)(s^2+w^2)}=\frac{As+B}{s^2+p^2}+\frac{Cs+D}{s^2+w^2}$.

Multiplying by $(s^2+p^2)(s^2+w^2)$, we get

$$s=(As+B)(s^2+w^2)+(Cs+D)(s^2+p^2).$$

Equating the coefficient of s^3, $0=A+C$. (a)

Equating the coefficient of s^2, $0=B+D$. (b)

Equating the coefficient of s, $1=Aw^2+Cp^2$. (c)

Equating the constant term, $0=Bw^2+Dp^2$. (d)

On solving, (a) gives $A=-C$.

(c) gives $Aw^2-Ap^2=1\Rightarrow A=\frac{1}{w^2-p^2}$

$$C=\frac{1}{p^2w^2}.$$

Also (b) give $B=-D$.

Also (d) gives $Bw^2-Dp^2=0\Rightarrow B=0$ and $D=0$.

$$\therefore \frac{s}{\left(s^2+p^2\right)\left(s^2+w^2\right)} = \frac{1}{w^2-p^2}\left[\frac{s}{s^2+p^2} - \frac{s}{s^2+w^2}\right].$$

Hence, from equation (D), we have

$$i_2 = \frac{a(p-w)}{(w+p)(w-p)} L^{-1}\left[\frac{s}{\left(s^2+p^2\right)} - \frac{s}{\left(s^2+w^2\right)}\right] = -\frac{a}{(w+p)} L^{-1}\left[\frac{s}{\left(s^2+p^2\right)} - \frac{s}{\left(s^2+w^2\right)}\right]$$

$$i_2 = -\frac{a}{w+p}(\cos pt - \cos wt).$$

Q.No.35.: A mechanical system with two degrees of freedom satisfies the equation $2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4$, $2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0$. Use Laplace transform to determine x and y at any instant given that x, y, $\frac{dx}{dt}$, $\frac{dy}{dt}$ vanish at t = 0.

Sol.: Taking Laplace transform on both sides, we get

$$L\left[2\frac{d^2x}{dt^2} + 3\frac{dy}{dt}\right] = L(4) \text{ and } L\left[2\frac{d^2y}{dt^2} - 3\frac{dx}{dt}\right] = L(0)$$

$$\Rightarrow 2\left[s^2\bar{x} - sx(0) - x'(0)\right] + 3\left[s\bar{y} - y(0)\right] = \frac{4}{s} \text{ and } 2\left[s^2\bar{y} - sy(0) - y'(0)\right] - 3\left[s\bar{x} - x(0)\right] = 0$$

$$\Rightarrow 2s^2\bar{x} + 3s\bar{y} = \frac{4}{s} \tag{i}$$

and $-3s\bar{x} + 2s^2\bar{y} = 0$. (ii)

Operating $(i)\times 2s - (ii)\times(iii)$, we get

$$\left(4s^3 + 9s\right)\bar{x} = 8 \Rightarrow \bar{x} = \frac{8}{s\left(4s^2+9\right)}. \tag{*}$$

Now consider $\frac{8}{s\left(4s^2+9\right)} = \frac{A}{s} + \frac{Bs+C}{4s^2+9}$. (**)

Multiplying by $s\left(4s^2+9\right)$ both sides, we get

$$8 = A\left(4s^2+9\right) + (Bs+C)s. \tag{iii}$$

Put s = 0 in (iii), $8 = 9A \Rightarrow A = \frac{8}{9}$.

Compare the coefficient of s^2 in (ii), $0 = 4A + B \Rightarrow B = -\frac{32}{9}$.

Compare the coefficient of s in (ii), $0 = C \Rightarrow C = 0$.

Therefore from (**), we get $\frac{8}{s(4s^2+9)} = \frac{8}{9}.\frac{1}{s} - \frac{32}{9}\frac{s}{4s^2+9}$

and hence, from (*) ,

$$\bar{x} = \frac{8}{9}.\frac{1}{s} - \frac{32}{9}.\frac{s}{4s^2+9} = \frac{8}{9}\left[\frac{1}{s} - \frac{s}{s^2+\frac{9}{4}}\right] \qquad \text{(iv)}$$

Again Operating $(i)\times 3 + (ii)\times 2s$, we get

$$(9s+4s^3)\bar{y} = \frac{12}{s} \Rightarrow \bar{y} = \frac{12}{s^2(4s^2+9)}. \qquad \text{(iv-a)}$$

Now consider $\frac{12}{s^2(4s^2+9)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{4s^2+9}$. (***)

Multiplying by $s^2(4s^2+9)$ both sides, we get

$$12 = As(4s^2+9) + B(4s^2+9) + (Cs+D)s^2 \qquad \text{(v)}$$

Put s = 0, in (v), $12 = 9B \Rightarrow B = \frac{4}{3}$.

Compare the coefficient of s^3 in (v), $0 = 4A + C$.

Compare the coefficient of s^2 in (v), $0 = 4B + D$.

Compare the coefficient of s in (v), $0 = 9A \Rightarrow A = 0$ and $C = 0$.

Also $D = -4B = -\frac{16}{3}$.

Therefore, using (***), we get

$$\frac{12}{s^2(4s^2+9)} = \frac{4}{3}.\frac{1}{s^2} - \frac{16}{3}\frac{1}{4s^2+9}$$

and from iv-a, $\bar{y} = \frac{4}{3}.\frac{1}{s^2} - \frac{16}{3}.\frac{1}{4s^2+9} = \frac{4}{3}\left[\frac{1}{s^2} - \frac{1}{s^2+\frac{9}{4}}\right]$. (vi)

Now taking inverse Laplace transform of (iv) and (vi), we get

$$x = \frac{8}{9}\left[L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{s}{s^2 + \frac{9}{4}}\right)\right] = \frac{8}{9}\left(1 - \cos\frac{3}{2}t\right)$$

$$y = \frac{4}{3}\left[L^{-1}\left(\frac{1}{s^2}\right) - L^{-1}\left(\frac{s}{s^2 + \frac{9}{4}}\right)\right] = \frac{4}{3}\left(t - \frac{2}{3}\sin\frac{3}{2}t\right)$$, are the required solution.

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Home Assignments

Solve the following equations by the Laplace Transform method:

Q.No.1.: Solve the equations by transforms method:

$$\frac{d^2y}{dx^2} + \omega^2 t = 0, \text{ where } y(0) = A, \left(\frac{dy}{dx}\right)_{x=0} = B.$$

Ans.: $y = A\cos\omega x + \frac{B}{\omega}\sin\omega x$ **.**

Q.No.2.: Solve the equations by transforms method:

$$\frac{dx}{dt} + x = \sin\omega t, \quad x(0) = 2.$$

Ans.: $x = \left(2 + \frac{\omega}{\omega^2 + 1}\right)e^{-t} + \frac{\sin\omega t - \omega\cos\omega t}{\omega^2 + 1}$

Q.No.3.: Solve the equations by transforms method:

$$y'' + 4y' + 3y = e^{-t}, \quad y(0) = y'(0) = 1.$$

Ans.: $y = \frac{7}{4}e^{-t} - \frac{3}{4}e^{-3t} - \frac{1}{2}te^{-t}$

Q.No.4.: Solve the equations by transforms method:

$$(D^2 - 1)x = a\cosh t, \quad x(0) = x'(0) = 0.$$

Ans.: $x = \frac{at}{2}\sinh t$.

Q.No.5.: Solve the equations by transforms method:

$$y'' + y = t,\ y(0) = 1,\ y'(0) = -2$$

Ans.: $y = t - 3\sin t + \cos t$.

Q.No.6.: Solve the equations by transforms method:

$$\left(D^2 + \omega^2\right)y = \cos \omega t,\ t > 0, \text{ given that } y = 0 \text{ and } Dy = 0 \text{ at } t = 0.$$

Ans.: $y = \frac{1}{2\omega}\sin \omega t$.

Q.No.7.: Solve the equations by transforms method:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t,\quad y = \frac{dy}{dt} = 0 \text{ when } t = 0$$

Ans.: $y = \frac{1}{8}e^t - \frac{1}{40}e^{-3t} - \frac{1}{10}(2\sin t + \cos t)$.

Q.No.8.: Solve the equations by transforms method:

$$\frac{d^4y}{dt^4} - k^4 y = 0, \text{ where } y(0) = 1,\ y'(0) = y''(0) = y'''(0) = 0.$$

Ans.: $y = \frac{1}{2}(\cos kt + \cosh kt)$.

Q.No.9.: Solve the equations by transforms method:

$$y''''(x) + 2y''(t) + y(t) = \sin t, \text{ when } y(0) = y'(0) = y''(0) = y'''(0) = 0$$

Ans.: $y = \frac{1}{8}\left(3 - t^2\right)\sin t - 3t\cos t$.

Q.No.10.: Solve the equations by transforms method:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^t \sin t, \text{ where } y(0) = 1,\ y'(0) = 1$$

Ans.: $y = \frac{11}{3}e^{-t}(\sin t + \sin 2t)$.

Q.No.11.: Solve the equations by transforms method:

$$\left(D^2 + 1\right)x = t\cos 2t,\qquad x = Dx = 0 \text{ at } t = 0.$$

Ans.: $x = \frac{4}{9}\sin 2t - \frac{5}{9}\sin t - \frac{1}{3}t\cos 2t$

Q.No.12.: Solve the equations by transforms method:

$$ty'' + 2y' + ty = \sin t, \text{ when } y(0) = 1.$$

Ans.: $y = \frac{1}{2}\left(\frac{3\sin t}{t} - \cos t\right)$

Q.No.13.: Solve the equations by transforms method:

$$ty'' + (1-2t)y' - 2y = 0, \text{ when } y(0) = 1, \quad y'(0) = 2.$$

Ans.: $y = e^{2t}$.

Q.No.14.: Solve the equations by transforms method:

$$\frac{d^2x}{dt^2} - t\frac{dx}{dt} + x = 1, \quad x(0) = 1, \quad x'(0) = 2.$$

Ans.: $x = 1 + 2t$.

Q.No.15.: Show by Laplace transforms the differential equation for the current I in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force $E\sin\omega t$ satisfies the equation

$$L\frac{di}{dt + Ri} = E\sin\omega t.$$

Ans.: $I = \frac{E\sin(\omega t - \phi)}{\sqrt{(R^2 + \omega^2 L^2)}} + \frac{e\sin\phi}{\sqrt{(R^2 + \omega^2 L^2)}}.e^{-Rt/L}$

Q.No.16.: Obtain the equation for the forced oscillation of a mass m attached to the lower end of an elastic spring whose upper end is fixed and whose stiffness is k, when the driving force is $F_0 \sin at$. Solve this equation (using the Laplace transforms)when $a^2 \neq k/m$, given that initial velocity and displacement (from equilibrium position) are zero.

Ans.: $(n\sin at - a\sin nt)\frac{F_0}{mn(n^2 - a^2)}$, where $n^2 = k/m$.

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6th Topic

Laplace Transforms

Laplace Transformation of some other useful functions
Unit Step Function,
Laplace Transformation of Unit Step Function, Second Shifting Theorem
Unit Impulse Function
Laplace Transformation of Unit Impulse function

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7th Topic

Laplace Transforms

Laplace Transformation of some other useful Functions
Unit Step Function,
Second Shifting Theorem
Unit Impulse Function

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Unit Step Function (or Heaviside's Unit Function):

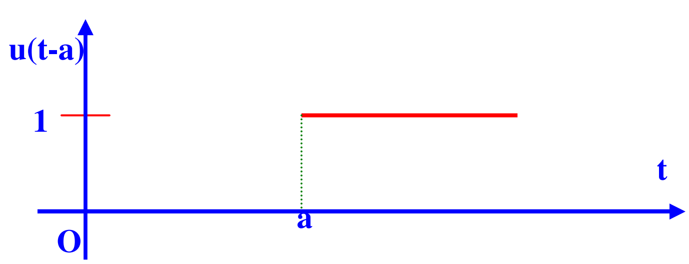
In engineering, we come across such functions of which the inverse transform cannot be determined from the formulae so far derived. In order to cover such cases, we introduce the **unit step function** (or **Heaviside's unit function***).

*** Named after the British Electrical Engineer Oliver Heaviside (1850-1925).**

Definition:

The unit step function u(t – a) is defined as

$$u(t-a) = \begin{cases} 0, & \text{for } t < a \\ 1, & \text{for } t \geq a \end{cases} \quad \text{where } a \geq 0.$$



Particular case:

As a particular case (when a = 0)

$$u(t)=\begin{cases}0, & \text{for } t<0\\ 1, & \text{for } t\geq 0\end{cases}$$

The product $f(t).u(t-a)=\begin{cases}0, & \text{for } t<a\\ f(t), & \text{for } t\geq a\end{cases}$

The function $f(t).u(t-a)$ represents the graph of f(t) shifted through a distance 'a' to the right.

Laplace Transform of Unit Step Function:

Since $u(t-a)=\begin{cases}0, & \text{for } t<a\\ 1, & \text{for } t\geq a\end{cases}$.

$$L\{u(t-a\}=\int_0^\infty e^{-st}u(t-a)\,dt=\int_0^a e^{-st}.0dt+\int_a^\infty e^{-st}.1dt=0+\left[\frac{e^{-st}}{-s}\right]_a^\infty=\frac{1}{s}e^{-as}.$$

In particular, $L\{u(t)\}=\frac{1}{s}$.

Second Shifting Theorem (Second Shifting Property):

Statement: If $L\{f(t)\}=\bar{f}(s)$, then $L\{f(t-a).u(t-a)\}=e^{-as}\bar{f}(s)$.

Proof: $L\{f(t-a).u(t-a)\}=\int_0^\infty e^{-st}f(t-a).u(t-a)dt$

$$=\int_0^a e^{-st}f(t-a)(0)\,dt+\int_a^\infty e^{-st}f(t-a)(1)\,dt$$

$$=\int_0^\infty e^{-s(u+a)}f(u)du \qquad \text{(Put } t-a=u\text{)}$$

$$=e^{-as}\int_0^\infty e^{-su}f(u)du=e^{-as}\bar{f}(s).$$

Important results:

1. $L^{-1}\{e^{-as}\bar{f}(s)\}=f(t-a).u(t-a)$.

2. If $a=0$, $L\{f(t).u(t)\}=\bar{f}(s)=L\{f(t)\}$.

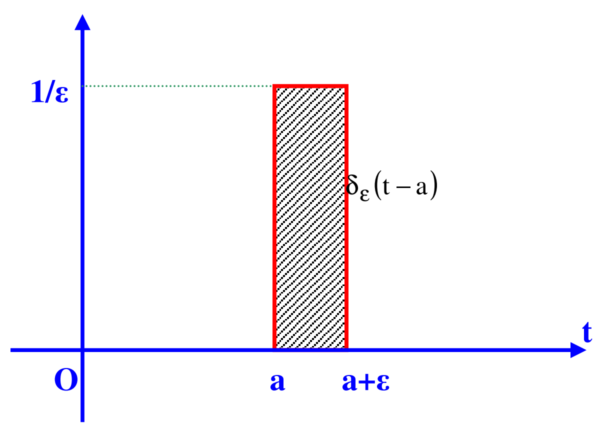
Unit Impulse Function (or Dirac-delta Function*):

In mechanics, we come across problems where a very large force acts for a very small time. In the study of bending of beams, we have point loads, which is equivalent to large pressure acting over a very small area. To deal with such problems, we introduce the unit impulse function.

*** After the English Physicist Paul Dirac (1902-1984), who was awarded Nobel Prize in 1933 for his work in Quantum Mechanics)**

Thus, Unit Impulse Function is considered as the limiting form of the function

$$\delta_\varepsilon(t-a)=\begin{cases}0, & \text{for } t<a\\ \dfrac{1}{\varepsilon}, & \text{for } a\le t\le a+\varepsilon\\ 0, & \text{for } t>a\end{cases}$$



- It is clear from the figure that as $\varepsilon\to 0$, the height of the strip increases indefinitely and the width decreases in such a way that its area is always unity.

  Integrating this function, we get

  $$\int_0^\infty \delta_\varepsilon(t-a)\,dt=\int_a^{a+\varepsilon}\frac{1}{\varepsilon}dt=\frac{1}{\varepsilon}(a+\varepsilon-a)=1.$$

- If $\delta_\varepsilon(t-a)$ represents a force acting for short duration ε at time t = a, then the integral

  $$\underset{\varepsilon\to 0}{\mathrm{Lt}}\int_a^{a+\varepsilon}\delta_\varepsilon(t-a)\,dt=1,$$

  represents unit impulse at t = a.

Hence, the limiting form of $\delta_\varepsilon(t-a)$ as $\varepsilon \to 0$ is expressed as unit impulse function denoted by $\delta(t-a)$.

Definition:

Thus, the Unit Impulse Function $\delta(t-a)$ is defined as follows:

$$\delta(t-a) = \begin{cases} \infty, & \text{for } t = a \\ 0, & \text{for } t \neq a \end{cases}$$

such that $\int_0^\infty \delta(t-a)dt = 1$. $(a \geq 0)$.

Laplace Transform of Unit Impulse Function:

Since, we know $\delta(t-a) = \begin{cases} \infty, & \text{for } t = a \\ 0, & \text{for } t \neq a \end{cases}$

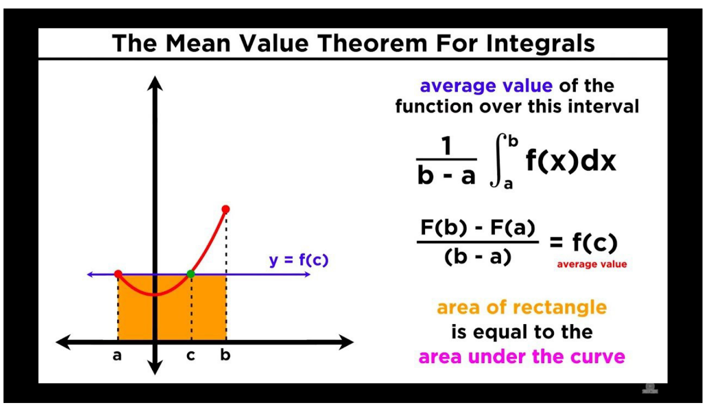
If f(t) be a function of t continuous at t = a, then

$$\int_0^\infty f(t)\delta_\varepsilon(t-a)dt = \int_a^{a+\varepsilon} f(t).\frac{1}{\varepsilon}dt = (a+\varepsilon-a)f(c).\frac{1}{\varepsilon} = f(c) \qquad \text{where } a < c < a+\varepsilon$$

(by mean value theorem for integrals)

If f is continuous on [a, b], then there exist a number c in [a, b] such that

$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx \Rightarrow \int_a^b f(x)dx = f(c).(b-a)$$



As $\varepsilon \to 0$, we get $\int_0^\infty f(t)\delta(t-a)dt = f(a)$.

Important results:

1. $L[\delta(t-a)] = \int_0^\infty e^{-st}\delta(t-a)dt = e^{-sa}$.

2. $L\{\delta(t)\} = e^{-s} = e^0 = 1$.

Now let find Laplace Transforms of Unit Step Functions:

Q.No.1.: Find the Laplace transforms of

(i). $(t-1)^2 u(t-1)$ **(ii).** $\sin tu(t-\pi)$ **(iii).** $e^{-3t}u(t-2)$.

Sol.: (i). Given function is $(t-1)^2 u(t-1)$.

Step No. 01: Comparing given expression with f(t – a).u(t – a).

Comparing $(t-1)^2 u(t-1)$ with $f(t-a)u(t-a)$, we have

$a = 1$ and $f(t) = t^2$.

Step No. 02: Laplace Transform of f(t):

$\therefore \bar{f}(s) = L\{f(t)\} = \frac{2}{s^3}$. $\left[\because L(t^n) = \frac{n!}{s^{n+1}}, \text{when } n = 0, 1, 2, 3, \ldots\ldots\right]$.

Step No. 03: Laplace Transform of given function by using Second Shifting Property:

[By second shifting property: If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(t-a).u(t-a)\} = e^{-as}\bar{f}(s)$.]

$$L\{(t-1)^2 u(t-1)\} = e^{-s}\bar{f}(s) = \frac{2e^{-s}}{s^3}.$$

(ii). Given function is $\sin t\ u(t-\pi)$.

Expressing $\sin t$ as a function of $(t-\pi)$, we have

$$\sin t = \sin[(t-\pi)+\pi] = -\sin(t-\pi)$$

Step No. 01: Comparing given expression with f(t – a).u(t – a).

Comparing $-\sin(t-\pi).u(t-\pi))$ with $f(t-a)u(t-a)$, we get

$a = \pi$ and $f(t) = -\sin t$.

Step No. 02: Laplace Transform of f(t):

$\therefore \bar{f}(s) = L\{f(t)\} = -\frac{1}{s^2+1}$ $\left[\because L(\sin at) = \frac{a}{s^2+a^2}\right]$

Step No. 03: Laplace Transform of given function by using Second Shifting Property:

[By second shifting property: If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(t-a).u(t-a)\} = e^{-as}\bar{f}(s)$.]

$$\therefore L\{\sin t\, u(t-\pi)\} = L\{-\sin(t-\pi)u(t-\pi)\}$$

$$= e^{-\pi s} f(s) \qquad \text{[By second shifting property]}$$

$$= -\frac{e^{-\pi s}}{s^2+1}.$$

(iii). Given function is $e^{-3t}u(t-2)$

Expressing e^{-3t} as a function of $(t-a)$, we have

$$e^{-3t} = e^{-3[(t-2)+2]} = e^{-6}.e^{-3(t-2)}$$

$$\therefore e^{-3t}u(t-2) = e^{-6}.e^{-3(t-2)}.u(t-2)$$

Step No. 01: Comparing given expression with f(t – a).u(t – a).

Comparing $e^{-3(t-2)}.u(t-2)$ with $f(t-a)u(t-a)$, we get

$a = 2$ and $f(t) = e^{-3t}$

Step No. 02: Laplace Transform of f(t):

$$\therefore \bar{f}(s) = L\{f(t)\} = \frac{1}{s+3}.$$

Step No. 03: Laplace Transform of given function by using Second Shifting Property:

[By second shifting property: If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(t-a).u(t-a)\} = e^{-as}\bar{f}(s)$.]

$$\therefore L\{e^{-3t}u(t-2)\} = e^{-6}L\{e^{-(3t-2)}u(t-2)\}$$

$$= e^{-6}.e^{-2s}\bar{f}(s) \quad .$$

$$= \frac{e^{-2(s+3)}}{s+3}$$

Q.No.2.: Find the Laplace transform of:

(i). $f(t)=\begin{cases} t-1 & (1<t<2) \\ 3-t & (2<t<3 \end{cases}$ **(ii).** $e^{-t}[1-u(t-2)]$.

Sol.: **(i).** $f(t)=(t-1)[u(t-1)-u(t-2)]+(3-t)[u(t-2)-u(t-3)]$

$$=(t-1)u(t-1)-2(t-2)u(t-2)+(t-3)u(t-3).$$

Since $L\{f(t-a)u(t-a)\}=e^{-as}\bar{f}(s)$

$$\therefore L[f(t)]=e^{-s}.\frac{1}{s^2}-2e^{-2s}.\frac{1}{s^2}+e^{-3s}.\frac{1}{s^2}=\frac{\left(e^{-s}-2e^{-2s}+e^{-3s}\right)}{s^2} \qquad [\because f(t)=t]$$

(ii). $L\{e^{-t}[1-u(t-2)]\}=L\left(e^{-t}\right)-L\left(e^{-t}u(t-2)\right)=\dfrac{1}{s+1}-e^{-2}L\left[e^{-(t-2)}u(t-2)\right]$

Taking $f(t)=e^{-t}$, $\bar{f}(s)=\dfrac{1}{s+1}$ and using the equation.

$L\{f(t-a)u(t-a)\}=e^{-as}\bar{f}(s)$.

$L\left\{e^{-(t-2)}u(t-2)\right\}=e^{-2s}.\dfrac{1}{s+1}$.

Hence, $Le^{-t}\{1-u(t-2)\}=\dfrac{\left\{1-e^{-2(s+1)}\right\}}{(s+1)}$.

Q.No.3.: Find the Laplace transform of

(i) $e^{t-2}u(t-2)$, **(ii)** $t^2u(t-3)$, **(iii)** $\sin 2tu(t-\pi)$, **(iv)** $t^{2t}u(t-3)$.

Sol.: **(i)** Compare $e^{t-2}u(t-2)$ with $f(t-a)u(t-a)$

Here $a=2$, $f(t-2)=e^{t-2}\Rightarrow f(t)=e^t$.

Now by second shifting property,

we know $L[f(t-a)u(t-a)]=e^{-as}\bar{f}(s)$, where $L(f(t))=\bar{f}(s)$.

Hence, $L\left(e^{t-2}u(t-2)\right)=e^{-2s}.\dfrac{1}{s-1}$. $\left[\text{Here } \bar{f}(s)=L(f(t))=L\left(e^t\right)=\dfrac{1}{s-1}\right]$

(ii). We first express t^2 in terms of $t-3$

Now $t^2=(t-3+3)^2=(t-3)^2+9+6(t-3)=(t-3)^2+6(t-3)+9$

Therefore $t^2u(t-3)=\left[(t-3)^2+6(t-3)+9\right]u(t-3)$

$$= (t-3)^2 u(t-3) + 6(t-3)u(t-3) + 9u(t-3)$$

$$\Rightarrow L\left[t^2 u(t-3)\right] = L\left[(t-3)^2 u(t-3\right] + 6[L(t-3)u(t-3)] + 9L[u(t-3)]. \qquad \text{(i)}$$

Now consider $L\left[(t-3)^2 u(t-3\right]$.

Here $a = 3$, $f(t-3) = (t-3)^2$, $f(t) = t^2$ and $\bar{f}(s) = Lf(t) = \frac{2}{s^2}$

Therefore $L\left[(t-3)^2 u(t-3)\right] = e^{-3s}.\frac{2}{s^3}$ [Using 2nd shifting theorem]

Also $L[(t-3)u(t-3)] = e^{-3s}.\frac{1}{s^2}$ and $L[u(t-3)] = e^{-3s}.\frac{1}{s}$.

Hence from (i), $L\left[t^2 u(t-3)\right] = e^{-3s}.\frac{2}{s^3} + 6e^{-3s}.\frac{1}{s^2} + 9e^{-3s}.\frac{1}{s} = \frac{e^{-3s}}{s^3}\left[2 + 6s + 9s^2\right]$.

(iii). We express $\sin 2t$ in terms of $t-\pi$

Here $\sin 2t = \sin(t-\pi+\pi) = \sin(2(t-\pi)+2\pi) = \sin 2(t-\pi)$.

Now $L[\sin 2tu(t-\pi)] = L[\sin 2(t-a)u(t-a)]$.

Compare $\sin 2(t-a)u(t-a)$ with $f(t-a)u(t-a)$.

Here $a = \pi$, $f(t-\pi) = \sin 2(t-\pi)$

$$\Rightarrow f(t) = \sin 2t \Rightarrow Lf(t) = \frac{2}{s^2+4} = \bar{f}(s)$$

Using 2nd shifting theorem,

$$L[\sin 2tu(t-\pi)] = L[\sin 2(t-\pi)u(t-\pi)] = e^{-\pi s}.\frac{2}{s^2+4}.$$

(iv) $L\left[e^{2t}u(t-3)\right] = L\left[e^{2(t-3+3)}u(t-3)\right]$

Compare $e^{2(t-3)}u(t-3)$ with $f(t-a)u(t-a)$ (i)

Here $a = 3$, $f(t-3) = e^{2(t-3)} \Rightarrow f(t) = e^{2t}$

Also $\bar{f}(s) = L(f(t) = L\left(e^{2t}\right) = \frac{1}{s-2}$.

From (i), using 2nd shifting theorem,

$$L\left[e^{2t}u(t-3)\right] = e^6 L\left[e^{2(t-3)}u(t-3)\right] = e^6.e^{-3s}.\frac{1}{s-2} = e^{-3(s-2)}.\frac{1}{s-2}.$$

Q.No.4.: Find the inverse Laplace transform of:

(i). $\dfrac{e^{-2s}}{s-3}$ **(ii).** $\dfrac{se^{-as}}{s^2-w^2}$, $a>0$ **(iii).** $\dfrac{se^{-s/2}+\pi e^{-8}}{s^2+\pi^2}$.

Sol.: By second shifting theorem, we have

$$L^{-1}\left\{e^{as}\bar{f}(s)\right\}=f(t-a).u(t-a).$$

(i). Given function is $\dfrac{e^{-2s}}{s-3}$

Let $\bar{f}(s)=\dfrac{1}{s-3}$, then $f(t)=e^{3t}$.

Remember: $L^{-1}\left\{e^{-as}\bar{f}(s)\right\}=f(t-a).u(t-a)$.

$$\therefore L^{-1}\left\{\frac{e^{-2s}}{s-3}\right\}=L^{-1}\left\{e^{-2s}\bar{f}(s)\right\}=f(t-2).u(t-2)=e^{3(t-2)}.u(t-2).$$

(ii). Given function is $\dfrac{se^{-as}}{s^2-w^2}$, $a>0$

Let $\bar{f}(s)=\dfrac{s}{s^2-w^2}$, then $f(t)=\cosh w$

Remember: $L^{-1}\left\{e^{-as}\bar{f}(s)\right\}=f(t-a).u(t-a)$.

$$\therefore L^{-1}\left\{\frac{se^{-as}}{s^2-w^2}\right\}=L^{-1}\left\{e^{-as}\bar{f}(s)\right\}=f(t-a).u(t-a)=\cosh w(t-a).u(t-a).$$

(iii). Given function is $\dfrac{se^{-s/2}+\pi e^{-8}}{s^2+\pi^2}$.

Since $L^{-1}\left\{\dfrac{s}{s^2+\pi^2}\right\}=\cos\pi t$ and $L^{-1}\left\{\dfrac{\pi}{s^2+\pi^2}\right\}=\sin\pi t$

$$\therefore L^{-1}\left\{\frac{se^{-s/2}+\pi e^{-8}}{s^2+\pi^2}\right\}=L^{-1}\left\{\frac{se^{-s/2}}{s^2+\pi^2}\right\}+L^{-1}\left\{\frac{\pi e^{-8}}{s^2+\pi^2}\right\}$$

Remember: $L^{-1}\left\{e^{-as}\bar{f}(s)\right\}=f(t-a).u(t-a)$.

$$=\cos\pi\left(t-\frac{1}{2}\right).u\left(t-\frac{1}{2}\right)+\sin\pi(t-1).u(t-1)$$

$$= \cos\left(\pi t - \frac{\pi}{2}\right).u\left(t - \frac{1}{2}\right) + \sin(\pi t - \pi).u(t-1)$$

$$= \sin \pi t.u\left(t - \frac{1}{2}\right) - \sin \pi t.u(t-1) = \sin \pi t\left[u\left(t - \frac{1}{2}\right) - u(t-1)\right].$$

Q.No.5.: Find inverse Laplace transform of: $\frac{e^{-cs}}{s^2(s+a)} (c > 0).$

Sol.: Given function is $\frac{e^{-cs}}{s^2(s+a)} (c > 0).$

$$L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = L^{-1}\left\{e^{-cs}\left(\frac{A}{s} + \frac{B}{s^2} + \frac{1}{a^2}.\frac{1}{(s+a)}\right)\right\}$$

$$= L^{-1}\left\{e^{-cs}\left(\left(-\frac{1}{a^2}\right).\frac{1}{s} + \left(\frac{1}{a}\right).\frac{1}{s^2} + \frac{1}{a^2}.\frac{1}{(s+a)}\right)\right\}$$

Remember: $L^{-1}\left\{e^{-as}\bar{f}(s)\right\} = f(t-a).u(t-a).$

$$\therefore L^{-1}\left[\frac{e^{-cs}}{s^2(s+a)}\right] = -\frac{1}{a^2}[1.u(t-c)] + \frac{1}{a}[(t-c).u(t-c)] + \frac{1}{a^2}\left[e^{-a(t-c)}.u(t-c)\right]$$

$$\left[\because L^{-1}\left(\frac{1}{s}\right) = 1,\ L(t^n) = \frac{n!}{s^{n+1}}, \text{when } n = 0, 1, 2, 3, \text{........ and } L^{-1}\left(\frac{1}{s-a}\right) = e^{at}.\right]$$

$$= \frac{1}{a^2}\left\{a(t-c) - 1 + e^{-a(t-c)}\right\}u(t-c).$$

Q.No.6.: Find the Inverse Laplace's transform of $\frac{e^{-s}}{\sqrt{s+1}}$.

Sol.: Let $F(s) = \frac{1}{\sqrt{s+1}}$ so that

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s+1)^{1/2}}\right\} = e^{-t}L^{-1}\left\{\frac{1}{s^{1/2}}\right\} = e^{-t}.\frac{t^{\frac{-1}{2}}}{\sqrt{\pi}}.$$

Using t-shift

$$L^{-1}\left\{\frac{e^{-s}}{\sqrt{s+1}}\right\} = L^{-1}\left\{e^{-s}.F(s)\right\} = f(t-1).u(t-1) = \left\{e^{-(t-1)}.\frac{(t-1)^{\frac{-1}{2}}}{\sqrt{\pi}}\right\}u(t-1)$$

$$=\begin{cases} e^{-(t-1)}.\dfrac{(t-1)^{\frac{-1}{2}}}{\pi}, & t>1 \\ 0, & 0<t<1 \end{cases}$$

Q.No.7.: Find the Inverse Laplace's transform of $\dfrac{\left(5-3e^{-3s}-2e^{-7s}\right)}{s}$.

Sol.: $L^{-1}\left\{\dfrac{\left(5-3e^{-3s}-2e^{-7s}\right)}{s}\right\}=5L^{-1}\left\{\dfrac{1}{s}\right\}-3L^{-1}\left\{e^{-3s}\dfrac{1}{s}\right\}-2L^{-1}\left\{e^{-7s}.\dfrac{1}{s}\right\}$

Applying the second shifting theorem

$$=5.1-3.u(t-3)-2u(t-7)$$

Since $L^{-1}\left\{\dfrac{1}{s}\right\}=1$.

Also $u(t-3)=0 \qquad 0<t<3$

$=1 \qquad t>3$

and $u(t-7)=0 \qquad 0<t<7$

$=1 \qquad t>7$

Therefore

$$L^{-1}\left\{\frac{\left(5-3e^{-3s}-2e^{-7s}\right)}{s}\right\}=\begin{cases} 5-3.0-2.0, & 0<t<3 \\ 5-3.1-2.0, & 3<t<7 \\ 5-3.1-2.1, & t>7 \end{cases}=\begin{cases} 5, & 0<t<3 \\ 2, & 3<t<7 \\ 0, & t>7 \end{cases}.$$

Q.No.8.: Find the Inverse Laplace's transform of $\dfrac{(2+5s)}{\left(s^2e^{4s}\right)}$.

Sol.: Take $F(s)=\dfrac{2+5s}{s^2}=\dfrac{2}{s^2}+\dfrac{5}{s}$. Then

$$f(t)=L^{-1}\{F(s)\}=L^{-1}\left\{\frac{2}{s^2}+\frac{5}{s}\right\}=2L^{-1}\left\{\frac{1}{s^2}\right\}+5L^{-1}\left\{\frac{1}{s}\right\}=2t+5.$$

Using result $L^{-1}\left\{e^{-as}F(s)\right\}=f(t-a)u(t-a)$.

We get (with $a=4$)

$$L^{-1}\left\{e^{-4s}\left(\frac{2}{s^2}+\frac{5}{s}\right)\right\}=f(t-4).u(t-4)=\begin{cases} 0, & 0<t<4 \\ f(t-4), & t>4 \end{cases}$$

Since $f(t) = 2t + 5$; so $f(t-4) = 2(t-4) + 5 = 2t - 3$

$$\therefore L^{-1}\left\{e^{-4s}\left(\frac{2}{s^2} + \frac{5}{s}\right)\right\} = \begin{cases} 0, & 0 < t < 4 \\ 2t - 3, & t > 4 \end{cases}$$

Q.No.9.: Find the Inverse Laplace's transform of $\dfrac{1}{s^2 - e^{-as}}$.

Sol.: $$\frac{1}{s^2 - e^{-as}} = \frac{1}{s^2}\left[1 - \frac{e^{as}}{s^2}\right]^{-1} = \frac{1}{s^2}\sum_{n=0}^{\infty}\left(\frac{e^{-as}}{s^2}\right)^n = \sum_{n=0}^{\infty}\frac{e^{-ans}}{s^{2n+2}} = \sum_{n=0}^{\infty}\frac{e^{-ans}}{s^{(2n+1)+1}}$$

Since $L^{-1}\left\{\dfrac{1}{s^{(2n+1)+1}}\right\} = \dfrac{t^{2n+1}}{(2n+1)!}$.

$$L^{-1}\left\{\frac{1}{s^2 - e^{-as}}\right\} = L^{-1}\left\{\sum_{n=0}^{\infty}\frac{e^{-ans}}{s^{2n+2}}\right\} = \sum_{n=0}^{\infty} L^{-1}\left\{\frac{e^{-ans}}{s^{2n+2}}\right\} = \sum_{n=0}^{\infty}\frac{(t-an)^{2n+1}}{(2n+1)!}u(t-an).$$

Q.No.10.: Find the Inverse Laplace transform of

(i) $\dfrac{e^{-\pi s}}{s^2+1}$ (ii) $\dfrac{e^{-\pi s}}{s^2}$ (iii) $\dfrac{se^{-2s}}{s^2-1}$ (iv) $\dfrac{e^{-cs}}{s^2(s+a)}$, $c > 0$.

Sol.: (i) By second shifting theorem, we know

$$L^{-1}(e^{-as}\bar{f}(s)) = f(t-a)u(t-a)$$

Comparing, $e^{-\pi s}.\dfrac{1}{s^2+1}$ with $e^{-as}.\bar{f}(s)$

Here $a = \pi$, $\bar{f}(s) = \dfrac{1}{s^2+1}$

$\therefore f(t) = L^{-1}\left(\bar{f}(s)\right) = \sin t$

Hence $$L^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right) = L^{-1}\left(e^{-\pi s}.\frac{1}{s^2+1}\right) = \sin(t-\pi)u(t-\pi)$$

$$= -\sin t(t-\pi) \qquad [\sin(\pi - t) = \sin t, \sin(-t) = -\sin t]$$

(ii) $L^{-1}\left(e^{-\pi s}\dfrac{1}{s^2}\right)$

Comparing $e^{-\pi s}\dfrac{1}{s^2}$ with $e^{-as}.\bar{f}(s)$, we have

Here $a=\pi$, $\bar{f}(s)=\dfrac{1}{s^2}$ and $f(t)=L^{-1}\left(\bar{f}(s)\right)=L^{-1}\left(\dfrac{1}{s^2}\right)=t$

Therefore by second shifting theorem, we know

$$L^{-1}\left(e^{-\pi s}\frac{1}{s^2}\right)=(t-\pi).u(t-\pi). \qquad \left[L^{-1}\left(e^{-as}.\bar{f}(s)\right)=f(t-a)u(t-a)\right]$$

(iii). $L^{-1}\left[\dfrac{se^{-2s}}{s^2-1}\right]=L^{-1}\left(se^{-2s}.\dfrac{1}{s^2-1}\right)=\cosh(t-2)\mu(t-2)$

Comparing with $L^{-1}\left(e^{-as}.\bar{f}(s)\right)$, we have

Here $a=2$, $\bar{f}(s)=\dfrac{s}{s^2-1}$, $f(t)=L^{-1}(\bar{f}(s))=\cosh t$

$$\left[L^{-1}\left(e^{-as}.\bar{f}(s)\right)=f(t-a)u(t-a)\right]$$

(iv) $e^{-cs}\dfrac{1}{s^2(s+a)}$,

Comparing with $L^{-1}\left(e^{-as}.\bar{f}(s)\right)$, we have

Here $a=c$, $\bar{f}(s)=\dfrac{1}{s^2(s-a)}$,

Now $f(t)=L^{-1}(\bar{f}(s))=L^{-1}\left[\dfrac{1}{s^2(s+a)}\right]$. (i)

Consider $\dfrac{1}{s^2(s+a)}=\dfrac{A}{s}+\dfrac{B}{s^2}+\dfrac{C}{s+a}$. (*)

Multiply by $s^2(s+a)$ both sides,

$\Rightarrow 1=As(s+a)+B(s+a)+Cs^2$. (ii)

Put $s=0$ in (ii), $1=Ba\Rightarrow B=\dfrac{1}{a}$.

Put $s=-a$ in (ii), $1=ca^2\Rightarrow c=\dfrac{1}{a^2}$.

Comparing the coefficient of s^2 in equation (ii), we get

$0=A+C\Rightarrow A=-\dfrac{1}{a^2}$.

Hence from (*), $\dfrac{1}{s^2(s+a)} = -\dfrac{1}{a^2}.\dfrac{1}{s} + \dfrac{1}{a}.\dfrac{1}{s} + \dfrac{1}{a^2}.\dfrac{1}{s+a}$.

Therefore from (i), using (*), we get

$$f(t) = L^{-1}\left[\frac{-1}{a^2}.\frac{1}{s} + \frac{1}{a}.\frac{1}{s^2} + \frac{1}{a^2}.\frac{1}{s+a}\right] = \frac{-1}{a^2}L^{-1}\left(\frac{1}{s}\right) + \frac{1}{a}L^{-1}\left(\frac{1}{s^2}\right) + \frac{1}{a^2}L^{-1}\left(\frac{1}{s+a}\right)$$

$$= \frac{-1}{a^2} + \frac{1}{a}t + \frac{1}{a^2}e^{-at}$$

$$L^{-1}\left[e^{-cs}\frac{1}{s^2(s+a)}\right] = \left[\frac{-1}{a^2} + \frac{1}{a}(t-c) + \frac{1}{a^2}e^{-a(t-c)}\right]u(t-c)$$

$$\left[L^{-1}\left(e^{-as}.\bar{f}(s)\right) = f(t-a)u(t-a)\right]$$

Q.No.11.: In an electric circuit with e.m.f. E(t), resistance R and inductance L, the current I builds up at the rate given by $L\dfrac{di}{dt} + Ri = E(t)$.

If the switch is connected at t = 0 and disconnected at t = a, find the current i at any instant.

Sol.: Here i = 0 at t = 0 and $E(t) = \begin{cases} E & \text{for } 0 < t < a \\ 0 & \text{for } t > a \end{cases}$

Taking Laplace transform of both sides of the given equation, we have

$$L\left[s\bar{i} - i(0)\right] + R\bar{i} = \int_0^{\pi} e^{st}E(t)$$

$$\Rightarrow (Ls + R)\bar{i} = \int_0^a e^{-st}Edt = E\left[\frac{e^{-st}}{-s}\right]_0^a = \frac{E}{s}\left(1 - e^{-as}\right)$$

$$\Rightarrow \bar{i} = \frac{E}{s(Ls+R)} - \frac{Ee^{-as}}{s(Ls+R)}.$$

Taking inverse Laplace transform of both sides, we have

$$\bar{i} = L^{-1}\left\{\frac{E}{s(Ls+R)}\right\} - L^{-1}\left\{\frac{Ee^{-as}}{s(Ls+R)}\right\}. \qquad \text{(i)}$$

Now $L^{-1}\left\{\dfrac{E}{Rs(Ls+R)}\right\} = L^{-1}\left\{\dfrac{E}{Rs} - \dfrac{LE}{R(Ls+R)}\right\} = L^{-1}\left\{\dfrac{E}{Rs} - \dfrac{E}{R\left(s+\dfrac{R}{L}\right)}\right\}$

$$= \frac{E}{R}\left[L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left\{\frac{1}{s+\frac{R}{L}}\right\}\right] = \frac{E}{R}\left(1-e^{-Rt/L}\right).$$

and $L^{-1}\left\{\dfrac{Ee^{-as}}{s(Ls+R)}\right\} = \dfrac{E}{R}\left(1-e^{-R(t-a)/L}\right)u(t-a).$

$\therefore$ From (i), we have

$$i = \frac{E}{R}\left(1-e^{-Rt/L}\right) - \frac{E}{R}\left(1-e^{-R(t-a)/L}\right)u(t-a).$$

Hence, for $0 < t < a$, $i = \dfrac{E}{R}\left(1-e^{-Rt/L}\right)$,

and for $t > a$, $i = \dfrac{E}{R}\left(1-e^{-Rt/L}\right) - \left(1-e^{-R(t-a)/L}\right) = \dfrac{E}{R}e^{-Rt/L}\left(e^{Ra/L}-1\right).$

Q.No.12.: Calculate the maximum deflection of an encastre beam 1 ft. long carrying a uniformly distributed load $w\ell b$/ft. on its central half length.

Sol.: Taking the origin at the end A, we have

$$EI\frac{d^4y}{dx^4} = w)x), \quad \text{where } w(x) = w\left[u\left(x-\frac{\ell}{4}\right) - u\left(x-\frac{3\ell}{4}\right)\right]$$

Taking the Laplace transform of both sides, we get

$$EI\left[s^4\bar{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\right]$$

$$= w\left(\frac{e^{-\ell s/4}}{s} - \frac{e^{-3\ell s/4}}{s}\right).$$

Using the conditions $y(0) = y'(0) = 0$ and taking $y''(0) = c_1$ and $y'''(0) = c_2$, we have

$$EI\bar{y} = w\left\{\frac{e^{-\ell s/4}}{s^5} - \frac{e^{-3\ell s/4}}{s^5}\right\} + \frac{c_1}{s^3} + \frac{c_2}{s^4}.$$

On inversion, we get

$$EIy = \frac{w}{24}\left[\left(x-\frac{\ell}{4}\right)^4 u\left(x-\frac{\ell}{4}\right)^4 - \left(x-\frac{3\ell}{4}\right)^4 u\left(x-\frac{3\ell}{4}\right)\right] + \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3. \qquad \text{(i)}$$

For $x > \frac{3\ell}{4}$, $\qquad EIy = \frac{w}{24}\left[\left(x-\frac{\ell}{4}\right)^4 - \left(x-\frac{3\ell}{4}\right)^4\right] + \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3$

and $EIy' = \frac{w}{6}\left[\left(x-\frac{\ell}{4}\right)^3 - \left(x-\frac{3\ell}{4}\right)^3\right] + c_1x + \frac{1}{2}c_2x^2$.

Using the conditions $y(\ell) = 0$ and $y'(\ell) = 0$, we get

$$0 = \frac{w}{24}\left\{\left(\frac{3\ell}{4}\right)^4 - \left(\frac{\ell}{4}\right)^4\right\} + \frac{1}{2}c_1\ell^2 + \frac{1}{6}c_2\ell^3$$

and $\quad 0 = \frac{w}{6}\left\{\left(\frac{3\ell}{4}\right)^3 - \left(\frac{\ell}{4}\right)^3\right\} + c_1\ell + \frac{1}{2}c_2\ell^2$.

Hence $c_1 = \frac{11w\ell^2}{192}$; $\qquad c_2 = -\frac{w\ell}{4}$.

Thus for $\frac{\ell}{4} < x < \frac{3\ell}{4}$, (i) gives $EIy = \frac{w}{24}\left(x+\frac{1}{4}\right)^4 + \frac{11w\ell^2}{384}x^2 - \frac{w\ell}{24}x^3$.

Hence the maximum deflection $y\left(\frac{\ell}{2}\right) = \frac{13w\ell^4}{6144EI}$.

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Home Assignments

Q.No.1. Evaluate **(i).** $L\{e^{t-1}u(t-1)\}$ **(ii).** $L\{(t-1)^2u(t-1)\}$

(iii). $L\{t^2u(t-3)\}$ **(iv).** $L\{t^2u(t-1)+\delta(t-1)\}$.

Sol.: (i). $\frac{e^{-s}}{(s-1)}$ **(ii).** $\frac{2e^{-s}}{s^3}$ **(iii).** $\frac{e^{-3s}(2+6s+9s)^2}{s^3}$ **(iv).** $\frac{e^{-s}\left(2+2s+s^2+s^3\right)}{s^3}$.

Q.No.2. Find the inverse Laplace transforms of:

(i). $\dfrac{se^{-as}}{s^2 - w^2}$, **a > 0.** **(ii).** $\dfrac{e^{-s}}{(s+1)^3}$

Sol.: (i). $\cosh w(t-a).u(t-a)$ **(ii).** $\dfrac{1}{2}e^{-(t-1)}(t-1)^2 u(t-1)$.

Q.No.3.: Using t-shift, find the inverse Laplace transform of $\dfrac{(s+1)e^{-\pi s}}{s^2+s+1}$.

Ans.: $\dfrac{e^{-\frac{1}{2}(t-\pi)}}{\sqrt{3}}\left\{\sqrt{3}\cos\dfrac{\sqrt{3}}{2}(t-\pi)+\sin\dfrac{\sqrt{3}}{2}(t-\pi)\right\}\times u(t-\pi)$.

Q.No.4.: Using t-shift, find the inverse Laplace transform of $\dfrac{\left(e^{-4s}-e^{-7s}\right)}{s^2}$.

Ans.: $\begin{cases} 0, & 0 < t < 4 \\ t-4, & 4 < t < 7 \\ 3, & t > 7 \end{cases}$.

Q.No.5.: Using t-shift, find the inverse Laplace transform of $\dfrac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}$.

Ans.: $\dfrac{4(t-3)^{3/2}e^{-4(t-4)}}{3\sqrt{\pi}}u(t-3)$.

Q.No.6.: Using t-shift, find the inverse Laplace transform of $\dfrac{e^{-3s}}{s^2-2s+5}$.

Ans.: $\dfrac{1}{2}e^{(t-3)}\sin 2(t-3).u(t-3)$

Q.No.7.: Using t-shift, find the inverse Laplace transform of $\dfrac{e^{-s}}{(s+1)^3}$.

Ans.: $\dfrac{1}{2}.e^{-(t-1)}.(t-1)^2.u(t-1)$.

Q.No.8.: Using t-shift, find the inverse Laplace transform of $\dfrac{se^{\frac{-1}{2}} + \pi e^{-s}}{s^2+\pi^2}$.

Ans.: $\sin\pi t\left\{u\left(t-\dfrac{1}{2}\right)-u(t-1)\right\}$.

Q.No.9.: Using t-shift, find the inverse Laplace transform of $\frac{s}{s^2-5s+6}e^{-2s}$.

Ans.: $\left\{-2e^{2(t-2)}+3e^{3(t-2)}\right\}u(t-2)$.

Q.No.10.: Using t-shift, find the inverse Laplace transform of $\frac{3}{s}-\frac{4e^{-s}}{s^2}+\frac{4e^{-3s}}{s^2}$.

Ans.: $3-4(t-1)u(t-1)+4(t-3)u(t-3)$.

Q.No.11.: $\frac{e^{-3s}}{(s+1)^3}$, Evaluate f(2), f(5), f(7).

Ans.: $f(2)=0$, $f(5)=2e^{-2}$, $f(7)=8e^{-4}$.

Q.No.12.: Using t-shift, find the inverse Laplace transform of $\frac{e^{-3s}}{(s-4)^2}$.

Ans.: $(t-3)e^{4(t-3)}u(t-3)$.

Q.No.13.: Using t-shift, find the inverse Laplace transform of $\frac{se^{-\pi s}}{s^2+9}$.

Ans.: $-\cos 3t.u(t-\pi)$.

Q.No.14.: Using t-shift, find the inverse Laplace transform of $\frac{1}{s+e^{-s}}$.

Ans.: $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(t-n)^n u(t-n)$.

Q.No.15. Sketch the graph of the following functions and express them in terms of unit step function. Hence find their Laplace transforms:

(i). $f(t)=2t$ for $0<t<\pi$, $f(t)=1$ for $t=\pi$.

(ii). $f(t)=t^2$ for $0<t\le 2$, $f(t)=0$ for $t>2$.

(iii). $f(t)=\cos(wt+\phi)$ for $0<t<T$, $F(t)=0$ for $t>T$.

Sol.: (i). $(1-2t)u(t-\pi)+2tu(t)$; $\frac{2}{s^2}+\left(\frac{1-2\pi}{s}-\frac{2}{s^2}\right)e^{-as}$.

(ii). $t^2[u(t)-u(t-2)]$; $\frac{2\left(I-e^{-2s}\right)}{s^3}-\frac{4e^{-2s}(1+s)}{s^2}$

(iii). $\{u(t)-u(t-T\}\cos(\omega t+\phi)$;

$$\left[(s\cos\phi-\omega\sin\phi)-e^{-sT}\times\{s\cos(\phi+\omega T)-\omega\sin(\phi+\omega T)\}\right]/\left(s^2+\omega^2\right)$$

Q.No.16. Using Laplace transforms, solve $x''(t) + x(t) = u(t)$, $x(0) = 1$, $x'(0 = 0)$,

$$\text{where} \quad u(t) = \begin{cases} 3, & 0 \le t \le 4 \\ 2t - 5, & t > 4 \end{cases}$$

Sol.: $x = 3 - 2\cos t + 2\{t - 4 - \sin(t-4)\}u(t-4)$.

Now let find Laplace Transforms of Unit Impulse Functions:

Q.No.1.: A beam is simply supported at its end x = 0 and is clamped at the other end $x = \ell$. It carries a load 'w' at $x = \frac{\ell}{4}$. Find the resulting deflection at any point.

Sol.: Since we know that for a uniformly loaded beam, the intensity of loading at any point $P(x,y) = EI\frac{d^4y}{dx^4}$

Step No. 01: Write the differential equation for deflection.

The differential equation for deflection is $\frac{d^4y}{dx^4} = \frac{w}{EI}\delta\left(x - \frac{\ell}{4}\right)$.

Step No. 02: Laplace Transform of the differential equation.

Taking the Laplace transform, we have

$$s^4\bar{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = \frac{w}{EI}e^{-\ell s/4}.$$

Remember: $L[\delta(t-a)] = \int_0^\infty e^{-st}\delta(t-a)dt = e^{-sa}$.

Step No. 03: Find the expression for $\bar{y} = \bar{f}(s)$ using initial conditions:

Using the conditions y(0) = 0, $y''(0) = 0$ and taking $y'(0) = c_1$, $y'''(0) = c_2$, we get

$$\bar{y} = \frac{c_1}{s^2} + \frac{c_2}{s^4} + \frac{w}{EI}\frac{e^{-\ell s/4}}{s^4}.$$

Step No. 04: Find the Inverse Laplace Transform for evaluating y:

Taking the Inverse Laplace transform of both sides, we get

$$y = c_1 x + c_2 \frac{x^3}{3!} + \frac{w}{EI} \frac{\left(x - \frac{\ell}{4}\right)^3}{3!} u\left(x - \frac{\ell}{4}\right)$$

$$\left.\begin{array}{lll} \text{i.e.} & y = c_1 x + \frac{1}{6} c_2 x^3, & 0 < x < \frac{\ell}{4} \\ \text{and} & y = c_1 x + \frac{1}{6} c_2 x^3 + \frac{w}{6EI}\left(x - \frac{\ell}{4}\right)^3, & \frac{\ell}{4} < x < \ell \end{array}\right\}. \qquad \text{(i)}$$

Step No. 05: Find the values of constants c_1 and c_2:

Using the conditions $y(\ell) = 0$, $y'(\ell) = 0$, we get

$$0 = c_1 \ell + \frac{1}{6} c_2 \ell^3 + \frac{w}{6EI}\left(\frac{27\ell^3}{64}\right) \Rightarrow 0 = c_1 \ell + \frac{1}{6} c_2 \ell^3 + \frac{9w\ell^3}{128EI} \qquad \text{(ii)}$$

Since $y' = c_1 + \frac{1}{6} c_2 3x^2 + \frac{w}{6EI} 3\left(x - \frac{\ell}{4}\right)^2 = c_1 + \frac{1}{2} c_2 x^2 + \frac{w}{2EI}\left(x - \frac{\ell}{4}\right)^2$

Now using the conditions $y'(\ell) = 0$, we get $0 = c_1 + \frac{1}{2} c_2 \ell^2 + \frac{9w\ell^2}{32EI}$ (iii)

Multiplying (iii) by ℓ and subtracting from (ii), we get $c_2 = -\frac{81w}{128EI}$.

Substituting the value of $c_2 = -\frac{81w}{128EI}$ in (iii), we get $c_1 = \frac{9w\ell^2}{256EI}$

Thus $c_1 = \frac{9w\ell^2}{256EI}$, $c_2 = -\frac{81w}{128EI}$

Step No. 06: Write the deflection at any point:

Substituting the values of c_1 and c_2 in (i), we get the deflection at any point.

$$\left.\begin{array}{lll} \text{i.e.} & y = \left(\frac{9w\ell^2}{256EI}\right)x - \left(\frac{27w}{256EI}\right)x^3, & 0 < x < \frac{\ell}{4} \\ \text{and} & y = \left(\frac{9w\ell^2}{256EI}\right)x - \left(\frac{27w}{256EI}\right)x^3 + \frac{w}{6EI}\left(x - \frac{\ell}{4}\right)^3, & \frac{\ell}{4} < x < \ell \end{array}\right\}$$

Q.No.2.: Obtain the deflection of a weightless beam of length ℓ and freely supported at ends, when a concentrated load W acts at x = a. The differential equation for deflection being $EI\frac{d^4y}{dx^4} = W\delta(x - a)$.

Sol.: Taking the Laplace transform of the differential equation, we get

$$s^4\bar{y} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{W}{EI} e^{-as}.$$

Using the conditions $y(0) = y''(0) = 0$ and taking $y'(0) = c_1$, $y'''(0) = c_2$, we get

$$s^4\bar{y} - c_1 s^2 - c_2 = \frac{W}{EI}.e^{-as} \Rightarrow \bar{y} = \frac{c_1}{s^2} + \frac{c_2}{s^4} + \frac{W}{EI}.\frac{e^{-as}}{s^4}.$$

Taking inverse transforms, we have

$$y = c_1 x + c_2.\frac{x^3}{3!} + \frac{W}{EI}.\frac{(x-a)^3}{3!} u(x-a) \tag{i}$$

$$\therefore y' = c_1 + \frac{1}{2} c_2 x^2 + \frac{W}{2EI}(x-a)^2 u(x-a)$$

$$y'' = c_2 x + \frac{W}{EI}(x-a) u(x-a).$$

Using the conditions $y(\ell) = 0$, $y'(\ell) = 0$, we get

$$0 = c_1 \ell + c_2 \frac{\ell^3}{6} + \frac{W}{EI}.\frac{b^3}{6} \Rightarrow 0 = c_2 \ell + \frac{W}{EI} b \quad [\because u(x-a) = 1, \text{ for } x \geq a \text{ and } \ell - a = b]$$

$$\therefore c_2 = -\frac{W}{EI}.\frac{b}{\ell}$$

and $c_1 = -\frac{1}{\ell}\left[c_2 \frac{\ell^3}{6} + \frac{W}{EI}.\frac{b^3}{6}\right] = -\frac{1}{\ell}\left[-\frac{W}{EI}.\frac{b\ell^2}{6} + \frac{W}{EI}.\frac{b^3}{6}\right]$

$$= \frac{WB}{6EI\ell}\left(\ell^2 - b^2\right) = \frac{Wb}{6EI\ell}(\ell + b)(\ell - b) = \frac{Wab}{6EI\ell}(\ell + b).$$

$\therefore$ From (i) the solution is

$$y = \frac{Wab}{6EI\ell}(\ell + b)x - \frac{W}{6EI}.\frac{b}{\ell}x^3 + \frac{W}{EI}.\frac{(x-a)^3}{3!} u(x-a)$$

$$= \frac{W}{6EI}\left[\frac{ab(\ell + b)}{\ell} x - \frac{b}{\ell} x^3 + (x-a)^3 u(x-a)\right].$$

When $0 < x < a$, $u(x-a) = 0$

$$\therefore y = \frac{W}{6EI}\left[\frac{ab(\ell + b)}{\ell} x - \frac{b}{\ell} x^3\right]. \tag{ii}$$

When $a < x < \ell$, $u(x-a) = 1$

$$\therefore y = \frac{W}{6EI}\left[\frac{ab(\ell+b)}{\ell}x - \frac{b}{\ell}x^3 + (x-a)^3\right]. \qquad \text{(iii)}$$

From (ii) and (iii), we have

$$y(a) = \frac{W}{6EI\ell}\left[a^2b(\ell+b) - ba^3\right]$$

$$= \frac{W}{6EI\ell}a^2b[(\ell+b)-a] = \frac{W}{6EI\ell}a^2b.(2b) = \frac{1}{3}\frac{W}{EI}.\frac{a^2b^2}{\ell}.$$

Q.No.3.: An impulsive voltage $E\delta(t)$ is applied to a circuit consisting of L, R, C in series with zero initial conditions. If i be the current at any subsequent time t, find the limit of I as $t \to 0$?

Sol.: The equation of the circuit governing the current i is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t idt = E\delta(t) \quad \text{where } i = 0, \text{ when } t = 0.$$

Taking Laplace transform of both sides, we get

$$L\left[s\,\bar{i} - i(0)\right] + R\bar{i} + \frac{1}{C}\frac{1}{s}\bar{i} = E$$

$$\Rightarrow \left(s^2 + \frac{R}{L}s + \frac{1}{CL}\right)\bar{i} = \frac{E}{L}s \Rightarrow \left(s^2 + 2as + a^2 + b^2\right)\bar{i} = \left(\frac{E}{L}\right)s$$

where $\frac{R}{L} = 2a$ and $\frac{1}{CL} = a^2 + b^2$

$$\Rightarrow \bar{i} = \frac{E}{L}\frac{(s+a)-a}{(s+a)^2+b^2} = \frac{E}{L}\left\{\frac{s+a}{(s+a)^2+b^2} - a\frac{1}{(s+a)^2+b^2}\right\}$$

On inversion, we get $i = \frac{E}{L}\left\{e^{-at}\cos bt - \frac{a}{b}e^{-at}\sin bt\right\}$

Taking limits as $t \to 0$, $i \to \frac{E}{L}$.

Although the current i = 0 initially, yet a large current will develop instantaneously ue to impulsive voltage applied at t = 0. In fact, we have determined the limit of this current which is $\frac{E}{L}$.

Q.No.4.: A light beam of length ℓ has its ends $x = 0$ and $x = \ell$ hinged. If concentrated load w acts at the point $x = \frac{\ell}{3}$. Show that the deflection is given by

$$EIy = \frac{W}{81}x\left(5\ell^2 - 9x^2\right) + \frac{W}{6}\left(x - \frac{\ell}{3}\right)^3 \mu\left(x - \frac{\ell}{3}\right).$$

Sol.: We know that the differential equation for the deflection is

$$\frac{d^4y}{dx^4} = \frac{W}{EI}\delta\left(x - \frac{\ell}{3}\right), \text{ satisfying,} \qquad \text{(i)}$$

$y(0) - 0, \quad y'(0) = c_1$

$y''(0) = 0, \quad y'''(0) = c_2$

and $\quad y(\ell) = 0, \quad y'(\ell) = 0$

Taking Laplace transform both sides of (i),

$$s^4\bar{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = \frac{W}{EI}e^{-\frac{t}{3}s} \qquad \left[L[\delta(t-a)] = e^{-as}\right]$$

$$\Rightarrow s^4\bar{y} - s^2c_1 - c_2 = \frac{W}{EI}e^{-\frac{t}{3}s}$$

$$\Rightarrow \bar{y} = \frac{c_1}{s^2} + \frac{c_2}{s^4} + \frac{W}{EI}e^{-\frac{t}{3}s}.\frac{1}{s^4}$$

Here $\bar{f}(s) = \frac{1}{s^4}$ and $f(x) = L^{-1}\left(\bar{f}(s)\right) = L^{-1}\frac{1}{s^4} = \frac{x^3}{3!}.\frac{x^3}{6}$

Also for $a = \frac{\ell}{3}$, $L^{-1}\left(e^{-as}\bar{f}(s)\right) = \frac{\left(x - \frac{\ell}{3}\right)^3}{6}.\mu\left(x - \frac{\ell}{3}\right)$

Taking Inverse Laplace transform,

$$y = c_1x + c_2\frac{x^3}{6} + \frac{W}{EI}\frac{\left(x - \frac{\ell}{3}\right)^3}{6}\mu\left(x - \frac{\ell}{3}\right) \qquad \text{(A)}$$

$$\begin{cases} c_1 x + \dfrac{c_2 x^3}{6}, 0 < x < \dfrac{\ell}{x} \\ c_1 x + \dfrac{c_2 x^3}{6} + \dfrac{W}{6EI}\left(x - \dfrac{\ell}{3}\right)^3 ; \dfrac{\ell}{3} < x < \ell \end{cases}$$

Now $0 = y(\ell) = c_1 \ell + c_2 \dfrac{\ell^3}{6} + \dfrac{W}{6EI} \cdot \dfrac{8\ell^3}{27}$ (ii)

And $0 = y'(\ell) = c_1 + \dfrac{1}{2} c_2 \ell^2 + \dfrac{W}{2EI} \cdot \dfrac{4\ell^2}{9}$ (iii)

Operating $(\text{ii}) \times \dfrac{1}{\ell} - (\text{iii})$, we get

$$-\frac{\ell^2}{3} c_2 + \frac{W}{27EL} \ell^2 \left(-\frac{14}{3}\right) = 0 \Rightarrow c_2 = -\frac{14W}{27EI}.$$

Again from (ii),

$$c_1 \ell = \frac{14W}{27EI} \cdot \frac{\ell^3}{6} - \frac{8W\ell^3}{6EI} \cdot \frac{1}{27} \Rightarrow c_1 = \frac{W\ell^2}{27EL}.$$

Hence from (A),

$$y = \frac{W\ell^2}{27EI} x - \frac{14W}{27EI} \cdot \frac{x^3}{6} + \frac{W}{6EI}\left(x - \frac{\ell}{3}\right)^3 \mu\left(x - \frac{\ell}{3}\right)$$

$$EIy = \frac{W}{81} x\left(3\ell^2 - 7x^3\right) + \frac{W}{6}\left(x - \frac{\ell}{3}\right)^3 \mu\left(x - \frac{\ell}{3}\right).$$

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Home Assignments

Q.No.1.: The deflection of beam of length ℓ, clamped horizontally at both ends loaded at $x = \dfrac{\ell}{4}$ by a weight W is given by $EI\dfrac{d^4 y}{dx^4} = W\delta\left(x - \dfrac{\ell}{4}\right)$.

Find the deflection curve, given $y = \dfrac{dy}{dx} = 0$, when $x = 0$ and $x = \ell$.

Ans.: $EIy = \frac{W}{6}\left(x - \frac{\ell}{4}\right)^3 u\left(x - \frac{\ell}{4}\right) + \frac{9}{128}WLx^2 - \frac{9}{64}W^2x^3$.

Q.No.2.: Abeam has its ends clamped at $x = 0$ and $x = \ell$. A concentrated load W acts vertically downwards at the point $x = \frac{\ell}{3}$. Find the resulting deflection.

Ans.: $$y(x) = \begin{cases} \dfrac{2Wx^2(3\ell - 5x)}{81\ EI}, & 0 < x < \dfrac{\ell}{3} \\ \dfrac{2Wx^2(3\ell - 5x)}{81\ EI} + \dfrac{W}{6EI}\left(x - \dfrac{\ell}{3}\right)^3, & \dfrac{\ell}{3} < x < \ell \end{cases}$$

Q.No.3.: A cantilever beam is clamped at the end $x = 0$ and is free at the end $x = \ell$. It carries a uniform load w per unit length from $x = 0$ to $x = \frac{\ell}{2}$. Calculate the deflection y at any point.

Ans.: $y(x) = \frac{w\ell^2}{16EI}x^2 - \frac{w\ell}{12EI}x^3 + \frac{w}{24EI}x^4 - \frac{w}{24EI}\left(x - \frac{\ell}{2}\right)^4 u\left(x - \frac{\ell}{2}\right)$.

Q.No.4.: An impulse I (kg-sec) is applied to a mass m attached to a spring having constant k. The system is damped with damping constant μ. Derive expression for displacement and velocity of the mass, assuming initial condition $x(0) = x'(0) = 0$.

Ans.: $x = \frac{1}{mn}e^{-\mu t/2m}\sin nt$; $\frac{dx}{dt} = \frac{I}{m}e^{-\mu t/2m}\left\{\cos nt - \frac{\mu}{2mn}\sin nt\right\}$,

where $n^2 = \frac{k}{m} - \frac{\mu^2}{4m^2}$.

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7th Topic

Laplace Transforms

Periodic functions, Special functions

[Bessel functions, Error function]

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1^{st} Topic

Fourier Transforms

Integral Transforms
Fourier Integral Theorem
(*Sufficient conditions for the validity of Fourier Integral*)
Fourier Sine and Cosine integrals
Complex Form of Fourier Integral

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Introduction:

In the previous topic, we have already discussed the use of Laplace Transforms in the solution of ordinary differential equations. Today, we will discuss the one more transform which is as known Fourier Transforms. We will also discuss their properties and its use in the solution of partial differential equations.

Note:

- **The choice of a particular transform to be employed for the solution of an equation depends on the boundary conditions of the problem and without difficulty with which the transform can be inverted.**
- **A Fourier Transform when applied to a partial differential equation reduces the number of its independent variables by one.**

Applications:

With the help of the theory of integral transforms, we can obtain solutions of numerous boundary value problems of engineering, e.g.

- **Conduction of heat,**
- **Transverse vibrations of a string,**
- **Transverse oscillations of an elastic beam,**

- **Free and forced vibrations of a membrane,**
- **Transmission lines etc.**

INTEGRAL TRANSFORMS:

The integral transform of a function f(x), denoted by $I\{f(x)\}$ is defined by

$$I\{f(x)\} = \bar{f}(s) = \int_a^b f(x)K(s, x)dx,$$

where K(s, x) is a known function of s and x, called **kernel** of the transform.

The function f(x) is called the **inverse transform** of $\bar{f}(s)$.

Three simple examples of a kernel are as follows:

1. Laplace Transform:

When $K(s, x) = e^{-sx}$, then it leads to Laplace transform of f(x), i.e.

$$L\{f(x)\} = \bar{f}(s) = \int_0^\infty f(x)e^{-sx}dx.$$

2. Fourier Transform:

When $K(s, x) = e^{isx}$, then we have the Fourier transform of f(x), i.e.

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx.$$

3. Mellin Transform:

If $K(s, x) = x^{s-1}$, then its gives the Mellin transform of f(x), i.e.

$$M(s) = \int_0^\infty f(x)x^{s-1}dx.$$

Other special transforms arise when kernel is a sine or cosine function or a Bessel's function. These lead to Fourier sine or cosine transforms and the Hankel transform, respectively.

4. Fourier Sine Transform:

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx dx.$$

5. Fourier Cosine Transform:

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx .$$

6. Hankel Transform:

If $K(s, x) = xJ_n(sx)$, then

$$H_n(s) = \int_0^{\infty} f(x)\, xJ_n(sx)\, dx ,$$

where $J_n(sx)$ is the Bessel function of the first kind and order n.

In order to introduce the Fourier transforms, we shall first derive the Fourier Integral theorem:

FOURIER INTEGRAL THEOREM:

[Sufficient conditions for the validity of Fourier Integral]

1. f(x) is **piecewise continuous** on every finite interval $[-c, c]$.
2. f(x) is **absolutely integrable** on the x-axis, that is, $\int_{-\infty}^{\infty} |f(x)| dx$ converges.
3. At every x on the real line f(x) has left and right hand derivatives.

If f(x) satisfies the conditions 1 to 3 stated above, then

(A) Fourier Integral of f converges to f(x) at every point x at which f is continuous,

and

(B) Fourier Integral of f(x) converges to the mean value $\frac{1}{2}[f(x_0+0)+f(x_0-0)]$ at every point x at which f is discontinuous, where $f(x+)$ and $f(x-)$ are the right and left hand limits respectively.

In other words, if f(x) satisfies the conditions 1 to 3 stated above, then f(x) can be represented by a Fourier integral

$$f(x) = \int_0^{\infty} [A(\alpha)\cos \alpha x + B(\alpha)\sin \alpha x] d\alpha ,$$

which is valid at all points of continuity.

At a point of discontinuity x_0, the Fourier integral $=\frac{1}{2}[f(x_0+0)+f(x_0-0)]$, i.e., average of the left and right hand limits.

Proof:

Consider a function f(x), which satisfies the Dirichlet's conditions in every interval $(-c,\ c)$.

Then, we can develop a Fourier series as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{c}+b_n\sin\frac{n\pi x}{c}\right), \tag{i}$$

where, $a_0=\frac{1}{c}\int_{-c}^{c} f(x)dx$, $a_n=\frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx$ and $b_n=\frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx$.

Substituting the values of a_0, a_n and b_n in (i), we get

$$f(x)=\left(\frac{1}{2c}\int_{-c}^{c} f(t)dt\right)+\frac{1}{c}\sum_{n=1}^{\infty}\left[\cos\frac{n\pi x}{c}\left(\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}dt\right)+\sin\frac{n\pi x}{c}\left(\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}dt\right)\right]$$

Set $\alpha_n=\frac{n\pi}{c}$ and $\delta\alpha=\alpha_{n-1}-\alpha_n=\frac{(n+1)\pi}{c}-\frac{n\pi}{c}=\frac{\pi}{c}$. Then

$$f(x)=\frac{1}{2c}\int_{-c}^{c} f(t)dt+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos\alpha_n x)\,\delta\alpha\int_{-c}^{c} f(t)\cos(\alpha_n t)dt+(\sin\alpha_n x)\,\delta\alpha\int_{-c}^{c} f(t)\sin(\alpha_n t)dt\right] \tag{ii}$$

We now let $c\to\infty$, and assume f(x) to be absolutely integrable over the interval $(-\infty,\infty)$, i.e. $\int_{-\infty}^{\infty}|f(x)|dx$ converges.

Then the value of the integral $\frac{1}{2c}\int_{-c}^{c} f(t)dt$ tends to zero as $c\to\infty$.

Also $\delta\alpha=\frac{\pi}{c}\to 0$ and the infinite series in (ii) becomes an integral from 0 to ∞, which represents f(x) as

$$f(x)=\frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos\alpha_n x)\,\delta\alpha\int_{-c}^{c} f(t)\cos(\alpha_n t)dt+(\sin\alpha_n x)\,\delta\alpha\int_{-c}^{c} f(t)\sin(\alpha_n t)dt\right]$$

$$\Rightarrow f(x)=\int_{0}^{\infty}\left[\left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\alpha t)dt\right)\cos\alpha x+\left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\alpha t)dt\right)\sin\alpha x\right]d\alpha$$

$$\Rightarrow f(x)=\int_0^{\infty}\left[A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x\right]d\alpha, \qquad \text{(iii)}$$

This is called the **Fourier Integral representation** of f(x).

Here $A(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\alpha t)dt$ and $B(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\alpha t)dt$, which are called **Fourier coefficients.**

Thus, a function f(x) can be represented by a Fourier integral

$$f(x)=\int_0^{\infty}[A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x]d\alpha,$$

which is valid at all points of continuity.

At a point of discontinuity x_0, the Fourier integral $=\frac{1}{2}[f(x_0+0)+f(x_0-0)]$, i.e., average of the left and right hand limits.

This completes the proof of **Fourier integral theorem.**

Another Form of Fourier Integral Representation:

Since we know the **Fourier integral** of f(x)

$$f(x)=\int_0^{\infty}[A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x]d\alpha.$$

Substituting the expressions for $A(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\alpha t)dt$ and $B(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\alpha t)dt$,

we obtain

$$f(x)=\int_0^{\infty}\left[\left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\alpha t)dt\right)\cos\alpha x+\left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\alpha t)dt\right)\sin\alpha x\right]d\alpha$$

$$\Rightarrow f(x)=\frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(t)\left[\cos\alpha t\cos\alpha x+\sin\alpha t\sin\alpha x\right]dt\right]d\alpha$$

$$\Rightarrow f(x)=\frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)dt\right]d\alpha, \qquad \text{(iii)}$$

which is known as **Fourier integral** of f(x).

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Fourier Sine and Cosine Integrals:

Since we know the **Fourier integral** of f(x)

$$f(x)=\int_{0}^{\infty}\left[A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x\right]d\alpha .$$

where $A(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\cos(\alpha t)dt$ and $B(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\sin(\alpha t)dt$,

Fourier Cosine Integral:

When f(x) is an even function,

then $B(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\sin(\alpha t)dt=0$ and $A(\alpha)=\frac{2}{\pi}\int_{0}^{\infty}f(t)\cos(\alpha t)dt$,

then the Fourier integral reduces to the Fourier cosine integral

$$f(x)=\int_{0}^{\infty}A(\alpha)\cos(\alpha x)d\alpha=\frac{2}{\pi}\int_{0}^{\infty}\cos(\alpha x)\int_{0}^{\infty}f(t)\cos(\alpha t)dtd\alpha .$$

This is called Fourier cosine integral.

Fourier Sine Integral:

When f(x) is an odd function,

then $A(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\cos(\alpha t)dt=0$ and $B(\alpha)=\frac{2}{\pi}\int_{0}^{\infty}f(t)\sin(\alpha t)dt$,

then the Fourier integral (I) reduces to the Fourier sine integral

$$f(x)=\int_{0}^{\infty}B(\alpha)\sin(\alpha x)d\alpha=\frac{2}{\pi}\int_{0}^{\infty}\sin(\alpha x)\int_{0}^{\infty}f(t)\sin(\alpha t)dtd\alpha .$$

This is called Fourier sine integral.

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Complex Form of Fourier Integrals:

Since we know the **Fourier integral** of f(x)

$$f(x)=\int_0^{\infty}[A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x]d\alpha .$$

Substituting the expressions for $A(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\alpha t)dt$ and $B(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\alpha t)dt$,

we obtain

$$f(x)=\int_0^{\infty}\left[\left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\alpha t)\,dt\right)\cos\alpha x+\left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\alpha t)\,dt\right)\sin\alpha x\right]d\alpha$$

$$\Rightarrow f(x)=\frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(t)[\cos\alpha t\cos\alpha x+\sin\alpha t\sin\alpha x]dt\right]d\alpha$$

$$\Rightarrow f(x)=\frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dt\right]d\alpha ,$$

Since $\cos\alpha(t-x)$ is an even function of α, we have

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dt\right]d\alpha=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dt\right]d\alpha$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dtd\alpha \qquad \left[\because 2\int_0^{a} f(x)dx=\int_{-a}^{a} f(x)dx,\ \text{if } f(x) \text{ is even}\right] \text{(i)}$$

Since $\sin\alpha(t-x)$ is an odd function of α, so that

$$0=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\alpha(t-x)dtd\alpha \qquad \left[\because 2\int_{-a}^{a} f(x)dx=0,\ \text{if } f(x) \text{ is odd}\right] \qquad \text{(ii)}$$

Multiplying (ii) by i and adding to (i), we get

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\{\cos\alpha(t-x)+i\sin\alpha(t-x)\}dtd\alpha$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{i\alpha(t-x)}dtd\alpha=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha x}\int_{-\infty}^{\infty} f(t)e^{i\alpha t}dtd\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t} dt \right] e^{-i\alpha x} d\alpha,$$

which is known as the **Complex form of Fourier integral.**

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Fourier integrals

Q.No.1.: Express the function $f(x) = \begin{cases} 1 & \text{for } |x| \le 1, \\ 0 & \text{for } |x| > 1, \end{cases}$

as a **Fourier integral**.

Hence evaluate $\displaystyle\int_0^{\infty} \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda$.

Also prove that $\displaystyle\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Sol.: Given $f(x) = \begin{cases} 1 & \text{for } |x| \le 1, \text{ i.e. for } -1 \le x \le 1 \\ 0 & \text{for } |x| > 1 \text{ i.e. for } -\infty < x < -1 \text{ and } 1 < x < \infty \end{cases}$

The Fourier integral for f(x) is $f(x) = \dfrac{1}{\pi} \displaystyle\int_0^{\infty} \left[\int_{-\infty}^{\infty} f(t) \cos\alpha(t - x) dt \right] d\alpha$

$$\Rightarrow f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos\lambda(t - x) dt d\lambda = \frac{1}{\pi} \int_0^{\infty} \left(\int_{-1}^{1} \cos\lambda(t - x) dt \right) d\lambda$$

$$\left[\begin{array}{l} \because f(t) = 0 \text{ when } -\infty < t < -1 \text{ and when } 1 < t < \infty \\ \text{and } f(t) = 1 \text{ when } -1 \le t \le 1 \end{array} \right]$$

$$= \frac{1}{\pi} \int_0^{\infty} \left[\frac{\sin\lambda(t - x)}{\lambda} \right]_{-1}^{1} d\lambda = \frac{1}{\pi} \int_0^{\pi} \frac{\sin\lambda(1 - x) - \sin\lambda(-1 - x)}{\lambda} d\lambda$$

$$= \frac{1}{\pi} \int_0^{\pi} \frac{\sin\lambda(1 + x) + \sin\lambda(1 - x)}{\lambda} d\lambda = \frac{2}{\pi} \int_0^{\infty} \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda$$

$\Rightarrow f(x) = \dfrac{2}{\pi} \displaystyle\int_0^{\infty} \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda$, which is required Fourier integral.

2nd Part: Since $f(x)=\frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda \Rightarrow \int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda=\frac{\pi}{2}f(x)$.

Case I: When $|x|<1$, then $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda=\frac{\pi}{2}(1)=\frac{\pi}{2}$.

Case II: When $|x|=1$

i.e. $x=\pm1$, f(x) is discontinuous and as we know at a point of discontinuity x_0, the Fourier integral $=\frac{1}{2}[f(x_0+0)+f(x_0-0)]$, i.e., average of the left and right hand limits.

Integral has the value $\frac{1}{2}\left(\frac{\pi}{2}(1)+\frac{\pi}{2}(0)\right)=\frac{\pi}{4}$.

$\therefore \int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda=\frac{\pi}{4}$.

Case III: When $|x|>1$, then $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda=\frac{\pi}{2}(0)=0$.

3rd Part: Putting x = 0, we get $\int_0^\infty \frac{\sin\lambda}{\lambda}d\lambda=\frac{\pi}{2} \Rightarrow \int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}$.

Q.No.2.: Express $f(x)=1$ for $0\le x\le\pi$

$=0$ for $x>\pi$,

as a **Fourier sine integral** and hence evaluate

$$\int_0^\infty \frac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)d\lambda .$$

Sol.: Here given $f(x)=\begin{cases}1, & \text{when } 0<x<\pi\\ 0, & \text{when } x>\pi\end{cases}$

1st Part: Fourier sine integral representation of f(x):

Since we know that the Fourier sine integral is

$$f(x)=\int_0^\infty B(\alpha)\sin(\alpha x)d\alpha, \text{ where } B(\alpha)=\frac{2}{\pi}\int_0^\infty f(t)\sin(\alpha t)dt.$$

or $f(x)=\int_0^\infty B(\lambda)\sin(\lambda x)d\lambda$, where $B(\lambda)=\dfrac{2}{\pi}\int_0^\infty f(t)\sin(\lambda t)dt$.

Using Fourier sine integral, we have

$$f(x)=\frac{2}{\pi}\int_0^\infty\left(\int_0^\infty f(t)\sin(\lambda t)dt\right)\sin(\lambda x)d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty\left[\int_0^\pi f(t)\sin(\lambda t)dt+\int_\pi^\infty f(t)\sin(\lambda t)dt\right]\sin\lambda x d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty\left(\int_0^\pi 1.\sin(\lambda t)dt\right)\sin(\lambda x)d\lambda \qquad \text{[On substituting for f(t)]}$$

$$=\frac{2}{\pi}\int_0^\infty\left[-\frac{\cos(\lambda t)}{\lambda}\right]_0^\pi\sin(\lambda x)d\lambda=\frac{2}{\pi}\int_0^\infty\frac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)d\lambda,$$

which is required Fourier sine integral.

2nd Part: Evaluate $\int_0^\infty\dfrac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)d\lambda$.

Since $f(x)=\begin{cases}1, & \text{when } 0<x<\pi\\ 0, & \text{when } x>\pi\end{cases}$

$$\therefore\int_0^\infty\frac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)d\lambda=\frac{\pi}{2}f(x).$$

At $x=\pi$, f(x) is discontinuous and as we know at a point of discontinuity x_0, the Fourier integral $=\dfrac{1}{2}\left[f(x_0+0)+f(x_0-0)\right]$, i.e., average of the left and right hand limits.

Thus, the value of the above integral

$$\int_0^\infty\frac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)d\lambda=\frac{\pi}{2}\left[\frac{f(\pi-0)+f(\pi+0)}{2}\right]=\frac{\pi}{2}.\frac{1+0}{2}=\frac{\pi}{4}.$$

Q.No.3.: Find the **(a)** **Fourier cosine integral** and **(b)** **Fourier sin integral** of

$$f(x) = \sin x \quad \text{if } 0 \le x \le \pi$$
$$= 0 \quad \text{if } x > \pi$$

Sol.: Here given

$$f(x) = \sin x \quad \text{if } 0 \le x \le \pi$$
$$= 0 \quad \text{if } x > \pi$$

1st Part: Fourier cosine integral representation of f(x):

Since we know that the Fourier cosine integral is

$$f(x) = \int_0^\infty A(\alpha)\cos\alpha x d\alpha, \text{ where } A(\alpha) = \frac{2}{\pi}\int_0^\infty f(t)\cos\alpha t dt.$$

$$\text{Now } A(\alpha) = \frac{2}{\pi}\left[\int_0^\pi \sin t.\cos\alpha t dt + \int_\pi^\infty 0\right] = \frac{1}{2}\frac{2}{\pi}\int_0^\pi [\sin(1+\alpha)t + \sin(1-\alpha)t]dt$$

$$= -\frac{1}{\pi}\left[\frac{\cos(1+\alpha)t}{1+\alpha} + \frac{\cos(1-\alpha)t}{1-\alpha}\right]_{t=0}^{\pi} = +\frac{1}{\pi}\left[\frac{1-\cos(1+\alpha)\pi}{1+\alpha} + \frac{1-\cos(1-\alpha)\pi}{1-\alpha}\right]$$

$$A(\alpha) = \frac{1}{\pi}\left[\frac{1+\cos\alpha\pi}{1+\alpha} + \frac{1+\cos\alpha\pi}{1-\alpha}\right] = \frac{2(1+\cos\alpha\pi)}{\pi(1-\alpha^2)}.$$

Substituting $A(\alpha)$, we get the Fourier cosine integral of f(x) as

$$f(x) = \int_0^\infty \frac{2(1+\cos\alpha\pi)}{\pi(1-\alpha^2)}.\cos\alpha x d\alpha = \frac{2}{\pi}\int_0^\infty \frac{(1+\cos\alpha\pi)}{(1-\alpha^2)}.\cos\alpha x d\alpha.$$

2nd Part: Fourier sine integral representation of f(x):

Since we know that the Fourier sine integral is

$$f(x) = \int_0^\infty B(\alpha)\sin\alpha x d\alpha, \text{ where } B(\alpha) = \frac{2}{\pi}\int_0^\infty f(t)\sin\alpha t dt.$$

$$\text{Now } B(\alpha) = \frac{2}{\pi}\left[\int_0^\pi f(t).\sin\alpha t dt\right] = \frac{2}{\pi}\left[\int_0^\pi \sin t.\sin\alpha t dt + \int_\pi^\infty 0 dt\right]$$

$$= \frac{2}{\pi}\frac{1}{2}\int_0^\pi [\cos(1-\alpha)t - \cos(1+\alpha)t]dt = \frac{1}{\pi}\left[\frac{\cos(1-\alpha)t}{1-\alpha} - \frac{\cos(1+\alpha)t}{1+\alpha}\right]_{t=0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin\alpha\pi}{1-\alpha} + \frac{\sin\alpha\pi}{1+\alpha}\right] = \frac{2\sin\alpha\pi}{\pi(1-\alpha^2)}.$$

Substituting $A(\alpha)$, we get the Fourier sine integral of f(x) as

$$f(x)=\int_0^\infty \frac{2\sin\alpha\pi}{\pi\left(1-\alpha^2\right)}.\sin\alpha x d\alpha=\frac{2}{\pi}\int_0^\infty \frac{\sin\alpha\pi}{\left(1-\alpha^2\right)}.\sin\alpha x d\alpha$$

Q.No.4.: Using **Fourier sine integral**, show that

$$\int_0^\infty \frac{1-\cos\pi\lambda}{\lambda}\sin x\lambda d\lambda=\begin{cases}\frac{1}{2}\pi, & \text{when } 0<x<\pi\\ 0, & \text{when } x>\pi\end{cases}.$$

Sol.: Let $f(x)=\begin{cases}\frac{1}{2}\pi, & \text{when } 0<x<\pi\\ 0, & \text{when } x>\pi\end{cases}$

To show: $\int_0^\infty \frac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)d\lambda=\begin{cases}\frac{1}{2}\pi, & \text{when } 0<x<\pi\\ 0, & \text{when } x>\pi\end{cases}$

i.e. $f(x)=\int_0^\infty \frac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)d\lambda$.

Now since we know that the Fourier sine integral is

$$f(x)=\int_0^\infty B(\alpha)\sin(\alpha x)d\alpha, \text{ where } B(\alpha)=\frac{2}{\pi}\int_0^\infty f(t)\sin(\alpha t)dt.$$

or $f(x)=\int_0^\infty B(\lambda)\sin(\lambda x)d\lambda$, where $B(\lambda)=\frac{2}{\pi}\int_0^\infty f(t)\sin(\lambda t)dt$.

Using Fourier sine integral, we have

$$f(x)=\frac{2}{\pi}\int_0^\infty\left(\int_0^\infty f(t)\sin(\lambda t)dt\right)\sin(\lambda x)d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty\left[\int_0^\pi f(t)\sin(\lambda t)dt+\int_\pi^\infty f(t)\sin(\lambda t)dt\right]\sin\lambda x d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty\left(\int_0^\pi \frac{\pi}{2}\sin(\lambda t)dt\right)\sin(\lambda x)d\lambda$$ [On substituting for f(t)]

$$=\int_0^\infty\left[-\frac{\cos(\lambda t)}{\lambda}\right]_0^\pi\sin(\lambda x)d\lambda=\int_0^\infty\frac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)d\lambda$$

Since $f(x) = \begin{cases} \frac{1}{2}\pi, & \text{when } 0 < x < \pi \\ 0, & \text{when } x > \pi \end{cases}$

$$\therefore \int_0^\infty \frac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)d\lambda = \begin{cases} \frac{1}{2}\pi, & \text{when } 0 < x < \pi \\ 0, & \text{when } x > \pi \end{cases}.$$

This completes the proof.

Q.No.5.: Using **Fourier integral** representation, show that

$$\int_0^\infty \frac{\cos x\alpha + \alpha\sin x\alpha}{1+\alpha^2}d\alpha = \begin{cases} 0 & \text{if } x < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0. \end{cases}$$

Sol.: Consider the function defined by $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ e^{-x} & \text{if } x > 0 \end{cases}$

Now, find the Fourier integral representation of f(x) in the exponential form:

By definition: $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t).e^{-i\alpha t}dt\right]e^{i\alpha x}d\alpha$.

Consider $I = \int_{-\infty}^{\infty} f(t)e^{-i\alpha t}dt = \int_{-\infty}^{0}(0)dt + \int_0^\infty \left(e^{-x}e^{-i\alpha t}\right)dt = \left.\frac{e^{-t(1+i\alpha)}}{-(1+i\alpha)}\right|_{t=0}^{\infty} = \frac{1}{1+i\alpha} = \frac{1-i\alpha}{1+\alpha^2}$.

$$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Ie^{i\alpha x}d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1-i\alpha}{1+\alpha^2}\right)e^{i\alpha x}d\alpha = \int_{-\infty}^{\infty}\frac{1}{2\pi\left(1+\alpha^2\right)}(1-i\alpha)\times(\cos\alpha x + i\sin\alpha x)d\alpha$$

$$= \int_{-\infty}^{\infty}\frac{1}{2\pi\left(1+\alpha^2\right)}[(\cos\alpha x + \alpha\sin\alpha x) + i(\sin\alpha x - \alpha\cos\alpha x)]d\alpha.$$

The second integral on the right side is zero because the integral is an odd function.

$$f(x) = \frac{2}{2\pi\left(1+\alpha^2\right)}\int_0^\infty (\cos\alpha x + \alpha\sin\alpha x)d\alpha.$$

For $x > 0,\ f(x) = e^{-x}$ so

$$e^{-x} = \frac{1}{\pi}\int_0^\infty \frac{\cos\alpha x + \alpha\sin\alpha x}{1+\alpha^2}d\alpha \Rightarrow \int_0^\infty \frac{\cos\alpha x + \alpha\sin\alpha x}{1+\alpha^2}d\alpha = \pi e^{-x}. \qquad \text{(i)}$$

For $x < 0,\ f(x) = 0$ so $0 = \frac{1}{\pi}\int_0^\infty \frac{\cos\alpha x + \alpha\sin\alpha x}{1+\alpha^2}d\alpha \Rightarrow \int_0^\infty \frac{\cos\alpha x + \alpha\sin\alpha x}{1+\alpha^2}d\alpha = 0$. (ii)

At $x = 0$, $f(x)$ has a discontinuity. So $f(x) = \frac{1}{2}[f(x+0)+f(x-0)] = \frac{1}{2}[1+0] = \frac{1}{2}$.

Thus, for $x = 0$, $\frac{1}{2} = \frac{1}{\pi}\int_0^\infty \frac{\cos\alpha x + \alpha\sin\alpha x}{1+\alpha^2}d\alpha \Rightarrow \int_0^\infty \frac{\cos\alpha x + \alpha\sin\alpha x}{1+\alpha^2}d\alpha = \frac{\pi}{2}$. (iii)

From (i), (ii) and (iii), we get

$$\int_0^\infty \frac{\cos x\alpha + \alpha\sin x\alpha}{1+\alpha^2}d\alpha = \begin{cases} 0 & \text{if } x < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0. \end{cases}.$$

Hence proved.

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Home Assignments

Fourier Integrals

Q.No.1.: Find the **Fourier integral** representation of the function $f(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ e^{-x}, & x > 0 \end{cases}$.

Ans.: $f(x) = \frac{1}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x + \cos\lambda x}{1+\lambda^2}d\lambda$.

Q.No.2.: Find the **Fourier integral** representation of f(x); $f(x)=\begin{cases} x, & |x|<1 \\ 0, & |x|>1 \end{cases}$

Ans.: $\displaystyle\int_{-\infty}^{\infty}\frac{\sin\alpha-\alpha\cos\alpha}{i\pi\alpha^2}e^{i\alpha x}d\alpha$.

Q.No.3.: Find the **Fourier integral** representation of f(x); $f(x)=\begin{cases} \cos x, & \frac{-\pi}{2}<x<\frac{\pi}{2} \\ 0, & |x|>\frac{\pi}{2} \end{cases}$

Ans.: $\displaystyle\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos\alpha\pi/2}{1-\alpha^2}e^{i\alpha x}d\alpha$.

Q.No.4.: Find the **Fourier cosine and sine integrals** of $f(x)=e^{-kx}$, for x > 0, k > 0.

Ans.: Fourier cosine integral (FCL) = $\displaystyle f(x)=e^{-kx}=\frac{2k}{\pi}\int_0^{\infty}\frac{\cos sx}{k^2+s^2}dx$.

Fourier sine integral (FSL) = $\displaystyle f(x)=e^{-kx}=\frac{2}{\pi}\int_0^{\infty}\frac{s\sin sx}{k^2+s^2}dx$.

Q.No.5.: Find the **Fourier cosine integral** of $f(x)=e^{-x}\cos x$.

Ans.: $\displaystyle f(x)=e^{-x}\cos x=\frac{2}{\pi}\int_0^{\infty}\frac{(x^2+2)\cos sxds}{s^4+4}$.

Q.No.6.: Find the **Fourier sine integral** of $f(x)=e^{-ax}-e^{-bx}$.

Ans.: $\displaystyle f(x)=e^{-ax}-e^{-bx}=\frac{2}{\pi}\int_0^{\infty}\frac{(b^2-a^2)s\sin sxds}{(a^2+s^2)(b^2+s^2)}$.

Q.No.7.: Using **Fourier integrals**, show that

(i). $\displaystyle\int_0^{\infty}\frac{\sin\pi\lambda\sin x\lambda}{1-\lambda^2}d\lambda=\begin{cases} \frac{1}{2}\pi\sin x, & \text{when } 0\le x\le\pi \\ 0, & \text{when } x>\pi \end{cases}$

(ii). $\displaystyle\int_0^{\infty}\frac{\lambda\sin\lambda x}{k^2+\lambda^2}d\lambda=\frac{\pi}{2}e^{-kx};\quad x>0,\ k>0.$

Q.No.8.: Using the **Fourier integral** representation, show that

$$\int_0^\infty \frac{\sin\omega\cos x\omega}{\omega}d\omega = \frac{\pi}{2} \quad \text{when} \quad 0 \le x < 1.$$

Q.No.9.: Using **Fourier integral** representation, show that

$$\int_0^\infty \frac{\sin s.\cos xs}{s}ds = \begin{cases} \frac{\pi}{2}, & \text{if } 0 \le x < 1 \\ \frac{\pi}{4}, & \text{if } x = 1 \\ 0, & \text{if } x > 1 \end{cases}$$

Hint: Find the Fourier integral of

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ 0, & \text{if } x \ge 1 \end{cases}$$

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1^{st} Topic

(Part-I)

$\mathbb{Z}$-Transforms

- Introduction
- Definition of the $\mathbb{Z}$-Transform
- Properties of $\mathbb{Z}$-Transforms
- $\mathbb{Z}$-Transforms of basic sequences

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Introduction:

- $\mathbb{Z}$-transform is basically discrete–time counter-part of the Laplace Transform, as well as generalization of the discrete-time Fourier Transform.
- The properties of $\mathbb{Z}$-transform are very close to the properties of the Laplace transform except the fundamental difference between the continuous and discrete signals in the time-domain.
- Basically, it simplifies the algebra of discrete systems to some extent.
- $\mathbb{Z}$-transform has various applications in the field of business, computer science, science and technology, etc. such as in signal processing circuit theory, coding theory, radar detection system, economic modeling, etc.

Definition of the $\mathbb{Z}$-Transform:

The $\mathbb{Z}$-transform of a given discrete sequence $\{f(n)\}$ is denoted by $F(z)$ or $\mathbb{Z}\{f(n)\}$ and is defined as

$$\mathbb{Z}\{f(n)\} = F(z) = \sum_{n=-\infty}^{\infty} f(n) z^{-n}, \qquad \text{(i)}$$

where z is an continuous complex variable and $\mathbb{Z}\{\,\}$ is $\mathbb{Z}$-transform operator.

The $\mathbb{Z}$-transform is termed as **bilateral** $\mathbb{Z}$-transform (two sided $\mathbb{Z}$-transform).

The **unilateral** $\mathbb{Z}$ -transform (one sided $\mathbb{Z}$ -transform) is defined as

$$\mathbb{Z}\{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}.$$

The **inverse** $\mathbb{Z}$ **-transform** of $\mathbb{Z}$ {f(n)} = F(z) is defined as $\mathbb{Z}^{-1}\{F(z)\} = \{f(n)\}$.

Example:

The $\mathbb{Z}$ -transform of a discrete signal $\{x(t)\} = \{x_0, x_1, x_2, ...\}$ is $X(z) = x_0 + \frac{x_1}{z} + \frac{x_2}{z^2} + ...$.

The sequence $\{x_0, x_1, x_2, ...\}$ is called the inverse $\mathbb{Z}$ -transform of X(z) and z is related to the Laplace transform by $z = e^{Ts}$.

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Convergence of series (Region of convergence):

For a given sequence $\{f(n)\}$, the set R of values of z for which the $\mathbb{Z}$ -transform of $\{f(n)\}$ converges is called the region of convergence (ROC).

Consider $\mathbb{Z}\{f(n)\} = \sum_{n=0}^{\infty} f(n)z^{-n}$. (i)

The region of the z-plane for which (i) converges absolutely is known as the region of convergence of $\mathbb{Z}$ -transform of $\{f(n)\}$.

Determination of Region of convergence:

The $\mathbb{Z}$ -transform does not necessarily exist for all values of z in the complex plane. For the existence of $\mathbb{Z}$ -transform, the summation of the series must converge. Therefore, the $\mathbb{Z}$ -transform is defined where it exists. Now, the range of convergence of the $\mathbb{Z}$ -transform is the range of values of z for which it is finite.

Consider $\mathbb{Z}\{u_n\} = \mathbb{Z}\{f(n)\} = \sum_{n=0}^{\infty} f(n)z^{-n} = \sum_{n=0}^{\infty} u_n z^{-n}$.

Now for convergence, we can use Cauchy's ratio test.

i.e. $\underset{n\to\infty}{\text{Lt}} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \underset{n\to\infty}{\text{Lt}} \left|\frac{f(n+1)z^{-1(n+1)}}{f(n)z^{-n}}\right| < 1 \Rightarrow \underset{n\to\infty}{\text{Lt}} \left|\frac{f(n+1)}{f(n)}\right| \left|z^{-1}\right| < 1$.

Thus, by Cauchy's ratio test, the series is convergent if $\underset{n\to\infty}{\text{Lt}} \left|\frac{f(n+1)}{f(n)}\right| < |z|$.

$\Rightarrow$ The region of convergence is the exterior of the radius R (say).

Hence, R is called the radius of convergence of the series $\sum_{n=0}^{\infty} f(n)z^{-n}$.

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Note: While finding $\mathbb{Z}$ -transforms

1. If f(n) is given simply replace f(n). It can also be denoted as u_n.
2. If f(t) is given simply replace t by nT.

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Properties of $\mathbb{Z}$ -Transforms:

Property No. 1:

Linear property:

Statement: If a, b, c, be any constants and u_n, v_n, w_n be any discrete functions, then

$$\mathbb{Z}\{au_n + bv_n - cw_n\} = a\mathbb{Z}\{u_n\} + b\mathbb{Z}\{v_n\} - c\mathbb{Z}\{w_n\}.$$

This means that $\mathbb{Z}$ -transform is linear.

Proof: Now since $\mathbb{Z}\{au_n + bv_n - cw_n\} = \sum_{n=-\infty}^{\infty} (au_n + bv_n - cw_n) z^{-n}$ (by definition)

$$= a\sum_{n=-\infty}^{\infty} u_n z^{-n} + b\sum_{n=-\infty}^{\infty} v_n z^{-n} - c\sum_{n=-\infty}^{\infty} w_n z^{-n}$$

$$= a\mathbb{Z}\{u_n\} + b\mathbb{Z}\{v_n\} - c\mathbb{Z}\{w_n\}.$$

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Property No. 2:

Change of Scale (or Damping Rule):

If $\mathbb{Z}\{u_n\} = U(z)$, then $\mathbb{Z}\{a^{-n}u_n\} = U(az)$.

Proof: Since $\mathbb{Z}\{a^{-n}u_n\} = \sum_{n=-\infty}^{\infty} a^{-n}u_n.z^{-n} = \sum_{n=-\infty}^{\infty} u_n.(az)^{-n} = U(az)$.

Corollary: $\mathbb{Z}\{a^n u_n\} = U\left(\frac{z}{a}\right)$.

Observation: (Reason behind the name damping rule)

The geometric function a^{-n} when $|a|<1$, damps the function u_n, hence the name damping rule.

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Property No. 3:

Multiplication by n:

If $\mathbb{Z}\{u_n\} = U(z)$, then $\mathbb{Z}\{nu_n\} = -z\dfrac{dU(z)}{dz}$.

Proof: Here $\mathbb{Z}\{nu_n\} = \sum_{n=0}^{\infty} n.u_n z^{-n} = -z\sum_{n=0}^{\infty} u_n(-n)z^{-n-1} = -z\sum_{n=0}^{\infty} u_n \dfrac{d}{dz}(z^{-n})$.

$$= -z\sum_{n=0}^{\infty} \frac{d}{dz}(u_n z^{-n}) = -z\frac{d}{dz}\left(\sum_{n=0}^{\infty} u_n z^{-n}\right) = -z\frac{d}{dz}U(z).$$

Extension: By mathematical induction, we obtain $\mathbb{Z}\{n^m u_n\} = (-z)^m \dfrac{d^m U(z)}{dz^m}$.

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Property No. 4:

Shift Property-Shifting Rule-Shifting Theorem:

(1) Shifting u_n to the right:

If $\mathbb{Z}\{u_n\} = U(z)$ and k is any positive integer, then

$$\mathbb{Z}\{u_{n-k}\} = z^{-k}U(z).$$

Proof: Now by definition, we have

$$\mathbb{Z}\{u_{n-k}\} = \sum_{n=0}^{\infty} u_{n-k}z^{-n} = u_0 z^{-k} + u_1 z^{-(k+1)} + ... = z^{-k}\sum_{n=0}^{\infty} u_n z^{-n} = z^{-k}U(z).$$

(2) Shifting u_n to the left:

If $\mathbb{Z}\{u_n\} = U(z)$ and k is any positive integer, then

$$\mathbb{Z}\{u_{n+k}\} = z^k\left[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - .... - u_{k-1}z^{-(k-1)}\right].$$

Proof: Now by definition, we have

$$\mathbb{Z}\{u_{n+k}\} = \sum_{n=0}^{\infty} u_{n+k}z^{-n} = z^k\sum_{n=0}^{\infty} u_{n+k}z^{-(n+k)}.$$

$$= z^k\left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n}\right]$$

Hence $\mathbb{Z}\{u_{n+k}\} = z^k\left[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - .... - u_{k-1} z^{-(k-1)}\right]$.

In particular, we have the following standard results:

- $\mathbb{Z}\{u_{n+1}\} = z\left[U(z) - u_0\right]$
- $\mathbb{Z}\{u_{n+2}\} = z^2\left[U(z) - u_0 - u_1 z^{-1}\right]$
- $\mathbb{Z}\{u_{n+3}\} = z^3\left[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}\right]$

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$\mathbb{Z}$ -Transform of basic sequences:

1. Prove that $\mathbb{Z}\{k\} = \dfrac{kz}{z-1}$, if $|z| > 1$

Proof: Since by definition, we have

$$\mathbb{Z}\{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}.$$

$\therefore \mathbb{Z}\{k\} = \sum_{n=0}^{\infty} k z^{-n}$, if $|z| > 1$.

$$= k\{1 + z^{-1} + z^{-2} + ... + \infty\} = k\frac{1}{1-z^{-1}} = \frac{kz}{z-1}.$$

Particular case: When k = 1, then $\mathbb{Z}\{1\} = \dfrac{z}{z-1}$, if $|z| > 1$.

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2. Prove that $\mathbb{Z}\{(-1)^n\} = \dfrac{z}{z+1}$, if $|z| > 1$

Proof: $\mathbb{Z}\{(-1)^n\} = \sum_{n=0}^{\infty} (-1)^n z^{-n}$ if $|z| > 1$.

$$= \{1 - z^{-1} + z^{-2} - .... + \infty\} = \frac{1}{1+\frac{1}{z}} = \frac{z}{z+1}.$$

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3. Prove that $\mathbb{Z}\{a^n\} = \frac{z}{z-a}$, if $|z| > |a|$.

Proof: $\mathbb{Z}\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n}$ if $|z| > |a|$.

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}.$$

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4. Prove that $\mathbb{Z}\{n\} = \frac{z}{(z-1)^2}$.

Proof: $\mathbb{Z}\{n\} = \sum_{n=0}^{\infty} (n) z^{-n} = -z\frac{d}{dz}\mathbb{Z}\{1\}$ $\left[\because \mathbb{Z}\{nu_n\} = -z\frac{d}{dz}U(z) \text{ where } U(z) = \mathbb{Z}\{u_n\}\right]$

$$= -z\frac{d}{dz}\left(\frac{z}{z-1}\right). \left[\because \mathbb{Z}\{1\} = \frac{z}{z-1}, \text{ if } |z| > 1\right].$$

$$= -z\left[\frac{(z-1).1-1.z}{(z-1)^2}\right] = \frac{z}{(z-1)^2}.$$

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5. Prove that $\mathbb{Z}\{na^n\} = \frac{az}{(z-a)^2}$.

Proof: $\mathbb{Z}\{na^n\} = -z\frac{d}{dz}\mathbb{Z}\{a^n\} = -z\frac{d}{dz}\left\{\frac{z}{z-a}\right\} = \frac{az}{(z-a)^2}$. $\left[\because \mathbb{Z}\{a^n\} = \frac{z}{z-a}, \text{if } |z| > |a|\right]$.

Note: $\mathbb{Z}\{n^{m+1}\} = (-z)^m \frac{d^m}{dz^m}\mathbb{Z}\{n\}$.

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6. Prove that **(a)** $\mathbb{Z}\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$;

(b) $\mathbb{Z}\{\sin n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$ if $|z| > 1$.

Proof: We know that $\mathbb{Z}\{a^n\} = \frac{z}{z-a}$, if $|z| > |a|$.

Let $a = e^{i\theta}$. $\therefore \mathbb{Z}\{e^{in\theta}\} = \dfrac{z}{z - e^{i\theta}} = \dfrac{z}{z - \{\cos\theta + i\sin\theta\}}$.

$$\Rightarrow \mathbb{Z}\{\cos n\theta + i\sin n\theta\} = \frac{z}{\{z - \cos\theta - i\sin\theta\}} \times \frac{\{z - \cos\theta + i\sin\theta\}}{\{z - \cos\theta + i\sin\theta\}} = \frac{z(z - \cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}.$$

Equating real and imaginary parts, we get

$$\mathbb{Z}\{\cos n\theta\} = \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}; \qquad \mathbb{Z}\{\sin n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}; \ |z| > 1.$$

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7. Prove that **(a)** $\mathbb{Z}\{r^n \cos n\theta\} = \dfrac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2}$ and

(b) $\mathbb{Z}\{r^n \sin n\theta\} = \dfrac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$, if $|z| > |r|$.

Proof: $\mathbb{Z}\{a^n\} = \dfrac{z}{z - a}$ if $|z| > |a|$.

Let $a = re^{i\theta}$ in $\mathbb{Z}\{a^n\}$ $\therefore \mathbb{Z}\{r^n e^{in\theta}\} = \dfrac{z}{z - re^{i\theta}}$.

$$\Rightarrow \mathbb{Z}\{r^n(\cos n\theta + i\sin n\theta)\} = \frac{z}{z - re^{i\theta}} = \frac{z}{z - r\{\cos\theta + i\sin\theta\}}.$$

$$\Rightarrow \mathbb{Z}\{r^n(\cos n\theta + i\sin n\theta)\} = \frac{z}{[z - r\cos\theta] - ir\sin\theta} \times \frac{[z - r\cos\theta] + ir\sin\theta}{[z - r\cos\theta] + ir\sin\theta}$$

$$\Rightarrow \mathbb{Z}\{r^n(\cos n\theta + i\sin n\theta)\} = \frac{z[z - r\cos\theta + ir\sin\theta]}{[z - r\cos\theta]^2 + r^2\sin^2\theta}$$

$$\Rightarrow \mathbb{Z}\{r^n(\cos n\theta + i\sin n\theta)\} = \frac{z(z - r\cos\theta) + izr\sin\theta}{z^2 - 2zr\cos\theta + r^2}.$$

Equating real and imaginary parts, we get

$$\mathbb{Z}\{r^n \cos n\theta\} = \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2}$$

and $\mathbb{Z}\{r^n \sin n\theta\} = \dfrac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}$, if $|z| > |r|$.

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1st Topic

(Part-II)

$\mathbb{Z}$-Transforms

- $\mathbb{Z}$-Transforms of Standard Discrete Functions
- First Shifting Theorem and deductions of some important results
- Second Shifting Theorem
- Two Basic Theorems
  Initial Value Theorem
  Final Value Theorem

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Discrete functions:

A discrete function is a function with distinct and separate values. This means that the values of the functions are not connected with each other.

i.e. A function that is defined only for a set of numbers that can be listed, such as the set of whole numbers or the set of integers.

For example, a discrete function can equal 1 or 2 but not 1.5.

Domain and Range: A discrete function is a function in which the domain and range are each a discrete set of values, rather than an interval in R.

Difference between continuous function and discrete function:

A continuous function allows the x-values to be ANY points in the interval, including fractions, decimals, and irrational values. A discrete function allows the x-values to be only certain points in the interval, usually only integers or whole numbers.

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$\mathbb{Z}$-Transform for discrete values of t:

If f(t) is a function defined for discrete values of t, where t = nT, n = 0, 1, 2, ….., T being the sampling period, then $\mathbb{Z}$-transform of f(t) is defined as

$$\mathbb{Z}\{f(t)\} = \sum_{n=0}^{\infty} f(nT)z^{-n} = F(z).$$

Note:

- The important element of discrete-time systems is the **samples** in which a switch close to admit an input signal in every T seconds.
- A **samples** is a **conversion device** which converts a continuous signal into a sequence of pulses occurring at sampling instants 0, T, 2T, ……, where T is the sampling period.

$\mathbb{Z}$-Transform of standard discrete functions:

1. Prove that $\mathbb{Z}\{t\} = \dfrac{Tz}{(z-1)^2}$.

Proof: $\mathbb{Z}\{t\} = \sum_{n=0}^{\infty} (nT)z^{-n}$

$$= T\sum_{n=0}^{\infty} nz^{-n} = T(-z)\frac{d}{dz}\mathbb{Z}\{1\} \quad \left[\because \mathbb{Z}\{nu_n\} = -z\frac{dU(z)}{dz}, \text{ where } U(z) = \mathbb{Z}\{u_n\}\right]$$

$$= -Tz\frac{d}{dz}\left(\frac{z}{z-1}\right) = (-Tz)\left[\frac{(z-1).1-1.z}{(z-1)^2}\right] = \frac{Tz}{(z-1)^2}. \left[\because \mathbb{Z}\{1\} = \frac{z}{z-1}, \text{ if } |z|>1\right]$$

****

2. Prove that (i) $\mathbb{Z}\{e^{-at}\} = \dfrac{z}{z-e^{-aT}}, \quad |z| > |e^{-aT}|$;

(ii) $\mathbb{Z}\{e^{at}\} = \dfrac{z}{z-e^{aT}}, \quad |z| > |e^{aT}|$.

Proof: (i) $\mathbb{Z}\{e^{-at}\} = \sum_{n=0}^{\infty} e^{-anT}z^{-n} = \sum_{n=0}^{\infty} (e^{-aT})^n z^{-n} = \dfrac{z}{z-e^{-aT}}$, provided $|z| > |e^{-aT}|$.

$$\left[\because \mathbb{Z}\{a^n\} = \frac{z}{z-a}, \text{ if } |z|>|a|\right].$$

(ii) $\mathbb{Z}\{e^{at}\} = \sum_{n=0}^{\infty} e^{anT}z^{-n} = \sum_{n=0}^{\infty} (e^{aT})^n z^{-n} = \dfrac{z}{z-e^{aT}}$, provided $|z| > |e^{aT}|$.

$$\left[\because \mathbb{Z}\{a^n\} = \frac{z}{z-a},\ \text{ if } |z|>|a|\right].$$

****

3. Prove that $\mathbb{Z}\{\cos\omega t\} = \dfrac{z(z-\cos\omega t)}{z^2 - 2z\cos\omega T + 1}$.

Proof: $\mathbb{Z}\{\cos\omega t\} = \sum \cos\omega(nT)z^{-n} = \mathbb{Z}\{\cos n\theta\}$, where $\theta = \omega T$

$$= \frac{z(z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}. \qquad \left[\because \mathbb{Z}\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}\right]$$

Aliter: $\mathbb{Z}\{\cos\omega t\} = \mathbb{Z}\left\{\dfrac{e^{i\omega t} + e^{-i\omega t}}{2}\right\} = \mathbb{Z}\left\{\dfrac{e^{i\omega nT} + e^{-i\omega nT}}{2}\right\} = \mathbb{Z}\left\{\dfrac{\left(e^{i\omega T}\right)^n + \left(e^{-i\omega T}\right)^n}{2}\right\}$

$$= \frac{1}{2}\left[\frac{z}{z-e^{i\omega T}} + \frac{z}{z-e^{-i\omega T}}\right] \quad \left[\because \mathbb{Z}\{a^n\} = \frac{z}{z-a},\ \text{if } |z|>|a|\right].$$

$$= \frac{z}{2}\left[\frac{2z - \left(e^{i\omega T} + e^{-i\omega T}\right)}{\left(z-e^{i\omega T}\right)\left(z-e^{-i\omega T}\right)}\right]$$

$$= \frac{z}{2}\left[\frac{2z - 2\cos\omega T}{z^2 - \left(e^{i\omega T} + e^{-i\omega T}\right)z + 1}\right] = \frac{z(z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}.$$

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4. Prove that $\mathbb{Z}\{\sin\omega t\} = \dfrac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}$.

Proof: $\mathbb{Z}\{\sin\omega t\} = \mathbb{Z}\left\{\dfrac{e^{i\omega t} - e^{-i\omega t}}{2i}\right\} = \mathbb{Z}\left\{\dfrac{e^{i\omega nT} - e^{-i\omega nT}}{2i}\right\}$

$$= \frac{1}{2i}\left[\frac{z}{z-e^{i\omega T}} - \frac{z}{z-e^{-i\omega T}}\right] \quad \left[\begin{array}{l} \because \mathbb{Z}\{a^n\} = \dfrac{z}{z-a},\ \text{if } |z|>|a| \\ \mathbb{Z}\{e^{-at}\} = \dfrac{z}{z-e^{-aT}}, |z|>\left|e^{-aT}\right| \\ \mathbb{Z}\{e^{at}\} = \dfrac{z}{z-e^{aT}},\ \ |z|>\left|e^{aT}\right| \end{array}\right]$$

$$= \frac{z}{2i}\left[\frac{e^{i\omega T} - e^{-i\omega T}}{z^2 - z\left(e^{i\omega T} + e^{-i\omega T}\right) + 1}\right] = \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}.$$

****

5. Prove that $\mathbb{Z}\{t^k\} = -Tz\frac{d}{dz}\mathbb{Z}\{t^{k-1}\}$.

Proof: $\mathbb{Z}\{t^k\} = \sum_{n=0}^{\infty} (nT)^k z^{-n} = Tz\sum_{n=0}^{\infty} n^k T^{k-1} z^{-(n+1)} = Tz\sum_{n=0}^{\infty} (nT)^{k-1} nz^{-(n+1)}$ (i)

Since we have $\mathbb{Z}\{t^{k-1}\} = \sum_{n=0}^{\infty} (nT)^{k-1} z^{-n}$. (by definition)

Differentiating w.r.t. z, we get

$$\therefore \frac{d}{dz}\left[\mathbb{Z}\{t^{k-1}\}\right] = \sum_{n=0}^{\infty} (nT)^{k-1}(-n)z^{-(n+1)} = -\sum_{n=0}^{\infty} (nT)^{k-1} nz^{-(n+1)}. \quad \text{(ii)}$$

Using (ii) in (i), we get $\mathbb{Z}\{t^k\} = Tz\left[\sum_{n=0}^{\infty} (nT)^{k-1} nz^{-(n+1)}\right] = -Tz\frac{d}{dz}\left[\mathbb{Z}\{t^{k-1}\}\right]$. (iii)

Setting k = 1, 2, 3, …….., we get $\mathbb{Z}\{t\}$, $\mathbb{Z}\{t^2\}$,……….

$$\mathbb{Z}\{t\} = -Tz\frac{d}{dz}\left[\mathbb{Z}\{1\}\right] = -Tz\frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{Tz}{(z-1)^2}, \quad \text{(iv)}$$

$$\mathbb{Z}\{t^2\} = -Tz\frac{d}{dz}\mathbb{Z}\{t\} = -Tz\frac{d}{dz}\left(\frac{Tz}{(z-1)^2}\right) = \frac{T^2z(z+1)}{(z-1)^3}, \quad \text{(v)}$$

and so on.

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FIRST SHIFTING THEOREM:

Statement: If $\mathbb{Z}\{f(t)\} = F(z)$ then $\mathbb{Z}\{e^{-at}f(t)\} = F\{ze^{aT}\}$.

Proof: $\mathbb{Z}\{e^{-at}f(t)\} = \sum_{n=0}^{\infty} \left[e^{-anT}f(nT)\right]z^{-n} = \sum_{n=0}^{\infty} f(nT)\left(ze^{aT}\right)^{-n}$

$$= F\{ze^{aT}\} \qquad \left[\because \mathbb{Z}\{f(t)\} = F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}\right]$$

Thus $\mathbb{Z}\{e^{-at}f(t)\} = \mathbb{Z}\{f(t)\}_{z \to ze^{aT}}$

$= F\{z\}$, where $z \to ze^{aT}$.

Deductions of some important results from this theorem:

1. Prove that $\mathbb{Z}\left\{e^{-at}\right\}=\dfrac{z}{z-e^{-aT}}$.

Proof: $\mathbb{Z}\left\{e^{-at}\right\}=\mathbb{Z}\left\{e^{-at}.1\right\}=\mathbb{Z}\{1\}_{z\to ze^{aT}}$

$$=\left(\frac{z}{z-1}\right)_{z\to ze^{aT}} \qquad \left[\because \mathbb{Z}\{1\}=\frac{z}{z-1},\text{ if } |z|>1\right]$$

$$=\frac{ze^{aT}}{ze^{aT}-1}.$$

$$\Rightarrow \mathbb{Z}\left\{e^{-at}\right\}=\frac{z}{z-e^{-aT}}.$$

2. Prove that $\mathbb{Z}\left\{te^{-at}\right\}=\dfrac{Tze^{-aT}}{\left(z-e^{-aT}\right)^2}$.

Proof: $\mathbb{Z}\left\{te^{-at}\right\}=\mathbb{Z}\left\{e^{-at}.t\right\}=\mathbb{Z}\{t\}_{z\to ze^{aT}}$

$$=\left[\frac{Tz}{(z-1)^2}\right]_{z\to ze^{aT}} \qquad \left[\because \mathbb{Z}\{t\}=\frac{Tz}{(z-1)^2}\right]$$

$$=\frac{Tze^{aT}}{\left(ze^{aT}-1\right)^2}$$

$$\Rightarrow \mathbb{Z}\left\{te^{-at}\right\}=\frac{Tze^{-aT}}{\left(z-e^{-aT}\right)^2}.$$

3. Prove that $\mathbb{Z}[\cos at]=\dfrac{z(z-\cos aT)}{z^2-2z\cos aT+1}$ and $\mathbb{Z}[\sin at]=\dfrac{z\sin aT}{z^2-2z\cos aT+1}$.

Proof: $\mathbb{Z}\left\{e^{-iat}\right\}=\mathbb{Z}\left\{e^{-iat}.1\right\}=\mathbb{Z}\{1\}_{z\to ze^{iaT}}$

$$=\left(\frac{z}{z-1}\right)_{z\to ze^{iaT}}=\frac{z}{z-e^{-iaT}}=\frac{z}{\left(z-e^{-iaT}\right)}\times\frac{\left(z-e^{iaT}\right)}{\left(z-e^{iaT}\right)}$$

$$=\frac{z\left[z-(\cos aT+i\sin aT)\right]}{z^2-z\left(e^{iaT}+e^{-iaT}\right)+1}$$

$$\Rightarrow \mathbb{Z}\left\{e^{-iat}\right\}=\frac{z\left[(z-\cos aT)-i\sin aT\right]}{z^2-2z\cos aT+1}$$

$$\Rightarrow \mathbb{Z}(\cos at - i\sin at) = \frac{z\left[(z-\cos aT) - i\sin aT\right]}{z^2 - 2z\cos aT + 1}$$

Equating real and imaginary parts, we get

$$\mathbb{Z}[\cos at] = \frac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1} \text{ and } \mathbb{Z}[\sin at] = \frac{z\sin aT}{z^2 - 2z\cos aT + 1}.$$

4. Prove that $\mathbb{Z}\left\{e^{-at}\cos bt\right\} = \dfrac{ze^{aT}\left\{ze^{aT} - \cos bT\right\}}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}$.

Proof: $\mathbb{Z}\left\{e^{-at}\cos bt\right\} = \mathbb{Z}\left\{\cos bt\right\}_{z\to ze^{aT}}$

$$= \left[\frac{z(z-\cos bT)}{z^2 - 2z\cos bT + 1}\right]_{z\to ze^{aT}}$$

$$= \frac{ze^{aT}\left\{ze^{aT} - \cos bT\right\}}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}.$$

5. Prove that $\mathbb{Z}\left\{e^{-at}\sin bt\right\} = \dfrac{ze^{aT}\sin bT}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}$.

Proof: $\mathbb{Z}\left\{e^{-at}\sin bt\right\} = \mathbb{Z}\left\{\sin bt\right\}_{z\to ze^{aT}}$

$$= \left[\frac{z\sin bT}{z^2 - 2z\cos \omega T + 1}\right]_{z\to ze^{aT}}$$

$$= \frac{ze^{aT}\sin bT}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}.$$

SECOND SHIFTING THEOREM:

Statement: If $\mathbb{Z}\{f(t)\} = F(z)$ then $\mathbb{Z}\{f(t+T)\} = z\left[F(z) - f(0)\right]$.

Proof: Since $\mathbb{Z}\{f(t+T)\} = \sum_{n=0}^{\infty} f(nT+T)z^{-n}$

$$= z\sum_{n=0}^{\infty} f[(n+1)T]z^{-(n+1)} = z\sum_{k=1}^{\infty} f(kT)z^{-k} \qquad [\text{Put } n+1=k\ ]$$

$$= z\left[\sum_{k=0}^{\infty} f(kT)z^{-k} - f(0)\right] = z[F(z) - f(0)].$$

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TWO BASIC THEOREMS

Initial Value Theorem: If $\mathbb{Z}\{f(t)\} = F(z)$ then $f(0) = \underset{z\to\infty}{\mathrm{Lt}} F(z)$.

Proof: $F(z) = \mathbb{Z}\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$

$$= f(0.T) + \frac{f(1.T)}{z} + \frac{f(2.T)}{z^2} + \ldots\ldots\ldots\ldots\ldots\infty$$

$$= f(0) + \frac{1}{z} f(T) + \frac{1}{z^2} f(2T) + \ldots\ldots\infty$$

$\therefore \underset{z\to\infty}{\mathrm{Lt}} F(z) = f(0)$.

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Final Value Theorem: If $\mathbb{Z}[f(t)] = F(z)$ then $\underset{t\to\infty}{\mathrm{Lt}} f(t) = \underset{z\to 1}{\mathrm{Lt}} (z-1)F(z)$.

Proof: $\mathbb{Z}[f(t+T) - f(t)] = \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n}$

$$\Rightarrow \mathbb{Z}[f(t+T)] - \mathbb{Z}[f(t)] = \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n}$$

$$\Rightarrow zF(z) - zf(0) - F(z) = \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n}$$

$$\therefore \underset{z\to 1}{\mathrm{Lt}} [(z-1)F(z) - zf(0)] = \underset{z\to 1}{\mathrm{Lt}} \left[\sum_{n=0}^{\infty} \{f(nT+T) - f(nT)\} z^{-n}\right]$$

$$\underset{z\to 1}{\mathrm{Lt}} [(z-1)F(z) - f(0)] = \sum_{n=0}^{\infty} \{f(nT+T) - f(nT)\}$$

$$= \underset{n\to\infty}{\mathrm{Lt}} [f(T) - f(0) + f(2T) - f(T) + \ldots.. + f\{(n+1)T\} - f(nT)]$$

$$= \underset{n\to\infty}{\mathrm{Lt}} f\{(n+1)T\} - f(0) = f(\infty) - f(0)$$

$$\Rightarrow \underset{z\to 1}{\mathrm{Lt}} [(z-1)F(z) - f(0)] = \underset{t\to\infty}{\mathrm{Lt}} f(t) - f(0)$$

$$\Rightarrow \underset{z\to 1}{\mathrm{Lt}} (z-1)F(z) = \underset{t\to\infty}{\mathrm{Lt}} f(t).$$

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1st Topic

(Part-II)

$\mathbb{Z}$-Transforms

- Initial value theorem and Final value theorem
- Convolution theorem

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Now let us find $\mathbb{Z}$-transform of various types of sequences:

Q.No.1.: Find the $\mathbb{Z}$-transform of

(i) n^2, **(ii)** $n(n-1)$, **(iii)** $\frac{1}{n}$,

(iv) $\frac{1}{n(n+1)}$, **(v)** $\cos\frac{n\pi}{2}$.

Sol.: (i) $\mathbb{Z}(n^2)=\mathbb{Z}(n.n)=-\frac{d}{dz}\mathbb{Z}(n)=-\frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\}=\frac{z(z+1)}{(z-1)^3}$. Ans.

(ii) $\mathbb{Z}\{n(n-1)\}=\mathbb{Z}\{n^2-n\}=\mathbb{Z}(n^2)-\mathbb{Z}(n)$

$$=\frac{z(z+1)}{(z-1)^3}-\frac{z}{(z-1)^2}=\frac{z^2+z-z(z-1)}{(z-1)^3}=\frac{2z}{(z-1)^3}.\text{ Ans.}$$

(iii) $\mathbb{Z}\left\{\frac{1}{n}\right\}=\sum_{n=1}^{\infty}\frac{1}{n}z^{-n}=\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3z^3}+\ldots\ldots\infty=-\log\left(1-\frac{1}{z}\right)$, if $\left(\frac{1}{z}\right)<1$

$$=\log\left(\frac{z}{z-1}\right),\text{ if } |z|>1.$$

(iv) $\mathbb{Z}\left\{\frac{1}{n(n+1)}\right\} = \mathbb{Z}\left\{\frac{1}{n} - \frac{1}{n+1}\right\} = \mathbb{Z}\left\{\frac{1}{n}\right\} - \mathbb{Z}\left\{\frac{1}{n+1}\right\}$

$$= \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} - \sum_{n=1}^{\infty} \frac{1}{n+1} z^{-n}$$

$$= \log\left(\frac{z}{z-1}\right) - \left\{1 + \frac{1}{2z} + \frac{1}{3z^2} + \ldots\ldots\ldots\ldots\infty\right\}$$

$$= \log\left(\frac{z}{z-1}\right) - z\left\{\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \ldots\ldots\ldots\infty\right\}$$

$$= \log\left(\frac{z}{z-1}\right) - z\left\{-\log\left(1 - \frac{1}{z}\right)\right\} = (z-1)\log\left\{\frac{z-1}{z}\right\}. \text{ Ans.}$$

(v) $\mathbb{Z}\left\{\cos\frac{n\pi}{2}\right\} = \sum_{n=0}^{\infty} \cos\frac{n\pi}{2} z^{-n} = 1 - \frac{1}{z^2} + \frac{1}{z^4} - \ldots\ldots\ldots\ldots\ldots\infty$

$$= \left[1 + \frac{1}{z^2}\right]^{-1} = \frac{z^2}{z^2+1}, \quad \text{if } |z| > 1. \text{ Ans.}$$

Q.No.2.: Find the $\mathbb{Z}$ -transform of

(i) $\frac{1}{2}(n+1)(n+2)$, **(ii)** $ab^n\ (a \neq 0,\ b \neq 0)$,

(iii) $f(n) = \begin{cases} 1 & ; \quad n = k \\ 0 & ; \quad n \neq k \end{cases}$, **(v)** $f(n) = \begin{cases} 0 & \text{if} \quad n > 0 \\ 1 & \text{if} \quad n \leq 0 \end{cases}$,

(vi) $f(n) = \begin{cases} \frac{a^n}{n!}, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$.

Sol.: (i) $\mathbb{Z}\left\{\frac{1}{2}(n+1)(n+2)\right\} = \frac{1}{2}\left\{\mathbb{Z}\left(n^2\right) + 3\mathbb{Z}(n) + 2\mathbb{Z}(1)\right\}$

$$= \frac{1}{2}\left[\frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{(z-1)}\right], \text{ if } |z| > 1. \text{ Ans.}$$

(ii) $\mathbb{Z}\left\{ab^n\right\} = a\mathbb{Z}\left(b^n\right) = a\sum_{n=0}^{\infty} b^n z^{-n} = a\sum_{n=0}^{\infty} \left(\frac{b}{z}\right)^n = a\left\{\frac{z}{z-b}\right\} = \frac{az}{z-b}$, if $|z| > |b|$. Ans.

(iii) $\mathbb{Z}\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} nz^{z-n} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \ldots\ldots\ldots\infty$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2} = \frac{z}{(z-1)^2}, \text{ if } |z| > 1. \text{ Ans.}$$

(iv) $\mathbb{Z}\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{\substack{n=0 \\ n\neq k}}^{\infty} (0).z^{-n} + .z^{-k} = \frac{1}{z^k}$. Ans.

(v) $\mathbb{Z}\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n} = \sum_{n=-\infty}^{\infty} (1) z^{-n} = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, if $|z| < 1$. Ans.

(vi) $\mathbb{Z}\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{\left(az^{-1}\right)^n}{n!} = e^{az^{-1}} = e^{a/z}$. Ans.

Q.No.3.: Find the $\mathbb{Z}$ -transform of

(i). $u(n-1)$ **(ii)** $4^n \delta(n-1)$ **(iii).** $\delta(n-k)$.

Sol.: (i) $\mathbb{Z}\{u(n-1)\} = \sum_{n=1}^{\infty} 1.z^{-n} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\ldots\ldots\infty = \frac{1}{z}\left\{1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\ldots\ldots\ldots\right\}$

$$= \frac{1}{z}\left\{\frac{1}{1 - \frac{1}{z}}\right\} \text{ if } \left|\frac{1}{z}\right| < 1 = \frac{1}{z-1}, \text{ if } |z| > 1. \text{ Ans.}$$

(ii) $\mathbb{Z}\{4^n \delta(n-1)\} = \sum_{n=0}^{\infty} 4^n \delta(n-1) z^{-n} = \frac{4}{z}$. Ans.

(iii) $\mathbb{Z}\{\delta(n-k)\} = \sum_{n=0}^{\infty} \delta(n-k) z^{-n} = \frac{1}{z^k}$, k is a positive integer. Ans.

Q.No.4.: Find the $\mathbb{Z}$ -transform of

(i) $e^{-t}t^2$, **(ii)** $e^{-2t}t^3$, **(iii)** e^{2t+5},

(iv) $e^{3t}\sin 2t$, **(v)** $e^{4t}\cos t$, **(vi)** $a^n \cos n\pi$.

Sol.: (i) $\mathbb{Z}\left[e^{-t}t^2\right] = \mathbb{Z}\{t^2\}_{z\to ze^T} = \left\{\frac{T^2 z(z+1)}{(z-1)^3}\right\}_{z\to ze^T} = \frac{T^2 ze^T\left(ze^T + 1\right)}{\left(ze^T - 1\right)^3}$. Ans.

(ii) $\mathbb{Z}\left[e^{-2t}t^3\right] = \mathbb{Z}\{t^3\}_{z\to ze^{2T}}$

$$=\left\{\frac{T^3z\left(1+4z+z^2\right)}{(z-1)^4}\right\}_{z\to ze^{2T}}=\frac{T^3ze^{2T}\left(1+4ze^{2T}+z^2e^{4t}\right)}{\left(ze^{2T}-1\right)^4}. \text{ Ans.}$$

(iii) $\mathbb{Z}\left\{e^{2t+5}\right\}=e^5\mathbb{Z}\left\{e^{2t}\right\}=e^5.\frac{z}{z-e^{2T}}$. Ans.

(iv) $\mathbb{Z}\left\{e^{3t}\sin 2t\right\}=\mathbb{Z}\{\sin 2t\}_{z\to ze^{-3t}}=\left[\frac{z\sin 2T}{z^2-2z\cos 2T+1}\right]_{z\to ze^{-3t}}$

$$=\frac{ze^{-3t}\sin 2T}{z^2e^{-6T}-2ze^{-3t}\cos 2T+1}. \text{ Ans.}$$

(v) $\mathbb{Z}\left\{e^{4t}\cos t\right\}=\mathbb{Z}\{\cos t\}_{z\to ze^{-4t}}=\left[\frac{z(z-\cos T)}{z^2-2z\cos T+1}\right]_{z\to ze^{-4t}}$

$$=\frac{ze^{-4t}\left(ze^{-4t}-\cos T\right)}{z^2e^{-8T}-2ze^{-4t}\cos T+1}. \text{ Ans.}$$

(vi) $\mathbb{Z}\left(a^n\cos n\pi\right)=\sum_{n=0}^{\infty} a^n\cos n\pi.z^{-n}$

$$=1-\frac{a}{z}+\left(\frac{a}{z}\right)^2-\left(\frac{a}{z}\right)^3+\ldots\ldots\ldots\infty=\left(1+\frac{a}{z}\right)^{-1} \text{ if } \left|\frac{a}{z}\right|<1$$

$$=\frac{z}{z+a}, \text{ if } |z|<|a|. \text{ Ans.}$$

Q.No.5.: If $F(z)=\frac{z(z-\cos aT)}{z^2-2z\cos aT+1}$, find f(0).

Sol.: From initial value theorem, $f(0)=\underset{z\to\infty}{\mathrm{Lt}}\, F(z)$.

Initial value theorem: If $\mathbb{Z}\{f(t)\}=F(z)$, then $f(0)=\underset{z\to\infty}{\mathrm{Lt}}\, f(z)$.

$\therefore f(0)=\underset{z\to\infty}{\mathrm{Lt}}\,\frac{z(z-\cos aT)}{z^2-2z\cos aT+1}=1$. (by L'-Hospital rule)

Q.No.6.: If $F(z)=\frac{z}{z-e^{-T}}$, find $\underset{t\to\infty}{\mathrm{Lt}}\, f(t)$.

Sol.: From final value theorem, we have

$$\underset{t\to\infty}{\mathrm{Lt}}\, f(t)=\underset{z\to 1}{\mathrm{Lt}}\,(z-1)F(z)=\underset{z\to 1}{\mathrm{Lt}}\,(z-1)\frac{z}{z-e^{-T}}=0.$$

Q.No.7.: Find the $\mathbb{Z}$-transform of

(i) $\sin(t+T)$, **(ii)** $e^{2(t+T)}$, **(iii)** $(t+T)e^{-(t+T)}$.

Sol.: (i) $\mathbb{Z}\{\sin(t+T)\} = \mathbb{Z}\{f(t+T)\}$ where $f(t) = \sin t$

$$= z\{F(z) - f(0)\} = z\left[\frac{z\sin T}{z^2 - 2z\cos T + 1} - 0\right] = \frac{z^2 \sin T}{z^2 - 2z\cos T + 1}. \text{ Ans.}$$

Aliter: $\mathbb{Z}\{\sin(t+T)\} = \mathbb{Z}\{\sin t\cos T + \cos t \sin T\}$

$$= \cos T.\mathbb{Z}(\sin t) + \sin T.\mathbb{Z}(\cos t)$$

$$= \cos T\left\{\frac{z\sin T}{z^2 - 2z\cos T + 1}\right\} + \left\{\frac{z(z - \cos T)}{z^2 - 2z\cos T + 1}\right\}$$

$$= \frac{z^2 \sin T}{z^2 - 2z\cos T + 1}. \text{ Ans.}$$

(ii) $\mathbb{Z}\{e^{2(t+T)}\} = \mathbb{Z}\{f(t+T)\}$, where $f(t) = e^{2t}$

SECOND SHIFTING THEOREM:

Statement: If $\mathbb{Z}\{f(t)\} = F(z)$ then $\mathbb{Z}\{f(t+T)\} = z[F(z) - f(0)]$.

$$= z[F(z) - f(0)] = z\left[\frac{z}{z - e^{2T}} - 1\right]$$

$$= \frac{ze^{2T}}{z - e^{2T}}. \text{ Ans.}$$

Aliter: $\mathbb{Z}\{e^{2(t+T)}\} = e^{2T}\mathbb{Z}\{e^{2t}\} = e^{2T}.\dfrac{z}{z - e^{2T}}$. Ans.

(iii) $\mathbb{Z}\left[(t+T)e^{-(t+T)}\right] = e^{-T}\mathbb{Z}\{(t+T)e^{-t}\} = e^{T}\left[\mathbb{Z}(te^{-t}) + T\mathbb{Z}(e^{-t})\right]$

$$= e^{T}\left[\frac{Tze^{T}}{(ze^{T} - 1)^2} + \frac{Tz}{z - e^{-T}}\right] = e^{-T}\left[\frac{Tze^{T}}{(ze^{T} - 1)^2} + \frac{Te^{T}z}{ze^{T} - 1}\right]$$

$$= e^{-T}.Tze^{T}\left[\frac{ze^{T}}{(ze^{T} - 1)^2}\right] = \frac{Tz^2e^{T}}{(ze^{T} - 1)^2}. \text{ Ans.}$$

Remember: $\mathbb{Z}^{-1}\left[z^{-m}F(z)\right] = f(n-m) = \mathbb{Z}^{-1}\{F(n)\}_{n \to n-m}$.

Q.No.8.: Find $\mathbb{Z}^{-1}\left\{\dfrac{1}{z+1}\right\}$ given $\mathbb{Z}^{-1}\left\{\dfrac{z}{z+1}\right\} = (-1)^n$.

Sol.: $\mathbb{Z}^{-1}\left\{\frac{1}{z+1}\right\}=\mathbb{Z}^{-1}\left\{z^{-1}.\frac{z}{z+1}\right\}$

$$=\mathbb{Z}^{-1}\left\{\frac{z}{z+1}\right\}_{n\to n-1}=\left\{(-1)^n\right\}_{n\to n-1}=(-1)^{n-1},\ \ n=1,2,3,.....$$

Q.No.9.: Find $\mathbb{Z}^{-1}\left\{\frac{3}{3z-1}\right\}$.

Sol.: $\mathbb{Z}^{-1}\left\{\frac{3}{3z-1}\right\}=\mathbb{Z}^{-1}\left\{\frac{1}{1-\frac{1}{3}}\right\}=\mathbb{Z}^{-1}\left[z^{-1}\left\{\frac{z}{z-\frac{1}{3}}\right\}\right]$

$$=\mathbb{Z}^{-1}\left\{\frac{z}{z-\frac{1}{3}}\right\}_{n\to n-1}=\left(\frac{1}{3}\right)^{n-1}\Rightarrow\left(\frac{1}{3}\right)^{n-1}u(n-1)\textbf{.}\text{ Ans.}$$

Differentiation:

Let $\mathbb{Z}\{f(n)\}=F(z)$. An infinite series can be differentiated term by term with in its region of convergence. F(z) may be treated as a function of z^{-1}.

$$F(z)=\sum_{n=0}^{\infty}f(n)z^{-n}=\sum_{n=0}^{\infty}f(n)\left(z^{-1}\right)^n.$$

Differentiating on both sides w.r.t. z^{-1}, we get

$$\frac{d}{dz^{-1}}F(z)=\sum_{n=0}^{\infty}nf(n)\left(z^{-1}\right)^{n-1} \qquad \text{(i)}$$

$$\Rightarrow z^{-1}\frac{dF(z)}{dz^{-1}}=\sum_{n=0}^{\infty}nf(n)z^{-n}=\mathbb{Z}\{nf(n)\}.$$

$$\therefore \mathbb{Z}\{nf(n)\}=z^{-1}\frac{dF(z)}{dz^{-1}}. \qquad \text{(ii)}$$

Differentiating (i) w.r.t. z^{-1} again, we get

$$\frac{d^2F(z)}{d\left(z^{-1}\right)^2}=\sum_{n=0}^{\infty}n(n-1)f(n)\left(z^{-1}\right)^{n-2}$$

$$\Rightarrow z^{-2}\frac{d^2F(z)}{d\left(z^{-1}\right)^2}=\sum_{n=0}^{\infty} n(n-1)f(n)z^{-n}=\mathbb{Z}\{n(n-1)f(n)\}.$$

$$\therefore \mathbb{Z}\{n(n-1)f(n)\}=z^{-2}\frac{d^2F(z)}{d\left(z^{-1}\right)^2}.$$

Q.No.1.: Find the $\mathbb{Z}$ -transforms of

(i) $na^n u(n)$, **(ii)** $n(n-1)a^n u(n)$.

Sol.: (i) $\mathbb{Z}\{na^n u(n)\}=z^{-1}\dfrac{d}{dz^{-1}}\left(\dfrac{z}{z-a}\right)=z^{-1}\dfrac{d}{dz^{-1}}\left[1-az^{-1}\right]^{-1}$

$$=z^{-1}\frac{a}{\left(1-az^{-1}\right)^2}=\frac{az^{-1}}{\left(1-az^{-1}\right)^2}. \text{ Ans.}$$

(ii) $\mathbb{Z}\{n(n-1)a^n u(n)\}=z^{-2}\dfrac{d^2}{d\left(z^{-1}\right)^2}\left\{1-az^{-1}\right\}^{-1}=\dfrac{2a^2z^2}{\left(1-az^{-1}\right)^3}$. Ans.

Note: If a = 1 then $\mathbb{Z}\{nu(n)\}=\dfrac{z^{-1}}{\left(1-z^{-1}\right)^2}$; $\mathbb{Z}\{n(n-1)u(n)\}=\dfrac{2z^2}{\left(1-z^{-1}\right)^3}$.

Convolution of sequences:

Definition: The convolution of two sequences {f(n)} and {g}n)} is defined as

$$\omega(n)=\sum_{k=-\infty}^{\infty} f(k)g(n-k)=f*g.$$

Remarks: If it is one sided (right) sequence, take f(k) = 0, g(k) = 0 for k < 0,

then $\omega(n)=\sum_{k=0}^{\infty} f(k)g(n-k)=f*g$.

Convolution Theorem:

Statement: If w(n) is the convolution of two sequences f(n) and g(n), then

$$\mathbb{Z}\{w(n)\}=W(z)=\mathbb{Z}\{f(n)\}.\mathbb{Z}\{g(n)\}=F(z).G(z)$$

Proof: $W(z)=\mathbb{Z}\{w(n)\}=\mathbb{Z}\left[\sum_{k=0}^{\infty} f(k)g(n-k)\right]=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty} f(k)g(n-k)\right]z^{-n}$

$$= \sum_{k=0}^{\infty} f(k)\left[\sum_{n=0}^{\infty} g(n-k)z^{-n}\right] \text{(by changing the order of summation)}$$

$$= \sum_{k=0}^{\infty} f(k)\left[\sum_{p=0}^{\infty} g(p)z^{-(p+k}\right] \text{(Putting } n-k=p\text{)}$$

$$= \left[\sum_{k=0}^{\infty} f(k)z^{-k}\right]\left[\sum_{p=0}^{\infty} g(p)z^{-p}\right] = F(z)G(z).$$

Note: This result will be true only for those values of z inside the region of convergence.

Another form of Convolution Theorem:

Statement: If $\mathbb{Z}\{f(t)\} = F(z)$, $\mathbb{Z}\{g(t)\} = G(z)$, then convolution product is

$$w(t) = \sum_{k=0}^{n} f(kT)g(nT-kT) = f * g$$

and $\mathbb{Z}\{w(t)\} = W(z) = \mathbb{Z}\{f(t)\}\mathbb{Z}\{g(t)\} = F(z)G(z)$

Proof: (Here we are dealing with one sided $\mathbb{Z}$ -transforms only)

$$F(z) = \sum_{m=0}^{\infty} f(mT)z^{-m} \quad ; \quad F(z) = \sum_{m=0}^{\infty} g(nT)z^{-n}$$

$$F(z)G(z) = \left\{\sum_{m=0}^{\infty} f(mT)z^{-m}\right\}\left\{\sum_{n=0}^{\infty} g(nT)z^{-n}\right\}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(mT)g(nT)z^{-m-n} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} f(pT)g\{(n-p)T\}\right)z^{-n}$$

$$= \mathbb{Z}\left(\sum_{p=0}^{\infty} f(pT)g\{(n-p)T\}\right)z^{-n} = \mathbb{Z}\{f * g\}.$$

Importance: The convolution theorem plays an important role in the solution of differential equations and in probability problems involving sums of two independent random variables.

Q.No.1.: Find the $\mathbb{Z}$ -transforms of f*g, where

(i) $f(n) = u(n)$, $g(n) = 2^n u(n)$.

(ii) $f(n) = 3^n u(n)$, $g(n) = 4^n u(n)$, using convolution theorem.

Sol.: (i) $F(z) = \mathbb{Z}\{u(n)\} = \sum_{n=0}^{\infty} 1.z^{-n} = \frac{z}{z-1}$ if $|z| > 1$

$$G(z) = \mathbb{Z}\{2^n u(n)\} = \sum_{n=0}^{\infty} 2^n z^{-n} = \frac{z}{z-2} \text{ if } |z| > |2|$$

By Convolution theorem, we get

$$\mathbb{Z}\{f * g\} = \mathbb{Z}\{w(n)\} = W(z) = F(z).G(z)$$

$$= \frac{z}{z-1} \cdot \frac{z}{z-2} = \frac{z^2}{(z-1)(z-2)}, \text{ if } |z| > |2|.$$

(ii) $F(z) = \mathbb{Z}\{3^n u(n)\} = \frac{z}{z-3}$, if $|z| > |3|$.

$$G(z) = \mathbb{Z}\{4^n u(n)\} = \frac{z}{z-4}, \text{ if } |z| > |4|.$$

By Convolution theorem, we get

$$\mathbb{Z}\{f * g\} = \mathbb{Z}\{w(n)\} = W(z) = F(z).G(z)$$

$$= \frac{z}{z-3} \cdot \frac{z}{z-4} = \frac{z^2}{(z-3)(z-4)}, \text{ if } |z| > |4|.$$

Some useful $\mathbb{Z}$ -Transforms:

$\mathbb{Z}$ -Transform of sequences:

Sr. No.	F(n)	$\mathbb{Z}\{f(n)\}$
1	1	$\frac{z}{z-1}$
2	n	$\frac{z}{(z-1)^2}$
3	n^2	$\frac{z^2+z}{(z-1)^3}$
4	z^3	$\frac{z^3+4z^2+z}{(z-1)^4}$
5	$n(n-1)$	$\frac{2z}{(z-1)^2}$

6	$n^{(k)}$	$\frac{k!z}{(z-1)^{k+1}}$
7	$(-1)^n$	$\frac{z}{z+1}$
8	$u(n)$	$\frac{z}{z-1}, \ \lvert z\rvert > 1$
9	$u(n-1)$	$\frac{1}{z-1}$
10	$u(n-k)$	$z^{-k}.\frac{z}{z-1}$
11	$\frac{1}{n}$	$\log\left(\frac{z}{z-1}\right), \ \lvert z\rvert > 1$
12	$\frac{1}{n+1}$	$z\log\left(\frac{z}{z-1}\right)$
13	$\frac{1}{n!}$	$e^{1/z}$
14	$\delta(n)$	1
15	$\delta(n-k)$	z^{-k}
16	$a^n u(n)$ or a^n	$\frac{z}{z-a}$
17	$na^n u(n)$ or na^n	$\frac{az}{(z-a)^2}$
18	$(n+1)a^n u(n)$ or $(n+1)a^n$	$\frac{z^2}{(z-a)^2}$
19	$n(n-1)a^n u(n)$ or $n(n-1)a^n$	$\frac{2a^2 z}{(z-a)^3}$
20	$\cos\frac{n\pi}{2}$ or $\cos\frac{n\pi}{2}.u(n)$	$\frac{z^2}{z^2+1}$
21	$\sin\frac{n\pi}{2}$ or $\sin\frac{n\pi}{2}.u(n)$	$\frac{z}{z^2+1}$

22	$a^n \cos\frac{n\pi}{2}.u(n)$ or $a^n \cos\frac{n\pi}{2}$	$\frac{z^2}{z^2+a^2}$
23	$a^n \sin\frac{n\pi}{2}.u(n)$ or $a^n \sin\frac{n\pi}{2}$	$\frac{az}{z^2+a^2}$
24	$\cos n\theta.u(n)$ or $\cos n\theta$	$\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$
25	$\sin n\theta.u(n)$ or $\sin n\theta$	$\frac{z\sin\theta}{z^2-2z\cos\theta+1}$
26	$r^n \cos n\theta.u(n)$ or $r^n \cos n\theta$	$\frac{z(z-r\cos\theta)}{z^2-2zr\cos\theta+r^2}$
27	$r^n \sin n\theta.u(n)$ or $r^n \sin n\theta$	$\frac{rz\sin\theta}{z^2-2zr\cos\theta+r^2}$
28	$a^n f(n)$	$F\left(\frac{z}{a}\right)$
29	$nf(n)$	$-z\frac{dF(z)}{dz}$
30	n^{m+1}	$(-z)^m \frac{d^m}{dz^m} z(n)$

$\mathbb{Z}$ -Transform of standard discrete functions:

Sr. No.	F(t)	$\mathbb{Z}\{f(t)\}$
1	t^k	$-Tz\frac{d}{dz}\left[Z\left(t^{k-1}\right)\right]$
2	t	$\frac{Tz}{(z-1)^2}$
3	t^2	$\frac{T^2z(z+1)}{(z-1)^3}$
4	t^3	$\frac{T^3z(1+4z-z^2)}{(z-1)^4}$

5	$a^n f(t)$	$F\left(\frac{z}{a}\right)$
6	$nf(nt) = nf(t)$	$-z\frac{d}{dz}F(z)$
7	e^{-at}	$\frac{z}{z-e^{-aT}}$ if $\lvert z\rvert > \lvert e^{-aT}\rvert$
8	e^{at}	$\frac{z}{z-e^{aT}}$ if $\lvert z\rvert > \lvert e^{-aT}\rvert$
9	$\sin \omega t$	$\frac{z\sin \omega t}{z^2 - 2z\cos \omega T + 1}$ if $\lvert z\rvert > 1$
10	$\cos \omega t$	$\frac{z(z-\cos \omega T)}{z^2 - 2z\cos \omega T + 1}$ if $\lvert z\rvert > 1$

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Home Assignments

Q.No.1.: Find the $\mathbb{Z}$-transforms (One sided) of the following sequences $\{f(n)\}$, where $f(n)$ is

(i) $\left(\frac{1}{4}\right)^n u(n)$, **(ii)** $\delta(n-3)$, **(iii)** $(-1)^n u(n)$,

(iv) $3^n \sin\frac{n\pi}{2}$, **(v)** $2^n \cos\frac{n\pi}{2}$, **(vi)** $4^n + \left(\frac{1}{2}\right)^n + u(n-3)$.

Ans.: (i) $\frac{4z}{4z-1}$ **(ii)** $\frac{1}{z^2}$ **(iii)** $\frac{z}{z+1}$

(iv) $\frac{3z}{z^2+9}$ **(v)** $\frac{z^2}{z^2+4}$ **(vi)** $\frac{z}{z-4} + \frac{2z}{2z-1} + \frac{1}{z^2(z-1)}$.

Q.No.2.: Find the $\mathbb{Z}$-transforms of $f(t)$

(i) e^{-3t+4}, **(ii)** $\sin 3t$, **(iii)** $\cos 2t$, **(iv)** $\sin^2 3t$,

(v) $\cos^3 t$ **,** **(vi)** $e^{-at}\sin bt$ **,** **(vii)** $e^{-t}.t^2$**.**

Ans.: (i) $e^4.\dfrac{z}{z-e^{-3T}}$, **(ii)** $\dfrac{z\sin 3T}{z^2-2z\cos 3T+1}$, **(iii)** $\dfrac{z(z-\cos 2T)}{z^2-2z\cos 2T+1}$,

(iv) $\dfrac{1}{2}\dfrac{z}{z-1}-\dfrac{1}{2}\left[\dfrac{z(z-\cos 6T)}{z^2-2z\cos 6T+1}\right]$,

(v) $\dfrac{3}{4}\left[\dfrac{z(z-\cos T)}{z^2-2z\cos T+1}\right]+\dfrac{1}{4}\left[\dfrac{z(z-\cos T)}{z^2-2z\cos 3T+1}\right]$,

(vi) $\dfrac{ze^{eT}\sin bt}{z^2 2^{2aT}-2ze^{aT}\cos bT+1}$, **(vii).** $\dfrac{T^2ze^T\left(ze^T+1\right)}{\left(2e^T-1\right)^3}$**.**

Q.No.3.: Given $\mathbb{Z}(u_n)=\dfrac{2z^2+3z+4}{(z-3)^3}$, $|z|>3$, show that $u_1 = 2$, $u_2 = 21$, $u_3 = 139$.

Q.No.4.: Using $\mathbb{Z}(n)=\dfrac{z}{(z-1)^2}$, show that $\mathbb{Z}(n\cos n\theta)=\dfrac{(z^3+z)\cos\theta-2z^2}{(z^2-2z\cos\theta+1)^2}$.

Q.No.5.: Find convolution of

(i) $n(n-1)*3^n$**,** **(ii)** $3^n*\cos n\theta$ **,** **(iii)** $\cos\dfrac{n\pi}{2}*\sin\dfrac{n\pi}{2}$**.**

Ans.: (i) $\dfrac{2z^2}{(z-1)^2(z-3)}$, **(ii)** $\dfrac{z^2(z-\cos\theta)}{(z-3)\left(z^2-2z\cos\theta+1\right)}$, **(iii).** $\dfrac{z^3}{\left(z^2+1\right)^2}$.

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2nd Topic

Fourier Transforms

- Sufficient conditions for the existence of Fourier Transform
- Sufficient conditions for the existence of Inverse Fourier Transform
- Definition of Fourier Transform of f(x)

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Fourier Transform:

Sufficient conditions for the existence of Fourier Transform:

1. f(x) is **piecewise continuous** on every finite interval, and
2. f(x) is **absolutely integrable** on the x-axis.

Sufficient conditions for the existence of Inverse Fourier Transform:

1. F(s) is **absolutely integrable** over $(-\infty,\infty)$.
2. $\lim\limits_{|s|\to\infty} F(s) = 0$

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Fourier transform of f(x):

The Fourier integral of f(x) in the complex form given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt\right]e^{-isx}ds$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{ist}dt\right]e^{-isx}ds \qquad \text{(i)}$$

The expression in bracket, a function of s denoted by $F(s)$, is called the Fourier Transform of f.

Since t is a dummy variable, we replace t by x.

Thus, we have

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds, \qquad \text{(ii)}$$

where $$F(s)=\int_{-\infty}^{\infty} f(x)e^{isx}dx. \qquad \text{(iii)}$$

$F(s)$ defined by (iii) is known as the **Complex Fourier Transform of f(x).**

$f(x)$ defined by (ii) is known as the **Inverse Complex Fourier Transform of** $F(s)$**.**

Remarks:

- $F(s)$ and $f(x)$ are known as Fourier transform pair, which differs only in the sign of the exponent.
- The factor $\frac{1}{2\pi}$ can multiply in the expression of $F(s)$ in integral (iii) instead of the expression of f(x) in integral (ii).
- Basically, the choice of normalizing factors $\frac{1}{\sqrt{2\pi}}$ or $\frac{1}{2\pi}$ in integrals (ii) and (iii) is optional. To make the two integrals as symmetric as possible, we can multiply $\frac{1}{\sqrt{2\pi}}$ in both the expressions.
- We call $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$ as an **inversion formula** corresponding to $F(s)=\int_{-\infty}^{\infty} f(x)e^{isx}dx$.
- Some times, we call $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{isx}ds$ as an **inversion formula** corresponding to $F(s)=\int_{-\infty}^{\infty} f(x)e^{-isx}dx$.
- Some times, we denote $\bar{f}(s)$ in place of $F(s)$

- Fourier transform breaks up the function into a continuous spectrum of frequencies s.

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Some times, the Fourier integral of f(x) in the complex form given by

$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{ist}dt\right]e^{-isx}ds$ can be written as $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$, where $F(s)=\int_{-\infty}^{\infty} f(x)e^{isx}dx$.	$f(x)=\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{ist}dt\right]e^{-isx}ds$ can be written as $f(x)=\int_{-\infty}^{\infty} F(s)e^{-isx}ds$, where $F(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{isx}dx$.

or

$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{ist}dt\right]e^{-isx}ds$ can be written as $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{isx}ds$, where $F(s)=\int_{-\infty}^{\infty} f(x)e^{-isx}dx$.	$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt\right]e^{-isx}ds$ can be written as $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$, where $F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$.

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Fourier Transforms

Q.No.1.: Find the **Fourier transform** of $f(x)=\begin{cases}1-x^2, & \text{if } |x|<1\\ 0, & \text{if } |x|>1\end{cases}$

and use it to evaluate $\int_0^{\infty}\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}dx$.

Sol.: The Fourier transform of f(x) in the complex form given by

$$F(s)=\int_{-\infty}^{\infty} f(x)e^{isx}dx\text{, where } f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds .$$

1st Part: Find Fourier transform of $f(x)=\begin{cases}1-x^2, & \text{if } |x|<1\\ 0, & \text{if } |x|>1\end{cases}$

The Fourier transform of given f(x) is given by

$$F(s)=\int_{-\infty}^{\infty} f(x)e^{isx}dx=\left[\int_{-\infty}^{-1} f(x).e^{isx}dx+\int_{-1}^{1} f(x).e^{isx}dx+\int_{1}^{\infty} f(x).e^{isx}dx\right]$$

$$=\int_{-1}^{1}(1-x^2)e^{isx}dx \qquad \left[\because f(x)=\begin{cases}1-x^2, & \text{if } |x|<1\\ 0, & \text{if } |x|>1\end{cases}\right]$$

$$=\left[(1-x^2).\frac{e^{isx}}{is}-(-2x)\frac{e^{isx}}{(is)^2}+(-2)\frac{e^{isx}}{(is)^3}\right]_{-1}^{1}$$

$$=\left[-\frac{2}{s^2}\left(e^{is}+e^{-is}\right)+\frac{2}{is^3}\left(e^{is}-e^{-is}\right)\right]$$

$$=\left[-\frac{4}{s^2}\left(\frac{e^{is}+e^{-is}}{2}\right)+\frac{4}{s^3}\left(\frac{e^{is}-e^{-is}}{2i}\right)\right]$$

$$\Rightarrow F(s)=4\left[-\frac{\cos s}{s^2}+\frac{\sin s}{s^3}\right]=-4\left(\frac{s\cos s-\sin s}{s^3}\right)$$

$$\Rightarrow F\{f(x)\}=-4\left(\frac{s\cos s-\sin s}{s^3}\right)$$, which is the required Fourier transform.

2nd Part: Evaluate $\int_{0}^{\infty}\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}dx$.

Now, by the inversion formula, we have

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F\{f(x)\}e^{-isx}ds=-\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)(\cos sx-i\sin sx)dx$$

$$=-\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sxds+\frac{2i}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\sin sxds$$

$$=-\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sxds$$ [Since the integrand in the second integral on RHS is odd]

$$\Rightarrow f(x)=-\frac{4}{\pi}\int_{0}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sxds,$$

$$\Rightarrow\int_{0}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sxds=-\frac{\pi}{4}f(x)=\left(-\frac{\pi}{4}\right)\begin{cases}1-x^2, & \text{if } |x|<1\\ 0, & \text{if } |x|>1\end{cases}$$

Taking $x=\frac{1}{2}$, we have

$$\int_0^\infty \left(\frac{s\cos s-\sin s}{s^3}\right)\cos\frac{s}{2}ds=\left(-\frac{\pi}{4}\right)\left[1-\left(\frac{1}{2}\right)^2\right]=\left(-\frac{\pi}{4}\right)\left(1-\frac{1}{4}\right)=-\frac{\pi}{4}.\frac{3}{4}=-\frac{3\pi}{16}.$$

Replacing s by x, we get

$$\int_0^\infty \left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}dx=-\frac{3\pi}{16}. \text{ Ans.}$$

Q.No.2.: Find the **Fourier transform** of $f(x)=\begin{cases}1 & \text{for } |x|<1\\ 0 & \text{for } |x|>1\end{cases}$.

Hence evaluate $\int_0^\infty \frac{\sin x}{x}dx$.

Sol.: The Fourier transform of f(x) in the complex form given by

$$F(s)=\int_{-\infty}^{\infty} f(x)e^{isx}dx\text{, where } f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds .$$

1st Part: Find Fourier transform of $f(x)=\begin{cases}1 & \text{for } |x|<1\\ 0 & \text{for } |x|>1\end{cases}$

The Fourier transform of given f(x) is given by

$$F(s)=\int_{-\infty}^{\infty} f(x)e^{isx}dx=\int_{-1}^{1}(1)e^{isx}dx=\left|\frac{e^{isx}}{is}\right|_{-1}^{1}=\frac{e^{is}-e^{-is}}{is}=2\frac{\sin s}{s}.$$

Thus $F(s)=2\frac{\sin s}{s}$, $s\neq 0$.

This is the required Fourier transform.

For s = 0, we have F(0) = 2.

2nd Part: Evaluate $\int_0^\infty \frac{\sin x}{x}dx$.

Now, by the inversion formula, we get

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds \Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin s}{s}e^{-isx}ds=\begin{cases}1 & \text{for } |x|<1\\ 0 & \text{for } |x|>1\end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\sin s}{s}e^{-isx}ds=\begin{cases}\pi & \text{for } |x|<1\\ 0 & \text{for } |x|>1\end{cases}$$

Putting x = 0, we get

$$\int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi \Rightarrow \int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}.$$ [since the integrand is even].

Replacing s by x, we get $\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Q.No.3.: Find the **Fourier transform** of

$$f(x) = \frac{1}{2a}, \quad \text{if } |x| \le a$$
$$= 0, \quad \text{if } |x| > a.$$

Sol.: By definition, the Fourier transform of f(x) is

$$F(s) = \int_{-\infty}^{\infty} f(x).e^{isx} dx$$

$$= \left[\int_{-\infty}^{-a} f(x).e^{isx} dx + \int_{-a}^{a} f(x).e^{isx} dx + \int_{a}^{\infty} f(x).e^{isx} dx \right]$$

$$= \left[\int_{-\infty}^{-a} 0dx + \int_{-a}^{a} \frac{1}{2a} e^{isx} dx + \int_{a}^{\infty} 0dx \right]$$

$$= \frac{1}{2a} \frac{e^{isx}}{(is)} \Bigg|_{x=-a}^{a} = \frac{1}{2asi}\left[e^{isa} - e^{-isa} \right]$$

$$= \frac{1}{as}\left[\frac{e^{isa} - e^{-isa}}{2i} \right] = \frac{\sin(sa)}{as}. \text{ Ans}$$

Q.No.4.: Represent f(x) as an exponential **Fourier transform** when

$$f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & \text{otherwise} \end{cases}$$

Show that the result can be written as

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos sx + \cos s(x - \pi)}{(1 - s^2)} ds.$$

Sol.: Fourier transform in the exponential form is given by

$$F(s) = \int_{-\infty}^{\infty} f(x).e^{isx} dx = \left[\int_{-\infty}^{0} 0dx + \int_{0}^{\pi} \sin x.e^{isx} dx + \int_{\pi}^{\infty} 0dx \right]$$

$$=\int_0^{\pi}\left(\frac{e^{ix}-e^{-ix}}{2i}\right)e^{isx}dx \quad \left[\because \sin x=\frac{\left(e^{ix}-e^{-ix}\right)}{2i}\right]$$

$$=\frac{1}{2i}\int_0^{\pi}\left[e^{i(1+s)x}-e^{-i(1-s)x}\right]dx=\frac{1}{2i}\left[\frac{e^{i(1+s)x}}{i(1+s)}-\frac{e^{-i(1-s)x}}{-i(1-s)}\right]_0^{\pi}=\frac{1}{2i}\left[\frac{e^{i(1+s)x}}{i(1+s)}+\frac{e^{-i(1-s)x}}{i(1-s)}\right]_0^{\pi}$$

$$=\frac{1}{2i}\left[\frac{e^{i(1+s)\pi}}{i(1+s)}+\frac{e^{-i(1-s)\pi}}{i(1-s)}-\frac{1}{i(1+s)}-\frac{1}{i(1-s)}\right]$$

$$=\frac{1}{2i}\left[\left(\frac{e^{i\pi}e^{is\pi}}{i(1+s)}+\frac{e^{-i\pi}e^{is\pi}}{i(1-s)}\right)-\left(\frac{1}{i(1+s)}+\frac{1}{i(1-s)}\right)\right]$$

$$=-\frac{1}{2i}\left[\left(\frac{2e^{is\pi}}{i\left(1-s^2\right)}\right)+\left(\frac{2}{i\left(1-s^2\right)}\right)\right] \text{ since } e^{\pm i\pi}=-1\,.$$

$$\Rightarrow F(s)=\frac{e^{i\pi s}+1}{1-s^2}$$

is the required exponential Fourier transform representation.

Now $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(s)e^{-isx}ds=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{e^{i\pi s}+1}{1-s^2}\right)e^{-isx}ds\,.$

$$\Rightarrow f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\cos\pi s+i\sin\pi s+1}{1-s^2}\right)(\cos sx-i\sin sx)ds$$

$$\Rightarrow f(x)=\frac{1}{2\pi}\int_{\infty}^{\infty}\frac{\cos sx+\cos s(x-\pi)}{\left(1-s^2\right)}ds-\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\sin sx+\sin s(x-\pi)}{\left(1-s^2\right)}ds$$

Since the 2nd integral is zero (odd function), we get

$$\Rightarrow f(x)=\frac{2}{2\pi}\int_0^{\infty}\frac{\cos sx+\cos s(x-\pi)}{\left(1-s^2\right)}ds$$

$$\Rightarrow f(x)=\frac{1}{\pi}\int_0^{\infty}\frac{\cos sx+\cos s(x-\pi)}{\left(1-s^2\right)}ds\,.$$

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Home Assignments

3rd Topic

Fourier Transforms

Properties of Fourier Transforms

- *Linearity Property (Linearity Theorem)*
- *Change of Scale Property*
- *First Shifting Property*
- *Second Shifting Property*
- *Modulation (Modulation Theorem)*
- *Fourier transforms of the Derivatives*
- *Fourier transforms of the Integrals*
- *Fourier transforms of Convolutions (Convolution Theorem)*

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Properties of Fourier Transforms:

- Linearity Property (Linearity Theorem)
- Change of Scale Property
- First Shifting Property
- Second Shifting Property
- Modulation (Modulation Theorem)
- Fourier transforms of the Derivatives
- Fourier transforms of the Integrals
- Fourier transforms of Convolutions (Convolution Theorem)

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1. Linearity Property (Linearity Theorem):

If F(s) and G(s) are Fourier transforms of f(x) and g(x) respectively, then

$$F[af(x)+bg(x)] = aF(s)+bG(s)$$

where a and b are constants.

Proof: The proof follows directly from the definition of Fourier Transform.

Since we have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx$ and $G(s) = \int_{-\infty}^{\infty} e^{isx} g(x)dx$.

$$\therefore F[af(x)+bg(x)] = \int_{-\infty}^{\infty} e^{isx}[af(x)+bg(x)]dx = a\int_{-\infty}^{\infty} e^{isx} f(x)dx + b\int_{-\infty}^{\infty} e^{isx} g(x)dx$$

$$= aF(s)+bG(s).$$

This completes the proof.

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2. Change of Scale Property:

If F(s) is the complex Fourier transform of f(x), then

$$F\{f(ax)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a \neq 0.$$

Proof: The proof follows directly from the definition of Fourier Transform.

Since $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx$.

$$\therefore F[f(ax)] = \int_{-\infty}^{\infty} e^{isx} f(ax)dx \qquad [\text{Put } ax = t, \text{ so that } dx = dt/a]$$

$$= \int_{-\infty}^{\infty} e^{ist/a} f(t)\frac{dt}{a} = \frac{1}{a}\int_{-\infty}^{\infty} e^{i(s/a)t} f(t)dt = \frac{1}{a}F\left(\frac{s}{a}\right) \qquad [\text{by(i)}].$$

This completes the proof.

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3. First-Shifting Property-Shifting x by a, i.e. shifting on x-axis by a:

If F(s) is the complex Fourier transform of f(x), then

$$F\{f(x-a)\} = e^{isa}F(s).$$

Proof: Since we have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx$. (i)

$$\therefore F\{f(x-a)\} = \int_{-\infty}^{\infty} e^{isx} f(x-a)dx \qquad [\text{Put } x-a=t, \text{ so that } dx = dt]$$

$$= \int_{-\infty}^{\infty} e^{is(t+a)} f(t)dt = e^{isa} \int_{-\infty}^{\infty} e^{ist} f(t)dt = e^{isa}F(s). \qquad [\text{By (i)}]$$

This completes the proof.

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4. Second-Shifting Property:

If F(s) is the Fourier transform of the f(x), then the function $g(x) = f(x)e^{-iax}$ has a Fourier transform $F(s-a)$, i.e. $F[g(x)] = F(s-a)$, where a is constant

Proof: Since we have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx$. (i)

$$\therefore F(s-a) = \int_{-\infty}^{\infty} e^{i(s-a)x} f(x)dx = \int_{-\infty}^{\infty} e^{isx}e^{-iax} f(x)dx = \int_{-\infty}^{\infty} \{e^{-iax} f(x)\}e^{isx}dx$$

$$= \int_{-\infty}^{\infty} g(x)e^{isx}dx = F[g(x)].$$

This completes the proof.

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5. Modulation Theorem:

Statement: If F(s) is the complex Fourier transform of f(x), then

$$F\{f(x)\cos ax\} = \frac{1}{2}\{F(s+a)+F(s-a)\}.$$

Proof: The proof follows directly from the definition of Fourier Transform.

Since $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx$. (i)

$$\therefore F\{f(x)\cos ax\} = \int_{-\infty}^{\infty} e^{isx}\{f(x)\cos ax\}dx = \int_{-\infty}^{\infty} e^{isx}.\left\{f(x).\frac{e^{iax}+e^{-iax}}{2}\right\}dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{i(s+a)x} f(x)dx + \int_{-\infty}^{\infty} e^{i(s-a)x} f(x)dx\right] = \frac{1}{2}[F(s+a)+F(s-a)].$$

Importance:

This theorem is of great importance in **radio** and **television**, where the harmonic carrier wave is modulated by an envelope.

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6. Fourier Transforms of the Derivatives:

If f(x) is continuous function of x with $f(x) \to 0$ as $|x| \to \infty$ and $f'(x)$ is absolutely integrable over $(-\infty, \infty)$, then

$$F\{f'(x)\} = F\left\{\frac{df}{dx}\right\} = (-is)F\{f(x)\}.$$

i.e. Fourier transform of a derivative of a function f(x) corresponds to multiplication of the Fourier transform by 'is' .

Proof:

By definition, $F\{f'(x)\} = \int_{-\infty}^{\infty} e^{isx} f'(x)dx$.

Integrating by parts, we obtain

$$F\{f'(x)\} = \left[e^{isx}.f(x)\Big|_{-\infty}^{\infty} - (is)\int_{-\infty}^{\infty} e^{isx}.f(x)dx\right].$$

Since $f(x) \to 0$ as $|x| \to \infty$, therefore

$$F\{f'(x)\}=(-is)F\{f(x)\}.$$

In general: $F\left[f^n(x)\right]=(-is)^n F\{f(x)\}\Rightarrow F\left\{\frac{d^n f}{dx^n}\right\}=(-is)^n F\{f(x)\}.$

In particular: $F\left\{\frac{d^2 f}{dx^2}\right\}=-s^2 F\{f(x)\}.$

This completes the proof.

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7. Fourier Transforms of the Integrals:

If $F\left[f(x)\right]=F(s)$, then

$$F\left\{\int_{-\infty}^{x} f(t)\,dt\right\}=\frac{i}{s}F\{f(x)\},$$

provided $F(s)$ satisfies $F(0)=0$.

Proof: Let $g(x)=\int_{-\infty}^{x} f(t)\,dt$, then $g'(x)=f(x)$, since $\lim_{x\to-\infty} f(x)=0$.

Also $F[g'(x)]=(-is)F\left[g(x)\right].$

Substituting for $g(x)$ and $g'(x)$, it becomes

$$F\{f(x)\}=-isF\left\{\int_{-\infty}^{x} f(t)\,dt\right\}\Rightarrow F\left\{\int_{-\infty}^{x} f(t)\,dt\right\}=\frac{i}{s}F\{f(x)\}.$$

This completes the proof.

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Convolution of Two Functions:

Definition: Convolution of two functions f(x) and g(x) denoted by f*g is defined as

$$h(x)=(f*g)(x)=\int_{-\infty}^{\infty} f(u)g(x-u)\,du=\int_{-\infty}^{\infty} f(x-u)g(u)\,du$$

Remarks:

1. Convolution is commutative, i.e. f*g = g*f.
2. Convolution is associative, i.e. f*(g*h) = (f*g)*h.

8. Fourier Transforms of Convolution:

Convolution Theorem:

Statement:

Let f(x) and g(x) be two **piecewise continuous**, **bounded** and **absolutely integrable** functions on the x-axis, then the Fourier Transform of $f*g$, the convolution of f(x) and g(x) is

$$F\{f*g\} = F\{f(x)\}.F\{g(x)\}.$$

Proof: By definition

$$F(f*g) = \int_{-\infty}^{\infty} (f*g)e^{isx}dx = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)g(x-u)du\right]e^{isx}dx.$$

Interchange the order of integration, we obtain

$$F(f*g) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)g(x-u)e^{isx}dx\right]du.$$

Now put $x-u=q$, so x = u + q, with q as the new variable of integration instead of x.

$$F(f*g) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)g(q)e^{is(u+q)}dqdu$$

$$= \left[\int_{-\infty}^{\infty} f(u)e^{isu}du\right]\left[\int_{-\infty}^{\infty} g(q)e^{isq}dq\right]$$

$$= \left[\int_{-\infty}^{\infty} f(x)e^{isx}dx\right]\left[\int_{-\infty}^{\infty} g(x)e^{isx}dx\right]$$

$= F(f(x)).F(g(x))$. Since u & q is a dummy variables, we replace u & q by x.

This completes the proof.

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Another way to represent convolution theorem for Fourier transform:

Statement: The Fourier transform of convolution of f(x) and g(x) is the product of their Fourier transforms,

i.e. $F\{f*g\} = F\{f(x)\}.F\{g(x)\}$ or $F\{f*g\} = \left(\sqrt{2\pi}\right)F\{f(x)\}.F\{g(x)\}$

or $F\{f*g\} = F\{f(x)\}.F\{g(x)\}$ or $F\{f*g\} = (2\pi)F\{f(x)\}.F\{g(x)\}$

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Q. No.1.: Find the Fourier Transform of $f(x)=e^{-ax^2}, a>0$.

Sol.: Since $F(s)=\int_{-\infty}^{\infty} f(x)e^{isx}dx=\int_{-\infty}^{\infty} e^{-ax^2}e^{isx}dx$

$$=\int_{-\infty}^{\infty} e^{-(ax^2-isx)}dx \quad =\int_{-\infty}^{\infty} e^{\left[-\left(\sqrt{a}x-\frac{is}{2\sqrt{a}}\right)^2+\left(\frac{is}{2\sqrt{a}}\right)^2\right]}dx$$

$$=e^{-\frac{s^2}{4a}}\int_{-\infty}^{\infty} e^{\left[-\left(\sqrt{a}x-\frac{is}{2\sqrt{a}}\right)^2\right]}dx$$

$$=\frac{1}{\sqrt{a}}e^{-\frac{s^2}{4a}}\int_{-\infty}^{\infty} e^{-t^2}dt \quad \left[\text{Setting } t=\sqrt{a}x-\frac{is}{2\sqrt{a}} \Rightarrow dx=\frac{dt}{\sqrt{a}}\right]$$

$$=\frac{1}{\sqrt{a}}e^{-\frac{s^2}{4a}}\sqrt{\pi} \quad \left[\because \int_{-\infty}^{\infty} e^{-t^2}dt=2\int_{0}^{\infty} e^{-t^2}dt=\sqrt{\pi}\right]$$

$$=\sqrt{\frac{\pi}{a}}.e^{-\frac{s^2}{4a}}. \text{ Ans.}$$

Q. No.2.: Find the Fourier Transform of $f(x)=e^{-a(x-5)^2}, a>0$, using **shifting property**.

Sol.: Since, we know $F(s)=\int_{-\infty}^{\infty} f(x)e^{isx}dx$.

If F(s) is the complex Fourier transform of f(x), then by shifting property

$$F\{f(x-a)\}=e^{isa}F(s).$$

Now $F\left(e^{-a(x-5)^2}\right)=e^{is5}F\left(e^{-ax^2}\right)$

$$=e^{is5}\left(\sqrt{\frac{\pi}{a}}.e^{-\frac{s^2}{4a}}\right) \quad \left[\because F(s)=\int_{-\infty}^{\infty} f(x)e^{isx}dx=\int_{-\infty}^{\infty} e^{-ax^2}e^{isx}dx=\sqrt{\frac{\pi}{a}}.e^{-\frac{s^2}{4a}}\right]$$

$$=\sqrt{\frac{\pi}{a}}.e^{-\left(\frac{s^2}{4a}-is5\right)}. \text{ Ans.}$$

Q. No.3.: Find the Fourier transform of $f(x)=xe^{-ax^2}, a>0$.

Sol.: Here $F(s)=F\left[f(x)\right]=F\left[xe^{-ax^2}\right]=F\left[-\frac{1}{2a}\left(e^{-ax^2}\right)'\right]$

$$= -\frac{1}{2a}F\left[\left(e^{-ax^2}\right)'\right]$$

$$= -\frac{1}{2a}(-is)F\left[e^{-ax^2}\right]\left[\because F\{f'(x)\} = (-is)F\{f(x)\}\right]$$

$$= \frac{is}{2a}\left(\sqrt{\frac{\pi}{a}}.e^{-\frac{s^2}{4a}}\right) \qquad \left[\because F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx = \int_{-\infty}^{\infty} e^{-ax^2}e^{isx}dx = \sqrt{\frac{\pi}{a}}.e^{-\frac{s^2}{4a}}\right]$$

$$= \frac{is}{2a}\sqrt{\frac{\pi}{a}}e^{-\frac{s^2}{4a}}$$

Q. No.4.: Using the transform of **integrals**, find the Fourier transform of

$f(x) = e^{-ax^2}, a > 0$.

Sol.: If $F[f(x)] = F(s)$, then

$$F\left\{\int_{-\infty}^{x} f(t)dt\right\} = \frac{i}{s}F\{f(x)\}, \text{ provided } F(s) \text{ satisfies } F(0) = 0.$$

Here $f(x) = e^{-ax^2}, a > 0$.

Now $F\left(e^{-ax^2}\right) = -2a\left[\int_{-\infty}^{x} xe^{-ax^2}dx\right] = -2a.\frac{i}{s}F\left(xe^{-ax^2}\right)$

$$= -2a.\frac{i}{s}\left(\frac{is}{2a}\sqrt{\frac{\pi}{a}}e^{-\frac{s^2}{4a}}\right) \quad \left[\because F\left[xe^{-ax^2}\right] = \frac{is}{2a}\sqrt{\frac{\pi}{a}}e^{-\frac{s^2}{4a}}\right]$$

$$= \sqrt{\frac{\pi}{a}}.e^{-\frac{s^2}{4a}}. \text{ Ans.}$$

Q. No.5.: Find the Fourier transform of $f(x) = e^{-|x|}$.

Sol.: The given function is $f(x) = \begin{cases} e^{x} & -\infty < x \le 0 \\ e^{-x} & 0 < x < \infty \end{cases}$.

By definition, $F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$= \left[\int_{-\infty}^{0} e^{x}e^{isx}dx + \int_{0}^{\infty} e^{-x}e^{isx}dx\right]$$

$$= \left[\int_{-\infty}^{0} e^{(1+is)x}dx + \int_{0}^{\infty} e^{-(1-is)x}dx\right]$$

$$=\left[\frac{e^{(1+is)x}}{(1+is)}\right]_{-\infty}^{0}-\left[\frac{e^{-(1-is)x}}{(1-is)}\right]_{0}^{\infty}$$

$$=\left[\frac{1}{(1+is)}\right]+\left[\frac{1}{(1-is)}\right]$$

$$=\frac{2}{(1+s^2)}$$

Q. No.6.: Find the inverse Fourier transform of $f(s)=\dfrac{1}{(4+s^2)(9+s^2)}$.

Sol.: Let $h(x)$ be the inverse Fourier transform, then

$$h(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(4+s^2)(9+s^2)}e^{-isx}ds=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{4+s^2}\right)\left(\frac{1}{9+s^2}\right)e^{-isx}ds$$

$$=\frac{1}{2\pi}(f*g)(x), \qquad \left[\because F\{f*g\}=F\{f(x)\}.F\{g(x)\}\right]$$

where $f(x)=F^{-1}\left(\dfrac{1}{4+s^2}\right)$, $g(x)=F^{-1}\left(\dfrac{1}{9+s^2}\right)$ and $(f*g)$ is the convolution of f and g.

Now since the Fourier transform of $f(x)\left(=e^{-|x|}\right)=\dfrac{2}{(1+s^2)}$ and scaling property, we have

$$f(x)=F^{-1}\left(\frac{1}{4+s^2}\right)=\frac{1}{4}e^{-2|x|}.$$

$$g(x)=F^{-1}\left(\frac{1}{9+s^2}\right)=\frac{1}{6}e^{-3|x|}.$$

Hence $h(x)=\dfrac{1}{2\pi}(f*g)(x)=\dfrac{1}{2\pi}\left(\dfrac{1}{4}e^{-2|x|}\right)\left(\dfrac{1}{6}e^{-3|x|}\right)=\dfrac{1}{2\pi}\dfrac{1}{24}\left(e^{-2|x|}\right)\left(e^{-3|x|}\right)$.

Using convolution theorem, $h(x)=(f*g)(x)=\displaystyle\int_{-\infty}^{\infty}f(x-u)g(u)du$, we get

$$h(x)=\frac{1}{2\pi}\frac{1}{24}\left(e^{-2|x|}\right)\left(e^{-3|x|}\right)=\frac{1}{2\pi}\frac{1}{24}\int_{-\infty}^{\infty}e^{-2|x-t|}.e^{-3|t|}dt.$$

Case I: When $x>0$,

$$\int_{-\infty}^{\infty}e^{-2|x-t|}.e^{-3|t|}dt=\int_{-\infty}^{0}e^{-2|x-t|}.e^{-3|t|}dt+\int_{0}^{x}e^{-2|x-t|}.e^{-3|t|}dt+\int_{x}^{\infty}e^{-2|x-t|}.e^{-3|t|}dt$$

$$=\int_{-\infty}^{0} e^{-2(x-t)}.e^{3t}dt+\int_{0}^{x} e^{-2(x-t)}.e^{-3t}dt+\int_{x}^{\infty} e^{-2(t-x)}.e^{-3t}dt$$

$$=\int_{-\infty}^{0} e^{-2x}.e^{5t}dt+\int_{0}^{x} e^{-2x}.e^{-t}dt+\int_{x}^{\infty} e^{2x}.e^{-5t}dt$$

$$=e^{-2x}.\left[\frac{e^{5t}}{5}\right]_{-\infty}^{0}+e^{-2x}.\left[\frac{e^{-t}}{-1}\right]_{0}^{x}+e^{2x}.\left[\frac{e^{-5t}}{-5}\right]_{x}^{\infty}$$

$$=e^{-2x}.\left[\frac{1}{5}-0\right]+e^{-2x}.\left[\frac{e^{-x}}{-1}-\frac{1}{-1}\right]+e^{2x}.\left[0-\frac{e^{-5x}}{-5}\right]$$

$$=e^{-2x}.\left[\frac{1}{5}+1\right]+e^{-3x}.\left[-1+\frac{1}{5}\right]$$

$$=\frac{6e^{-2x}}{5}-\frac{4e^{-3x}}{5}.$$

Case II: When $x<0$, $\int_{-\infty}^{\infty} e^{-2|x-t|}.e^{-3|t|}dt=\frac{6e^{2x}}{5}-\frac{4e^{3x}}{5}$.

Case III: When $x=0$, $\int_{-\infty}^{\infty} e^{-2|x-t|}.e^{-3|t|}dt=\frac{2}{5}$.

Therefore
$$h(x)=\begin{cases}\frac{1}{2\pi}\frac{1}{24}\left(\frac{6e^{-2x}}{5}-\frac{4e^{-3x}}{5}\right) & x<0\\ \frac{1}{2\pi}\frac{1}{24}\frac{2}{5} & x=0\\ \frac{1}{2\pi}\frac{1}{24}\left(\frac{6e^{2x}}{5}-\frac{4e^{3x}}{5}\right) & x>0\end{cases}$$

$$\Rightarrow h(x)=\begin{cases}\frac{1}{2\pi}\left(\frac{e^{-2x}}{20}-\frac{e^{-3x}}{30}\right) & x<0\\ \frac{1}{120\pi} & x=0\\ \frac{1}{2\pi}\left(\frac{e^{2x}}{20}-\frac{e^{3x}}{30}\right) & x>0\end{cases}$$

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Home Assignments

Fourier Transforms

Q.No.1.: Show that the **Fourier transform** of $f(x) = e^{-x^2/2}$ is $e^{-s^2/2}$.

Q.No.2.: Find the **Fourier transform** of

(i). $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$

(ii). $f(x) = \begin{cases} x, & |x| < a \\ 0, & |x| > a \end{cases}$

Ans.: (i). $\sqrt{\frac{2}{\pi}} \frac{\sin as}{s}$ **(ii).** $i\sqrt{\frac{2}{\pi}}\left(\frac{ascoas - \sin as}{s^2}\right)$.

Q.No.3.: Find the **Fourier transform** of f(x): $f(x) = \begin{cases} 1, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$.

Hence evaluate $\int_0^\infty \frac{\sin ax}{x} dx$.

Ans.: $F\{f(x)\} = \frac{2\sin sa}{s}$, for $s \neq 0$, for s = 0, F(s) = 2, Integral $= \frac{\pi}{2}$

Q.No.4.: Find the **Fourier transform** of f(x): $f(x) = \begin{cases} x, & \text{for } |x| \le a \\ 0, & \text{for } |x| > a \end{cases}$.

Ans.: $\frac{2i}{i^2}(as\cos sa - \sin sa)$

Q.No.5.: Find the **Fourier transform** of f(x): $f(x) = e^{-x^2/2}$, $-\infty < x < \infty$.

Ans.: $\sqrt{2\pi}e^{-x^2/2}$.

Q.No.6.: Find the **Fourier transform** of f(x): $f(x) = \begin{cases} 0, & -\infty < x < a \\ x, & a \le x \le b \\ 0, & x > b \end{cases}$.

Ans.: $\frac{1}{s}\left(ae^{isa} - be^{isb}\right) + \frac{1}{s^2}\left(e^{isb} - e^{isa}\right)$

Q.No.7.: Find the **Fourier transform** of f(x): $f(x) = xe^{-x}$, $0 \le x < \infty$.

Ans.: $\frac{1}{2\pi} \cdot \frac{(1+is)^2}{\left(1+s^2\right)^2}$.

Q.No.8.: Find the **Fourier transform** of f(x): $f(x) = \begin{cases} \cos x, & \text{if } \; 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$.

Ans.: $\frac{1}{4\pi}\left[\frac{\sin(s+1)}{s+1} + \frac{\sin(s-1)}{s-1} + i\left\{\frac{1-\cos(s+1)}{s+1} + \frac{1-\cos(s-1)}{s-1}\right\}\right]$.

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4th Topic

Fourier Transform

- Fourier Cosine and Sine Transforms and their Inversion Formulae

  Properties of Fourier Cosine and Sine Transforms
- Linearity Property (Linearity Theorem)
- Change of Scale Property
- Modulation Theorem
- Fourier cosine and sine transforms of the Derivatives

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Fourier cosine and sine Transforms and their inversion formulae:

The Fourier cosine and sine transforms can be considered as special cases of the Fourier Transform of f(x) when f(x) is even or odd function of x over the real axis.

Sufficient conditions for the existence of Fourier cosine and sine Transforms are:

1. f(x) is **piecewise continuous** on every finite interval $[0, \ell]$, and
2. f(x) is **absolutely integrable** on the positive real axis, that is, $\int_0^\infty |f(x)|dx$ converges.

Sufficient conditions for the existence of Inverse Fourier cosine and sine Transforms are:

1. $F_c(s)$ and $F_s(s)$ must be **absolutely integrable** over $(0,\infty)$, that is, $\int_0^\infty |f(x)|dx$ converges.

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Fourier Cosine Transform and its inversion formula:

We know that the Fourier integral of an **even function** f(x) reduces to the Fourier cosine integral given by

$$f(x)=\int_0^\infty A(\alpha)\cos(\alpha x)d\alpha$$

$$=\frac{2}{\pi}\int_0^\infty\left[\int_0^\infty f(t)\cos(\alpha t)dt\right]\cos(\alpha x)d\alpha \qquad \left[\because A(\alpha)=\frac{2}{\pi}\int_0^\infty f(t)\cos(\alpha t)dt\right]$$

Put $F_c(s)=\int_0^\infty f(x)\cos(sx)dx$. (Here replacing α by s and t by x)

Then $f(x)=\frac{2}{\pi}\int_0^\infty F_c(s)\cos(sx)ds$.

$F_c(s)$ given by $F_c(s)=\int_0^\infty f(x)\cos(sx)dx$ is called the **Fourier Cosine Transform** of f(x) in the interval $0<x<\infty$ and f(x) given by $f(x)=\frac{2}{\pi}\int_0^\infty F_c(s)\cos(sx)ds$ is called the **inverse Fourier Cosine Transform** of $F_c(s)$.

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Fourier Sine Transform and its inversion formula:

We know that the Fourier integral of an **odd function** f(x) reduces to the Fourier sine integral given by

$$f(x)=\int_0^\infty B(\alpha)\sin(\alpha x)d\alpha$$

$$=\frac{2}{\pi}\int_0^\infty\left[\int_0^\infty f(t)\sin(\alpha t)\,dt\right]\sin(\alpha x)\,d\alpha \qquad \left[\because B(\alpha)=\frac{2}{\pi}\int_0^\infty f(t)\sin(\alpha t)dt\right]$$

Put $F_s(s)=\int_0^\infty f(x)\sin(sx)\,dx$. (Here replacing α by s and t by x)

Then $f(x)=\frac{2}{\pi}\int_0^\infty F_s(s)\sin(sx)\,ds$.

$F_s(s)$ given by $F_s(s)=\int_0^\infty f(x)\sin(sx)\,dx$ is called the **Fourier Sine Transform** of f(x) in the interval $0<x<\infty$ and f(x) given by $f(x)=\frac{2}{\pi}\int_0^\infty F_s(s)\sin(sx)\,ds$ is called the **inverse Fourier Sine Transform** of $F_s(s)$.

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Properties of Fourier Cosine and Sine Transforms:

- **Linearity Property (Linearity Theorem)**
- **Change of Scale Property for Fourier Cosine and Sine Transforms**
- **Modulation Theorem for Fourier Cosine and Sine Transforms**
- **Fourier Cosine and Sine Transforms of the Derivatives**

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1. Linearity Property (Linearity Theorem):

For any two functions f(x) and g(x) whose Fourier cosine and sine transformations exist and for any constants a and b

$$F_c[af(x)+bg(x)]=aF_c[f(x)]+bG_c[g(x)]=aF_c(s)+bG_c(s)$$

$$\text{and } F_s[af(x)+bg(x)]=aF_s[f(x)]+bG_s[g(x)]=aF_s(s)+bG_s(s)$$

Proof: The proof follows directly from the definition of Fourier cosine and sine Transforms.

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2. Change of Scale Property of Fourier cosine and sine Transforms

If F(s) is the complex Fourier transform of f(x), then by change of scale property of Fourier Transform, we have

$$F\{f(ax)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a \neq 0.$$

Similarly the change of scale property of Fourier cosine and sine Transforms is

(i) $F_c\{f(ax)\} = \frac{1}{a}F_c\left(\frac{s}{a}\right), \quad a \neq 0.$

(ii) $F_s\{f(ax)\} = \frac{1}{a}F_s\left(\frac{s}{a}\right), \quad a \neq 0.$

Proof: The proof follows directly from the definition of Fourier cosine and sine Transforms.

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3. Modulation Theorem for Fourier Cosine and Sine Transforms of f(x):

If $F_c(s)$ and $F_s(s)$ are Fourier cosine and sine transforms of f(x) respectively, then

(i). $F_c\{f(x)\cos ax\} = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

(ii). $F_c\{f(x)\sin ax\} = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$

(iii). $F_s\{f(x)\cos ax\} = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

(iv). $F_s\{f(x)\sin ax\} = \frac{1}{2}[F_c(s-a) - F_c(s+a)].$

Proof:

(i): To prove: $F_c\{f(x)\cos ax\} = \frac{1}{2}[F_c(s+a) + F_c(s-a)].$

$$\text{LHS}=F_c\{f(x)\cos ax\} = \int_0^\infty (f(x)\cos ax)\cos(sx)\,dx \quad \left[\because F_c(s) = \int_0^\infty f(x)\cos(sx)\,dx\right]$$

$$= \int_0^\infty [\cos sx \cos ax]\, f(x)\,dx$$

$$=\frac{1}{2}\left[\int_0^\infty \cos(s+a)xf(x)dx+\int_0^\infty \cos(s-a)xf(x)dx\right]$$

$$\left[\because \cos\alpha\cos\beta=\frac{1}{2}\{\cos(\alpha-\beta)+\cos(\alpha+\beta)\}\right]$$

$$=\frac{1}{2}\left[\int_0^\infty f(x)\cos(s+a)xdx+\int_0^\infty f(x)\cos(s-a)xdx\right]$$

$$=\frac{1}{2}[F_c(s+a)+F_c(s-a)] \qquad \left[\because F_c(s)=\int_0^\infty f(x)\cos(sx)dx\right]$$

(ii): To prove $F_c\{f(x)\sin ax\}=\frac{1}{2}[F_s(s+a)-F_s(s-a)]$.

$$\text{LHS}=F_c\{f(x)\sin ax\}=\int_0^\infty (f(x)\sin ax)\cos(sx)dx \qquad \left[\because F_c(s)=\int_0^\infty f(x)\cos(sx)dx\right]$$

$$=\int_0^\infty (\cos sx\sin ax)f(x)dx$$

$$=\frac{1}{2}\left[\int_0^\infty \sin(s+a)xf(x)dx-\int_0^\infty \sin(s-a)xf(x)dx\right]$$

$$\left[\because \cos\alpha\sin\beta=\frac{1}{2}\{\sin(\alpha+\beta)-\sin(\alpha-\beta)\}\right]$$

$$=\frac{1}{2}\left[\int_0^\infty f(x)\sin(s+a)xdx-\int_0^\infty f(x)\sin(s-a)xdx\right]$$

$$=\frac{1}{2}[F_s(s+a)-F_s(s-a)] \qquad \left[\because F_s(s)=\int_0^\infty f(x)\sin(sx)dx\right]$$

(iii): To prove: $F_s\{f(x)\cos ax\}=\frac{1}{2}[F_s(s+a)+F_s(s-a)]$

$$\text{LHS}=F_s\{f(x)\cos ax\}=\int_0^\infty (f(x)\cos ax)\sin(sx)dx \qquad \left[\because F_s(s)=\int_0^\infty f(x)\sin(sx)dx\right]$$

$$=\int_0^\infty (\sin sx\cos ax)f(x)dx$$

$$=\frac{1}{2}\left[\int_0^\infty \sin(s+a)xf(x)dx+\int_0^\infty \sin(s-a)xf(x)dx\right]$$

$$\left[\because \sin\alpha\cos\beta = \frac{1}{2}\{\sin(\alpha+\beta)+\sin(\alpha-\beta)\}\right]$$

$$= \frac{1}{2}[F_s(s+a)+F_s(s-a)] \qquad \left[\because F_s(s)=\int_0^\infty f(x)\sin(sx)\,dx\right]$$

(iv): To prove: $F_s\{f(x)\sin ax\} = \frac{1}{2}[F_c(s-a)-F_c(s+a)]$

$$\text{LHS}=F_s\{f(x)\sin ax\} = \int_0^\infty (f(x)\sin ax)\sin(sx)\,dx \qquad \left[\because F_s(s)=\int_0^\infty f(x)\sin(sx)\,dx\right]$$

$$= \int_0^\infty (\sin sx \sin ax) f(x)\,dx$$

$$= \frac{1}{2}\left[\int_0^\infty \cos(s-a)x f(x)\,dx - \int_0^\infty \cos(s+a)x f(x)\,dx\right]$$

$$\left[\because \sin\alpha\sin\beta = \frac{1}{2}\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}\right]$$

$$= \frac{1}{2}[F_c(s-a)-F_c(s+a)] \qquad \left[\because F_c(s)=\int_0^\infty f(x)\cos(sx)\,dx\right]$$

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4. Fourier Cosine and Sine Transforms of the Derivatives:

Prove that **(a)** $F_c\{f'(x)\} = sF_s\{f(x)\} - f(0)$

(b) $F_s\{f'(x)\} = -sF_c\{f(x)\}$

Proof:

(a). By definition and applying integration by parts

$$F_c\{f'(x)\} = \int_0^\infty f'(x)\cos(sx)\,dx = f(x).\cos sx\Big|_0^\infty + s\int_0^\infty f(x)\sin(sx)\,dx$$

$$\Rightarrow F_c\{f'(x)\} = -f(0) + sF_s\{f(x)\}.$$

(b). Similarly

$$F_s\{f'(x)\} = \int_0^\infty f'(x)\sin(sx)\,dx = f(x).\sin sx\Big|_0^\infty - s\int_0^\infty f(x)\cos(sx)\,dx$$

$$\Rightarrow F_s\{f'(x)\} = 0 - sF_c\{f(x)\}$$

Corollary 1:

$$F_c\{f''(x)\} = sF_s\{f'(x)\} - f'(0) = s\left(-sF_c\{f(x)\}\right) - f'(0) = -s^2F_c\{f(x)\} - f'(0)$$

Corollary 2:

$$F_s\{f''(x)\} = -sF_c\{f'(x)\} = -s\left[-f(0) + sF_s\{f(x)\}\right] = -s^2F_s\{f(x)\} + sf(0).$$

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Now let us solve some problems related to these topics:

FOURIER COSINE AND SINE TRANSFORMS

Q.No.1.: Find the **Fourier sine transform** of $e^{-|x|}$.

Hence evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2} dx$.

Sol.:

Given: We know that the Fourier sine transform of f(x) is

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin(sx)dx = \int_0^\infty e^{-x}\sin(sx)dx$$

since in the interval $(0, \infty)$, x is positive so that $e^{-|x|} = e^{-x}$.

Step No. 01: Find Fourier sine transform of $e^{-|x|}$.

Thus, Fourier sine transform of $f(x) = e^{-x}$ is given by

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin(sx)dx = \int_0^\infty e^{-x}\sin(sx)dx$$

$$= \left[\frac{e^{-x}}{1+s^2}(-\sin sx - s\cos sx)\right]_0^\infty = \left(\frac{s}{1+s^2}\right),$$

$$\Rightarrow F_s\{f(x)\} = \left(\frac{s}{1+s^2}\right),$$

which is the required Fourier sine transform.

Step No. 02: Evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2} dx$.

Using inversion formula for Fourier sine transform, we get

$$f(x)=\frac{2}{\pi}\int_0^\infty F_s\{f(x)\}\sin(sx)dx \Rightarrow e^{-x}=\frac{2}{\pi}\int_0^\infty\left(\frac{s}{1+s^2}\right)\sin sxds.$$

Changing x to m, we get

$$e^{-m}=\frac{2}{\pi}\int_0^\infty\left(\frac{s}{1+s^2}\right)\sin msds$$

$$\Rightarrow e^{-m}=\frac{2}{\pi}\int_0^\infty\frac{x\sin mx}{1+x^2}dx$$

Hence $\int_0^\infty\frac{x\sin mx}{1+x^2}dx=\frac{\pi}{2}e^{-m}$. Ans.

Q.No.2.: Find the **Fourier sine transform** of $\frac{e^{-ax}}{x}$.

Sol.:

Given: Now since we know that the Fourier sine transform of f(x) is

$$F_s\{f(x)\}=\int_0^\infty f(x)\sin(sx)dx\text{ , where }f(x)=\frac{e^{-ax}}{x}$$

To Find: Fourier sine transform of $\frac{e^{-ax}}{x}$.

Fourier sine transform of f(x) i.e. $F_s\{f(x)\}$is given by

$$F_s\{f(x)\}=\int_0^\infty f(x)\sin(sx)dx=\int_0^\infty\frac{e^{-ax}}{x}\sin(sx)dx=F(s)\text{, (say)}$$

Differentiating both sides w.r.t. s, we get

$$\frac{d}{ds}\{F(s)\}=\int_0^\infty\frac{xe^{-ax}\cos sx}{x}dx=\int_0^\infty e^{-ax}\cos sxdx=\frac{a}{s^2+a^2}.$$

Integrating w.r.t. s, we obtain $F(s)=\int_0^\infty\frac{a}{s^2+a^2}ds=\tan^{-1}\frac{s}{a}+c$.

But F(s) = 0, when s = 0, $\therefore c=0$.

Hence, Fourier sine transform of $\frac{e^{-ax}}{x}$ is $F(s)=\tan^{-1}\left(\frac{s}{a}\right)$. Ans.

Q.No.3.: Find the **Fourier cosine transform** of e^{-x^2}.

Sol.:

Given: We know that the Fourier cosine transform of f(x) is

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos(sx)dx \quad \text{where } f(x) = e^{-x^2}$$

To Find: Fourier cosine transform of e^{-x^2}.

Fourier cosine transform of e^{-x^2} is given by

$$F_c\left\{e^{-x^2}\right\} = \int_0^\infty e^{-x^2}\cos sxdx = I \text{ (say).} \qquad \text{(i)}$$

Differentiating w.r.t. to s, we have

$$\frac{dI}{ds} = -\int_0^\infty xe^{-x^2}\sin sxdx = \frac{1}{2}\int_0^\infty (\sin sx)\left(-2xe^{-x^2}\right)dx$$

$$= \frac{1}{2}\left[\left\{\sin(sx).e^{-x^2}\right\}_0^\infty - s\int_0^\infty \cos(sx).e^{-x^2}dx\right] \qquad \text{[Integrating by parts]}$$

$$= -\frac{s}{2}\left[\int_0^\infty e^{-x^2}\cos sxdx\right] = -\frac{s}{2}I \Rightarrow \frac{dI}{I} = -\frac{s}{2}ds.$$

Integrating, we have

$$\log I = -\frac{s^2}{4} + \log c \Rightarrow I = ce^{-s^2/4}. \qquad \text{(ii)}$$

Now when s = 0, then from (i), we get $I = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

$\therefore$ From (ii), we get $\frac{\sqrt{\pi}}{2} = c$.

Hence, $I = F_c\left\{e^{-x^2}\right\} = \frac{\sqrt{\pi}}{2}e^{-s^2/4}$. Ans.

Q.No.4.: Find the **Fourier cosine transform** of $f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2 - x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$

Sol.:

Given: We know that the Fourier cosine transform of f(x) is

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos(sx)dx \text{, where } f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2 - x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$$

To Find: Fourier cosine transform of $f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2 - x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$.

Fourier cosine transform of f(x) i.e. $F_c\{f(x)\}$is given by

$$= \int_0^\infty f(x)\cos sx dx$$

$$= \int_0^1 x\cos sx dx + \int_1^2 (2 - x)\cos sx dx + \int_2^\infty 0.dx$$

$$= \left| x\frac{\sin sx}{s} - \left(\frac{-\cos sx}{s^2}\right) \right|_0^1 + \left| (2 - x)\frac{\sin sx}{s} - (-1)\frac{-\cos sx}{s^2} \right|_1^2$$

$$= \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2}\right) + \left(-\frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2}\right)$$

$$= \frac{2\cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2}\text{. Ans.}$$

Q.No.5.: Find the **Fourier cosine transform** of $f(x) = \frac{1}{(1 + x^2)}$.

Hence derive Fourier sine transform of $\phi(x) = \frac{x}{(1 + x^2)}$.

Sol.:

Given: We know that the Fourier cosine transform of f(x) is

$$B(s) = F_c\{f(x)\} = \int_0^\infty f(x)\cos(sx)dx \text{, where } f(x) = \frac{2}{\pi}\int_0^\infty F_c(s)\cos(sx)ds$$

Step No 01: Find Fourier cosine transform of $f(x) = \frac{1}{(1 + x^2)}$.

Fourier cosine transform of f(x) i.e. $F_c\{f(x)\}$is given by

$$F_c\{f(x)\} = \int_0^\infty \frac{\cos sx}{1+x^2}\,dx = I,\ (\text{say}) \tag{i}$$

Differentiating both sides w.r.t. s, we get

$$\therefore \frac{dI}{ds} = \int_0^\infty \frac{-x\sin sx}{1+x^2}\,dx = -\int_0^\infty \frac{x^2\sin sx}{x\left(1+x^2\right)}\,dx \tag{ii}$$

$$= -\int_0^\infty \frac{\left[\left(1+x^2\right)-1\right]\sin sx}{x\left(1+x^2\right)}\,dx = -\int_0^\infty \frac{\sin sx}{x}\,dx + \int_0^\infty \frac{\sin sx}{x\left(1+x^2\right)}\,dx$$

$$\Rightarrow \frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x\left(1+x^2\right)}\,dx\,. \tag{iii}$$

Again, differentiating both sides w.r.t. s, we get

$$\therefore \frac{d^2I}{ds^2} = \int_0^\infty \frac{x\cos sx}{x\left(1+x^2\right)}\,dx = I\,.$$

$$\Rightarrow \frac{d^2I}{ds^2} - I = 0 \Rightarrow \left(D^2-1\right)I = 0,\ \text{where } D = \frac{dI}{ds}.$$

Its solution is $I = c_1e^{s} + c_2e^{-s}$ (iv)

$$\therefore \frac{dI}{ds} = c_1e^{s} - c_2e^{-s}\,. \tag{v}$$

When s = 0, (i) and (iv) give $c_1 + c_2 = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$.

Also when s = 0, (iii) and (v) give $c_1 - c_2 = -\frac{\pi}{2}$.

Solving these, $c_1 = 0$, and $c_2 = \frac{\pi}{2}$.

Thus, from (i) and (iv), we have $F_c\{f(x)\} = I = \left(\frac{\pi}{2}\right)e^{-s}$,

which is required Fourier cosine transform.

Step No 02: Derive Fourier sine transform of $\phi(x) = \frac{x}{\left(1+x^2\right)}$.

Now since we know that the Fourier sine transform of f(x) is

$B(\alpha)=F_s\{f(x)\}=\int_0^\infty f(x)\sin(\alpha x)dx$, where $f(x)=\frac{2}{\pi}\int_0^\infty F_s(\alpha)\sin(\alpha x)d\alpha$

Fourier cosine transform of f(x) i.e. $F_s\{f(x)\}$is given by

$$F_s[\phi(x)]=\int_0^\infty \frac{\sin sx}{x\left(1+x^2\right)}dx=-\frac{dI}{ds}, \text{(from (ii))}$$

$$=\left(\frac{\pi}{2}\right)e^{-s}, \text{ from (v), with } c_1=0, \text{ and } c_2=\frac{\pi}{2}.$$

Q.No.6.: Find the **(a)** **Fourier cosine** and **(b)** **sine transform** of $f(x)=e^{-ax}$ for $x\geq 0$ and $a>0$.

Deduce the integrals known as "Laplace integrals"

$$\int_0^\infty \frac{\cos sx}{a^2+s^2}ds \quad \text{and} \quad \int_0^\infty \frac{s\sin sx}{a^2+s^2}ds$$

Sol.: (a). By definition Fourier cosine transform of f(x) is

$$F_c\{f(x)\}=F_c(s)=\int_0^\infty f(x)\cos sxdx$$

$$F_c\{f\}=\int_0^\infty e^{-ax}\cos sxdx=\frac{a}{a^2+s^2}$$

The inverse Fourier cosine transform

$$f(x)=\frac{2}{\pi}\int_0^\infty F_c(s)\cos sxdx=\frac{2}{\pi}\int_0^\infty \frac{a}{a^2+s^2}\cos sxds.$$

Since $f(x)=e^{-ax}$, the above integral can be rewritten as

$$\int_0^\infty \frac{\cos sx}{a^2+s^2}d\alpha=\frac{\pi e^{-ax}}{2a}.$$

(b). The Fourier sine transform of f(x) is

$$F_s\{f(x)\}=F_s(s)=\int_0^\infty f(x)\sin sxdx=\int_0^\infty e^{-ax}\sin sxdx=\frac{s}{a^2+s^2}.$$

Now the inverse Fourier sine transform

$$f(x)=\frac{2}{\pi}\int_0^\infty F_s(s)\sin sxdx=\frac{2}{\pi}\int_0^\infty \frac{s}{a^2+s^2}\sin sxds.$$

With $f(x) = e^{-ax}$, the above integral takes the form

$$\int_0^\infty \frac{s \sin sx}{a^2 + s^2} d\alpha = \frac{\pi e^{-ax}}{2}.$$

Note: For a = 0, $\int_0^\infty \frac{s \sin sx}{s^2} ds = \int_0^\infty \frac{\sin sx}{s} ds = \frac{\pi}{2}$.

INVERSE FOURIER TRANSFORM

Q.No.1.: Find the **inverse Fourier sine transform** of $\frac{1}{s} e^{-as}$.

Sol.: Find: Inverse Fourier sine transform of $\frac{1}{s} e^{-as}$.

$$\text{i.e. } f(x) = F_s^{-1}\left\{\frac{e^{-as}}{s}\right\}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s} \sin sx ds \qquad \text{(i)}$$

Differentiating w.r.t. x, we get

$$\frac{df}{dx} = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s}(\sin sx)ds \quad = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s}.s.\cos s.xds$$

$$\frac{df}{dx} = \frac{2}{\pi}\int_0^\infty e^{-as} \cos sxds = \frac{2}{\pi}.\frac{a}{x^2 + a^2}$$

Integrating, we get

$$f(x) = \frac{2}{\pi}\int a \frac{dx}{x^2 + a^2} \quad = \frac{2}{\pi}\tan^{-1}\frac{x}{a} + A \qquad \text{(ii)}$$

From (i) at x = 0, f(0) = 0.

Using this in (ii), we get

0 = f(0) = 0 +A $\quad\therefore$ A = 0

Thus $f(x) = \frac{2}{\pi}\tan^{-1}\frac{x}{a}$, is the required inverse Fourier transform.

Note: When a = 0, $f(x) = F_s^{-1}\left\{\frac{1}{s}\right\} = \frac{2}{\pi}\tan^{-1}\infty = 1$.

Q.No.2.: Find f(x) whose **Fourier cosine transform** is $\frac{\sin as}{s}$.

Sol.: Given: $F_c\{f(x)\} = \frac{\sin sa}{s}$.

To find: The expression for f(x).

$$f(x) = F_c^{-1}\left\{\frac{\sin sa}{s}\right\} = \frac{2}{\pi}\int_0^\infty \frac{\sin sa}{s}\cos sx ds = \frac{2}{\pi}\frac{1}{2}\int_0^\infty \frac{\sin(s(a+x)) + \sin(s(a-x))}{s} ds$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin s(a+x)}{s} ds + \frac{1}{\pi}\int_0^\infty \frac{\sin s(a-x)}{s} ds = \frac{1}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) \quad \text{if } a - x > 0 \text{ i.e } x < a$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{2}\right) \quad \text{if } x - a > 0 \text{ i.e } x > a$$

Since $\int_0^\infty \frac{\sin ax}{x} dx = \frac{\pi}{2}$ when $a > 0$.

Thus $f(x) = \begin{cases} 1, & \text{if } x < a \\ 0, & \text{if } x > a \end{cases}$.

INTEGRAL EQUATIONS

Q.No.1.: Solve for f(x) the **integral equation**

$$\int_0^\infty f(x)\sin xt dx = \begin{cases} 1, & 0 \le t < 1 \\ 2, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$$

Sol.: By definition $F_s(t) = F_s\{f(s)\} = \int_0^\infty f(x)\sin xt dx = \begin{cases} 1, & 0 \le t < 1 \\ 2, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(t)\sin tsdt = \frac{2}{\pi}\left[\int_0^1 1.\sin txdt + \int_1^2 2\sin txdt + \int_2^\infty 0\right] = \frac{2}{\pi}\left[\left.\frac{-\cos tx}{x}\right|_{t=0}^{1} - 2\left.\frac{\cos tx}{x}\right|_{t=0}^{2}\right]$$

$$f(x) = \frac{2}{\pi x}[1 + \cos x - 2\cos 2x]. \text{ Ans.}$$

Q.No.2.: Solve the **integral equation** $\int_0^\infty F(x)\cos px dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$.

Hence deduce that $\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$.

Sol.: **Ist Part:** Solve the integral equation $\int_0^\infty F(x)\cos px dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$.

This means we have to find the result of F(x).

Now let $\int_0^\infty F(x)\cos px dx = f(p)$, then $f(p) = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$.

Now since we know that the Fourier cosine transform of f(x) is

$$B(\alpha) = F_c\{f(x)\} = \frac{2}{\pi}\int_0^\infty f(t)\cos(\alpha t)dt \text{ where } f(x) = \int_0^\infty B(\alpha)\cos(\alpha x)d\alpha$$

or $$B(\alpha) = F_c\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos(\alpha t)dt \text{ where } f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty B(\alpha)\cos(\alpha x)d\alpha$$

$$\therefore F_c\{F(x)\} = \bar{f}(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(x)\cos px dx = \sqrt{\frac{2}{\pi}} f(p).$$

By inversion formula for Fourier cosine transform, we have

$$F(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}(p)\cos px dp = \sqrt{\frac{2}{\pi}}\int_0^\infty \left(\sqrt{\frac{2}{\pi}} f(p)\right)\cos px dp = \frac{2}{\pi}\int_0^\infty f(p)\cos px dp$$

$$= \frac{2}{\pi}\left[\int_0^1 (1-p)\cos px dp + \int_1^\infty 0.\cos px dp\right]$$

$$= \frac{2}{\pi}\left[(1-p).\frac{\sin px}{x} - (-1).\frac{-\cos px}{x^2}\right]_0^1 = \frac{2}{\pi}\left[-\frac{\cos x}{x^2} + \frac{1}{x^2}\right] = \frac{2(1-\cos x)}{\pi x^2}. \text{ Ans}$$

IInd Part: To show: $\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$.

Since $\int_0^\infty F(x)\cos px dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$.

But $F(x) = \frac{2(1-\cos x)}{\pi x^2}$.

$$\therefore \int_0^\infty \left(\frac{2(1-\cos x)}{\pi x^2}\right)\cos px dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{(1-\cos x)}{x^2}.\cos px dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$$

When $p = 0$, we have $\frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2}dx = 1 \Rightarrow \int_0^\infty \frac{2\sin^2\frac{x}{2}}{x^2}dx = \frac{\pi}{2}$.

Putting $x = 2t$ so that $dx = 2dt$, we get $\int_0^\infty \frac{\sin^2 t}{t^2}dt = \frac{\pi}{2}$. Hence prove.

Home Assignments

Convolution theorem

Q.No.1.: Verify Convolution theorem for $f(x) = g(x) = e^{-x^2}$.

FOURIER COSINE AND SINE TRANSFORMS

Q.No.1.: Find the **Fourier cosine transform** of the function f(x), if

$$f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}.$$

Ans.: $\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(1+s)a}{1+s} + \frac{\sin(1-p)a}{1-p}\right]$

Q.No.2.: Find the **Fourier cosine transform** of $f(u) = \begin{cases} u^2, & |u| < u_0 \\ 0, & |u| > u_0 \end{cases}$

Ans.: $\frac{2\{(u_0^2 s^2 - 2)\sin u_0 s + 2u_0 s\cos u_0 s\}}{s^3}$.

Q.No.3.: Find the **Fourier cosine transform** of $f(x) = e^{-x^2/2}$.

Ans.: $\frac{\sqrt{\pi}}{2}e^{-s^2/4}$.

Q.No.4.: Find **Fourier cosine transform** of $f(x) = e^{-ax}\cos ax$.

Ans.: $\frac{a\left(s^2+2a^2\right)}{s^4+4a^4}$.

Q.No.5.: Obtain **Fourier sine transform** of $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$.

Ans.: $\frac{1}{2}\left\{\frac{\sin[a(1-s)]}{1-s} - \frac{\sin[a(1+s)]}{1+s}\right\}$.

Q.No.6.: Find **Fourier sine transform** of $\frac{1}{\left[x(x^2+a^2)\right]}$.

Ans.: $\frac{\pi}{2a^2}\left(1-e^{-as}\right)$

Q.No.7.: Find the **Fourier sine and cosine transforms** of $f(x) = e^{-ax}$ $(x > 0)$.

Ans.: $\frac{s}{a^2+s^2}$, $\frac{a}{a^2+s^2}$.

Q.No.8.: Find the **Fourier sine and cosine transforms** of

(i). $2e^{-5x} + 3e^{-2x}$

(ii). $f(x) = \begin{cases} 1, & \text{for } 0 \le x < a \\ 0, & \text{for } x > a \end{cases}$.

Ans.: (i). $s\sqrt{\frac{2}{\pi}}\left(\frac{2}{s^2+25} + \frac{5}{s^2+4}\right)$; $10\sqrt{\frac{2}{\pi}}\left(\frac{1}{s^2+25} + \frac{1}{s^2+4}\right)$,

(ii). $\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos as}{s}\right)$; $\sqrt{\frac{2}{\pi}}\left(\frac{\sin as}{s}\right)$.

Q.No.9.: Find **Fourier sine and cosine transforms** of

$$f(x) = \begin{cases} k, & \text{if } 0 < x < a \\ 0, & \text{if } x > a \end{cases}.$$

Ans.: $F_c\{f(x)\} = k.\frac{\sin as}{s}$, $F_s\{f(x)\} = \frac{k(1-\cos as)}{s}$

Q.No.10.: Find **Fourier sine and cosine transforms** of $2e^{-5x} + 5e^{-2x}$.

Ans.: FST $= \dfrac{2s}{s^2+25} + \dfrac{5s}{s^2+4}$, FCT $= \dfrac{10}{s^2+4} + \dfrac{10}{s^2+25}$.

Q.No.11.: If the **Fourier sine transform** of $f(x) = \dfrac{1-\cos n\pi}{n^2\pi^2}(0 < x < \pi)$, find f(x).

Ans.: $\dfrac{2}{\pi^3}\sum_{n=1}^{\infty} \dfrac{1-\cos n\pi}{n^2}\sin n\pi$.

INVERSE FOURIER TRANSFORM

Q.No.1.: Find f(x) if its **Fourier cosine transform** is $\dfrac{1}{1+s^2}$ and **Fourier sine transform** is $\dfrac{s}{1+s^2}$.

Ans.: $f(x) = e^{-x}$, $f(x) = e^{-x}$

Q.No.2.: Find f(x) if its **Fourier cosine transform** is $\dfrac{1}{2\pi}\left(a - \dfrac{s}{2}\right)$ if $s < 2a$ and zero if $s \ge 2a$.

Ans.: $\dfrac{\left(2\sin^2 ax\right)}{\pi^2 x^2}$.

Q.No.3.: Find the inverse **Fourier sine transform** of $s^n e^{-as}$.

Ans.: $\dfrac{2.n!\sin[(n+1)x]}{\pi.\left(a^2+x^2\right)^{\frac{n+1}{2}}}$.

Q.No.4.: Find the inverse **Fourier transform** of $e^{-|s|y}$.

Ans.: $\dfrac{y}{\left[\pi\left(y^2+x^2\right)\right]}$.

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5th Topic

Fourier Transform

- Finite Fourier Transforms
- Finite Fourier Cosine and Sine Transforms
- Inverse Finite Fourier Cosine and Sine Transforms
- Properties of Finite Fourier Cosine and Sine Transforms
  Finite Fourier Cosine and Finite Fourier Sine Transforms of the Derivatives
- Parseval's Identity
  For Fourier Transforms
  For Fourier Cosine and Sine Transforms
- Relation between Fourier and Laplace Transforms

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Finite Fourier Transform of f(x):

The **finite Fourier transform** of f(x) in $-c < x < c$ is defined as

$$F(n) = \int_{-c}^{c} f(x) e^{i\frac{n\pi x}{c}} dx ,$$

where n is an integer.

Finite Fourier Sine Transform of f(x):

Let f(x) be a function defined in a finite interval $0 < x < c$ i.e., when the range of one of the variables say x is finite. Suppose f(x) is neither periodic nor even nor odd. Now by redefining f(x) as an odd function in $-c < x < c$, the half range Fourier sine series expansion of f(x) can be obtained as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{c}\right), \text{ where } b_n = \frac{2}{c}\int_0^c f(x)\sin\left(\frac{n\pi x}{c}\right)dx .$$

Definition: The **finite Fourier sine transform** of f(x) in $0 < x < c$ is defined as

$$F_s(n) = \int_0^c f(x) \sin\left(\frac{n\pi x}{c}\right) dx,$$

where n is an integer.

Inverse Finite Fourier Sine Transform of $F_s(n)$:

Definition: The **inverse finite Fourier sine transform** of $F_s(n)$ is given by

$$f(x) = \frac{2}{c}\sum_{n=1}^{\infty} F_s(n).\sin\left(\frac{n\pi x}{c}\right).$$

where n is an integer.

Finite Fourier Cosine Transform of f(x):

Now by redefining f(x) as an even function in $-c < x < c$, the half range Fourier cosine series expansion of f(x) can be obtained as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right), \text{ where } a_0 = \frac{1}{c}\int_0^c f(x)dx \text{ and } a_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{n\pi x}{c}\right)dx.$$

Definition: The **finite Fourier cosine transform** of f(x) in $0 < x < c$ is defined as

$$F_c(n) = \int_0^c f(x)\cos\left(\frac{n\pi x}{c}\right)dx.$$

where n is an integer.

Inverse Finite Fourier Cosine Transform of $F_c(n)$:

Definition: The **inverse finite Fourier cosine transform** of $F_c(n)$ is given by

$$f(x) = \frac{1}{c}F_c(0) + \frac{2}{c}\sum_{n=1}^{\infty} F_c(n)\cos\left(\frac{n\pi x}{c}\right).$$

Importance:

- These transforms are very useful for such a boundary-value problems in which at least two of the boundaries are parallel and separated by a finite distance.
- The finite Fourier sine transform is very useful for problems involving boundary conditions of heat distribution on two parallel boundaries.
- The finite Fourier cosine transform is very useful for problems in which the velocities normal to two parallel boundaries are among the boundary conditions.

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Properties of Finite Fourier Cosine and Finite Fourier Sine Transforms:

Since we know the Fourier Cosine and Sine Transforms of the Derivatives:

(i) $F_c\{f'(x)\} = sF_s\{f(x)\} - f(0)$,

(ii) $F_c\{f''(x)\} = -s^2F_c\{f(x)\} - f'(0)$,

(iii) $F_s\{f'(x)\} = -sF_c\{f(x)\}$,

(iv) $F_s\{f''(x)\} = -s^2F_s\{f(x)\} + sf(0)$.

Finite Fourier Cosine and Finite Fourier Sine Transforms of the Derivatives:

Result 1: $F_s\left\{\dfrac{\partial^2 f}{\partial x^2}\right\} = -\dfrac{n^2\pi^2}{c^2}F_s(n) - \dfrac{n\pi}{c}\left[(-1)^n f(c,t) - f(0,t)\right]$.

Proof: By definition, the **finite Fourier sine transform** of f(x,t) in $0 < x < c$ is defined as

$$F_s(n) = \int_0^c f(x,t)\sin\left(\frac{n\pi x}{c}\right)dx,$$

Thus by definition, $F_s\left\{\dfrac{\partial^2 f}{\partial x^2}\right\} = \displaystyle\int_0^c \left(\dfrac{\partial^2 f}{\partial x^2}\right).\sin\left(\dfrac{n\pi x}{c}\right)dx$.

Integrating by parts, we get

$$F_s\left\{\frac{\partial^2 f}{\partial x^2}\right\} = \frac{\partial f}{\partial x}.\sin\left(\frac{n\pi x}{c}\right)\Bigg|_0^c - \frac{n\pi}{c}\int_0^c \frac{\partial f}{\partial x}\cos\left(\frac{n\pi x}{c}\right)dx$$

$$\Rightarrow F_s\left\{\frac{\partial^2 f}{\partial x^2}\right\} = 0 - \frac{n\pi}{c}\left\{\left[f(x,t).\cos\left(\frac{n\pi x}{c}\right)\right]_0^c + \frac{n\pi}{c}\int_0^c f(x,t).\cos\left(\frac{n\pi x}{c}\right)dx\right\}$$

$$\Rightarrow F_s\left\{\frac{\partial^2 f}{\partial x^2}\right\} = -\frac{n\pi}{c}\left[(-1)^n f(c,t) - f(0,t)\right] - \frac{n^2\pi^2}{c^2}F_s(n).$$

$$\Rightarrow F_s\left\{\frac{\partial^2 f}{\partial x^2}\right\} = -\frac{n^2\pi^2}{c^2}F_s(n) - \frac{n\pi}{c}\left[(-1)^n f(c,t) - f(0,t)\right].$$

This completes the proof.

Result 2: $F_c\left\{\dfrac{\partial^2 f}{\partial x^2}\right\} = -\left[f_x(0,t) - (-1)^n f_x(c,t)\right] - \dfrac{n^2\pi^2}{c^2} F_c(n)$.

Proof: By definition, the **finite Fourier cosine transform** of f(x,t) in $0 < x < c$ is defined as

$$F_c(n) = \int_0^c f(x,t)\cos\left(\frac{n\pi x}{c}\right)dx\,,$$

Thus, $F_c\left\{\dfrac{\partial^2 f}{\partial x^2}\right\} = \displaystyle\int_0^c \left(\dfrac{\partial^2 f}{\partial x^2}\right).\cos\left(\dfrac{n\pi x}{c}\right)dx$.

Integrating by parts, we get

$$F_c\left\{\frac{\partial^2 f}{\partial x^2}\right\} = \frac{\partial f}{\partial x}.\cos\left(\frac{n\pi x}{c}\right)\Bigg|_0^c + \frac{n\pi}{c}\int_0^c \frac{\partial f}{\partial x}.\sin\left(\frac{n\pi x}{c}\right)dx$$

$$\Rightarrow F_c\left\{\frac{\partial^2 f}{\partial x^2}\right\} = \left[f_x(0,t) + (-1)^n f_x(c,t)\right] + \frac{n\pi}{c}\left\{f(x,t).\sin\left(\frac{n\pi x}{c}\right)\Bigg|_0^c - \frac{n\pi}{c}\int_0^c f(x,t).\cos\left(\frac{n\pi x}{c}\right)dx\right\}$$

$$\Rightarrow F_c\left\{\frac{\partial^2 f}{\partial x^2}\right\} = \left[(-1)^n f_x(c,t) - f_x(0,t)\right] - \frac{n^2\pi^2}{c^2}F_c(n).$$

This completes the proof.

Remarks:

1. If in a problem, $f(x,t)_{x=0}$ is given, then we use infinite Fourier sine transform to remove $\dfrac{\partial^2 f}{\partial x^2}$ from the differential equation.

2. If in a problem, $\left[\dfrac{\partial f(x,t)}{\partial x}\right]_{x=0}$ is given, then we use infinite Fourier cosine transform to remove $\dfrac{\partial^2 f}{\partial x^2}$ from the differential equation.

3. If in a problem, $f(0,t)$ and $f(c,t)$ are given, then we use finite Fourier sine transform to remove $\dfrac{\partial^2 f}{\partial x^2}$ from the differential equation.

4. If in a problem, $\left[\frac{\partial f}{\partial x}\right]_{x=0}$ and $\left[\frac{\partial f}{\partial x}\right]_{x=c}$ are given, then we use finite Fourier cosine transform to remove $\frac{\partial^2 f}{\partial x^2}$ from the differential equation.

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Parseval's Identities for Fourier Transforms:

Statement:

Let $F(s)$ and $G(s)$ be respectively, the Fourier transforms of f(x) and g(x).

Then (i) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\overline{G}(s)ds = \int_{-\infty}^{\infty} f(x)\overline{g}(x)dx, \qquad \text{(i)}$$

(ii) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx, \qquad \text{(ii)}$$

where bar implies the complex conjugate.

(A French Mathematician, Marc Antoine Parseval (1755-1836) discovered these identities).

Proof:

Using the inversion formula for Fourier transform for $\overline{g}(x)$ in the RHS of (i), we have

$$\int_{-\infty}^{\infty} f(x)\overline{g}(x)dx = \int_{-\infty}^{\infty} f(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G}(s)e^{isx}ds\right\}dx.$$

Interchanging the order of integration in RHS, we get

$$\int_{-\infty}^{\infty} f(x)\overline{g}(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G}(s)\left\{\int_{-\infty}^{\infty} f(x)e^{isx}dx\right\}ds.$$

Observing that inner most integral is the Fourier transform of f(x), we have

$$\int_{-\infty}^{\infty} f(x)\overline{g}(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G}(s)F(s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\overline{G}(s)ds.$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\overline{G}(s)dx = \int_{-\infty}^{\infty} f(x)\overline{g}(x)dx.$$

This completes the proof.

IInd Part:

Taking g(x) = f(x) and note that $z\overline{z} = |z|^2$. Then (i) i.e.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\overline{G}(s)ds = \int_{-\infty}^{\infty} f(x)\overline{g}(x)dx$$

reduces to

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx . \qquad \text{(ii)}$$

Results (ii) and its generalization (i) are known as **Parseval' Identity for Fourier transforms (integrals).**

Parseval's Identities for Fourier Cosine and Sine Transforms:

Corollary 1: The Parseval's identity for Fourier cosine transform are

a). $\frac{2}{\pi}\int_{0}^{\infty} F_c(s)G_c(s)ds = \int_{0}^{\infty} f(x)g(x)dx$ **,**

b). $\frac{2}{\pi}\int_{0}^{\infty} |F_c(s)|^2 ds = \int_{0}^{\infty} |f(x)|^2 dx$ **.**

Corollary 2: The Parseval's identity for Fourier sine transform are

c). $\frac{2}{\pi}\int_{0}^{\infty} F_s(s)G_s(s)ds = \int_{0}^{\infty} f(x)g(x)dx$ **,**

d). $\frac{2}{\pi}\int_{0}^{\infty} |F_s(s)|^2 ds = \int_{0}^{\infty} |f(x)|^2 dx$ **.**

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Relation between Fourier and Laplace transforms:

Statement: If $f(t) = \begin{cases} e^{-xt}g(t), & t>0 \\ 0, & t<0 \end{cases}$ (i)

Then $F\{f(t)\} = L\{g(t)\}$.

Proof: We have $F\{f(t)\} = \int_{-\infty}^{\infty} e^{ist}f(t)\,dt = \int_{-\infty}^{0} e^{ist}.0dt + \int_{0}^{\infty} e^{ist}.e^{-xt}g(t)dt$

$$= \int_{0}^{\infty} e^{(is-x)t}g(t)dt = \int_{0}^{\infty} e^{-pt}g(t)dt \quad \text{where } p = x - is$$

$$= L\{g(t)\}.$$

Hence, the Fourier transform of f(t) [defined by (i)] is the Laplace transform of g(t).

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Now let us solve some problems related to these topics:

Finite Fourier sine and cosine transform

Q.No.1.: Find the **finite Fourier sine and cosine transform** of $f(x) = x(\pi - x)$ in $0 < x < \pi$.

Sol.:

Part-I:

Remember: Finite Fourier Sine Transform of f(x):

The **finite Fourier sine transform** of f(x) in $0 < x < c$ is defined as

$$F_s(n) = \int_0^c f(x) \sin\left(\frac{n\pi x}{c}\right) dx,$$

where n is an integer.

Thus, Finite Fourier sine transform of $f(x) = x(\pi - x)$ is

$$F_s(n) = \int_0^c f(x) \sin \frac{n\pi x}{c} dx$$

$$= \int_0^\pi x(\pi - x).\sin nx dx$$

$$= \left[x(\pi - x)\int_0^\pi \sin nx dx\right] - \int_0^\pi .\left[\left(\frac{d}{dx} x(\pi - x)\right)\int_0^\pi \sin nx dx\right] dx$$

$$= \left[x(\pi - x)\left(\frac{-\cos nx}{n}\right)\Big|_0^\pi\right] - \int_0^\pi .\left[(\pi - 2x)\left(\frac{-\cos nx}{n}\right)\right] dx$$

$$= 0 - \left[(\pi - 2x)\int_0^\pi \left(\frac{-\cos nx}{n}\right) - \int_0^\pi \frac{d}{dx}(\pi - 2x)\int_0^\pi \left(\frac{-\cos nx}{n}\right) dx\right]$$

$$= -\left[(\pi - 2x).\left(\frac{-\sin nx}{n^2}\right)\Big|_0^\pi - (-2).\int_0^\pi \left(\frac{-\sin nx}{n^2}\right) dx\right]$$

$$= -\left[0 + (2).\left(\frac{\cos nx}{n^3}\right)\right]_0^\pi.$$

$$F_s(n) = -\frac{2}{n^3}[\cos n\pi - 1] = \frac{2}{n^3}\left[1 - (-1)^n\right].$$ Ans.

Part-II:

Remember: Finite Fourier Cosine Transform of f(x):

The **finite Fourier cosine transform** of f(x) in $0 < x < c$ is defined as

$$F_c(n) = \int_0^c f(x)\cos\left(\frac{n\pi x}{c}\right)dx .$$

where n is an integer.

Thus, Finite Fourier cosine transform of $f(x) = x(\pi - x)$ is

$$F_c(n) = \int_0^c f(x)\cos\frac{n\pi x}{c}dx$$

$$= \int_0^\pi x(\pi - x).\cos nx dx$$

$$= \left[x(\pi-x)\int_0^\pi \cos nx dx\right] - \int_0^\pi .\left[\left(\frac{d}{dx}x(\pi-x)\right)\int_0^\pi \cos nx dx\right]dx$$

$$= \left[x(\pi-x)\left(\frac{\sin nx}{n}\right)\Big|_0^\pi\right] - \int_0^\pi .\left[(\pi-2x)\left(\frac{\sin nx}{n}\right)\right]dx$$

$$= 0 - \left[(\pi-2x)\int_0^\pi \left(\frac{\sin nx}{n}\right) - \int_0^\pi \frac{d}{dx}(\pi-2x)\int_0^\pi \left(\frac{\sin nx}{n}\right)dx\right]$$

$$= -\left[(\pi-2x).\left(\frac{-\cos nx}{n^2}\right)\Big|_0^\pi - (-2).\int_0^\pi \left(\frac{-\cos nx}{n^2}\right)dx\right]$$

$$= -\left[(\pi-2\pi).\left(\frac{-\cos n\pi}{n^2}\right) - (\pi-2.0).\left(\frac{-\cos n0}{n^2}\right) - (-2)\left(\frac{-\sin nx}{n^3}\right)\Big|_0^\pi\right]$$

$$= -\left[\left(\frac{\pi\cos n\pi}{n^2}\right) + \left(\frac{\pi}{n^2}\right) - 0\right]$$

$$F_c(n) = -\frac{(\pi\cos n\pi + \pi)}{n^2} = -\frac{\pi}{n^2}\left[(-1)^n + 1\right].\ \text{Ans.}$$

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Inverse finite Fourier cosine and sine transforms

Q.No.1.: Find the **inverse finite Fourier cosine transform** of $F_c(n)$ if

$$F_c(n) = \frac{\sin\left(\frac{n\pi}{2}\right)}{2n} \text{ for } n = 1, 2, 3, \ldots\ldots\ldots \text{ and}$$

$$= \frac{\pi}{4} \text{ when } n = 0$$

in $0 < x < \pi$.

Sol.:

Remember: Inverse Finite Fourier Cosine Transform of $F_c(n)$:

The **inverse finite Fourier cosine transform** of $F_c(n)$ is given by

$$f(x) = \frac{1}{c}F_c(0) + \frac{2}{c}\sum F_c(n)\cos\left(\frac{n\pi x}{c}\right).$$

Thus, f(x) = Inverse finite Fourier cosine transform of $F_c(n)$

$$= \frac{1}{\pi}F_c(0) + \frac{2}{\pi}\sum_{n=1}^{\infty} F_c(n).\cos\left(\frac{n\pi x}{\pi}\right)$$

$$= \frac{1}{\pi}.\frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right).\left(\frac{1}{2n}\right).\cos(nx)$$

Thus $f(x) = \frac{1}{4} + \frac{1}{\pi}\sum \frac{1}{n}.\sin\left(\frac{n\pi}{2}\right).\cos nx$. Ans.

Q.No.2.: Find the **inverse finite Fourier sine transform** of $F_s(n)$ if

$$F_s(n) = \frac{2\pi(-1)^{n-1}}{n^2} \text{ for } n = 1, 2, 3, \ldots\ldots\ldots$$

in $0 < x < \pi$.

Sol.:

Remember: Inverse Finite Fourier Sine Transform of $F_s(n)$:

The **inverse finite Fourier sine transform** of $F_s(n)$ is given by

$$f(x) = \frac{2}{c}\sum F_s(n).\sin\left(\frac{n\pi x}{c}\right).$$

where n is an integer.

Thus, $f(x) = \frac{2}{c}\sum_{n=1}^{\infty} F_s(n).\sin\left(\frac{n\pi x}{c}\right)$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{2\pi(-1)^{n-1}}{n^2}\right\}.\sin nx$$

$$= 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\sin nx \text{ . Ans.}$$

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Parseval's Identities for Fourier Transforms

Q.No.1.: Using Parseval's identity, evaluate $\int_0^{\infty}\frac{dx}{(a^2+x^2)(b^2+x^2)}$ and

hence find that $\int_0^{\infty}\frac{dx}{(x^2+1)^2}$.

Sol.:

Step-I: Evaluate $\int_0^{\infty}\frac{dx}{(a^2+x^2)(b^2+x^2)}$.

Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$.

Then $F_c(s) = \frac{a}{a^2+s^2}$ and $G_c(s) = \frac{b}{b^2+s^2}$

Now, using Parseval's identity for Fourier cosine transform, i.e.

$$\frac{2}{\pi}\int_0^{\infty}F_c(s)G_c(s)ds = \int_0^{\infty}f(x)g(x)dx \qquad \text{(i)}$$

Thus, we have $\frac{2}{\pi}\int_0^{\infty}\frac{ab}{(a^2+s^2)(b^2+s^2)}ds = \int_0^{\infty}e^{-(a+b)x}dx$

$$\Rightarrow \frac{2ab}{\pi}\int_0^{\infty}\frac{ds}{(a^2+s^2)(b^2+s^2)} = \left|\frac{e^{-(a+b)x}}{-(a+b)}\right|_0^{\infty} = \frac{1}{a+b}$$

$$\Rightarrow \int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \frac{\pi}{2ab(a+b)}$$

Replacing s by x, we get

$$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)}.$$

Step-II: Hence find that $\int_0^\infty \frac{dx}{(x^2+1)^2}$.

Since $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)}$.

Put $a=1, b=1$, we get $\int_0^\infty \frac{dx}{(1^2+x^2)(1^2+x^2)} = \frac{\pi}{2(1+1)} \Rightarrow \int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$.

Q.No.2.: Using Parseval's identity, evaluate $\int_0^\infty \frac{\sin at}{t(a^2+t^2)}dt$.

Sol.: Let $f(x) = e^{-ax}$ and $g(x) = \begin{cases} 1, & 0<x<a \\ 0, & x>a \end{cases}$.

Then $F_c(s) = \frac{a}{a^2+s^2}$, $G_c(s) = \frac{\sin as}{s}$

Now using Parseval's identity for Fourier cosine transform, i.e.

$$\frac{2}{\pi}\int_0^\infty F_c(s)G_c(s)ds = \int_0^\infty f(x)g(x)dx. \qquad \text{(i)}$$

Thus, we have $\frac{2}{\pi}\int_0^\infty \frac{a\sin as}{s(a^2+s^2)}ds = \int_0^a e^{-ax}.1dx = \left[\frac{e^{-ax}}{-a}\right]_0^a = \frac{1-e^{-a^2}}{a}$

$$\Rightarrow \int_0^\infty \frac{\sin at}{t(a^2+t^2)}dt = \frac{\pi}{2a^2}\left(1-e^{-a^2}\right).$$

Q.No.3.: Using Parseval's identity, prove that $\int_0^\infty \frac{(x\cos x - \sin x)^2}{x^6}dx = \frac{\pi}{15}$.

Sol.: Consider the function f(x) defined as $f(x) = \begin{cases} a^2 - x^2, & \text{if } |x| \le a \\ 0 & \text{if } |x| > a \end{cases}$.

Since we know the Fourier transform of f(x) is defines as

$$F\{f(x)\} = F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx = \left[\int_{-\infty}^{-a} f(x)e^{isx}dx + \int_{-a}^{a} f(x)e^{isx}dx + \int_{a}^{\infty} f(x)e^{isx}dx\right].$$

Since f(x) = 0 in $(-\infty,-a)$ and (a,∞), we have

$$F(s) = \int_{-a}^{a} (a^2 - x^2)e^{isx}dx = \int_{-a}^{a} a^2e^{isx}dx - \int_{-a}^{a} x^2e^{isx}dx = I_1 - I_2\ .$$

Here $I_1 = \int_{-a}^{a} a^2e^{isx}dx = a^2\left(\frac{e^{isx}}{is}\right)\Bigg|_{-a}^{a} = \frac{a^2}{is}\left[e^{isa} - e^{-isa}\right].$

And integrating by parts, we get

$$I_2 = \int_{-a}^{a} x^2e^{isx}dx = \left[x^2\frac{e^{isx}}{is}\right]_{-a}^{a} - \int_{-a}^{a} 2x\frac{e^{isx}}{is}dx = \left[x^2\frac{e^{isx}}{is}\right]_{-a}^{a} - \left[2x\frac{e^{isx}}{i^2s^2}\right]_{-a}^{a} + \int_{-a}^{a} 2\frac{e^{isx}}{i^2s^2}dx$$

$$= \left(x^2\frac{e^{isx}}{is} - 2x\frac{e^{isx}}{i^2s^2} + 2\frac{e^{isx}}{i^3s^3}\right)\Bigg|_{-a}^{a}$$

$$= \frac{a^2}{is}\left[e^{isa} - e^{-isa}\right] + \frac{2a}{s^2}\left[e^{isa} + e^{-isa}\right] - \frac{2}{is^3}\left[e^{isa} - e^{-isa}\right]$$

So $F(s) = I_1 - I_2$

$$= \left\{\frac{a^2}{is}\left[e^{isa} - e^{-isa}\right]\right\} - \left\{\frac{a^2}{is}\left[e^{isa} - e^{-isa}\right] + \frac{2a}{s^2}\left[e^{isa} + e^{-isa}\right] - \frac{2}{is^3}\left[e^{isa} - e^{-isa}\right]\right\}$$

$$= \left\{\frac{2a}{s^2}\left[2\cos sa\right] - \frac{2}{is^3}\left[2i\sin sa\right]\right\}$$

$$\Rightarrow F(s) = -\frac{4}{s^3}\left[\sin sa - as\cos sa\right].$$

Parseval's Identities for Fourier Transforms:

Statement:

Let $F(s)$ and $G(s)$ be respectively the Fourier transforms of f(x) and g(x).

Then (i) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\overline{G}(s)ds = \int_{-\infty}^{\infty} f(x)\overline{g}(x)dx\,, \qquad \text{(i)}$$

(ii) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|F(s)\right|^2 ds = \int_{-\infty}^{\infty} \left|f(x)\right|^2 dx\,, \qquad \text{(ii)}$$

where bar implies the complex conjugate.

Apply Parseval's identity for f(x) and $F(s)$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{2\pi}\left(-\frac{4}{s^3}[\sin sa - as\cos sa]\right)^2 ds = \int_{-a}^{a} \left(\left|a^2 - x^2\right|\right)^2 dx = \int_{-a}^{a} \left(a^4 + x^4 - 2a^2x^2\right)dx$$

$$= \left[a^4x + \frac{x^5}{5} - 2a^2\frac{x^3}{3}\right]_{-a}^{a}$$

$$= 2a^5\left(1 + \frac{1}{5} - \frac{2}{3}\right) = \frac{16a^5}{15}.$$

Replacing s by x, we get

$$2\int_0^{\infty} \frac{8}{\pi}\left(\frac{ax\cos ax - \sin ax}{x^3}\right)^2 dx = \frac{16a^5}{15}.$$

$$\Rightarrow \int_0^{\infty} \frac{1}{x^6}(ax\cos ax - \sin ax)^2 dx = \frac{\pi a^5}{15}.$$

For a = 1, we get $\int_0^{\infty} \frac{(x\cos x - \sin x)^2}{x^6} dx = \frac{\pi}{15}$.

Hence proved.

Q.No.4.: Using Parseval's identity, evaluate $\int_0^{\infty} \frac{x^2}{\left(a^2 + x^2\right)\left(b^2 + x^2\right)} dx$ and

hence find $\int_0^{\infty} \left(\frac{x}{x^2 + 1}\right)^2 dx$.

Sol.:

Part-I: Evaluate $\int_0^{\infty} \frac{x^2}{\left(a^2 + x^2\right)\left(b^2 + x^2\right)} dx$

We know that if $f(x) = e^{-ax}$, then Fourier sine transform of

$$f(x) = F_s\{f(x)\} = F_s(s) = \frac{s}{a^2 + s^2}.$$

Similarly, for $g(x)=e^{-bx}$ then $G_s\{g(x)\}=G_s(s)=\dfrac{s}{b^2+s^2}$.

Now by Parseval's identity for Fourier sine transform, we get

$$\frac{2}{\pi}\int_0^\infty F_s(s)G_s(s)ds=\int_0^\infty f(x)g(x)dx$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \left(\frac{s}{a^2+s^2}\right)\left(\frac{s}{b^2+s^2}\right)ds=\int_0^\infty e^{-ax}.e^{-bx}dx$$

$$\Rightarrow \int_0^\infty \left(\frac{s}{a^2+s^2}\right)\left(\frac{s}{b^2+s^2}\right)ds=\frac{\pi}{2}\int_0^\infty e^{-ax}.e^{-bx}dx=\frac{\pi}{2}\int_0^\infty e^{-(a+b)x}dx$$

$$\Rightarrow \int_0^\infty \frac{s^2}{(a^2+s^2)(b^2+s^2)}ds=\frac{\pi}{2}\frac{e^{-(a+b)x}}{-(a+b)}\bigg|_{x=0}^{\infty}=0-\frac{\pi}{2}\frac{1}{(a+b)}.$$

Replacing s by x, we get

Thus $\displaystyle\int_0^\infty \frac{x^2}{(a^2+x^2)(b^2+x^2)}dx=\frac{\pi}{2(a+b)}$. Ans.

Part-II: Find $\displaystyle\int_0^\infty \left(\frac{x}{x^2+1}\right)^2 dx$.

Put a = b =1, we get $\displaystyle\int_0^\infty \left(\frac{x}{x^2+1}\right)^2 dx=\frac{\pi}{4}$. Ans.

INTEGRAL EQUATIONS

Q.No.1.: Solve for f(x) the **integral equation**

$$\int_0^\infty f(x)\sin xt dx=\begin{cases}1, & 0\le t<1\\ 2, & 1\le t<2\\ 0, & t\ge 2\end{cases}$$

Sol.: By definition $F_s(t)=F_s\{f(s)\}=\displaystyle\int_0^\infty f(x)\sin xt dx=\begin{cases}1, & 0\le t<1\\ 2, & 1\le t<2\\ 0, & t\ge 2\end{cases}$

$$f(x)=\frac{2}{\pi}\int_0^\infty F_s(t)\sin ts dt=\frac{2}{\pi}\left[\int_0^1 1.\sin txdt+\int_1^2 2\sin txdt+\int_2^\infty 0\right]=\frac{2}{\pi}\left[\frac{-\cos tx}{x}\bigg|_{t=0}^{1}-2\frac{\cos tx}{x}\bigg|_{t=0}^{2}\right]$$

$f(x) = \frac{2}{\pi x}[1 + \cos x - 2\cos 2x]$. Ans.

Q.No.2.: Solve the **integral equation** $\int_0^\infty F(x)\cos pxdx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$.

Hence deduce that $\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$.

Sol.: Ist Part: Solve the integral equation $\int_0^\infty F(x)\cos pxdx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$.

This means we have to find the result of F(x).

Now let $\int_0^\infty F(x)\cos pxdx = f(p)$, then $f(p) = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$.

Now since we know that the Fourier cosine transform of f(x) is

$$B(\alpha) = F_c\{f(x)\} = \frac{2}{\pi}\int_0^\infty f(t)\cos(\alpha t)dt \quad \text{where} \quad f(x) = \int_0^\infty B(\alpha)\cos(\alpha x)d\alpha$$

or $B(\alpha) = F_c\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos(\alpha t)dt$ where $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty B(\alpha)\cos(\alpha x)d\alpha$

$$\therefore F_c\{F(x)\} = \bar{f}(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(x)\cos pxdx = \sqrt{\frac{2}{\pi}} f(p).$$

By inversion formula for Fourier cosine transform, we have

$$F(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}(p)\cos pxdp = \sqrt{\frac{2}{\pi}}\int_0^\infty \left(\sqrt{\frac{2}{\pi}} f(p)\right)\cos pxdp = \frac{2}{\pi}\int_0^\infty f(p)\cos pxdp$$

$$= \frac{2}{\pi}\left[\int_0^1 (1-p)\cos pxdp + \int_1^\infty 0.\cos pxdp\right]$$

$$= \frac{2}{\pi}\left[(1-p).\frac{\sin px}{x} - (-1).\frac{-\cos px}{x^2}\right]_0^1 = \frac{2}{\pi}\left[-\frac{\cos x}{x^2} + \frac{1}{x^2}\right] = \frac{2(1-\cos x)}{\pi x^2}. \text{ Ans}$$

IInd Part: To show: $\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$.

Since $\int_0^\infty F(x)\cos px dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$.

But $F(x) = \frac{2(1-\cos x)}{\pi x^2}$.

$$\therefore \int_0^\infty \left(\frac{2(1-\cos x)}{\pi x^2}\right)\cos px dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{(1-\cos x)}{x^2}.\cos px dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$$

When p = 0, we have $\frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2}dx = 1 \Rightarrow \int_0^\infty \frac{2\sin^2\frac{x}{2}}{x^2}dx = \frac{\pi}{2}$.

Putting x = 2t so that dx = 2dt, we get $\int_0^\infty \frac{\sin^2 t}{t^2}dt = \frac{\pi}{2}$. Hence prove.

Home Assignments

INTEGRAL EQUATIONS

Q.No.1.: Solve the **integral equation** $\int_0^\infty F(x)\sin px dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$.

Ans.: $F(x) = \frac{2(x-\sin x)}{\pi x^2}$.

Q.No.2.: Solve the **integral equation** $\int_0^\infty f(x)\cos\lambda x dx = e^{-\lambda}$.

Ans.: $f(x) = \frac{2}{\pi(1+x^2)}$.

Q.No.3.: Solve the **integral equation** $\int_{-\infty}^\infty \frac{f(u)du}{(s-a^2)+a^2} = \frac{1}{x^2+b^2}$., $0 < a < b$

Ans.: $f(x) = \frac{(b-a)\alpha}{b\pi[x^2+(b-a)^2]}$.

Q.No.4.: Solve the **integral equation** $\int_{-\infty}^{\infty} f(x)\cos\alpha x dx = e^{-\alpha}$.

Ans.: $\frac{2}{\pi(1+x^2)}$.

Finite Fourier sine and cosine transform

Q.No.1.: Find the **finite Fourier sine and cosine transform** of f(x) = 2x, 0 < x < 4.

Ans.: $F_s(p) = \frac{-32(-1)^p}{p\pi}$; $F_c(p) = \frac{32(-1)^p - 1}{p^2\pi^2}$.

Q.No.2.: Find the **finite Fourier sine transform** of

$$f(x) = \begin{cases} -x, & x < c \\ \pi - x, & x > c \end{cases} \quad \text{where } 0 \le c \le \pi$$

Ans.: $\left(\frac{\pi}{s}\right)\cos sc$.

Q.No.3.: Find the **finite Fourier sine and cosine transform** of $f(x) = 2x$ in $0 < x < 4$.

Ans.: FFST : $\frac{32}{s\pi}(-1)^{n+1}$, FFCT : $\frac{32}{s\pi}\left((-1)^n - 1\right)$.

Q.No.4.: Find the **finite Fourier sine and cosine transform** of

$$f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ -1 & \pi/2 < x < \pi \end{cases}.$$

Ans.: FFST $= \frac{\left(-2\cos\frac{s\pi}{2} + 1 + \cos\frac{s\pi}{2}\right)}{2}$.

FFCT $= \frac{2\sin\left(\frac{s\pi}{2}\right)}{2}$, s = 1, 2, 3,……..

Q.No.5.: Find the **finite Fourier sine and cosine transform** of

$$f(x) = x^2 \text{ in } 0 < x < L$$

Ans.: FFST : $\frac{-L^2\cos n\pi}{n\pi} + \frac{2L^3}{n^3\pi^3}(\cos n\pi - 1)$, n = 1, 2, 3,……..

FFCT: $\dfrac{2L^3(\cos n\pi - 1)}{n^2\pi^2}$.

Q.No.6.: Find **finite Fourier cosine transform** of $f(x) = e^{ax}$ in (0, L)

Ans.: $\dfrac{s\pi L\left[(-1)^{s+1}e^{aL} + 1\right]}{\left[a^2L^2 + s^2\pi^2\right]}$.

Q.No.7.: Find **finite Fourier cosine transform** of $f(x) = \dfrac{x^2}{2\pi} - \dfrac{\pi}{6}$ in $(0, \pi)$.

Ans.: $\dfrac{(-1)^n}{n^2}$ for n = 1, 2, 3, and zero for n = 0.

Q.No.8.: Determine the inverse **finite Fourier sine transform** of $\dfrac{16(-1)^{n-1}}{n^3}$,

n =1, 2, 3, ….. and zero $0 < x < 8$.

Ans.: $f(x) = \dfrac{2}{8}\sum_{n=1}^{\infty} \dfrac{16(-1)^{n-1}}{n^3}\sin\dfrac{n\pi x}{8}$.

Q.No.9.: Determine the inverse **finite Fourier cosine transform** of $\dfrac{6\sin\frac{\pi}{2} - \cos n\pi}{(2n + \pi)}$ for

n = 1, 2, 3, ….. and equal to $\dfrac{2}{\pi}$ for n = 0 in $0 < x < 4$.

Ans.: $f(x) = \dfrac{1}{4}.\dfrac{2}{\pi} + \dfrac{2}{4}\sum_{n=1}^{\infty} \dfrac{6\sin\frac{\pi}{2} - \cos n\pi}{(2n + \pi)}\cos\left(\dfrac{n\pi}{4}\right)$.

Q.No.10.: Determine **finite Fourier sine transform** of $\dfrac{(1 - \cos n\pi)}{n^2\pi^2}$ where $0 < x < \pi$

Ans.: $f(x) = \dfrac{2}{\pi^2}\sum_{n=1}^{\infty} \dfrac{(1 - \cos n\pi)}{n^2}\sin nx$.

Parseval's Identity for Fourier Transforms

Solve the following problems using Parseval's identity for Fourier transform:

Q.No.1.: Using Parseval's identities, show that $\int_0^{\infty} \dfrac{\sin^2 ax}{x^2}dx = \dfrac{\pi a}{2}$.

Hint: Consider f(x) = 1, $|x| < a$ and f(x) = 0 for $|x| > a$. Then $F\{f(x)\} = F(\alpha) = \dfrac{2\sin a\alpha}{\alpha}$,

apply PI then $\displaystyle\int_{-a}^{a} (1)^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\sin^2 a\alpha}{\alpha^2} d\alpha$.

Ans.: (a). $\dfrac{\pi}{2ab(a+b)}$ **(b).** For a = b = 1, $\dfrac{\pi}{4}$.

Q.No.2.: Using Parseval's identities, prove that $\displaystyle\int_0^{\infty} \left(\frac{\sin x}{x}\right)^4 dx = \frac{\pi}{3}$.

Hint: Take $f(x) = 1 - |x|$, for $|x| < 1$, 0, otherwise $F\{f(x)\} = \sqrt{\dfrac{2}{\pi}}\left(\dfrac{1-\cos\alpha}{\alpha^2}\right)$, use PI.

Q.No.3.: Using Parseval's identities, evaluate **(a)** $\displaystyle\int_0^{\infty} \left(\frac{1-\cos x}{x}\right)^2 dx$,

(b) $\displaystyle\int_0^{\infty} \frac{\sin^4 x}{x^2} dx$.

Hint: Take f(x) = 1, for $0 \le x < 1$, 0 otherwise, then $F_c\{f(x)\} = \dfrac{1-\cos\alpha}{\alpha}$,

$F_s\{f(x)\} = \dfrac{\sin\alpha}{\alpha}$, use PI.

Ans.: (a). $\dfrac{\pi}{2}$ **(b).** $\dfrac{\pi}{2}$.

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