1^{st} Topic

Vector Calculus

Differentiation of vectors, Space curves (Curves in Space), Curvature, Torsion, Radius of curvature and radius of torsion, Frenet's Formulae

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Introduction:

Many physical quantities that occur in engineering and science require more than a single number to characterize them. When describing quantities such as force and velocity it is necessary to specify both a magnitude and a direction, and these are examples of vector quantities, whereas the air temperature, which can be specified by giving a single number, is an example of a scalar quantity. Physical problems are often best described in terms of vectors, so the objective of this topic is to develop the most important aspects of vector differential calculus.

Differentiation of vectors:

If a vector **R** varies continuously as a scalar variable t changes, then **R** is said to be a function of t and is written as $\mathbf{R}=\mathbf{F}(t)$.

Definition: The derivative of a vector function $\mathbf{R}=\mathbf{F}(t)$ is defined as

$$\underset{\delta t\to 0}{\mathrm{Lt}}\ \frac{\mathbf{F}(t+\delta t)-\mathbf{F}(t)}{\delta t}$$

and write it as $\frac{d\mathbf{R}}{dt}$ or $\frac{d\mathbf{F}}{dt}$ or $\mathbf{F}'(t)$.

General rules of differentiation:

General rules of differentiation are similar to those of ordinary calculus provided the order of factors in vector products is maintained.

Thus, if φ is scalar and **F, G, H** are vector functions of a scalar variable t, then we have

(i) $\frac{d}{dt}(\mathbf{F}+\mathbf{G}-\mathbf{H})=\frac{d\mathbf{F}}{dt}+\frac{d\mathbf{G}}{dt}-\frac{d\mathbf{H}}{dt}$

(ii) $\frac{d}{dt}(\mathbf{F}\varphi)=\frac{d\mathbf{F}}{dt}\varphi+\mathbf{F}\frac{d\varphi}{dt}$

(iii) $\frac{d}{dt}(\mathbf{F}.\mathbf{G})=\frac{d\mathbf{F}}{dt}.\mathbf{G}+\mathbf{F}.\frac{d\mathbf{G}}{dt}$

(iv) $\frac{d}{dt}(\mathbf{F}\times\mathbf{G})=\frac{d\mathbf{F}}{dt}\times\mathbf{G}+\mathbf{F}\times\frac{d\mathbf{G}}{dt}$

(v) $\frac{d}{dt}(\mathbf{FGH})=\left[\frac{d\mathbf{F}}{dt}\mathbf{GH}\right]+\left[\mathbf{F}\frac{d\mathbf{G}}{dt}\mathbf{H}\right]+\left[\mathbf{FG}\frac{d\mathbf{H}}{dt}\right]$

(vi) $\frac{d}{dt}\left[(\mathbf{F}\times\mathbf{G})\times\mathbf{H}\right]=\left(\frac{d\mathbf{F}}{dt}\times\mathbf{G}\right)\times\mathbf{H}+\left(\mathbf{F}\times\frac{d\mathbf{G}}{dt}\right)\times\mathbf{H}+(\mathbf{F}\times\mathbf{G})\times\frac{d\mathbf{H}}{dt}$.

(iv) Prove that $\frac{d}{dt}(\mathbf{F}\times\mathbf{G})=\mathbf{F}\times\frac{d\mathbf{G}}{dt}+\frac{d\mathbf{F}}{dt}\times\mathbf{G}$.

Proof: $\frac{d}{dt}(\mathbf{F}\times\mathbf{G})=\underset{\delta t\to 0}{\mathrm{Lt}}\frac{(\mathbf{F}+\delta\mathbf{F})\times(\mathbf{G}+\delta\mathbf{G})-\mathbf{F}\times\mathbf{G}}{\delta t}$

$$=\underset{\delta t\to 0}{\mathrm{Lt}}\frac{(\mathbf{F}\times\mathbf{G})+(\mathbf{F}\times\delta\mathbf{G})+(\delta\mathbf{F}\times\mathbf{G})+(\delta\mathbf{F}\times\delta\mathbf{G})-(\mathbf{F}\times\mathbf{G})}{\delta t}$$

$$=\underset{\delta t\to 0}{\mathrm{Lt}}\frac{(\mathbf{F}\times\delta\mathbf{G})+(\delta\mathbf{F}\times\mathbf{G})+(\delta\mathbf{F}\times\delta\mathbf{G})}{\delta t}$$

$$=\underset{\delta t\to 0}{\mathrm{Lt}}\left[\mathbf{F}\times\frac{\delta\mathbf{G}}{\delta t}+\frac{\delta\mathbf{F}}{\delta t}\times\mathbf{G}+\frac{\delta\mathbf{F}}{\delta t}\times\delta\mathbf{G}\right]=\mathbf{F}\times\frac{d\mathbf{G}}{dt}+\frac{d\mathbf{F}}{dt}\times\mathbf{G}\ \left[\because \delta\mathbf{G}\to 0 \text{ as } \delta t\to 0\right]$$

This completes the proof.

Result No.1.: If **F**(t) has a constant magnitude, then show that $\mathbf{F}.\frac{d\mathbf{F}}{dt}=0$.

Proof: Given: **F**(t) has a constant magnitude $\Rightarrow |\mathbf{F}(t)|=$ constant .

To show: $\mathbf{F}.\frac{d\mathbf{F}}{dt}=0$.

Since $|\mathbf{F}(t)|=$ constant $\Rightarrow |\mathbf{F}(t)|^2=$ constant $\Rightarrow \mathbf{F}(t).\mathbf{F}(t)=$ constant .

Differentiate w.r.t. t, we get

$$\frac{d}{dt}(\mathbf{F}.\mathbf{F})=0\Rightarrow \mathbf{F}.\frac{d\mathbf{F}}{dt}+\frac{d\mathbf{F}}{dt}.\mathbf{F}=0\Rightarrow 2\mathbf{F}.\frac{d\mathbf{F}}{dt}=0\Rightarrow \mathbf{F}.\frac{d\mathbf{F}}{dt}=0.$$

This completes the proof.

Note: $\mathbf{F}.\frac{d\mathbf{F}}{dt}=0\Rightarrow\frac{d\mathbf{F}}{dt}\perp\mathbf{F}$.

Result No.2.: If **F**(t) has a constant direction, then show that $\mathbf{F}\times\frac{d\mathbf{F}}{dt}=\mathbf{0}$.

Proof: Given: **F**(t) has a constant direction.

Therefore, assume that $\hat{\mathbf{G}}(t)$ be a **unit vector** in the direction of **F**(t).

Let $|\mathbf{F}(t)|=f(t)$, then $\mathbf{F}(t)=f(t)\hat{\mathbf{G}}(t)$.

Differentiate w.r.t. t, we get

$$\frac{d\mathbf{F}}{dt} = f(t)\frac{d\hat{\mathbf{G}}}{dt} + \frac{df}{dt}\hat{\mathbf{G}}. \qquad \text{(i)}$$

Since $\mathbf{F}(t)$ has a constant direction $\Rightarrow \hat{\mathbf{G}}(t)$ has also a constant direction.

$\Rightarrow \hat{\mathbf{G}}(t)$ is a constant vector $\Rightarrow \dfrac{\hat{\mathbf{G}}(t)}{dt} = \mathbf{0}$.

From (i), we have $\dfrac{d\mathbf{F}}{dt} = f(t)\dfrac{d\hat{\mathbf{G}}}{dt} + \dfrac{df}{dt}\hat{\mathbf{G}} \Rightarrow \dfrac{d\mathbf{F}}{dt} = \dfrac{df}{dt}\hat{\mathbf{G}}$

Now $\mathbf{F}\times\dfrac{d\mathbf{F}}{dt} = f\,\hat{\mathbf{G}}\times\left(\dfrac{df}{dt}\hat{\mathbf{G}}\right) = f\dfrac{df}{dt}\hat{\mathbf{G}}\times\hat{\mathbf{G}} = \mathbf{0}$

$$\Rightarrow \mathbf{F}\times\frac{d\mathbf{F}}{dt} = \mathbf{0}.$$

This completes the proof.

Now let us solve few problems using the general rule of differentiation:

Q.No.1.: If $\mathbf{A} = 5t^2\hat{\mathbf{I}} + t\hat{\mathbf{J}} - t^3\hat{\mathbf{K}}$, $\mathbf{B} = \sin t\,\hat{\mathbf{I}} - \cos t\,\hat{\mathbf{J}}$,

find **(i)** $\dfrac{d}{dt}(\mathbf{A}.\mathbf{B})$, **(ii)** $\dfrac{d}{dt}(\mathbf{A}\times\mathbf{B})$.

Sol.: (i) $\dfrac{d}{dt}(\mathbf{A}.\mathbf{B}) = \mathbf{A}.\dfrac{d\mathbf{B}}{dt} + \dfrac{d\mathbf{A}}{dt}.\mathbf{B}$

$$= \left(5t^2\hat{\mathbf{I}} + t\hat{\mathbf{J}} - t^3\hat{\mathbf{K}}\right).\left[\cos t\,\hat{\mathbf{I}} - (-\sin t)\hat{\mathbf{J}}\right] + \left(10t\,\hat{\mathbf{I}} + \hat{\mathbf{J}} - 3t^2\hat{\mathbf{K}}\right).\left(\sin t\,\hat{\mathbf{I}} - \cos t\,\hat{\mathbf{J}}\right)$$

$$= \left(5t^2\cos t + t\sin t\right) + \left(10t\sin t - \cos t\right) = 5t^2\cos t + 11t\sin t - \cos t. \text{ Ans.}$$

(ii): $\dfrac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\dfrac{d\mathbf{B}}{dt} + \dfrac{d\mathbf{A}}{dt}\times\mathbf{B}$

$$= \left(5t^2\hat{\mathbf{I}} + t\hat{\mathbf{J}} - t^3\hat{\mathbf{K}}\right)\times\left(\cos t\,\hat{\mathbf{I}} + \sin t\,\hat{\mathbf{J}}\right) + \left(10t\,\hat{\mathbf{I}} + \hat{\mathbf{J}} - 3t^2\hat{\mathbf{K}}\right)\times\left(\sin t\,\hat{\mathbf{I}} - \cos t\,\hat{\mathbf{J}}\right)$$

$$= \left[5t^2\sin t\,\hat{\mathbf{K}} + t\cos t\left(-\hat{\mathbf{K}}\right) - t^3\cos t\,\hat{\mathbf{J}} - t^3\sin t\left(-\hat{\mathbf{I}}\right)\right]$$

$$+ \left[-10t\cos t\,\hat{\mathbf{K}} + \sin t\left(-\hat{\mathbf{K}}\right) - 3t^2\sin t\,\hat{\mathbf{J}} + 3t^2\cos t\left(-\hat{\mathbf{I}}\right)\right]$$

$$= \left(t^3 \sin t - 3t^2 \cos t\right)\hat{\mathbf{I}} - t^2\left(t\cos t + 3\sin t\right)\hat{\mathbf{J}} + \left[\left(5t^2 - 1\right)\sin t - 11t\cos t\right]\hat{\mathbf{K}}. \text{ Ans.}$$

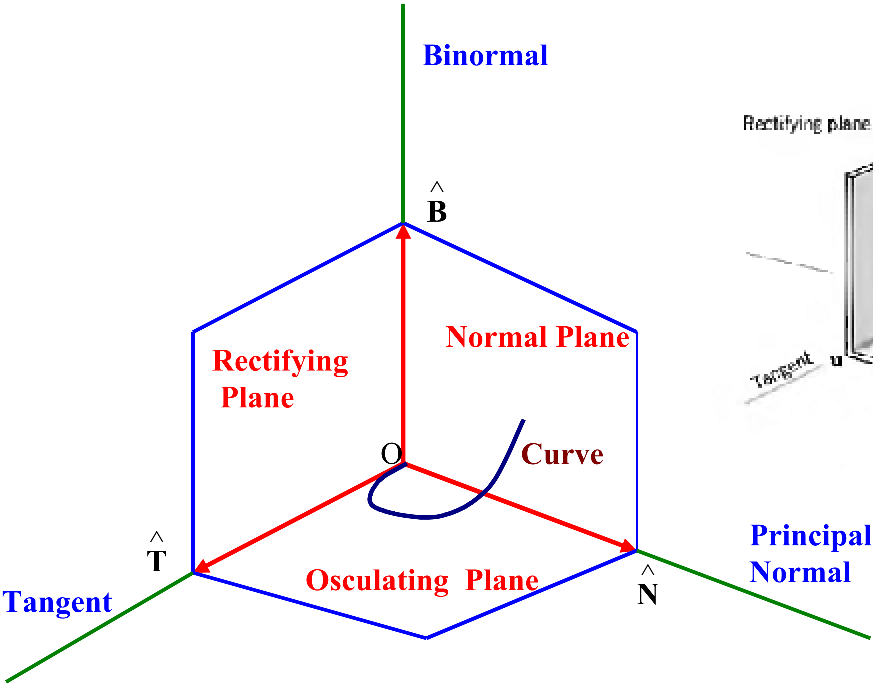
Space curves (Curves in Space):

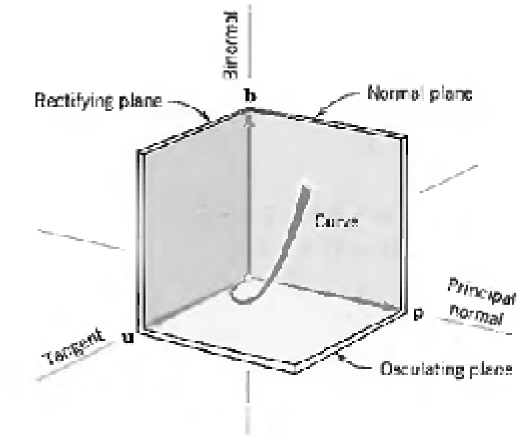
Associated with each point on a curve, there is a set of three mutually perpendicular lines known as

Tangent, Principal normal, Binormal

and three mutually perpendicular planes determined by these in pairs and known as

Osculating plane, Normal plane, Rectifying plane.





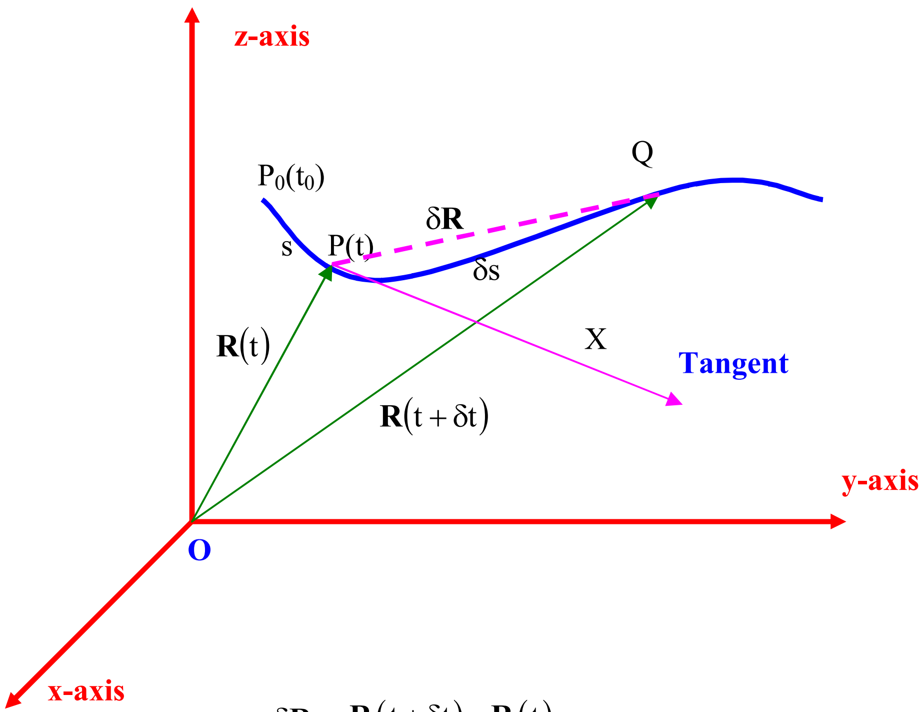
(1) Tangent at a point:

Let $\mathbf{R}(t) = x(t)\hat{\mathbf{I}} + y(t)\hat{\mathbf{J}} + z(t)\hat{\mathbf{K}}$ be the position vector of a point P.

We observe that, as the scalar parameter t takes different values, then the point P traces out a ***curve in space.***

If the neighbouring point Q corresponds to $t + \delta t$, then

$$\delta\mathbf{R} = \mathbf{R}(t + \delta t) - \mathbf{R}(t).$$



As δt is scalar, then the vector $\dfrac{\delta\mathbf{R}}{\delta t}=\dfrac{\mathbf{R}(t+\delta t)-\mathbf{R}(t)}{\delta t}$ is **parallel** to $\vec{PQ}$ or **directed along** the chord PQ.

Now, taking the limit, when $Q\to P$, and consequently $\delta t\to 0$, we get $\lim\limits_{\substack{Q\to P\\ \delta t\to 0}}\dfrac{\delta\mathbf{R}}{\delta t}$, which becomes the **tangent (vector)** to the curve at P, whenever it exists and is not zero.

But $\lim\limits_{\substack{Q\to P\\ \delta t\to 0}}\dfrac{\delta\mathbf{R}}{\delta t}=\dfrac{d\mathbf{R}}{dt}=\mathbf{R}'$

Thus, the vector $\mathbf{R}'=\dfrac{d\mathbf{R}}{dt}$ is a **tangent** to the space curve $\mathbf{R}=\mathbf{F}(t)$.

Vector equation of the tangent:

If u is the scalar parameter, $\mathbf{R}$ is the position vector of the point P and $\mathbf{X}$ is the position vector of any point on the tangent.

Then, the vector equation of the tangent at P is $\mathbf{X}=\mathbf{R}+u\dfrac{d\mathbf{R}}{dt}$

$$\Rightarrow \mathbf{X}=\mathbf{F}(t)+u\mathbf{F}'(t).$$

Formula for evaluating arc length:

Let P_0 be a fixed point of this curve corresponding to $t = t_0$.

If s be the length of the arc P_0P, then

$$\frac{\delta s}{\delta t} = \frac{\delta s}{|\delta \mathbf{R}|} \cdot \frac{|\delta \mathbf{R}|}{\delta t} = \frac{\text{arc PQ}}{\text{chord PQ}} \left|\frac{\delta \mathbf{R}}{\delta t}\right|.$$

Now, as $Q \to P$ along the curve QP i.e., $\delta t \to 0$, and consequently $\frac{\text{arc PQ}}{\text{chord PQ}} \to 1$.

Then $\lim\limits_{\substack{Q\to P\\ \delta t\to 0}} \frac{\delta s}{\delta t} = \lim\limits_{\substack{Q\to P\\ \delta t\to 0}} \left|\frac{\delta \mathbf{R}}{\delta t}\right|$

$$\Rightarrow \frac{ds}{dt} = \left|\frac{d\mathbf{R}}{dt}\right| = |\mathbf{R}'(t)|.$$

If $\mathbf{R}'(t)$ is continuous, then by taking integration, the arc P_0P is given by

$$s = \int_{t_0}^{t} |\mathbf{R}'|\, dt = \int_{t_0}^{t} \sqrt{(x')^2 + (y')^2 + (z')^2}\, dt.$$

This is the required formula for evaluating the length of an arc.

Remarks:

Since, we know $\frac{ds}{dt} = \left|\frac{d\mathbf{R}}{dt}\right|$.

If we take s as the parameter in place of t, then the magnitude of the tangent vector is equal to one, i.e.

$$\left|\frac{d\mathbf{R}}{ds}\right| = 1.$$

Thus, if we denote the unit tangent vector by $\hat{\mathbf{T}}$, then we have $\hat{\mathbf{T}} = \frac{d\mathbf{R}}{ds}$.

(2) Principal Normal:

Since $\hat{\mathbf{T}}$ is unit tangent vector, then we have $\frac{d\hat{\mathbf{T}}}{ds} \cdot \hat{\mathbf{T}} = 0$.

$\Rightarrow \frac{d\hat{\mathbf{T}}}{ds}$ is **perpendicular** to $\hat{\mathbf{T}}$ and $\frac{d\hat{\mathbf{T}}}{ds} = 0$

$\Rightarrow \hat{\mathbf{T}}$ is constant vector w.r.t. the arc length s and so has a fixed direction.

i.e. the curve is a straight line.

If we denote a unit normal vector to the curve at P by $\hat{\mathbf{N}}$,

then $\dfrac{d\hat{\mathbf{T}}}{ds}$ is in the **direction** of $\hat{\mathbf{N}}$,

which is known as the principal normal to the space curve at P.

Osculating plane:

The plane between $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ is called the **osculating plane** of the curve at P.

(3) Binormal:

A third unit vector $\hat{\mathbf{B}}$ defined by $\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$, is called the **binormal** at P.

Since $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ are unit vectors

$\Rightarrow \hat{\mathbf{B}}$ is also a unit vector perpendicular to both $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$.

Hence $\hat{\mathbf{B}}$ is **normal** to the **osculating plane** at P.

Final Conclusions:

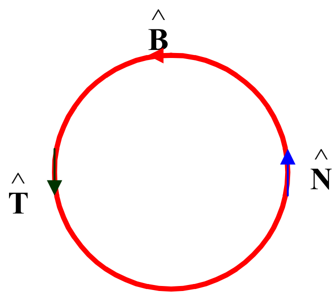
Thus, at each point P of a space curve, there are three mutually perpendicular **unit vectors** $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, $\hat{\mathbf{B}}$ which form a **moving trihedral** such that

$$\hat{\mathbf{T}} = \hat{\mathbf{N}} \times \hat{\mathbf{B}},$$

$$\hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}},$$

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}.$$

This moving trihedral determines the following three fundamentals planes at each point of the curve:



(i) The **osculating plane** containing $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$.

(ii) The **normal plane** containing $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$.

(iii) The **rectifying plane** containing $\hat{\mathbf{B}}$ and $\hat{\mathbf{T}}$.

Remarks:

(i) The tangent is parallel to the vectors $\frac{d\mathbf{R}}{ds}$

The principal normal is parallel to the vector $\frac{d^2\mathbf{R}}{ds^2}$

The binomial is parallel to the vector $\frac{d\mathbf{R}}{ds} \times \frac{d^2\mathbf{R}}{ds^2}$.

(ii) Equation of normal plane at a point P with position vector **R,** is

$$(\mathbf{X}-\mathbf{R}).\frac{d\mathbf{R}}{dt} = 0,$$

where **X** is the position vector of any point on the plane.

(iii) Equation of osculating plane is

$$(\mathbf{X}-\mathbf{R}).\frac{d\mathbf{R}}{dt} \times \frac{d^2\mathbf{R}}{dt^2} = 0$$

$$\Rightarrow \mathbf{X}.\frac{d\mathbf{R}}{dt} \times \frac{d^2\mathbf{R}}{dt^2} = \mathbf{R}\frac{d\mathbf{R}}{dt} \times \frac{d^2\mathbf{R}}{dt^2}.$$

Curvature:

The arc rate of rotation of the tangent (i.e. the magnitude of $\frac{d\hat{\mathbf{T}}}{ds}$) is called **curvature** of the curve and is denoted by k.

Thus, $\left|\frac{d\hat{\mathbf{T}}}{ds}\right| = k$.

or

The arc rate at which the tangent changes its direction as the point moves along the curve is known as curvature.

Torsion:

The arc rate of rotation of the bi-normal (i.e. the magnitude of $\frac{d\hat{\mathbf{B}}}{ds}$) is called **torsion** of the curve and is denoted by τ. Thus, $\left|\frac{d\hat{\mathbf{B}}}{ds}\right| = |\tau|$.

or

The arc rate at which the bi-normal changes its direction as the point moves along the curve is known as torsion.

Remarks: Here, the torsion τ is positive or negative according as the vectors $\hat{\mathbf{N}}$ and $\frac{d\hat{\mathbf{B}}}{ds}$ have the same or opposite senses.

Radius of curvature and radius of torsion:

The **reciprocal of curvature** is called **radius of curvature** and is denoted by ρ.

Thus $\rho = \frac{1}{k}$.

The **reciprocal of torsion** is called the **radius of torsion** and is denoted by σ.

Thus $\sigma = \frac{1}{\tau}$.

Frenet's Formulae:

We shall now establish the following important results, known as Frenet's formulae.

(i) $\frac{d\hat{\mathbf{T}}}{ds} = k\hat{\mathbf{N}}$, **(ii)** $\frac{d\hat{\mathbf{B}}}{ds} = \tau\hat{\mathbf{N}}$, **(iii)** $\frac{d\hat{\mathbf{N}}}{ds} = -\tau\hat{\mathbf{B}} - k\hat{\mathbf{T}}$

Proof:

(i) To prove: $\frac{d\hat{\mathbf{T}}}{ds} = k\hat{\mathbf{N}}$.

Since, we know that $\hat{\mathbf{N}}$ is the unit vector having the sense and direction of $\frac{d\hat{\mathbf{T}}}{ds}$.

i.e. $\frac{d\hat{\mathbf{T}}}{ds} \parallel \hat{\mathbf{N}}$.

And moreover, we also know that $\left|\dfrac{d\hat{\mathbf{T}}}{ds}\right| = k$.

Combining these two facts, we obtain $\dfrac{d\hat{\mathbf{T}}}{ds} = k\hat{\mathbf{N}}$.

(ii) To prove: $\dfrac{d\hat{\mathbf{B}}}{ds} = \tau\hat{\mathbf{N}}$.

Since $\hat{\mathbf{B}}$ is a unit vector, we have $\dfrac{d\hat{\mathbf{B}}}{ds}.\hat{\mathbf{B}} = 0$.

$\Rightarrow \dfrac{d\hat{\mathbf{B}}}{ds}$ is **perpendicular** to $\hat{\mathbf{B}}$.

Also $\hat{\mathbf{B}}.\hat{\mathbf{T}} = 0 \Rightarrow \dfrac{d}{ds}\left(\hat{\mathbf{B}}.\hat{\mathbf{T}}\right) = 0$

$$\Rightarrow \frac{d\hat{\mathbf{B}}}{ds}.\hat{\mathbf{T}} + \hat{\mathbf{B}}.\frac{d\hat{\mathbf{T}}}{ds} = 0$$

$$\Rightarrow \frac{d\hat{\mathbf{B}}}{ds}.\hat{\mathbf{T}} + \hat{\mathbf{B}}.\left(k\hat{\mathbf{N}}\right) = 0. \quad \left[\because \frac{d\hat{\mathbf{T}}}{ds} = k\hat{\mathbf{N}}\right]$$

$$\Rightarrow \frac{d\hat{\mathbf{B}}}{ds}.\hat{\mathbf{T}} = 0 \qquad \left[\because \hat{\mathbf{B}}.\hat{\mathbf{N}} = 0\right]$$

$\Rightarrow \dfrac{d\hat{\mathbf{B}}}{ds}$ is **perpendicular** to $\hat{\mathbf{T}}$.

Thus, $\dfrac{d\hat{\mathbf{B}}}{ds} \perp \hat{\mathbf{B}}$ and $\dfrac{d\hat{\mathbf{B}}}{ds} \perp \hat{\mathbf{T}}$.

Hence, $\dfrac{d\hat{\mathbf{B}}}{ds}$ is parallel to the vector $\hat{\mathbf{N}} \Rightarrow \dfrac{d\hat{\mathbf{B}}}{ds} \parallel \hat{\mathbf{N}}$.

And moreover, we also know that $\left|\dfrac{d\hat{\mathbf{B}}}{ds}\right| = |\tau|$.

Combining these two facts, we obtain $\dfrac{d\hat{\mathbf{B}}}{ds} = \tau\hat{\mathbf{N}}$.

Here, the torsion τ is positive or negative according as the vectors $\hat{\mathbf{N}}$ and $\dfrac{d\hat{\mathbf{B}}}{ds}$ as defined above, have the same or opposite senses.

Remarks: If $\dfrac{d\hat{\mathbf{B}}}{ds}$ has direction of $-\hat{\mathbf{N}}$, then $\dfrac{d\hat{\mathbf{B}}}{ds} = -\tau\hat{\mathbf{N}}$.

(iii) To prove: $\dfrac{d\hat{\mathbf{N}}}{ds} = -\tau\hat{\mathbf{B}} - k\hat{\mathbf{T}}$.

Since we know that $\hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}}$.

Differentiate w.r.t. s, we get

$$\frac{d\hat{\mathbf{N}}}{ds} = \frac{d\hat{\mathbf{B}}}{ds} \times \hat{\mathbf{T}} + \hat{\mathbf{B}} \times \frac{d\hat{\mathbf{T}}}{ds}.$$

$$= \tau\hat{\mathbf{N}} \times \hat{\mathbf{T}} + \hat{\mathbf{B}} \times k\hat{\mathbf{N}} \quad \left[\because \frac{d\hat{\mathbf{B}}}{ds} = \tau\hat{\mathbf{N}}, \frac{d\hat{\mathbf{T}}}{ds} = k\hat{\mathbf{N}}\right]$$

$$= -\tau\hat{\mathbf{B}} - k\hat{\mathbf{T}} \left[\because \hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}},\ \hat{\mathbf{T}} = \hat{\mathbf{N}} \times \hat{\mathbf{B}}\right].$$

Now let us solve some more problems using the general rule of differentiation:

Q.No.2.: Show that, if $\mathbf{R} = \mathbf{A}\sin\omega t + \mathbf{B}\cos\omega t$, where **A, B** and ω are constants, then

$$\frac{d^2\mathbf{R}}{dt^2} = -\omega^2\mathbf{R} \quad \text{and} \quad \mathbf{R}\times\frac{d\mathbf{R}}{dt} = -\omega\mathbf{A}\times\mathbf{B}.$$

Sol.: $\mathbf{R} = \mathbf{A}\sin\omega t + \mathbf{B}\cos\omega t$, **A, B** and ω are constants

(i) $\mathbf{R} = \mathbf{A}\sin\omega t + \mathbf{B}\cos\omega t$

Differentiating w. r. t. t, we get

$$\therefore \frac{d\mathbf{R}}{dt} = \omega\mathbf{A}\cos\omega t + \mathbf{B}(-\omega\sin\omega t)$$

Differentiating again w. r. t. t, we get

$$\frac{d^2\mathbf{R}}{dt^2} = -\omega^2\mathbf{A}\cos\omega t - \mathbf{B}\omega^2\cos\omega t = -\omega^2(\mathbf{A}\sin\omega t + \mathbf{B}\cos\omega t)$$

$$\frac{d^2\mathbf{R}}{dt^2} = -\omega^2\mathbf{R}. \text{ Ans.}$$

(ii) $\mathbf{R} = \mathbf{A}\sin\omega t + \mathbf{B}\cos\omega t$

$$\Rightarrow \frac{d\mathbf{R}}{dt} = \omega\mathbf{A}\cos\omega t - \omega\mathbf{B}\sin\omega t$$

$$\mathbf{R}\times\frac{d\mathbf{R}}{dt} = (\mathbf{A}\sin\omega t - \mathbf{B}\cos\omega t)\times(\omega\mathbf{A}\cos\omega t - \omega\mathbf{B}\sin\omega t)$$

$$= -\omega\mathbf{A}\times\mathbf{B}\sin^2\omega t - \omega\mathbf{A}\times\mathbf{B}\cos^2\omega t = -\omega(\mathbf{A}\times\mathbf{B})(\sin^2\omega t + \cos^2\omega t)$$

$$= -\omega(\mathbf{A}\times\mathbf{B}). \text{ Ans.}$$

Q.No.3.: $\mathbf{R} = t^m\mathbf{A} + t^n\mathbf{B}$, where **A, B** are constant vectors,

show that, if **R** and $\frac{d^2\mathbf{R}}{dt^2}$ are parallel vectors, then m + n =1, unless m = n.

Sol.: $\mathbf{R} = t^m\mathbf{A} + t^n\mathbf{B}$

Differentiating w. r. t 't', we get $\frac{d\mathbf{R}}{dt} = mt^{m-1}\mathbf{A} + nt^{n-1}\mathbf{B} + t^m\frac{d\mathbf{A}}{dt} + t^n\frac{d\mathbf{B}}{dt}$

Differentiating again w. r. t 't', we get $\frac{d^2\mathbf{R}}{dt^2} = m(m-1)t^{m-2}\mathbf{A} + n(n-1)t^{n-2}\mathbf{B}$

Since **R** and $\frac{d^2\mathbf{R}}{dt^2}$ are parallel vectors, then $\mathbf{R}\times\frac{d^2\mathbf{R}}{dt^2}=0$

$$\Rightarrow\left(t^m\mathbf{A}+t^n\mathbf{B}\right)\times\left[m(m-1)t^{m-2}\mathbf{A}+n(n-1)t^{n-2}\mathbf{B}\right]=0$$

$$\Rightarrow n(n-1)t^{m-n-2}\mathbf{A}\times\mathbf{B}+m(m-1)t^{m+n-2}\mathbf{B}\times\mathbf{A}=0$$

$$\Rightarrow n(n-1)t^{m-n-2}\mathbf{A}\times\mathbf{B}=-(m)(m-1)t^{m+n-2}\mathbf{B}\times\mathbf{A}$$

$$\Rightarrow n(n-1)t^{m-n-2}\mathbf{A}\times\mathbf{B}=(m)(m-1)t^{m+n-2}\mathbf{A}\times\mathbf{B}$$

$$\Rightarrow(m)(m-1)=n(n-1)\Rightarrow m^2-m=n^2-n\Rightarrow m^2-n^2=m-n$$

$$\Rightarrow(m-n)(m+n)=m-n\Rightarrow m+n=1$$

$$\Rightarrow(m-n)\left[(m+n)-1\right]=0$$

R and $\frac{d^2\mathbf{R}}{dt^2}$ are parallel if m + n =1, unless m = n.

Hence this proved the result.

Q.No.4.: If $\mathbf{P}=5t^2\,\hat{\mathbf{I}}+t^3\,\hat{\mathbf{J}}-t\,\hat{\mathbf{K}}$, $\mathbf{Q}=2\,\hat{\mathbf{I}}\sin t-\hat{\mathbf{J}}\cos t+5t\,\hat{\mathbf{K}}$,

find **(i)** $\frac{d}{dt}(\mathbf{P}.\mathbf{Q})$ **(ii)** $\frac{d}{dt}(\mathbf{P}\times\mathbf{Q})$.

Sol.:(i) $\frac{d}{dt}(\mathbf{P}.\mathbf{Q})=\mathbf{P}.\frac{d\mathbf{Q}}{dt}+\frac{d\mathbf{P}}{dt}.\mathbf{Q}$

$$=\left(5t^2\,\hat{\mathbf{I}}+t^3\,\hat{\mathbf{J}}-t\,\hat{\mathbf{K}}\right)\left(2\cos t\,\hat{\mathbf{I}}+\sin t\,\hat{\mathbf{J}}+5t\,\hat{\mathbf{K}}\right)+\left(10t\,\hat{\mathbf{I}}+3t^2\,\hat{\mathbf{J}}-\hat{\mathbf{K}}\right)\left(2\sin t\,\hat{\mathbf{I}}-\cos t\,\hat{\mathbf{J}}+5t\,\hat{\mathbf{K}}\right)$$

$$=10t^2\cos t+t^3\sin t-5t+20t\sin t-3t^2\cos t-5t$$

$=t^3\sin t+7t^2\cos t+20t\sin t-10t$. Ans.

(ii) $\frac{d}{dt}(\mathbf{P}\times\mathbf{Q})$

Now $\mathbf{P}\times\mathbf{Q}=\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ 5t^2 & t^3 & -t\\ 2\sin t & -\cos t & 5t\end{vmatrix}$

$$=\hat{\mathbf{I}}\left(5t^4-t\cos t\right)+\hat{\mathbf{J}}\left(-2t\sin t-25t^3\right)+\hat{\mathbf{K}}\left(-5t^2\cos t-2t^3\sin t\right)$$

$$\therefore \frac{d}{dt}(\mathbf{P}\times\mathbf{Q}) = \left[20t^3 - (-t\sin t + \cos t)\right]\mathbf{I} + \left[-2(t\cos t + \sin t) - 75t^2\right]\mathbf{J}$$

$$+\left[-5\left(-t^2\sin t - 2\sin t\cos t\right) - 2\left(t^3\cos t + \sin t 3t^2\right)\right]\hat{\mathbf{K}}$$

$$= \left[20t^3 + t\sin t - \cos t\right]\hat{\mathbf{I}} - \left[2t\cos t + 75t^2 + 2\sin t\right]\hat{\mathbf{J}}$$

$$-t\left[2t^2\cos t + t\sin t + 10\cos t\right]\hat{\mathbf{K}}. \text{ Ans.}$$

Q.No.5.: If $\frac{d\mathbf{U}}{dt} = \mathbf{W}\times\mathbf{U}$ and $\frac{d\mathbf{V}}{dt} = \mathbf{W}\times\mathbf{V}$, prove that $\frac{d}{dt}(\mathbf{U}\times\mathbf{V}) = \mathbf{W}\times(\mathbf{U}\times\mathbf{V})$.

Sol.: Since $\frac{d}{dt}(\mathbf{U}\times\mathbf{V}) = \mathbf{U}\times\frac{d\mathbf{V}}{dt} + \frac{d\mathbf{U}}{dt}\times\mathbf{V}$

Given $\frac{d\mathbf{V}}{dt} = \mathbf{W}\times\mathbf{V}$, $\frac{d\mathbf{U}}{dt} = \mathbf{W}\times\mathbf{U}$

Then $\frac{d}{dt}(\mathbf{U}\times\mathbf{V}) = \mathbf{U}\times(\mathbf{W}\times\mathbf{V}) + (\mathbf{W}\times\mathbf{U})\times\mathbf{V}$

Now using the formulae

$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}.\mathbf{C})\mathbf{B} - (\mathbf{B}.\mathbf{C})\mathbf{A}$

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}.\mathbf{C})\mathbf{B} - (\mathbf{A}.\mathbf{B})\mathbf{C} = \left[(\mathbf{U}.\mathbf{V})\mathbf{W} - (\mathbf{U}.\mathbf{W})\mathbf{V}\right] + \left[(\mathbf{W}.\mathbf{V})\mathbf{U} - (\mathbf{U}.\mathbf{W})\mathbf{W}\right]$

$$= (\mathbf{W}.\mathbf{V})\mathbf{U} - (\mathbf{U}.\mathbf{W})\mathbf{V}$$

$$\Rightarrow \frac{d}{dt}(\mathbf{U}\times\mathbf{V}) = \mathbf{W}\times(\mathbf{U}\times\mathbf{V})$$

Hence this proved the result.

Q.No.6.: If $\mathbf{A} = x^2yz\,\hat{\mathbf{I}} - 2xz^3\,\hat{\mathbf{J}} + xz^2\,\hat{\mathbf{K}}$ and $\mathbf{B} = 2z\,\hat{\mathbf{I}} + y\,\hat{\mathbf{J}} - x^2\,\hat{\mathbf{K}}$, find $\frac{\partial^2}{\partial x\partial y}(\mathbf{A}\times\mathbf{B})$

at $(1,\ 0,\ -2)$

Sol.: Given $\mathbf{A} = x^2yz\,\hat{\mathbf{I}} - 2xz^3\,\hat{\mathbf{J}} + xz^2\,\hat{\mathbf{K}}$, $\mathbf{B} = 2z\,\hat{\mathbf{I}} + y\,\hat{\mathbf{J}} - x^2\,\hat{\mathbf{K}}$.

$$(\mathbf{A}\times\mathbf{B}) = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix} = \left(2x^3z^3 - xyz^2\right)\hat{\mathbf{I}} - \left(-x^4yz - 2xz^3\right)\hat{\mathbf{J}} + \left(x^2y^2z + 4xz^4\right)\hat{\mathbf{K}}$$

$$\frac{\partial}{\partial x}(\mathbf{A}\times\mathbf{B})=\left(6x^2z^3-yz^2\right)\hat{\mathbf{I}}-\left(-4x^3yz-2z^3\right)\hat{\mathbf{J}}+\left(2xy^2z+4z^4\right)\hat{\mathbf{K}}$$

$$\frac{\partial^2}{\partial x\partial y}(\mathbf{A}\times\mathbf{B})=-z^2\hat{\mathbf{I}}+4x^3z\hat{\mathbf{J}}+4xyz\hat{\mathbf{K}}$$

At $(1,\ 0,\ -2)$, we get

$$\frac{\partial^2}{\partial x\partial y}(\mathbf{A}\times\mathbf{B})=-(-2)^2\hat{\mathbf{I}}+4(1)(-2)\hat{\mathbf{J}}+4(1)(0)(-1)\hat{\mathbf{K}}=-4\hat{\mathbf{I}}-8\hat{\mathbf{J}}=-4\left(\hat{\mathbf{I}}+2\hat{\mathbf{J}}\right).$$ Ans.

Problem on angle between the tangents:

Q.No.7.: Find the angle between the tangents to the curve $\mathbf{R}=t^2\hat{\mathbf{I}}+2t\hat{\mathbf{J}}-t^3\hat{\mathbf{K}}$ at the point $t=\pm 1$.

Sol.: Let $\mathbf{T}_1,\ \mathbf{T}_2$ be two tangents at $t=+1$ and $t=-1$ are respectively.

To find: Angle between the tangents $\mathbf{T}_1$ and $\mathbf{T}_2$.

Since we know $\mathbf{T}_1.\,\mathbf{T}_2=|\mathbf{T}_1|.|\mathbf{T}_2|\cos\theta$.

i.e. we have to find: θ

Now given vector equation of the curve is $\mathbf{R}=t^2\hat{\mathbf{I}}+2t\hat{\mathbf{J}}-t^3\hat{\mathbf{K}}$.

Then the tangent at any point t is given by $\dfrac{d\mathbf{R}}{dt}=2t\hat{\mathbf{I}}+2\hat{\mathbf{J}}-3t^2\hat{\mathbf{K}}$.

$\therefore$ The tangents $\mathbf{T}_1,\ \mathbf{T}_2$ at $t=+1$ and $t=-1$ are respectively, given by

$$\mathbf{T}_1=2\hat{\mathbf{I}}+2\hat{\mathbf{J}}-3\hat{\mathbf{K}}\quad\Rightarrow|\mathbf{T}_1|=\sqrt{17}.$$

$$\mathbf{T}_2=-2\hat{\mathbf{I}}+2\hat{\mathbf{J}}-3\hat{\mathbf{K}}\Rightarrow|\mathbf{T}_2|=\sqrt{17}.$$

Then the required $\angle\theta$ is given by the relation

$$\mathbf{T}_1.\,\mathbf{T}_2=|\mathbf{T}_1|.|\mathbf{T}_2|\cos\theta\Rightarrow 2(-2)+2.2+(-3)(-3)=\sqrt{17}\sqrt{17}\cos\theta$$

i. e. $9=\sqrt{17}\sqrt{17}\cos\theta\quad\therefore\theta=\cos^{-1}\left(\dfrac{9}{17}\right)$. Ans.

Problems for evaluating unit tangent vector:

Q.No.8.: Find the **unit tangent vector** at any point on the curve $x=t^2+2$,

$y = 4t - 5$, $z = 2t^2 - 6t$, where t is any variable.

Also determine the **unit tangent vector** at any point t = 2.

Sol.: The vector equation of curve is $\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}$

$$= \left(t^2 + 2\right)\hat{\mathbf{I}} + \left(4t - 5\right)\hat{\mathbf{J}} + \left(2t^2 - 6t\right)\hat{\mathbf{K}}.$$

To find: Unit tangent vector $\hat{\mathbf{T}}$.

Since unit tangent vector $\hat{\mathbf{T}} = \dfrac{\frac{d\mathbf{R}}{dt}}{\left|\frac{d\mathbf{R}}{dt}\right|}$.

$$\therefore \frac{d\mathbf{R}}{dt} = (2t)\hat{\mathbf{I}} + (4)\hat{\mathbf{J}} + (4t - 6)\hat{\mathbf{K}}$$

and $\left|\frac{d\mathbf{R}}{dt}\right| = \sqrt{4t^2 + 16 + 16t^2 + 36 - 48t} = \sqrt{20t^2 - 48t + 52} = 2\sqrt{5t^2 - 12t + 13}$

Thus $\hat{\mathbf{T}} = \dfrac{\frac{d\mathbf{R}}{dt}}{\left|\frac{d\mathbf{R}}{dt}\right|} = \dfrac{1}{2\sqrt{5t^2 - 12t + 13}}.\left(2t\hat{\mathbf{I}} + 4\hat{\mathbf{J}} + (4t - 6)\hat{\mathbf{K}}\right) = \dfrac{t\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + (2t - 3)\hat{\mathbf{K}}}{\sqrt{5t^2 - 12t + 13}}$. Ans

2nd Part:

To find: The **unit tangent vector** at any point t = 2.

At point t = 2, we get

$$\hat{\mathbf{T}} = \frac{\frac{d\mathbf{R}}{dt}}{\left|\frac{d\mathbf{R}}{dt}\right|} = \frac{2\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + \hat{\mathbf{K}}}{3}, \text{ Ans.}$$

which is the required **unit tangent vector** at any point t = 2.

Q.No.9.: If $\mathbf{R} = (a\cos t)\hat{\mathbf{I}} + (a\sin t)\hat{\mathbf{J}} + (at\tan\alpha)\hat{\mathbf{K}}$, find the value of

$$\text{(i) } \left|\frac{d\mathbf{R}}{dt} \times \frac{d^2\mathbf{R}}{dt^2}\right| \quad \text{(ii) } \left[\frac{d\mathbf{R}}{dt}, \frac{d^2\mathbf{R}}{dt^2}, \frac{d^3\mathbf{R}}{dt^3}\right] \text{ or } \left[\frac{d\mathbf{R}}{dt}.\left[\frac{d^2\mathbf{R}}{dt^2} \times \frac{d^3\mathbf{R}}{dt^3}\right]\right].$$

Also find the **unit tangent vector** at any point t of the curve.

Sol.: Given $\mathbf{R} = (a\cos t)\hat{\mathbf{I}} + (a\sin t)\hat{\mathbf{J}} + (at\tan\alpha)\hat{\mathbf{K}}$.

$$\therefore \frac{d\mathbf{R}}{dt} = (-a\sin t)\hat{\mathbf{I}} + (a\cos t)\hat{\mathbf{J}} + (a\tan\alpha)\hat{\mathbf{K}}$$

$$\Rightarrow \left|\frac{d\mathbf{R}}{dt}\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + a^2\tan^2\alpha} = a\sqrt{1+\tan^2\alpha} = a\sec\alpha .$$

Also $\dfrac{d^2\mathbf{R}}{dt^2} = (-a\cos t)\hat{\mathbf{I}} + (-a\sin t)\hat{\mathbf{J}}$.

(i) $$\left|\frac{d\mathbf{R}}{dt} \times \frac{d^2\mathbf{R}}{dt^2}\right| = \left|\left[(-a\sin t)\hat{\mathbf{I}} + (a\cos t)\hat{\mathbf{J}}\right] \times \left[(-\cos t)\hat{\mathbf{I}} - (a\sin t)\hat{\mathbf{J}}\right] + a\tan\alpha\,\hat{\mathbf{K}}\right|$$

$$= \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= \left|(a^2\tan\alpha\sin t)\hat{\mathbf{I}} - (a^2\cos t\tan\alpha)\hat{\mathbf{J}} + (a^2)\hat{\mathbf{K}}\right| = \sqrt{a^4(\tan^2\alpha+1)} = a^2\sec\alpha .$$

(ii) $$\frac{d^3\mathbf{R}}{dt^3} = (a\sin t)\hat{\mathbf{I}} - (a\cos t)\hat{\mathbf{J}} .$$

$$\left[\frac{d\mathbf{R}}{dt}, \frac{d^2\mathbf{R}}{dt^2}, \frac{d^3\mathbf{R}}{dt^3}\right] = \frac{d\mathbf{R}}{dt}\cdot\left[\frac{d^2\mathbf{R}}{dt^2} \times \frac{d^3\mathbf{R}}{dt^3}\right]$$

Now $$\left[\frac{d^2\mathbf{R}}{dt^2} \times \frac{d^3\mathbf{R}}{dt^3}\right] = \begin{bmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{bmatrix} = (0)\hat{\mathbf{I}} + (0)\hat{\mathbf{J}} + a^2\,\hat{\mathbf{K}}$$

Putting the values, we get

$$\frac{d\mathbf{R}}{dt}\cdot\left[\frac{d^2\mathbf{R}}{dt^2} \times \frac{d^3\mathbf{R}}{dt^3}\right] = \left[(-a\sin t)\hat{\mathbf{I}} + (a\cos t)\hat{\mathbf{J}} + (a\tan\alpha)\hat{\mathbf{K}}\right]\cdot\left[(0)\hat{\mathbf{I}} + (0)\hat{\mathbf{J}} + a^2\,\hat{\mathbf{K}}\right] = a^3\tan\alpha .$$

3rd Part: Tangent vector at any point of curve is $\dfrac{d\mathbf{R}}{dt} =$ $(-a\sin t)\hat{\mathbf{I}} + (a\cos t)\hat{\mathbf{J}} + (a\tan\alpha)\hat{\mathbf{K}}$.

$$\therefore \text{Unit tangent vector is } = \frac{\frac{d\mathbf{R}}{dt}}{\left|\frac{d\mathbf{R}}{dt}\right|} = \frac{1}{a\sec\alpha}\left[(-a\sin t)\hat{\mathbf{I}} + (a\cos t)\hat{\mathbf{J}} + (a\tan\alpha)\hat{\mathbf{K}}\right]$$

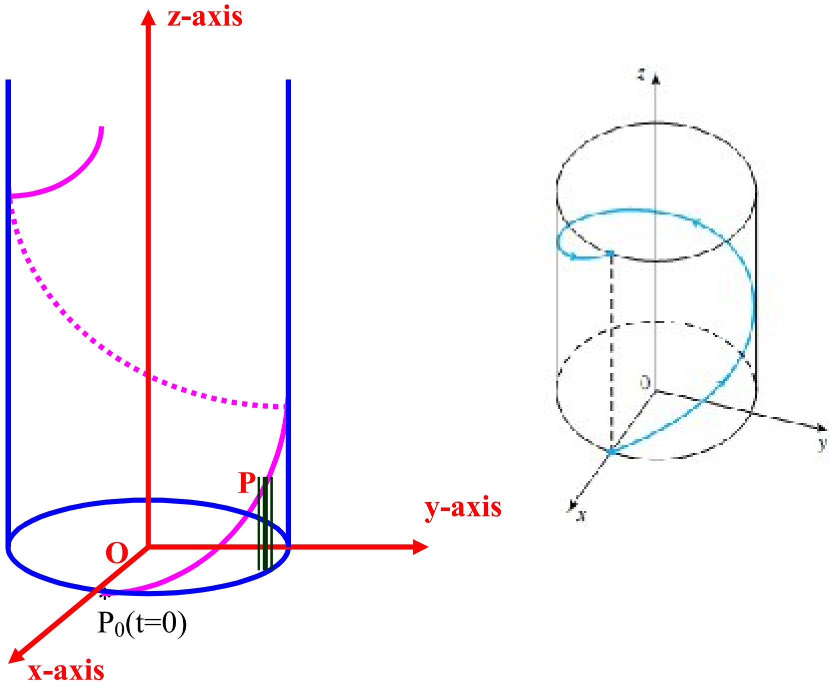
$$=(-\sin t.\cos\alpha)\hat{\mathbf{I}}+(\cos t.\cos\alpha)\hat{\mathbf{J}}+(\sin\alpha)\hat{\mathbf{K}}$$

This is the required unit tangent vector at any point of the curve.

Problems on curvature and torsion:

Q.No.10.: Find the **curvature** and **torsion** of the curve $x = a\cos t,\quad y = a\sin t,\quad z = bt$.

Sol.:



This curve is drawn on a circular cylinder cutting its generators at a constant angle and is known as **circular helix**.

The vector equation of the curve is $\mathbf{R} = a\cos t\,\hat{\mathbf{I}} + a\sin t\,\hat{\mathbf{J}} + bt\,\hat{\mathbf{K}}$.

$$\therefore \frac{d\mathbf{R}}{dt} = -a\sin t\,\hat{\mathbf{I}} + a\cos t\,\hat{\mathbf{J}} + b\,\hat{\mathbf{K}}.$$

To find: (i) $k\ (\text{curvature}) = \left|\dfrac{d\hat{\mathbf{T}}}{ds}\right|$ **(ii)** $|\tau|\ (\text{torsion}) = \left|\dfrac{d\hat{\mathbf{B}}}{ds}\right|$.

Consequently, we have to evaluate:

(i) $$\frac{d\hat{\mathbf{T}}}{ds} = \frac{\frac{d\hat{\mathbf{T}}}{dt}}{\frac{ds}{dt}},\quad \hat{\mathbf{T}} = \frac{d\mathbf{R}}{ds} = \frac{\frac{d\mathbf{R}}{dt}}{\frac{ds}{dt}},\quad s = \int_0^t \left|\frac{d\mathbf{R}}{dt}\right| dt$$

(ii) $$\frac{d\hat{\mathbf{B}}}{ds} = \frac{\frac{d\hat{\mathbf{B}}}{dt}}{\frac{ds}{dt}},\ \hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}},\ \frac{d\hat{\mathbf{T}}}{ds} = \frac{\frac{d\hat{\mathbf{T}}}{dt}}{\frac{ds}{dt}} = k\hat{\mathbf{N}} \Rightarrow \hat{\mathbf{N}} = ?.$$

Its arc length from $P_0(t=0)$ to any point P(t) is given by

$$s = \int_0^t \left|\frac{d\mathbf{R}}{dt}\right| dt = \sqrt{\left(a^2+b^2\right)}\ t \Rightarrow \frac{ds}{dt} = \sqrt{\left(a^2+b^2\right)}.$$

Then $$\hat{\mathbf{T}} = \frac{d\mathbf{R}}{ds} = \frac{\frac{d\mathbf{R}}{dt}}{\frac{ds}{dt}} = \frac{-a\sin t\,\hat{\mathbf{I}} + a\cos t\,\hat{\mathbf{J}} + b\,\hat{\mathbf{K}}}{\sqrt{\left(a^2+b^2\right)}}.$$

$$\therefore \frac{d\hat{\mathbf{T}}}{ds} = \frac{\frac{d\hat{\mathbf{T}}}{dt}}{\frac{ds}{dt}} = \frac{-a\left(\cos t\,\hat{\mathbf{I}} + \sin t\,\hat{\mathbf{J}}\right)}{a^2+b^2} = k\hat{\mathbf{N}} \Rightarrow \hat{\mathbf{N}} = -\left(\cos t\,\hat{\mathbf{I}} + \sin t\,\hat{\mathbf{J}}\right).$$

Thus $$k\ (\text{curvature}) = \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \frac{a}{a^2+b^2}.$$

Also $$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \left[\frac{-a\sin t\,\hat{\mathbf{I}} + a\cos t\,\hat{\mathbf{J}} + b\,\hat{\mathbf{K}}}{\sqrt{\left(a^2+b^2\right)}}\right] \times \left[-\left(\cos t\,\hat{\mathbf{I}} + \sin t\,\hat{\mathbf{J}}\right)\right]$$

$$= \frac{\left(b\sin t\,\hat{\mathbf{I}} - b\cos t\,\hat{\mathbf{J}} + a\,\hat{\mathbf{K}}\right)}{\sqrt{\left(a^2+b^2\right)}}$$

$$\therefore \frac{d\hat{\mathbf{B}}}{ds} = \frac{\frac{d\hat{\mathbf{B}}}{dt}}{\frac{ds}{dt}} = \frac{b}{a^2+b^2}\left(\cos t\,\hat{\mathbf{I}} + \sin t\,\hat{\mathbf{J}}\right) = -\tau\hat{\mathbf{N}} = \tau\left(\cos t\,\hat{\mathbf{I}} + \sin t\,\hat{\mathbf{J}}\right)$$

Hence $\tau = \dfrac{b}{a^2+b^2}$. Ans.

Q.No.11.: A circular helix is given by the equation $\mathbf{R}(t) = (2\cos t)\hat{\mathbf{I}} + (2\sin t)\hat{\mathbf{J}} + \hat{\mathbf{K}}$. Find the **curvature** and **torsion** of the curve at any point and show that they are constant.

Sol.: The vector equation of circular helix is $\mathbf{R}(t) = (2\cos t)\hat{\mathbf{I}} + (2\sin t)\hat{\mathbf{J}} + \hat{\mathbf{K}}$

$$\therefore \frac{d\mathbf{R}}{dt} = (-2\sin t)\hat{\mathbf{I}} + (2\cos t)\hat{\mathbf{J}}$$

$$\frac{ds}{dt} = \left|\frac{d\mathbf{R}}{dt}\right| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$$

$$\therefore \text{The unit tangent vector} = \hat{\mathbf{T}} = \frac{d\mathbf{R}}{ds} = \frac{\frac{d\mathbf{R}}{dt}}{\frac{ds}{dt}} = \frac{(-2\sin t)\hat{\mathbf{I}} + 2\cos t\,\hat{\mathbf{J}}}{2} = \frac{-\sin t\,\hat{\mathbf{I}} + \cos\hat{\mathbf{J}}}{1}$$

Now $\frac{d\hat{\mathbf{T}}}{dt} = -\cos t\,\hat{\mathbf{I}} - \sin t\,\hat{\mathbf{J}}$

$$\therefore \frac{d\hat{\mathbf{T}}}{ds} = \frac{\frac{d\hat{\mathbf{T}}}{dt}}{\frac{ds}{dt}} = \frac{-\cos t\,\hat{\mathbf{I}} - \sin t\,\hat{\mathbf{J}}}{2} \qquad \therefore \hat{\mathbf{N}} = \frac{\frac{d\hat{\mathbf{T}}}{dt}}{\left|\frac{d\hat{\mathbf{T}}}{dt}\right|} = \frac{\frac{-\cos t\,\hat{\mathbf{I}} - \sin t\,\hat{\mathbf{J}}}{2}}{\frac{1}{2}} = -\cos t\,\hat{\mathbf{I}} - \sin t\,\hat{\mathbf{J}}$$

$k = \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \frac{1}{2}$, which is constant. (i)

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \frac{1}{2}\begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \left[\hat{\mathbf{I}}(0) - (0)\hat{\mathbf{J}} + \left(\sin^2 t + \cos^2 t\right)\hat{\mathbf{K}}\right] = \hat{\mathbf{K}}$$

$$\therefore \frac{d\hat{\mathbf{B}}}{dt} = 0$$

Hence $|\tau| = \left|\frac{d\hat{\mathbf{B}}}{ds}\right| = \left|\frac{\frac{d\hat{\mathbf{B}}}{dt}}{\frac{ds}{dt}}\right| = \left|\frac{0}{2}\right| = 0$, which is constant. (ii)

From (i) and (ii), we get

Curvature k and Torsion τ are constant.

Hence this proves the result.

Remarks: Another way to calculate curvature and torsion

Curvature $k = \dfrac{\left|\dfrac{d\mathbf{R}}{dt} \times \dfrac{d^2\mathbf{R}}{dt^2}\right|}{\left|\dfrac{d\mathbf{R}}{dt}\right|^3}$, **Torsion** $\tau = \dfrac{\left(\dfrac{d\mathbf{R}}{dt} \times \dfrac{d^2\mathbf{R}}{dt^2}\right) \cdot \dfrac{d^3\mathbf{R}}{dt^3}}{\left|\dfrac{d\mathbf{R}}{dt} \times \dfrac{d^2\mathbf{R}}{dt^2}\right|^2}$.

Q.No.12.: Show that the curve $\mathbf{R} = a\left(3t - t^3\right)\hat{\mathbf{I}} + \left(3at^2\right)\hat{\mathbf{J}} + a\left(3t + t^3\right)\hat{\mathbf{K}}$, the **curvature** equals **torsion**.

Sol.: The vector equation of the curve is $\mathbf{R} = a\left(3t - t^3\right)\hat{\mathbf{I}} + \left(3at^2\right)\hat{\mathbf{J}} + a\left(3t + t^3\right)\hat{\mathbf{K}}$

$$\therefore \frac{d\mathbf{R}}{dt} = 3a\left(1 - t^2\right)\hat{\mathbf{I}} + (6at)\hat{\mathbf{J}} + 3a\left(1 + t^2\right)\hat{\mathbf{K}}$$

$$\frac{d^2\mathbf{R}}{dt^2} = -6at\,\hat{\mathbf{I}} + 6a\,\hat{\mathbf{J}} + 6at\,\hat{\mathbf{K}}$$

$$\frac{d^3\mathbf{R}}{dt^3} = -6a\,\hat{\mathbf{I}} + 6a\,\hat{\mathbf{K}}$$

Now $\dfrac{d\mathbf{R}}{dt} \times \dfrac{d^2\mathbf{R}}{dt^2} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 3a\left(1-t^2\right) & 6at & 3a\left(1+t^2\right) \\ -6at & 6a & 6at \end{vmatrix} = 18a^2\left[\left(t^2 - 1\right)\hat{\mathbf{I}} - 2t\,\hat{\mathbf{J}} + \left(1 + t^2\right)\hat{\mathbf{K}}\right]$

$$\therefore \left|\frac{d\mathbf{R}}{dt} \times \frac{d^2\mathbf{R}}{dt^2}\right| = 18a^2\sqrt{\left(t^2 - 1\right) + (2t)^2 + \left(1 + t^2\right)} = 18a^2\sqrt{\left(t^4 + 1 - 2t^2 + 4t^2 + 1 + t^2 + 2t^2\right)}$$

$$= 18a^2\sqrt{2\left(t^4 + 1 + 2t^2\right)} = 18\sqrt{2}\ a^2\sqrt{\left(t^2 + 1\right)^2} = 18\sqrt{2}a^2\left(t^2 + 1\right)$$

And $\left|\dfrac{d\mathbf{R}}{dt}\right| = 3a\sqrt{\left(1 - t^2\right)^2 + 4t^2 + \left(1 + t^2\right)^2} = 3a\sqrt{1 + t^4 - 2t^2 + 4t^2 + 1 + t^4 + 2t^2}$

$$= 3a\sqrt{2\left(t^4 + 2t^2 + 1\right)} = 3\sqrt{2}\ a\sqrt{\left(t^2 + 1\right)^2} = 3\sqrt{2}\ a\left(1 + t^2\right)$$

$$\left(\frac{d\mathbf{R}}{dt} \times \frac{d^2\mathbf{R}}{dt^2}\right) \cdot \frac{d^3\mathbf{R}}{dt^3} = 18a^2\left[\left(t^2 - 1\right)\hat{\mathbf{I}} - 2t\,\hat{\mathbf{J}} + \left(1 + t^2\right)\hat{\mathbf{K}}\right] \cdot \left[-6a\,\hat{\mathbf{I}} + 6a\,\hat{\mathbf{K}}\right]$$

$$= 18a^2 . 6a\left(1 - t^2 + 1 + t^2\right) = 216a^3$$

Since we know that curvature $k = \dfrac{\left|\dfrac{d\mathbf{R}}{dt} \times \dfrac{d^2\mathbf{R}}{dt^2}\right|}{\left|\dfrac{d\mathbf{R}}{dt}\right|^3} = \dfrac{18\sqrt{2}\ a^2(1+t^2)}{54\sqrt{2}\ a^3\left(1+t^2\right)^3}$

Curvature $k = \dfrac{1}{3a\left(1+t^2\right)^2}$ (i)

Also $\tau = \dfrac{\left(\dfrac{d\mathbf{R}}{dt} \times \dfrac{d^2\mathbf{R}}{dt^2}\right) . \dfrac{d^3\mathbf{R}}{dt^3}}{\left|\dfrac{d\mathbf{R}}{dt} \times \dfrac{d^2\mathbf{R}}{dt^2}\right|^2} = \dfrac{216a^3}{18.18.2a^4\left(1+t^2\right)^2}$

Torsion $\tau = \dfrac{1}{3a\left(1+t^2\right)^2}$ (ii)

From (i) and (ii), we have

$k = \tau$

This shows that curvature equals torsion

Hence this proves the result.

Q.No.13.: Find the **curvature** of the (i) ellipse $\mathbf{R}(t) = a\cos t\,\hat{\mathbf{I}} + b\sin t\,\hat{\mathbf{J}}$

(ii) Parabola $\mathbf{R}(t) = 2t\,\hat{\mathbf{I}} + t^2\,\hat{\mathbf{J}}$ at the point $t = 1$.

Sol.: **(i)** The vector equation of the ellipse is $\mathbf{R}(t) = a\cos t\,\hat{\mathbf{I}} + b\sin t\,\hat{\mathbf{J}}$

$\therefore \dfrac{d\mathbf{R}}{dt} = -a\sin t\,\hat{\mathbf{I}} + b\cos t\,\hat{\mathbf{J}}$

Also $\dfrac{ds}{dt} = \left|\dfrac{d\mathbf{R}}{dt}\right| = \left(a^2\sin^2 t + b^2\cos^2 t\right)^{1/2}$

$\hat{\mathbf{T}} = \dfrac{d\mathbf{R}}{ds} = \dfrac{\dfrac{d\mathbf{R}}{dt}}{\dfrac{ds}{dt}} = \left|\dfrac{\dfrac{d\mathbf{R}}{dt}}{\dfrac{ds}{dt}}\right| = \dfrac{-a\sin t\,\hat{\mathbf{I}} + b\cos t\,\hat{\mathbf{J}}}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$

$$\frac{d\hat{\mathbf{T}}}{dt}=\frac{\left[\begin{array}{l}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\left(-a\cos t\,\hat{\mathbf{I}}-b\sin t\,\hat{\mathbf{J}}\right)\\ -\left(-a\sin t\,\hat{\mathbf{I}}+b\cos t\,\hat{\mathbf{J}}\right)\frac{1}{2}\left(a^2\sin^2 t+b^2\cos^2 t\right)^{-1/2}\left(2a^2\sin t\cos t-2b^2\cos t\sin t\right)\end{array}\right]}{\left(a^2\sin^2 t+b^2\cos^2 t\right)}$$

$$\Rightarrow \frac{d\hat{\mathbf{T}}}{dt}=\frac{\left[\begin{array}{l}\left(a^2\sin^2 t+b^2\cos^2 t\right)^{1/2}\left(-a\cos t\,\hat{\mathbf{I}}-b\sin t\,\hat{\mathbf{J}}\right)\\ -\frac{\left(-a\sin t\,\hat{\mathbf{I}}+b\cos t\,\hat{\mathbf{J}}\right)}{2}\left(a^2\sin^2 t+b^2\cos^2 t\right)^{-1/2}\left(2a^2\sin t\cos t-2b^2\cos t\sin t\right)\end{array}\right]}{\left(a^2\sin^2 t+b^2\cos^2 t\right)}$$

$$\Rightarrow \frac{d\hat{\mathbf{T}}}{dt}=\frac{\left[\begin{array}{l}\left(a^2\sin^2 t+b^2\cos^2 t\right)\left(-a\cos t\,\hat{\mathbf{I}}-b\sin t\,\hat{\mathbf{J}}\right)\\ -\left(-a\sin t\,\hat{\mathbf{I}}+b\cos t\,\hat{\mathbf{J}}\right)\left(a^2\sin t\cos t-b^2\cos t\sin t\right)\end{array}\right]}{\left(a^2\sin^2 t+b^2\cos^2 t\right)^{3/2}}$$

$$\Rightarrow \frac{d\hat{\mathbf{T}}}{dt}=\frac{-ab^2\cos t\,\hat{\mathbf{I}}-a^2 b\sin t\,\hat{\mathbf{J}}}{\left(a^2\sin^2 t+b^2\cos^2 t\right)^{3/2}}$$

$$k=\left|\frac{d\hat{\mathbf{T}}}{ds}\right|,\text{ where }\frac{d\hat{\mathbf{T}}}{ds}=\frac{\frac{d\hat{\mathbf{T}}}{dt}}{\frac{ds}{dt}}$$

$$\Rightarrow \frac{d\hat{\mathbf{T}}}{ds}=\frac{\left(a^2\sin^2 t+b^2\cos^2 t\right)\left(-a\cos t\,\hat{\mathbf{I}}-b\sin t\,\hat{\mathbf{J}}\right)-\left(-a\sin t\,\hat{\mathbf{I}}+b\cos t\,\hat{\mathbf{J}}\right)\left(a^2\sin 2t-b^2\sin 2t\right)}{\left(a^2\sin^2 t+b^2\cos^2 t\right)^2}$$

$$\frac{d\hat{\mathbf{T}}}{ds}=\frac{\frac{d\hat{\mathbf{T}}}{dt}}{\frac{ds}{dt}}=\frac{-ab^2\cos t\,\hat{\mathbf{I}}-a^2 b\sin t\,\hat{\mathbf{J}}}{\left(a^2\sin^2 t+b^2\cos^2 t\right)^{4/2}}.$$

$$\therefore\ k=\left|\frac{d\hat{\mathbf{T}}}{ds}\right|=\sqrt{\frac{a^2b^2\left(b^2\cos^2 t-a^2\sin^2 t\right)}{\left(a^2\sin^2 t+b^2\cos^2 t\right)^4}}=\frac{ab}{\left(a^2\sin^2 t+b^2\cos^2 t\right)^{3/2}}\text{. Ans.}$$

(ii) The vector equation of parabola is $\mathbf{R}(t)=2t\hat{\mathbf{I}}+t^2\hat{\mathbf{J}}$

$$\therefore \frac{d\mathbf{R}}{dt}=2\hat{\mathbf{I}}+2t\hat{\mathbf{J}}$$

Now $\dfrac{ds}{dt}=\left|\dfrac{d\mathbf{R}}{dt}\right|=\sqrt{4+4t^2}=2\sqrt{1+t^2}$

$$\hat{\mathbf{T}}=\frac{d\mathbf{R}}{ds}=\frac{\frac{d\mathbf{R}}{dt}}{\frac{ds}{dt}}=\frac{\frac{d\mathbf{R}}{dt}}{\left|\frac{d\mathbf{R}}{dt}\right|}=\frac{2\hat{\mathbf{I}}+2t\hat{\mathbf{J}}}{2\sqrt{1+t^2}}$$

$$\therefore \frac{d\hat{\mathbf{T}}}{dt}=\frac{\left[\sqrt{1+t^2}\left(\hat{\mathbf{J}}\right)-\left(\hat{\mathbf{I}}+t\hat{\mathbf{J}}\right)\ \frac{1}{2}\left(1+t^2\right)^{-1/2}(2t)\right]}{1+t^2}=\frac{\left[\sqrt{1+t^2}\left(\hat{\mathbf{J}}\right)-\left(\hat{\mathbf{I}}+t\hat{\mathbf{J}}\right)t\left(1+t^2\right)^{-1/2}\right]}{1+t^2}$$

$$=\frac{\left[\sqrt{1+t^2}\,\hat{\mathbf{J}}-\left(\hat{\mathbf{I}}+t\hat{\mathbf{J}}\right)t\right]}{\left(1+t^2\right)^{3/2}}=\frac{\left[-t\hat{\mathbf{I}}+\hat{\mathbf{J}}\right]}{\left(1+t^2\right)^{3/2}}$$

$$\frac{d\hat{\mathbf{T}}}{ds}=\frac{\frac{d\hat{\mathbf{T}}}{dt}}{\frac{ds}{dt}}=\frac{\frac{-t\hat{\mathbf{I}}+\hat{\mathbf{J}}}{\left(1+t^2\right)}}{2\sqrt{1+t^2}}=\frac{-t\hat{\mathbf{I}}+\hat{\mathbf{J}}}{2\left(1+t^2\right)^2}=\frac{\sqrt{t^2+1}}{2\left(1+t^2\right)^2}$$

Put $t = 1$, we get $\dfrac{d\hat{\mathbf{T}}}{ds}=\dfrac{\sqrt{2}}{2(1+1)^2}=\dfrac{\sqrt{2}}{2\times 4}=\dfrac{1}{4\sqrt{2}}$. Ans.

$$\therefore k=\left|\frac{d\hat{\mathbf{T}}}{ds}\right|=\frac{1}{4\sqrt{2}}\text{. Ans.}$$

Problems on equations of the tangent line, the osculating plane and binormal:

Q.No.14.: Find the **equation of the tangent line** to the curve $x=a\cos\theta$, $y=a\sin\theta$,

$$z = a\theta\tan\alpha \quad \text{at} \quad \theta = \frac{\pi}{4}.$$

Sol.: The vector equation of the curve is $\mathbf{R} = a\cos\theta\,\hat{\mathbf{I}} + a\sin\theta\,\hat{\mathbf{J}} + a\theta\tan\alpha\,\hat{\mathbf{K}}$

$$\frac{d\mathbf{R}}{d\theta} = -a\sin\theta\,\hat{\mathbf{I}} + a\cos\theta\,\hat{\mathbf{J}} + a\tan\alpha\,\hat{\mathbf{K}}$$

$$\frac{ds}{d\theta} = \left|\frac{d\mathbf{R}}{d\theta}\right| = \left(a^2\sin^2\theta + a^2\cos^2\theta + a^2\tan^2\alpha\right)^{1/2} = \left[a^2\left(1+\tan^2\alpha\right)\right]^{1/2} = a\sec\alpha$$

$$\hat{\mathbf{T}} = \frac{d\mathbf{R}}{ds} = \frac{\frac{d\mathbf{R}}{d\theta}}{\frac{ds}{d\theta}} = \frac{-a\sin\theta\,\hat{\mathbf{I}} + a\cos\theta\,\hat{\mathbf{J}} + a\tan\alpha\,\hat{\mathbf{K}}}{a\sec\alpha}$$

At $\theta = \frac{\pi}{4}$, $x = \frac{a}{\sqrt{2}}$, $y = \frac{a}{\sqrt{2}}$, $z = \frac{a\pi}{4}\tan\alpha$

Now $\hat{\mathbf{T}} = \frac{-1}{\sqrt{2}\sec\alpha}\hat{\mathbf{I}} + \frac{1}{\sqrt{2}\sec\alpha}\hat{\mathbf{J}} + \frac{\tan\alpha}{\sec\alpha}\hat{\mathbf{K}}$

D. C. of tangent line are $\left\langle \frac{-1}{\sqrt{2}\sec\alpha}, \ \frac{1}{\sqrt{2}\sec\alpha}, \ \frac{\tan\alpha}{\sec\alpha} \right\rangle$

and it passes through the point $\left(\frac{a}{\sqrt{2}}, \ \frac{a}{\sqrt{2}}, \ \frac{a\pi}{4}\tan\alpha\right)$.

Then the equation of the tangent line is

$$-\frac{x - \frac{a}{\sqrt{2}}}{\frac{1}{\sqrt{2}}\sec\alpha} = \frac{y - \frac{a}{\sqrt{2}}}{\frac{1}{\sqrt{2}}\sec\alpha} = \frac{z - \frac{a\pi}{4}\tan\alpha}{\frac{\tan\alpha}{\sec\alpha}}$$

$$\Rightarrow -\left(x - \frac{a}{\sqrt{2}}\right) = \left(y - \frac{a}{\sqrt{2}}\right) = \frac{z - \frac{a\pi}{4}\tan\alpha}{\sqrt{2}\tan\alpha}. \text{ Ans.}$$

Q.No.15.: Find the **equation of the osculating plane** and **binormal** to the curve

(i) $x = 2\cosh\left(\frac{t}{2}\right), y = 2\sinh\left(\frac{t}{2}\right), \ z = 2t$ at $t = 0$.

(ii) $x = e^t\cos t, y = e^t\sin t, \ z = e^t$ at $t = 0$.

Sol.: (i) The vector equation of the curve is $\mathbf{R} = 2\cosh\left(\frac{t}{2}\right)\hat{\mathbf{I}} + 2\sinh\left(\frac{t}{2}\right)\hat{\mathbf{J}} + 2t\,\hat{\mathbf{K}}$

$$\frac{d\mathbf{R}}{dt}=\frac{d}{dt}\left[2\cosh\left(\frac{t}{2}\right)\hat{\mathbf{I}}+2\sinh\left(\frac{t}{2}\right)\hat{\mathbf{J}}+2t\,\hat{\mathbf{K}}\right]=2\sinh\left(\frac{t}{2}\right)\left(\frac{1}{2}\right)\hat{\mathbf{I}}+2\cosh\left(\frac{t}{2}\right)\left(\frac{1}{2}\right)\hat{\mathbf{J}}+2\,\hat{\mathbf{K}}$$

$$=\sinh\left(\frac{t}{2}\right)\hat{\mathbf{I}}+\cosh\left(\frac{t}{2}\right)\hat{\mathbf{J}}+2\,\hat{\mathbf{K}}$$

$$\therefore \frac{ds}{dt}=\left|\frac{d\mathbf{R}}{dt}\right|=\sqrt{\sinh^2\frac{t}{2}+\cosh^2\frac{t}{2}+4}$$

$$\hat{\mathbf{T}}=\frac{d\mathbf{R}}{ds}=\frac{\frac{d\mathbf{R}}{dt}}{\frac{ds}{dt}}=\frac{\sinh\left(\frac{t}{2}\right)\hat{\mathbf{I}}+\cosh\left(\frac{t}{2}\right)\hat{\mathbf{J}}+2\,\hat{\mathbf{K}}}{\sqrt{\sinh^2\frac{t}{2}+\cosh^2\frac{t}{2}+4}}$$

$$\frac{d\hat{\mathbf{T}}}{dt}=\frac{\left[\begin{array}{l}\left(\sqrt{\sinh^2\frac{t}{2}+\cosh^2\frac{t}{2}+4}\right)\left(\cosh\left(\frac{t}{2}\right)\left(\frac{1}{2}\right)\hat{\mathbf{I}}+\sinh\left(\frac{t}{2}\right)\left(\frac{1}{2}\right)\hat{\mathbf{J}}\right)-\left(\sinh\left(\frac{t}{2}\right)\hat{\mathbf{I}}+\cosh\left(\frac{t}{2}\right)\hat{\mathbf{J}}+2\,\hat{\mathbf{K}}\right)\\ \qquad .\frac{1}{2}\left(\sinh^2\frac{t}{2}+\cosh^2\frac{t}{2}+4\right)^{-1/2}\left\{2\sinh\left(\frac{t}{2}\right)\cosh\left(\frac{t}{2}\right)\left(\frac{1}{2}\right)+2\cosh\frac{t}{2}\sinh\frac{t}{2}\left(\frac{1}{2}\right)\right\}\end{array}\right]}{\left(\sqrt{\sinh^2\frac{t}{2}+\cosh^2\frac{t}{2}+4}\right)^2}$$

$$=\frac{\left[\begin{array}{l}\left(\sinh^2\frac{t}{2}+\cosh^2\frac{t}{2}+4\right)\left(\cosh\left(\frac{t}{2}\right)\hat{\mathbf{I}}+\sinh\left(\frac{t}{2}\right)\hat{\mathbf{J}}\right)\frac{1}{2}-\left(\sinh\left(\frac{t}{2}\right)\hat{\mathbf{I}}+\cosh\left(\frac{t}{2}\right)\hat{\mathbf{J}}+2\,\hat{\mathbf{K}}\right)\\ \qquad .\frac{1}{2}\left(\sinh^2\frac{t}{2}\cosh^2\frac{t}{2}+4\right)^{-1/2}\left(2\sinh\frac{t}{2}\cosh\frac{t}{2}\right)\end{array}\right]}{\left(\sinh^2\frac{t}{2}+\cosh^2\frac{t}{2}+4\right)}$$

$$=\frac{\left[\begin{array}{l}\left(\sinh^2\frac{t}{2}+\cosh^2\frac{t}{2}+4\right)\left(\cosh\left(\frac{t}{2}\right)\hat{\mathbf{I}}+\sinh\left(\frac{t}{2}\right)\hat{\mathbf{J}}\right)\frac{1}{2}-\left(\sinh\left(\frac{t}{2}\right)\hat{\mathbf{I}}-\cosh\left(\frac{t}{2}\right)\hat{\mathbf{J}}+2\,\hat{\mathbf{K}}\right)\\ \qquad .\left(\sinh\frac{t}{2}\cosh\frac{t}{2}\right)\end{array}\right]}{\left(\sinh^2\frac{t}{2}+\cosh^2\frac{t}{2}+4\right)^{3/2}}$$

At t = 0, we have $\hat{\mathbf{T}} = \dfrac{\hat{\mathbf{J}} + 2\hat{\mathbf{K}}}{\sqrt{5}}$

$$\therefore \frac{d\hat{\mathbf{T}}}{ds} = \frac{\frac{d\hat{\mathbf{T}}}{dt}}{\frac{ds}{dt}} = \frac{\left[\begin{array}{l}\left(\sinh^2\frac{t}{2} + \cosh^2\frac{t}{2} + 4\right)\left(\cosh\left(\frac{t}{2}\right)\hat{\mathbf{I}} + \sinh\left(\frac{t}{2}\right)\hat{\mathbf{J}}\right) \\ \quad -\left(\sinh\left(\frac{t}{2}\right)\hat{\mathbf{I}} + \cosh\left(\frac{t}{2}\right)\hat{\mathbf{J}} + 2\hat{\mathbf{K}}\right).\left(\sinh\left(\frac{t}{2}\right).\cosh\left(\frac{t}{2}\right)\right)\end{array}\right]}{\left(\sinh^2\frac{t}{2} + \cosh^2\frac{t}{2} + 4\right)^2}$$

At time t =0, we have

$$\frac{d\hat{\mathbf{T}}}{ds} = \frac{5\hat{\mathbf{I}} - 0}{\left(\sqrt{5}\right)^2} = \frac{5\hat{\mathbf{I}}}{5} = \hat{\mathbf{I}}. \qquad \therefore \hat{\mathbf{N}} = \frac{\frac{d\hat{\mathbf{T}}}{dt}}{\left|\frac{ds}{dt}\right|} = \frac{\hat{\mathbf{I}}}{1} = \hat{\mathbf{I}}$$

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \frac{\hat{\mathbf{J}} + 2\hat{\mathbf{K}}}{\sqrt{5}} \times \hat{\mathbf{I}} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \end{vmatrix} = \hat{\mathbf{I}}(0) + \hat{\mathbf{J}}\left(\frac{2}{\sqrt{5}}\right) + \hat{\mathbf{K}}\left(-\frac{1}{\sqrt{5}}\right) = \left(\frac{2}{\sqrt{5}}\right)\hat{\mathbf{J}} - \left(\frac{1}{\sqrt{5}}\right)\hat{\mathbf{K}}$$

$\therefore \hat{\mathbf{B}} = \dfrac{2\hat{\mathbf{J}} - \hat{\mathbf{K}}}{\sqrt{5}}$. Ans.

which is required equation of binormal. Since we know that any vector $\mathbf{r}$ in the plane containing two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by $\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b}$, where α, β are arbitrary constants. Thus

$$\mathbf{R} = p\hat{\mathbf{N}} + q\hat{\mathbf{T}} = p\hat{\mathbf{I}} + \frac{q}{\sqrt{5}}\left(\hat{\mathbf{J}} + 2\hat{\mathbf{K}}\right)$$

$\mathbf{R} = p\hat{\mathbf{I}} + q'\hat{\mathbf{J}} + 2q'\hat{\mathbf{K}}$, where $q' = \dfrac{q}{\sqrt{5}}$,

which is the required equation of osculating plane.

(ii): Given $x = e^t\cos t$, $y = e^t\sin t$, $z = e^t$.

$$\therefore \mathbf{R} = e^t\left(\cos t\,\hat{\mathbf{I}} + \sin t\,\hat{\mathbf{J}} + \hat{\mathbf{K}}\right)$$

$$\therefore \frac{d\mathbf{R}}{dt} = e^t(\cos t - \sin t)\hat{\mathbf{I}} + e^t(\cos t - \sin t)\hat{\mathbf{J}} + e^t\,\hat{\mathbf{K}}$$

$$\Rightarrow \frac{ds}{dt} = \left|\frac{d\mathbf{R}}{dt}\right| = e^t\sqrt{2\left(\cos^2 t + \sin^2 t\right) + 1} = \sqrt{3}\; e^t$$

$$\hat{\mathbf{T}} = \frac{d\mathbf{R}}{ds} = \frac{\frac{d\mathbf{R}}{dt}}{\frac{ds}{dt}} = \frac{(\cos t - \sin t)\hat{\mathbf{I}} + (\cos t - \sin t)\hat{\mathbf{J}} + \hat{\mathbf{K}}}{\sqrt{3}}$$

$$\therefore \frac{d\hat{\mathbf{T}}}{dt} = \frac{-(\sin t + \cos t)\hat{\mathbf{I}} + (\cos t - \sin t)\hat{\mathbf{J}}}{\sqrt{3}}$$

$$\frac{d\hat{\mathbf{T}}}{ds} = \frac{\frac{d\hat{\mathbf{T}}}{dt}}{\frac{ds}{dt}} \frac{-(\sin t + \cos t)\hat{\mathbf{I}} + (\cos t - \sin t)\hat{\mathbf{J}}}{3e^t}.$$

At $t = 0$, we get

$$\frac{d\hat{\mathbf{T}}}{ds} = \frac{\frac{d\hat{\mathbf{T}}}{dt}}{\frac{ds}{dt}} = \frac{-\hat{\mathbf{I}} + \hat{\mathbf{J}}}{3}.$$

$$\Rightarrow \hat{\mathbf{N}} = \frac{\frac{d\hat{\mathbf{T}}}{ds}}{\left|\frac{d\hat{\mathbf{T}}}{ds}\right|} = \frac{\frac{-\hat{\mathbf{I}} + \hat{\mathbf{J}}}{3}}{\frac{\sqrt{1+1}}{3}} = \frac{-\hat{\mathbf{I}} + \hat{\mathbf{J}}}{\sqrt{2}}.$$

Thus, at $t = 0$, we get $\hat{\mathbf{T}} = \dfrac{\hat{\mathbf{I}} + \hat{\mathbf{J}} + \hat{\mathbf{K}}}{\sqrt{3}}$ and $\hat{\mathbf{N}} = \dfrac{-\hat{\mathbf{I}} + \hat{\mathbf{J}}}{\sqrt{2}}$

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \frac{1}{\sqrt{6}}\begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \frac{1}{\sqrt{6}}\left[(0-1)\hat{\mathbf{I}} + (-1-0)\hat{\mathbf{J}} + (1+1)\hat{\mathbf{K}}\right]$$

$$\hat{\mathbf{B}} = \frac{-\hat{\mathbf{I}} - \hat{\mathbf{J}} + 2\hat{\mathbf{K}}}{\sqrt{6}}. \text{ Ans.}$$

Also equation of plane through $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ is $\mathbf{R} = p'\hat{\mathbf{T}} + q'\hat{\mathbf{N}} = p'\left(\frac{\hat{\mathbf{I}}+\hat{\mathbf{J}}+\hat{\mathbf{K}}}{\sqrt{3}}\right) + q'\left(\frac{-\hat{\mathbf{I}}+\hat{\mathbf{J}}}{\sqrt{2}}\right)$.

$\hat{\mathbf{R}} = (p-q)\hat{\mathbf{I}} + (p+q)\hat{\mathbf{J}} + p\hat{\mathbf{K}}$.

which is the required equation of osculating plane.

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2nd Topic

Vector Calculus

Velocity and Acceleration,

Tangential and normal acceleration,

Relative velocity and acceleration

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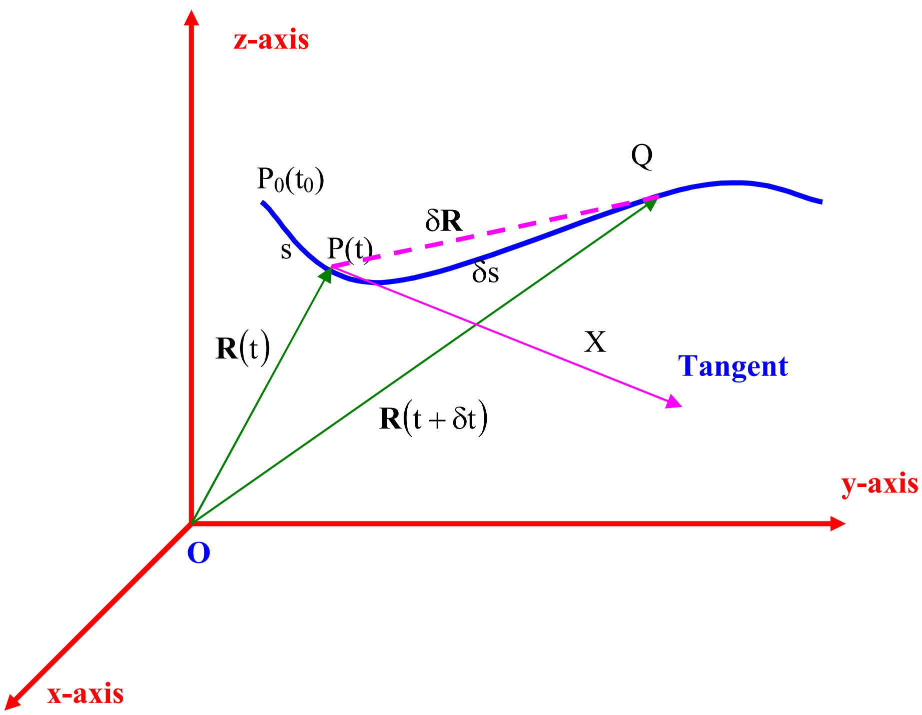
Latest update available at: http://www.freewebs.com/sunilnit/

Velocity:



Let the position of a particle P at time t (scalar variable) on a path C be $\mathbf{R}(t)$. At time $t+\delta t$, let the particle be at Q, then $\delta\mathbf{R} = \mathbf{R}(t+\delta t) - \mathbf{R}(t)$.

Now $\frac{\delta \mathbf{R}}{\delta t}$ is directed along PQ.

As $Q \to P$ along C, the line PQ becomes the tangent at P to C.

$$\therefore \underset{\delta t \to 0}{\mathrm{Lt}} \frac{\delta \mathbf{R}}{\delta t} = \frac{d\mathbf{R}}{dt} = \mathbf{V}$$

is the tangent vector of C at P which is the velocity (vector) $\mathbf{V}$ of the motion and its magnitude is the speed $v = \frac{ds}{dt}$, where s is the arc length of P from a fixed point P_0 $(s=0)$ on C.

Acceleration:

The derivative of the velocity vector $\mathbf{V}(t)$ is called the acceleration (vector) $\mathbf{A}(t)$, which is given by

$$\mathbf{A}(t) = \underset{\delta t \to 0}{\mathrm{Lim}} \frac{\delta \mathbf{V}}{\delta t} = \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{R}}{dt^2}.$$

Tangential and normal acceleration:

It is important to note that the magnitude of acceleration is not always the rate of change of $|\mathbf{V}|$ because $\mathbf{A}(t)$ is not always tangential to the path C. In fact

$$\mathbf{V}(t) = \frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}}{ds}.\frac{ds}{dt}, \text{ where } \frac{d\mathbf{R}}{ds} \text{ is a unit tangent vector to C.}$$

$$\mathbf{A}(t) = \frac{d\mathbf{V}}{dt} = \frac{d}{dt}\left[\frac{ds}{dt}.\frac{d\mathbf{R}}{ds}\right] = \frac{d^2s}{dt^2}.\frac{d\mathbf{R}}{ds} + \left(\frac{ds}{dt}\right)^2 \frac{d^2\mathbf{R}}{ds^2}.$$

Now since $\frac{d\mathbf{R}}{dt}.\frac{d^2\mathbf{R}}{dt^2} = 0 \quad \Rightarrow \frac{d^2\mathbf{R}}{dt^2}$ is perpendicular to $\frac{d\mathbf{R}}{dt}$.

Hence the acceleration $\mathbf{A}(t)$ is comprised of

(i) The tangent component $\frac{d^2s}{dt^2}.\frac{d\mathbf{R}}{ds}$, called the **tangent acceleration**.

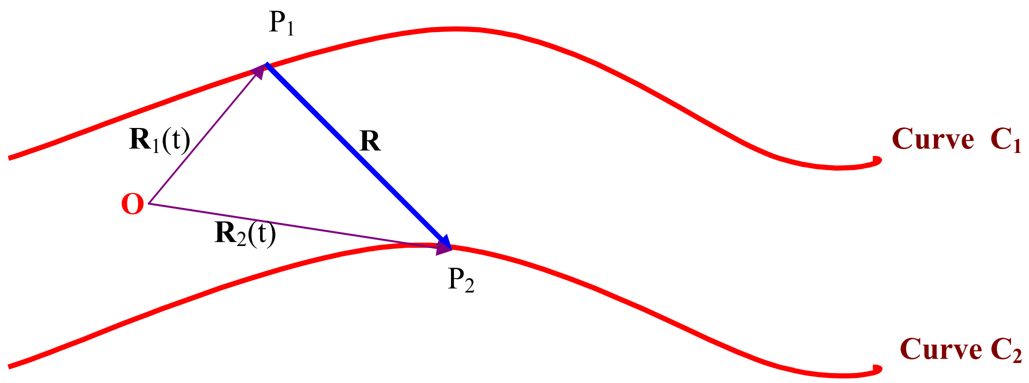
(ii) The normal component $\left(\frac{ds}{dt}\right)^2.\frac{d^2\mathbf{R}}{ds^2}$, called the **normal acceleration**.

Remark:

The acceleration is the time rate change of $|\mathbf{V}| = \frac{ds}{dt}$, if and only if the normal acceleration is zero, for then

$$|\mathbf{A}| = \left|\frac{d^2s}{dt^2}\right| \cdot \left|\frac{d\mathbf{R}}{ds}\right| = \left|\frac{d^2s}{dt^2}\right|$$

Relative velocity and acceleration:



Let two particles P_1 and P_2 moving along the curves C_1 and C_2 have position vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ at time t, respectively so that $\mathbf{R} = \overrightarrow{P_1 P_2} = \mathbf{R}_2 - \mathbf{R}_1$.

Differentiating w. r. t. t, we get

$$\frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}_2}{dt} - \frac{d\mathbf{R}_1}{dt}.$$

This defines the relative velocity (vector) of P_2 w. r. t. P_1 and states that the velocity (vector) of P_2 relative to P_1 = velocity (vector) of P_2 − velocity (vector) of P_1

Again differentiating, we have

$$\frac{d^2\mathbf{R}}{dt^2} = \frac{d^2\mathbf{R}_2}{dt^2} - \frac{d^2\mathbf{R}_1}{dt^2}$$

i. e., acceleration (vector) of P_2 relative to P_1 = acceleration (vector) of P_2 − acceleration (vector) of P_1

Remarks:

Component of $\mathbf{A}$ along the tangent $= \mathbf{A}.\hat{\mathbf{V}} = \mathbf{A}.\frac{\mathbf{V}}{|\mathbf{V}|}$.

Component of $\mathbf{A}$ along the normal $= |\mathbf{A} - \text{Resolved part of } \mathbf{A} \text{ along the tangent}|$

$$= \left|\mathbf{A} - \left(\mathbf{A}.\hat{\mathbf{V}}\right)\frac{\mathbf{V}}{|\mathbf{V}|}\right|.$$

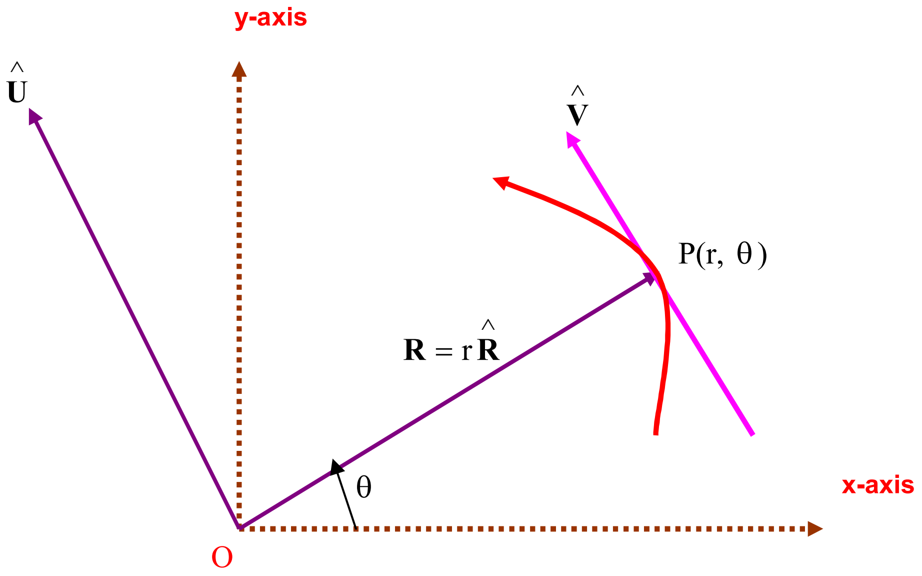
Now let us solve few problems:

Problems related with radial and transverse components of acceleration:

Q.No.1.: Find the **radial** and **transverse acceleration** of a particle moving in a plane curve.

Sol.: At any time t, let the position vector of the moving particle $P(r, \theta)$ be **R.**

Then $\mathbf{R} = r\hat{\mathbf{R}} = r(\cos\theta\hat{\mathbf{I}} + \sin\theta\hat{\mathbf{J}})$.



$\therefore$ Its velocity $\mathbf{V} = \frac{d\mathbf{R}}{dt} = \frac{dr}{dt}\hat{\mathbf{R}} + r\frac{d\hat{\mathbf{R}}}{dt}$. (i)

Since $\hat{\mathbf{R}} = (\cos\theta\hat{\mathbf{I}} + \sin\theta\hat{\mathbf{J}})$ then $\frac{d\hat{\mathbf{R}}}{dt} = (-\sin\theta\hat{\mathbf{I}} + \cos\theta\hat{\mathbf{J}})\frac{d\theta}{dt}$

and $\left|\frac{d\hat{\mathbf{R}}}{dt}\right| = \frac{d\theta}{dt}$.

Also $\frac{d\hat{\mathbf{R}}}{dt}.\hat{\mathbf{R}} = 0 \Rightarrow \frac{d\hat{\mathbf{R}}}{dt} \perp \hat{\mathbf{R}}$,

Let $\hat{\mathbf{U}}$ is a unit vector $\perp \mathbf{R}$, then $\frac{d\hat{\mathbf{R}}}{dt} = \frac{d\theta}{dt}\hat{\mathbf{U}}$.

$\therefore$ (i) becomes $\mathbf{V} = \frac{dr}{dt}\hat{\mathbf{R}} + r\frac{d\theta}{dt}\hat{\mathbf{U}}$.

Thus, the radial and transverse components of the velocity are $\frac{dr}{dt}$ and $r\frac{d\theta}{dt}$.

Also $\mathbf{A} = \frac{d\mathbf{V}}{dt} = \left(\frac{d^2r}{dt^2}\hat{\mathbf{R}} + \frac{dr}{dt}\frac{d\hat{\mathbf{R}}}{dt}\right) + \left(\frac{dr}{dt}\frac{d\theta}{dt}\hat{\mathbf{U}} + r\frac{d^2\theta}{dt^2}\hat{\mathbf{U}} + r\frac{d\theta}{dt}\frac{d\hat{\mathbf{U}}}{dt}\right)$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{R}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{\mathbf{U}} \qquad \left[\because \hat{\mathbf{U}} = -\sin\theta\,\hat{\mathbf{I}} + \sin\theta\,\hat{\mathbf{J}} \Rightarrow \frac{d\hat{\mathbf{U}}}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{R}}\right]$$

Thus, the radial component of acceleration $= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$.

And, the transverse component of acceleration is $= 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$.

Problem related with tangential and normal components of acceleration:

Q.No.2.: Find the tangential and normal acceleration of a point moving in plane curve.

Sol.: Since we know that the velocity of a point moving in a plane curve is given by $\mathbf{V} = v\hat{\mathbf{T}}$

Differentiating w. r. t. t, we get

$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d}{dt}\left(v\hat{\mathbf{T}}\right) = v.\frac{d\hat{\mathbf{T}}}{dt} + \frac{d}{dt}v\hat{\mathbf{T}} = v.\frac{d\hat{\mathbf{T}}}{ds}\frac{ds}{dt} + \frac{d}{dt}v\hat{\mathbf{T}}.$$

Now, by Franet's formula $\frac{d\hat{\mathbf{T}}}{ds} = k\hat{\mathbf{N}}$ and also we know that $v = \frac{ds}{dt}$, we get

$$\Rightarrow \mathbf{A} = v.k\hat{\mathbf{N}}v + \frac{d}{dt}\left(\frac{ds}{dt}\right)\hat{\mathbf{T}}$$

$\Rightarrow \mathbf{A} = v^2 k \hat{\mathbf{N}} + \frac{d^2s}{dt^2}\hat{\mathbf{T}} = \frac{d^2s}{dt^2}\hat{\mathbf{T}} + \frac{v^2}{\rho}\hat{\mathbf{N}}$. $\left[\because k = \frac{1}{\rho},\ \text{where } \rho = \text{radius of curvature.}\right]$

Thus, the tangential component of $\mathbf{A}$ (Tangential acceleration) $= \frac{d^2s}{dt^2}$.

And normal component of $\mathbf{A}$ (Normal acceleration) $= \frac{v^2}{\rho}$.

Problems related with velocity and acceleration:

Q.No.3.: A particle moves along a curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time variable. Determine its **velocity** and **acceleration** vectors and also the magnitude of velocity and acceleration at t = 0.

Sol.: The vector equation of the curve is $\mathbf{R}(t) = e^{-t}\hat{\mathbf{I}} + 2\cos 3t\,\hat{\mathbf{J}} + 2\sin 3t\,\hat{\mathbf{K}}$.

$\therefore$ Velocity vector $\mathbf{V}(t) = \frac{d}{dt}[\mathbf{R}(t)] = \frac{d}{dt}\left[e^{-t}\hat{\mathbf{I}} + 2\cos 3t\,\hat{\mathbf{J}} + 2\sin 3t\,\hat{\mathbf{K}}\right]$

$$= -e^{-t}\hat{\mathbf{I}} - 6\sin 3t\,\hat{\mathbf{J}} + 6\cos 3t\,\hat{\mathbf{K}}. \text{ Ans.}$$

Magnitude of velocity vector $= \sqrt{\left(-e^{-t}\right)^2 + (-6\sin 3t)^2 + (6\cos 3t)^2}$

$$= \sqrt{e^{-2t} + 36\sin^2 3t + 36\cos^2 3t}$$

$$= \sqrt{e^{-2t} + 36\left(\sin^2 3t + \cos^2 3t\right)} = \sqrt{e^{-2t} + 36}.$$

At time $t = 0$, we get $|\mathbf{V}| = \sqrt{1+36} = \sqrt{37}$. Ans.

Acceleration vector $[\mathbf{A}(t)] = \frac{d}{dt}[\mathbf{V}(t)] = \frac{d}{dt}\left[-e^{-t}\hat{\mathbf{I}} - 6\sin 3t\,\hat{\mathbf{J}} + 6\cos 3t\,\hat{\mathbf{K}}\right]$

$$= -e^{-t}\hat{\mathbf{I}} - 18\cos 3t\,\hat{\mathbf{J}} - 18\sin 3t\,\hat{\mathbf{K}}. \text{ Ans.}$$

Magnitude of acceleration $= \sqrt{\left(e^{-t}\right)^2 + (-18\cos 3t)^2 + (-18\sin 3t)^2}$

$$= \sqrt{e^{-2t} + 324\cos^2 3t + 324\sin^2 3t}$$

$$= \sqrt{e^{-2t} + 324\left(\cos^2 3t + \sin^2 3t\right)} = \sqrt{e^{-2t} + 324}.$$

At $t = 0$, $|\mathbf{A}| = \sqrt{1+324} = \sqrt{325}$. Ans.

Q.No.4.: A particle moves on the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of **velocity** and **acceleration** at t = 1 in the direction $\hat{\mathbf{I}} - 3\hat{\mathbf{J}} + 2\hat{\mathbf{K}}$.

Sol.: The vector equation of the curve is $\mathbf{R} = 2t^2\,\hat{\mathbf{I}} + \left(t^2 - 4t\right)\hat{\mathbf{J}} + (3t - 5)\hat{\mathbf{K}}$.

$$\therefore \mathbf{V} = \frac{d\mathbf{R}}{dt} = 4t\,\hat{\mathbf{I}} + (2t - 4)\hat{\mathbf{J}} + 3\hat{\mathbf{K}}.$$

At time t = 1, we get $\mathbf{V} = \dfrac{d\mathbf{R}}{dt} = 4\hat{\mathbf{I}} - 2\hat{\mathbf{J}} + 3\hat{\mathbf{K}}$.

$\therefore$ Component of velocity **V** at time t =1 in the direction $\hat{\mathbf{I}} - 3\hat{\mathbf{J}} + 2\hat{\mathbf{K}}\left[= \mathbf{D}\text{ (say)}\right] = \mathbf{V}.\dfrac{\mathbf{D}}{|\mathbf{D}|}$

$$= 4\hat{\mathbf{I}} - 2\hat{\mathbf{J}} + 3\hat{\mathbf{K}}.\frac{\hat{\mathbf{I}} - 3\hat{\mathbf{J}} + 2\hat{\mathbf{K}}}{\sqrt{1+9+4}} = \frac{4+6+6}{\sqrt{14}} = \frac{16}{14}\sqrt{14} = \frac{8}{7}\sqrt{14}. \text{ Ans.}$$

$$\text{Now } \mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d}{dt}\left[4t\,\hat{\mathbf{I}} + (2t-4)\hat{\mathbf{J}} + 3\hat{\mathbf{K}}\right] = 4\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + 0\hat{\mathbf{K}}.$$

Components of **A** along $\hat{\mathbf{I}} - 3\hat{\mathbf{J}} + 2\hat{\mathbf{K}}$ $= \mathbf{A}.\dfrac{\mathbf{D}}{|\mathbf{D}|} = 4\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + 0\hat{\mathbf{K}}.\dfrac{\left(\hat{\mathbf{I}} - 3\hat{\mathbf{J}} + 2\hat{\mathbf{K}}\right)}{\sqrt{14}} = \dfrac{4-6}{\sqrt{14}}$

$$= \frac{-2}{\sqrt{14}} = \frac{-2\sqrt{14}}{14} = \frac{-\sqrt{14}}{7}. \text{ Ans.}$$

Q.No.5.: The position vector of a particle at time t is $\mathbf{R} = \cos(t-1)\hat{\mathbf{I}} + \sinh(t-1)\hat{\mathbf{J}} + at^3\,\hat{\mathbf{K}}$. Find the condition imposed on 'a' by requiring that at time t = 1, the acceleration is normal to position vector.

Sol.: The position vector of a particle at time t is

$$\mathbf{R} = \cos(t-1)\hat{\mathbf{I}} + \sinh(t-1)\hat{\mathbf{J}} + at^3\,\hat{\mathbf{K}}.$$

$$\text{Velocity vector } \mathbf{V} = \frac{d\mathbf{R}}{dt} = \frac{d}{dt}\left[\cos(t-1)\hat{\mathbf{I}} + \sinh(t-1)\hat{\mathbf{J}} + at^3\,\hat{\mathbf{K}}\right]$$

$$= -\sin(t-1)\hat{\mathbf{I}} + \cosh(t-1)\hat{\mathbf{J}} + 3at^2\,\hat{\mathbf{K}}.$$

Acceleration vector $\mathbf{A} = \dfrac{d\mathbf{V}}{dt} = \dfrac{d}{dt}\left[-\sin(t-1)\hat{\mathbf{I}} + \cosh(t-1)\hat{\mathbf{J}} + 3at^2\hat{\mathbf{K}}\right]$

$$= -\cos(t-1)\hat{\mathbf{I}} + \sinh(t-1)\hat{\mathbf{J}} + 6at\,\hat{\mathbf{K}}.$$

But the given condition is that the acceleration is normal to the position vector i.e.

$\mathbf{A}\,.\,\mathbf{R} = 0$

$$\Rightarrow \left[-\cos(t-1)\hat{\mathbf{I}} + \sinh(t-1)\hat{\mathbf{J}} + 6at\,\hat{\mathbf{K}}\right].\left[\cos(t-1)\mathbf{I} + \sinh(t-1)\mathbf{J} + at^3\mathbf{K}\right] = 0$$

$$\Rightarrow -\cos^2(t-1) + \sinh^2(t-1) + 6a^2t^4 = 0.$$

At time $t = 0$, we get

$-1 + 0 + 6a^2 = 0 \Rightarrow 6a^2 = 1 \Rightarrow a = \pm\dfrac{1}{\sqrt{6}}$. Ans.

Q.No.6.: A particle moves along the curve $\mathbf{R} = \left(t^3 - 4t\right)\hat{\mathbf{I}} + \left(t^2 + 4t\right)\hat{\mathbf{J}} + \left(8t^2 - 3t^3\right)\hat{\mathbf{K}}$, where t denotes time. Find magnitudes of acceleration along the tangent and normal at time $t = 2$.

Sol.: The vector equation of the curve is $\mathbf{R} = \left(t^3 - 4t\right)\hat{\mathbf{I}} + \left(t^2 + 4t\right)\hat{\mathbf{J}} + \left(8t^2 - 3t^3\right)\hat{\mathbf{K}}$.

Velocity $\dfrac{d\mathbf{R}}{dt} = \left(3t^2 - 4\right)\hat{\mathbf{I}} + (2t + 4)\hat{\mathbf{J}} + \left(16t - 9t^2\right)\hat{\mathbf{K}}$

and acceleration $\dfrac{d^2\mathbf{R}}{dt^2} = 6t\,\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + (16 - 18t)\hat{\mathbf{K}}$.

$\therefore$ At time $t = 2$, $\mathbf{V} = 8\hat{\mathbf{I}} + 8\hat{\mathbf{J}} - 4\hat{\mathbf{K}}$ and acceleration $\mathbf{A} = 12\hat{\mathbf{I}} + 2\hat{\mathbf{J}} - 20\hat{\mathbf{K}}$.

Since, the velocity is along the tangent to the curve, therefore the component of **A** along the tangent $= \mathbf{A}.\hat{\mathbf{V}} = \mathbf{A}.\dfrac{\mathbf{V}}{|\mathbf{V}|} = \left(12\hat{\mathbf{I}} + 2\hat{\mathbf{J}} - 20\hat{\mathbf{K}}\right).\dfrac{8\hat{\mathbf{I}} + 8\hat{\mathbf{J}} - 4\hat{\mathbf{K}}}{\sqrt{(64 + 64 + 16)}}$

$$= \frac{12\times 8 + 2\times 8 + (-20)\times(-4)}{12} = 16.$$

Now the component of $\mathbf{A}$ along the normal $= |\mathbf{A} - \text{Resolved part of } \mathbf{A} \text{ along the tangent}|$

$$= \left|\mathbf{A} - \left(\mathbf{A}.\hat{\mathbf{V}}\right)\frac{\mathbf{V}}{|\mathbf{V}|}\right| = \left|12\hat{\mathbf{I}} + 2\hat{\mathbf{J}} - 20\hat{\mathbf{K}} - 16.\frac{8\hat{\mathbf{I}} + 8\hat{\mathbf{J}} - 4\hat{\mathbf{K}}}{12}\right|$$

$$= \frac{1}{3}\left|\frac{36\hat{\mathbf{I}} + 6\hat{\mathbf{J}} - 60\hat{\mathbf{K}} - 4.\left(8\hat{\mathbf{I}} + 8\hat{\mathbf{J}} - 4\hat{\mathbf{K}}\right)}{3}\right| = \frac{1}{3}\left|4\hat{\mathbf{I}} - 26\hat{\mathbf{J}} - 44\hat{\mathbf{K}}\right| = 2\sqrt{(73)}.\ \text{Ans.}$$

Q.No.7.: The position vector of a moving particle at time t is $\mathbf{R} = t^2\hat{\mathbf{I}} - t^3\hat{\mathbf{J}} + t^4\hat{\mathbf{K}}$.

Find the tangential and normal components of its acceleration at time t.

Sol.: The vector equation of the curve is $\mathbf{R} = t^2\hat{\mathbf{I}} - t^3\hat{\mathbf{J}} + t^4\hat{\mathbf{K}}$.

$$\therefore \mathbf{V} = \frac{d\mathbf{R}}{dt} = 2t\hat{\mathbf{I}} - 3t^2\hat{\mathbf{J}} + 4t^3\hat{\mathbf{K}}.$$

At time $t = 1$, we get $\mathbf{V} = 2\hat{\mathbf{I}} - 3\hat{\mathbf{J}} + 4\hat{\mathbf{K}}$. (i)

Also $\mathbf{A} = \dfrac{d\mathbf{V}}{dt} = 2\hat{\mathbf{I}} - 6t\hat{\mathbf{J}} + 12t^2\hat{\mathbf{K}}$.

At $t = 1$, we have $\mathbf{A} = 2\hat{\mathbf{I}} - 6\hat{\mathbf{J}} + 12\hat{\mathbf{K}}$. (ii)

$\therefore$ The component $\mathbf{A}$ along the tangent $= \mathbf{A}\hat{\mathbf{V}}$

$$= \mathbf{A}.\frac{\mathbf{V}}{|\mathbf{V}|} = 2\hat{\mathbf{I}} - 6\hat{\mathbf{J}} + 12\hat{\mathbf{K}}.\frac{2\hat{\mathbf{I}} - 3\hat{\mathbf{J}} + 4\hat{\mathbf{K}}}{\sqrt{4+9+16}} = \frac{70}{\sqrt{29}}.\ \text{Ans.}$$

Now the components of $\mathbf{A}$ along normal

$$= |\mathbf{A} - \text{Resolved part of } \mathbf{A} \text{ along the tangent}| = \left|\vec{\mathbf{A}} - \left(\vec{\mathbf{A}}.\hat{\mathbf{V}}\right).\frac{\vec{\mathbf{V}}}{|\vec{\mathbf{V}}|}\right|$$

$$= \left|2\hat{\mathbf{I}} - 6\hat{\mathbf{J}} + 12\hat{\mathbf{K}} - \frac{70}{\sqrt{29}}\frac{2\hat{\mathbf{I}} - 3\hat{\mathbf{J}} + 4\hat{\mathbf{K}}}{\sqrt{29}}\right| = \left|2\hat{\mathbf{I}} - 6\hat{\mathbf{J}} + 12\hat{\mathbf{K}} - \frac{140\hat{\mathbf{I}} + 210\hat{\mathbf{J}} - 280\hat{\mathbf{K}}}{29}\right|$$

$$= \frac{1}{29}\left|58\hat{\mathbf{I}} - 174\hat{\mathbf{J}} + 348\hat{\mathbf{K}} - 140\hat{\mathbf{I}} + 210\hat{\mathbf{J}} - 280\hat{\mathbf{K}}\right|$$

$$= \frac{1}{29}\left|-82\hat{\mathbf{I}} + 36\hat{\mathbf{J}} + 68\hat{\mathbf{K}}\right| = \frac{1}{29}\sqrt{12644} = \frac{1}{\sqrt{29}}.\sqrt{436} = \sqrt{\frac{436}{29}}\textbf{.}\ \text{Ans.}$$

(i) $\therefore$ Tangent component of acceleration $= \dfrac{70}{\sqrt{29}}$.

(ii) The normal component of acceleration $\sqrt{\dfrac{436}{29}}$.

Q.No.8.: A particle moves so that its position vector is given by $\mathbf{R} = \mathbf{I}\cos\omega t + \mathbf{J}\sin\omega t$. Show that the **velocity V** of the particle is perpendicular to **R** and $\mathbf{R}\times\mathbf{V}$ is a constant vector.

Sol.: (i) The position vector of a particle is given by

$$\mathbf{R} = \cos\omega t\,\hat{\mathbf{I}} + \sin\omega t\,\hat{\mathbf{J}}. \tag{i}$$

$$\therefore \text{Velocity } \mathbf{V} = \frac{d\mathbf{R}}{dt} = \frac{d}{dt}\left(\cos\omega t\,\hat{\mathbf{I}} + \sin\omega t\,\hat{\mathbf{J}}\right) = -\omega\cos\omega t\,\hat{\mathbf{I}} + \omega\sin\omega t\,\hat{\mathbf{J}}$$

$$= \omega\left(-\sin\omega t\,\hat{\mathbf{I}} + \cos\omega t\,\hat{\mathbf{J}}\right). \tag{ii}$$

$$\mathbf{R.V} = \cos\omega t\,\hat{\mathbf{I}} + \sin\omega t\,\hat{\mathbf{J}}.\left(-\omega\sin\omega t\,\hat{\mathbf{I}} + \omega\cos\omega t\,\hat{\mathbf{J}}\right) = -\omega\sin\omega t\cos\omega t + \omega\sin\omega t\cos\omega t = 0$$

$\therefore \mathbf{R} \perp \mathbf{V} \Rightarrow$ Velocity **V** of the particle is perpendicular to **R.**

$$\textbf{(ii)}\ \mathbf{R}\times\mathbf{V} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix} = (0-0)\hat{\mathbf{I}} + (0-0)\hat{\mathbf{J}} + \left(\omega\cos^2\omega t + \omega\sin^2\omega t\right)\hat{\mathbf{K}}$$

$$= \omega\left(\cos^2\omega t + \sin^2\omega t\right)\hat{\mathbf{K}} = \omega\hat{\mathbf{K}}.$$

Thus $\mathbf{R}\times\mathbf{V}$ is a constant vector.

Q.No.9.: A particle (position vector **R**) moving in a circle with constant angular velocity ω, show by vector method, **acceleration** is equal to $-\omega^2\mathbf{R}$.

Sol.: Let the position vector of a particle in a circle is.

$$\mathbf{R} = \hat{\mathbf{I}}\cos\omega t + \hat{\mathbf{J}}\sin\omega t$$

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} = \frac{d}{dt}\left(\cos\omega t\,\hat{\mathbf{I}} + \sin\omega t\,\hat{\mathbf{J}}\right) = -\omega\cos\omega t\,\hat{\mathbf{I}} + \omega\sin\omega t\,\hat{\mathbf{J}},$$

and acceleration $\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{R}}{dt^2} = \frac{d}{dt}\left(-\omega\sin\omega t\,\hat{\mathbf{I}} + \omega\cos\omega t\,\hat{\mathbf{J}}\right)$

$$= -\omega^2\left(\cos\omega t\,\hat{\mathbf{I}} + \sin\omega t\,\hat{\mathbf{J}}\right) = -\omega^2\mathbf{R}$$

$\Rightarrow \mathbf{A} = -\omega^2\mathbf{R}$.

Hence this proves the result.

Q.No.10.: A particle moves along a catenary $s = c\tan\psi$. The direction of acceleration at any point makes equal angels with the tangent and normal to the path at that point. If the speed at the vertex $(\psi = 0)$ be v_0, show that the magnitude of **velocity** and **acceleration** at any point are given by $v_0 e^{\psi}$ and $\frac{\sqrt{2}}{c} v_0^{\,2} e^{2\psi}\cos^2\psi$ respectively.

Sol.: $s = c\tan\psi$, $v = \frac{ds}{dt} = c\sec^2\psi\frac{d\psi}{dt}$

$$a_T = \frac{dv}{dt} = c^2\sec^2\psi\tan\psi\left(\frac{d\psi}{dt}\right)^2 + c\sec^2\psi\frac{d^2\psi}{dt^2}$$

$$a_N = \frac{v^2}{\rho}, \qquad \rho = \text{radius of curvature}$$

$$\rho = c\sec^2\psi$$

$$\Rightarrow a_N = \frac{c^2\sec^4\psi\left(\frac{d\psi}{dt}\right)^2}{c\sec^2\psi} = c\sec^2\psi\left(\frac{d\psi}{dt}\right)^2$$

$a_T = a_N$ {given}

$$\Rightarrow c^2 \sec^2\psi \tan\psi \left(\frac{d\psi}{dt}\right)^2 + c\sec^2\psi \frac{d^2\psi}{dt^2} = c\sec^2\psi \left(\frac{d\psi}{dt}\right)^2$$

$$\Rightarrow 2\tan\psi \left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d^2\psi}{dt^2}\right) = \left(\frac{d\psi}{dt}\right)^2 \Rightarrow (1 - 2\tan\psi)\left(\frac{d\psi}{dt}\right)^2 = \frac{d^2\psi}{dt^2}$$

Put $\frac{d\psi}{dt} = P \Rightarrow \frac{d^2\psi}{dt^2} = P\frac{dP}{d\psi} \Rightarrow (1 - 2\tan\psi)P^2 = P\frac{dP}{d\psi}$

$$\Rightarrow (1 - 2\tan\psi)d\psi = \frac{dP}{P} \Rightarrow \int (1 - 2\tan\psi)d\psi = \int \frac{dP}{P}$$

$$\Rightarrow \psi - 2\ell n\sec\psi + c = \ell nP \Rightarrow P = e^{\psi - 2\ell n\sec\psi + c} \qquad \text{(i)}$$

$$v = \frac{ds}{dt} = c\sec^2\psi \frac{d\psi}{dt}$$

at $\psi = 0 \Rightarrow v_0 = c\sec^2 0 \frac{d\psi}{dt} \Rightarrow \frac{d\psi}{dt} = \frac{v_0}{c}$. $\qquad \left[\because P = \frac{d\psi}{dt}\right]$

Putting in (i), we get

$$\Rightarrow \frac{v_0}{c} = e^c \Rightarrow c = \ell n\frac{v_0}{c}$$

$$\Rightarrow v = c\sec^2\psi . e^{\psi - 2\ell n\sec\psi + \ell n v_0/c}$$

$$\Rightarrow v = c\sec^2\psi \times \frac{e^\psi}{e^{2\ell n\sec\psi}} \times e^{\ell n\, v_0/c}$$

$$\Rightarrow v = c\sec^2\psi \times \frac{e^\psi}{\sec^2} \times \frac{v_0}{c} \Rightarrow v_0 e^\psi$$

$$a_T = a_N$$

$$\Rightarrow A = \sqrt{a_T{}^2 + a_N{}^2}$$

$$a_N = \frac{v^2}{\rho} = \frac{v_0{}^2 e^{2\psi}}{c\sec^2\psi} = \frac{v_0{}^2 e^{2\psi}}{c}\cos^2\psi$$

$$\Rightarrow A = \sqrt{a_T{}^2 + a_N{}^2} = \sqrt{2a_N{}^2} = a_N\sqrt{2} = \frac{\sqrt{2}}{c} v_0{}^2 e^{2\psi}\cos^2\psi \text{ . Ans.}$$

Problems related with actual velocity/ relative velocity:

Q.No.11.: A person going east wards with a velocity of 4 kmph, finds that the wind appears to blow directly from the north. He doubles his speed and wind seems to come from north-east. Find the **actual velocity** of the wind.

Sol.: Let the actual velocity of the wind be $x\hat{\mathbf{I}}+y\hat{\mathbf{J}}$, where $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$ represent velocities of 1 kmph towards the east and north respectively.

As the person is going east wards with a velocity of 4 kmph, his actual velocity is $4\hat{\mathbf{I}}$.

Then the velocity of the wind relative to the man is $\left(x\hat{\mathbf{I}}+y\hat{\mathbf{J}}\right)-4\hat{\mathbf{I}}$, which is parallel to $-\mathbf{J}$, as it appears to blow from the north.

$$\therefore \frac{x-4}{0}=\frac{y-0}{-1}\Rightarrow\frac{x-4}{0}=\frac{y}{-1}\Rightarrow(-1)(x-4)=0\Rightarrow x=4.$$

Hence x = 4. (i)

When the velocity of the person becomes $8\hat{\mathbf{I}}$, the velocity of the wind relative to man is $\left(x\hat{\mathbf{I}}+y\hat{\mathbf{J}}\right)-8\hat{\mathbf{I}}$, but this is parallel to $-\left(\hat{\mathbf{I}}+\hat{\mathbf{J}}\right)$.

$$\therefore \frac{x-8}{-1}=\frac{y-0}{-1}\Rightarrow\frac{4-8}{1}=\frac{y}{1}\Rightarrow y=-4.$$

Hence, the actual velocity of the wind is $4\left(\hat{\mathbf{I}}-\hat{\mathbf{J}}\right)$.

i.e. $4\sqrt{2}$ kmph towards the south east.

Q.No.12.: A person traveling towards the north-east with a velocity of 6 kmph, finds that the wind appears to blow from the north but when he doubles his speed it seems to come from a direction inclined at an angle $\tan^{-1}2$ to the north of east. Show that the **actual velocity** of the wind is $3\sqrt{2}$ kmph towards the east.

Sol.: 1st case: Let the actual velocity of the wind be $\mathbf{V}_w=x\hat{\mathbf{I}}+y\hat{\mathbf{J}}$,

where $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$ represents velocities of 1 kmph towards the east and north respectively.

As the person traveling towards the north-east with a velocity of 6 kmph, his actual velocity is

$$\mathbf{V}_n=6\cos45°\hat{\mathbf{I}}+6\sin45°\hat{\mathbf{J}}.$$

Then, the velocity of the wind relative to person is $\mathbf{V}_w - \mathbf{V}_p$.

$$\mathbf{V}_{wp} = \mathbf{V}_w - \mathbf{V}_p = \left(x - \frac{6}{\sqrt{2}}\right)\hat{\mathbf{I}} + \left(y - \frac{6}{\sqrt{2}}\right)\hat{\mathbf{J}}.$$

It is parallel to $-\hat{\mathbf{J}}$, as it appear to blow from the north.

$$\therefore \frac{x - \frac{6}{\sqrt{2}}}{0} = \frac{y - \frac{6}{\sqrt{2}}}{-1}.$$

$$\Rightarrow -x + \frac{6}{\sqrt{2}} = 0 \Rightarrow x = \frac{6}{\sqrt{2}} . \qquad \text{(i)}$$

2nd case: When he double his speed then

$$\mathbf{V}_p = 12\cos 45°\,\hat{\mathbf{I}} + 12\sin 45°\,\hat{\mathbf{J}} = \frac{12}{\sqrt{2}}\hat{\mathbf{I}} + \frac{12}{\sqrt{2}}\hat{\mathbf{J}}.$$

$\therefore$ The velocity of the wind relative to the person is $\mathbf{V}_w - \mathbf{V}_p$

$$\mathbf{V}_{wp} = \mathbf{V}_w - \mathbf{V}_p = \left(x - \frac{12}{\sqrt{2}}\right)\hat{\mathbf{I}} + \left(y - \frac{12}{\sqrt{2}}\right)\hat{\mathbf{J}}.$$

But this is parallel to $+\hat{\mathbf{I}} + 2\hat{\mathbf{J}}$

$$\therefore \frac{x - \frac{12}{\sqrt{2}}}{1} = \frac{y - \frac{12}{\sqrt{2}}}{2} \Rightarrow 2x - \frac{24}{\sqrt{2}} = y - \frac{12}{\sqrt{2}} \Rightarrow 2x - y = \frac{-12}{\sqrt{2}} + \frac{24}{\sqrt{2}}$$

$$\Rightarrow 2x - y = \frac{1}{\sqrt{2}}(12)$$

But $x = \frac{6}{\sqrt{2}} \Rightarrow \frac{2\times 6}{\sqrt{2}} = \frac{12}{\sqrt{2}} \Rightarrow y = 0$.

Hence, the actual velocity of the wind is $\mathbf{V}_w = \frac{6}{\sqrt{2}}\hat{\mathbf{I}}$.

$$\therefore |\mathbf{V}_w| = \sqrt{\left(\frac{6}{\sqrt{2}}\right)^2} = \sqrt{18} = 3\sqrt{2} \text{ kmph toward the East.}$$

Hence, this proves the result.

Q.No.13.: The velocity of a boat relative to water is represented by $3\hat{\mathbf{I}} + 4\hat{\mathbf{J}}$ and that of water relative to earth is $\hat{\mathbf{I}}-3\hat{\mathbf{J}}$. What is the **velocity** of the boat relative to the earth if $\hat{\mathbf{I}}$ and $\hat{\mathbf{J}}$ represent one km an hour east and north respectively.

Sol.: Given the velocity of a boat relative to water = $3\hat{\mathbf{I}}+4\hat{\mathbf{J}}$.

And velocity of water relative to earth = $\hat{\mathbf{I}}-3\hat{\mathbf{J}}$.

Let velocity of boat $= x_1\hat{\mathbf{I}}+y_1\hat{\mathbf{J}}$.

Let velocity of water $= x_2\hat{\mathbf{I}}+y_2\hat{\mathbf{J}}$.

Let velocity of earth $= x_3\hat{\mathbf{I}}+y_3\hat{\mathbf{J}}$.

The velocity of a boat relative to water $= x_1\hat{\mathbf{I}}-y_1\hat{\mathbf{J}}-x_2\hat{\mathbf{I}}-y_2\hat{\mathbf{J}}=3\hat{\mathbf{I}}+4\hat{\mathbf{J}}$. (i)

Velocity of water relative to earth $= x_2\hat{\mathbf{I}}+y_2\hat{\mathbf{J}}-x_3\hat{\mathbf{I}}-y_3\hat{\mathbf{J}} = \hat{\mathbf{I}}-3\hat{\mathbf{J}}$. (ii)

On adding (i) and (ii), we get

$= x_1\hat{\mathbf{I}}+y_1\hat{\mathbf{J}}-x_3\hat{\mathbf{I}}-y_3\hat{\mathbf{J}}=4\hat{\mathbf{I}}+\hat{\mathbf{J}}$. (iii)

$\therefore$ The velocity of a boat relative to earth = $4\hat{\mathbf{I}}+\hat{\mathbf{J}}$.

Now its magnitude $=\sqrt{4^2+1^2}=\sqrt{17}$mph

and $\tan\theta=\frac{1}{4}=0.25 \Rightarrow \theta=\tan^{-1}(0.25)$.

Thus the velocity of boat relative to earth is $\sqrt{17}$ mph in the direction $\tan^{-1}(0.25)$ north to east.

Q.No.14.: A vessel A is sailing with a velocity of 11 knots/hour in the direction of South East and a second vessel B is sailing with a velocity of 13 knots/hour in a direction of $30°$ East of North. Find the **velocity** of A relative to B.

Sol.: We have

$V_1 = 11\cos 45°\hat{\mathbf{I}}-11\sin 45°\hat{\mathbf{J}}$.

$V_2 = 13\cos 60°\hat{\mathbf{I}}-13\sin 60°\hat{\mathbf{J}}$.

$\therefore$ Relative velocity will be $V_{21} = V_1 - V_2$

$$\Rightarrow V_{21} = \left(\frac{11}{\sqrt{2}}\hat{\mathbf{I}} - \frac{11}{\sqrt{2}}\hat{\mathbf{J}}\right) - \left(\frac{13}{2}\hat{\mathbf{I}} + \frac{13\sqrt{3}}{2}\hat{\mathbf{J}}\right) = \left(\frac{11}{\sqrt{2}} - \frac{13}{2}\right)\hat{\mathbf{I}} - \left(\frac{11}{\sqrt{2}} + \frac{13\sqrt{3}}{2}\right)\hat{\mathbf{J}}$$

$$= (7.78 - 6.5)\hat{\mathbf{I}} - (7.78 + 11.26)\hat{\mathbf{J}} = 1.28\hat{\mathbf{I}} - 19.04\hat{\mathbf{J}}.$$

$$|\mathbf{V}_{21}| = \sqrt{(1.28)^2 - (19.04)^2} = \sqrt{364.16} = 19.08 \text{ knots per hour}$$

To find the direction, we have

$$\tan\theta = -\left(\frac{19.028}{1.27}\right) \Rightarrow \theta = -86.18^\circ \text{ South of East.}$$

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3rd Topic

Vector Calculus

Scalar point function,

Vector point function,

Vector operator 'del',

Del applied to scalar point functions (Gradient) and its Geometrical interpretation,

Directional derivative

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Scalar point function:

If to each point $P(\mathbf{R})$ of a region E in space there corresponds a definite scalar denoted by $f(\mathbf{R})$, then $f(\mathbf{R})$ is called a **scalar point function** in E. The region E so defined is called a scalar field.

The **temperature** at any instant, **density** of a body and **potential** due to gravitational matter are all examples of scalar point functions.

Vector point function:

If to each point $P(\mathbf{R})$ of a region E in space there corresponds a definite vector denoted by **F(R),** then it is called the **vector point function** in E. The region E so defined is called a vector field.

The **velocity** of a moving fluid at any instant, the **gravitational intensity** of force are examples of vector point functions.

Thus, if **F** (x, y, z) be a vector point function, then

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{F}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{F}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{F}}{\partial z}\frac{dz}{dt}$$

and $d\mathbf{F} = \frac{\partial \mathbf{F}}{\partial x}dx + \frac{\partial \mathbf{F}}{\partial y}dy + \frac{\partial \mathbf{F}}{\partial z}dz = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right)\mathbf{F}$. (i)

Vector operator del:

Definition: The operator on the right side of the equation (i)

i.e. $\left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right)$ is in the form of a scalar product of $\left(\hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}\right)$

and $\left(\hat{\mathbf{I}}\,dx + \hat{\mathbf{J}}\,dy + \hat{\mathbf{K}}\,dz\right)$.

If ∇ (read as del) be defined by the equation $\nabla = \hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}$.

Then (i) $\Rightarrow d\mathbf{F} = (\nabla . d\mathbf{R})\mathbf{F}$, where $\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}$, $d\mathbf{R} = \hat{\mathbf{I}}\,dx + \hat{\mathbf{J}}\,dy + \hat{\mathbf{K}}\,dz$.

History of del operator: In vector calculus, **del** is a vector differential operator represented by the nabla symbol: ∇.

The name comes from the Greek word for a Hebrew harp, which had a similar shape.

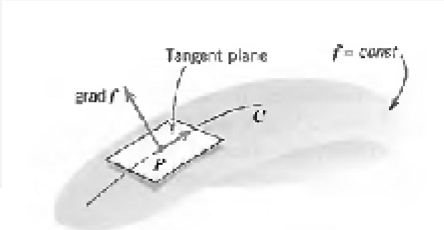
The harp, the instrument after which the nabla symbol is named.

The symbol was first used by William Rowan Hamilton.

Del applied to scalar point functions: (Gradient)

Definition: The vector function ∇f is defined as the gradient of the scalar point function f and is written as grad f.

Thus, $\text{grad } f = \nabla f = \hat{\mathbf{I}}\frac{\partial f}{\partial x} + \hat{\mathbf{J}}\frac{\partial f}{\partial y} + \hat{\mathbf{K}}\frac{\partial f}{\partial z} = \sum \hat{\mathbf{I}}\frac{\partial f}{\partial x}$.



Gradient as Surface Normal Vector

Let f be a differentiable scalar function in space. Let $f(x, y, z) = c = const$ represent a surface S. Then if the gradient of f at a point P of S is not the zero vector, it is a normal vector of S at P.

Geometrical interpretation:

1. $\nabla f = |\nabla f|\hat{N} \Rightarrow \nabla f$ is normal to the surface $f(x, y, z) = c$.

2. $|\nabla f| = \dfrac{\partial f}{\partial n} \Rightarrow$ magnitude of ∇f is equal to the rate of change of f along this normal.

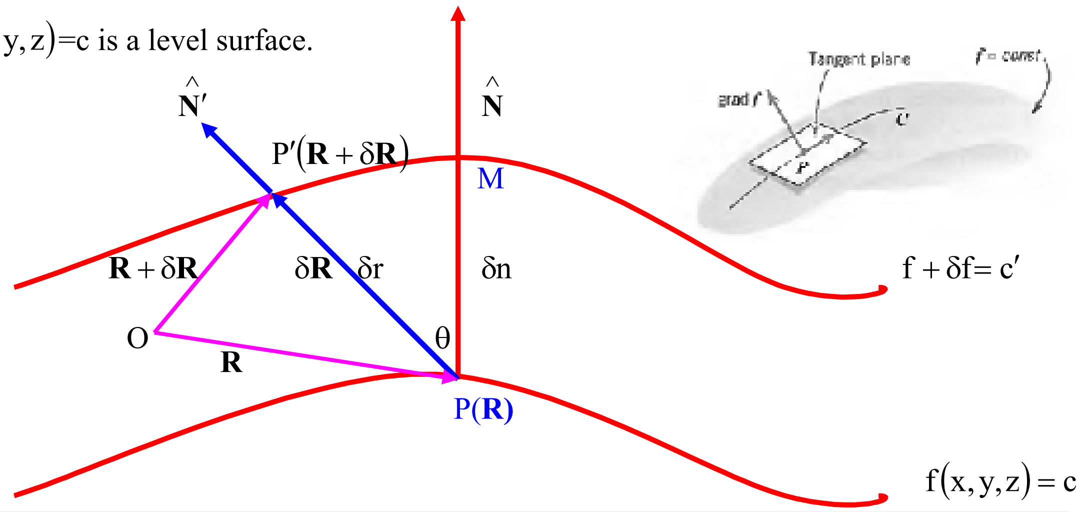
Consider the scalar point function $f(\mathbf{R})$, where $\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}$.

Draw a surface f (x, y, z) = c through any point $P(\mathbf{R})$ s.t. at each point on it, the function $f(x, y, z)$ has the same value as at P.

This type of surface is called a **level surface** of the function f through P.

Examples: **Equipotential** or **isothermal surfaces** are examples of level surfaces.

Thus, if $f(x, y, z)$ represents potential at the point (x, y, z), the equipotential surface $f(x, y, z) = c$ is a level surface.



1. $\nabla f = |\nabla f|\hat{N} \Rightarrow \nabla f$ is normal to the surface $f(x, y, z) = c$.

Let $P'(\mathbf{R} + \delta\mathbf{R})$ be a point on a neighbouring level surface $f + \delta f = c'$, where the function is $f + \delta f$.

Then $\nabla f.\delta\mathbf{R} = \left[\hat{\mathbf{I}}\frac{\partial f}{\partial x} + \hat{\mathbf{J}}\frac{\partial f}{\partial y} + \hat{\mathbf{K}}\frac{\partial f}{\partial z}\right].\left(\hat{\mathbf{I}}\,\delta x + \hat{\mathbf{J}}\,\delta y + \hat{\mathbf{K}}\,\delta z\right) = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z = \delta f$.

Now, if P' lies on the same level surface as P, then $\Rightarrow \delta f = 0$

Now since $\nabla f.\delta\mathbf{R} = \delta f \;\Rightarrow \nabla f.\delta\mathbf{R} = 0$.

This means that ∇f is $\perp$ to every $\delta\mathbf{R}$ lying on this surface.

Thus ∇f is normal to the surface f (x, y, z) =c.

Now, if $\hat{\mathbf{N}}$ is unit vector normal to the surface f (x, y, z) =c, then we can write

$$\nabla f = |\nabla f|\,\hat{\mathbf{N}} .$$

$$\Rightarrow \nabla f \text{ is normal to the surface } f(x, y, z) = c.$$

2. $|\nabla f| = \frac{\partial f}{\partial n} \Rightarrow$ magnitude of ∇f is equal to the rate of change of f along this normal.

Let the perpendicular distance PM between the surfaces through P and P' is δn .

Then the rate of change of f normal to the surface through $P = \frac{\partial f}{\partial n}$.

$$\text{Now } \frac{\partial f}{\partial n} = \underset{\delta n \to 0}{\mathrm{Lt}}\frac{\delta f}{\delta n} = \underset{\delta n \to 0}{\mathrm{Lt}}\,\nabla f.\frac{\delta\mathbf{R}}{\delta n} \qquad [\because \delta f = \nabla f.\delta\mathbf{R}]$$

$$= |\nabla f|\underset{\delta n \to 0}{\mathrm{Lt}}\frac{\hat{\mathbf{N}}.\delta\mathbf{R}}{\delta n} \qquad \left[\because \nabla f = |\nabla f|\,\hat{\mathbf{N}}\right]$$

$$= |\nabla f|\underset{\delta n \to 0}{\mathrm{Lt}}\frac{|\hat{\mathbf{N}}|.|\delta\mathbf{R}|\cos\theta}{\delta n} = |\nabla f|\underset{\delta n \to 0}{\mathrm{Lt}}\frac{\delta r\cos\theta}{\delta n}$$

$$= |\nabla f|\underset{\delta n \to 0}{\mathrm{Lt}}\frac{\delta n}{\delta n} = |\nabla f| \qquad \left[\because \frac{\delta n}{\delta r} = \cos\theta\right]$$

Hence, the magnitude of $\nabla f = \frac{\partial f}{\partial n}$.

Thus the magnitude of ∇f is equal to the rate of change of f along this normal.

Thus, grad (f) is a vector normal to the surface (f = constant) and has a magnitude equal to the rate of change of f along the normal.

Directional derivative:

Definition: Let δr denotes the length PP' and $\hat{\mathbf{N}}'$ is a unit vector in the direction of PP' .

Then the limiting value of $\frac{\delta f}{\delta r}$ as $\delta r \to 0$ $\left(\text{i.e. } \frac{\partial f}{\partial r}\right)$ is known as **directional derivative** of f at P along the direction PP'.

Now since $\delta r = \frac{\delta n}{\cos\theta} = \frac{\delta n}{\hat{\mathbf{N}}.\hat{\mathbf{N}}'}$ $\left[\because \cos\theta = \frac{\delta n}{\delta r}\right]$

$$\therefore \frac{\partial f}{\partial r} = \lim_{\delta r\to 0}\frac{\delta f}{\delta r} = \lim_{\delta n\to 0}\left[\hat{\mathbf{N}}.\hat{\mathbf{N}}'\frac{\delta f}{\delta n}\right]$$

$$= \hat{\mathbf{N}}'.\frac{\partial f}{\partial n}\hat{\mathbf{N}} = \hat{\mathbf{N}}'.|\nabla f|\hat{\mathbf{N}} = \hat{\mathbf{N}}'.\nabla f . \quad \left[\because |\nabla f| = \frac{\partial f}{\partial n} \text{ and } |\nabla f|\hat{\mathbf{N}} = \nabla f\right]$$

Thus, directional derivative of f in the direction of $\hat{\mathbf{N}}'$ is the resolved part of ∇f in the direction $\hat{\mathbf{N}}'$.

Since $\nabla f.\hat{\mathbf{N}}' = |\nabla f|\cos\alpha \le |\nabla f| \Rightarrow \frac{\partial f}{\partial r} \le |\nabla f|$.

$\Rightarrow \nabla f$ gives the maximum rate of change of f, and the magnitude of this maximum is $|\nabla f|$.

Now let us solve some problems related to these topics:

Q.No.1.: Find the **unit vector normal** to the surface $xy^3z^2 = 4$, at the point $(-1, -1, 2)$.

Sol.: A vector normal to the given surface is $\nabla(xy^3z^2)$

$$= \hat{\mathbf{I}}\frac{\partial}{\partial x}(xy^3z^2) + \hat{\mathbf{J}}\frac{\partial}{\partial y}(xy^3z^2) + \hat{\mathbf{K}}\frac{\partial}{\partial z}(xy^3z^2)$$

$$= \hat{\mathbf{I}}(y^3z^2) + \hat{\mathbf{J}}(3xy^2z^2) + \hat{\mathbf{K}}(2xy^3z)$$

$$= -4\hat{\mathbf{I}} - 12\hat{\mathbf{J}} + 4\hat{\mathbf{K}} \text{ at the point } (-1, -1, 2).$$

Hence, the desired unit normal to the surface $\hat{\mathbf{N}} = \frac{\nabla f}{|\nabla f|} = \frac{-4\hat{\mathbf{I}} - 12\hat{\mathbf{J}} + 4\hat{\mathbf{K}}}{\sqrt{16+144+16}}$

$$= -\frac{1}{\sqrt{11}}\left(\hat{\mathbf{I}} + 3\hat{\mathbf{J}} - \hat{\mathbf{K}}\right). \text{ Ans.}$$

Q.No.2: Find a **unit vector normal** to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1,\ 2,\ -1)$.

Sol.: A vector normal to the given surface is ∇f, where $f = x^3 + y^3 + 3xyz - 3$.

$$\nabla f = \hat{\mathbf{I}}\frac{\partial f}{\partial x} + \hat{\mathbf{J}}\frac{\partial f}{\partial y} + \hat{\mathbf{K}}\frac{\partial f}{\partial z}$$

$$= \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(x^3 + y^3 + 3xyz - 3\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(x^3 + y^3 + 3xyz - 3\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(x^3 + y^3 + 3xyz - 3\right)$$

$$= \hat{\mathbf{I}}\left(3x^2 + 3yz\right) + \hat{\mathbf{J}}\left(3y^2 + 3xz\right) + \hat{\mathbf{K}}(3xy)$$

Therefore, ∇f at the point $(1,\ 2,\ -1)$ is $-3\hat{\mathbf{I}} + 9\hat{\mathbf{J}} + 6\hat{\mathbf{K}}$

Since, we know that unit vector normal to the surface is $\hat{\mathbf{N}} = \dfrac{\nabla f}{|\nabla f|}$.

Hence, the desired unit vector normal to the surface is

$$\hat{\mathbf{N}} = \frac{\nabla f}{|\nabla f|} = \frac{-3\hat{\mathbf{I}} + 9\hat{\mathbf{J}} + 6\hat{\mathbf{K}}}{\sqrt{(-3)^2 + 9^2 + 6^2}} = \frac{-3\hat{\mathbf{I}} + 9\hat{\mathbf{J}} + 6\hat{\mathbf{K}}}{\sqrt{126}} = \frac{-3\hat{\mathbf{I}} + 9\hat{\mathbf{J}} + 6\hat{\mathbf{K}}}{3\sqrt{14}} = \frac{-\hat{\mathbf{I}} + 3\hat{\mathbf{J}} + 2\hat{\mathbf{K}}}{\sqrt{14}}. \text{ Ans.}$$

Q.No.3: Find the **directional derivatives** of $f(x,\ y,\ z) = xy^2 + yz^3$ at the point $(2,\ -1,\ 1)$ in the direction of vector $\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + 2\hat{\mathbf{K}}$.

Sol.: Given $f(x,\ y,\ z) = xy^2 + yz^3$

Then $\nabla f = \left[\hat{\mathbf{I}}\dfrac{\partial f}{\partial x} + \hat{\mathbf{J}}\dfrac{\partial f}{\partial y} + \hat{\mathbf{K}}\dfrac{\partial f}{\partial y}\right]$

$$= \hat{\mathbf{I}}\left(y^2\right) + \hat{\mathbf{J}}\left(2xy + z^3\right) + \hat{\mathbf{K}}\left(3yz^2\right) = \hat{\mathbf{I}} - 3\hat{\mathbf{J}} - 3\hat{\mathbf{K}} \text{ at the point } (2,\ -1,\ 1).$$

$\therefore$ Directional derivative of f in the direction $\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + 2\hat{\mathbf{K}} = \hat{\mathbf{N}}'.\nabla f = \dfrac{\mathbf{N}'}{|\mathbf{N}'|}.\nabla f$

$$= \frac{\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + 2\hat{\mathbf{K}}}{\sqrt{\left(1^2 + 2^2 + 2^2\right)}}.\left(\hat{\mathbf{I}} - 3\hat{\mathbf{J}} - 3\hat{\mathbf{K}}\right) = -3\frac{2}{3}. \text{ Ans.}$$

Q.No.4: Find the **directional derivative** of $\phi = x^2yz + 4xz^2$ at the point $(1,\ -2,\ -1)$ in the direction of the vector $2\hat{\mathbf{I}} - \hat{\mathbf{J}} - 2\hat{\mathbf{K}}$.

Sol.: Here $\nabla\phi = \hat{\mathbf{I}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{J}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{K}}\frac{\partial\phi}{\partial z}$

$$= \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(x^2yz + 4xz^2\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(x^2yz + 4xz^2\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(x^2yz + 4xz^2\right)$$

$$= \hat{\mathbf{I}}\left(2xyz + 4z^2\right) + \hat{\mathbf{J}}\left(x^2z\right) + \hat{\mathbf{K}}\left(x^2y + 8xz\right)$$

$$= 8\hat{\mathbf{I}} - \hat{\mathbf{J}} - 10\hat{\mathbf{K}} \qquad \text{at } (1,\ -2,\ -1)$$

Hence, the directional derivative of f in the direction of $2\hat{\mathbf{I}} - \hat{\mathbf{J}} - 2\hat{\mathbf{K}}$

$$= \hat{\mathbf{N}}'.\nabla f = \frac{\mathbf{N}'}{|\mathbf{N}'|}.\nabla f$$

$$= \nabla\phi.\hat{\mathbf{N}}' = \nabla\phi.\frac{\mathbf{N}'}{|\mathbf{N}'|} = \left(8\hat{\mathbf{I}} - \hat{\mathbf{J}} - 10\hat{\mathbf{K}}\right).\frac{2\hat{\mathbf{I}} - \hat{\mathbf{J}} - 2\hat{\mathbf{K}}}{\sqrt{4+1+4}} = \frac{16+1+20}{\sqrt{9}} = \frac{37}{3} = 12\frac{1}{3}. \text{ Ans.}$$

Q.No.5: What is the **directional derivative** of $\phi = xy^2 + yz^3$ at the point $(2,\ -1,\ 1)$ in the direction of normal the surface $x\log z - y^2 = -4$ at $(-1,\ 2,\ 1)$?

Sol.: The directional derivative of $\phi = xy^2 + yz^3$ in the direction of normal the surface $f = x\log z - y^2 + 4 = 0$ is $\nabla\phi.\hat{\mathbf{N}}' = \nabla\phi.\frac{\nabla f}{|\nabla f|}$,

where $\hat{\mathbf{N}}'$ is the unit vector normal to the surface f.

Given $\phi = xy^2 + yz^3$

Then $\nabla\phi = \hat{\mathbf{I}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{J}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{K}}\frac{\partial\phi}{\partial z}$

$$= \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(xy^2 + yz^3\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(xy^2 + yz^3\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(xy^2 + yz^3\right)$$

$$= \hat{\mathbf{I}}\left(y^2\right) + \hat{\mathbf{J}}\left(2xy + z^3\right) + \hat{\mathbf{K}}\left(3yz^2\right)$$

$$= \hat{\mathbf{I}} - 3\hat{\mathbf{J}} - 3\hat{\mathbf{K}} \qquad \text{at } (-1,\ 2,\ 1)$$

Also given $f = x\log z - y^2 + 4$

Then $\nabla f = \hat{\mathbf{I}}\frac{\partial f}{\partial x} + \hat{\mathbf{J}}\frac{\partial f}{\partial y} + \hat{\mathbf{K}}\frac{\partial f}{\partial z}$

$$= \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(x\log z - y^2 + 4\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(x\log z - y^2 + 4\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(x\log z - y^2 + 4\right)$$

$$= \hat{\mathbf{I}}(\log z) + \hat{\mathbf{I}}(-2y) + \hat{\mathbf{K}}\left(\frac{x}{2}\right)$$

$$= -4\hat{\mathbf{J}} - \hat{\mathbf{K}} \qquad \text{at } (-1,\ 2,\ 1)$$

Then the directional derivative of $\phi = xy^2 + yz^3$ in the direction of normal the surface $f = x\log z - y^2 + 4 = 0$ is

$$\nabla\phi.\hat{\mathbf{N}}' = \nabla\phi.\frac{\nabla f}{|\nabla f|} = \left(\hat{\mathbf{I}} - 3\hat{\mathbf{J}} - 3\hat{\mathbf{K}}\right).\frac{-4\hat{\mathbf{J}} - \hat{\mathbf{K}}}{\sqrt{16+1}} = \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}. \text{ Ans.}$$

Q.No.6: Find the **directional derivative** of $f(x,y,z) = 2xy + z^2$ at the point $(1,\ -1,\ 3)$ in the direction of the vector $\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + 2\check{\mathbf{K}}$**.**

Sol.: Here $\nabla f = \hat{\mathbf{I}}\frac{\partial f}{\partial x} + \hat{\mathbf{J}}\frac{\partial f}{\partial y} + \hat{\mathbf{K}}\frac{\partial f}{\partial z}$

$$= \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(2xy + z^2\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(2xy + z^2\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(2xy + z^2\right)$$

$$= \hat{\mathbf{I}}(2y) + \hat{\mathbf{J}}(2x) + \hat{\mathbf{K}}(2z)$$

$$= -2\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + 6\hat{\mathbf{K}} \qquad \text{at } (1,\ -1,\ 3)$$

Hence, the directional derivative of f in the direction of $\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + 2\hat{\mathbf{K}}$

$$= \nabla f.\hat{\mathbf{N}}' = \left(-2\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + 6\hat{\mathbf{K}}\right).\frac{\mathbf{N}'}{|\mathbf{N}'|}$$

$$= \left(-2\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + 6\hat{\mathbf{K}}\right).\frac{\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + 2\hat{\mathbf{K}}}{\sqrt{1+4+4}} = \frac{-2+4+12}{\sqrt{9}} = \frac{14}{3} = 4\frac{2}{3}. \text{Ans.}$$

Q.No.7: In what direction from $(3,\ 1,\ -2)$ is the **directional derivative** of $\phi = x^2y^2z^4$ maximum ? Find also the magnitude of this maximum**.**

Sol.: Given $\phi = x^2y^2z^4$

Then $\nabla\phi = \hat{\mathbf{I}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{J}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{K}}\frac{\partial\phi}{\partial z} = \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(x^2y^2z^4\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(x^2y^2z^4\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(x^2y^2z^4\right)$

$= \hat{\mathbf{I}}\left(2xy^2z^4\right) + \hat{\mathbf{J}}\left(2yx^2z^4\right) + \hat{\mathbf{K}}\left(4z^3x^2y^2\right)$

$= 96\hat{\mathbf{I}} + 288\hat{\mathbf{J}} - 288\hat{\mathbf{K}}$ at $(3, 1, -2)$

$= 96\left(\hat{\mathbf{I}} + 3\hat{\mathbf{J}} - 3\hat{\mathbf{K}}\right)$. Ans.

Thus, the directional derivative of $\phi = x^2y^2z^4$ is maximum in the direction of $96\left(\hat{\mathbf{I}} + 3\hat{\mathbf{J}} - 3\hat{\mathbf{K}}\right)$ from the point $(3, 1, -2)$.

IInd Part: Since we know that the directional derivative of $\phi = x^2y^2z^4$ is maximum along its normal, so its magnitude is $|\nabla\phi| = 96\sqrt{1+9+9} = 96\sqrt{19}$. Ans.

Q.No.8: Prove that $\nabla r^n = nr^{n-2}\mathbf{R}$, where $\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}$.

Sol.: We have $f(x, y, z) = r^n = \left(x^2 + y^2 + z^2\right)^{n/2}$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial\left(r^n\right)}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{n/2} = \frac{n}{2}\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}-1}.2x = n\,x\,r^{n-2}.$$

Similarly $\frac{\partial f}{\partial y} = ny\,r^{n-2}$ and $\frac{\partial f}{\partial z} = nz\,r^{n-2}$

Thus $\nabla r^n = \mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z} = nr^{n-2}\left(x\mathbf{I} + y\mathbf{J} + z\mathbf{K}\right) = nr^{n-2}\mathbf{R}$. Ans.

Q.No.9: **(a)** Find $\nabla\phi$, if $\phi = \log\left(x^2 + y^2 + z^2\right)$, **(b)** Show that $\text{grad}\left(\frac{1}{r}\right) = -\frac{\mathbf{R}}{r^3}$.

Sol.: **(a)** Given $\phi = \log\left(x^2 + y^2 + z^2\right)$, then

$$\nabla\phi = \hat{\mathbf{I}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{J}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{K}}\frac{\partial\phi}{\partial z}$$

$$= \hat{\mathbf{I}}\frac{\partial}{\partial x}\log\left(x^2+y^2+z^2\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\log\left(x^2+y^2+z^2\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\log\left(x^2+y^2+z^2\right)$$

$$= \hat{\mathbf{I}}\frac{2x}{x^2+y^2+z^2} + \hat{\mathbf{J}}\frac{2y}{x^2+y^2+z^2} + \hat{\mathbf{K}}\frac{2z}{x^2+y^2+z^2}$$

$$= 2\frac{\left(\hat{\mathbf{I}}\,x + \hat{\mathbf{J}}\,y + \hat{\mathbf{K}}\,z\right)}{x^2+y^2+z^2}. \text{ Ans.}$$

(b) Now since $\mathbf{R} = \hat{\mathbf{I}}\,x + \hat{\mathbf{J}}\,y + \hat{\mathbf{K}}\,z$, then $r^2 = x^2+y^2+z^2 \Rightarrow \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$

Then $\text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$$= \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{-\hat{\mathbf{I}}\,x - \hat{\mathbf{J}}\,y - \hat{\mathbf{K}}\,z}{\left(x^2+y^2+z^2\right)^{3/2}} = -\frac{\hat{\mathbf{I}}\,x + \hat{\mathbf{J}}\,y + \hat{\mathbf{K}}\,z}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = -\frac{\mathbf{R}}{r^3}$$

This completes the proof.

Q.No.10: Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ is orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1,\ -1,\ 2)$.

Sol.: Let the given surfaces be f_1 and f_2 , therefore

$f_1 = ax^2 - byz - (a+2)x$ and $f_2 = 4x^2y + z^3 - 4$

$$\nabla f_1 = \hat{\mathbf{I}}\frac{\partial f_1}{\partial x} + \hat{\mathbf{J}}\frac{\partial f_1}{\partial y} + \hat{\mathbf{K}}\frac{\partial f_1}{\partial z}$$

$$= \hat{\mathbf{I}}\frac{\partial}{\partial x}\left[ax^2 - byz - (a+2)x\right] + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left[ax^2 - byz - (a+2)x\right] + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left[ax^2 - byz - (a+2)x\right]$$

$$= \hat{\mathbf{I}}[2ax - (a+2)] - \hat{\mathbf{J}}(bz) - \hat{\mathbf{K}}(by)$$

Since, we know that unit vector normal to the surface f_1 is $\hat{\mathbf{N}} = \frac{\nabla f_1}{|\nabla f_1|}$.

Now ∇f_1 at the point $(1,\ -1,\ 2)$ is $(a-2)\hat{\mathbf{I}}-2b\hat{\mathbf{J}}+b\hat{\mathbf{K}}$

Hence, the desired unit vector normal to the surface is

$$\hat{\mathbf{N}}_1=\frac{\nabla f_1}{|\nabla f_1|}=\frac{(a-2)\hat{\mathbf{I}}-2b\hat{\mathbf{J}}+b\hat{\mathbf{K}}}{\sqrt{(a-2)^2+4b^2+b^2}}=\frac{(a-2)\hat{\mathbf{I}}-2b\hat{\mathbf{J}}+b\hat{\mathbf{K}}}{\sqrt{(a-2)^2+5b^2}}$$

Similarly,

$$\nabla f_2=\hat{\mathbf{I}}\frac{\partial f_2}{\partial x}+\hat{\mathbf{J}}\frac{\partial f_2}{\partial y}+\hat{\mathbf{K}}\frac{\partial f_2}{\partial z}$$

$$=\hat{\mathbf{I}}\frac{\partial}{\partial x}\left(4x^2y+z^3-4\right)+\hat{\mathbf{J}}\frac{\partial}{\partial y}\left(4x^2y+z^3-4\right)+\hat{\mathbf{K}}\frac{\partial}{\partial z}\left(4x^2y+z^3-4\right)$$

$$=\hat{\mathbf{I}}(8xy)+\hat{\mathbf{J}}\left(4x^2\right)+\hat{\mathbf{K}}\left(3z^2\right)$$

Now ∇f_{21} at the point $(1,\ -1,\ 2)$ is $-8\hat{\mathbf{I}}+4\hat{\mathbf{J}}+12\hat{\mathbf{K}}$

Hence, the desired unit vector normal to the surface is

$$\hat{\mathbf{N}}_2=\frac{\nabla f_2}{|\nabla f_2|}=\frac{-8\hat{\mathbf{I}}+4\hat{\mathbf{J}}+12\hat{\mathbf{K}}}{\sqrt{(-8)^2+4^2+12^2}}=\frac{-8\hat{\mathbf{I}}+4\hat{\mathbf{J}}+12\hat{\mathbf{K}}}{\sqrt{224}}=\frac{-2\hat{\mathbf{I}}+\hat{\mathbf{J}}+3\hat{\mathbf{K}}}{\sqrt{14}}.$$

Since the surfaces are orthogonal to each other. Hence $\hat{\mathbf{N}}_1.\hat{\mathbf{N}}_2=0$.

$$\Rightarrow \frac{(a-2)\hat{\mathbf{I}}-2b\hat{\mathbf{J}}+b\hat{\mathbf{K}}}{\sqrt{(a-2)^2+5b^2}}.\frac{-2\hat{\mathbf{I}}+\hat{\mathbf{J}}+3\hat{\mathbf{K}}}{\sqrt{14}}=0$$

$$\Rightarrow \frac{(a-2)(-2)}{\sqrt{(a-2)^2+5b^2}\sqrt{14}}-\frac{2b(1)}{\sqrt{(a-2)^2+5b^2}\sqrt{14}}+\frac{b(3)}{\sqrt{(a-2)^2+5b^2}\sqrt{14}}=0$$

$$\Rightarrow -2a+4-2b+3b=0 \Rightarrow b-2a+4=0 \Rightarrow 2a-b=4$$

$$\Rightarrow b=2a-4$$

Putting this value of b in f_1, we get

$$f_1=ax^2-byz-(a+2)x=ax^2-(2a-4)yz-(a+2)x=0$$

$$\Rightarrow a.(1)^2-(2a-4)(-1).2-(a+2).1=0 \quad \text{at } (1,\ -1,\ 2)$$

$$\Rightarrow a-(-4a+8)-(a+2)=0$$

$\Rightarrow 4a - 10 = 0$

$\Rightarrow a = \frac{10}{4} = 2.5$ Ans.

And $b = 2a - 4 = (2 \times 2.5) - 4 = 5 - 4 = 1$. Ans.

Hence the values of a and b are $a = 2.5$ and $b = 1$.

Q.No.11: What is the greatest rate of increase of $u = x^2 + yz^2$ at the point $(1, -1, 3)$?

Sol.: Since we know that ∇u gives the greatest rate of change of a scalar point function u.

$$\text{Then } \nabla u = \hat{\mathbf{I}}\frac{\partial u}{\partial x} + \hat{\mathbf{J}}\frac{\partial u}{\partial y} + \hat{\mathbf{K}}\frac{\partial u}{\partial z} = \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(x^2 + yz^2\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(x^2 + yz^2\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(x^2 + yz^2\right)$$

$$= \hat{\mathbf{I}}(2x) + \hat{\mathbf{J}}\left(z^2\right) + \hat{\mathbf{K}}(2yz)$$

$$= 2\hat{\mathbf{I}} + 9\hat{\mathbf{J}} - 6\hat{\mathbf{K}} \qquad \text{at } (1, -1, 3)$$

Thus $\nabla u = 2\hat{\mathbf{I}} + 9\hat{\mathbf{J}} - 6\hat{\mathbf{K}}$ is the greatest rate of increase of $u = x^2 + yz^2$ at the point $(1, -1, 3)$.

Q.No.12: The temperature of points in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at $(1, 1, 2)$ desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move ?

Sol.: The mosquito wants to move to the region where the temperature is maximum as soon as possible. So, it will transverse its direction where the directional derivative of temperature is maximum i.e. the direction normal to the isothermal surface on which it is situated. As we know $\left(\frac{\partial T}{\partial r}\right)_{max} = \nabla T$, which is normal to the surface.

$$\text{Then } \nabla T = \hat{\mathbf{I}}\frac{\partial T}{\partial x} + \hat{\mathbf{J}}\frac{\partial T}{\partial y} + \hat{\mathbf{K}}\frac{\partial T}{\partial z} = \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(x^2 + y^2 - z\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(x^2 + y^2 - z\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(x^2 + y^2 - z\right)$$

$$= \hat{\mathbf{I}}(2x) + \hat{\mathbf{J}}(2y) - \hat{\mathbf{K}}(1) = 2\hat{\mathbf{I}} + 2\hat{\mathbf{J}} - \hat{\mathbf{K}} \qquad \text{at } (1, 1, 2)$$

Thus unit vector normal gives the direction in which the mosquito ought to move is

$$\hat{\mathbf{N}} = \frac{\nabla T}{|\nabla T|} = \frac{2\hat{\mathbf{I}} + 2\hat{\mathbf{J}} - \hat{\mathbf{K}}}{\sqrt{4+4+1}} = \frac{1}{3}\left(2\hat{\mathbf{I}} + 2\hat{\mathbf{J}} - \hat{\mathbf{K}}\right). \text{ Ans.}$$

Q.No.13: Calculate the **angle** between the normals to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$.

Sol.: Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be any vectors normal to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$ respectively.

Now ∇f gives the vector normal to the surface $f = xy - z^2 = 0$

$$\text{Then } \nabla f = \hat{\mathbf{I}}\frac{\partial f}{\partial x} + \hat{\mathbf{J}}\frac{\partial f}{\partial y} + \hat{\mathbf{K}}\frac{\partial f}{\partial z} = \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(xy - z^2\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(xy - z^2\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(xy - z^2\right)$$

$$= \hat{\mathbf{I}}(y) + \hat{\mathbf{J}}(x) - \hat{\mathbf{K}}(2z)$$

$$\Rightarrow \mathbf{N}_1 = \hat{\mathbf{I}} + 4\hat{\mathbf{J}} - 4\hat{\mathbf{K}} \qquad \text{at } (4, 1, 2)$$

$$\text{and} \quad \mathbf{N}_2 = 3\hat{\mathbf{I}} + 3\hat{\mathbf{J}} + 6\hat{\mathbf{K}} \qquad \text{at } (3, 3, -3)$$

$$\text{Also } |\mathbf{N}_1| = \sqrt{1+16+16} = \sqrt{33}, \text{ and } |\mathbf{N}_2| = \sqrt{9+9+36} = \sqrt{54} = 3\sqrt{6}$$

$$\therefore \cos\theta = \frac{\mathbf{N}_1.\mathbf{N}_2}{|\mathbf{N}_1|.|\mathbf{N}_2|} = \frac{3+12-24}{3\sqrt{6}.\sqrt{33}} = \frac{-9}{9\sqrt{22}} = -\frac{1}{\sqrt{22}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-1}{\sqrt{22}}\right). \text{ Ans.}$$

Q.No.14: Find the **angle** between the tangent planes to the surfaces $x\log z = y^2 - 1$, $x^2y = 2 - z$ at the point $(1, 1, 1)$.

Sol.: Let $f = x\log z - y^2 + 1 = 0$ and $g = x^2y + z - 2 = 0$ be two surfaces.

$$\text{Then } \nabla f = \hat{\mathbf{I}}\frac{\partial f}{\partial x} + \hat{\mathbf{J}}\frac{\partial f}{\partial y} + \hat{\mathbf{K}}\frac{\partial f}{\partial z}$$

$$= \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(x\log z - y^2 + 1\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(x\log z - y^2 + 1\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(x\log z - y^2 + 1\right)$$

$$= \hat{\mathbf{I}}(\log z) - \hat{\mathbf{J}}(2y) + \hat{\mathbf{K}}\left(\frac{x}{z}\right)$$

$$\Rightarrow \mathbf{N}_1 = -2\hat{\mathbf{J}} + \hat{\mathbf{K}} \qquad \text{at } (1, 1, 1)$$

Also, $\nabla g = \hat{\mathbf{I}}\frac{\partial g}{\partial x} + \hat{\mathbf{J}}\frac{\partial g}{\partial y} + \hat{\mathbf{K}}\frac{\partial g}{\partial z} = \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(x^2y+z-2\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(x^2y+z-2\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(x^2y+z-2\right)$

$= \hat{\mathbf{I}}(2xy) + \hat{\mathbf{J}}\left(x^2\right) + \hat{\mathbf{K}}(1)$

$\Rightarrow \mathbf{N}_2 = 2\hat{\mathbf{I}} + \hat{\mathbf{J}} + \hat{\mathbf{K}}$ at $(1,\ 1,\ 1)$

where $\mathbf{N}_1$ and $\mathbf{N}_2$ be normals of the surfaces f and g, respectively.

Now angle between the two tangents planes = angle between their normals

$$\therefore \cos\theta = \frac{\mathbf{N}_1.\mathbf{N}_2}{|\mathbf{N}_1|.|\mathbf{N}_2|} = \frac{-2.1+1.1}{\sqrt{4+1}.\sqrt{4+1+1}} = \frac{-1}{\sqrt{5}\sqrt{6}} = -\frac{1}{\sqrt{30}}$$

$\Rightarrow \theta = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)$. Ans.

Q.No.15: Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at $(2,-1,\ 2)$.

Sol.: Let $f = x^2+y^2+z^2-9=0$ and $g = x^2+y^2-z-3=0$ be two surfaces.

Then $\nabla f = \hat{\mathbf{I}}\frac{\partial f}{\partial x} + \hat{\mathbf{J}}\frac{\partial f}{\partial y} + \hat{\mathbf{K}}\frac{\partial f}{\partial z}$

$= \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(x^2+y^2+z^2-9\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(x^2+y^2+z^2-9\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(x^2+y^2+z^2-9\right)$

$= \hat{\mathbf{I}}(2x) + \hat{\mathbf{J}}(2y) + \hat{\mathbf{K}}(2z)$

$\Rightarrow \mathbf{N}_1 = 4\hat{\mathbf{I}} - 2\hat{\mathbf{J}} + 4\hat{\mathbf{K}}$ at $(2,-1,\ 2)$

Also, $\nabla g = \hat{\mathbf{I}}\frac{\partial g}{\partial x} + \hat{\mathbf{J}}\frac{\partial g}{\partial y} + \hat{\mathbf{K}}\frac{\partial g}{\partial z}$

$= \hat{\mathbf{I}}\frac{\partial}{\partial x}\left(x^2+y^2-z-3\right) + \hat{\mathbf{J}}\frac{\partial}{\partial y}\left(x^2+y^2-z-3\right) + \hat{\mathbf{K}}\frac{\partial}{\partial z}\left(x^2+y^2-z-3\right)$

$= \hat{\mathbf{I}}(2x) + \hat{\mathbf{J}}(2y) - \hat{\mathbf{K}}(1)$

$\Rightarrow \mathbf{N}_2 = 4\hat{\mathbf{I}} - 2\hat{\mathbf{J}} - \hat{\mathbf{K}}$ at $(2,\ -1,\ 2)$

where $\mathbf{N}_1$ and $\mathbf{N}_2$ be normals of the surfaces f and g, respectively.

Now angle between the two surfaces = angle between their normals

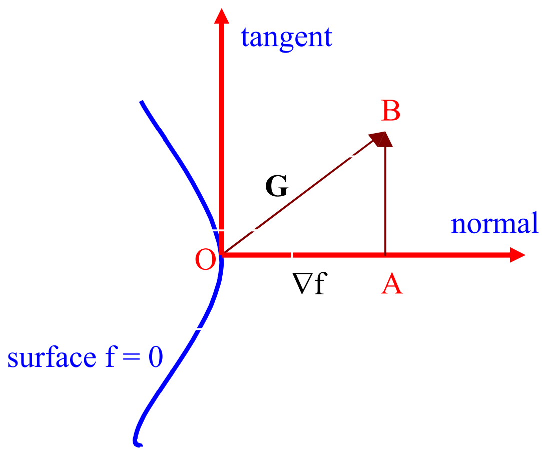
$$\therefore\ \cos\theta=\frac{\mathbf{N}_1.\mathbf{N}_2}{|\mathbf{N}_1|.|\mathbf{N}_2|}=\frac{4.4+(-2).(-2)+4.(-1)}{\sqrt{16+4+16}.\sqrt{16+4+1}}=\frac{16}{\sqrt{36}\sqrt{21}}=\frac{16}{6\sqrt{21}}=\frac{8}{3\sqrt{21}}=\frac{8\sqrt{21}}{63}$$

$$\Rightarrow \theta=\cos^{-1}\left(\frac{8\sqrt{21}}{63}\right).\ \text{Ans.}$$

Q.No.16: If f and **G** are point functions, prove that the components of the latter normal and tangential to the surface $f=0$ are

$$\frac{(\mathbf{G}.\nabla f)\nabla f}{(\nabla f)^2}\ \text{and}\ \frac{\nabla f\times(\mathbf{G}\times\nabla f)}{(\nabla f)^2}$$

Sol.:



Normal component of $\mathbf{G}$ = OA (unit vector along ∇f)

$$=\left(\mathbf{G}.\hat{\nabla f}\right)\hat{\nabla f}=\left(\mathbf{G}.\frac{\nabla f}{|\nabla f|}\right)\frac{\nabla f}{|\nabla f|}$$

$$=\frac{(\mathbf{G}.\nabla f)\nabla f}{(\nabla f)^2}\qquad\left[\because |\nabla f|^2=(\nabla f)^2\right]$$

Tangential component of $\mathbf{G}=\ \mathbf{AB}=\mathbf{OB}-\mathbf{OA}=\mathbf{G}-\dfrac{(\mathbf{G}.\nabla f)\nabla f}{(\nabla f)^2}$

$$=\frac{(\nabla f.\nabla f)\mathbf{G}-(\mathbf{G}.\nabla f)\nabla f}{\nabla f.\nabla f}=\frac{\nabla f\times(\mathbf{G}\times\nabla f)}{(\nabla f)^2}$$

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Home Assignments

1–6 CALCULATION OF GRADIENTS

Find ∇f. Graph some level curves $f = const.$ Indicate ∇f by arrows at some points of these curves.

1. $f = x^2 + y^2$
2. $f = x^2 + \frac{1}{9}y^2$
3. $f = \dfrac{x}{y}$
4. $f = x^4 + y^4$
5. $f = (x - 2)(y + 2)$
6. $f = (x - 3)^2 + (y - 1)^2$

7–12 USE OF GRADIENTS. VELOCITY FIELDS

Given the velocity potential f of a flow, find the velocity $\mathbf{v} = \nabla f$ of the flow and its value at P. Make a sketch of $\mathbf{v}(P)$.

7. $f = x^2 + y^2 + z^2$, P: $(3, 2, 2)$
8. $f = \ln(x^2 + y^2)$, P: $(4, 3)$
9. $f = \cos x \cosh y$, P: $(\frac{1}{2}\pi, \ln 2)$
10. $f = x^2 + 4y^2 + 9z^2$, P: $(3, 2, 1)$
11. $f = e^x \sin y$, P: $(1, \pi)$
12. $f = (x^2 + y^2 + z^2)^{-1/2}$, P: $(2, 1, 2)$

13–18 HEAT FLOW

Experiments show that in a temperature field, heat flows in the direction of maximum decrease of temperature T. Find this direction in general and at a given point P. Sketch that direction at P as an arrow.

13. $T = x^2 - y^2$, P: $(2, 1)$
14. $T = \arctan \dfrac{y}{x}$, P: $(2, 2)$
15. $T = x^3 - 3xy^2$, P: $(\sqrt{8}, \sqrt{2})$
16. $T = x/(x^2 + y^2)$, P: $(4, 0)$
17. $T = 3x^2y - y^3$, P: $(4, -2)$
18. $T = \sin x \cosh y$, P: $(\frac{1}{4}\pi, \ln 5)$

19–24 ELECTRIC FORCE

The force in an electrostatic field $f(x, y, z)$ has the direction of the gradient of f. Find ∇f and its value at P.

19. $f = (x-1)^2 - (y+1)^2$, P: $(4, -3)$

20. $f = y/(x^2 + y^2)$, P: $(5, 3)$

21. $f = x^2 - 2x - y^2$, P: $(-2, 6)$

22. $f = \ln(x^2 + y^2)$, P: $(3, 3)$

23. $f = (x^2 + y^2 + z^2)^{-1/2}$, P: $(12, 0, 16)$

24. $f = x^2y - \frac{1}{3}y^3$, P: $(2, 3)$

25. **(Gradient)** What does it mean if $|\text{grad } f(P)| < |\text{grad } f(Q)|$ at two points P and Q in a scalar field?

26. **(Landscape)** If $z(x, y) = 2000 - 4x^2 - y^2$ [meters] gives the elevation of a mountain above sea level, what is the direction of steepest ascent at P: $(3, -6)$? What does the mountain look like?

27–32 SURFACE NORMAL

Find a normal vector of the surface at the given point P.

27. $ax + by + cz = d$, any P

28. $x^2 + 3y^2 + z^2 = 28$, P: $(4, 1, 3)$

29. $x^2 + y^2 = 25$, P: $(4, 3, 8)$

30. $x^2 - y^2 + 4z^2 = 67$, P: $(-2, 1, 4)$

31. $x^4 + y^4 + z^4 = 243$, P: $(3, 3, 3)$

32. $z = x^2 + y^2$, P: $(3, 4, 25)$

33–38 DIRECTIONAL DERIVATIVE

Find the directional derivative of f at P in the direction of $\mathbf{a}$.

33. $f = x^2 + y^2 - z$, P: $(1, 1, -2)$, $\mathbf{a} = [1, 1, 2]$

34. $f = x^2 + y^2 + z^2$, P: $(2, -2, 1)$, $\mathbf{a} = [-1, -1, 0]$

35. $f = xyz$, P: $(-1, 1, 3)$, $\mathbf{a} = [1, -2, 2]$

36. $f = (x^2 + y^2 + z^2)^{-1/2}$, P: $(4, 2, -4)$, $\mathbf{a} = [1, 2, -2]$

37. $f = e^x \sin y$, P: $(2, \frac{1}{2}\pi, 0)$, $\mathbf{a} = [2, 3, 0]$

38. $f = 4x^2 + y^2 + 9z^2$, P: $(2, 4, 0)$, $\mathbf{a} = [-2, -4, 3]$

39–41 POTENTIALS

for a given vector field—if they exist!—can be obtained by a method to be discussed in Sec. 9.9. In simpler cases, use inspection. Find a potential $f = \text{grad } \mathbf{v}$ for given $\mathbf{v}(x, y, z)$.

39. $\mathbf{v} = [3x, \quad 5y, \quad -4z]$

40. $\mathbf{v} = [ye^x, e^x, 2z]$

41. $\mathbf{v} = [4x^3, 3y^2, -6z]$

42. Project. Useful Formulas for Gradients and Laplacians. Prove the following formulas and give for each of them two examples showing when they are advantageous.

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(f^n) = nf^{n-1}\nabla f$$

$$\nabla(f/g) = (1/g^2)(g\nabla f - f\nabla g)$$

$$\nabla^2(fg) = g\nabla^2 f + 2\nabla f \cdot \nabla g + f\nabla^2 g$$

43. CAS PROJECT. Equipotential Curves. Graph some **isotherms** (curves of constant temperature) and indicate directions of heat flow by arrows when the temperature $T(x, y)$ equals:

(a) $x^3 - 3xy^2$ (b) $\sin x \sinh y$ (c) $e^x \sin y$.

4^{th} Topic

Vector Calculus

Del applied to vector point function (Divergence, Curl)

Physical interpretation of divergence

Physical interpretation of curl Irrotational motion

Del Applied Twice to Point Function

Del applied to products of point functions

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Del applied to vector point function:

Divergence:

The divergence of a continuously differentiable vector point function $\mathbf{F}$ is denoted by div $\mathbf{F}$ and is defined by the equation

$$\operatorname{div}\mathbf{F} = \nabla.\mathbf{F} = \hat{\mathbf{I}}.\frac{\partial \mathbf{F}}{\partial x} + \hat{\mathbf{J}}.\frac{\partial \mathbf{F}}{\partial y} + \hat{\mathbf{K}}.\frac{\partial \mathbf{F}}{\partial z} = \sum \hat{\mathbf{I}}.\frac{\partial \mathbf{F}}{\partial x}.$$

Let $\mathbf{F} = f\,\hat{\mathbf{I}} + \phi\,\hat{\mathbf{J}} + \psi\,\hat{\mathbf{K}}$.

Then $\operatorname{div}\mathbf{F} = \nabla.\mathbf{F} = \left[\hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}\right].\left[f\,\hat{\mathbf{I}} + \phi\,\hat{\mathbf{J}} + \psi\,\hat{\mathbf{K}}\right] = \frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}$.

This means $\nabla.\mathbf{F}$ is a scalar function.

Curl:

The curl of a continuously differentiable vector point function **F** is defined by the equation $\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \hat{\mathbf{I}} \times \frac{\partial \mathbf{F}}{\partial x} + \hat{\mathbf{J}} \times \frac{\partial \mathbf{F}}{\partial y} + \hat{\mathbf{K}} \times \frac{\partial \mathbf{F}}{\partial z} = \sum \hat{\mathbf{I}} \times \frac{\partial \mathbf{F}}{\partial x}$.

Let $\mathbf{F} = f\,\hat{\mathbf{I}} + \phi\,\hat{\mathbf{J}} + \psi\,\hat{\mathbf{K}}$.

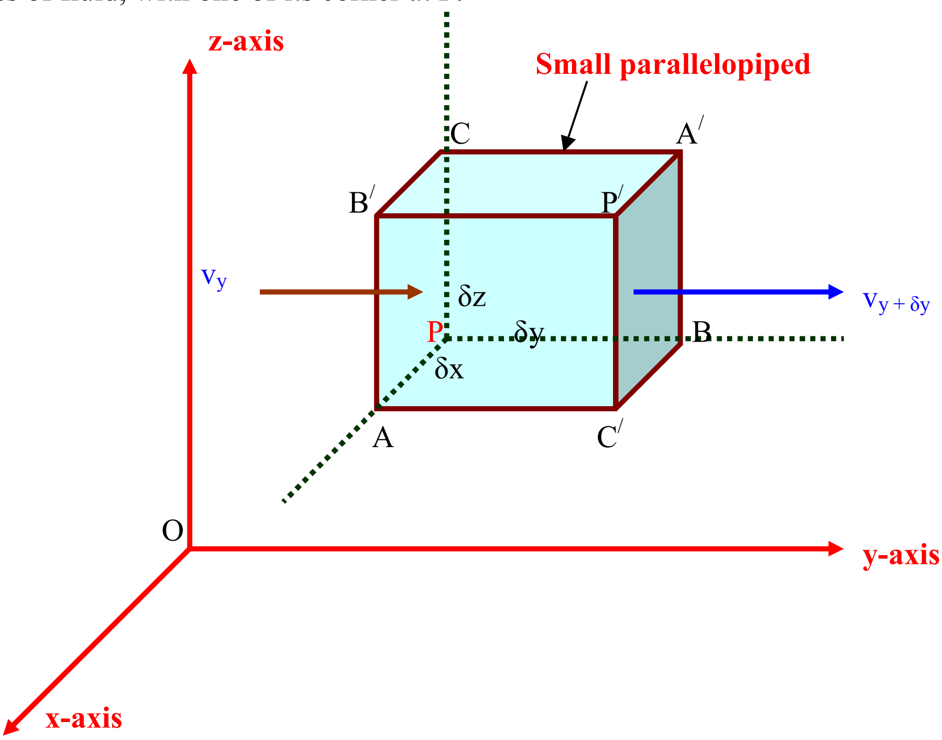
Then $\text{curl}\mathbf{F} = \nabla \times \mathbf{F} = \left[\hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}\right] \times \left[f\,\hat{\mathbf{I}} + \phi\,\hat{\mathbf{J}} + \psi\,\hat{\mathbf{K}}\right]$

$$= \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & \phi & \psi \end{vmatrix} = \hat{\mathbf{I}}\left(\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial z}\right) + \hat{\mathbf{J}}\left(\frac{\partial f}{\partial z} - \frac{\partial \psi}{\partial x}\right) + \hat{\mathbf{K}}\left(\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial y}\right).$$

Physical interpretation of divergence: (div V)

Consider the motion of the fluid having velocity $\mathbf{V} = v_x\,\hat{\mathbf{I}} + v_y\,\hat{\mathbf{J}} + v_z\,\hat{\mathbf{K}}$ at a point $\mathbf{P}(x, y, z)$.

Then consider a small parallelopiped with edges δx, δy, δz parallel to the axes, in the mass of fluid, with one of its corner at P.



$\therefore$ The amount of fluid **entering** the face PB' in unit time $= v_y \delta z \delta x$

and the amount of fluid **leaving** the face $P'B$ in unit time $= v_{y+\delta y} \delta z \delta x$

$$= \left[v_y + \frac{\partial v_y}{\partial y} \delta y \right] \delta z \, \delta x \quad \text{(nearly)}$$

Therefore, the **net decrease** of the amount of fluid due to flow across these two faces

$$= \frac{\partial v_y}{\partial y} \delta x \, \delta y \, \delta z .$$

Similarly, we can find the contributions of other two pairs of faces.

i.e. the contributions of other two pairs of faces are $\frac{\partial v_x}{\partial x} \delta x \ \delta y \ \delta z$ and $\frac{\partial v_z}{\partial z} \delta x \ \delta y \ \delta z$.

Then, the **total decrease** of amount of fluid inside the parallelopiped per unit time

$$= \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right] \delta x \, \delta y \, \delta z .$$

Thus, the rate of loss of fluid **per unit volume** in unit time $= \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right] = \operatorname{div} \mathbf{V}$.

Hence, if $\mathbf{V}$ is the **velocity of fluid**, then $\operatorname{div} \mathbf{V}$ gives the rate at which fluid is originating at a point per unit volume in unit time.

Similarly, if $\mathbf{V}$ represents an **electric flux**, then $\operatorname{div} \mathbf{V}$ is the amount of flux which **diverges** per unit volume in unit time.

If $\mathbf{V}$ represents **heat flux**, then $\operatorname{div} \mathbf{V}$ is the rate at which heat is **issuing** from a point per unit volume.

In general, ***the divergence of a vector point function representing any physical quantity gives at each point, the rate per unit volume at which the physical quantity is issuing from that point.*** **This explains the justification for the name divergence of vector point function.**

If the fluid is incompressible fluid, then there can be no gain or no loss in the volume element. Hence, $\operatorname{div} \mathbf{V} = 0$.

This is known as the **equation of continuity** for incompressible fluid in hydrodynamics.

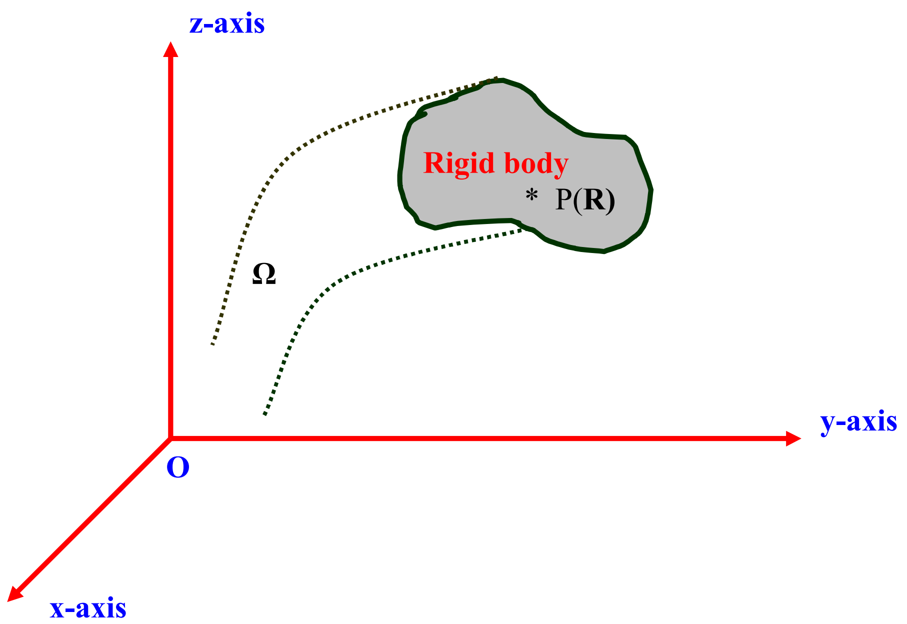
From this discussion, we should conclude and remember that, roughly speaking, ***the divergence measures outflow minus inflow***.

Solenoidal vector function:

Definition: If the flux entering any element of space is the same as that leaving it, i.e. $\text{div}\,\mathbf{V} = 0$, everywhere, then such a point function is called **solenoidal vector function**.

Physical interpretation of curl:

Consider the motion of a rigid body rotating about a fixed axis through O. If $\mathbf{\Omega}$ be its angular velocity, then the velocity $\mathbf{V}$ of any particle $P(\mathbf{R})$ of the body is given by $\mathbf{V} = \mathbf{\Omega} \times \mathbf{R}$.



Let $\mathbf{\Omega} = \omega_1 \hat{\mathbf{I}} + \omega_2 \hat{\mathbf{J}} + \omega_3 \hat{\mathbf{K}}$ and $\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}$.

Then $\mathbf{V} = \mathbf{\Omega} \times \mathbf{R} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \hat{\mathbf{I}}(\omega_2 z - \omega_3 y) + \hat{\mathbf{J}}(\omega_3 x - \omega_1 z) + \hat{\mathbf{K}}(\omega_1 y - \omega_2 x)$.

Taking curl on both sides, we get

$$\text{curl}\,\mathbf{V} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y, & \omega_3 x - \omega_1 z, & \omega_1 y - \omega_2 x \end{vmatrix} = \hat{\mathbf{I}}(\omega_1 + \omega_1) + \hat{\mathbf{J}}(\omega_2 + \omega_2) + \hat{\mathbf{K}}(\omega_3 + \omega_3).$$

$$= 2\left(\omega_1 \hat{\mathbf{I}} + \omega_2 \hat{\mathbf{J}} + \omega_3 \hat{\mathbf{K}}\right) = 2\mathbf{\Omega}.$$

Hence, $\mathbf{\Omega} = \frac{1}{2}\,\text{curl}\,\mathbf{V}$.

Thus, the angular velocity of rotation at any point is equal to half the curl of the velocity vector, which justifies the name rotation used for curl.

In general, *the curl of any vector point function gives the measure of the angular velocity at any point of the vector field.*

Irrotational motion:

Definition: Any motion in which the curl of the velocity vector is zero is said to be irrotational, otherwise, rotational.

Del Applied Twice to Point Function:

∇f and $\nabla \times \mathbf{F}$ being vector point functions, we can form their divergence and curl, whereas $\nabla.\mathbf{F}$ being a scalar point function, we can have its gradient only.

Thus we have the following five formulae:

(1) div grad $f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

(2) curl grad $f = \nabla \times \nabla f = \mathbf{0}$

(3) div curl $\mathbf{F} = \nabla.\nabla \times \mathbf{F} = 0$

(4) curl curl $\mathbf{F}$ = grad div $\mathbf{F} - \nabla^2 \mathbf{F}$ i.e. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla.\mathbf{F}) - \nabla^2 \mathbf{F}$

(5) grad div $\mathbf{F}$ = curl curl $\mathbf{F} + \nabla^2 \mathbf{F}$ i.e. $\nabla(\nabla.\mathbf{F}) = \nabla \times (\nabla \times \mathbf{F}) + \nabla^2 \mathbf{F}$

Proofs:

(1) Prove that div grad f $= \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

Proof: Here $\nabla^2 f = \nabla.\nabla f = \nabla.\left(\hat{\mathbf{I}}\frac{\partial f}{\partial x} + \hat{\mathbf{J}}\frac{\partial f}{\partial y} + \hat{\mathbf{K}}\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operator and $\nabla^2 f = 0$ is called the Laplace's equation.

(2) Prove that curl grad f $= \nabla \times \nabla f = \mathbf{0}$.

Proof: Here $\nabla \times \nabla f = \nabla \times \left(\hat{\mathbf{I}}\frac{\partial f}{\partial x} + \hat{\mathbf{J}}\frac{\partial f}{\partial y} + \hat{\mathbf{K}}\frac{\partial f}{\partial z}\right)$

$$= \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{I}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \sum \hat{\mathbf{I}}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) = \mathbf{0}.$$

(3). Prove that div curl $\mathbf{F} = \nabla.\nabla \times \mathbf{F} = 0$.

Proof: $\nabla.\nabla \times \mathbf{F} = \left(\sum \hat{\mathbf{I}}\frac{\partial}{\partial x}\right).\left(\hat{\mathbf{I}} \times \frac{\partial \mathbf{F}}{\partial x} + \hat{\mathbf{J}} \times \frac{\partial \mathbf{F}}{\partial y} + \hat{\mathbf{K}} \times \frac{\partial \mathbf{F}}{\partial z}\right)$

$$= \sum \hat{\mathbf{I}}\left(\hat{\mathbf{I}} \times \frac{\partial^2 \mathbf{F}}{\partial x^2} + \hat{\mathbf{J}} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial y} + \hat{\mathbf{K}} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)$$

$$= \sum \left(\hat{\mathbf{I}} \times \hat{\mathbf{I}}.\frac{\partial^2 \mathbf{F}}{\partial x^2} + \hat{\mathbf{I}} \times \hat{\mathbf{J}}\frac{\partial^2 \mathbf{F}}{\partial x \partial y} + \hat{\mathbf{I}} \times \hat{\mathbf{K}}.\frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)$$

$$= \sum \left(\hat{\mathbf{K}}.\frac{\partial^2 \mathbf{F}}{\partial x \partial y} - \hat{\mathbf{J}}\frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right) = 0.$$

(4). Prove that curl curl $\mathbf{F}$ = grad div $\mathbf{F} - \nabla^2 \mathbf{F}$ i. e. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla.\mathbf{F}) - \nabla^2 \mathbf{F}$.

Proof: $\nabla\times(\nabla\times\mathbf{F})=\left(\sum \hat{\mathbf{I}}\frac{\partial}{\partial x}\right)\times\left(\hat{\mathbf{I}}\times\frac{\partial \mathbf{F}}{\partial x}+\hat{\mathbf{J}}\times\frac{\partial \mathbf{F}}{\partial y}+\hat{\mathbf{K}}\times\frac{\partial \mathbf{F}}{\partial z}\right)$

$$=\sum \hat{\mathbf{I}}\times\left(\hat{\mathbf{I}}\times\frac{\partial^2 \mathbf{F}}{\partial x^2}+\hat{\mathbf{J}}\times\frac{\partial^2 \mathbf{F}}{\partial x\partial y}+\hat{\mathbf{K}}\times\frac{\partial^2 \mathbf{F}}{\partial x\partial z}\right)$$

$$=\sum\left[\begin{array}{l}\left\{\left(\hat{\mathbf{I}}.\frac{\partial^2 \mathbf{F}}{\partial x^2}\right)\hat{\mathbf{I}}-\left(\hat{\mathbf{I}}.\hat{\mathbf{I}}\right)\frac{\partial^2 \mathbf{F}}{\partial x^2}\right\}+\left\{\left(\hat{\mathbf{I}}.\frac{\partial^2 \mathbf{F}}{\partial x\partial y}\right)\hat{\mathbf{J}}-\left(\hat{\mathbf{I}}.\hat{\mathbf{J}}\right)\frac{\partial^2 \mathbf{F}}{\partial x\partial y}\right\}\\ \qquad+\left\{\left(\hat{\mathbf{I}}.\frac{\partial^2 \mathbf{F}}{\partial x\partial z}\right)\hat{\mathbf{K}}-\left(\hat{\mathbf{I}}.\hat{\mathbf{K}}\right)\frac{\partial^2 \mathbf{F}}{\partial x\partial z}\right\}\end{array}\right]$$

$$=\sum\left[\left(\hat{\mathbf{I}}.\frac{\partial^2 \mathbf{F}}{\partial x^2}\right)\hat{\mathbf{I}}+\left(\hat{\mathbf{I}}.\frac{\partial^2 \mathbf{F}}{\partial x\partial y}\right)\hat{\mathbf{J}}\left\{-\left(\hat{\mathbf{I}}.\hat{\mathbf{I}}\right)\frac{\partial^2 \mathbf{F}}{\partial x^2}\right\}+\left(\hat{\mathbf{I}}.\frac{\partial^2 \mathbf{F}}{\partial x\partial z}\right)\hat{\mathbf{K}}\right]-\sum\frac{\partial^2 \mathbf{F}}{\partial x^2}$$

$$=\left(\sum \hat{\mathbf{I}}\frac{\partial}{\partial x}\right)\left(\hat{\mathbf{I}}.\frac{\partial \mathbf{F}}{\partial x}+\hat{\mathbf{J}}.\frac{\partial \mathbf{F}}{\partial y}+\hat{\mathbf{K}}.\frac{\partial \mathbf{F}}{\partial z}\right)-\sum\frac{\partial^2 \mathbf{F}}{\partial x^2}=\nabla(\nabla.\mathbf{F})-\nabla^2\mathbf{F}.$$

(5). is just another way of writing (4) above.

Del applied to products of point functions:

To prove that

(1). $\nabla(fg)=f\nabla g+g\nabla f$

(2). $\nabla.(f\mathbf{G})=\nabla f.\mathbf{G}+f\nabla.\mathbf{G}$

(3). $\nabla\times(f\mathbf{G})=\nabla f\times\mathbf{G}+f\nabla\times\mathbf{G}$

(4). $\nabla(\mathbf{F}.\mathbf{G})=(\mathbf{F}.\nabla)\mathbf{G}+(\mathbf{G}.\nabla)\mathbf{F}+\mathbf{F}\times(\nabla\times\mathbf{G})+\mathbf{G}\times(\nabla\times\mathbf{F})$

(5). $\nabla.(\mathbf{F}\times\mathbf{G})=\mathbf{G}.(\nabla\times\mathbf{F})-\mathbf{F}.(\nabla\times\mathbf{G})$

(6). $\nabla\times(\mathbf{F}\times\mathbf{G})=\mathbf{F}(\nabla.\mathbf{G})-\mathbf{G}(\nabla.\mathbf{F})+(\mathbf{G}.\nabla)\mathbf{F}-(\mathbf{F}.\nabla)\mathbf{G}$

Proofs:

(2). To prove that $\nabla.(f\mathbf{G})=\nabla f.\mathbf{G}+f\nabla.\mathbf{G}$.

Proof: $\nabla.(f\mathbf{G})=\sum \hat{\mathbf{I}}.\frac{\partial}{\partial x}f\mathbf{G}=\sum \hat{\mathbf{I}}\left(\frac{\partial f}{\partial x}\mathbf{G}+f\frac{\partial \mathbf{G}}{\partial x}\right)$

$$=\left(\sum\frac{\partial f}{\partial x}\right).\mathbf{G}+f\left(\sum \hat{\mathbf{I}}.\frac{\partial \mathbf{G}}{\partial x}\right)=\nabla f.\mathbf{G}+f\nabla.\mathbf{G}$$

(4). To prove that $\nabla(\mathbf{F}.\mathbf{G})=(\mathbf{F}.\nabla)\mathbf{G}+(\mathbf{G}.\nabla)\mathbf{F}+\mathbf{F}\times(\nabla\times\mathbf{G})+\mathbf{G}\times(\nabla\times\mathbf{F})$.

Proof: $\nabla(\mathbf{F}.\mathbf{G})=\sum\hat{\mathbf{I}}\frac{\partial}{\partial x}(\mathbf{F}.\mathbf{G})=\sum\hat{\mathbf{I}}\left(\frac{\partial\mathbf{F}}{\partial x}.\mathbf{G}+\mathbf{F}.\frac{\partial\mathbf{G}}{\partial x}\right)=\sum\hat{\mathbf{I}}\frac{\partial\mathbf{F}}{\partial x}.\mathbf{G}+\sum\hat{\mathbf{I}}\left(\mathbf{F}.\frac{\partial\mathbf{G}}{\partial x}\right)$ (i)

Now $\mathbf{G}\times\left(\hat{\mathbf{I}}\times\frac{\partial\mathbf{F}}{\partial x}\right)=\left(\mathbf{G}.\frac{\partial\mathbf{F}}{\partial x}\right)\hat{\mathbf{I}}-\left(\mathbf{G}.\hat{\mathbf{I}}\right)\frac{\partial\mathbf{F}}{\partial x}$

or $\left(\mathbf{G}.\frac{\partial\mathbf{F}}{\partial x}\right)\hat{\mathbf{I}}=\mathbf{G}\times\left(\hat{\mathbf{I}}\times\frac{\partial\mathbf{F}}{\partial x}\right)+\left(\mathbf{G}.\hat{\mathbf{I}}\right)\frac{\partial\mathbf{F}}{\partial x}$

$$\therefore\sum\left(\mathbf{G}.\frac{\partial\mathbf{F}}{\partial x}\right)\hat{\mathbf{I}}=\mathbf{G}\times\sum\hat{\mathbf{I}}\times\frac{\partial\mathbf{F}}{\partial x}+\sum\left(\mathbf{G}.\hat{\mathbf{I}}\right)\frac{\partial\mathbf{F}}{\partial x}=\mathbf{G}\times(\nabla\times\mathbf{F})+(\mathbf{G}.\nabla)\mathbf{F}\qquad\text{(ii)}$$

Interchanging **F** and **G,** we get

$$\sum\left(\mathbf{F}.\frac{\partial\mathbf{G}}{\partial x}\right)\hat{\mathbf{I}}=\mathbf{F}\times(\nabla\times\mathbf{G})+(\mathbf{F}.\nabla)\mathbf{G}\qquad\text{(iii)}$$

Substituting in (i) from (ii) and (iii), we get

$\nabla(\mathbf{F}.\mathbf{G})=(\mathbf{F}.\nabla)\mathbf{G}+(\mathbf{G}.\nabla)\mathbf{F}+\mathbf{F}\times(\nabla\times\mathbf{G})+\mathbf{G}\times(\nabla\times\mathbf{F})$

(6). To prove that $\nabla\times(\mathbf{F}\times\mathbf{G})=\mathbf{F}(\nabla.\mathbf{G})-\mathbf{G}(\nabla.\mathbf{F})+\mathbf{G}.\nabla\mathbf{F}-\mathbf{F}.\nabla\mathbf{G}$ **.**

Proof: $\nabla\times(\mathbf{F}\times\mathbf{G})=\sum\hat{\mathbf{I}}\frac{\partial}{\partial x}(\mathbf{F}\times\mathbf{G})=\sum\hat{\mathbf{I}}\left(\frac{\partial\mathbf{F}}{\partial x}\times\mathbf{G}+\mathbf{F}\times\frac{\partial\mathbf{G}}{\partial x}\right)$

$$=\sum\left[\left(\hat{\mathbf{I}}.\mathbf{G}\right)\frac{\partial\mathbf{F}}{\partial x}-\left(\hat{\mathbf{I}}.\frac{\partial\mathbf{F}}{\partial x}\right)\mathbf{G}\right]+\sum\left[\left(\hat{\mathbf{I}}.\frac{\partial\mathbf{G}}{\partial x}\right)\mathbf{F}-\left(\hat{\mathbf{I}}.\mathbf{F}\right)\frac{\partial\mathbf{G}}{\partial x}\right]$$

$$=\sum\left(\mathbf{G}.\hat{\mathbf{I}}\right)\frac{\partial\mathbf{F}}{\partial x}-\mathbf{G}\sum\hat{\mathbf{I}}.\frac{\partial\mathbf{F}}{\partial x}+\mathbf{F}\sum\hat{\mathbf{I}}.\frac{\partial\mathbf{G}}{\partial x}-\sum\left(\mathbf{F}.\hat{\mathbf{I}}\right)\frac{\partial\mathbf{G}}{\partial x}$$

$$=\mathbf{F}\left(\sum\hat{\mathbf{I}}.\frac{\partial\mathbf{G}}{\partial x}\right)-\mathbf{G}\sum\left(\hat{\mathbf{I}}.\frac{\partial\mathbf{F}}{\partial x}\right)+\sum\left(\mathbf{G}.\hat{\mathbf{I}}\right)\frac{\partial\mathbf{F}}{\partial x}-\sum\left(\mathbf{F}.\hat{\mathbf{I}}\right)\frac{\partial\mathbf{G}}{\partial x}$$

Thus, $\nabla\times(\mathbf{F}\times\mathbf{G})=\mathbf{F}(\nabla.\mathbf{G})-\mathbf{G}(\nabla.\mathbf{F})+\mathbf{G}.\nabla\mathbf{F}-\mathbf{F}.\nabla\mathbf{G}$ **.**

Now let us solve some problems related to these topics:

Q.No.1.: If $\mathbf{R}=x\hat{\mathbf{I}}+y\hat{\mathbf{J}}+z\hat{\mathbf{K}}$, show that **(i)** $\nabla.\mathbf{R}=3$, **(ii)** $\nabla\times\mathbf{R}=\mathbf{0}$.

Sol.: (i) $\nabla.\mathbf{R}=\left[\hat{\mathbf{I}}\frac{\partial}{\partial x}+\hat{\mathbf{J}}\frac{\partial}{\partial y}+\hat{\mathbf{K}}\frac{\partial}{\partial z}\right].\left[x\hat{\mathbf{I}}+y\hat{\mathbf{J}}+z\hat{\mathbf{K}}\right]=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=3$.

(ii) $\nabla\times\mathbf{R}=\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=\hat{\mathbf{I}}\left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)-\hat{\mathbf{J}}\left(\frac{\partial z}{\partial x}-\frac{\partial x}{\partial z}\right)+\hat{\mathbf{K}}\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right)$

$$=\hat{\mathbf{I}}(0-0)+\hat{\mathbf{J}}(0-0)+\hat{\mathbf{K}}(0-0)=\mathbf{0}.$$

This completes the proof.

Q.No.2.: Find div $\mathbf{F}$ and curl $\mathbf{F}$, where $\mathbf{F}=\text{grad}\left(x^3+y^3+z^3-3xyz\right)$.

Sol.: If $u=\left(x^3+y^3+z^3-3xyz\right)$

$$\mathbf{F}=\nabla u=\hat{\mathbf{I}}\frac{\partial u}{\partial x}+\hat{\mathbf{J}}\frac{\partial u}{\partial y}+\hat{\mathbf{K}}\frac{\partial u}{\partial z}=\hat{\mathbf{I}}\left(3x^2-3yz\right)+\hat{\mathbf{J}}\left(3y^2-3zx\right)+\hat{\mathbf{K}}\left(3z^2-3xy\right)$$

$$\therefore \text{div }\mathbf{F}=\frac{\partial}{\partial x}\left(3x^2-3yz\right)+\frac{\partial}{\partial y}\left(3y^2-3zx\right)+\frac{\partial}{\partial z}\left(3z^2-3xy\right)=6(x+y+z).\ \text{Ans.}$$

$$\text{curl }\mathbf{F}=\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 3\left(x^2-yz\right) & 3\left(y^2-zx\right) & 3\left(z^2-xy\right)\end{vmatrix}=\hat{\mathbf{I}}(-3x+3x)+\hat{\mathbf{J}}(-3y+3)+\hat{\mathbf{K}}(-3z+3z)=\mathbf{0}.\ \text{Ans.}$$

Q.No.3.: Show that $\nabla^2\left(r^n\right)=n(n+1)r^{n-2}$.

Sol.: $\nabla^2\left(r^n\right)=\nabla.\left(\nabla r^n\right)=\nabla.\left(n.r^{n-2}\mathbf{R}\right)$ $\left[\because \nabla r^n=n.r^{n-2}\mathbf{R}\right]$

Prove that $\nabla r^n=nr^{n-2}\mathbf{R}$, where $\mathbf{R}=x\hat{\mathbf{I}}+y\hat{\mathbf{J}}+z\hat{\mathbf{K}}$.

Sol.: We have $f(x,\ y,\ z)=r^n=\left(x^2+y^2+z^2\right)^{n/2}$

$$\therefore \frac{\partial f}{\partial x}=\frac{\partial\left(r^n\right)}{\partial x}=\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{n/2}=\frac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}.2x=n\,x\,r^{n-2}.$$

Similarly $\frac{\partial f}{\partial y}=ny\,r^{n-2}$ and $\frac{\partial f}{\partial z}=nz\,r^{n-2}$

Thus $\nabla r^n=\mathbf{I}\frac{\partial f}{\partial x}+\mathbf{J}\frac{\partial f}{\partial y}+\mathbf{K}\frac{\partial f}{\partial z}=nr^{n-2}\left(x\mathbf{I}+y\mathbf{J}+z\mathbf{K}\right)=nr^{n-2}\mathbf{R}$. Ans.

$$= \nabla.\left(nr^{n-2}\mathbf{R}\right) = n\nabla.\left(r^{n-2}\mathbf{R}\right)$$

$$= n\left[\left(\nabla r^{n-2}\right)\mathbf{R} + r^{n-2}(\nabla.\mathbf{R})\right]$$

$[\because \nabla.(f\mathbf{G}) = \nabla f.\mathbf{G} + f\nabla.\mathbf{G}]$

$$= n\left[(n-2)r^{n-4}\mathbf{R}.\mathbf{R} + r^{n-2}(3)\right] = n\left[(n-2)r^{n-4}\left(r^2\right) + 3r^{n-2}\right] = n(n+1)r^{n-2}$$

Otherwise, we can evaluate this as follow:

$$\nabla^2\left(r^n\right) = \frac{\partial^2\left(r^n\right)}{\partial x^2} + \frac{\partial^2\left(r^n\right)}{\partial y^2} + \frac{\partial^2\left(r^n\right)}{\partial z^2}. \qquad \text{(i)}$$

Now $\frac{\partial\left(r^n\right)}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\frac{x}{r} = nr^{n-2}x$.

$$\therefore \frac{\partial^2\left(r^n\right)}{\partial x^2} = n\left[r^{n-2} + (n-2)r^{n-3}\frac{\partial r}{\partial x}x\right] = n\left[r^{n-2} + (n-2)r^{n-3}\frac{x}{r}x\right]$$

$$= n\left[r^{n-2} + (n-2)r^{n-4}x^2\right]. \qquad \text{(ii)}$$

Similarly, $\frac{\partial^2\left(r^n\right)}{\partial y^2} = n\left[r^{n-2} + (n-2)r^{n-4}y^2\right].$ (iii)

$$\frac{\partial^2\left(r^n\right)}{\partial z^2} = n\left[r^{n-2} + (n-2)r^{n-4}z^2\right]. \qquad \text{(iv)}$$

Adding (ii), (iii) and (iv) and (i), we get

$\nabla^2\left(r^n\right) = n\left[3r^{n-2} + (n-2)r^{n-4}\left(x^2 + y^2 + z^2\right)\right] = n\left[3r^{n-2} + (n-2)r^{n-4}r^2\right] = n(n+1)r^{n-2}$. Ans.

Q.No.4.: If $u\mathbf{F} = \nabla v$, where u, v are scalar fields and **F** is a vector field, show that

$\mathbf{F}.\,\text{curl}\,\mathbf{F} = 0$.

Sol.: Since $\mathbf{F} = \frac{1}{u}\nabla v$.

$\therefore \text{curl}\,\mathbf{F} = \nabla\times\left(\frac{1}{u}\nabla v\right) = \nabla\frac{1}{u}\times\nabla v + \frac{1}{u}\nabla\times(\nabla v) = \nabla\frac{1}{u}\times\nabla v$ $[\because \nabla\times\nabla f = 0,\ \therefore \nabla\times\nabla v = 0\,]$

Hence, $\mathbf{F}.\,\text{curl}\mathbf{F} = \frac{1}{u}\nabla v.\left(\nabla\frac{1}{u}\times\nabla v\right) = 0$.

[$\because$ it is a scalar triple product in which two factors are equal].

Q.No.5.: If r and **R** have their usual meanings and **A** constant vector, prove that

$$\nabla \times \left(\frac{\mathbf{A} \times \mathbf{R}}{r^n}\right) = \frac{2-n}{r^n}\mathbf{A} + \frac{n(\mathbf{A}.\mathbf{R})}{r^{n+2}}\mathbf{R}.$$

Sol.: Since $\nabla \times \left[r^{-n}(\mathbf{A} \times \mathbf{R})\right] = r^{-n}\left[\nabla \times (\mathbf{A} \times \mathbf{R})\right] + \left[\nabla r^{-n} \times (\mathbf{A} \times \mathbf{R})\right]$

$$[\because \nabla \times (f\mathbf{G}) = \nabla f \times \mathbf{G} + f\nabla \times \mathbf{G}]$$

$$= r^{-n}\left[(\nabla.\mathbf{R})\mathbf{A} - (\mathbf{A}.\nabla)\mathbf{R}\right] + \left(-nr^{-(n+1)}\frac{\mathbf{R}}{r}\right) \times (\mathbf{A} \times \mathbf{R})$$

$$= r^{-n}(3\mathbf{A} - \mathbf{A}) - nr^{-(n+2)}\mathbf{R} \times (\mathbf{A} \times \mathbf{R})$$

$[\because \nabla.\mathbf{R} = 3]$

$$= 2\mathbf{A}r^{-n} - nr^{-(n+2)}\left[(\mathbf{R}.\mathbf{R})\mathbf{A} - (\mathbf{A}.\mathbf{R})\mathbf{R}\right]$$

$$= \frac{2\mathbf{A}}{r^n} - \frac{n}{r^{n+2}}\left[r^2\mathbf{A} - (\mathbf{A}.\mathbf{R})\mathbf{R}\right] = \frac{2-n}{r^n}\mathbf{A} + \frac{n(\mathbf{A}.\mathbf{R})}{r^{n+2}}\mathbf{R}. \text{Ans.}$$

Q.No.6.: Evaluate **(i)** $\text{div}\left(3x^2\hat{\mathbf{I}} + 5xy^2\hat{\mathbf{J}} + xyz^3\hat{\mathbf{K}}\right)$ at the point $(1,\ 2,\ 3)$.

(ii) $\text{curl}\left[e^{xyz}\left(\hat{\mathbf{I}} + \hat{\mathbf{I}} + \hat{\mathbf{K}}\right)\right]$.

(iii) $\text{curl}\left[xyz\hat{\mathbf{I}} + 3x^2y\hat{\mathbf{J}} + \left(xz^2 - y^2z\right)\hat{\mathbf{K}}\right]$.

Sol.: (i) $\text{div}\left(3x^2\hat{\mathbf{I}} + 5xy^2\hat{\mathbf{J}} + xyz^3\hat{\mathbf{K}}\right) = \nabla.\left(3x^2\hat{\mathbf{I}} + 5xy^2\hat{\mathbf{J}} + xyz^3\hat{\mathbf{K}}\right)$

$$= \left(\hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}\right).\left(3x^2\hat{\mathbf{I}} + 5xy^2\hat{\mathbf{J}} + xyz^3\hat{\mathbf{K}}\right)$$

$$= 6x + 10xy + 3xyz^2$$

$$= 6 + 20 + 54 \quad \text{at the point } (1,\ 2,\ 3)$$

$$= 80. \text{ Ans.}$$

(ii) $\text{curl}\left[e^{xyz}\left(\hat{\mathbf{I}} + \hat{\mathbf{J}} + \hat{\mathbf{K}}\right)\right] = \nabla \times \left[e^{xyz}\left(\hat{\mathbf{I}} + \hat{\mathbf{J}} + \hat{\mathbf{K}}\right)\right] = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xyz} & e^{xyz} & e^{xyz} \end{vmatrix}$

$$= e^{xyz}(xz - xy)\hat{\mathbf{I}} + e^{xyz}(xy - yz)\hat{\mathbf{J}} + e^{xyz}(yz - xz)\hat{\mathbf{K}}$$

$$= e^{xyz}\left[x(z-y)\hat{\mathbf{I}} + y(x-z)\hat{\mathbf{J}} + z(y-x)\hat{\mathbf{K}}\right]. \text{ Ans.}$$

(iii) $\text{curl}\left[xyz\hat{\mathbf{I}} + 3x^2y\hat{\mathbf{J}} + (xz^2 - y^2z)\hat{\mathbf{K}}\right] = \nabla \times \left[xyz\hat{\mathbf{I}} + 3x^2y\hat{\mathbf{J}} + (xz^2 - y^2z)\hat{\mathbf{K}}\right]$

$$= \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & (xz^2 - y^2z) \end{vmatrix} = (-2yz - 0)\hat{\mathbf{I}} + (xy - z^2)\hat{\mathbf{J}} + (6xy - xz)\hat{\mathbf{K}}$$

$$= (-2yz)\hat{\mathbf{I}} + (xy - z^2)\hat{\mathbf{J}} + x(6y - z)\hat{\mathbf{K}}. \text{ Ans.}$$

Q.No.7.: If $\mathbf{V} = \dfrac{x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}}{\sqrt{x^2 + y^2 + z^2}}$, show that **(i)** $\nabla.\mathbf{V} = \dfrac{2}{\sqrt{x^2 + y^2 + z^2}}$,

and **(ii)** $\nabla \times \mathbf{V} = \mathbf{0}$.

Sol.: (i) $\nabla.\mathbf{V} = \nabla.\left(\dfrac{x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}}{\sqrt{x^2 + y^2 + z^2}}\right) = \left(\hat{\mathbf{I}}\dfrac{\partial}{\partial x} + \hat{\mathbf{J}}\dfrac{\partial}{\partial y} + \hat{\mathbf{K}}\dfrac{\partial}{\partial z}\right).\left(\dfrac{x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}}{\sqrt{x^2 + y^2 + z^2}}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2 + y^2 + z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$= \frac{(x^2 + y^2 + z^2)^{\frac{1}{2}} - \frac{1}{2}x(x^2 + y^2 + z^2)^{-\frac{1}{2}}.2x}{(x^2 + y^2 + z^2)} + \frac{(x^2 + y^2 + z^2)^{\frac{1}{2}} - \frac{1}{2}y(x^2 + y^2 + z^2)^{-\frac{1}{2}}.2y}{(x^2 + y^2 + z^2)}$$

$$+ \frac{(x^2 + y^2 + z^2)^{\frac{1}{2}} - \frac{1}{2}z(x^2 + y^2 + z^2)^{-\frac{1}{2}}.2z}{(x^2 + y^2 + z^2)}$$

$$= \frac{(x^2 + y^2 + z^2) - x^2 + (x^2 + y^2 + z^2) - y^2 + (x^2 + y^2 + z^2) - z^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$= \frac{2(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = \frac{2}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}. \text{ Ans.}$$

(ii) $\nabla \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\right]\hat{\mathbf{I}} + \left[\frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right]\hat{\mathbf{J}}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right]\hat{\mathbf{K}}$$

$$= \left(\frac{-yz+yz}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)\hat{\mathbf{I}} + \left(\frac{-xz+xz}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)\hat{\mathbf{J}} + \left(\frac{-xy+xy}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)\hat{\mathbf{K}}$$

$= 0\hat{\mathbf{I}} + 0\hat{\mathbf{J}} - 0\hat{\mathbf{K}} = \mathbf{0}$. Ans.

Q.No.8.: If $\mathbf{F} = (x+y+1)\hat{\mathbf{I}} + \hat{\mathbf{J}} - (x+y)\hat{\mathbf{K}}$, show that $\mathbf{F}.\text{curl}\,\mathbf{F} = 0$.

Sol.: $\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+1) & 1 & -(x+y) \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}\{-(x+y)\} - \frac{\partial}{\partial z}(1)\right]\hat{\mathbf{I}} + \left[\frac{\partial}{\partial z}(x+y+1) - \frac{\partial}{\partial x}\{-(x+y)\}\right]\hat{\mathbf{J}}$$

$$+ \left[\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(x+y+1)\right]\hat{\mathbf{K}}$$

$$= (-1-0)\hat{\mathbf{I}} + (0+1)\hat{\mathbf{J}} + (0-1)\hat{\mathbf{K}} = -\hat{\mathbf{I}} + \hat{\mathbf{J}} - \hat{\mathbf{K}}$$

$\therefore$ $\mathbf{F}.\text{curl}\,\mathbf{F} = \left[(x+y+1)\hat{\mathbf{I}} + \hat{\mathbf{J}} - (x+y)\hat{\mathbf{K}}\right].\left(-\hat{\mathbf{I}} + \hat{\mathbf{J}} - \hat{\mathbf{K}}\right) = -(x+y+1) + 1 + (x+y) = 0$. Ans.

Q.No.9.: Find the value of 'a' if the vector $\left(ax^2y + yz\right)\hat{\mathbf{I}} + \left(xy^2 - xz^2\right)\hat{\mathbf{J}} + \left(2xyz - 2x^2y^2\right)\hat{\mathbf{K}}$ has zero divergence. Find the curl of the above vector, which has zero divergence.

Sol.: 1st part: $\mathbf{F} = \left(ax^2y + yz\right)\hat{\mathbf{I}} + \left(xy^2 - xz^2\right)\hat{\mathbf{J}} + \left(2xyz - 2x^2y^2\right)\hat{\mathbf{K}}$

$$\text{div}\,\mathbf{F} = 0 \Rightarrow \left[\hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}\right].\left(ax^2y + yz\right)\hat{\mathbf{I}} + \left(xy^2 - xz^2\right)\hat{\mathbf{J}} + \left(2xyz - 2x^2y^2\right)\hat{\mathbf{K}} = 0$$

$\Rightarrow 2axy + 4xy = 0 \Rightarrow xy(a+2) = 0 \Rightarrow a = -2$. Ans.

2nd Part: $\mathbf{F} = \left(ax^2y + yz\right)\hat{\mathbf{I}} + \left(xy^2 - xz^2\right)\hat{\mathbf{J}} + \left(2xyz - 2x^2y^2\right)\hat{\mathbf{K}}$

$$\therefore \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(-2x^2y + yz\right) & \left(xy^2 - xz^2\right) & \left(2xyz - 2x^2y^2\right) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(2xyz - 2x^2y^2\right) - \frac{\partial}{\partial z}\left(xy^2 - xz^2\right)\right]\hat{\mathbf{I}} + \left[\frac{\partial}{\partial z}\left(-2x^2y + yz\right) - \frac{\partial}{\partial x}\left(2xyz - 2x^2y\right)\right]\hat{\mathbf{J}}$$

$$+ \left[\frac{\partial}{\partial x}\left(xy^2 - xz^2\right) - \frac{\partial}{\partial y}\left(-2x^2y + yz\right)\right]\hat{\mathbf{K}}$$

$$= \left[\left(2xz - 4x^2y\right) + 2xz\right]\hat{\mathbf{I}} + \left[\left(y + 2yz + 4xy^2\right)\right]\hat{\mathbf{J}} + \left[\left(y^2 - z^2 + 2x^2 - z\right)\right]\hat{\mathbf{K}}$$

$$= \left(4xz - 4x^2y\right)\hat{\mathbf{I}} + \left(4xy^2 - 2yz + y\right)\hat{\mathbf{J}} + \left(2x^2 + y^2 - z - z^2\right)\hat{\mathbf{K}}$$

$$= 4x(z - xy)\hat{\mathbf{I}} + \left(y - 2yz + 4xy^2\right)\hat{\mathbf{J}} + \left(2x^2 + y^2 - z^2 - z\right)\hat{\mathbf{K}}$$

Q.No.10.: Show that each of the following vectors are solenoidal

(i) $(x + 3y)\hat{\mathbf{I}} + (y - 3z)\hat{\mathbf{J}} + (x - 2z)\hat{\mathbf{K}}$,

(ii) $3y^4z^2\,\hat{\mathbf{I}} + 4x^3z^2\,\hat{\mathbf{J}} + 3x^2y^2\,\hat{\mathbf{K}}$,

(iii) $\nabla u \times \nabla v$.

Sol.: (i) Let $\mathbf{F} = (x + 3y)\hat{\mathbf{I}} + (y - 3z)\hat{\mathbf{J}} + (x - 2z)\hat{\mathbf{K}}$

$$\nabla.\mathbf{F} = \nabla.\left[(x + 3y)\hat{\mathbf{I}} + (y - 3z)\hat{\mathbf{J}} + (x - 2z)\hat{\mathbf{K}}\right] = \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 3z) + \frac{\partial}{\partial z}(x - 2z)$$

$$= 1 + 1 - 2 = 0.$$

This shows that the given vector is solenoidal.

(ii) Let $\mathbf{F} = 3y^4z^2\,\hat{\mathbf{I}} + 4x^3z^2\,\hat{\mathbf{J}} + 3x^2y^2\,\hat{\mathbf{K}}$

$$\nabla.\mathbf{F} = \frac{\partial}{\partial x}\left(3y^4z^2\right) + \frac{\partial}{\partial y}\left(4x^3z^2\right) + \frac{\partial}{\partial z}3x^2y^2 = 0.$$

This shows that the given vector is solenoidal.

(iii) $\nabla u \times \nabla v$, where u and v are scalar functions.

Since u and v are scalar point functions,

But ∇u and ∇v are vector functions.

Since we know that

$$\operatorname{curl}(\operatorname{grad} f) = \nabla \times (\nabla f) = 0, \because \nabla \times \nabla f = \nabla \times \left[\frac{\partial f}{\partial x}\hat{\mathbf{I}} + \frac{\partial f}{\partial y}\hat{\mathbf{J}} + \frac{\partial f}{\partial z}\hat{\mathbf{K}}\right] = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = 0$$

If $\nabla.(\nabla u \times \nabla v) = 0$, then $\nabla u \times \nabla v$ is solenoidal.

$$\therefore \nabla.(\nabla u \times \nabla v) = \nabla v.(\nabla \times \nabla u) - \nabla u.(\nabla \times \nabla v) = 0 - 0 = 0$$

$$\left[\because \nabla.(\mathbf{F} \times \mathbf{G}) = \mathbf{G}(\nabla \times \mathbf{F}) - \mathbf{F}.(\nabla \times \mathbf{G})\right]$$

Q.No.11: If $u = x^2 + y^2 + z^2$ and $\mathbf{V} = x\,\hat{\mathbf{I}} + y\,\hat{\mathbf{J}} + z\,\hat{\mathbf{K}}$, show that div (u**V**) = 5u.

Sol.: Given $u = x^2 + y^2 + z^2$, $\mathbf{V} = x\,\hat{\mathbf{I}} + y\,\hat{\mathbf{J}} + z\,\hat{\mathbf{K}}$

$$\Rightarrow \nabla.(u\mathbf{V}) = \nabla u.\mathbf{V} + u\nabla.\mathbf{V} \qquad \left[\because \nabla.(f\mathbf{G}) = \nabla f.\mathbf{G} + f\nabla.\mathbf{G}\right]$$

$$= \left(\hat{\mathbf{I}}\frac{\partial u}{\partial x} + \hat{\mathbf{J}}\frac{\partial u}{\partial y} + \hat{\mathbf{K}}\frac{\partial u}{\partial z}\right).\left(x\,\hat{\mathbf{I}} + y\,\hat{\mathbf{J}} + z\,\hat{\mathbf{K}}\right) + u\left(\hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}\right)\left(x\,\hat{\mathbf{I}} + y\,\hat{\mathbf{J}} + z\,\hat{\mathbf{K}}\right)$$

$$= \left(2x\,\hat{\mathbf{I}} + 2y\,\hat{\mathbf{J}} + 2z\,\hat{\mathbf{K}}\right).\left(x\,\hat{\mathbf{I}} + y\,\hat{\mathbf{J}} + z\,\hat{\mathbf{K}}\right) + u\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)$$

$$= 2\left(x^2 + y^2 + z^2\right) + 3u = 2u + 3u = 5u.$$

Q.No.12: If $\mathbf{R} = x\,\hat{\mathbf{I}} + y\,\hat{\mathbf{J}} + z\,\hat{\mathbf{K}}$, show that **(i)** $\nabla\left(\frac{1}{r^2}\right) = -\frac{2\mathbf{R}}{r^4}$, **(ii)** $\nabla.\left(\frac{\mathbf{R}}{r^2}\right) = \frac{1}{r^2}$,

$$\textbf{(iii)}\ \nabla^2\left[\nabla.\frac{\mathbf{R}}{r^2}\right] = -\frac{6}{r^4}.$$

Sol.: (i) Given $\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}$

Then $\nabla\left(\frac{1}{r^2}\right) = \frac{\partial}{\partial x}\frac{1}{\left(x^2+y^2+z^2\right)}\hat{\mathbf{I}} + \frac{\partial}{\partial y}\frac{1}{\left(x^2+y^2+z^2\right)}\hat{\mathbf{J}} + \frac{\partial}{\partial z}\frac{1}{\left(x^2+y^2+z^2\right)}\hat{\mathbf{K}}$

$$= -\frac{2x}{\left(x^2+y^2+z^2\right)^2}\hat{\mathbf{I}} - \frac{2y}{\left(x^2+y^2+z^2\right)^2}\hat{\mathbf{J}} - \frac{2x}{\left(x^2+y^2+z^2\right)}\hat{\mathbf{K}}$$

$$= -\frac{2}{r^4}\left(x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}\right) = -\frac{2\mathbf{R}}{r^4}. \text{ Ans.}$$

(ii) $\nabla.\left(\frac{\mathbf{R}}{r^2}\right) = \left(\frac{\partial}{\partial x}\hat{\mathbf{I}} + \frac{\partial}{\partial y}\hat{\mathbf{J}} + \frac{\partial}{\partial z}\hat{\mathbf{K}}\right).\left[\frac{x}{\left(x^2+y^2+z^2\right)}\hat{\mathbf{I}} + \frac{y}{\left(x^2+y^2+z^2\right)}\hat{\mathbf{J}} + \frac{z}{\left(x^2+y^2+z^2\right)}\hat{\mathbf{K}}\right]$

$$= \frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)} + \frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)} + \frac{\partial}{\partial z}\frac{z}{\left(x^2+y^2+z^2\right)}$$

$$= \frac{r^2-2x^2}{r^4} + \frac{r^2-2y}{r^4} + \frac{r^2-2z^2}{r^4} = \frac{3r^2}{r^4} - \frac{2\left(x^2+y^2+z^2\right)}{r^4}$$

$$= \frac{3}{r^2} - \frac{2r^2}{r^4} = \frac{1}{r^2} \text{ Ans.}$$

(iii) $\nabla^2\left[\nabla.\left(\frac{\mathbf{R}}{r^2}\right)\right] = \nabla^2\left(\frac{1}{r^2}\right)$ $\quad\left[\because \nabla.\left(\frac{\mathbf{R}}{r^2}\right) = \frac{1}{r^2}\right]$

$$= \nabla.\nabla\left(\frac{1}{r^2}\right) = \nabla.\left(-\frac{2\mathbf{R}}{r^4}\right)$$

$$= \left(\frac{\partial}{\partial x}\hat{\mathbf{I}} + \frac{\partial}{\partial y}\hat{\mathbf{J}} + \frac{\partial}{\partial z}\hat{\mathbf{K}}\right).\left[-\frac{2}{r^4}\left(x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}\right)\right] = \frac{-6}{r^4}.\text{Ans.}$$

Q.No.13: If $\mathbf{V}_1$ and $\mathbf{V}_2$ be the vector joining the fixed points $\left(x_1, y_1, z_1\right)$ and $\left(x_2, y_2, z_2\right)$ respectively to a variable point (x, y, z), prove that

(i) $\operatorname{div}\left(\mathbf{V}_1 \times \mathbf{V}_2\right) = 0$,

(ii) $\operatorname{grad}\left(\mathbf{V}_1.\mathbf{V}_2\right) = \left(\mathbf{V}_1 + \mathbf{V}_2\right)$,

(iii) $\operatorname{curl}\left(\mathbf{V}_1 \times \mathbf{V}_2\right) = 2\left(\mathbf{V}_1 - \mathbf{V}_2\right)$.

Sol.: (i) $\text{div}\left(\mathbf{V}_1 \times \mathbf{V}_2\right) = 0$

Now $\mathbf{V}_1 = (x - x_1)\hat{\mathbf{I}} + (y - y_1)\hat{\mathbf{J}} + (z - z_1)\hat{\mathbf{K}}$

and $\mathbf{V}_2 = (x - x_2)\hat{\mathbf{I}} + (y - y_2)\hat{\mathbf{J}} + (z - z_2)\hat{\mathbf{K}}$

$\therefore \text{div}\left(\mathbf{V}_1 \times \mathbf{V}_2\right) = \nabla.\left(\mathbf{V}_1 \times \mathbf{V}_2\right)$

$$\text{Now } \left(\mathbf{V}_1 \times \mathbf{V}_2\right) = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ (x - x_1) & (y - y_1) & (z - z_1) \\ (x - x_2) & (y - y_2) & (z - z_2) \end{vmatrix}$$

$$= \left[(y - y_1)(z - z_2) - (y - y_2)(z - z_1)\right]\hat{\mathbf{I}} + \left[(x - x_1)(z - z_2) - (x - x_2)(z - z_2)\right]\hat{\mathbf{J}}$$

$$+ \left[(x - x_1)(y - y_2) - (x - x_2)(y - y_1)\right]\hat{\mathbf{K}}$$

$\therefore \nabla.\left(\mathbf{V}_1 \times \mathbf{V}_2\right)$

$$= \frac{\partial}{\partial x}\left[(y - y_1)(z - z_2) - (y - y_2)(z - z_1)\right] - \frac{\partial}{\partial y}\left[(x - x_1)(z - z_2) - (x - x_2)(z - z_2)\right]$$

$$+ \frac{\partial}{\partial z}\left[(x - x_1)(y - y_2) - (x - x_2)(y - y_1)\right]$$

= 0. Ans.

(ii) $\nabla\left(\mathbf{V}_1.\mathbf{V}_2\right)$

Now $\mathbf{V}_1.\mathbf{V}_2 = (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2)$

$\therefore \nabla\left(\mathbf{V}_1.\mathbf{V}_2\right) = (2x - x_2 - x_1)\hat{\mathbf{I}} + (2y - y_2 - y_1)\hat{\mathbf{J}} + (2z - z_2 - z_1)\hat{\mathbf{K}} = \left(\mathbf{V}_1 + \mathbf{V}_2\right)$

(iii) Since $\text{curl}\left(\mathbf{V}_1 \times \mathbf{V}_2\right) = \mathbf{V}_1[\nabla.\mathbf{V}_2] - \mathbf{V}_2[\nabla.\mathbf{V}_1] + (\mathbf{V}_2.\nabla)\mathbf{V}_1 - (\mathbf{V}_1.\nabla)\mathbf{V}_2$

$$= 3\mathbf{V}_1 - 3\mathbf{V}_2 + \mathbf{V}_2 - \mathbf{V}_1 = 2\left(\mathbf{V}_1 - \mathbf{V}_2\right)$$

Q.No.14.: Show that **(i)** $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$,

(ii) $\nabla.(\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$.

Sol.: Since $\nabla^2[f(r)] = \frac{\partial^2}{\partial x^2}f(r) + \frac{\partial^2}{\partial y^2}f(r) + \frac{\partial^2}{\partial z^2}f(r)$

$r^2 = x^2 + y^2 + z^2$

Differentiating partially w. r. t. x, we get

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$\therefore \frac{\partial}{\partial x}\{f(r)\} = f'(r)\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}$

$$\frac{\partial^2}{\partial x^2}[f(r)] = f''(r)\left(\frac{\partial r}{\partial x}\right).\frac{x}{r} + f'(r)\left(\frac{r - \frac{\partial r}{\partial x}.x}{r^2}\right) = f''(r)\left(\frac{x^2}{r^2}\right).+f'(r)\left(\frac{r^2 - x^2}{r^3}\right) \quad \left[\because \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

$$= f''(r)\left(\frac{x^2}{r^2}\right).+f'(r)\left(\frac{1}{r} - \frac{x^2}{r^3}\right) \qquad \text{(ii)}$$

Similarly $\frac{\partial^2}{\partial y^2}[f(r)] = f''(r)\left(\frac{x^2}{r^2}\right).+f'(r)\left(\frac{1}{r} - \frac{y^2}{r^3}\right) \qquad \text{(iii)}$

$$\frac{\partial^2}{\partial z^2}[f(r)] = f''(r)\left(\frac{z^2}{r^2}\right).+f'(r)\left(\frac{1}{r} - \frac{z^2}{r^3}\right) \qquad \text{(iv)}$$

Adding (ii), (iii) and (iv), we get

$$\nabla^2[f(r)] = f''(r)\left(\frac{x^2+y^2+z^2}{r^2}\right) + f'(r)\left(\frac{3}{r} - \frac{x^2+y^2+z^2}{r^3}\right) = f''(r) + \frac{2}{r}f'(r) = \text{R. H. S.}$$

(ii) $\nabla.(\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$

L.H.S. $= \nabla.(\phi\nabla\psi - \psi\nabla\phi) = \nabla(\phi\nabla\psi) - \nabla(\psi\nabla\phi) = (\nabla\phi.\nabla\psi + \phi\nabla.\nabla\psi) - (\nabla\psi.\nabla\phi + \psi\nabla.\nabla\phi)$

$$= \phi\nabla^2\psi - \psi\nabla^2\phi = \text{R. H. S.}$$

Q.No.15.: If $\mathbf{A}$ is a constant vector and $\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}$, prove that

(i) $\text{grad}(\mathbf{A.R}) = \mathbf{A}$, (ii) $\text{div}(\mathbf{A}\times\mathbf{R}) = 0$, (iii) $\text{curl}(\mathbf{A}\times\mathbf{R}) = 2\mathbf{A}$,

(iv) $\text{grad}[(\mathbf{A.R})\mathbf{R}] = \mathbf{A}\times\mathbf{R}$

Sol.: (i) To Show: $\text{grad}(\mathbf{A.R}) = \mathbf{A}$

Now since $\mathbf{A} = \text{constant vector} = A_1\hat{\mathbf{I}} + A_2\hat{\mathbf{J}} + A_3\hat{\mathbf{K}}$

and $\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}$

$\therefore$ L. H. S. $= \nabla(\mathbf{A}.\mathbf{R}) = \sum \hat{\mathbf{I}}\dfrac{\partial}{\partial x}(\mathbf{A}.\mathbf{R}) = \sum \hat{\mathbf{I}}\dfrac{\partial}{\partial x}\mathbf{A}.\mathbf{R} + \sum \hat{\mathbf{I}}\left(\mathbf{A}.\dfrac{\partial \mathbf{R}}{\partial x}\right) = \sum \hat{\mathbf{I}}\left(\mathbf{A}..\hat{\mathbf{I}}\right)$

$$= A_1\hat{\mathbf{I}} + A_2\hat{\mathbf{J}} + A_3\hat{\mathbf{K}} = \mathbf{A}. = \text{R. H. S.}$$

(ii) To Show: $\text{div}(\mathbf{A}\times\mathbf{R}) = 0$

L. H. S. $= \text{div}(\mathbf{A}\times\mathbf{R}) = \mathbf{R}.(\nabla\times\mathbf{A}) - \mathbf{A}(\nabla\times\mathbf{R})$

Now $\nabla\times\mathbf{A} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \end{vmatrix} = 0$ and $\nabla\times\mathbf{R} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$

Hence $\text{div}(\mathbf{A}\times\mathbf{R}) = 0 - 0 = 0$

(iii) To Show: $\text{curl}(\mathbf{A}\times\mathbf{R}) = 2\mathbf{A}$

L. H. S. $= \nabla\times(\mathbf{A}\times\mathbf{R}) = \mathbf{A}(\nabla.\mathbf{R}) - \mathbf{R}(\nabla.\mathbf{A} + \mathbf{R}\nabla\mathbf{A} - \mathbf{A}.\nabla\mathbf{R}) = 3\mathbf{A} - \mathbf{A} = 2\mathbf{A}$.= R. H. S.

(iv) To Show: $\text{grad}\left[(\mathbf{A}.\mathbf{R})\mathbf{R}\right] = \mathbf{A}\times\mathbf{R}$

L. H. S. $= \nabla\times\left[(\mathbf{A}.\mathbf{R})\mathbf{R}\right] = \nabla(\mathbf{A}.\mathbf{R})\times\mathbf{R} + \mathbf{A}.\mathbf{R}(\nabla\times\mathbf{R}) = \mathbf{A}\times\mathbf{R} + \mathbf{A}.\mathbf{R}\times 0 = \mathbf{A}\times\mathbf{R}$. =R. H.S.

Q.No.16.: Prove that **(i)** $\nabla\mathbf{A}^2 = 2(\mathbf{A}.\nabla)\mathbf{A} + 2\mathbf{A}\times(\nabla\times\mathbf{A})$

(ii) $\nabla\times(\mathbf{R}\times\mathbf{U}) = \mathbf{R}(\nabla.\mathbf{U}) - 2\mathbf{U} - (\mathbf{R}.\nabla)\mathbf{U}$

Sol.: (i) $\nabla\mathbf{A}^2 = \nabla(\mathbf{A}.\mathbf{A}) = (\mathbf{A}.\nabla)\mathbf{A} + (\mathbf{A}.\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{A})$

$$= 2(\mathbf{A}.\nabla)\mathbf{A} + 2\mathbf{A}\times(\nabla\times\mathbf{A})$$

$$\Rightarrow \nabla\mathbf{A}^2 = 2(\mathbf{A}.\nabla)\mathbf{A} + 2\mathbf{A}\times(\nabla\times\mathbf{A})$$

(ii) $\nabla\times(\mathbf{R}\times\mathbf{U}) = \mathbf{R}\nabla.\mathbf{U} - (\mathbf{R}.\nabla)\mathbf{U} + (\mathbf{U}.\nabla)\mathbf{R} - \mathbf{U}\nabla.\mathbf{R}$

$$= \mathbf{R}(\nabla.\mathbf{U}) - (\mathbf{R}.\nabla)\mathbf{U} + \mathbf{U} - 3\mathbf{U} \qquad \left[\begin{array}{l}\because (\mathbf{U}.\nabla) = 0 \\ \nabla.\mathbf{R} = 3\end{array}\right]$$

$$= \mathbf{R}(\nabla.\mathbf{U}) - (\mathbf{R}.\nabla)\mathbf{U} - 2\mathbf{U}$$

$\nabla \times (\mathbf{R} \times \mathbf{U}) = \mathbf{R}(\nabla.\mathbf{U}) - 2\mathbf{U} - (\mathbf{R}.\nabla)\mathbf{U}$. Ans.

Q.No.17: (a) If $f = (x^2 + y^2 + z^2)^{-n}$, find div grad f and determine n if div grad f = 0.

(b) Show that div $(\text{grad}\, r^n) = n(n+1)r^{n-2}$, where $r^2 = x^2 + y^2 + z^2$.

Sol.: (a) Given $f = (x^2 + y^2 + z^2)^{-n}, \quad r^2 = x^2 + y^2 + z^2$

$\therefore f = (r^2)^{-n} = r^{-2n}$

Now div grad f $= \nabla.\nabla f = \nabla^2 f$

$$\Rightarrow \nabla^2 f = \nabla^2(r^{-2n}) = \frac{\partial^2}{\partial x^2}(r^{-2n}) + \frac{\partial^2}{\partial y^2}(r^{-2n}) + \frac{\partial^2}{\partial z^2}(r^{-2n}).$$

Now $\frac{\partial}{\partial x}(r^{-2n}) = -2n(r)^{-2n-1}\frac{\partial r}{\partial x} = -2n(r)^{-2n-1}\frac{x}{r} = -2nr^{-2n-2}x$

$$\therefore \frac{\partial^2}{\partial x^2}(r^{-2n}) = -2n\left[r^{-2n-2} + (-2n-2)r^{-2n-3}\frac{\partial r}{\partial x}.x\right] = -2n\left[r^{-2n-2} + (-2n-2)r^{-2n-4}x^2\right] \quad \text{(ii)}$$

Similarly $\frac{\partial^2}{\partial y^2}(r^{-2n}) = -2n\left[r^{-2n-2} + (-2n-2)r^{-2n-4}y^2\right]$ (iii)

$$\frac{\partial^2}{\partial z^2}(r^{-2n}) = -2n\left[r^{-2n-2} + (-2n-2)r^{-2n-4}z^2\right] \quad \text{(iv)}$$

Adding (ii), (iii) and (iv), we get

$$\nabla^2(r^{-2n}) = -2n\left[3(r^{-2n-2}) + (-2n-2)r^{-2n-4}(x^2 + y^2 + z^2)\right]$$

$$= -2n(-2n-2+3).r^{-2n-2} = -2n(-2n+1).r^{-2(n+1)} = -2n(2n-1).r^{-2(n+1)}$$

$$= \frac{2n(2n-1)}{r^{2(n+1)}} = \frac{2n(2n-1)}{(x^2 + y^2 + z^2)^{(n+1)}}$$

If $\nabla^2(r^{-2n}) = 0 \Rightarrow \frac{2n(2n-1)}{(x^2 + y^2 + z^2)^{(n+1)}} = 0 \Rightarrow 2n(2n-1) = 0$

$\therefore n = \frac{1}{2}$. Ans.

(b) $\nabla^2 r^n = \text{div}(\text{grad}\, r^n) \Rightarrow \nabla^2 r^n = \frac{\partial^2}{\partial x^2}r^n + \frac{\partial^2}{\partial y^2}r^n + \frac{\partial^2}{\partial z^2}r^n$

Now $\frac{\partial^2}{\partial x^2} r^n = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\frac{x}{r} = nr^{n-2}.x$

$$\frac{\partial^2}{\partial x^2} r^n = n\left[r^{n-2} + (n-2)r^{n-3}\frac{\partial r}{\partial x}x\right] = n\left[r^{n-2} + (n-2)r^{n-3}\frac{x}{r}x\right]$$

$$= n\left[r^{n-2} + (n-2)r^{n-4}x^2\right] \qquad \text{(ii)}$$

Similarly $\frac{\partial^2}{\partial y^2}(r^n) = n\left[r^{n-2} + (n-2)r^{n-4}y^2\right]$ (iii)

$$\frac{\partial^2}{\partial z^2}(r^n) = n\left[r^{n-2} + (n-2)r^{n-4}z^2\right] \qquad \text{(iv)}$$

Adding (ii), (iii) and (iv), we get

$$\nabla^2(r^n) = n\left[3r^{n-2} + (n-2)r^{n-4}(x^2+y^2+z^2)\right] = n\left[3r^{n-2} + (n-2)r^{n-4}r^2\right]$$

$$= n(n-2+3)r^{n-2} = n(n+1)r^{n-2}.$$

Hence, this proves the result.

Q.No.18: For a solenoidal vector **F,** show that curl curl curl curl $\mathbf{F} = \nabla^4\mathbf{F}$.

Sol.: Since **F** is a solenoidal vector, then $\nabla.\mathbf{F} = 0$.

R. H. S. $= \nabla\times\nabla\times\nabla\times\nabla\times\mathbf{F} = \nabla\times\nabla\times\left[\nabla(\nabla\times\mathbf{F}) - \mathbf{F}(\nabla.\nabla)\right]$

$$= \nabla\times\nabla\times\left[-\nabla^2.\mathbf{F}\right] \qquad [\because \nabla.\mathbf{F} = 0]$$

$$= \nabla\times\left[\nabla\times\left(-\nabla^2\mathbf{F}\right)\right] = \nabla\left(\nabla.\nabla^2\mathbf{F}\right) + \nabla^2\mathbf{F}(\nabla.\nabla)$$

$$= \nabla\times 0 + \nabla^2\mathbf{F}\nabla^2 = \nabla^4\mathbf{F} = \text{L. H. S.}$$

Hence this proves the result.

Q.No.19: If $u = x^2y^2$, $v = xy - 3z^2$, find **(i)** $\nabla(\nabla u.\nabla v)$,

(ii) $\nabla.(\nabla u\times\nabla v)$.

Sol.: Given $u = x^2y^2$, $v = xy - 3z^2$

$$\Rightarrow \nabla u = \frac{\partial}{\partial x}u\,\hat{\mathbf{I}} + \frac{\partial}{\partial y}u\,\hat{\mathbf{J}} + \frac{\partial}{\partial z}u\,\hat{\mathbf{K}} = 2xyz\hat{\mathbf{I}} + x^2z\hat{\mathbf{J}} + x^2y\hat{\mathbf{K}} \qquad \text{(i)}$$

$$\nabla v = \frac{\partial}{\partial x}v\,\hat{\mathbf{I}} + \frac{\partial}{\partial y}v\,\hat{\mathbf{J}} + \frac{\partial}{\partial z}v\,\hat{\mathbf{K}} = \frac{\partial}{\partial x}(xy-3z^2)\hat{\mathbf{I}} + \frac{\partial}{\partial y}(xy-3z^2)\hat{\mathbf{J}} + \frac{\partial}{\partial z}(xy-3z^2)\hat{\mathbf{K}}$$

$$= y\hat{\mathbf{I}} + x\hat{\mathbf{J}} - 6z\hat{\mathbf{K}}.$$

(i) $\therefore (\nabla u.\nabla v) = \left(2xyz\hat{\mathbf{I}} + x^2z\hat{\mathbf{J}} + x^2y\hat{\mathbf{K}}\right).\left(y\hat{\mathbf{I}} + x\hat{\mathbf{J}} - 6z\hat{\mathbf{K}}\right) = \left(2xy^2z + x^3z - 6x^2yz\right).$ (ii)

$$\nabla(\nabla u.\nabla v) = \frac{\partial}{\partial x}(\nabla u.\nabla v)\hat{\mathbf{I}} + \frac{\partial}{\partial y}(\nabla u.\nabla v)\hat{\mathbf{J}} + \frac{\partial}{\partial z}(\nabla u.\nabla v)\hat{\mathbf{K}}$$

$$= \frac{\partial}{\partial x}\left(2xy^2z + x^5z - 6x^2yz\right)\hat{\mathbf{I}} + \frac{\partial}{\partial y}\left(2xy^2z + x^3z - 6x^2yz\right)\hat{\mathbf{J}}$$

$$+ \frac{\partial}{\partial z}\left(2xy^2z + x^3z - 6x^2yz\right)\hat{\mathbf{K}}$$

$$= \left(2y^2z + 3x^2z - 12xyz\right)\hat{\mathbf{I}} + \left(4xyz - 6x^2z\right)\hat{\mathbf{J}} + \left(2x^2y^2 + x^3 - 6x^2y\right)\hat{\mathbf{K}}$$

$$= \left(2y^2 + 3x^2 - 12xy\right)z.\hat{\mathbf{I}} + \left(4xy - 6x^2\right)z.\hat{\mathbf{J}} + \left(2x^2y^2 + x^3 - 6x^2y\right)\hat{\mathbf{K}}$$

$$\nabla(\nabla u.\nabla v) = \left(2y^2 + 3x^2 - 12xy\right)\hat{\mathbf{I}} + \left(4xy - 6x^2\right)\hat{\mathbf{J}} + \left(2x^2y^2 + x^3 - 6x^2y\right)\hat{\mathbf{K}}$$

(ii) Find $\nabla.(\nabla u \times \nabla v)$

Given $u = x^2y^2, \quad v = xy - 3z^2$

$\therefore \nabla u = 2xy^2z\hat{\mathbf{I}} + 2x^2yz\hat{\mathbf{J}} + x^2y^2\hat{\mathbf{K}}$ and $\nabla v = y\hat{\mathbf{I}} + x\hat{\mathbf{J}} - 6z\hat{\mathbf{K}}$

$$\therefore \nabla u \times \nabla v = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 2xy^2z & 2x^2yz & x^2y^2 \\ y & x & -6z \end{vmatrix}$$

$$= \hat{\mathbf{I}}\left(-12x^2yz^2 - x^3y^2\right) - \hat{\mathbf{J}}\left(-12xy^2z^2 - x^2y^3\right) + \hat{\mathbf{K}}\left(2x^2y^2z - 2x^2y^2z\right)$$

$$\nabla.(\nabla u \times \nabla v) = \frac{\partial}{\partial x}\left(-12x^2yz^2 - x^3y^2\right) - \frac{\partial}{\partial y}\left(-12xy^2z^2 - x^2y^3\right) + \frac{\partial}{\partial z}\left(2x^2y^2z - 2x^2y^2z\right)$$

$$= \left(-24xyz^2 - 3x^2y^2\right) - \left(24xyz^2 - 3x^2y^2\right) + 0 = 0.$$

Q.No.20: Find directional derivative of $\nabla.(\nabla\phi)$ at the point $(1, -2, 1)$ in the directional of the normal to the surface $xy^2z = 3x + z^2$, where $\phi = 2x^3y^2z^4$.

Sol.: $f = \nabla.(\nabla\phi) \Rightarrow \nabla\phi = \left(\frac{\partial}{\partial x}\hat{\mathbf{I}} + \frac{\partial}{\partial y}\hat{\mathbf{J}} + \frac{\partial}{\partial z}\hat{\mathbf{K}}\right)(2x^3y^2z^4)$

$$= 6x^2y^2z^4\,\hat{\mathbf{I}} + 4x^2yz^4\,\hat{\mathbf{J}} + 8x^3y^2z^3\,\hat{\mathbf{K}}.$$

$\nabla.\nabla\phi = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$

$$\nabla(\nabla.\nabla\phi) = (12y^2z^2 + 12x^2z^4 + 72x^2y^2z^2)\hat{\mathbf{I}} + (24xyz^4 + 48x^3yz^2)\hat{\mathbf{J}}$$
$$+ (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\hat{\mathbf{K}}.$$

At $(1,-2,1)$, $\nabla f = (48+12+288)\hat{\mathbf{I}} + (-48-96)\hat{\mathbf{J}} + (192+16+192)\hat{\mathbf{K}}$

$$\Rightarrow \nabla f = 348\,\hat{\mathbf{I}} - 144\,\hat{\mathbf{J}} + 400\,\hat{\mathbf{K}}.$$

Normal to surface $xy^2z = 3x + z^2$ at $(1,-2,1)$ is $\nabla(xy^2z = 3x + z^2)$

$= (y^2z - 3)\hat{\mathbf{I}} + 2xyz\hat{\mathbf{J}} + (xy^2 - 22)\hat{\mathbf{K}}.$

At $(1,-2,1)$,

$= \hat{\mathbf{I}} - 4\hat{\mathbf{J}} + 2\hat{\mathbf{K}}.$

Directional derivative at f in the direction of normal to surface

$$= 348\hat{\mathbf{I}} - 144\hat{\mathbf{J}} + 400\hat{\mathbf{K}}\,\frac{\hat{\mathbf{I}} - 4\hat{\mathbf{J}} + 2\hat{\mathbf{K}}}{\sqrt{1+16+4}} = \frac{1724}{\sqrt{21}}.\ \text{Ans.}$$

Q.No.21: If r is the distance of a point (x, y, z) from the origin, prove that

$\text{curl}\left(\hat{\mathbf{K}} \times \text{grad}\frac{1}{r}\right) + \text{grad}\left(\hat{\mathbf{K}}.\text{grad}\frac{1}{r}\right) = 0$, where $\hat{\mathbf{K}}$ is a unit vector in the direction of OZ.

Sol.: Let the position vector of point (x, y, z) is $\mathbf{R} = x\,\hat{\mathbf{I}} + y\,\hat{\mathbf{J}} + z\,\hat{\mathbf{K}}$

So $r = (x^2 + y^2 + z^2)^{1/2}$, $\quad \therefore \frac{1}{r} = (x^2 + y^2 + z^2)^{-1/2}$.

$$\text{grad}\frac{1}{r} = (x^2+y^2+z^2)^{-3/2}(-x)\hat{\mathbf{I}} + (x^2+y^2+z^2)^{-3/2}(-y)\hat{\mathbf{J}} + (x^2+y^2+z^2)^{-3/2}(-z)\hat{\mathbf{K}}.$$

Now $\hat{K}\times \text{grad}\frac{1}{r} = \begin{vmatrix} \hat{I} & \hat{J} & \hat{K} \\ 0 & 0 & 1 \\ -x\left(x^2+y^2+z^2\right)^{-3/2} & -y\left(x^2+y^2+z^2\right)^{-3/2} & -z\left(x^2+y^2+z^2\right)^{-3/2} \end{vmatrix}$

$$= y\left(x^2+y^2+z^2\right)^{-3/2}\hat{I} - x\left(x^2+y^2+z^2\right)^{-3/2}\hat{J}$$

Now $\text{curl}\left(\hat{K}\times \text{grad}\frac{1}{r}\right) = \begin{vmatrix} \hat{I} & \hat{J} & \hat{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\left(x^2+y^2+z^2\right)^{-3/2} & -x\left(x^2+y^2+z^2\right)^{-3/2} & 0 \end{vmatrix}$

$$= \frac{\partial}{\partial z}\left[x\left(x^2+y^2+z^2\right)^{-3/2}\right]\hat{I} + \frac{\partial}{\partial z}\left[y\left(x^2+y^2+z^2\right)^{-3/2}\right]\hat{J}$$
$$-\left[\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-3/2}(x) + \frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{-3/2}(y)\right]\hat{K}$$

$$= x\left(-\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}\times 2z\hat{I} + y\left(-\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}\times 2z\hat{J}$$
$$-\left[\begin{matrix}\left(x^2+y^2+z^2\right)^{-3/2} + x\left(-\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}.2x \\ +\left(x^2+y^2+z^2\right)^{-3/2} + y\left(-\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}.2y\end{matrix}\right]\hat{K}$$

$$= \hat{I}\left[-3xzr^{-5}\right] + \hat{J}\left[-3yzr^{-5}\right] - \hat{K}\left[r^{-3} - 3x^2r^{-5} + r^{-3} - 3y^2r^{-5}\right]$$
$$= r^{-5}\left[-3xz\hat{I} - 3yz\hat{J} + \hat{K}\left(3x^2+3y^2-2r^{-3}\right)\right].$$

Also $\hat{K}.\left(\text{grad}\frac{1}{r}\right) = -z\left(x^2+y^2+z^2\right)^{-3/2}$.

$$\therefore \text{grad}\left(\hat{K}.\text{grad}\frac{1}{r}\right) = \frac{\partial}{\partial x}\left[-z\left(x^2+y^2+z^2\right)^{-3/2}\right]\hat{I} + \frac{\partial}{\partial y}\left[-z\left(x^2+y^2+z^2\right)^{-3/2}\right]\hat{J}$$

$$+\frac{\partial}{\partial z}\left[-z\left(x^2+y^2+z^2\right)^{-3/2}\right]\hat{\mathbf{K}}$$

$$=\left[-z\left(-\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}.2x\right]\hat{\mathbf{I}}+\left[-z\left(-\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}.2y\right]\hat{\mathbf{J}}$$

$$+\left[-z\left(-\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}\times 2z-\left(x^2+y^2+z^2\right)^{-3/2}\right]\hat{\mathbf{K}}$$

$$=r^{-5}\left[3xz\hat{\mathbf{I}}+3yz\hat{\mathbf{J}}+\left(3z^2-r^{-3}\right)\hat{\mathbf{K}}\right].$$

$$\operatorname{curl}\left(\hat{\mathbf{K}}\times\operatorname{grad}\frac{1}{r}\right)+\operatorname{grad}\left(\hat{\mathbf{K}}.\operatorname{grad}\frac{1}{r}\right)$$

$$=-3xzr^{-5}\hat{\mathbf{I}}-3yzr^{-5}\hat{\mathbf{J}}+3xzr^{-5}\hat{\mathbf{I}}+3yzr^{-5}\hat{\mathbf{J}}+\left[3r^{-5}\left(x^2+y^2+z^2\right)-2r^{-3}-r^{-3}\right]\hat{\mathbf{K}}=\mathbf{0}.$$

Hence, this proves the result.

Q.No.22: In electromagnetic theory, we have $\nabla.\mathbf{D}=\rho$, $\nabla.\mathbf{H}=0$, $\nabla\times\mathbf{D}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}$,

$$\nabla\times\mathbf{H}=\frac{1}{c}\left(\rho\mathbf{V}+\frac{\partial\mathbf{D}}{\partial t}\right).$$

Prove that $\nabla^2\mathbf{H}-\frac{1}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2}=\nabla\rho+\frac{1}{c^2}\frac{\partial}{\partial t}(\rho\mathbf{V})$ and $\nabla^2\mathbf{H}-\frac{1}{c}\frac{\partial^2\mathbf{H}}{\partial t^2}=-\frac{1}{c}\nabla\times(\rho\mathbf{V})$

Sol.: Consider $\nabla\times(\nabla\times\mathbf{D})=\nabla(\nabla\mathbf{D})-\nabla^2\mathbf{D}=\nabla\rho-\nabla^2\mathbf{D}$

$$\Rightarrow\nabla^2\mathbf{D}+\nabla\times(\nabla\times\mathbf{D})=\nabla\rho.$$

Now $\nabla\times\mathbf{D}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}$ (Given)

$$\therefore\nabla^2\mathbf{D}+\nabla\times\left(-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right)=\nabla\rho$$

$$\Rightarrow\nabla^2\mathbf{D}-\frac{1}{c}\left[\frac{\partial}{\partial t}\left\{\frac{1}{c}\left(\rho\mathbf{V}+\frac{\partial\mathbf{D}}{\partial t}\right)\right\}\right]=\nabla\rho$$

$$\Rightarrow\nabla^2\mathbf{D}-\frac{1}{c^2}\frac{\partial}{\partial t}(\rho\mathbf{V})-\frac{1}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2}=\nabla\rho$$

$$\Rightarrow\nabla^2\mathbf{D}-\frac{1}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2}=\nabla\rho+\frac{1}{c^2}\frac{\partial}{\partial t}(\rho\mathbf{V}).$$

Hence this proves the result.

Now consider $\nabla\times(\nabla\times\mathbf{H})=\nabla(\nabla.\mathbf{H})-\nabla^2\mathbf{H}$

$$\Rightarrow \nabla\times\left(\frac{1}{c}\rho\mathbf{V}+\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}\right)=\nabla(0)-\nabla^2\mathbf{H}, \qquad \left[\because \nabla\times\mathbf{H}=\frac{1}{c}\left(\rho\mathbf{V}+\frac{\partial\mathbf{D}}{\partial t}\right)\right]$$

$$\Rightarrow \left(\nabla\times\frac{1}{c}\rho\mathbf{V}\right)+\left(\nabla\times\frac{1}{c}\frac{\partial\mathbf{D}}{\partial \mathbf{t}}\right)=-\nabla^2\mathbf{H}$$

$$\Rightarrow \left(\frac{1}{c}\frac{\partial(\nabla\times\mathbf{D})}{\partial t}\right)+\left(\nabla^2\mathbf{H}\right)=-\frac{1}{c}\nabla\times(\rho\mathbf{V})$$

$$\Rightarrow \nabla^2\mathbf{H}-\frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}=-\frac{1}{c}\nabla\times(\rho\mathbf{V}). \qquad \left[\because \nabla\times\mathbf{H}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right]$$

Hence, this proves the result.

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5^{th} Topic

Vector Calculus

Integration of vectors

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Latest update available at: http://www.freewebs.com/sunilnit/

Integration of vectors:

If two vector functions $\mathbf{F}(t)$ and $\mathbf{G}(t)$ be such that $\frac{d\mathbf{G}(t)}{dt} = \mathbf{F}(t)$,

then $\mathbf{G}(t)$ is called an integral of $\mathbf{F}(t)$ w.r.t. the scalar variable t and we write

$$\int \mathbf{F}(t)dt = \mathbf{G}(t).$$

If $\mathbf{C}$ be an arbitrary constant vector, we have

$$\mathbf{F}(t) = \frac{d\mathbf{G}(t)}{dt} = \frac{d}{dt}[\mathbf{G}(t) + \mathbf{C}], \text{ then } \int \mathbf{F}(t)dt = \mathbf{G}(t) + \mathbf{C}.$$

This is called the **indefinite integral** of $\mathbf{F}(t)$ and its definite integral is

$$\int_a^b \mathbf{F}(t)dt = [\mathbf{G}(t) + \mathbf{C}]_a^b = \mathbf{G}(b) - \mathbf{G}(a).$$

Now let us solve some problems related to this topic.

Q.No.1.: Given $\mathbf{F}(t) = \left(5t^2 - 3t\right)\hat{\mathbf{I}} + 6t^3\hat{\mathbf{J}} - 7t\hat{\mathbf{K}}$, evaluate $\int_{t=2}^{t=4} \mathbf{F}(t)dt$.

Sol.: $\mathbf{F}(t) = \left(5t^2 - 3t\right)\hat{\mathbf{I}} + 6t^3\hat{\mathbf{J}} - 7t\hat{\mathbf{K}}$

$$\therefore \int_{t=2}^{t=4} \mathbf{F}(t)dt = \int_{t=2}^{t=4}\left[\left(5t^2 - 3t\right)\hat{\mathbf{I}} + 6t^3\hat{\mathbf{J}} - 7t\hat{\mathbf{K}}\right]dt$$

$$= \int_{t=2}^{t=4}\left(5t^2 - 3t\right)\hat{\mathbf{I}}\,dt + \int_{t=2}^{t=4} 6t^3\hat{\mathbf{J}}\,dt + \int_{t=2}^{t=4}\left(-7t\hat{\mathbf{K}}\right)dt$$

$$= \left[\frac{5t^3}{3} - \frac{3t^2}{2}\right]_2^4\hat{\mathbf{I}} + \left[\frac{3}{2}t^4\right]_2^4\hat{\mathbf{J}} + \left[-\frac{7t^2}{2}\right]_2^4\hat{\mathbf{K}}$$

$$= \left[\frac{320}{3} - 24 - \frac{40}{3} + 6\right]\hat{\mathbf{I}} + \left[\frac{3}{2}(256-16)\right]\hat{\mathbf{J}} - \frac{7}{2}(16-4)\hat{\mathbf{K}}$$

$$= \left[\frac{280}{3} - 18\right]\hat{\mathbf{I}} + \left[\frac{3}{2}.240\right]\hat{\mathbf{J}} - \frac{7}{2}.12\hat{\mathbf{K}} = \frac{226}{3}\hat{\mathbf{I}} + 360\hat{\mathbf{J}} - 42\hat{\mathbf{K}}. \text{ Ans.}$$

Q.No.2.: If $\frac{d^2\mathbf{P}}{dt^2} = 6t\hat{\mathbf{I}} - 12t^2\hat{\mathbf{J}} + 4\cos t\hat{\mathbf{K}}$, find **P,** given that $\frac{d\mathbf{P}}{dt} = -\hat{\mathbf{I}} - 3\hat{\mathbf{K}}$ and

$\mathbf{P} = 2\hat{\mathbf{I}} + \hat{\mathbf{J}}$, when $t = 0$.

Sol.: Given $\frac{d^2\mathbf{P}}{dt^2} = 6t\hat{\mathbf{I}} - 12t^2\hat{\mathbf{J}} + 4\cos t\hat{\mathbf{K}}$

$$\therefore \frac{d\mathbf{P}}{dt} = \int\left(\frac{d^2\mathbf{P}}{dt^2}\right)dt = \int\left(6t\hat{\mathbf{I}} - 12t^2\hat{\mathbf{J}} + 4\cos t\hat{\mathbf{K}}\right)dt$$

$$\Rightarrow \frac{d\mathbf{P}}{dt} = \frac{6t^2}{2}\hat{\mathbf{I}} - \frac{12t^3}{3}\hat{\mathbf{J}} + 4\sin t\hat{\mathbf{K}} + C_1 = 3t^2\hat{\mathbf{I}} - 4t^3\hat{\mathbf{J}} + 4\sin t\hat{\mathbf{K}} + C_1$$

At $t = 0$, we get $\frac{d\mathbf{P}}{dt} = C_1$.

Now at $t = 0$, given $\frac{d\mathbf{P}}{dt} = -\hat{\mathbf{I}} - 3\hat{\mathbf{K}}$

$$\Rightarrow C_1 = -\hat{\mathbf{I}} - 3\hat{\mathbf{K}}$$

$$\therefore \frac{d\mathbf{P}}{dt} = \left(3t^2 - 1\right)\hat{\mathbf{I}} - 4t^3\hat{\mathbf{J}} + \left(4\sin t - 3\right)\hat{\mathbf{K}}$$

Now $\mathbf{P} = \int \frac{d\mathbf{P}}{dt}dt = \int \left[\left(3t^2 - 1\right)\hat{\mathbf{I}} - 4t^3\hat{\mathbf{J}} + \left(4\sin t - 3\right)\hat{\mathbf{K}}\right]dt$

$$= \left(\frac{3t^3}{3} - t\right)\hat{\mathbf{I}} - \frac{4t^4}{4}\hat{\mathbf{J}} + \left(-4\cos t - 3t\right)\hat{\mathbf{K}} + C_2$$

$$\Rightarrow \mathbf{P} = \left(t^3 - 1\right)\hat{\mathbf{I}} - t^4\hat{\mathbf{J}} + \left(-4\cos t - 3t\right)\hat{\mathbf{K}} + C_2$$

At $t = 0$, we get $\mathbf{P} = -4\hat{\mathbf{K}} + C_2$

Now at $t = 0$, given $\mathbf{P} = 2\hat{\mathbf{I}} + \hat{\mathbf{J}}$

$$\Rightarrow 2\hat{\mathbf{I}} + \hat{\mathbf{J}} = -4\hat{\mathbf{K}} + C_2 \Rightarrow C_2 = 2\hat{\mathbf{I}} + \hat{\mathbf{J}} + 4\hat{\mathbf{K}}.$$

Hence $\mathbf{P} = \left(t^3 - t + 2\right)\hat{\mathbf{I}} + \left(1 - t^4\right)\hat{\mathbf{J}} + \left(4 - 4\cos t - 3t\right)\hat{\mathbf{K}}$. Ans.

Q.No.3.: The acceleration of a particle at any instant $t \geq 0$ is given by

$12\cos 2t\,\hat{\mathbf{I}} - 8\sin 2t\,\hat{\mathbf{J}} + 16t\,\hat{\mathbf{K}}$, the velocity and displacement are initially zero. Find **velocity** and **displacement** at any time.

Sol.: Given $\mathbf{A} = 12\cos 2t\,\hat{\mathbf{I}} - 8\sin 2t\,\hat{\mathbf{J}} + 16t\,\hat{\mathbf{K}}$

Now velocity $\mathbf{V} = \int \mathbf{A}dt = \int \left\{12\cos 2t\,\hat{\mathbf{I}} - 8\sin 2t\,\hat{\mathbf{J}} + 16t\,\hat{\mathbf{K}}\right\}dt$

$$= \left\{\left(\frac{12\sin 2t}{2}\right)\hat{\mathbf{I}} - 8\left(-\frac{\cos 2t}{2}\right)\hat{\mathbf{J}} + 16\frac{t^2}{2}\hat{\mathbf{K}} + C_1\right\}$$

$$= \left\{6\sin 2t\,\hat{\mathbf{I}} + 4\cos 2t\,\hat{\mathbf{J}} + 8t^2\,\hat{\mathbf{K}} + C_1\right\}.$$

Initially, i.e. $t = 0$, velocity is zero $\Rightarrow \mathbf{V} = \mathbf{0}$, we have

$$\mathbf{0} = 0\hat{\mathbf{I}} + 4\hat{\mathbf{J}} + 0\hat{\mathbf{K}} + C_1 \Rightarrow C_1 = -4\hat{\mathbf{J}}.$$

Hence $\mathbf{V} = 6\sin 2t\,\hat{\mathbf{I}} + 4\left(\cos 2t - 1\right)\hat{\mathbf{J}} + 8t^2\,\hat{\mathbf{K}}$

Now displacement, $\mathbf{R} = \int \mathbf{V}dt = \int \left[6\sin 2t\,\hat{\mathbf{I}} + 4(\cos 2t - 1)\hat{\mathbf{J}} + 8t^2\,\hat{\mathbf{K}}\right]dt$

$$\Rightarrow \mathbf{R} = \left[6\left(-\frac{\cos 2t}{2}\right)\hat{\mathbf{I}} + 4\left(\frac{\sin 2t}{2} - t\right)\hat{\mathbf{J}} + 8\left(\frac{t^3}{3}\right)\hat{\mathbf{K}} + C_2\right]$$

$$\Rightarrow \mathbf{R} = (-3\cos 2t)\hat{\mathbf{I}} + (2\sin 2t - 4t)\hat{\mathbf{J}} + \left(\frac{8}{3}t^3\right)\hat{\mathbf{K}} + C_2.$$

Initially i. e. $t = 0$, displacement is also zero

$\Rightarrow \mathbf{R} = \mathbf{0}$ at $t = 0$

$\therefore \mathbf{0} = (-3)\hat{\mathbf{I}} + (0)\hat{\mathbf{J}} + (0)\hat{\mathbf{K}} + C_2 \Rightarrow C_2 = 3\hat{\mathbf{I}}$

Hence $\mathbf{R} = 3(1 - \cos 2t)\hat{\mathbf{I}} + 2(\sin 2t - 2t)\hat{\mathbf{J}} + \frac{8}{3}t^3\,\hat{\mathbf{K}}$. Ans.

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6th Topic

Vector Calculus

Tangential line integral, Circulation, Work

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(Last updated on 01-08-2009)

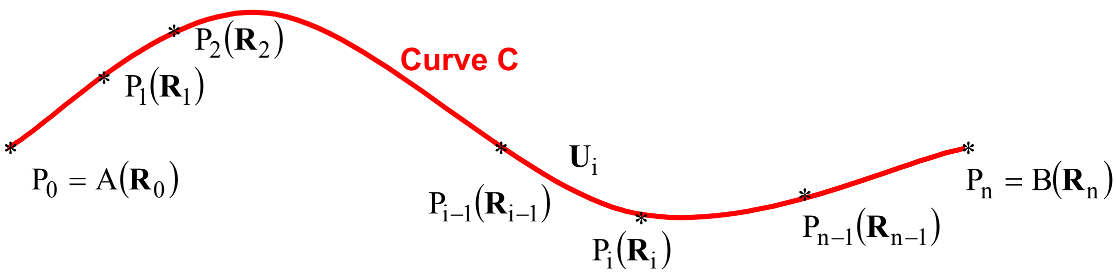
Latest update available at: http://www.freewebs.com/sunilnit/

TANGENTIAL LINE INTEGRAL:

Definition: Let a continuous vector function $\mathbf{F}(\mathbf{R})$ which is defined at each point of the curve C in space**.** Divide this curve C into n parts at the points $A = P_0, P_1, \ldots\ldots, P_{i-1}, P_i, \ldots\ldots, P_n = B$.

Let their position vectors be $\mathbf{R}_{0,}\ \mathbf{R}_1, \ldots\ldots, \mathbf{R}_{i-1}, \mathbf{R}_i, \ldots\ldots, \mathbf{R}_n$**.**

Let $\mathbf{U}_i$, be the position vector of any point on the arc $\mathbf{P}_{i-1}\ \mathbf{P}_i$.



Now consider the sum $S = \sum_{i=0}^{n} \mathbf{F}(\mathbf{U}_i).\delta\mathbf{R}_i$, where $\delta\mathbf{R}_i = \mathbf{R}_i - \mathbf{R}_{i-1}$.

The limit of this sum as $n \to \infty$ in such a way that $|\delta\mathbf{R}_i| \to 0$, provided it exists, is called the **tangential line integral** of $\mathbf{F}(\mathbf{R})$ along C, and is symbolically written as

$$\int_C \mathbf{F}(\mathbf{R}).d\mathbf{R} = \int_C \mathbf{F}.\frac{d\mathbf{R}}{dt}dt = \lim_{\substack{n\to\infty \\ |\delta\mathbf{R}_i|\to 0}} \sum_{i=0}^{n} \mathbf{F}(\mathbf{U}_i).\delta\mathbf{R}_i .$$

When the path of integration is a closed curve, then this fact is denoted by using $\oint$ in place of $\int$.

If $\mathbf{F}(\mathbf{R}) = \hat{\mathbf{I}}\, f(x,y,z) + \hat{\mathbf{J}}\, \phi(x,y,z) + \hat{\mathbf{K}}\, \psi(x,y,z)$ and $d\mathbf{R} = \hat{\mathbf{I}}\,dx + \hat{\mathbf{J}}\,dy + \hat{\mathbf{K}}\,dz$

Then $\int_C \mathbf{F}(\mathbf{R}).d\mathbf{R} = \int_C (f dx + \phi dy + \psi dz)$.

Similarly, two other types of line integrals are $\int_C \mathbf{F} \times d\mathbf{R}$ and $\int_C f d\mathbf{R}$, which are both vectors.

Circulation:

Definition: If $\mathbf{F}$ represents the velocity of a fluid particle, then the integral $\int_C \mathbf{F}.d\mathbf{R}$ is called the circulation of F around the curve. When circulation of $\mathbf{F}$ around every closed curve in a region E vanishes, then $\mathbf{F}$ is said to be irrotational in E.

Work:

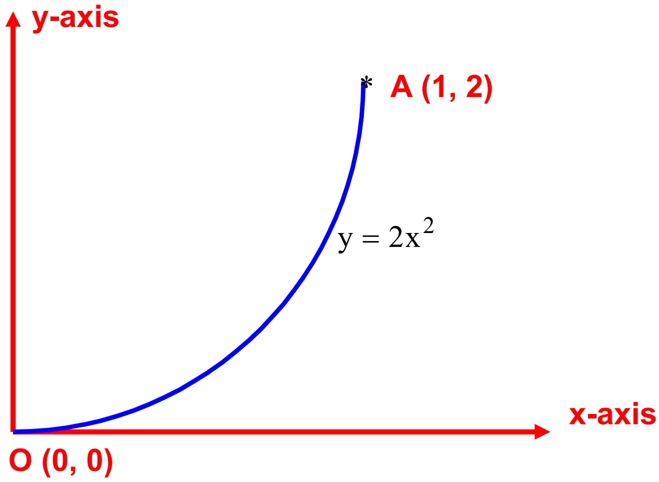
Definition: If $\mathbf{F}$ represents the force acting on a particle moving along an arc AB then the integral $\int_A^B \mathbf{F}.d\mathbf{R}$ is the total work done during the displacement from A to B.

Now let us evaluate some line integrals:

Q.No.1: If $\mathbf{F} = 3xy\,\hat{\mathbf{I}} - y^2\,\hat{\mathbf{J}}$, evaluate $\int_C \mathbf{F}.d\mathbf{R}$, where C is the curve in the xy-plane

$y = 2x^2$ from (0, 0) to (1, 2).

Sol.: Since the particle moving in the xy-plane $(z = 0)$, $\therefore$ we take $\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}}$.



Then $\int_C \mathbf{F}.d\mathbf{R} = \int_C \left(3xy\hat{\mathbf{I}} - y^2\hat{\mathbf{J}}\right).\left(dx\hat{\mathbf{I}} + dy\hat{\mathbf{J}}\right)$, where C is the parabola $y = 2x^2$.

$$= \int_C \left(3xydx - y^2dy\right). \qquad \text{(i)}$$

Substituting $y = 2x^2$, where x goes from 0 to 1, then (i) becomes

$$\int_C \mathbf{F}.d\mathbf{R} = \int_{x=0}^{1}\left[3x\left(2x^2\right)dx - \left(2x^2\right)^2 d\left(2x^2\right)\right] = \int_0^1 \left(6x^3 - 16x^5\right)dx = -\frac{7}{6}. \text{ Ans.}$$

Q.No.2.: Evaluate $\int_c \left(y^2dx - 2x^2dy\right)$ along the parabola $y = x^2$ from (0, 0) to (2, 4).

Sol.: To find: $\int_c \left(y^2dx - 2x^2dy\right)$ along the parabola $y = x^2$ from (0, 0) to (2, 4).

Since $\int_c \left(y^2dx - 2x^2dy\right) = \int_0^2 \left[\left(x^2\right)^2 - 2x^2d\left(x^2\right)\right] = \int_0^2 \left(x^4dx - 4x^3dx\right)$

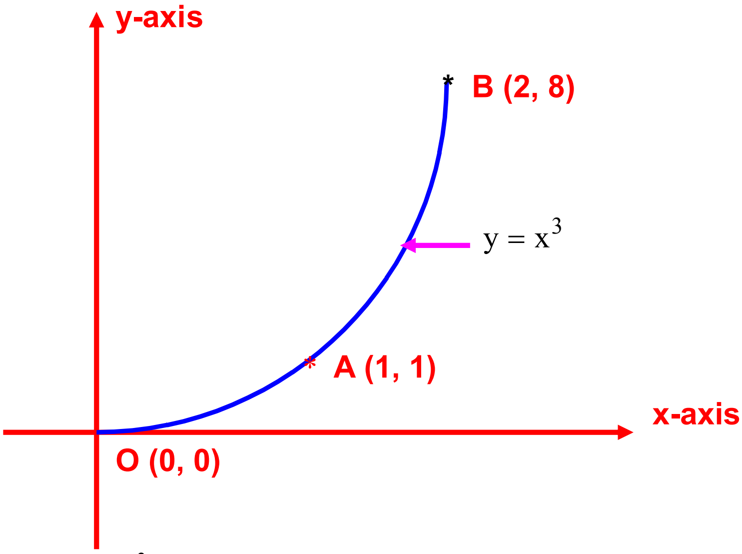
$$= \left[\frac{x^5}{5} - 4\frac{x^4}{4}\right]_0^2 = -\frac{48}{5}. \text{ Ans.}$$

Q.No.3.: If $\mathbf{F} = \left(5xy - 6x^2\right)\hat{\mathbf{I}} + \left(2y - 4x\right)\hat{\mathbf{J}}$, evaluate $\int_C \mathbf{F}.d\mathbf{R}$ along the curve C in the xy-plane, $y = x^3$ from the point (1, 1) to (2, 8).

Sol.: Here we take $\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}}$

Then

$$\int_C \mathbf{F}.d\mathbf{R} = \int_C \left[\left(5xy - 6x^2\right)\hat{\mathbf{I}} + (2y - 4x)\hat{\mathbf{J}}\right].\left(dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}}\right) = \int_C \left[\left(5xy - 6x^2\right)dx + (2y - 4x)dy\right],$$



where C is the curve $y = x^3$ in the xy-plane from the point (1, 1) to (2, 8).

Substituting $y = x^3$, where x varies from x = 1 to x = 2, then above line integral becomes

$$\int_C \mathbf{F}.d\mathbf{R} = \int_C \left[\left(5xy - 6x^2\right)dx + (2y - 4x)dy\right] = \int_1^2 \left[\left(5xx^3 - 6x^2\right)dx + \left(2x^3 - 4x\right)3x^2dx\right]$$

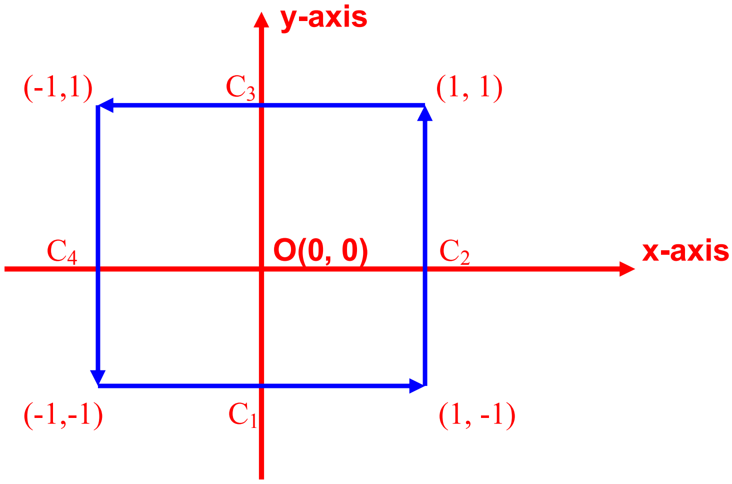
$$= \int_1^2 \left(5x^4 - 6x^2 + 6x^5 - 12x^3\right)dx = \left(5\frac{x^5}{5} - 6\frac{x^3}{3} + 6\frac{x^6}{6} - 12\frac{x^4}{4}\right)_1^2$$

$$= \left(x^5 - 2x^3 + x^6 - 3x^4\right)_1^2 = (32 - 16 + 64 - 48) - (1 - 2 + 1 - 3) = 35. \text{Ans.}$$

Q.No.4.: Evaluate the line integral $\int_C \left[\left(x^2 + xy\right)dx + \left(x^2 + y^2\right)dy\right]$ where C is the square formed by the lines $x = \pm 1,\ y = \pm 1$.

Sol.: Here the line integral is

$\int_C \mathbf{F}.d\mathbf{R} = \int_C \left[\left(x^2 + xy\right)dx + \left(x^2 + y^2\right)dy\right]$, where C is the square formed by the lines $x = \pm 1,\ y = \pm 1$.



Here $\mathbf{F} = \left(x^2 + xy\right)\hat{\mathbf{I}} + \left(x^2 + y^2\right)\hat{\mathbf{J}}$.

$$\therefore \int_C \mathbf{F}.d\mathbf{R} = \int_{C_1} \mathbf{F}.d\mathbf{R} + \int_{C_2} \mathbf{F}.d\mathbf{R} + \int_{C_3} \mathbf{F}.d\mathbf{R} + \int_{C_4} \mathbf{F}.d\mathbf{R} .$$

Along C_1, $y = -1$ and x varies from -1 to 1

$$\therefore \int_{C_1} \mathbf{F}.d\mathbf{R} = \int_{C_1} \left[\left(x^2 + xy\right)dx + \left(x^2 + y^2\right)dy\right] = \int_{-1}^{1} \left(x^2 - x\right)dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1} = \frac{2}{3}.$$

Along C_2, $x = 1$ and y varies from -1 to 1

$$\therefore \int_{C_2} \mathbf{F}.d\mathbf{R} = \int_{C_2} \left[\left(x^2 + xy\right)dx + \left(x^2 + y^2\right)dy\right] = \int_{-1}^{1} \left(1 + y^2\right)dy = \left[y + \frac{y^3}{3}\right]_{-1}^{1} = \frac{8}{3}.$$

Along C_3, $y = +1$ and x varies from 1 to -1

$$\therefore \int_{C_3} \mathbf{F}.d\mathbf{R} = \int_{C_3} \left[\left(x^2 + xy\right)dx + \left(x^2 + y^2\right)dy\right] = \int_{1}^{-1} \left(x^2 + x\right)dx = \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{1}^{-1} = -\frac{2}{3}.$$

Along C_4, $x = -1$ and y varies from 1 to -1

$$\therefore \int_{C_4} \mathbf{F}.d\mathbf{R} = \int_{C_4} \left[\left(x^2 + xy\right)dx + \left(x^2 + y^2\right)dy\right] = \int_{1}^{-1} \left(1 + y^2\right)dy = \left[y + \frac{y^3}{3}\right]_{1}^{-1} = -\frac{8}{3}.$$

Hence $\int_C \mathbf{F}.d\mathbf{R} = \int_{C_1} \mathbf{F}.d\mathbf{R} + \int_{C_2} \mathbf{F}.d\mathbf{R} + \int_{C_3} \mathbf{F}.d\mathbf{R} + \int_{C_4} \mathbf{F}.d\mathbf{R} = \frac{2}{3} + \frac{8}{3} - \frac{2}{3} - \frac{8}{3} = 0$. Ans.

Q.No.5.: Compute the line integral $\int_C \left(y^2 dx - x^2 dy\right)$ about the triangle whose vertices are (1, 0), (0, 1) and $(-1, 0)$.

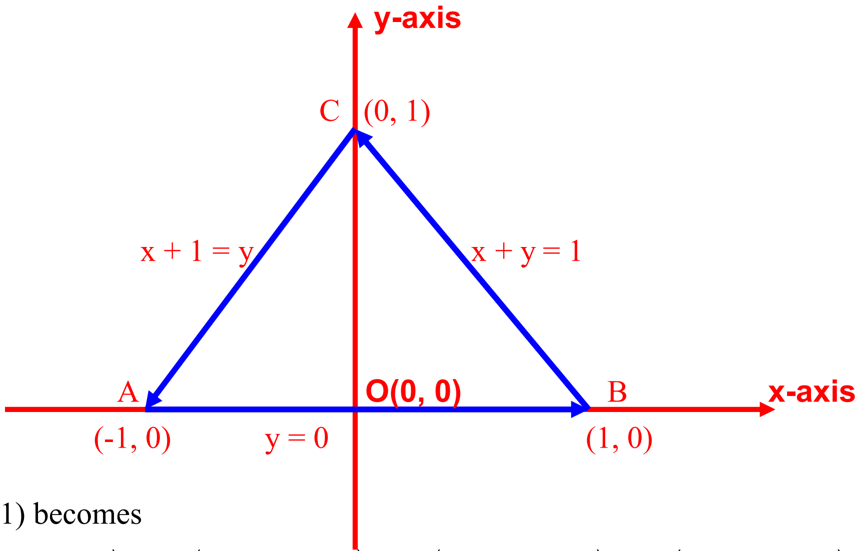
Sol.: Given the vertices of the triangle are(1, 0), (0, 1) and $(-1, 0)$, so the line integral becomes

$$\int_C \left(y^2dx - x^2dy\right) = \int_{AB} \left(y^2dx - x^2dy\right) + \int_{BC} \left(y^2dx - x^2dy\right) + \int_{CA} \left(y^2dx - x^2dy\right) \qquad (1)$$

Along AB: y = 0 and so dy = 0 and x varies from –1 to 1,

along BC: x + y =1, then put y = 1 – x and x varies from 1 to 0 and

along CA: x +1 = y, then put y = x + 1 and x varies from 0 to –1.



Then, (1) becomes

$$\int_C \left(y^2dx - x^2dy\right) = \int_{AB} \left(y^2dx - x^2dy\right) + \int_{BC} \left(y^2dx - x^2dy\right) + \int_{CA} \left(y^2dx - x^2dy\right)$$

$$= \int_{-1}^{1} \left(0dx - x^2 0\right) + \int_{1}^{0} \left[(1-x)^2dx - x^2d(1-x)\right] + \int_{0}^{-1} \left[(x+1)^2dx - x^2d(x+1)\right]$$

$$= 0 + \int_{1}^{0} \left[1 + x^2 - 2x + x^2\right]dx + \int_{0}^{-1} \left[x^2 + 1 + 2x - x^2\right]dx$$

$$= \int_{1}^{0} \left[1 + 2x^2 - 2x\right]dx + \int_{0}^{-1} \left[1 + 2x\right]dx = \left[x + 2\frac{x^3}{3} - 2\frac{x^2}{2}\right]_1^0 + \left[x + 2\frac{x^2}{2}\right]_0^{-1}$$

$$= \left[0 - \left(1 + 2.\frac{1}{3} - 1\right)\right] + \left[(-1+1) - 0\right] = -\frac{2}{3}.\text{ Ans.}$$

Q.No.6.: A vector field is given by $\mathbf{F} = (\sin y)\hat{\mathbf{I}} + x(1 + \cos y)\hat{\mathbf{J}}$. Evaluate the line integral over a circular path given by $x^2 + y^2 = a^2,\ z = 0$.

Sol.: Here we have given $\mathbf{F} = (\sin y)\hat{\mathbf{I}} + x(1+\cos y)\hat{\mathbf{J}}$

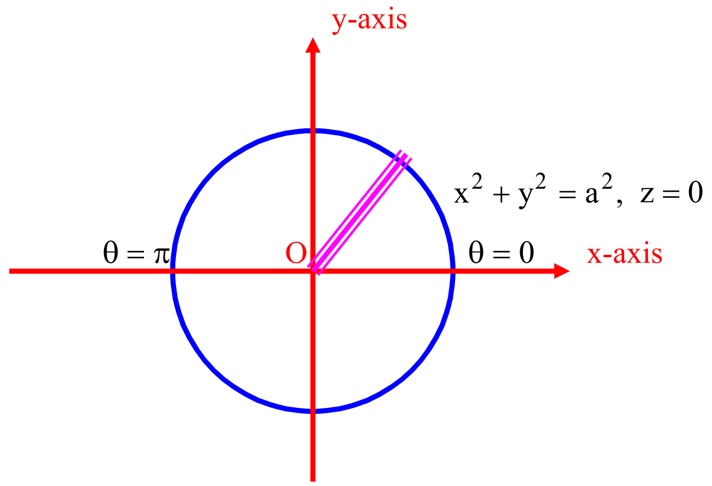
Now since $z = 0$, so we can take $\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} \Rightarrow d\mathbf{R} = dx\hat{\mathbf{I}} + dy\hat{\mathbf{J}}$

The line integral is given by

$$\int_C \mathbf{F}.d\mathbf{R} = \int_C \left[\sin y\hat{\mathbf{I}} + x(1+\cos y)\hat{\mathbf{J}}\right].\left[dx\hat{\mathbf{I}} + dy\hat{\mathbf{J}}\right] = \int_C \left[\sin y dx + x(1+\cos y)dy\right]$$

$$= \int_C \left[\sin y dx + x\cos y dy + x dy\right] = \int_C \left[d(x\sin y) + x dy\right],$$

where C is a circular path given by $x^2 + y^2 = a^2,\ z = 0$.



Put $x = a\cos t \Rightarrow dx = -a\sin t$ and $y = a\sin t \Rightarrow dy = a\cos t$

Then t varies from 0 to 2π.

$$\int_C \mathbf{F}.d\mathbf{R} = \int_C \left[d(x\sin y) + x dy\right] = \int_0^{2\pi} \left[d[a\cos t\sin(a\sin t)] + a^2\cos^2 t dt\right]$$

$$= \left|a\cos t\sin(a\sin t)\right|_0^{2\pi} + 4a^2\int_0^{\pi/2}\cos^2 t dt = 0 + 4a^2.\left(\frac{1}{2}\times\frac{\pi}{2}\right) = \pi a^2.\text{ Ans.}$$

Q.No.7.: If $\mathbf{A} = (3x^2 + 6y)\hat{\mathbf{I}} - 14yz\hat{\mathbf{J}} + 20xz^2\hat{\mathbf{K}}$, evaluate $\int \mathbf{A}.d\mathbf{R}$ from (0, 0, 0) to (1, 1, 1) along the path $x = t,\ y = t^2,\ z = t^3$.

Sol.: Let $d\mathbf{R} = dx\hat{\mathbf{I}} + dy\hat{\mathbf{J}} + dz\hat{\mathbf{K}}$

Then $\int \mathbf{A}.d\mathbf{R} = \int \left[\left(3x^2+6y\right)\hat{\mathbf{I}} - 14yz\hat{\mathbf{J}} + 20xz^2\hat{\mathbf{K}}\right].\left(dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}} + dz\,\hat{\mathbf{K}}\right)$

$$= \int \left[\left(3x^2+6y\right)dx - 14yzdy + 20xz^2dz\right].$$

Given $x = t$, $y = t^2$, $z = t^3$ and as this integral is evaluated from (0, 0, 0) to (1, 1, 1), then t varies from 0 to 1. Now putting $x = t$, $y = t^2$, $z = t^3$, we get

$$\int \mathbf{A}.d\mathbf{R} = \int \left[\left(3x^2+6y\right)dx - 14yzdy + 20xz^2dz\right]$$

$$= \int_0^1 \left[\left(3t^2+6t^2\right)dt - 14t^2t^3d\left(t^2\right) + 20t\left(t^3\right)^2d\left(t^3\right)\right]$$

$$= \int_0^1 \left[\left(3t^2+6t^2\right)dt - 28t^2t^3tdt + 60tt^6t^2dt\right] = \int_0^1 \left[9t^2 - 28t^6 + 60t^9\right]dt$$

$$= \left[9\frac{t^3}{3} - 28\frac{t^7}{7} + 60\frac{t^{10}}{10}\right]_0^1 = \left(\frac{9}{3} - \frac{28}{7} + \frac{60}{10}\right) - 0 = 3 - 4 + 6 = 5. \text{ Ans.}$$

Q.No.8.: Evaluate $\int_C \left(xy+z^2\right)ds$,where C is the arc of the helix $x = \cos t$, $y = \sin t$, $z = t$, which joins the points (1, 0, 0) and $(-1,\ 0,\ \pi)$.

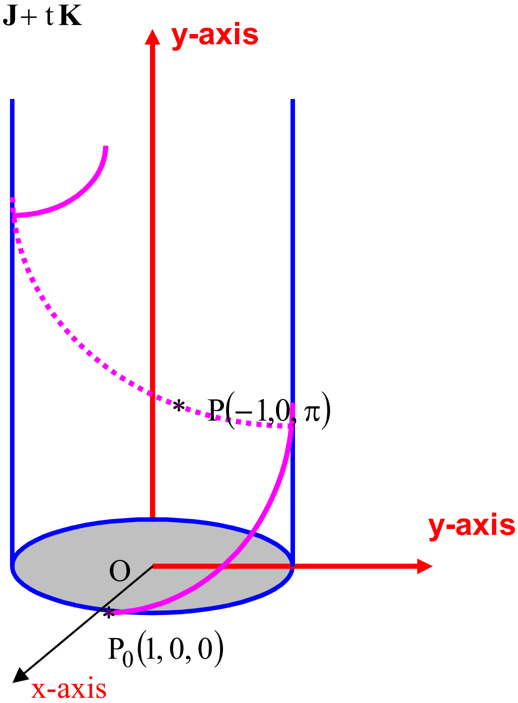
Sol.: The vector equation of the curve is $\mathbf{R} = \cos t\,\hat{\mathbf{I}} + \sin t\,\hat{\mathbf{J}} + t\,\hat{\mathbf{K}}$

$$\therefore \frac{d\mathbf{R}}{dt} = -\sin t\,\hat{\mathbf{I}} + \cos t\,\hat{\mathbf{J}} + \hat{\mathbf{K}}.$$

Its arc length from P_0 (t = 0) to any point P(t) is given by

$$s = \int_0^t \left|\frac{d\mathbf{R}}{dt}\right| dt = \sqrt{\left(1^2+1^2\right)}\ t = \sqrt{2}\ t.$$

$\frac{ds}{dt} = \sqrt{2} \Rightarrow ds = \sqrt{2}\,dt$, then the line integral becomes



$$\int_C \left(xy+z^2\right)ds = \int_0^{\pi}\left(\cos t\ \sin t + t^2\right)\sqrt{2}\,dt = \frac{1}{2}\int_0^{\pi}\sqrt{2}\sin 2t\,dt + \int_0^{\pi} t^2\sqrt{2}\,dt$$

$$= \frac{1}{\sqrt{2}}\left|-2\cos 2t\right|_0^{\pi} + \sqrt{2}\left|\frac{t^3}{3}\right|_0^{\pi} = 0 + \frac{\sqrt{2}}{3}\pi^3 = \frac{\sqrt{2}}{3}\pi^3.\text{ Ans.}$$

Q.No.9.: Evaluate the integral $\int_c x^{-1}(y+z)ds$, where c the arc of circle $x^2+y^2=4$, $z=0$ from (2, 0, 0) to $\left(\sqrt{2},\ \sqrt{2},\ 0\right)$ in the counterclockwise direction.

Sol.: Equation of the circle in parametric form is $x = 2\cos t$, $y = 2\sin t$

when x = 2, then t = 0 and when $x = \sqrt{2}$, then $t = \frac{\pi}{4}$

$$\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + \hat{\mathbf{K}} = 2\cos t\,\hat{\mathbf{I}} + 2\sin t\,\hat{\mathbf{J}} + 0\,\hat{\mathbf{K}}$$

$$\frac{d\mathbf{R}}{dt} = -2\sin t\,\hat{\mathbf{I}} + 2\cos t\,\hat{\mathbf{J}}$$

$$\mathbf{R}.\mathbf{R} = \frac{d\mathbf{R}}{dt}.\frac{d\mathbf{R}}{dt} = 4\sin^2 t + 4\cos^2 t = 4$$

$$\frac{ds}{dt} = \sqrt{\mathbf{R}.\mathbf{R}} = \sqrt{4} = 2.$$

Along c: $z = 0$ and ds = 2dt so that

$$\int_c x^{-1}(y+z)ds = \int \frac{y+0}{x}ds = \int_c \frac{2\sin t}{2\cos t}.2dt$$

$$= 2\int_0^{\pi/4}\tan t.dt = 2\ \log\sec t\Big|_0^{\pi/4} = 2\ \log\sqrt{2} = \log\ 2\text{ . Ans.}$$

Q.No.10.: If $\mathbf{F} = \left(2x+y^2\right)\hat{\mathbf{I}} + (3y-4x)\hat{\mathbf{J}}$, evaluate $\oint_c \mathbf{F}.d\mathbf{R}$ along triangle ABC in the xy-plane with A(0, 0), B(2, 0), C(2, 1).

(a). in the counterclockwise direction

(b). what is the value in opposite direction ?

Sol.: (a). in the counterclockwise direction

$$I = \oint_c \mathbf{F}.d\mathbf{R} = \int_{c_1}\mathbf{F}.d\mathbf{R} + \int_{c_2}\mathbf{F}.d\mathbf{R} + \int_{c_3}\mathbf{F}.d\mathbf{R} = I_1 + I_2 + I_3$$

Along c_1: The straight line from A(0, 0) to B (2, 0), y = 0, z = 0 and x to 2.

Thus $\mathbf{R} = x\hat{\mathbf{I}}$, $d\mathbf{R} = \hat{\mathbf{I}}\,dx$, dy = 0.

So with y = 0,

$$I_1 = \int_{c_1} \mathbf{F}.d\mathbf{R} = \int_0^2 \left(2x\hat{\mathbf{I}}\right).\hat{\mathbf{I}}\,dx = \int_0^2 2x dx = x^2\Big|_0^2 = 4$$

Along c_2: The straight line from B(2,0) to C(2, 1), x = 2, z = 0, y varies from 0 to t.

Thus $\mathbf{R} = 2\hat{\mathbf{I}} + y\hat{\mathbf{J}}$, $d\mathbf{R} = \hat{\mathbf{J}}\,dy$

So with x = 2,

$$I_2 = \int_{c_2} \mathbf{F}.d\mathbf{R} = \int_0^1 \left[\left(4+y^2\right)\hat{\mathbf{I}} + \left(3y-8\right)\hat{\mathbf{J}}\right] = \left[\hat{\mathbf{J}}\,dy\right] = \int_0^1 \left(3y-8\right)dy$$

$$= \left[\frac{3y^2}{2} - 8y\right]_0^1 = \frac{3}{2} - 8 = -\frac{13}{2}.$$

Along c_3: Straight line from C(2, 1) to A(0, 0) is $y = \frac{1}{2}x$, $dy = \frac{1}{2}dx$, x varies from 2 to 0

Thus $\mathbf{R} = x\hat{\mathbf{I}} + \frac{1}{2}x\hat{\mathbf{J}}$, $d\mathbf{R} = \left(\hat{\mathbf{I}} + \frac{1}{2}\hat{\mathbf{J}}\right)dx$

So with $y = \frac{x}{2}$,

$$I_3 = \int_{c_3} \mathbf{F}.d\mathbf{R} = \int_2^0 \left[\left(2x + \frac{x^2}{4}\right)\hat{\mathbf{I}} + \left(\frac{3x}{2} - 4x\right)\hat{\mathbf{J}}\right].\left[\hat{\mathbf{I}} + \frac{\hat{\mathbf{J}}}{2}\right]dx$$

$$= \int_2^0 \left(2x + \frac{x^2}{4} + \frac{3x}{4} - \frac{4x}{2}\right)dx = \left[\frac{x^3}{12} + \frac{3x^2}{8}\right]_2^0 = -\left(\frac{8}{12} + \frac{12}{8}\right) = -\frac{13}{6}.$$

The required line integral in the counterclockwise direction

$$I = \oint_c \mathbf{F}.d\mathbf{R} = I_1 + I_2 + I_3 = 4 - \frac{13}{2} - \frac{13}{6} = -\frac{14}{3}. \text{ Ans.}$$

(b). Line integral value in the opposite direction is $\frac{14}{3}$. Ans.

Q.No.11.: If $\mathbf{A} = (x-y)\hat{\mathbf{I}} + (x+y)\hat{\mathbf{J}}$, evaluate $\oint_c \mathbf{A}.d\mathbf{R}$ around the curve c consisting of

$y = x^2$ and $y^2 = x$.

Sol.: $\because I = \oint_c \mathbf{F}.d\mathbf{R} = \int_{c_1} \mathbf{F}.d\mathbf{R} + \int_{c_2} \mathbf{F}.d\mathbf{R} = I_1 + I_2$

where c_1: $y = x^2$ and c_2: $y^2 = x$ as shown in Fig. meet at points A(0, 0) and B(1, 1).

Along the curve c_1: $y = x^2$, so that

$$\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} = x\hat{\mathbf{I}} + x^2\hat{\mathbf{J}} = t\hat{\mathbf{I}} + t^2\hat{\mathbf{J}}$$

$d\mathbf{R} = \left(\hat{\mathbf{I}} + 2t\hat{\mathbf{J}}\right)dt$ with t varying from 0 to 1.

$$I_1 = \int_{c_1} (\mathbf{A}.d\mathbf{R})$$

$$= \int_0^1 \left[(t-t^2)\hat{\mathbf{I}} + (t+t^2)\hat{\mathbf{J}}\right].=\left[\hat{\mathbf{I}} + 2t\hat{\mathbf{J}}\right]dt$$

$$= \int_0^1 \left[(t-t^2) + 2t(t+t^2)\right]dt = \int_0^1 (2t^3 + t^2 + t)dt$$

$$I_1 = \frac{2t^4}{4} + \frac{t^3}{3} + \frac{t^2}{2}\Bigg|_0^1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{2} = 1 + \frac{1}{3} = \frac{4}{3}.$$

Along the curve c_2: $y^2 = x$, $y = \sqrt{x}$

$$\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} = x\hat{\mathbf{I}} + \sqrt{x}\hat{\mathbf{J}} = t\hat{\mathbf{I}} + \sqrt{t}\hat{\mathbf{J}}$$

$$d\mathbf{R} = \left(\hat{\mathbf{I}} + \frac{1}{2\sqrt{t}}\hat{\mathbf{J}}\right)dt$$

with t varying from 1 to 0.

$$I_2 = \int_{c_2} \mathbf{A}.d\mathbf{R}$$

$$= \int_0^1 \left[(t-\sqrt{t})\hat{\mathbf{I}} + (t+\sqrt{t})\hat{\mathbf{J}}\right].=\left[\hat{\mathbf{I}} + \frac{1}{2\sqrt{t}}\hat{\mathbf{J}}\right]dt$$

$$= \int_0^1 \left[(t-\sqrt{t}) + \frac{1}{2}(\sqrt{t}+1)\right]dt = \int_0^1 \left(t - \frac{\sqrt{t}}{2} + \frac{1}{2}\right)dt$$

$$I_2 = \frac{t^2}{2} + \frac{t^{3/2}}{3} + \frac{1}{2}t\Big|_0^1 = \frac{1}{2} + \frac{1}{3} - \frac{1}{2} = -\frac{2}{3}$$

Line integral $I = I_1 + I_2 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$**.** Ans.

Q.No.12: Find the work done in moving a particle in the force field

$$\mathbf{F} = 3x^2\hat{\mathbf{I}} + (2xz - y)\hat{\mathbf{J}} + z\hat{\mathbf{K}}, \text{ along}$$

(a) the straight line from (0, 0, 0) to (2, 1, 3).

(b) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$**.**

Sol.: We know that $\int_C \mathbf{F}.d\mathbf{R} = \int_C \left[3x^2\hat{\mathbf{I}} + (2xz - y)\hat{\mathbf{J}} + z\hat{\mathbf{K}}\right].\left(dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}} + dz\,\hat{\mathbf{K}}\right)$

$$= \int_C \left[3x^2dx + (2xz - y)dy + zdz\right]. \qquad \text{(i)}$$

(a) The equation of the straight line from (0, 0, 0) to (2, 1, 3) are $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$ (say)

$\therefore x = 2t,\ y = t,\ z = 3t$ are its parametric equations.

The points (0, 0, 0) and (2, 1, 3) correspond to $t = 0$ and $t = 1$, respectively.

$$\therefore \text{Work done} = \int_C \mathbf{F}.d\mathbf{R} = \int_C \left[3(2t)^2 2dt + \{(4t)(3t) - t\}dt + (3t)3dt\right]$$

$$= \int_0^1 \left(36t^2 + 8t\right)dt = 16\textbf{. Ans.}$$

(b) Let $x = t$ in $x^2 = 4y, 3x^3 = 8z$.

Then the parametric equation of C are $x = t,\ y = \frac{t^2}{4},\ z = \frac{3t^3}{8}$ and t varies from 0 to 2.

$$\therefore \text{Work done} = \int_C \mathbf{F}.d\mathbf{R} = \int_0^2 \left[3t^2dt + \left\{2t\left(\frac{3t^3}{8}\right) - \frac{t^2}{4}\right\}d\left(\frac{t^2}{4}\right) + \frac{3t^3}{8}d\left(\frac{3t^3}{8}\right)\right].$$

$$= \int_0^2 \left[3t^2 + \left(\frac{6t^4}{8} - \frac{t^2}{4}\right)\frac{2t}{4} + \frac{3t^3}{8}.\frac{9t^2}{8}\right]dt = \int_0^2 \left[3t^2 + \frac{6t^5}{16} - \frac{t^3}{8} + \frac{27t^5}{64}\right]dt$$

$$= \int_0^2 \left(3t^2 - \frac{t^3}{8} + \frac{51}{64}t^5\right)dt = \left| t^3 - \frac{t^4}{32} + \frac{17}{128}t^6 \right|_0^2 = 16. \text{ Ans.}$$

Q.No.13.: Using the line integral, compute the work done by the force

$\mathbf{F} = (2y+3)\hat{\mathbf{I}} + xz\hat{\mathbf{J}} + (yz - x)\hat{\mathbf{K}}$, when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2$, $y = t$, $z = t^3$.

Sol.: We know that the work done by a force is given by $W = \int_C \mathbf{F}.d\mathbf{R}$

Here given $\mathbf{F} = (2y+3)\hat{\mathbf{I}} + xz\hat{\mathbf{J}} + (yz - x)\hat{\mathbf{K}}$ and let $d\mathbf{R} = dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}} + dz\,\hat{\mathbf{K}}$

$$\therefore W = \int_C \mathbf{F}.d\mathbf{R} = \int_C \left[(2y+3)\hat{\mathbf{I}} + xz\hat{\mathbf{J}} + (yz-x)\hat{\mathbf{K}}\right].\left(dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}} + dz\,\hat{\mathbf{K}}\right)$$

$$= \int_C \left[(2y+3)dx + xzdy + (yz - x)dz\right]$$

Given $x = 2t^2$, $y = t$, $z = t^3$ and as this integral is evaluated from (0, 0, 0) to (2, 1, 1), then t varies from 0 to 1. Now putting $x = 2t^2$, $y = t$, $z = t^3$, we get

$$\therefore W = \int_C \left[(2y+3)dx + xzdy + (yz-x)dz\right] = \int_0^1 \left[(2t+3)d\left(2t^2\right) + 2t^2t^3dt + \left(tt^3 - 2t^2\right)d\left(t^3\right)\right]$$

$$= \int_0^1 \left[(2t+3)4t + 2t^2t^3 + \left(tt^3 - 2t^2\right)3t^2\right]dt = \int_0^1 \left[8t^2 + 12t + 2t^5 + 3t^6 - 6t^4\right]dt$$

$$= \left| 8\frac{t^3}{3} + 12\frac{t^2}{2} + 2\frac{t^6}{6} + 3\frac{t^7}{7} - 6\frac{t^5}{5} \right|_0^1 = \left(\frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5}\right) - 0 = \frac{288}{35}. \text{ Ans.}$$

Q.No.14.: Find the work done in moving a particle in the force field

$\mathbf{F} = 3x^2\,\hat{\mathbf{I}} + (2xz - y)\hat{\mathbf{J}} + z\hat{\mathbf{K}}$ along

(a). Straight line from A(0, 0, 0) to B(2, 1, 3).

(b). Space curve c : $x = 2t^2$, $y = t$, $z = 4t^2 - t$ from $t = 0$ to $t = 1$.

(c). Curve c: defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.

Sol.: Work done along a curve is $\int_c \mathbf{F}.d\mathbf{R}$:

(a). Vector equation of straight line from A(0, 0, 0) to B(2, 1, 3) is $\mathbf{R} = 2t\,\hat{\mathbf{I}} + t\,\hat{\mathbf{J}} + 3t\,\hat{\mathbf{K}}$

Thus $d\mathbf{R} = \left(2\hat{\mathbf{I}} + \hat{\mathbf{J}} + 3\hat{\mathbf{K}}\right)dt$.

Since $\mathbf{F} = 3x\hat{\mathbf{I}} + (2xz - y)\hat{\mathbf{J}} + z\hat{\mathbf{K}} = 12t^2\hat{\mathbf{I}} + \left(12t^2 - t\right)\hat{\mathbf{J}} + 3t\hat{\mathbf{K}}$.

Thus work done by $\mathbf{F}$ in moving along the straight line from A(0, 0, 0) to B(2, 1, 3)

$$= \int_A^B \mathbf{F}.d\mathbf{R} = \int_0^1 \left[12t^2\hat{\mathbf{I}} + \left(12t^2 - t\right)\hat{\mathbf{J}} + 3t\hat{\mathbf{K}}\right].\left[2\hat{\mathbf{I}} + \hat{\mathbf{J}} + 3\hat{\mathbf{K}}\right]dt$$

$$= \int_0^1 \left(24t^2 + 12t^2 - t + 9t\right)dt = 12t^2 + 4t^2\Big|_0^1 = 16 \text{. Ans.}$$

(b). Vector equation of space curve c : $x = 2t^2$, $y = t$, $z = 4t^2 - t$ from t = 0 to t = 1 is

$\mathbf{R} = 2t^2\hat{\mathbf{I}} + t\hat{\mathbf{J}} + \left(4t^2 - t\right)\hat{\mathbf{K}}$

Thus $d\mathbf{R} = 4t\hat{\mathbf{I}} + \hat{\mathbf{J}} + (8t - 1)\hat{\mathbf{K}}$.

Since $\mathbf{F} = 3\left(2t^2\right)^2\hat{\mathbf{I}} + \left[2.\left(2t^2\right)\left(4t^2 - t\right) - t\right]\hat{\mathbf{J}} + \left(4t^2 - t\right)\hat{\mathbf{K}}$

Work done

$$= \int_c \mathbf{F}.d\mathbf{R} = \int_0^1 \left[12t^4\hat{\mathbf{I}} + \left[4t^2\left(4t^2 - t\right) - t\right]\hat{\mathbf{J}} + \left(4t^2 - t\right)\hat{\mathbf{K}}\right].\left[4t\hat{\mathbf{I}} + \hat{\mathbf{J}} + (8t - 1)\hat{\mathbf{K}}\right]$$

$$= \int_0^1 \left[48t^5 + \left(16t^4 - 4t^3 - t\right) + (8t - 1)\left(4t^2 - t\right)\right]dt$$

$$= 8t^2 + 16\frac{t^5}{5} + 7t^4 - 4t^3\Big|_0^1 = 8 + \frac{16}{5} + 7 - 4 = 14.2$$

(c). Vector equation of curve c defined by $x^2 = 4y$, $3x^3 = 8z$ from x = 0 to x = 2 is

$\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}} = x\hat{\mathbf{I}} + \frac{x^2}{4}\hat{\mathbf{J}} + \frac{3}{8}x^3\hat{\mathbf{K}} = t\hat{\mathbf{I}} + \frac{t^2}{4}\hat{\mathbf{J}} + \frac{3}{8}t^3\hat{\mathbf{K}}$

Thus $d\mathbf{R} = \left(\hat{\mathbf{I}} + \frac{t}{2}\hat{\mathbf{J}} + \frac{9}{8}t^2\hat{\mathbf{K}}\right)$

Since $\mathbf{F} = 3x^2\hat{\mathbf{I}} + \left(2.x.\frac{3}{8}x^3 - \frac{x^2}{4}\right)\hat{\mathbf{J}} + \frac{3}{8}x^3\hat{\mathbf{K}}$

$$\text{Work done} = \int \mathbf{F}.d\mathbf{R} = \int_0^2 \left[3t^2 + \frac{t}{2}\left(\frac{3}{4}t^4 - \frac{t^2}{4}\right) + \frac{27}{64}t^5\right]dt$$

$$= t^3 + \frac{t^6}{16} - \frac{t^4}{32} + \frac{27t^6}{384}\Bigg|_0^2 = 16.\ \text{Ans.}$$

Q.No.15.: If $\mathbf{A} = (y-2x)\hat{\mathbf{I}} + (3x+2y)\hat{\mathbf{J}}$, compute the circulation of **A** about a circle c in the xy plane with centre at the origin and radius 2, if c is traversed in the positive direction.

Sol.: Given c : circle : $x^2 + y^2 = 4$

In parametric form $x = 2\cos t$, $y = 2\sin t$ with t varying 0 to 2π.

$$\mathbf{A} = (2\sin t - 2(2\cos t))\hat{\mathbf{I}} + (3(2\cos t) + 2(2\sin t))\hat{\mathbf{J}}$$

$$d\mathbf{R} = d\left(x\hat{\mathbf{I}} + y\hat{\mathbf{J}}\right) = dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}}$$

$$d\mathbf{R} = \left(-2\sin t\,\hat{\mathbf{I}} + 2\cos t\,\hat{\mathbf{J}}\right)dt$$

By definition, circulation along curve c $= \int_c \mathbf{F}.d\mathbf{R}$

$$= \int_0^{2\pi} \left[(2\sin t - 4\cos t)\hat{\mathbf{I}} + (6\cos t + 4\sin t)\hat{\mathbf{J}}\right].\left[-2\sin t\,\hat{\mathbf{I}} + 2\cos t\,\hat{\mathbf{J}}\right]dt$$

$$= 4\int_0^{2\pi} \left[-\sin^2 t + 2\sin t\cos t + 3\cos^2 t + 2\sin t\cos t\right]dt$$

$$= 16\int_0^{2\pi} \sin t\, d(\sin t) - 4\int_0^{2\pi} \frac{1-\cos 2t}{2}dt + 12\int_0^{2\pi} \frac{1+\cos 2t}{2}dt = 8\pi.\ \text{Ans.}$$

Q.No.16.: Evaluate $\int_c f d\mathbf{R}$ where $f = 2xy^2z + x^2y$ and c is the curve $x = t,\ y = t^2,\ z = t^3$ from $t = 0$ to 1.

Sol.: Since $\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}} = t\hat{\mathbf{I}} + t^2\hat{\mathbf{J}} + t^3\hat{\mathbf{K}}$

$$d\mathbf{R} = \left(\hat{\mathbf{I}} + 2t\hat{\mathbf{J}} + 3t^2\hat{\mathbf{K}}\right)dt.$$

Along c: $f = 2t.\left(t^2\right)^2\left(t^3\right)+\left(t^2\right).t^2 = 2t^8+t^4$

$$\int_c f d\mathbf{R} = \int_0^1 \left(2t^8+t^4\right)\left(\hat{\mathbf{I}}+2t\,\hat{\mathbf{J}}+3t^2\,\hat{\mathbf{K}}\right)dt$$

$$= \hat{\mathbf{I}}\int_0^1 \left(2t^8+t^4\right)dt + \hat{\mathbf{J}}\int_0^1 \left(4t^9+2t^5\right)dt + \hat{\mathbf{K}}\int_0^1 \left(6t^{10}+3t^6\right)dt$$

$$= \hat{\mathbf{I}}\left(\frac{2t^9}{9}+\frac{t^5}{5}\right)\Bigg|_0^1 + \hat{\mathbf{J}}\left(\frac{4t^{10}}{10}+\frac{2t^6}{6}\right)\Bigg|_0^1 + \hat{\mathbf{K}}\left(\frac{6t^{11}}{11}+\frac{3t^7}{7}\right)\Bigg|_0^1$$

$$= \frac{19}{45}\hat{\mathbf{I}}+\frac{11}{15}\hat{\mathbf{J}}+\frac{75}{77}\hat{\mathbf{K}}. \text{ Ans.}$$

Q.No.17.: If $\mathbf{F} = 2y\,\hat{\mathbf{I}} - z\,\hat{\mathbf{J}} + x\,\hat{\mathbf{K}}$, evaluate $\int_C \mathbf{F}\times d\mathbf{R}$ along the curve x =cost, y = sint, z = 2cost from t = 0 to $t = \frac{\pi}{2}$.

Sol.: Given $\mathbf{F} = 2y\,\hat{\mathbf{I}} - z\,\hat{\mathbf{J}} + x\,\hat{\mathbf{K}}$ and let $d\mathbf{R} = dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}} + dz\,\hat{\mathbf{K}}$.

Here $$\mathbf{F}\times d\mathbf{R} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix} = \hat{\mathbf{I}}(-zdz - xdy) - \hat{\mathbf{J}}(2ydz - xdx) + \hat{\mathbf{K}}(2ydy + zdx)$$

Given x =cost, y = sint, z = 2cost

$$\int_C \mathbf{F}\times d\mathbf{R} = \int_C \hat{\mathbf{I}}(-zdz - xdy) - \hat{\mathbf{J}}(2ydz - xdx) + \hat{\mathbf{K}}(2ydy + zdx)$$

$$= \int_0^{\frac{\pi}{2}} \left[\begin{array}{l} \hat{\mathbf{I}}(2\cos t.2\sin t - \cos t\cos t) \\ -\hat{\mathbf{J}}(-2\sin t.2\sin t + \cos t.\sin t) + \hat{\mathbf{K}}(2\sin t.\cos t - 2\cos t.\sin t) \end{array}\right]dt$$

$$= \int_0^{\frac{\pi}{2}} \left[\hat{\mathbf{I}}\left(4\cos t.\sin t - \cos^2 t\right) + \hat{\mathbf{J}}\left(4\sin^2 t - \cos t.\sin t\right) + \hat{\mathbf{K}}(0)\right]dt$$

$$=\left(4.\frac{1}{2}-\frac{1}{2}.\frac{\pi}{2}\right)\hat{\mathbf{I}}+\left(4.\frac{1}{2}.\frac{\pi}{2}-\frac{1}{2}\right)\hat{\mathbf{J}} \qquad \text{(by using reduction formulae)}$$

$$=\left(2-\frac{\pi}{4}\right)\hat{\mathbf{I}}+\left(\pi-\frac{1}{2}\right)\hat{\mathbf{J}}\text{. Ans.}$$

Home Assignments

Q.No.1.: Evaluate the line integral $\int_c xy^3 ds$ where c is the segment of the line y = 2x in the xy-plane from $A(-1,-2,\ 0)$ to B(1, 2, 0).

Ans.: $\frac{16}{\sqrt{5}}$.

Q.No.2.: Evaluate the line integral $\int_c \left(x^2 y dx + (x-z)dy + xyzdz\right)$ where c is the arc of parabola $y = x^2$ in plane z = 2 from A(0, 0, 2) to B (1, 1, 2).

Ans.: $-\frac{17}{15}$.

Q.No.3.: Evaluate the line integral $\int_c \left(x^2 + y^2 + z^2\right)^2 ds$ where c is the arc of circular helix $r(t) = \cos t\,\hat{\mathbf{I}} + \sin t\,\hat{\mathbf{J}} + 3t\,\hat{\mathbf{K}}$ from $A(1,0,0)$ to $B(1,\ 0,\ 6\pi)$.

Ans.: $\sqrt{10}\left(2\pi + 6(2\pi)^3 + \frac{81}{5}(2\pi)^5\right)$

Q.No.4.: If $\mathbf{A}(t) = t\,\hat{\mathbf{I}} - t^2\,\hat{\mathbf{J}} + (t-1)\,\hat{\mathbf{K}}$, $\mathbf{B}(t) = 2t^2\,\hat{\mathbf{I}} - 6t\,\hat{\mathbf{K}}$, evaluate $\int_0^2 \mathbf{A}.\mathbf{B}dt$.

Ans.: 12.

Q.No.5.: If $\mathbf{A}(t) = t\,\hat{\mathbf{I}} - 3\,\hat{\mathbf{J}} + 2t\,\hat{\mathbf{K}}$, $\mathbf{B}(t) = \hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$, $\mathbf{C}(t) = 3\,\hat{\mathbf{I}} + t\,\hat{\mathbf{J}} - \hat{\mathbf{K}}$ evaluate

(a). $\int_1^2 \mathbf{A}.(\mathbf{B}\times\mathbf{C})dt$

(b). $\int_1^2 \left(\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\right)dt$.

Ans.:(a). 0 **(b).** $-\frac{87}{2}\hat{\mathbf{I}} - \frac{44}{3}\hat{\mathbf{J}} + \frac{15}{2}\hat{\mathbf{K}}$.

Q.No.6.: If $\mathbf{A}(2) = 2\hat{\mathbf{I}} - \hat{\mathbf{J}} + 2\hat{\mathbf{K}}$, $\mathbf{A}(3) = 4\hat{\mathbf{I}} - 2\hat{\mathbf{J}} + 3\hat{\mathbf{K}}$, then evaluate $\int_2^3 \mathbf{A}.\frac{d\mathbf{A}}{dt}dt$.

Ans.: 10.

Q.No.7.: Evaluate $\int_c \mathbf{A} \times d\mathbf{R}$ where $\mathbf{A} = 2y\hat{\mathbf{I}} - z\hat{\mathbf{J}} + x\hat{\mathbf{K}}$ and c is the curve $x = \cos t$,

$y = \sin t$, $z = 2\cos t$ from $t = 0$ to $\frac{\pi}{2}$.

Ans.: $\hat{\mathbf{I}}\left(2 - \frac{\pi}{4}\right) + \hat{\mathbf{J}}\left(\pi - \frac{1}{2}\right)$

Q.No.8.: Evaluate $\int_c \mathbf{A}.d\mathbf{R}$ where

(a). $\mathbf{A} = 2x\hat{\mathbf{I}} + 4y\hat{\mathbf{J}} + 3z\hat{\mathbf{K}}$. c: curve : $\mathbf{R}(t) = \cos t\,\hat{\mathbf{I}} + \sin t\,\hat{\mathbf{J}} + t\,\hat{\mathbf{K}}$ from $t = 0$ to π

.

(b). $\mathbf{A} = y\hat{\mathbf{I}} + z\hat{\mathbf{J}} + \hat{\mathbf{K}}$. c: circle $y^2 + z^2 = 1$, $x = 0$.

(c). $\mathbf{A} = yz\hat{\mathbf{I}} + zx\hat{\mathbf{J}} + xy\hat{\mathbf{K}}$. c: curve from (0, 0, 0) to (1, 1, 0) along the curve $x = y^2$, $z = 0$ in xy-plane , followed by the straight line path from (1, 1, 0) to (1, 1, 1).

Ans.: (a). $\frac{-3\pi^2}{2}$. **(b).** $-\pi$. **(c).** $\frac{3}{4}$.

Q.No.9.: Find the total work done in moving a particle in force field

$\mathbf{A} = 3xy\hat{\mathbf{I}} - 5z\hat{\mathbf{J}} + 10x\hat{\mathbf{K}}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$

from $t = 1$ to $t = 2$.

Ans.: 303.

Q.No.10.: Prove that the work done in moving an object from P_1 to P_2 in a conservative force field $\mathbf{F}$ is independent of the path joining the two points P_1 and P_2

Hint: Since $\mathbf{F}$ is conservative, $\mathbf{F} = \nabla f$

$$\int_{P_1}^{P_2} \mathbf{F}.d\mathbf{R} = \int_{P_1}^{P_2} \nabla f.d\mathbf{R} = \int_{P_1}^{P_2} \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \int_{P_1}^{P_2} df = f(P_2) - f(P_1).$$

Hint: Use $\frac{1}{2}\int_c \ xdy - ydx$ to compute area enclosed by c.

Q.No.11.: If $\mathbf{A} = (2x - y + 2z)\hat{\mathbf{I}} + (x + y - z)\hat{\mathbf{J}} + (3x - 2y - 5z)\hat{\mathbf{K}}$, calculate the **circulation** of **A** along the circle in the xy-plane of radius 2 and centre of origin.

Ans.: Circulation $= \text{f}\mathbf{A}.d\mathbf{R} = 8\pi$

Q.No.12.: Determine the **circulation** of $\mathbf{A} = y\,\hat{\mathbf{I}} + z\,\hat{\mathbf{J}} + x\,\hat{\mathbf{K}}$ around the curve $x^2 + y^2 = 1$, $z = 0$.

Ans.: $-\pi$

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7th Topic

Vector Calculus

Normal Surface integral, Flux across a surface, Solenoidal vector point function

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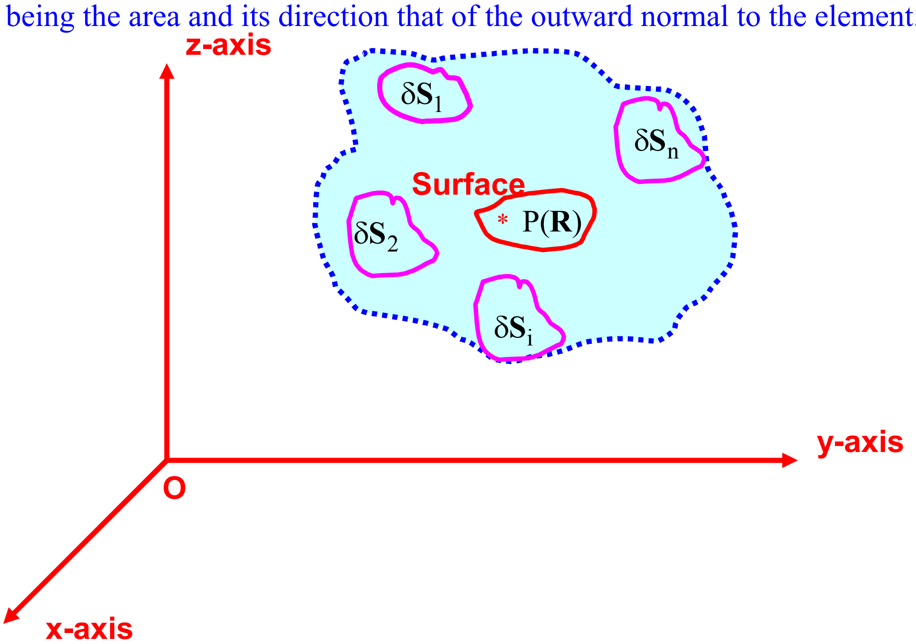
NORMAL SURFACE INTEGRAL:

Consider a continuous function $\mathbf{F}(\mathbf{R})$ and a two sided surface S. Now divide this surface S into a finite number of sub-surfaces $\delta\mathbf{S}_1, \delta\mathbf{S}_2........,\delta\mathbf{S}_i,...........\delta\mathbf{S}_n$. Let the surface element surrounding any point $\mathrm{P}(\mathbf{R})$ be $\delta\mathbf{S}_i$, which can be regarded as a vector; its magnitude being the area and its direction that of the outward normal to the element.



Now consider the sum

$$\sum_{i=1}^{n} \mathbf{F}(\mathbf{R}).\delta \mathbf{S}_i \,,$$

where the summation extends over all the sub-surfaces.

The limit of this sum, as the number of sub-surfaces tends to infinity $(\text{i.e. } n \to \infty)$ and consequently the area of each sub-surface tends to zero $\left(\text{i.e. } |\delta \mathbf{S}_i| \to 0\right)$, is called the **normal surface integral** of $\mathbf{F}(\mathbf{R})$ over S and is denoted by

$\int_S \mathbf{F}.d\mathbf{S}$ or $\int_S \mathbf{F}.\hat{\mathbf{N}}\, ds$, where $\hat{\mathbf{N}}$ is a unit outward normal at P to S.

Thus $\lim\limits_{\substack{n\to\infty \\ |\delta \mathbf{S}_i| \to 0}} \sum \mathbf{F}(\mathbf{R}).\delta \mathbf{S}_i = \int_S \mathbf{F}.d\mathbf{S} = \int_S \mathbf{F}.\hat{\mathbf{N}}\, ds$.

Other types of surface integral are $\int_S \mathbf{F} \times d\mathbf{S}$ and $\int_S f\, d\mathbf{S}$ which are both vectors.

Notation: Only one integral is sign is used when there is one differential (say $d\mathbf{R}$ or $d\mathbf{S}$) and two (or three) signs when there are two (or three) differentials.

Flux across a surface:

Definition: If $\mathbf{F}$ represents the flux, then the total outward flux of $\mathbf{F}$ across a closed surface S is the surface integral $\int_S \mathbf{F}.d\mathbf{S}$.

Solenoidal vector point function:

Definition: When the flux of $\mathbf{F}$ across every closed surface S in a region E vanishes, then $\mathbf{F}$ is said to be a **Solenoidal vector point function** in E.

Evaluation of a Surface Integral:

A surface integral is evaluated by reducing it to a double integral by projecting the given surface S onto one of the coordinate planes. Let D be the projection of S onto the xy-plane.

Then $ds = \dfrac{dydx}{\left|\hat{\mathbf{N}}.\hat{\mathbf{K}}\right|}$

Then $\iint_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \iint_D \mathbf{F}.\hat{\mathbf{N}}\dfrac{dxdy}{|\hat{\mathbf{N}}.\hat{\mathbf{K}}|}$

where $\hat{\mathbf{N}}$ is unit outward drawn normal to S. The RHS double integral in x, y over the plane region D is evaluated as a two-fold iterated integral. In a similar way the surface integral can be evaluated by projecting S onto the yz-plane as D_1 and xz-plane as D_2 as follows

$$\iint_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \iint_{D_1} \mathbf{F}.\hat{\mathbf{N}}\frac{dydz}{|\hat{\mathbf{N}}.\hat{\mathbf{I}}|}$$

$$\iint_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \iint_{D_2} \mathbf{F}.\hat{\mathbf{N}}\frac{dxdz}{|\hat{\mathbf{N}}.\hat{\mathbf{J}}|}.$$

Surface Area of a Curved Surface:

Let S be a surface represented by the equation

$$f(x, y, z) = 0 \tag{i}$$

Then the unit normal to the surface S is given by

$$\hat{\mathbf{N}} = \frac{\nabla f}{|\nabla f|} = \frac{f_x\hat{\mathbf{I}} + f_y\hat{\mathbf{J}} + f_z\hat{\mathbf{K}}}{\sqrt{f_x^2 + f_y^2 + f_z^2}}$$

where f_x, f_y, f_z are partial derivatives of f w.r.t. x, y, z respectively.

Let D be the projection of S onto plane xy-plane. Then

$$\text{Surface area of S} = \iint_S ds = \iint_D \frac{dxdy}{|\hat{\mathbf{N}}.\hat{\mathbf{K}}|} = \iint_D \frac{\sqrt{f_x^2 + f_y^2 + f_z^2}}{|f_z|}dxdy$$

and $\hat{\mathbf{N}}.\hat{\mathbf{K}} = \dfrac{f_z}{\sqrt{f_x^2 + f_y^2 + f_z^2}}$.

Corollary 1: If the equation of the surface S is z = f(x, y) then

Surface area $= \iint \sqrt{1 + z_x^2 + z_y^2}\,dxdy$.

Flux:

The normal component $\mathbf{F}.\hat{\mathbf{N}}$ is a scalar. Let ρ be the density, $\mathbf{V}$ be the velocity of a fluid and $\mathbf{F} = \rho\mathbf{V}$. Then flux $\mathbf{F}$ represents the total quantity of fluid flowing in unit time through (across) the surface S in the positive direction. The flux of $\mathbf{F}$ across S is given by the flux integral.

Flux of $\mathbf{F}$ across $S = \iint_S \mathbf{F}.\hat{\mathbf{N}}\,ds$.

Now let us solve some problems related to surface integral:

Q.No.1.: If velocity vector is $\mathbf{F} = y\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + xz\hat{\mathbf{K}}$ m/sec., show that the flux of water through the parabolic cylinder $y = x^2$, $0 \le x \le 3$, $0 \le z \le 2$ is 69 m^3/sec.

or

Calculate the flux of water through the parabolic cylinder $y = x^2$, between the planes x = 0, z = 0, x = 3, z = 2 if the velocity vector is $\mathbf{A} = y\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + xz\hat{\mathbf{K}}$ m/sec.

Sol.: Since we know that, the flux of water through the parabolic cylinder is evaluated by the formula

$$\int_S \mathbf{F}.d\mathbf{S} = \int_S \mathbf{F}.\hat{\mathbf{N}}\,ds,$$

where $\hat{\mathbf{N}}$ is a unit vector normal to the parabolic surface i.e. $f(x,y) = x^2 - y = 0$.

$$\therefore \hat{\mathbf{N}} = \frac{\nabla f}{|\nabla f|} = \frac{2x\hat{\mathbf{I}} - \hat{\mathbf{J}}}{\sqrt{4x^2+1}}.$$

$$\therefore \mathbf{F}.\hat{\mathbf{N}} = \left(y\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + zx\hat{\mathbf{K}}\right).\frac{\left(2x\hat{\mathbf{I}} - \hat{\mathbf{J}}\right)}{\sqrt{4x^2+1}} = \frac{2xy-2}{\sqrt{4x^2+1}}.$$

Also $ds = \dfrac{dxdz}{\left|\hat{\mathbf{N}}.\hat{\mathbf{J}}\right|} = \dfrac{dxdz}{1/\sqrt{4x^2+1}}$. $\qquad \left[\because \left|\hat{\mathbf{N}}.\hat{\mathbf{J}}\right| = \dfrac{1}{\sqrt{4x^2+1}}\right]$

Hence $\int_S \mathbf{F}.d\mathbf{S} = \int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_0^2 \int_0^3 \left(\frac{2xy-2}{\sqrt{4x^2+1}}\right)\frac{dxdz}{1/\sqrt{4x^2+1}} = \int_0^2 \left(\int_0^3 (2xy-2)\,dx\right)dz$

$$= \int_0^2 \left(\int_0^3 (2xx^2-2)\,dx\right)dz \qquad (\because y = x^2)$$

$$= \int_0^2 \left[\int_0^3 (2x^3-2)dx\right]dz = \int_0^2 \left[2\frac{x^4}{4} - 2x\right]_0^3 dz = \int_0^2 \left[\frac{81}{2} - 6\right]dz$$

$$= \frac{69}{2}\int_0^2 dz = \frac{69}{2}[z]_0^2 = 69\, m^3/\text{sec. Ans.}$$

Q.No.2.: Find the flux of the vector field $\mathbf{A} = (x-2z)\hat{\mathbf{I}} + (x+3y+z)\hat{\mathbf{J}} + (5x+y)\hat{\mathbf{K}}$ through the upper side of the triangle ABC with vertices at the points A(1,0,0), B(0, 1, 0), C(0, 0, 1)

Sol.: Equation of the plane in which the triangle ABC lies is x + y + z =1.

Unit normal $\hat{\mathbf{N}}$ to ABC is

$$\frac{\nabla(x+y+z-1)}{|\nabla(x+y+z-1)|} = \frac{\hat{\mathbf{I}}+\hat{\mathbf{J}}+\hat{\mathbf{K}}}{\sqrt{3}}.$$

$$\mathbf{A}.\hat{\mathbf{N}} = \frac{1}{\sqrt{3}}\left[(x-2z)+(x+3y+z)+(5x+y)\right] = \frac{7x+4y-z}{\sqrt{3}}.$$

Let AOB be the projection of ABC onto the xy-plane, then

$$ds = \frac{dxdy}{\left|\hat{\mathbf{N}}.\hat{\mathbf{K}}\right|} = \sqrt{3}dxdy.$$

Flux across the triangle ABC $= \iint_S \mathbf{A}.\hat{\mathbf{N}}\,ds$

$$= \iint_{AOB} \frac{7x+4y-z}{\sqrt{3}}\sqrt{3}dxdy$$

Replace z by $1-x-y$

$$= \iint \left[7x+4y-(1-x-y)\right]dxdy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (8x + 5y - 1)\, dydx = \frac{5}{3}.$$

Q.No.3.: Evaluate $\int_S \mathbf{F}.\hat{\mathbf{N}}\, ds$, where $\mathbf{F} = 6z\hat{\mathbf{I}} - 4\hat{\mathbf{J}} + y\hat{\mathbf{K}}$ and S is the portion of the plane $2x + 3y + 6z = 12$ in the first octant.

Sol.: Since $\hat{\mathbf{N}}$ is a unit vector normal to the surface $2x + 3y + 6z - 12 = 0$

i.e. $f(x, y) = 2x + 3y + 6z - 12 = 0$.

$$\therefore \hat{\mathbf{N}} = \frac{\nabla f}{|\nabla f|} = \frac{2\hat{\mathbf{I}} + 3\hat{\mathbf{J}} + 6\hat{\mathbf{K}}}{7}.$$

$$\therefore \mathbf{F}.\hat{\mathbf{N}} = \left(6z\hat{\mathbf{I}} - 4\hat{\mathbf{J}} + y\hat{\mathbf{K}}\right).\frac{\left(2\hat{\mathbf{I}} + 3\hat{\mathbf{J}} + 6\hat{\mathbf{K}}\right)}{7} = \frac{12z - 12 + 6y}{7}.$$

Also $ds = \dfrac{dxdy}{\left|\hat{\mathbf{N}}.\hat{\mathbf{K}}\right|} = \dfrac{dxdy}{6/7}$. $\left[\because \left|\hat{\mathbf{N}}.\hat{\mathbf{K}}\right| = \dfrac{6}{7}\right]$

Hence $\int_S \mathbf{F}.d\mathbf{S} = \int_S \mathbf{F}.\hat{\mathbf{N}}\, ds$,

where S is the region bounded by x-axis, y-axis and the planes $2x + 3y = 12$, $z = 0$.

$$\text{Thus } \int_S \mathbf{F}.d\mathbf{S} = \int_S \mathbf{F}.\hat{\mathbf{N}}\, ds = \int_0^4 \int_0^{\left(\frac{12-3y}{2}\right)} \frac{(12z - 12 + 6y)}{7} \frac{dxdy}{6/7} = \int_0^4 \int_0^{\left(\frac{12-3y}{2}\right)} (2z - 2 + y)dxdy$$

$$= \int_0^4 \int_0^{\left(\frac{12-3y}{2}\right)} \left[2\left(\frac{12 - 3y - 2x}{6}\right) - 2 + y\right] dxdy \qquad \left(\because z = \frac{12 - 3y - 2x}{6}\right)$$

$$= \frac{1}{3}\int_0^4 \left[\int_0^{\left(\frac{12-3y}{2}\right)} (6 - 2x)dx\right] dy = \frac{1}{3}\int_0^4 \left[6x - 2\frac{x^2}{2}\right]_0^{\left(\frac{12-3y}{2}\right)} dy$$

$$= \frac{1}{3}\int_0^4 \left[6\left(\frac{12-3y}{2}\right) - \left(\frac{12-3y}{2}\right)^2\right] dy = \frac{1}{12}\int_0^4 \left[36y - 9y^2\right] dy$$

$$= \frac{1}{12}\left[36\frac{y^2}{2} - 9\frac{y^3}{3}\right]_0^4 = \frac{1}{12}\left[36\frac{4^2}{2} - 9\frac{4^3}{3}\right] = \frac{1}{12}[288-192]$$

$$= \frac{1}{12}[96] = 8 \text{. Ans.}$$

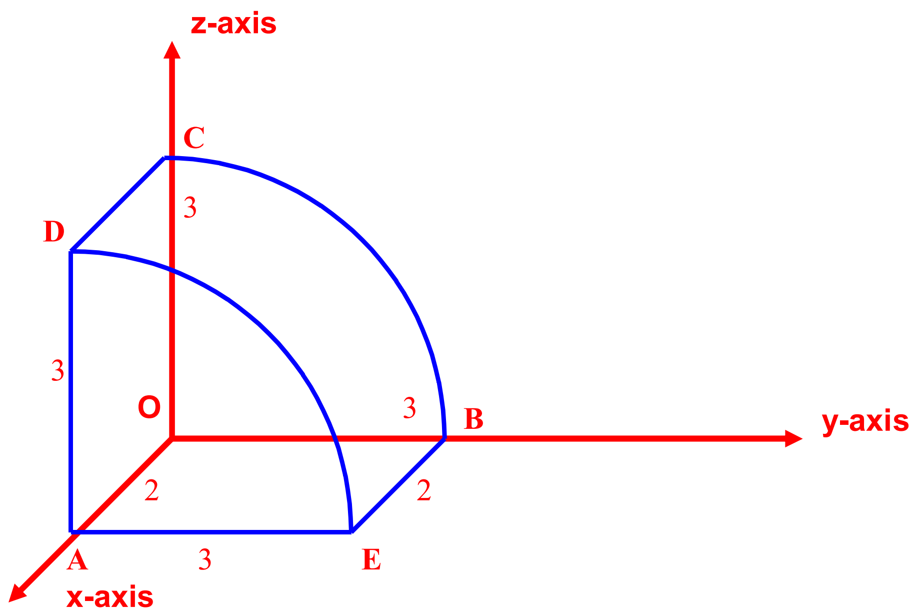
Q.No.4: Evaluate $\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds$ where $\mathbf{F} = 2x^2y\,\hat{\mathbf{I}} - y^2\,\hat{\mathbf{J}} + 4xz^2\,\hat{\mathbf{K}}$ and S is the closed surface of the region in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0,\ x = 2,\ y = 0,\ z = 0.$

Sol.: The given closed surface S is piecewise smooth and is comprised of

S_1 – the rectangular face OAEB in xy-plane; S_2 – the rectangular face OADC in xz-plane;

S_3 – the circular quadrant OBC in yz-plane; S_4 – the circular quadrant AED and

S_5 – the curved surface BCDE of the cylinder in the first octant.



$$\therefore \int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_{S_1} \mathbf{F}.\hat{\mathbf{N}}\,ds + \int_{S_2} \mathbf{F}.\hat{\mathbf{N}}\,ds + \int_{S_3} \mathbf{F}.\hat{\mathbf{N}}\,ds + \int_{S_4} \mathbf{F}.\hat{\mathbf{N}}\,ds + \int_{S_5} \mathbf{F}.\hat{\mathbf{N}}\,ds. \qquad \text{(i)}$$

Now $\int_{S_1} \mathbf{F}.\hat{\mathbf{N}}\, ds = \int_{S_1} \left(2x^2y\,\hat{\mathbf{I}} - y^2\,\hat{\mathbf{J}} + 4xz^2\,\hat{\mathbf{K}}\right).\left(-\hat{\mathbf{K}}\right) ds = -4\int_{S_1} xz^2 ds = 0$.

$[\because z = 0 \text{ in the xy - plane}]$

Similarly $\int_{S_2} \mathbf{F}.\hat{\mathbf{N}}\, ds = \int_{S_2} \left(2x^2y\,\hat{\mathbf{I}} - y^2\,\hat{\mathbf{J}} + 4xz^2\,\hat{\mathbf{K}}\right)\left(-\hat{\mathbf{J}}\right) ds = -\int_{S_2} y^2 ds = 0$.

$[\because y = 0 \text{ in the xz - plane}]$

and $\int_{S_3} \mathbf{F}.\hat{\mathbf{N}}\, ds = \int_{S_3} \left(2x^2y\,\hat{\mathbf{I}} - y^2\,\hat{\mathbf{J}} + 4xz^2\,\hat{\mathbf{K}}\right)\left(-\hat{\mathbf{I}}\right) ds = -2\int_{S_3} x^2 y ds = 0$.

$[\because x = 0 \text{ in the yz - plane}]$

$$\int_{S_4} \mathbf{F}.\hat{\mathbf{N}}\, ds = \int_{S_4} \left(2x^2y\,\hat{\mathbf{I}} - y^2\,\hat{\mathbf{J}} + 4xz^2\,\hat{\mathbf{K}}\right).\hat{\mathbf{I}}\, ds = \int_{S_4} 2x^2 y ds = \int_0^3 \int_0^{\sqrt{9-z^2}} 8y\, dy\, dz$$

$[\therefore x = 2 \text{ in } S_4]$

$$= 4\int_0^3 \left(9 - z^2\right) dz = 72.$$

To find $\hat{\mathbf{N}}$ in S_5, we note that $\nabla\left(y^2 + z^2\right) = 2y\,\hat{\mathbf{J}} + 2z\,\hat{\mathbf{K}}$.

$$\therefore \quad \hat{\mathbf{N}} = \frac{\nabla f}{|\nabla f|} = \frac{2y\,\hat{\mathbf{j}} + 2z\,\hat{\mathbf{K}}}{\sqrt{\left(4y^2 + 4z^2\right)}} = \frac{y\hat{\mathbf{J}} + z\hat{\mathbf{K}}}{3}. \qquad \left[\because y^2 + z^2 = 9\right]$$

$$\therefore \mathbf{F}.\hat{\mathbf{N}} = \left(2x^2y\,\hat{\mathbf{I}} - y^2\,\hat{\mathbf{J}} + 4xz^2\,\hat{\mathbf{K}}\right).\left(\frac{y\hat{\mathbf{J}} + z\hat{\mathbf{K}}}{3}\right) = \frac{\left(-y^3 + 4xz^3\right)}{3}.$$

Also $\left|\hat{\mathbf{N}}.\hat{\mathbf{K}}\right| = \left|\left(\frac{y\hat{\mathbf{J}} + z\hat{\mathbf{K}}}{3}\right).\hat{\mathbf{K}}\right| = \frac{z}{3}$ so that $ds = \frac{dxdy}{\left|\hat{\mathbf{N}}.\hat{\mathbf{K}}\right|} = \frac{dxdy}{\left(\frac{z}{3}\right)}$.

$$\int_{S_5} \mathbf{F}.\hat{\mathbf{N}}\, ds = \int_0^2 \int_0^3 \frac{\left(-y^3 + 4xz^3\right)}{3} . \frac{dxdy}{\left(\frac{z}{3}\right)} = \int_0^2 \left[\int_0^3 \left(\frac{-y^3}{z} + 4xz^2\right) dy\right] dx.$$

Put $y = 3\sin\theta, \quad z = 3\cos\theta, \quad \therefore dy = 3\cos\theta\, d\theta$.

$$\int_{S_5} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_0^2 \int_0^{\pi/2} \left[\frac{-27\sin^3\theta}{3\cos\theta} + 4x\left(9\cos^2\theta\right)\right] 3\cos\theta\, d\theta\, dx$$

$$= \int_0^2 \left\{\int_0^{\pi/2} \left(-27\sin^3\theta + 108x\cos^3\theta\right) d\theta\right\} dx = \int_0^2 \left[-27\times\frac{2}{3} + 108x\times\frac{2}{3}\right] dx$$

$$= \int_0^2 (-18 + 72x)\,dx = \left[-18x + \frac{72x^2}{2}\right]_0^2 = -36 + 144 = 108\,.$$

Hence, (i) gives $\int_S \mathbf{F}.\hat{\mathbf{N}}\, ds = 0 + 0 + 0 + 72 + 108 = 180$. **Ans.**

Q.No.5.: Evaluate $\iint_S \mathbf{A}.\hat{\mathbf{N}}\,ds$ over the entire surface S of the region bounded by the cylinder $x^2 + z^2 = 9$, x = 0, y = 0, z = 0 and y = 8

where $\mathbf{A} = 6z\hat{\mathbf{I}} + (2x + y)\hat{\mathbf{J}} - x\hat{\mathbf{K}}$.

Sol.: Here the entire surface S consists of 5 surfaces namely S_1 the curved (lateral) surface of the cylinder, ABDCA, S_2: AOEC, S_3: OBDE, S_4: OAB, S_5: CDE. Thus

$$\iint_S \mathbf{A}.\hat{\mathbf{n}}\,ds = \iint_{S_1+S_2+\ldots\ldots+S_5} = \iint_{S_1} + \iint_{S_2} + \ldots\ldots + \iint_{S_5} = SI_1 + SI_2 + SI_3 + SI_4 + SI_5\,.$$

For the curved (lateral) surface S_1 of the cylinder

$$\text{Unit normal } \hat{\mathbf{N}} = \frac{\nabla\left(x^2 + z^2\right)}{\left|\nabla\left(x^2 + z^2\right)\right|} = \frac{2x\hat{\mathbf{I}} + 2z\hat{\mathbf{K}}}{\sqrt{4x^2 + 4z^2}}\,\frac{x\hat{\mathbf{I}} + z\hat{\mathbf{K}}}{3}$$

$$\text{So } \mathbf{A}.\hat{\mathbf{N}} = \left(6z\hat{\mathbf{I}} + (2x + y)\hat{\mathbf{J}} - x\hat{\mathbf{K}}\right).\left(\frac{x\hat{\mathbf{I}} + z\hat{\mathbf{K}}}{3}\right) = \frac{5}{3}xz$$

$$\hat{\mathbf{N}}.\hat{\mathbf{K}} = \frac{z}{3}$$

$$SI_1 = \iint_{S_1} \mathbf{A}.\hat{\mathbf{N}}\,ds = \iint \frac{\mathbf{A}.\hat{\mathbf{N}}}{\hat{\mathbf{N}}.\hat{\mathbf{k}}}\,dxdy$$

$$= \iint \frac{5}{3}\frac{xz}{\left(\frac{z}{3}\right)} dxdy = 5\int_0^8 \int_0^3 xdxdy = 180$$

On the plane S_2: AOEC : $z = 0$, $\overset{\wedge}{\mathbf{N}} = -\overset{\wedge}{\mathbf{K}}$, $\mathbf{A}.\overset{\wedge}{\mathbf{N}} = x$

$$SI_2 = \iint_{S_2} \mathbf{A}.\overset{\wedge}{\mathbf{N}}\, ds = \int_0^8 \int_0^3 xdxdy = 36$$

On the plane S_3: OBDE: $x = 0$, $\overset{\wedge}{\mathbf{N}} = -\overset{\wedge}{\mathbf{I}}$, $\mathbf{A}.\overset{\wedge}{\mathbf{N}} = -6z$

$$SI_3 = \iint_{S_3} \mathbf{A}.\overset{\wedge}{\mathbf{N}}\, ds = \int_0^8 \int_0^3 -6zdzdy = -216$$

On the Sector S_4: OAB: $y = 0$, $\overset{\wedge}{\mathbf{N}} = -\overset{\wedge}{\mathbf{J}}$, $\mathbf{A}.\overset{\wedge}{\mathbf{N}} = -(2x + y) = -2x$

$$SI_4 = \iint_{S_4} \mathbf{A}.\overset{\wedge}{\mathbf{N}}\, ds = \iint_{OAB} -2xdxdz$$

In polar coordinates

$$SI_4 = \int_0^{\pi/2} \int_0^3 -2.r\cos t.rdrdt = -18$$

On the Sector S_5: CDE: $y = 8$, $\overset{\wedge}{\mathbf{N}} = \overset{\wedge}{\mathbf{J}}$, $\mathbf{A}.\overset{\wedge}{\mathbf{N}} = 2x + y = 2x + 8$

$$SI_5 = \iint_{S_5} \mathbf{A}.\overset{\wedge}{\mathbf{N}}\, ds = \iint_{CDE} (2x + 8)dxdz$$

In polar coordinates

$$SI_5 = \int_0^{\pi/2} \int_0^3 (2.r\cos t + 8)drdt = 18 + 18\pi$$

Thus the required surface integral is

$$SI = (180) + (36) + (-216) + (-18) + (18 + 18\pi) = 18\pi\,.$$

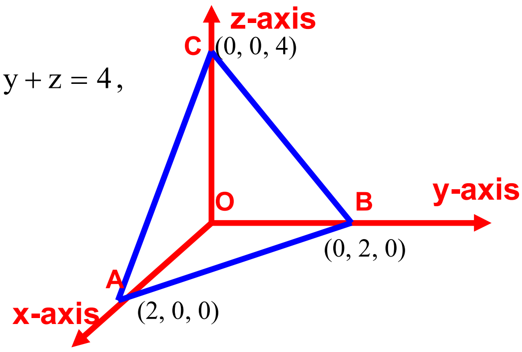
Q.No.6.: Evaluate $\int_S \mathbf{F}.d\mathbf{S}$, where $\mathbf{F} = x\overset{\wedge}{\mathbf{I}} + (z^2 - zx)\overset{\wedge}{\mathbf{J}} - xy\overset{\wedge}{\mathbf{K}}$ and S is the triangular surface with vertices $(2,\ 0,\ 0), (0,\ 2,\ 0)$ and $(0,\ 0,\ 4)$.

or

Evaluate $\int_S \mathbf{F}.d\mathbf{S}$, where $\mathbf{F} = x\hat{\mathbf{I}} + (z^2 - zx)\hat{\mathbf{J}} - xy\hat{\mathbf{K}}$ and S is the triangular surface $2x + 2y = 4 - z$ lying in the first octant.

Sol.: Since, we know that the equation of the triangular surface with vertices

$$(2,\ 0,\ 0), (0,\ 2,\ 0) \text{ and } (0,\ 0,\ 4) \quad \text{is } \frac{x}{2} + \frac{y}{2} + \frac{z}{4} = 1 \Rightarrow 2x + 2y + z = 4.$$



Since $\hat{\mathbf{N}}$ is a unit vector normal to the surface $2x + 2y + z = 4$,

i.e. $f(x, y, y) = 2x + 2y + z - 4 = 0$.

$$\therefore \hat{\mathbf{N}} = \frac{\nabla f}{|\nabla f|} = \frac{2\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + \hat{\mathbf{K}}}{3}.$$

$$\therefore \mathbf{F}.\hat{\mathbf{N}} = \left[x\hat{\mathbf{I}} + (z^2 - zx)\hat{\mathbf{J}} - xy\hat{\mathbf{K}}\right].\frac{\left(2\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + \hat{\mathbf{K}}\right)}{3} = \frac{2x + 2(z^2 - zx) - xy}{3}.$$

Also $ds = \dfrac{dxdy}{\left|\hat{\mathbf{N}}.\hat{\mathbf{K}}\right|} = \dfrac{dxdy}{1/3}$. $\qquad \left[\because \left|\hat{\mathbf{N}}.\hat{\mathbf{K}}\right| = \frac{1}{3}\right]$

Hence $\int_S \mathbf{F}.d\mathbf{S} = \int_S \mathbf{F}.\hat{\mathbf{N}}\, ds$,

where S is the surface $2x + 2y + z = 4$ and this surface lying in the first octant.

$$= \int_0^2 \int_0^{2-x} \frac{\left[2x + 2(z^2 - zx) - xy\right]}{3} \frac{dydx}{\frac{1}{3}} = \int_0^2 \int_0^{2-x} \left[2x + 2(z^2 - zx) - xy\right] dydx$$

$$= \int_0^2 \int_0^{2-x} \left[2x + 2(4 - 2x - 2y)^2 - 2(4 - 2x - 2y)x - xy\right] dydx \qquad (\because z = 4 - 2x - 2y)$$

$$= \int_0^2 \int_0^{2-x} \left[2x + 2(4x^2 + 4y^2 - 16x - 16y + 8xy + 16) - 2(4x - 2x^2 - 2xy) - xy\right] dydx$$

$$= \int_0^2 \left[\int_0^{2-x} \left[12x^2 + 8y^2 - 38x - 32y + 19xy + 32\right] dy \right] dx$$

$$= \int_0^2 \left[12x^2 y + 8\frac{y^3}{3} - 38xy - 32\frac{y^2}{2} + 19x\frac{y^2}{2} + 32y \right]_0^{2-x} dx$$

$$= \int_0^2 \left[12x^2(2-x) + 8\frac{(2-x)^3}{3} - 38x(2-x) - 32\frac{(2-x)^2}{2} + 19x\frac{(2-x)^2}{2} + 32(2-x) \right] dx$$

$$= \int_0^2 \left[-\frac{31}{6}x^3 + 24x^2 - 38x + \frac{64}{3} \right] dx = \left[-\frac{31}{6}\frac{x^4}{4} + 24\frac{x^3}{3} - 38\frac{x^2}{2} + \frac{64}{3}x \right]_0^2$$

$$= \left[-\frac{31}{6}\frac{2^4}{4} + 24\frac{2^3}{3} - 38\frac{2^2}{2} + \frac{64}{3}2 \right] = \left[-\frac{62}{3} + 64 - 76 + \frac{128}{3} \right] = \left[\frac{66}{3} - 12 \right]$$

$= [22 - 12] = 10$. Ans.

Q.No.7.: Evaluate **(a).** $\iint_S (\nabla \times \mathbf{F}).\hat{\mathbf{N}}\, ds$ and **(b).** $\iint \varphi \hat{\mathbf{N}}\, ds$ if $\mathbf{F} = (x + 2y)\hat{\mathbf{I}} - 3z\hat{\mathbf{J}} + x\hat{\mathbf{K}}$, $\varphi = 4x + 3y - 2z$ and S is the surface of the plane $2x + y + 2z = 6$ bounded by the coordinate planes x = 0, y = 0 and z = 0.

Sol.: The unit normal $\hat{\mathbf{N}}$ to the surface S is

$$\hat{\mathbf{N}} = \frac{\nabla(2x + y + 2z)}{|\nabla(2x + y + 2z)|} = \frac{2\hat{\mathbf{I}} + \hat{\mathbf{J}} + 2\hat{\mathbf{K}}}{\sqrt{4+1+4}} = \frac{2\hat{\mathbf{I}} + \hat{\mathbf{J}} + 2\hat{\mathbf{K}}}{3}$$

(a). $\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y & -3z & x \end{vmatrix} = 3\hat{\mathbf{I}} - \hat{\mathbf{J}} - 2\hat{\mathbf{K}}$

$$(\nabla \times \mathbf{F}).\hat{\mathbf{N}} = \left(3\hat{\mathbf{I}} - \hat{\mathbf{J}} - 2\hat{\mathbf{K}}\right).\left(\frac{2\hat{\mathbf{I}} + \hat{\mathbf{J}} + 2\hat{\mathbf{K}}}{3}\right) = \frac{1}{3}$$

$$SI = \iint_S (\nabla \times \mathbf{F}).\hat{\mathbf{N}}\, ds = \frac{1}{3}\iint_S \quad = \frac{1}{3}\iint \frac{dxdy}{|\hat{\mathbf{N}}.\hat{\mathbf{K}}|}$$

$$=\frac{1}{3}.\int_{x=0}^{3}\int_{y=0}^{6-2x}\frac{dydx}{\frac{2}{3}}=\frac{1}{2}\int_{0}^{3}(6-2x)dx=\frac{9}{2}.$$

(b). $SI=\iint \varphi\hat{N}ds=\iint (4x+3y-2z)\frac{2\hat{I}+\hat{J}+2\hat{K}}{3}ds$

Eliminating z using, $z=\frac{6-2x-y}{2}$

$$SI=\frac{2\hat{I}+\hat{J}+2\hat{K}}{3}.\int_{x=0}^{3}\int_{y=0}^{6-2x}(6x+4y-6)\frac{dydx}{\frac{2}{3}}$$

$$=\left(2\hat{I}+\hat{J}+2\hat{K}\right)\int_{0}^{3}\left[3(x-1)(6-2x)+(6-2x)^{2}\right]dx$$

$SI=72\hat{I}+36\hat{J}+72\hat{K}$.

Q.No.8.: Find the surface area of the plane $x+2y+2z=12$ cut off by x = 0, y = 0, and $x^2-y^2=16$.

Sol.: Rewriting equation of plane $z=\frac{12-x-2y}{2}$

We have $z_x=-\frac{1}{2}$, $z_y=-1$

Surface area $=\iint_R \sqrt{1+z_x^2+z_y^2}dxdy=\iint \sqrt{1+1+\frac{1}{4}}dxdy=\frac{3}{2}\iint dxdy$

In polar coordinates $=\frac{3}{2}\int_{0}^{\pi/2}\int_{0}^{4}rdrd\theta=6\pi$.

Aliter: $F(x,y,z)=x+2y+2z-12=0$, so $F_x=1$, $F_y=2$, $F_z=2$

$\sqrt{F_x^2+F_y^2+F_z^2}=\sqrt{1+4+4}=3$

Surface area $=\iint \frac{\sqrt{F_x^2+F_y^2+F_z^2}}{|F_z|}dxdy=\iint \frac{3}{2}dxdy=6\pi$.

Home Assignments

Q.No.1.: If S is the surface $2x+y+2z=6$ bounded by x = 0, x = 1, y = 1, y = 2,

Evaluate **(a).** $\iint (\nabla\times\mathbf{F}).\hat{\mathbf{N}}\,ds$ **(b).** $\iint_S \varphi\hat{\mathbf{N}}\,ds$.

Ans.: (a). 1 **(b).** $2\hat{\mathbf{I}}+\hat{\mathbf{J}}+2\hat{\mathbf{K}}$

Q.No.2.: Evaluate $\iint_S \mathbf{A}.\hat{\mathbf{N}}\,ds$, where $\mathbf{A}=18z\hat{\mathbf{I}}-12\hat{\mathbf{J}}+3y\hat{\mathbf{K}}$ and S is that part of the plane $2x+3y+6z=12$ which is located in the first octant.

Ans.: 24.

Q.No.3.: Evaluate $\iint_S (\nabla\times\mathbf{F}).\hat{\mathbf{N}}\,ds$ where $\mathbf{F}=y\hat{\mathbf{I}}+(x-2xz)\hat{\mathbf{J}}+xy\hat{\mathbf{K}}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane.

Ans.: 0.

Q.No.4.: If S is the entire surface of the cube bounded by x = 0, x = b, y = 0, y = b, z = 0, and z = b and $\mathbf{A}=4xz\hat{\mathbf{I}}-y^2\hat{\mathbf{J}}+yz\hat{\mathbf{K}}$ then evaluate $\iint_S \mathbf{F}.\hat{\mathbf{N}}\,ds$.

Ans.: $\frac{3b^4}{2}$.

Q.No.5.: Let S be the surface of the cylinder $x^2+y^2=16$ included in the first octant between z = 0 and z = 5. Evaluate $\iint_S \mathbf{A}.\hat{\mathbf{N}}\,ds$, where $\mathbf{A}=z\hat{\mathbf{I}}+x\hat{\mathbf{J}}-3y^2z\hat{\mathbf{K}}$.

Ans.: 90.

Q.No.6.: Let S be the surface of the cylinder $x^2+y^2=16$ included in the first octant between z = 0 and z = 5, Evaluate $\iint_S \varphi.\hat{\mathbf{N}}\,ds$, where $\varphi=\frac{3}{8}xyz$.

Ans.: $100\left(\hat{\mathbf{I}}+\hat{\mathbf{J}}\right)$.

Q.No.7.: Find the surface integral over the parallelopiped x = 0, y = 0, z = 0, x = 1, y = 2, z = 3 when $\mathbf{A} = 2xy\hat{\mathbf{I}} + yz^2\hat{\mathbf{J}} + xz\hat{\mathbf{K}}$.

Ans.: 33.

Q.No.8.: If S is the surface of the sphere $x^2 + y^2 + z^2 = d^2$ and $\mathbf{A} = ax\hat{\mathbf{I}} + by\hat{\mathbf{J}} + cz\hat{\mathbf{K}}$, evaluate $\iint_S \mathbf{F}.\hat{\mathbf{N}}\,ds$.

Ans.: $2.\dfrac{2\pi d^3}{3}(a+b+c)$.

Q.No.9.: Let S be the surface of the cylinder $x^2 + y^2 = a^2$ in the first octant between the planes z = 0 and z = h. Evaluate $\iint \mathbf{A}.\hat{\mathbf{N}}\,ds$ where $\mathbf{A} = z\hat{\mathbf{I}} + x\hat{\mathbf{J}} - 3zy^2\hat{\mathbf{K}}$.

Ans.: $\dfrac{ah(a+h)}{2}$.

Flux:

Q.No.10.: Find the flux across the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes y = 4 and z = 6 when the velocity vector $\mathbf{V} = 2y\hat{\mathbf{I}} - z\hat{\mathbf{J}} + x^2\hat{\mathbf{K}}$.

Ans.: 132.

Q.No.11.: Find the flux of $\mathbf{A} = \hat{\mathbf{I}} - \hat{\mathbf{J}} + xyz\hat{\mathbf{K}}$ through the circular region S obtained by cutting the sphere $x^2 + y^2 + z^2 = a^2$ with a plane y = x (take the side of S facing the positive side of the x-axis).

Hint: S is bounded by the ellipse $2x^2 + z^2 = a^2$, $\hat{\mathbf{N}} = \dfrac{\left(\hat{\mathbf{I}} - \hat{\mathbf{J}}\right)}{\sqrt{2}}$, $dS = \sqrt{2}dxdz$ area of the ellipse with semi axis $\dfrac{a}{\sqrt{2}}$ and a is $\dfrac{\pi a^2}{\sqrt{2}}$.

Ans.: $\sqrt{2}\pi a^2$.

Q.No.12.: Compute the flux of the vector $\mathbf{A} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + \sqrt{x^2+y^2-1}\,\hat{\mathbf{K}}$ through the outer side of the hyperboloid of one sheet $z = \sqrt{x^2+y^2-1}$ bounded by the planes z = 0 and $z = \sqrt{3}\pi$.

Hint: $\hat{\mathbf{N}} = \dfrac{x\hat{\mathbf{I}} + y\hat{\mathbf{J}}}{\sqrt{x^2+y^2-1}} - k$, $\mathbf{A}.\hat{\mathbf{N}} = \dfrac{1}{\sqrt{x^2+y^2-1}}$ with polar coordinates,

Flux $= \displaystyle\int_0^{2\pi}\int_1^{2} \frac{r\,dr\,d\theta}{\sqrt{r^2-1}} = 2\sqrt{3}\pi$

Ans.: $z = \sqrt{3}\pi$.

Q.No.13.: Evaluate $\iint_S \mathbf{F}.\hat{\mathbf{N}}\,ds$ where $\mathbf{F} = \dfrac{\hat{\mathbf{R}}}{r^3}$ and S is the sphere $x^2+y^2+z^2=b^2$.

Ans.: 4π.

Q.No.14.: Calculate the surface integral of the vector function $\mathbf{A} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}}$ over the portion of the surface of the unit sphere S: $x^2+y^2+z^2=1$ above the xy-plane $z \geq 0$.

Ans.: $\dfrac{4\pi}{3}$.

Q.No.15.: If S is the triangular surface with vertices (2, 0, 0), (0, 2, 0) and $\mathbf{A} = x\hat{\mathbf{I}} + \left(z^2 - zx\right)\hat{\mathbf{J}} + xy\hat{\mathbf{K}}$ then evaluate $\iint_S \mathbf{F}.\hat{\mathbf{N}}\,ds$.

Ans.: $-\dfrac{22}{3}$.

Surface Area:

Q.No.16.: What is the surface area of the surface S whose equation is F(x, y, z) = 0?

Ans.: $\displaystyle\iint_R \frac{\sqrt{F_x^2+F_y^2+F_z^2}}{|F_z|}dxdy$, where R is the projection of S on xy-plane.

Q.No.17.: Find the surface area of the plane $x+2y+2z=12$ cut off by x = 0, y = 0, x = 1, y =1.

Ans.: $\frac{3}{2}$.

Q.No.18.: Find the surface area of $z = x^2 + y^2$ included between z = 0 and z = 1.

Ans.: $\frac{\pi}{6}\left(\sqrt{125} - 1\right)$.

Q.No.19.: Find the surface area of the region common to the intersecting cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Ans.: $16a^2$.

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8th Topic

Vectors Calculus

Green's Theorem in the plane

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(Last updated on 12-11-2009)

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Green's Theorem in the plane:

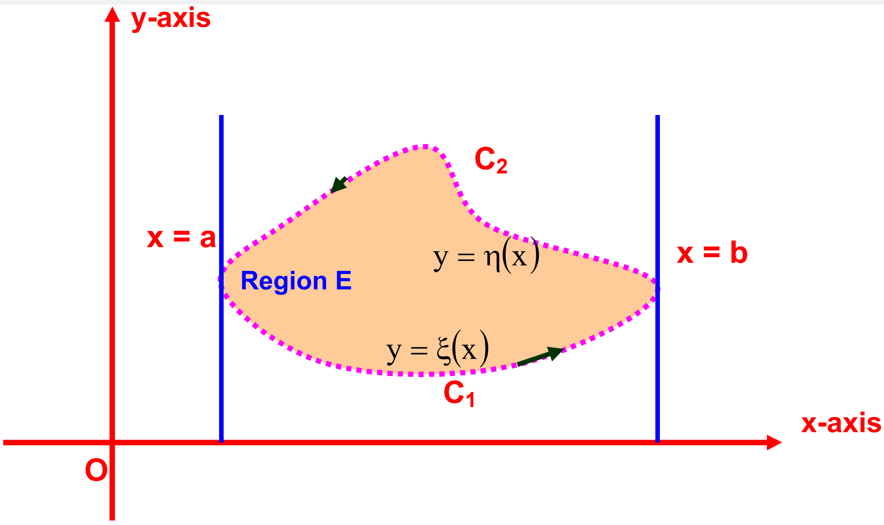
(Conversion of a line integral around a closed curve into a double integral)

(Relation between line and surface integral in two dimensions)

Statement: If $\phi(x, y), \psi(x, y), \phi_y$ and ψ_x be continuous in a region E of the xy-plane bounded by closed curve C, then

$$\int_C (\phi dx + \psi dy) = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy. \qquad \text{(i)}$$

i.e.
$$\int_C \mathbf{F}.d\mathbf{R} = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy.$$



Proof: Consider the region E bounded by a single closed curve C, which is cut by any line parallel to the axis at the most in two parts.

Let E be bounded by $x = a,\ y = \xi(x),\ x = b$ and $y = \eta(x)$, where, $\eta \geq \xi$, so that C is divided into curves C_1 and C_2 (see fig.).

$$\therefore \iint_E \frac{\partial \phi}{\partial y} dxdy = \int_a^b \left[\int_\xi^\eta \frac{\partial \phi}{\partial y} dy\right] dx = \int_a^b \left[\phi\Big|_\xi^\eta\right] dx = \int_a^b [\phi(x,\eta) - \phi(x,\xi)] dx$$

$$= -\int_{C_2} \phi(x, y)dx - \int_{C_1} \phi(x, y)dx$$

$$= -\int_C \phi(x, y)dx. \qquad \text{(ii)}$$

Similarly, it can be shown that

$$\iint_E \frac{\partial \psi}{\partial x} dxdy = \int_C \psi(x, y)dy. \qquad \text{(iii)}$$

Subtracting (ii) from (iii), we get

$$\int_c (\phi dx + \psi dy) = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy.$$

This completes the proof of Green's theorem in the plane.

Remarks:

1. This result can be extended to regions which may be divided into a finite number of sub-regions such that the boundary of each is cut at the most in two points by any line parallel to either axis. Applying (i) to each of these sub-regions and adding the results, the surface integrals combine into an integral over the whole region; the line integrals over the common boundaries cancel (for each is covered twice but in opposite directions), whereas the remaining line integrals combine into the line integral over the external curve C.

2. This theorem converts a line integral around a closed curve into a double integral and is a special case of Stoke's theorem, which we will discuss later on.

Corollary 1: Vector notation of Green's theorem:

Let $\mathbf{F} = \phi\hat{\mathbf{I}} + \psi\hat{\mathbf{J}}$ and $\mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}}$ so that

$\mathbf{F}.d\mathbf{R} = \phi dx + \psi dy$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi & \psi & O \end{vmatrix} = \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right)\hat{\mathbf{K}} \Rightarrow (\nabla \times \mathbf{F}).\hat{\mathbf{K}} = \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right).$$

Since $\int_c (\phi dx + \psi dy) = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy . \Rightarrow \oint_C \mathbf{F}.d\mathbf{R} = \iint_R (\nabla \times \mathbf{F}).\hat{\mathbf{K}}\, dxdy$

where $dR = dxdy$.

Corollary 2: Area A of the plane region R bounded by the simple closed curve c:

Let $\psi = x, \;\; \phi = -y$ so that

$$\oint_c (xdy - ydx) = \iint_R (1+1)dxdy = 2\iint_R dxdy = 2A .$$

Thus $A = \frac{1}{2}\oint_c (xdy - ydx)$.

Corollary 3: Area A in polar co-ordinates

Let $x = r\cos t, \;\; y = r\sin t$ so that

$dx = \cos tdr - r\sin tdt$

$dy = \sin tdr + r\sin tdt$.

Thus $A = \frac{1}{2}\int_c r^2 dt$.

Corollary 4: Green's theorem is valid for a doubly (multiply) connected domain R where c is the boundary of the region R consisting of c_1 and c_2 (several) curves all traversed in the positive direction.

Corollary 5: If $\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$, then by Green's theorem

$$\oint_c (\phi dx + \psi dy) = 0 .$$

Let us verify Green's theorem:

Q.No.1.: Verify Green's theorem for $\int_C \left[\left(xy + y^2\right)dx + x^2 dy\right]$, where C is bounded by

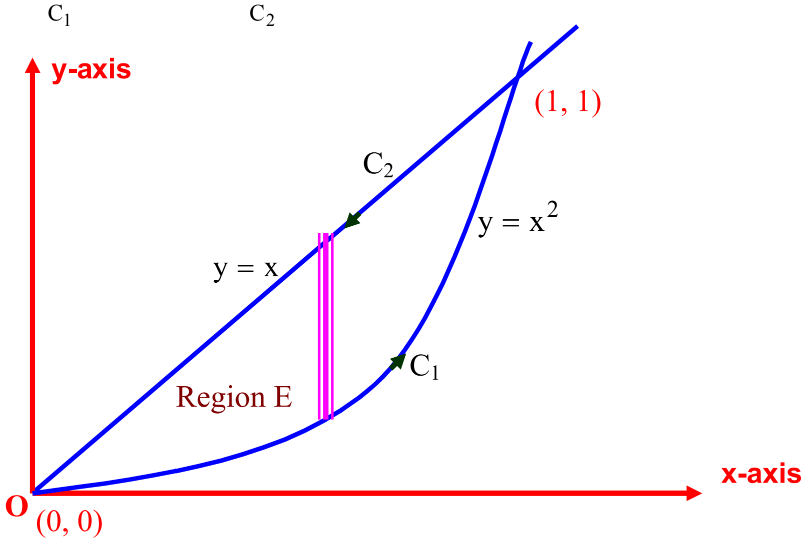
$y = x$ and $y = x^2$.

Proof: Green's theorem states that, if $\phi(x, y), \psi(x, y), \phi_y$ and ψ_x be continuous in a region E of the x y-plane bounded by closed curve C, then

$$\int_C (\phi dx + \psi dy) = \iint_E \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)dxdy .$$

Here $\phi = xy + y^2$ and $\psi = x^2$.

$$\therefore \int_C (\phi dx + \psi dy) = \int_{C_1} (\phi dx + \psi dy) + \int_{C_2} (\phi dx + \psi dy)$$



Along C_1, $y = x^2$ and x varies from 0 to 1

$$\therefore \int_{C_1} (\phi dx + \psi dy) = \int_0^1 \left[\left\{x\left(x^2\right) + \left(x^2\right)^2\right\}dx + x^2 d\left(x^2\right)\right]$$

$$= \int_0^1 \left(3x^3 + x^4\right)dx = \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \left(\frac{3}{4} + \frac{1}{5}\right) - 0 = \frac{15+4}{20} = \frac{19}{20}.$$

Along $C_{2,}$ $y = x$ and x varies from 1 to 0.

$$\therefore \int_{C_2} (\phi dx + \psi dy) = \int_1^0 \left[\left\{x(x) + (x)^2\right\}dx + x^2 d(x)\right] = \int_1^0 3x^2 dx = \left[\frac{3x^3}{3}\right]_0^3 = [0 - 1] = -1.$$

Thus $\int_C (\phi dx + \psi dx) = \frac{19}{20} - 1 = -\frac{1}{20}$. (i)

Also $\iint_E \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)dxdy = \iint_E \left[\frac{\partial}{\partial x}\left(x^2\right) - \frac{\partial}{\partial y}\left(xy + y^2\right)\right]dxdy$

$$= \int_0^1 \left(\int_{x^2}^{x} (2x - x - 2y)dy\right)dx = \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[\left(x.x - x^2\right) - \left(x.x^2 - x^4\right)\right]dx = \int_0^1 \left(0 - x^3 + x^4\right)dx$$

$$= \int_0^1 \left(x^4 - x^3\right)dx = \left(\frac{x^5}{5} - \frac{x^4}{4}\right)_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = -\frac{1}{20}. \qquad \text{(ii)}$$

From (i) and (ii), we get

$$\int_c (\phi dx + \psi dy) = \iint_E \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)dxdy.$$

Hence, Green's theorem is verified.

Q.No.2.: **Verify Green's theorem** for

$$\int_C \left[\left(3x - 8y^2\right)dx + (4y - 6xy)dy\right],$$

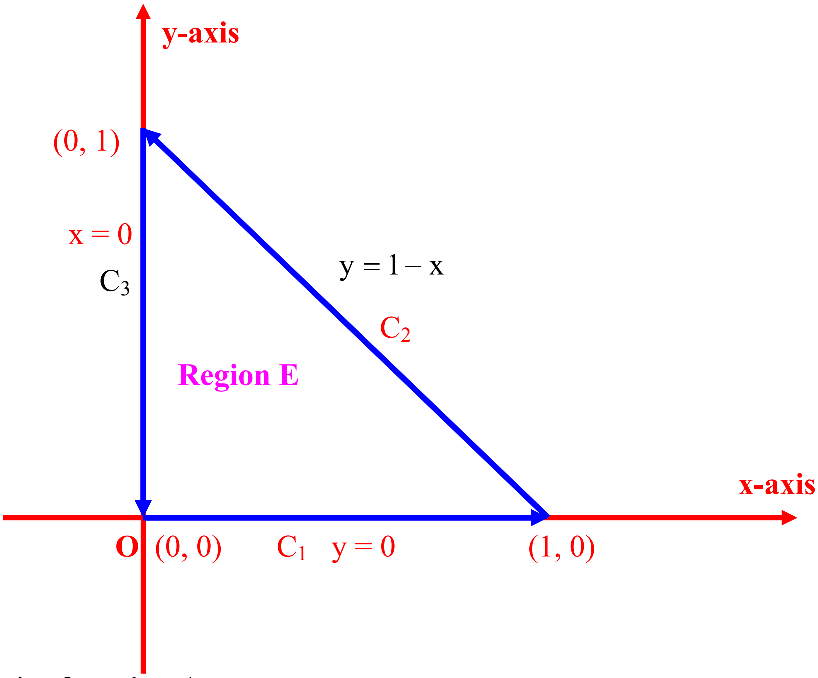
where C is the boundary of region bounded by $x = 0$, $y = 0$ and $x + y = 1$.

Sol.: Since we know that, if $\phi(x,y)$, $\psi(x,y)$, ϕ_y and ψ_x be continuous in a region E of the xy-plane bounded by a closed curve C, then

$$\int_C (\phi dx + \psi dy) = \iint_E \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)dxdy. \quad \text{[Green's theorem]}$$

Here $\phi = 3x - 8y^2$ and $\psi = 4y - 6xy$.

$$\therefore \int_C (\phi dx + \psi dy) = \int_{C_1} (\phi dx + \psi dy) + \int_{C_2} (\phi dx + \psi dy) + \int_{C_3} (\phi dx + \psi dy).$$



Along C_1, $y = 0$ and x varies from 0 to 1

$$\therefore \int_{C_1} (\phi dx + \psi dy) = \int_{C_1} \left[\left(3x - 8y^2\right)dx + (4y - 6xy)dy\right] = \int_0^1 3dx = 3\left[\frac{x^2}{2}\right]_0^1 = \frac{3}{2}.$$

Along C_2, $y = 1 - x$ and x varies from 1 to 0

$$\therefore \int_{C_2} (\phi dx + \psi dy) = \int_{C_2} \left[3xdx - 8(1-x)^2 dx + 4(1-x)d(1-x) - 6x(1-x)d(1-x)\right]$$

$$= \int_0^1 \left[3xdx - 8dx + 16xdx - 8x^2dx - 4dx + 4xdx + 6xdx - 6x^2dx\right]$$

$$= \int_0^1 \left[29xdx - 14x^2dx - 12dx\right]$$

$$= 29\left[\frac{x^2}{2}\right]_1^0 - 14\left[\frac{x^3}{3}\right]_1^0 - 12[x]_1^0 = -\frac{29}{2} + \frac{14}{3} + 12 = \frac{-87 + 28 + 72}{6}$$

$$= \frac{13}{6}.$$

Along C_3, $x = 0$ and y varies from 1 to 0

$$\therefore \int_{C_3} (\phi dx + \psi dy) = \int_{C_3} \left[\left(3x - 8y^2\right)dx + (4y - 6xy)dy\right] = \int_1^0 4ydy = 4\left[\frac{y^2}{2}\right]_1^0 = -2.$$

Hence $\int\limits_{C}(\phi dx+\psi dy)=\int\limits_{C_1}(\phi dx+\psi dy)+\int\limits_{C_2}(\phi dx+\psi dy)+\int\limits_{C_3}(\phi dx+\psi dy)$.

$$=\frac{3}{2}+\frac{13}{6}-2=\frac{5}{3}. \qquad ....(i)$$

Also $\iint\limits_{E}\left(\frac{\partial\psi}{\partial x}-\frac{\partial\phi}{\partial y}\right)dxdy=\iint\limits_{E}\left(\frac{\partial(4y-6xy)}{\partial x}-\frac{\partial\left(3x-8y^2\right)}{\partial y}\right)dxdy=\iint\limits_{E}(-6y+16y)dxdy$

$$=\iint\limits_{E}(-6y+16y)dxdy=\left[\iint\limits_{E}10ydy\right]dx=10\int_0^1\left[\frac{y^2}{2}\right]_0^{1-x}dx$$

$$=5\int_0^1(1-x)^2dx=5\int_0^1\left(1-2x+x^2\right)dx=5\left[x-x^2+\frac{x^3}{3}\right]_0^1$$

$$=5\left[1-1+\frac{1}{3}\right]=\frac{5}{3}. \qquad ...(ii)$$

Hence, Green's theorem is verified from the equality of (i) and (ii).

Q.No.3.: **Verify Green's theorem** in plane for $\oint\limits_{c}\left(x^2-2xy\right)dx+\left(x^2y+3\right)dy$, where c is the boundary of the region defined by $y^2=8x$ and $x=2$.

Sol.: Green's theorem states that Line integral = Double integral

(a). The LHS of the Green's theorem result is the line integral

$$=\text{LI}\oint\limits_{c}\left(x^2-2xy\right)dx+\left(x^2y+3\right)dy$$

Here c consists of the curves OA, ADB, BO, so

$$\text{LI}=\oint\limits_{c}=\int\limits_{OA+ADB+BO}=\int\limits_{OA}+\int\limits_{ADB}+\int\limits_{BO}=\text{LI}_1+\text{LI}_2+\text{LI}_3$$

Along OA: $y=-2\sqrt{2}\sqrt{x}$, so $dy=-\sqrt{\frac{2}{x}}dx$

$$\text{LI}_1=\int\limits_{OA}\left(x^2-2xy\right)dx+\left(x^2y+3\right)dy$$

$$=\int_0^2\left[x^2-2x\left(-2\sqrt{2}\sqrt{x}\right)\right]dx+\left[x^2-\sqrt{2}\sqrt{x}+3\right]\left(-\sqrt{\frac{2}{x}}\right)dx$$

$$=\int_0^2 \left(5x^2+4\sqrt{2}x^{3/2}-3\sqrt{2}x^{-1/2}\right)dx = \frac{5x^3}{3}+4\sqrt{3}.\frac{2}{5}.x^{5/2}-3\sqrt{2}.2\sqrt{x}\Big|_0^2$$

$$=\frac{40}{3}+\frac{64}{5}-12.$$

Along ADB: $x = 2,\ dx = 0$

$$LI_2 = \int_{ADB} \left(x^2-2xy\right)dx+\left(x^2y+3\right)dy = \int_{-4}^{4} \left(4y+3\right)dy = 24$$

Along BO: $y = 2\sqrt{2}\sqrt{x}$ with x: 2 to 0, $dy = \sqrt{\frac{2}{x}}dx$

$$LI_3 = \int_{BO} \left(x^2-2xy\right)dx+\left(x^2y+3\right)dy = \int_2^0 \left(5x^2-4\sqrt{2}x^{3/2}+3\sqrt{2}x^{-1/2}\right)dx = -\frac{40}{3}+\frac{64}{5}-12.$$

$$LI = LI_1 + LI_2 + LI_3 = \left(\frac{40}{3}+\frac{64}{5}-12\right)+(24)+\left(-\frac{40}{3}+\frac{64}{5}-12\right)=\frac{128}{5}.$$

(b). Here $M = x^2-2xy,\quad N = x^2y+3$

$$\frac{\partial M}{\partial y} = -2x,\ \frac{\partial N}{\partial x} = 2xy$$

So the RHS of the Green's theorem is the double integral given by

$$DI = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy = \iint_R \left[\left(2xy-(-2x)\right)\right]dxdy.$$

The region R is covered with y varying from $-2\sqrt{2}\sqrt{x}$ of the lower branch of the parabola to its upper branch $2\sqrt{2}\sqrt{x}$ while x varies from 0 to 2. Thus

$$DI = \int_{x=0}^{2}\int_{y=-\sqrt{8x}}^{\sqrt{8x}} \left(2xy+2x\right)dxdy = \int_0^2 xy^2+2xy\Big|_{-\sqrt{8x}}^{\sqrt{8x}} dx = 8\sqrt{2}\int_0^2 x^{3/2}dx = \frac{128}{5}$$

Since LI = DI, the Green's theorem is thus verified.

Q.No.4.: **Verify Green's theorem** in the plane for $\oint_c \left(2x-y^3\right)dx-xydy$, where c is the boundary of the annulus (doubly connected) region enclosed by the circles $x^2+y^2=1$ and $x^2+y^2=9$.

Sol.: Here $M = 2x-y^3,\quad N = xy$ so that

$$\frac{\partial M}{\partial y} = -3y^2, \quad \frac{\partial N}{\partial x} = y$$

Thus RHS of the Green's theorem is

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R \left(y + 3y^2\right) dxdy$$

where R is the annulus region

Put $x = r\cos t$, $y = r\sin t$, so that t varies from 0 to 2π and r from 1 to 3.

$$\text{RHS} = \int_0^{2\pi}\int_1^3 \left(r\sin t + 3r^2\sin^2 t\right) r dr dt = \frac{26}{3}\int_0^{2\pi} \sin t dt + 60\int_0^{2\pi} \frac{1-\sin 2t}{2} dt = 60t$$

$$\text{LHS} = \int_c Mdx + Ndy = \int_{c_1+c_2} \left(2x - y^3\right)dx - xydy$$

Changing to polar coordinate r, t.

$$= \int \left(2r\cos t - r^3\sin^3 t\right)\left(-r\sin tdt\right) - \int r^3\cos^2 t\sin t tdt = r^4 \frac{3\pi}{4}\Bigg|_1^3 = 60\pi.$$

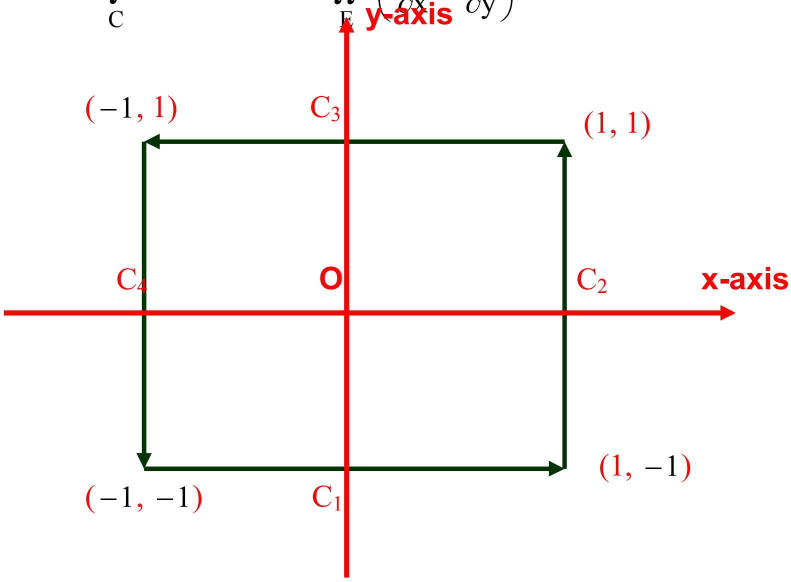
Now we will solve some line integrals problems by using Green's theorem:

Q.No.5.: Using Green's theorem, evaluate $\int_C \left[\left(x^2 + xy\right)dx + \left(x^2 + y^2\right)dy\right]$, where C is the square formed by the lines $x = \pm 1$, $y = \pm 1$.

Check the result with direct calculation.

Sol.: Green's theorem states that, if $\phi(x,y)$, $\psi(x,y)$, ϕ_y and ψ_x be continuous in a region E of the xy-plane bounded by a closed curve C, then

$$\int_C (\phi dx + \psi dy) = \iint_E \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dxdy.$$



By Green's theorem:

Also $$\int_C (\phi dx + \psi dy) = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy = \iint_E \left(\frac{\partial\left(x^2+y^2\right)}{\partial x} - \frac{\partial\left(x^2+xy\right)}{\partial y}\right) dxdy$$

$$= \iint_E (2x - x) dxdy = \iint_E xdxdy = \int_{-1}^{1}\left[\int_{-1}^{1} dy\right] xdx = \int_{-1}^{1} [y]_{-1}^{1} xdx$$

$$= 2\int_{-1}^{1} xdx = 2\left|\frac{x^2}{2}\right|_{-1}^{+1} = 0 \text{. Ans.}$$

Check the result with direct calculation. (by line integral):

Here $\phi = x^2 + xy$ and $\psi = x^2 + y^2$.

$$\therefore \int_C (\phi dx + \psi dy) = \int_{C_1} (\phi dx + \psi dy) + \int_{C_2} (\phi dx + \psi dy) + \int_{C_3} (\phi dx + \psi dy) + \int_{C_4} (\phi dx + \psi dy)$$

Along C_1, $y = -1$ and x varies from -1 to 1

$$\therefore \int_{C_1} (\phi dx + \psi dy) = \int_{C_1} \left[\left(x^2 + xy\right)dx + \left(x^2 + y^2\right)dy\right] = \int_{-1}^{1} \left(x^2 - x\right)dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1} = \frac{2}{3}.$$

Along C_2, $x = 1$ and y varies from -1 to 1

$$\therefore \int_{C_2} (\phi dx + \psi dy) = \int_{C_2} \left[\left(x^2 + xy\right)dx + \left(x^2 + y^2\right)dy\right] = \int_{-1}^{1} \left(1 + y^2\right)dy = \left[y + \frac{y^3}{3}\right]_{-1}^{1} = \frac{8}{3}.$$

Along C_3, $y = +1$ and x varies from 1 to -1

$$\therefore \int_{C_3} (\phi dx + \psi dy) = \int_{C_3} \left[\left(x^2 + xy\right)dx + \left(x^2 + y^2\right)dy\right] = \int_{1}^{-1} \left(x^2 + x\right)dx = \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{1}^{-1} = -\frac{2}{3}.$$

Along C_4, $x = -1$ and y varies from 1 to -1

$$\therefore \int_{C_4} (\phi dx + \psi dy) = \int_{C_4} \left[\left(x^2 + xy\right)dx + \left(x^2 + y^2\right)dy\right] = \int_{1}^{-1} \left(1 + y^2\right)dy = \left[y + \frac{y^3}{3}\right]_{1}^{-1} = -\frac{8}{3}.$$

Hence

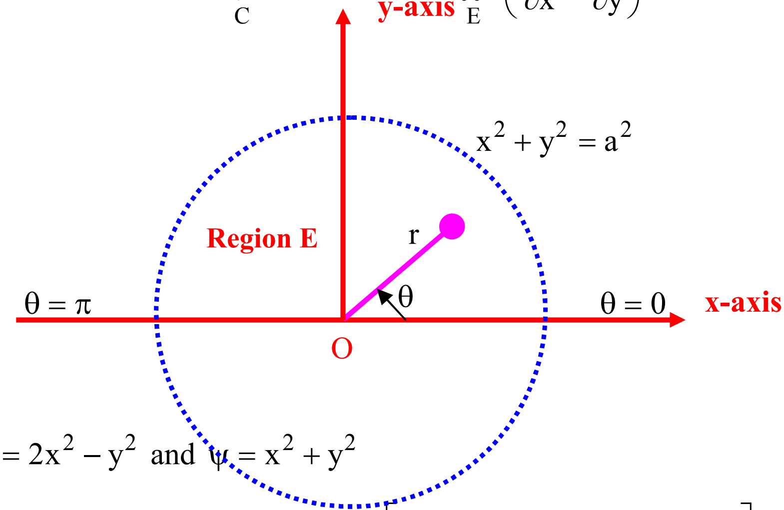
$$\int_C (\phi dx + \psi dy) = \int_{C_1} (\phi dx + \psi dy) + \int_{C_2} (\phi dx + \psi dy) + \int_{C_3} (\phi dx + \psi dy) \int_{C_4} (\phi dx + \psi dy)$$

$$=\frac{2}{3}+\frac{8}{3}-\frac{2}{3}-\frac{8}{3}=0. \text{ Ans.}$$

Q.No.6.: Apply Green's theorem to evaluate $\int_C \left[\left(2x^2-y^2\right)dx+\left(x^2+y^2\right)dy\right]$, where C is the boundary of the area enclosed by the x-axis and the upper half of the circle $x^2+y^2=a^2$.

Proof: Green's theorem states that, if $\phi(x,y), \psi(x,y), \phi_y$ and ψ_x be continuous in a region E of the x y-plane bounded by closed curve C, then

$$\int_C (\phi dx+\psi dy) = \iint_E \left(\frac{\partial\psi}{\partial x}-\frac{\partial\phi}{\partial y}\right)dxdy.$$



Here $\phi=2x^2-y^2$ and $\psi=x^2+y^2$

$$\therefore \int_C \left[\left(2x^2-y^2\right)dx+\left(x^2+y^2\right)dy\right]=\iint_E \left[\frac{\partial}{\partial x}\left(x^2+y^2\right)-\frac{\partial}{\partial y}\left(2x^2-y^2\right)\right]dxdy$$

$$=2\iint_E (x+y)dxdy, \text{ where E is the region of the figure.}$$

Now, changing to polar coordinates (r, θ), put $x=r\cos\theta$ and $y=r\sin\theta$;

and r varies from 0 to a and θ varies from 0 to π. Then

$$\int_C \left[\left(2x^2-y^2\right)dx+\left(x^2+y^2\right)dy\right]=2\int_0^a\int_0^\pi r(\cos\theta+\sin\theta).rd\theta\, dr$$

$$=2\int_0^a r^2dr\int_0^\pi (\cos\theta+\sin\theta)d\theta=2\int_0^a \left([\sin\theta-\cos\theta]_0^\pi\right)r^2dr.$$

$$= 2\int_0^a \left[\{0-(-1)\}-\{0-1\}\right]r^2dr = 2\int_0^a (1+1)r^2dr$$

$$= 4\int_0^a r^2dr = 4\left[\frac{r^3}{3}\right]_0^a = \frac{4}{3}a^3. \text{ Ans.}$$

Q.No.7.: Applying Green's theorem, evaluate

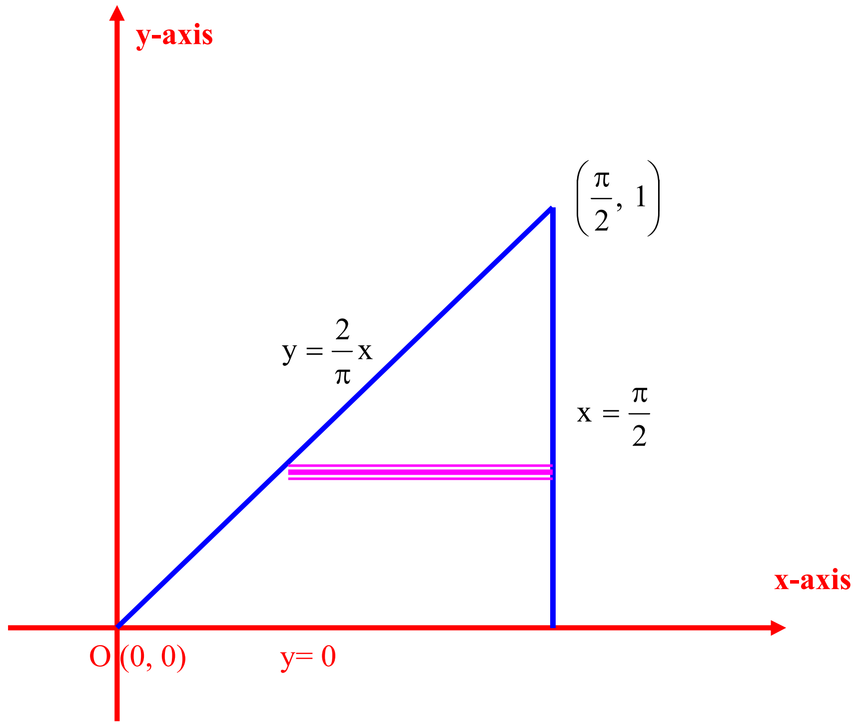
$$\int_C \left[(y-\sin x)dx + \cos x dy\right],$$

where C is the plane triangle enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$ and $y = \frac{2}{\pi}x$.

Sol.: Green's theorem states that, if $\phi(x,y)$, $\psi(x,y)$, ϕ_y and ψ_x be continuous in a region E of the xy-plane bounded by a closed curve C, then

$$\int_C (\phi dx + \psi dy) = \iint_E \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)dxdy.$$

Here $\phi = y - \sin x$ and $\psi = \cos x$.



$$\therefore \int_C (\phi dx + \psi dy) = \iint_E \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)dxdy = \iint_E \left(\frac{\partial(\cos x)}{\partial x} - \frac{\partial(y-\sin x)}{\partial y}\right)dxdy$$

$$= -\iint_E (\sin x + 1)dxdy = -\int_0^1 \left[\int_{\frac{\pi}{2}y}^{\frac{\pi}{2}} (1+\sin x)dx\right]dy = -\int_0^1 [x - \cos x]_{\frac{\pi}{2}y}^{\frac{\pi}{2}} dy$$

$$= -\int_0^1 \left[\left(\frac{\pi}{2} - \frac{\pi}{2}y\right) - \left(\cos\frac{\pi}{2} - \cos\frac{\pi}{2}y\right)\right]dy$$

$$= -\int_0^1 \left[\left(\frac{\pi}{2} - \frac{\pi}{2}y\right) + \left(\cos\frac{\pi}{2}y\right)\right]dy$$

$$= -\left[\frac{\pi}{2}(y)_0^1 - \frac{\pi}{2}\left(\frac{y^2}{2}\right)_0^1 + \frac{2}{\pi}\left(\sin\frac{\pi}{2}y\right)_0^1\right] = -\left[\frac{\pi}{2} - \frac{\pi}{4} + \frac{2}{\pi}\right]$$

$$= -\left[\frac{\pi}{4} + \frac{2}{\pi}\right]. \text{ Ans.}$$

Area:

Q.No.8.: Applying Green's theorem to prove that the area enclosed by a plane curve is

$$\frac{1}{2}\int_C [xdy - ydx].$$

Hence find the **area** of an ellipse whose semi-major and semi-minor axes are of lengths a and b.

Sol.: Ist Part: Since the area by the enclosed curve is $\iint_E dxdy$.

$$\therefore \iint_E dxdy = \frac{1}{2}\iint_E \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right)dxdy$$

$$= \frac{1}{2}\iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right)dxdy \qquad \text{where } \psi = x, \phi = -y$$

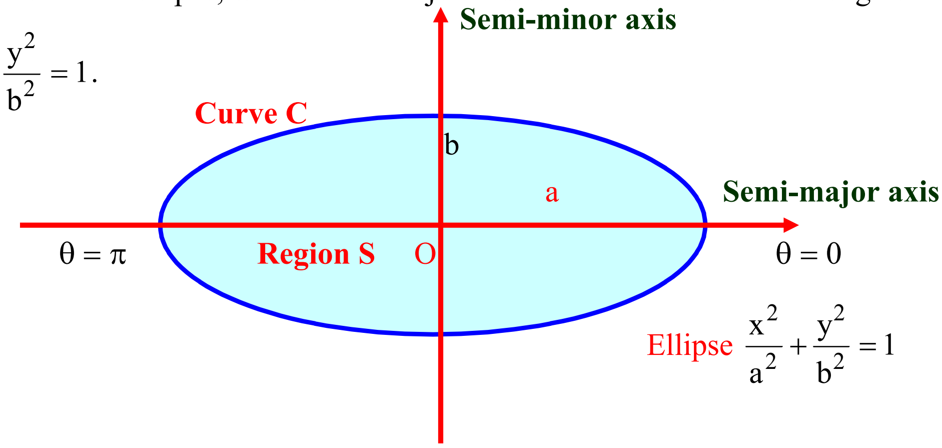
$$= \frac{1}{2}\int_C (\psi dy + \phi dx) \qquad \text{[by Green's theorem].}$$

$$= \frac{1}{2}\int_C (xdy - ydx).$$

Hence, this completes the proof.

IInd Part: Area of an ellipse $= \iint\limits_{E} dxdy = \frac{1}{2}\int\limits_{C}(xdy - ydx)$.

The equation of an ellipse, whose semi-major and semi-minor axes are of lengths a and b is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Put $x = a\cos\theta, y = b\sin\theta$. $\therefore dx = -a\sin\theta d\theta$ and $dy = b\cos\theta d\theta$.

$\therefore$ Area of an ellipse $= \frac{1}{2}\int\limits_{C}(xdy - ydx)$

$$= \frac{1}{2}\int_0^{2\pi}(a\cos\theta . b\cos\theta d\theta + b\sin\theta . a\sin\theta d\theta)$$

$$= \frac{ab}{2}\int_0^{2\pi} d\theta = \frac{ab}{2}[\theta]_0^{2\pi} = \frac{ab}{2}2\pi = \pi ab \text{ square units. Ans.}$$

Q.No.9.: Using Green's theorem, find **area** of region in the first quadrant bounded by the curves $y = x$, $y = \frac{1}{x}$, $y = \frac{x}{4}$.

Sol.: By Green's theorem area A of the region bounded by a closed curve c is given by

$$A = \frac{1}{2}\oint\limits_{c}(xdy - ydx)$$

Here c consists of the curves c_1: $y = \frac{x}{4}$, c_2: $y = \frac{1}{x}$ and c_3 : $y = x$, so

$$A = \frac{1}{2}\oint\limits_{c} = \frac{1}{2}\left[\int\limits_{c_1} + \int\limits_{c_2} + \int\limits_{c_3}\right] = \frac{1}{2}[I_1 + I_2 + I_3]$$

Along c_1: $y = \frac{x}{4}$, $dy = \frac{1}{4}dx$, x: 0 to 2

$$I_1 = \int_{c_1} (xdy - ydx) = \int_{c_1} \left(x\frac{1}{4}dx - \frac{x}{4}dx \right) = 0$$

Along c_2 : $y = \frac{1}{x}$, $dy = -\frac{1}{x^2}dx$, $x : 2$ to 1

$$I_2 = \int_{c_2} xdy - ydx = \int_2^1 x.\left(-\frac{1}{x^2}\right)dx - \frac{1}{x}dx$$

$$= -2 \ \ln x\Big|_2^1 = 2 \ \ln 2 .$$

Along c_3 : $y = x$, $dy = dx$, $x : 1$ to 0

$$I_3 = \int_{c_3} xdy - ydx = \int xdx - xdy = 0$$

$$A = \frac{1}{2}[I_1 + I_2 + I_3] = \frac{1}{2}(0 + 2 \ \ln 2 + 0) = \ln 2 .$$

Q.No.10.: Find the **area** bounded by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ with $a > 0$.

Sol.: Parametric equation of hypocycloid are

$x = a\cos^3 t$, $\quad y = a\sin^3 t$

$dx = -3a\cos^2 t\sin tdt$, $\quad dy = 3a\sin^2 t\cos tdt$

Area bounded by the hypocycloid

= 4. area under one leaf AB

= 4. area of the region AOB

Area of region AOB $= \frac{1}{2}\int_{ABOA} (xdy - ydx)$

$$= \frac{1}{2}\int_{AB} + \int_{BO} + \int_{OA} = \frac{1}{2}\int_{AB} + 0 + 0$$

Since x = 0 along BO and y = 0 along OA

$$= \frac{1}{2}\int_0^{\pi/2} \left[a\cos^3 t.\left(3a\sin^2 t\cos tdt\right) - a\sin^3 t\left(-3a\right)\cos^2 t\sin tdt\right]$$

$$= \frac{3a^2}{2}\int_0^{\pi/2} \sin^2 t\cos^4 tdt + \cos^2 t\sin^4 tdt$$

$$= \frac{3a^2}{2}\left[\frac{1.3.1}{6.4.2}.\frac{\pi}{2}\right] + \frac{3a^2}{2}\left[\frac{3.1.1}{6.4.2}.\frac{\pi}{2}\right] = 3\frac{\pi a^2}{32}$$

Area bounded by hypocycloid = $4.\dfrac{3\pi a^2}{32} = \dfrac{3\pi a^2}{8}$.

Q.No.11.: Compute the area of the region bounded by one arch of a cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ and the x-axis.

Sol.: Area $A = \dfrac{1}{2}\int_c (xdy - ydx)$.

$$A = \frac{1}{2}\int_0^{2\pi} a(1-\sin t).[a\sin tdt] - a(1-\cos t)[a(1-\cos t)dt]$$

$$= \frac{a^2}{2}\int_0^{2\pi} (t\sin t - \sin^2 t - 1 - \cos^2 t + 2\cos t)dt = \frac{a^2}{2}\int_0^{2\pi} (-2 + t\sin t + 2\cos t)dt$$

$$= \frac{a^2}{2}\left[-4\pi - 2\pi + 0 = -\frac{-6\pi a^2}{2}\right]. \text{ Ans.}$$

Other forms of Green's theorem:

Transformation of a double integral of the Laplacian of a function into a line integral of its normal derivative:

Q.No.11.: Show that $\iint_E \left(\dfrac{\partial \psi}{\partial x} - \dfrac{\partial \phi}{\partial y}\right)dxdy = \int_C (\phi dx + \psi dy)$ may be written in the form

$$\iint_E \nabla^2 w dxdy = \int_C \frac{\partial w}{\partial n} ds,$$

where w(x, y) be a function which is continuous and has continuous first and second partial derivatives in a domain of the xy-plane containing a region E of the type indicated in Green's theorem. Here s is the arc length of C.

Hint: Set $\phi = -\dfrac{\partial w}{\partial y}$ and $\psi = \dfrac{\partial w}{\partial x}$.

Sol.: Let w(x, y) be a function which is continuous and has continuous first and second partial derivatives in a domain of the xy-plane containing a region E of the type indicated in Green's theorem.

We set $\phi = -\dfrac{\partial w}{\partial y}$ and $\psi = \dfrac{\partial w}{\partial x}$.

Then $\dfrac{\partial \phi}{\partial y}$ and $\dfrac{\partial \psi}{\partial y}$ are continuous in E.

Now since $\iint\limits_E \left(\dfrac{\partial \psi}{\partial x} - \dfrac{\partial \phi}{\partial y}\right) dxdy = \int\limits_C (\phi dx + \psi dy)$.

Using the expressions for $\phi = -\dfrac{\partial w}{\partial y}$ and $\psi = -\dfrac{\partial w}{\partial x}$, we obtain

$$\iint\limits_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy = \iint\limits_E \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) dxdy = \iint\limits_E \nabla^2 w dxdy, \text{ the Laplacian of w.} \qquad \text{(i)}$$

$$\int\limits_C (\phi dx + \psi dy) = \int\limits_C \left(\phi \frac{dx}{ds} + \psi \frac{dy}{ds}\right) ds = \int\limits_C \left(-\frac{\partial w}{\partial y}\frac{dx}{ds} + \frac{\partial w}{\partial x}\frac{dy}{ds}\right) ds,$$

where s is the arc length of C.

The integrand of the last integral may be written as dot product of the vectors

$$\text{grad } w = \frac{\partial w}{\partial x}\hat{\mathbf{I}} + \frac{\partial w}{\partial y}\hat{\mathbf{J}} \quad \text{and} \quad \hat{\mathbf{N}} = \frac{dy}{ds}\hat{\mathbf{I}} - \frac{dx}{ds}\hat{\mathbf{J}}.$$

Thus $\displaystyle\int\limits_C (\phi dx + \psi dy) = \int\limits_C \left(-\frac{\partial w}{\partial y}\frac{dx}{ds} + \frac{\partial w}{\partial x}\frac{dy}{ds}\right) ds = \int\limits_C \left[(\text{grad } w).\hat{\mathbf{N}}\right] ds$. (ii)

The vector $\hat{\mathbf{N}}$ is a unit normal vector to C, because the vector $\hat{\mathbf{T}} = \dfrac{d\mathbf{R}}{ds} = \dfrac{dx}{ds}\hat{\mathbf{I}} + \dfrac{dy}{ds}\hat{\mathbf{J}}$ is a unit tangent vector to C, and $\hat{\mathbf{T}}.\hat{\mathbf{N}} = 0$.

It follows that the expression on the RHS of (ii) is the derivative of w in the direction of the outward normal to C.

Denoting this directional derivative by $\dfrac{\partial w}{\partial n}$.

Then we obtain from Green's theorem the useful integral formula

$$\iint\limits_E \nabla^2 w dxdy = \int\limits_C \frac{\partial w}{\partial n} ds.$$

This completes the proof.

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Home Assignments

Evaluation by Green's theorem:

Q.No.1.: Using Green's theorem, evaluate the line integral $\int_C (\phi dx + \psi dy)$ counterclockwise around the given contour, where $\phi dx + \psi dy$ equals $\left(3x^2 + y\right)dx + 4y^2 dy$, C: the boundary of the triangle with vertices (0, 0), (1,0), (0, 2).

Ans.: -1

Q.No.2.: Using Green's theorem, evaluate the line integral $\int_C (\phi dx + \psi dy)$ counterclockwise around the given contour, where $\phi dx + \psi dy$ equals $\left(x^3 - 3y\right)dx + \left(x + \sin y\right)dy$, C: the boundary of the triangle with vertices (0, 0), (1,0), (0, 2).

Ans.: 4.

Q.No.3.: Using Green's theorem, evaluate the line integral $\int_C (\phi dx + \psi dy)$ counterclockwise around the given contour, where $\phi dx + \psi dy$ equals $\left(x - y\right)dx - x^2 dy$, C: the boundary of the square $0 \le x \le 2,\ 0 \le y \le 2$.

Q.No.4.: Using Green's theorem, evaluate the line integral $\int_C (\phi dx + \psi dy)$ counterclockwise around the given contour, where $\phi dx + \psi dy$ equals $ydx - xdy$, C: the boundary of the square $0 \le x \le 1,\ 0 \le y \le 1$.

Q.No.5.: Using Green's theorem, evaluate the line integral $\int_C (\phi dx + \psi dy)$ counterclockwise around the given contour, where $\phi dx + \psi dy$ equals $\left(x^2 + y^2\right)dy$, C: the boundary of the square $2 \le x \le 4,\ 2 \le y \le 4$.

Ans.: 24.

Q.No.6.: Using Green's theorem, evaluate the line integral $\int_C (\phi dx + \psi dy)$ counterclockwise around the given contour, where $\phi dx + \psi dy$ equals $2xy^3 dx + 3x^2y^2 dy$, C: $x^2 + y^2 = 1$.

Ans.: 0.

Q.No.7.: Using Green's theorem, evaluate the line integral $\int_C (\phi dx + \psi dy)$ counterclockwise around the given contour, where $\phi dx + \psi dy$ equals $(2x - y)dx + (x + 3y)dy$, C: $x^2 + 4y^2 = 4$.

Q.No.8.: Using Green's theorem, evaluate the line integral $\int_C (\phi dx + \psi dy)$ counterclockwise around the given contour, where $\phi dx + \psi dy$ equals $\left(\frac{1}{3}e^x - y\right)dx + \left(\frac{1}{3}e^y + 2x\right)dy$, C: $x^2 + 4y^2 = 4$.

Ans.: 0.

Q.No.9.: Use Green's theorem to evaluate the line integral $\oint_c Mdx + Ndy$ when $Mdx + Ndy$ equal to $-y^3dx + x^3dy$ where c: circle $x^2 + y^2 = 1$.

Ans.: $\frac{3\pi}{2}$.

Q.No.10.: Use Green's theorem to evaluate the line integral $\oint_c Mdx + Ndy$ when $Mdx + Ndy$ equal to $x^{-1}e^y dx + \left(e^y \ \text{In}\, x + 2x\right)dy$ where c: the boundary of the region bounded by $y = 2,\ y = x^4 + 1$.

Ans.: $\frac{16}{5}$

Q.No.11.: Use Green's theorem to evaluate the line integral $\oint_c Mdx + Ndy$ when $Mdx + Ndy$ equal to $(\cos x \sin y - xy)dx + \sin x \cos y.dy$, where c : circle $x^2 + y^2 = 1$.

Ans.: 0.

Q.No.12.: Use Green's theorem to evaluate the line integral $\oint_c Mdx + Ndy$ when $Mdx + Ndy$ equal to $\left(x^2 - \cosh y\right)dx + \left(y + \sin x\right)dy$, where c: the boundary of the rectangle $0 \le x \le \pi,\ \ 0 \le y \le 1$.

Ans.: $\pi\left(\cosh 1 - 1\right)$.

Q.No.13.: Use Green's theorem to evaluate the line integral $\oint_c Mdx + Ndy$ when $Mdx + Ndy$ equal to $e^{-x}\left(\sin y dx + \cos y dy\right)$ where c : rectangle with vertices at $(0, 0)$, $\left(\pi,\ 0\right)$, $\left(\pi,\ \frac{\pi}{2}\right)$, $\left(0,\ \frac{\pi}{2}\right)$.

Ans.: $2\left(e^{-\pi} - 1\right)$.

Verification of Green's theorem

Q.No.14.: Verify Green's theorem or evaluate the line integral $\oint_c Mdx + Ndy$ **(a).** directly **(b).** using Green's theorem where $Mdx + Ndy$ is $\left(3x^2 - 8y^2\right)dx + \left(4y - 6x\right)dy$ with c : boundary of the region defined by $y = \sqrt{x}$ and $y = x^2$.

Ans.: Common value : $\frac{3}{2}$.

Q.No.15.: Verify Green's theorem or evaluate the line integral $\oint_c Mdx + Ndy$ **(a).** directly **(b).** using Green's theorem where $Mdx + Ndy$ is $\left(2x - y^3\right)dx - xydy$ with c : boundary of the region enclosed by the circle $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

Ans.: Common value : 60π.

Q.No.16.: Verify Green's theorem or evaluate the line integral $\oint_c Mdx + Ndy$ **(a).** directly **(b).** using Green's theorem where $Mdx + Ndy$ is $\left(3x + 4y\right)dx + \left(2x - 3y\right)dy$ with c : $x^2 + y^2 = 4$.

Ans.: Common value : -8π.

Area

Q.No.17.: Find the area of the region bounded by $y = x^2$ and $y = x + 2$.

Ans.: $\frac{9}{2}$.

Q.No.18.: Calculate the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Deduce the area bounded by the circle $x^2 + y^2 = a^2$.

Hint: Put $x = a\cos t$, $y = b\sin t$.

Ans.: Area of ellipse = πab, Put a = b, area of circle : πa^2.

Q.No.19.: Find the area of the loop of the folium of desecrates $x^2 + y^3 = 3axy$, $a > 0$.

Hint: Put y = tx, t : 0 to ∞.

Ans.: $A = \frac{1}{2}\int x^2 dt = \frac{9}{2}\int_0^{\infty} \frac{a^2t^2}{\left(1+t^3\right)^2} dt = \frac{3a^2}{2}$.

Q.No.20.: Find the area of a loop of the four-leafed rose $\rho = 3\sin 2\varphi$.

Hint: $A = \frac{1}{2}\int_0^{\pi/2} \rho^2 d\varphi = \frac{9\pi}{8}$.

Q.No.21.: Find the area of cardioid $\rho = a(1 - \cos\theta)$, with $0 \le \theta \le 2\pi$.

Ans.: $\frac{3\pi a^2}{2}$.

Q.No.22.: Find area bounded by one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $a > 0$ and the axis.

Ans.: $3\pi a^2$.

Q.No.13.: Compute the **area** of ellipse $x = a\cos t$, $y = b\sin t$.

Ans.: πab.

Q.No.14.: Find the **area** under of one arch of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$.

Ans.: $\frac{3\pi a^2}{8}$.

Q.No.15.: Find the **area** of the loop of folium of Descartes $x = \frac{3at}{1+t^3}$, $y = \frac{3at^2}{1+t^3}$.

Ans.: $\frac{3a^2}{2}$.

Other forms

Q.No.23.: Show that $\iint_E \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dxdy = \int_C (\phi dx + \psi dy)$ may be written in the form

$$\iint_E \operatorname{div}\mathbf{F} dxdy = \int_C \mathbf{F}.\hat{\mathbf{N}} ds \quad , \qquad (*)$$

where $\hat{\mathbf{N}}$ is the outward normal vector to the curve C and s is the arc length of C.

Verify (*) when $\mathbf{F} = 7x\hat{\mathbf{I}} - 3y\hat{\mathbf{J}}$ and C is the circle $x^2 + y^2 = 4$.

Hint: Introduce $\mathbf{F} = \psi\hat{\mathbf{I}} - \phi\hat{\mathbf{J}}$.

or

Show that Green's theorem can be written in the form $\int_c \mathbf{F}.\hat{\mathbf{N}} ds = \iint_R \nabla.\mathbf{F} dxdy$

where $\mathbf{F} = M\hat{\mathbf{I}} - N\hat{\mathbf{J}}$ and $\hat{\mathbf{N}}$ is the outer unit normal to the curve v.

Q.No.24.: Show that $\iint_E \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dxdy = \int_C (\phi dx + \psi dy)$ may be written in the form

$$\iint_E (\operatorname{curl}\mathbf{F}).\hat{\mathbf{K}} dxdy = \int_C \mathbf{F}.\hat{\mathbf{U}} ds \quad , \qquad (*)$$

where $\hat{\mathbf{K}}$ is a unit vector perpendicular to the xy-plane, $\hat{\mathbf{U}}$ is the unit tangent vector to the curve C and s is the arc length of C.

Verify (*) when $\mathbf{F} = y\hat{\mathbf{I}} + 4x\hat{\mathbf{J}}$ and C is the boundary of the triangle with vertices at (0, 0), (2, 0), (2, 1).

Q.No.25.: Show that $\int_c fdg = \iint_R \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) dxdy$ where R is the region bounded by the simple closed curve c.

Hint.: Use Green's theorem with $M = f\frac{\partial g}{\partial x}$ and $N = f\frac{\partial g}{\partial y}$.

Q.No.26.: Prove that $\int\limits_c \frac{dF}{dn} dS = \iint\limits_R \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) dxdy$ where $\frac{dF}{dn}$ is the directional derivative of F in the direction of the outer normal $\hat{\mathbf{N}}$ to the curve c bounding the region R.

Hint: Choose $M = -\frac{\partial F}{\partial y}$, $N = \frac{\partial F}{\partial x}$ and note that $\hat{\mathbf{N}} = \frac{dy}{dx}\hat{\mathbf{I}} - \frac{dx}{ds}\hat{\mathbf{J}}$.

Q.No.27.: If $\nabla^2 f = 0$ in R, show that $\iint\limits_R \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] dxdy = \int\limits_c f\frac{\partial f}{\partial n} ds$.

Hint: Take $M = -f\frac{\partial F}{\partial y}$, $N = f\frac{\partial F}{\partial x}$ and note that $\hat{\mathbf{N}} = \frac{dy}{dx}\hat{\mathbf{I}} - \frac{dx}{ds}\hat{\mathbf{J}}$.

Q.No.28.: Evaluate $\int\limits_c \mathbf{A}.d\mathbf{R}$ where

$\mathbf{A} = \alpha\left[-3a\sin^2 t\cos t\,\hat{\mathbf{I}} + a\left(2\sin t - 3\sin^3 t\right)\hat{\mathbf{J}} + b\sin 2t\,\hat{\mathbf{K}}\right]$ and the curve c is given by $\mathbf{R} = a\cos t\,\hat{\mathbf{I}} + a\sin t\,\hat{\mathbf{J}} + bt\,\hat{\mathbf{K}}$ and t varying from $\frac{\pi}{4}$ to $\frac{\pi}{2}$.

Ans.: $\frac{\alpha}{2}\left(a^2 + b^2\right)$.

9^{th} Topic

Vector Calculus

Stoke's Theorem

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(Last updated on 18-11-2009)
Latest update available at: http://www.freewebs.com/sunilnit/

STOKE'S THEOREM:
(Relation between line and surface integrals in three dimensions)

Statement: If S be an open surface bounded by a closed curve C and

$\mathbf{F} = f_1 \hat{\mathbf{I}} + f_2 \hat{\mathbf{J}} + f_3 \hat{\mathbf{K}}$ be any continuously differentiable vector point function,

then $\int_C \mathbf{F}.d\mathbf{R} = \int_S \text{curl}\, \mathbf{F}.\hat{\mathbf{N}}\, ds$,

where $\hat{\mathbf{N}} = \cos\alpha\, \hat{\mathbf{I}} + \cos\beta\, \hat{\mathbf{J}} + \cos\gamma\, \hat{\mathbf{K}}$ is a unit vector external normal at any point of S.

Proof: Let $d\mathbf{R} = dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}} + dz\,\hat{\mathbf{K}}$, then we have to show that

$$\int_C \mathbf{F}.d\mathbf{R} = \int_S \text{curl}\, \mathbf{F}.\hat{\mathbf{N}}\, ds$$

$$\Rightarrow \int_C (f_1 dx + f_2 dy + f_3 dz) = \int_S \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma\right] ds \tag{i}$$

Let us first prove that

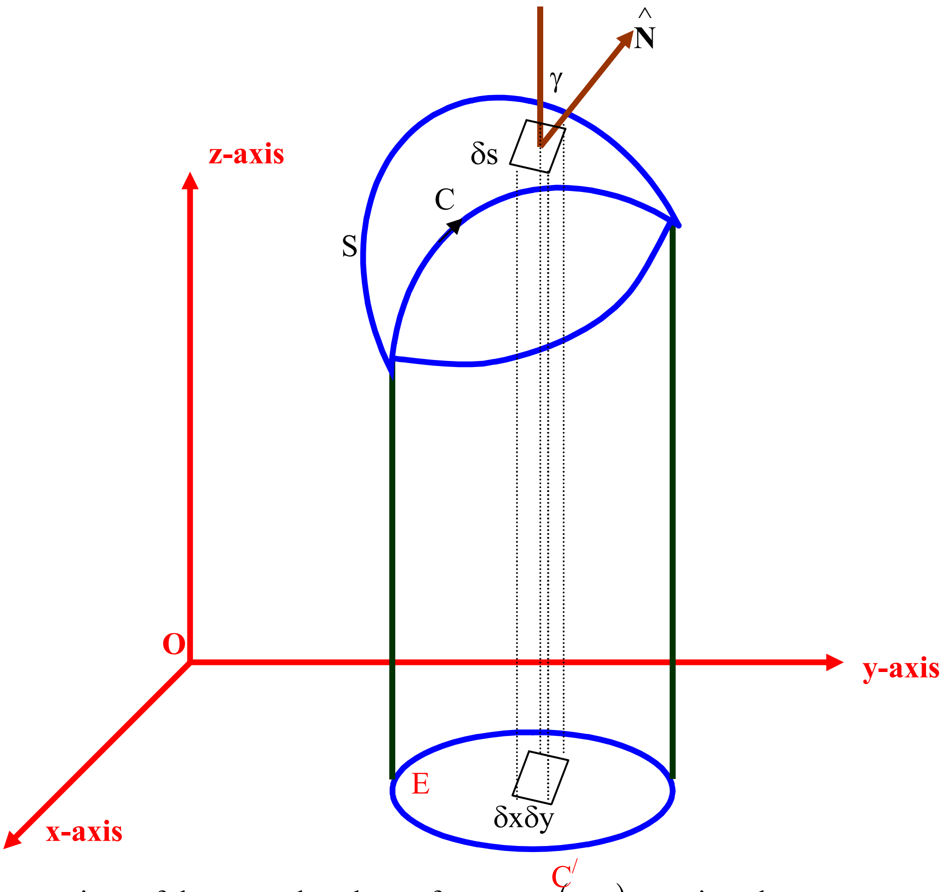
$$\oint_C f_1 dx = \int_C \left(\frac{\partial f_1}{\partial z}\cos\beta - \frac{\partial f_1}{\partial x}\cos\gamma \right) ds \qquad \text{(ii)}$$

Let $z = g(x, y)$ be the equation of the surface S whose projection on the xy-plane is the region E. Then the projection of C on the xy-plane is the curve C' enclosing region E.

$$\therefore \int_C f_1(x, y, z)dx = \int_C f_1\{x, y, g(x, y)\}dx = -\iint_E \frac{\partial}{\partial y} f_1(x, y, g)dxdy$$

$$\left[\text{by Green's theorem i.e.} \int_C (\phi dx + \psi dy) = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dxdy \right]$$

$$= -\iint_E \left(\frac{\delta f_1}{\delta y} + \frac{\delta f_1}{\delta z}\frac{\delta g}{\delta y} \right) dxdy . \qquad \text{(iii)}$$



The direction cosines of the normal to the surface $z = g(x, y)$ are given by

$$\frac{\cos\alpha}{-\frac{\partial g}{\partial x}} = \frac{\cos\beta}{-\frac{\partial g}{\partial y}} = \frac{\cos\gamma}{1} . \qquad \text{(iv)}$$

Moreover $dxdy$ = projection of ds on the xy-plane $= ds\cos\gamma \Rightarrow ds = \dfrac{dxdy}{\cos\gamma}$.

$$\therefore \text{ R. H. S. of (ii)} = \int_C \left(\frac{\partial f_1}{\partial z}\cos\beta - \frac{\partial f_1}{\partial x}\cos\gamma\right)ds = \iint_E \left(\frac{\partial f_1}{\partial z}\frac{\cos\beta}{\cos\gamma} - \frac{\partial f_1}{\partial y}\right)dxdy$$

$$= -\iint_E \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\frac{\partial g}{\partial y}\right)dxdy = \oint_C f_1 dx, \qquad \text{[by (iii)]}.$$

$$\text{Hence } \oint_C f_1 dx = \int_S \left(\frac{\partial f_1}{\partial z}\cos\beta - \frac{\partial f_1}{\partial x}\cos\gamma\right)ds.$$

Similarly, we can prove the other corresponding relations for f_2 and f_3.

$$\text{i.e. } \oint_C f_2 dy = \int_S \left(\frac{\partial f_2}{\partial x}\cos\gamma - \frac{\partial f_2}{\partial z}\cos\alpha\right)ds,$$

$$\text{and } \oint_C f_3 dz = \int_S \left(\frac{\partial f_3}{\partial y}\cos\alpha - \frac{\partial f_3}{\partial x}\cos\beta\right)ds.$$

Adding these three results, we get

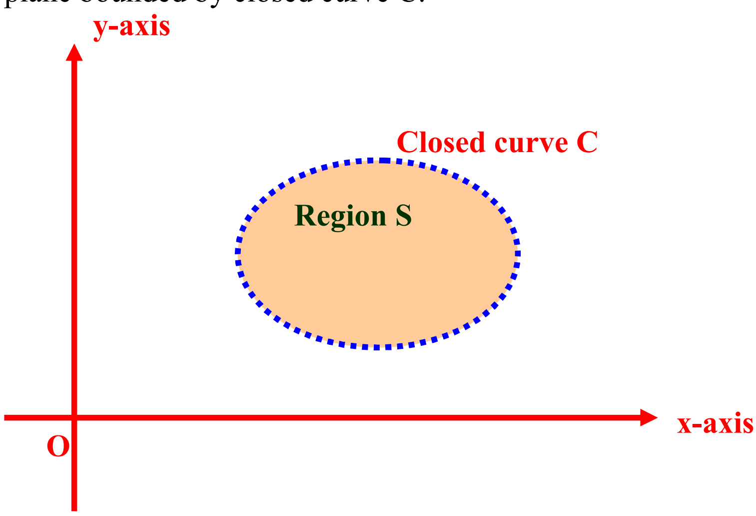
$$\int_C (f_1dx + f_2dy + f_3dz) = \int_S \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma\right]ds$$

This completes the proof of the Stoke's theorem.

Result:

Show that the Green's theorem in a plane is a special case of Stoke's theorem.

Proof: Let $\mathbf{F} = \phi\hat{\mathbf{I}} + \psi\hat{\mathbf{J}}$ be a vector function which is continuously differentiable in a region S of the xy-plane bounded by closed curve C.



Then $\int_C \mathbf{F}.d\mathbf{R} = \int_C \left(\phi\,\hat{\mathbf{I}} + \psi\,\hat{\mathbf{J}}\right).\left(dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}}\right) = \int_C (\phi dx + \psi dy)$.

Also $\text{curl}\,\mathbf{F}\,.\,\hat{\mathbf{N}} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ \phi & \psi & 0 \end{vmatrix} .\,\hat{\mathbf{K}} = \dfrac{\partial\psi}{\partial x} - \dfrac{\partial\phi}{\partial y}$. $\left[\because \hat{\mathbf{N}} = \hat{\mathbf{K}}\right]$

Hence, expression of Stoke's theorem, i.e. $\int_C \mathbf{F}.d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds$ takes the form

$$\int_C (\phi dx + \psi dy) = \iint_S \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dxdy,$$

which is the Green's theorem in a plane.

Thus, the Green's theorem in a plane is a special case of Stoke's theorem.

This completes the proof of this result.

Now let us verify Stoke's theorem in the following problems:

Q.No.1: **Verify Stoke's theorem** for $\mathbf{F} = \left(x^2 + y^2\right)\hat{\mathbf{I}} - 2xy\,\hat{\mathbf{J}}$ taken around the rectangle bounded by the lines $x = \pm a,\ y = 0,\ y = b$.

Sol.: Stoke's theorem states that, if S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\,\hat{\mathbf{I}} + f_2\,\hat{\mathbf{J}} + f_3\,\hat{\mathbf{K}}$ be any continuously differentiable vector point function, then

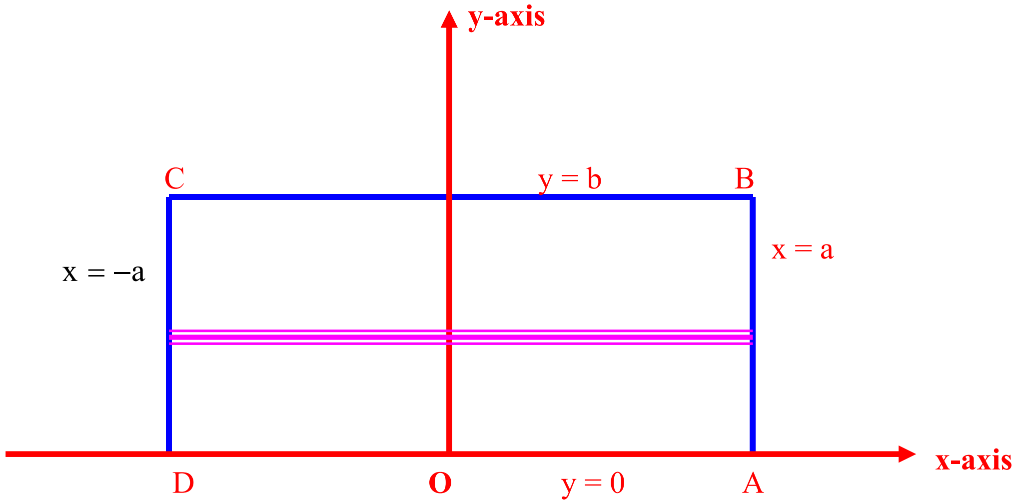
$$\int_C \mathbf{F}.\,d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}\,.\,\hat{\mathbf{N}}\,ds,$$

where $\hat{\mathbf{N}} = \cos\alpha\,\hat{\mathbf{I}} + \cos\beta\,\hat{\mathbf{J}} + \cos\gamma\,\hat{\mathbf{K}}$ is a unit vector external normal at any point of S.

Let ABCD be the rectangle as shown in figure.

$$\int_{ABCD} \mathbf{F}.d\mathbf{R} = \int_{AB} \mathbf{F}.d\mathbf{R} + \int_{BC} \mathbf{F}.d\mathbf{R} + \int_{CD} \mathbf{F}.d\mathbf{R} + \int_{DA} \mathbf{F}.d\mathbf{R}.$$

Now $\mathbf{F}.d\mathbf{R} = \left[\left(x^2 + y^2\right)\hat{\mathbf{I}} - 2xy\,\hat{\mathbf{J}}\right].\left(\hat{\mathbf{I}}dx + \hat{\mathbf{J}}dy\right) = \left(x^2 + y^2\right)dx - 2xydy$.



Along AB, $x = a$ (i.e. $dx = 0$) and y varies from 0 to b.

$$\therefore \int_{AB} \mathbf{F}.d\mathbf{R} = -2a\int_0^b ydy = -2a\left[\frac{y^2}{2}\right]_a^b = -2a.\frac{b^2}{2} = -ab^2.$$

Along BC, $y = b$ (i.e. $dy = 0$) and x varies from a to $-a$.

$$\therefore \int_{BC} \mathbf{F}.d\mathbf{R} = \int_a^{-a}\left(x^2+y^2\right)dx = \left[\frac{x^3}{3}+b^2x\right]_a^{-a} = -\frac{2a^3}{3} - 2ab^2.$$

Along CD, $x = -a$ (i.e. $dx = 0$) y varies from b to 0.

$$\therefore \int_{CD} \mathbf{F}.d\mathbf{R} = 2a\int_b^0 (y)dy = 2a\left[\frac{y^2}{2}\right]_b^0 = -ab^2.$$

Along DA, $y = 0$ (i.e. $dy = 0$) and x varies from $-a$ to a.

$$\therefore \int_{DA} \mathbf{F}.d\mathbf{R} = \int_{-a}^{a} x^2dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{2a^3}{3}.$$

Thus $\displaystyle\int_{ABCD} \mathbf{F}.d\mathbf{R} = \int_{AB} \mathbf{F}.d\mathbf{R} + \int_{BC} \mathbf{F}.d\mathbf{R} + \int_{CD} \mathbf{F}.d\mathbf{R} + \int_{DA} \mathbf{F}.d\mathbf{R}$

$$= -ab^2 - \frac{2a^3}{3} - -2ab^2 - ab^2 + \frac{2a^3}{3}$$

$$= -4ab^2. \qquad \text{(i)}$$

Also $\text{curl}\,\mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = \hat{\mathbf{I}}(0) + \hat{\mathbf{J}}(0) + \hat{\mathbf{K}}(-2y-2y) = (-4y)\hat{\mathbf{K}}$.

$$\therefore \int_S \text{curl}\mathbf{F}.\hat{\mathbf{N}}\,ds = \int_0^b \int_{-a}^{a} (-4y)\hat{\mathbf{K}}.\hat{\mathbf{K}}\,dxdy = -4\int_0^b \int_{-a}^{a} y\,dxdy$$

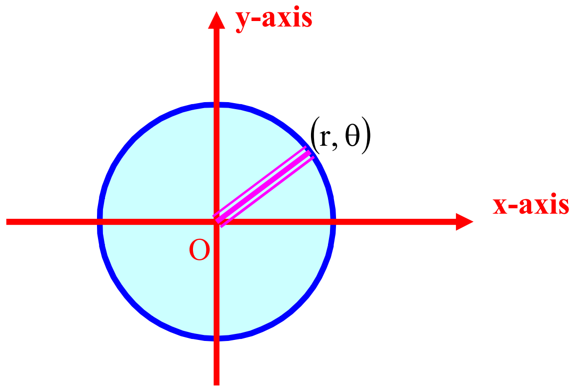
$$= -4\int_0^b |x|_{-a}^{a}\,ydy = -8a\left|\frac{y^2}{2}\right|_0^b = -4ab^2. \qquad \text{(ii)}$$

Hence, from (i) and (ii), we get $\int_C \mathbf{F}.d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds$.

Thus, Stoke's theorem is verified**.**

Q.No.2:Verify Stoke's theorem for the vector field defined by $\mathbf{F} = -y^3\hat{\mathbf{I}} + x^3\hat{\mathbf{J}}$, in the region $x^2+y^2 \le 1,\ z = 0$.

Sol.:



Stoke's theorem states that, if S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\hat{\mathbf{I}} + f_2\hat{\mathbf{J}} + f_3\hat{\mathbf{K}}$ be any continuously differentiable vector point function, then

$$\int_C \mathbf{F}.d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds,$$

where $\hat{\mathbf{N}} = \cos\alpha\,\hat{\mathbf{I}} + \cos\beta\,\hat{\mathbf{J}} + \cos\gamma\,\hat{\mathbf{K}}$ is a unit vector external normal at any point of S.

Here $\mathbf{F} = -y^3 \hat{\mathbf{I}} + x^3 \hat{\mathbf{J}}$.

Then $\int_C \mathbf{F}.d\mathbf{R} = \int_C \left(-y^3 \hat{\mathbf{I}} + x^3 \hat{\mathbf{J}}\right).\left(dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}}\right) = \int_C \left(-y^3 dx + x^3 dy\right)$.

Put $x = \cos\theta,\ y = \sin\theta \ \Rightarrow dx = -\sin\theta\, d\theta$ and $dy = \cos\theta\, d\theta$

$$\therefore \int_C \mathbf{F}.d\mathbf{R} = \int_C \left(-y^3 dx + x^3 dy\right) = \int_0^{2\pi} \sin^4\theta\, d\theta + \int_0^{2\pi} \cos^4\theta\, d\theta$$

$$= 4\left(\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta + \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta\right) = 4\left[\left(\frac{3\times1}{4\times2}\times\frac{\pi}{2}\right) + \left(\frac{3\times1}{4\times2}\times\frac{\pi}{2}\right)\right]$$

$$= 4\left[\frac{3}{16} + \frac{3}{16}\right]\pi = \frac{3}{2}\pi.$$

Now, we have to find the R.H.S. of Stoke's theorem.

$$\text{Now curl}\,\mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = (0)\hat{\mathbf{I}} - (0)\hat{\mathbf{J}} + \left(3x^2 + 3y^2\right)\hat{\mathbf{K}} = 3\left(x^2 + y^2\right)\hat{\mathbf{K}}.$$

Then, R.H.S. of Stoke's expression becomes

$$\int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\, ds = \int_S 3\left(x^2 + y^2\right)\hat{\mathbf{K}}.\hat{\mathbf{K}}\, ds = 3\iint_S \left(x^2 + y^2\right)dxdy.$$

Since the integral is evaluated over the region $x^2 + y^2 \le 1,\ z = 0$.

Put $x = r\cos\theta,\ y = r\sin\theta \ \Rightarrow dx\,dy = |J|\,dr\,d\theta = r\,dr\,d\theta$

$$\int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\, ds = 3\int_0^{2\pi}\int_0^1 \left(r^2\right) r\,dr\,d\theta = 3\int_0^{2\pi}\left(\int_0^1 \left(r^3\right) dr\right) d\theta$$

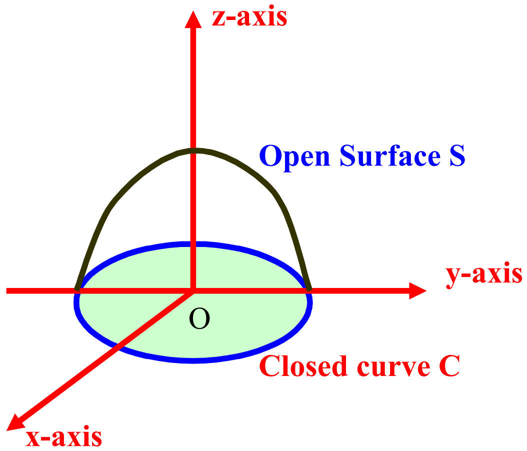
$$= 3\int_0^{2\pi}\left(\frac{r^4}{4}\right)_0^1 d\theta = \frac{3}{4}\int_0^{2\pi} d\theta = \frac{3}{4}(\theta)_0^{2\pi} = \frac{3}{2}\pi.$$

From both the results, we get $\int_C \mathbf{F}.d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\, ds$.

Hence, the Stoke's theorem is verified.

Q.No.3: **Verify Stoke's theorem** for the vector field $\mathbf{F} = (2x - y)\hat{\mathbf{I}} - yz^2\hat{\mathbf{J}} - y^2z\hat{\mathbf{K}}$ over the upper half surface of $x^2 + y^2 + z^2 = 1$, bounded by its projection on the xy-plane.

Sol.:



Let S be the upper half of the surface $x^2 + y^2 + z^2 = 1$.

The boundary C of open surface S is a circle in the xy-plane of radius unity and centre O.

The equations of C are $x^2 + y^2 = 1$ and $z = 0$, whose parametric form is

$x = \cos\theta,\ y = \sin\theta,\ z = 0,\ \ 0 \le \theta \le 2\pi$.

Here $\mathbf{F} = (2x - y)\hat{\mathbf{I}} - yz^2\hat{\mathbf{J}} - y^2z\hat{\mathbf{K}}$, then L.H.S. becomes

$$\int_C \mathbf{F}.d\mathbf{R} = \int_C \left[(2x-y)\hat{\mathbf{I}} - yz^2\hat{\mathbf{J}} - y^2z\hat{\mathbf{K}}\right].\left(dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}} + dz\,\hat{\mathbf{K}}\right)$$

$$= \int_C \left[(2x-y)\,dx - yz^2\,dy - y^2z\,dz\right]$$

$$= \int_C (2x-y)\,dx,\ \text{since on C},\ z = 0, dz = 0.$$

Put $x = \cos\theta,\ y = \sin\theta,\ z = 0,\ dx = -\sin\theta d\theta$, and $0 \le \theta \le 2\pi$.

$$\int_C \mathbf{F}.d\mathbf{R} = \int_C (2x-y)\,dx = \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta d\theta)$$
$$= \int_0^{2\pi} \left(-2\cos\theta\sin\theta + \sin^2\theta\right)d\theta = \int_0^{2\pi} \left(-\sin 2\theta + \frac{1-\cos 2\theta}{2}\right)d\theta$$
$$= \left(\frac{\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)_0^{2\pi} = \left(\frac{1}{2}+\pi\right) - \frac{1}{2} = \pi .$$

Now, we have to find the R.H.S. of Stoke's theorem.

Now $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & -yz^2 & -y^2z \end{vmatrix} = (-2yz+2yz)\hat{\mathbf{I}} - (0-0)\hat{\mathbf{J}} + (0+1)\hat{\mathbf{K}} = \hat{\mathbf{K}}$.

Then, R.H.S. of Stoke's expression becomes

$$\int_S \operatorname{curl}\mathbf{F}.\hat{\mathbf{N}}\,ds = \int_S \hat{\mathbf{K}}.\hat{\mathbf{N}}\,ds = \iint_R \hat{\mathbf{K}}.\hat{\mathbf{N}}\frac{dxdy}{\left|\hat{\mathbf{N}}.\hat{\mathbf{K}}\right|} = \iint_R dxdy, \qquad \left(\because \hat{\mathbf{N}} = \hat{\mathbf{K}}\right)$$

where R is the projection of S on xy-plane which is a circle of radius unity.

$$\int_S \operatorname{curl}\mathbf{F}.\hat{\mathbf{N}}\,ds = \iint_R dxdy = \pi(1)^2 = \pi .$$

From both the results, we get $\int_C \mathbf{F}.d\mathbf{R} = \int_S \operatorname{curl}\mathbf{F}.\hat{\mathbf{N}}\,ds$.

Hence, the Stoke's theorem is verified.

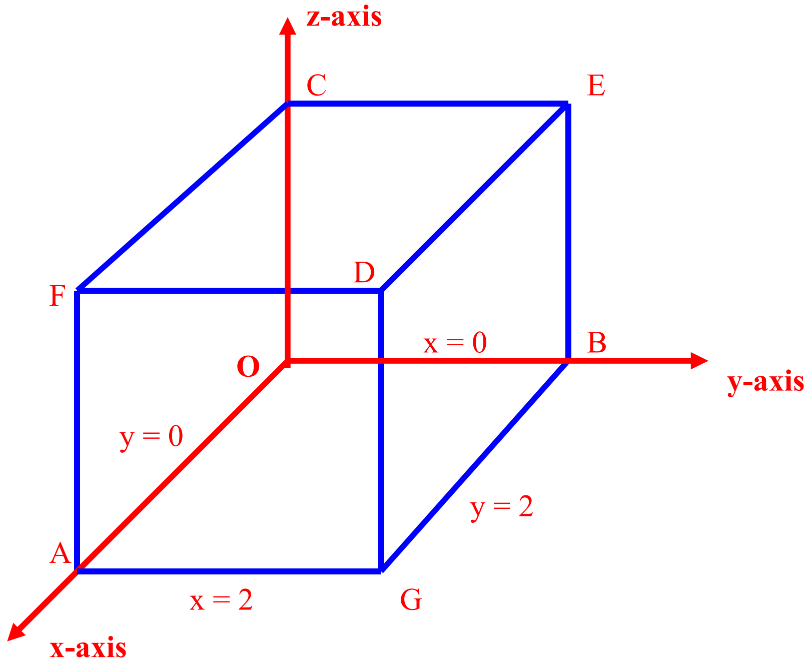
Q.No.4: **Verify Stoke's theorem** for $\mathbf{F} = (y-z+2)\hat{\mathbf{I}} + (yz+4)\hat{\mathbf{J}} - xz\hat{\mathbf{K}}$, where S is the surface of the cube x = 0, y = 0, z = 0, x = 2, y = 2, z = 2 above the xy-plane.

Sol.: The surface of the cube is bounded by the planes x = 0, x = 2 ; y = 0, y = 2 ; z = 0, z = 2 above the xy-plane and bounded by a closed curve OAGBO. Then we can verify Stoke's theorem.

Stoke's theorem states that, if S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\hat{\mathbf{I}} + f_2\hat{\mathbf{J}} + f_3\hat{\mathbf{K}}$ be any continuously differentiable vector point function, then

$$\int_C \mathbf{F}.\,d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}\,.\,\hat{\mathbf{N}}\,ds\,,$$

where $\hat{\mathbf{N}}$ is a unit vector external normal at any point of S.



Here $\mathbf{F} = (y - z + 2)\hat{\mathbf{I}} + (yz + 4)\hat{\mathbf{J}} - xz\hat{\mathbf{K}}$ and $d\mathbf{R} = dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}} + dz\,\hat{\mathbf{K}}$

$$\therefore \mathbf{F}.d\mathbf{R} = \left[(y - z + 2)\hat{\mathbf{I}} + (yz + 4)\hat{\mathbf{J}} - xz\hat{\mathbf{K}}\right].\left[dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}} + dz\,\hat{\mathbf{K}}\right]$$
$$= (y - z + 2)dx + (yz + 4)dy - xzdz\,.$$

Then $\displaystyle\int_{OABC} \mathbf{F}.\,d\mathbf{R} = \int_{OA} \mathbf{F}.\,d\mathbf{R} + \int_{AG} \mathbf{F}.\,d\mathbf{R} + \int_{GB} \mathbf{F}.\,d\mathbf{R} + \int_{BO} \mathbf{F}.\,d\mathbf{R}\,.$

Along OA, y = 0, z = 0 (i. e. dy = 0, dz = 0) and x varies from 0 to 2,

Along AG, x = 2, z = 0 (i. e. dx = 0, dz = 0) and y varies from 0 to 2,

Along GB, y = 2, z = 0 (i. e. dy = 0, dz = 0) and x varies from 2 to 0,

Along BO, x = 0, z = 0 (i. e. dx = 0, dz = 0) and y varies from 2 to 0.

Now, we have to find the R.H.S. of Stoke's expression.

Then $\int_{OABC} \mathbf{F}.d\mathbf{R} = \int_0^2 2dx + \int_0^2 4dy + \int_2^0 4dx + \int_2^0 4dy = 4+8-8-8 = -4$.

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y-z+2) & (yz+4) & -xz \end{vmatrix} = (-y)\hat{\mathbf{I}} + (z-1)\hat{\mathbf{J}} - \hat{\mathbf{K}}.$$

Then, R.H.S. of Stoke's expression becomes

$$\int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds = \int_{AGDF} \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds + \int_{GBED} \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds + \int_{OBEC} \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds$$
$$+ \int_{OCFA} \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds + \int_{CFDE} \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds$$

$$\int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds = \int_{AGDF} \left(-y\hat{\mathbf{I}}\right).\hat{\mathbf{I}}\,ds + \int_{GBED} (z-1)\hat{\mathbf{J}}.\hat{\mathbf{J}}\,ds + \int_{OBEC} \left(-y\hat{\mathbf{I}}\right).\left(-\hat{\mathbf{I}}\right)ds$$
$$+ \int_{OCFA} (z-1)\hat{\mathbf{J}}.\left(-\hat{\mathbf{J}}\right)ds + \int_{CFDE} \left(-\hat{\mathbf{K}}\right).\hat{\mathbf{K}}\,ds$$

$$\int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds = -\int_0^2\int_0^2 (y)\,dydz + \int_0^2\int_0^2 (z-1)dxdz + \int_0^2\int_0^2 (y)\,dydz$$
$$-\int_0^2\int_0^2 (z-1)dxdz + \int_0^2\int_0^2 (-1)\,dxdy$$

$$\Rightarrow \int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds = -\int_0^2\int_0^2 dxdy = -[\text{Area of the square, whose length is 2}] = -4.$$

From both the results, we get $\int_C \mathbf{F}.d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds$.

Hence, the Stoke's theorem is verified.

Q.No.5: **Verify Stoke's theorem** for the vector field $\mathbf{F} = \left(x^2 - y^2\right)\hat{\mathbf{I}} + 2xy\hat{\mathbf{J}}$

(i) integrated round the rectangle in the plane z = 0 and bounded by the lines x = 0, y = 0, x = a and y = b.

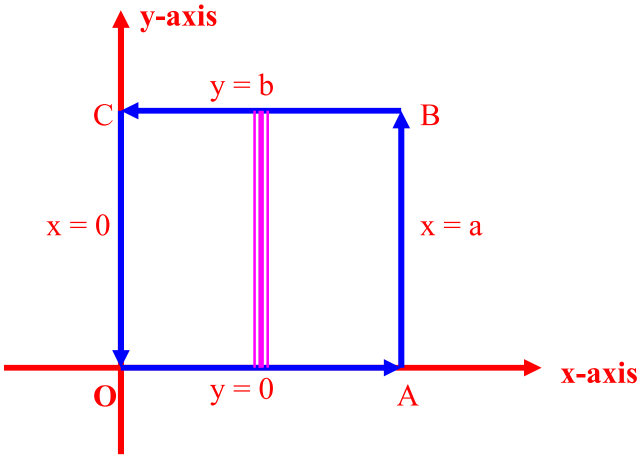
(ii) over the box bounded by the planes x = 0, x = a ; y = 0, y = b ; z = 0,

$z = c$; if the face $z = 0$ is cut.

Sol.: (i) Stoke's theorem states that, if S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1 \hat{\mathbf{I}} + f_2 \hat{\mathbf{J}} + f_3 \hat{\mathbf{K}}$ be any continuously differentiable vector point function, then

$$\int_C \mathbf{F}.d\mathbf{R} = \int_S \operatorname{curl} \mathbf{F} . \hat{\mathbf{N}} \, ds,$$

where $\hat{\mathbf{N}} = \cos\alpha \hat{\mathbf{I}} + \cos\beta \hat{\mathbf{J}} + \cos\gamma \hat{\mathbf{K}}$ is a unit vector external normal at any point of S.



Let OABC be the given rectangular as shown in figure.

Here $\mathbf{F} = \left(x^2 - y^2\right)\hat{\mathbf{I}} + 2xy\hat{\mathbf{J}}$ and $d\mathbf{R} = dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}}$.

$\therefore \mathbf{F}.d\mathbf{R} = \left[\left(x^2 - y^2\right)\hat{\mathbf{I}} + 2xy\hat{\mathbf{J}}\right].\left[dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}}\right] = \left(x^2 - y^2\right)dx + (2xy)dy$.

Then $\displaystyle\int_{OABC} \mathbf{F}.d\mathbf{R} = \int_{OA} \mathbf{F}.d\mathbf{R} + \int_{AB} \mathbf{F}.d\mathbf{R} + \int_{BC} \mathbf{F}.d\mathbf{R} + \int_{CO} \mathbf{F}.d\mathbf{R}$.

Along OA, $y = 0$ (i. e. $dy = 0$) and x varies from 0 to a,

Along AB, $x = a$ (i. e. $dx = 0$) and y varies from 0 to b,

Along BC, $y = b$ (i. e. $dy = 0$) and x varies from a to 0,

Along CO, $x = 0$ (i. e. $dx = 0$) and y varies from b to 0.

Then $\displaystyle\int_{OABC} \mathbf{F}.d\mathbf{R} = \int_0^a x^2 dx + \int_0^b 2ay\,dy + \int_a^0 \left(x^2 - b^2\right)dx + \int_b^0 0dy$

$$= \left(\frac{a^3}{3}\right) + \left(\frac{2ab^2}{2}\right) - \left(\frac{a^3}{3} - b^2 a\right) = 2ab^2.$$

Now, we have to find the R.H.S. of Stoke's theorem.

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - y^2) & 2xy & 0 \end{vmatrix} = (0)\hat{\mathbf{I}} - (0)\hat{\mathbf{J}} + (2y + 2y)\hat{\mathbf{K}} = (4y)\hat{\mathbf{K}}.$$

Then, R.H.S. of Stoke's expression becomes

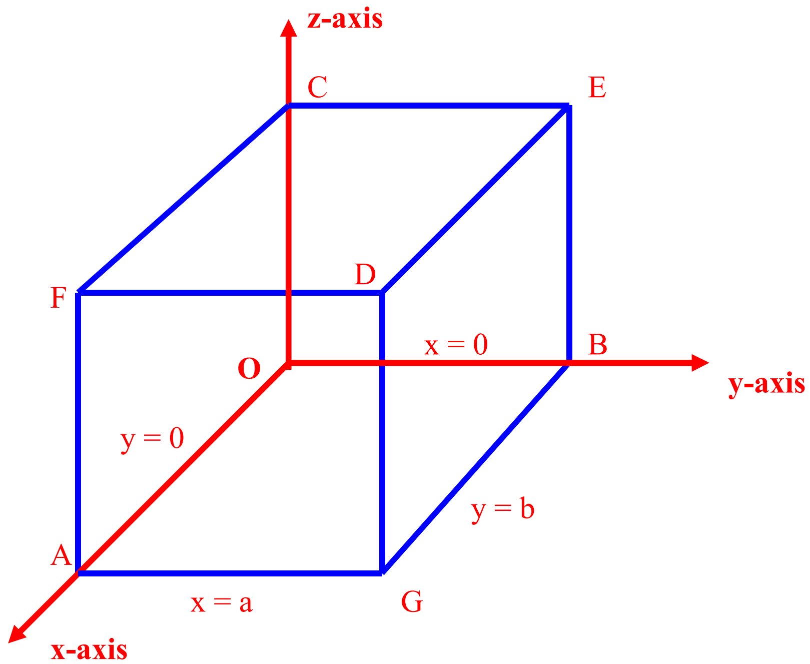
$$\int_S \text{curl}\,\mathbf{F}\,.\,\hat{\mathbf{N}}\,ds = \int_S (4y)\hat{\mathbf{K}}.\hat{\mathbf{K}}\,ds = 4\int_S y\,ds = 4\int_0^a \left(\int_0^b y\,dy\right)dx = 4\int_0^a \left(\frac{y^2}{2}\right)_0^b dx$$

$$= 2b^2 \int_0^a dx = 2b^2 (x)_0^a = 2ab^2.$$

From both the results, we get $\int_C \mathbf{F}.d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}\,.\,\hat{\mathbf{N}}\,ds$.

Hence, the Stoke's theorem is verified.

(ii)



This box is bounded by the planes x = 0, x = a ; y = 0, y = b ; z = 0, z = c. If the face z = 0 is cut, then this surface is open at the bottom and bounded by a closed curve OAGBO. Then we can verify Stoke's theorem.

Stoke's theorem states that, if S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\hat{\mathbf{I}} + f_2\hat{\mathbf{J}} + f_3\hat{\mathbf{K}}$ be any continuously differentiable vector point function, then

$$\int_C \mathbf{F}.d\mathbf{R} = \int_S \operatorname{curl}\mathbf{F}.\hat{\mathbf{N}}\,ds,$$

where $\mathbf{N} = \cos\alpha\hat{\mathbf{I}} + \cos\beta\hat{\mathbf{J}} + \cos\gamma\hat{\mathbf{K}}$ is a unit vector external normal at any point of S.

Here $\mathbf{F} = (x^2 - y^2)\hat{\mathbf{I}} + 2xy\hat{\mathbf{J}}$ and $d\mathbf{R} = dx\hat{\mathbf{I}} + dy\hat{\mathbf{J}}$.

$\therefore \mathbf{F}.d\mathbf{R} = \left[(x^2 - y^2)\hat{\mathbf{I}} + 2xy\hat{\mathbf{J}}\right].\left[dx\hat{\mathbf{I}} + dy\hat{\mathbf{J}}\right] = (x^2 - y^2)dx + 2xydy$, then L.H.S. becomes

$$\int_{OAGB} \mathbf{F}.d\mathbf{R} = \int_{OA} \mathbf{F}.d\mathbf{R} + \int_{AG} \mathbf{F}.d\mathbf{R} + \int_{GB} \mathbf{F}.d\mathbf{R} + \int_{BO} \mathbf{F}.d\mathbf{R}.$$

Along OA, y = 0 (i. e. dy = 0) and x varies from 0 to a,

Along AG, x = a (i. e. dx = 0) and y varies from 0 to b,

Along GB, y = b (i. e. dy = 0) and x varies from a to 0,

Along BO, x = 0 (i. e. dx = 0) and y varies from b to 0.

$$\int_{OAGB} \mathbf{F}.d\mathbf{R} = \int_0^a x^2dx + \int_0^b 2aydy + \int_a^0 (x^2 - b^2)dx + \int_b^0 0dy$$

$$= \left(\frac{a^3}{3}\right) + \left(\frac{2ab^2}{2}\right) - \left(\frac{a^3}{3} - b^2a\right) = 2ab^2.$$

Now, we have to find the R.H.S. of Stoke's theorem.

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - y^2) & 2xy & 0 \end{vmatrix} = (0)\hat{\mathbf{I}} - (0)\hat{\mathbf{J}} + (2y + 2y)\hat{\mathbf{K}} = (4y)\hat{\mathbf{K}}.$$

Then, R.H.S. of Stoke's theorem becomes

$$\int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds = \int_{AGDF} \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds + \int_{GBED} \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds + \int_{OBEC} \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds$$
$$+ \int_{OCFA} \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds + \int_{CFDE} \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds$$

$$\int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds = \int_{AGDF} \left(4y\hat{\mathbf{K}}\right).\hat{\mathbf{I}}\,ds + \int_{GBED} \left(4y\hat{\mathbf{K}}\right).\hat{\mathbf{J}}\,ds + \int_{OBEC} \left(4y\hat{\mathbf{K}}\right).\left(-\hat{\mathbf{I}}\right)ds$$
$$+ \int_{OCFA} \left(4y\hat{\mathbf{K}}\right).\left(-\hat{\mathbf{J}}\right)ds + \int_{CFDE} \left(4y\hat{\mathbf{K}}\right).\hat{\mathbf{K}}\,ds$$

$$\int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds = 0+0+0+0+\int_{CFDE} (4y)\hat{\mathbf{K}}.\hat{\mathbf{K}}\,ds = 4\int_{CFDE} y\,ds$$
$$= 4\int_0^a \left(\int_0^b y\,dy\right)dx = 4\int_0^a \left(\frac{y^2}{2}\right)_0^b dx = 2b^2\int_0^a dx = 2b^2(x)_0^a = 2ab^2$$

From both the results, we get $\int_C \mathbf{F}.d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds$.

Hence, the Stoke's theorem is verified.

Q.No.6.: **Verify Stokes theorem** for $\mathbf{A} = xz\hat{\mathbf{I}} - y\hat{\mathbf{J}} + x^2y\hat{\mathbf{K}}$ where S is the surface of the region bounded by x = 0, y = 0, z = 0, 2x + y + 2z = 8 which is not included in the xz-plane.

Sol.: Stokes theorem states that $\oint_c \mathbf{A}.d\mathbf{R} = \iint_S (\nabla\times\mathbf{A}).\hat{\mathbf{N}}\,ds$

Here c is curve consisting of the straight lines AO, OD and DA.

$$\text{LHS} = \oint_c \mathbf{A}.d\mathbf{R} = \int_{AO+OD+DA} = \int_{AO} + \int_{OD} + \int_{DA} = LI_1 + LI_2 + LI_3$$

On the straight line AO : y = 0, z = 0, **A** = 0, so

$$LI_1 = \int_{AO} \mathbf{A}.d\mathbf{R} = 0$$

On the straight line OD : x = 0, y = 0, **A** = 0, so

$$LI_2 = \int_{OD} \mathbf{A}.d\mathbf{R} = 0$$

On the straight line DA : x + z = 0, y = 0, so $\mathbf{A} = xz\hat{\mathbf{I}} = x(4-x)\hat{\mathbf{I}}$

$$LI_3 = \int_{OD} \mathbf{A}.d\mathbf{R} = \int_0^4 x(4-x)\hat{\mathbf{I}}.dx\,\hat{\mathbf{I}} = \int_0^4 x(4-x)\,dx = \frac{32}{3}.$$

$$LI = 0+0+\frac{32}{3} = \frac{32}{3}.$$

Here the surface S consists of 3 surface (plans)

S_1 : OAB, S_2 : OBD, S_3 : ABD, so that

$$RHS = \iint_S (\nabla\times\mathbf{A}).\hat{\mathbf{N}}\,ds = \iint_{S_1+S_2+S_3} = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} = SI_1 + SI_2 + SI_3$$

$$\nabla\times\mathbf{A} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -y & x^2y \end{vmatrix} = x^2\hat{\mathbf{I}} + x(1-2y)\hat{\mathbf{J}}$$

On the surface S_1 : plane OAB: $z = 0$, $\hat{\mathbf{N}} = -\hat{\mathbf{K}}$, so

$$(\nabla\times\mathbf{A}).\hat{\mathbf{N}} = \left[x^2\hat{\mathbf{I}} + x(1-2y)\hat{\mathbf{J}}\right].\left(-\hat{\mathbf{K}}\right) = 0$$

$$SI_1 = \iint_{S_1} (\nabla\times\mathbf{A}).\hat{\mathbf{N}}\,ds = 0.$$

On the surface S_2 : plane OBD: $x = 0$, $\hat{\mathbf{N}} = -\hat{\mathbf{I}}$, so

$$\nabla\times\mathbf{A} = 0$$

$$SI_2 = \iint_{S_2} (\nabla\times\mathbf{A}).\hat{\mathbf{N}}\,ds = 0$$

On the surface S_3: plane ABD: $2x+y+2z = 8$

Unit normal $\hat{\mathbf{N}}$ to the surface S_3 $= \dfrac{\nabla(2x+y+2z)}{|\nabla(2x+y+2z)|}$

$$\hat{\mathbf{N}} = \frac{2\hat{\mathbf{I}}+\hat{\mathbf{J}}+2\hat{\mathbf{K}}}{\sqrt{4+1+4}} = \frac{2\hat{\mathbf{I}}+\hat{\mathbf{J}}+2\hat{\mathbf{K}}}{3}$$

$$(\nabla\times\mathbf{A}).\hat{\mathbf{N}} = \frac{2}{3}x^2 + \frac{1}{3}x(1-2y)$$

To evaluate the surface integral on the surface S_3, Project S_3 on the say xz-plane i.e., projection of ABD on xz-plane is AOD

$$dS = \frac{dxdz}{\hat{\mathbf{N}}.\hat{\mathbf{J}}} = \frac{dxdz}{\frac{1}{3}} = 3dxdz$$

$$\text{Thus } SI_3 = \iint_{S_3} (\nabla \times \mathbf{A}).\hat{\mathbf{N}}\, ds = 0 = \iint_{AOD} \left[\frac{2}{3}x^2 + \frac{1}{3}x(1-2y)\right]3dxdz$$

$$= \int_{x=0}^{4} \int_{z=0}^{4} \left[2x^2 + x(1-2y)\right]dzdx$$

Since the region AOD is covered by varying z from 0 to $4-x$, while x varies from 0 to 4. Using the equation of the surface S_3, $2x + y + 2z = 8$, eliminate y, then

$$SI_3 = \int_0^4 \int_0^{4-x} \left\{2x^2 + x\left[1-2(8-2x-2z)\right]\right\} dzdx = \int_0^4 \int_0^{4-x} \left\{6x^2 - 15 + 4xz\right\} dzdx$$

$$= \int_0^4 \left. 6x^2z - 15xz + \frac{4xz^2}{2} \right|_0^{4-x} dx = \int_0^4 \left(23x^2 - 4x^3 - 28x\right)dx = \frac{32}{3}$$

Thus LHS = LI = RHS = SI

Hence Stokes theorem is verified.

Q.No.7.: **Verify Stokes theorem** for $\mathbf{A} = y^2\hat{\mathbf{I}} + xy\hat{\mathbf{J}} - xz\hat{\mathbf{K}}$ where S is the hemisphere

$$x^2 + y^2 + z^2 = a^2, \quad z \geq 0.$$

Sol.: The curve c which is the boundary of the given hemisphere is the base circle $x^2 + y^2 = a^2$

On curve c: $z = 0$, $x^2 + y^2 = a^2$

$$\text{LHS} = \text{LI} = \oint_c \mathbf{A}.d\mathbf{R} = \int y^2dx + xydy - xzdz = \int y^2dx + xydy$$

Introducing polar coordinates $x = a\cos t$, $y = a\sin t$, with t varying from 0 to 2π

$$\text{LI} = \int_0^{2\pi} a^2\sin^2 td(a\cos t) + a\cos t.a\sin t.d(a\sin t) = a^3\int_0^{2\pi} \left(-\sin^2 t + \cos^2 t\sin t\right)dt = 0$$

Now $\nabla\times\mathbf{A}=\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y^2 & xy & -xz\end{vmatrix}=z\hat{\mathbf{J}}-y\hat{\mathbf{K}}$

Unit normal $\hat{\mathbf{N}}$ to the sphere is

$$\hat{\mathbf{N}}=\frac{\nabla\left(x^2+y^2+z^2\right)}{\left|\nabla\left(x^2+y^2+z^2\right)\right|}=\frac{2x\hat{\mathbf{I}}+2y\hat{\mathbf{J}}+2z\hat{\mathbf{K}}}{\sqrt{4x^2+4y^2+4z^2}}=\frac{x\hat{\mathbf{I}}+y\hat{\mathbf{J}}+z\hat{\mathbf{K}}}{a}$$

$$(\nabla\times\mathbf{A}).\hat{\mathbf{N}}=\left(z\hat{\mathbf{J}}-y\hat{\mathbf{K}}\right).\left(\frac{x\hat{\mathbf{I}}+y\hat{\mathbf{J}}+z\hat{\mathbf{K}}}{a}\right)=\frac{1}{a}(zy-zy)=0$$

So RHS = SI $=\iint_S (\nabla\times\mathbf{A}).\hat{\mathbf{N}}\,ds=0$

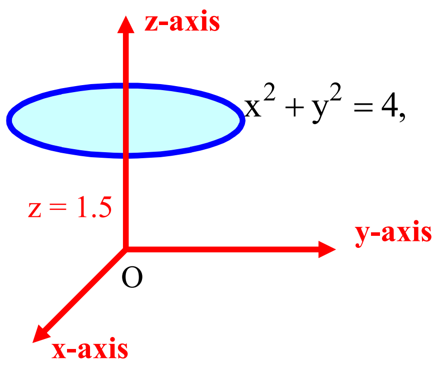
Thus LHS = LI = 0 = SI = RHS

Hence the Stokes theorem.

Now let us solve few line integrals by using Stoke's theorem:

Q.No.1: Evaluate **by Stoke's theorem** $\int_C \mathbf{F}.d\mathbf{R}$ where $\mathbf{F}=y\hat{\mathbf{I}}+xz^3\hat{\mathbf{J}}-zy^3\hat{\mathbf{K}}$, C is the circle $x^2+y^2=4,\ z=1.5$.

Sol.:



Stoke's theorem states that, if S be an open surface bounded by a closed curve C and $\mathbf{F}=f_1\hat{\mathbf{I}}+f_2\hat{\mathbf{J}}+f_3\hat{\mathbf{K}}$ be any continuously differentiable vector point function, then

$$\int_C \mathbf{F}.d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds,$$

where $\hat{\mathbf{N}}$ is a unit vector external normal at any point of S.

Here $\mathbf{F} = y\hat{\mathbf{I}} + xz^3\hat{\mathbf{J}} - zy^3\hat{\mathbf{K}}$, then

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz^3 & -zy^3 \end{vmatrix} = -3\left(zy^2 + xz^2\right)\hat{\mathbf{I}} + (0)\hat{\mathbf{J}} + \left(z^3 - 1\right)\hat{\mathbf{K}}.$$

$$\therefore \int_C \mathbf{F}.d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds = \int_S \left[-3\left(zy^2 + xz^2\right)\hat{\mathbf{I}} + (0)\hat{\mathbf{J}} + \left(z^3 - 1\right)\hat{\mathbf{K}}\right].\hat{\mathbf{K}}\,ds$$

$$= \int_S \left(z^3 - 1\right)ds = \int_S \left[(1.5)^3 - 1\right]ds = \int_S [3.325 - 1]ds = \int_S [2.375]ds$$

$$= (2.375)\int_S ds = (2.375) \times [\text{Area of the circle whose radius is } 2]$$

$$= (2.375) \times \left[\pi(2)^2\right] = 9.5\pi. \text{ Ans.}$$

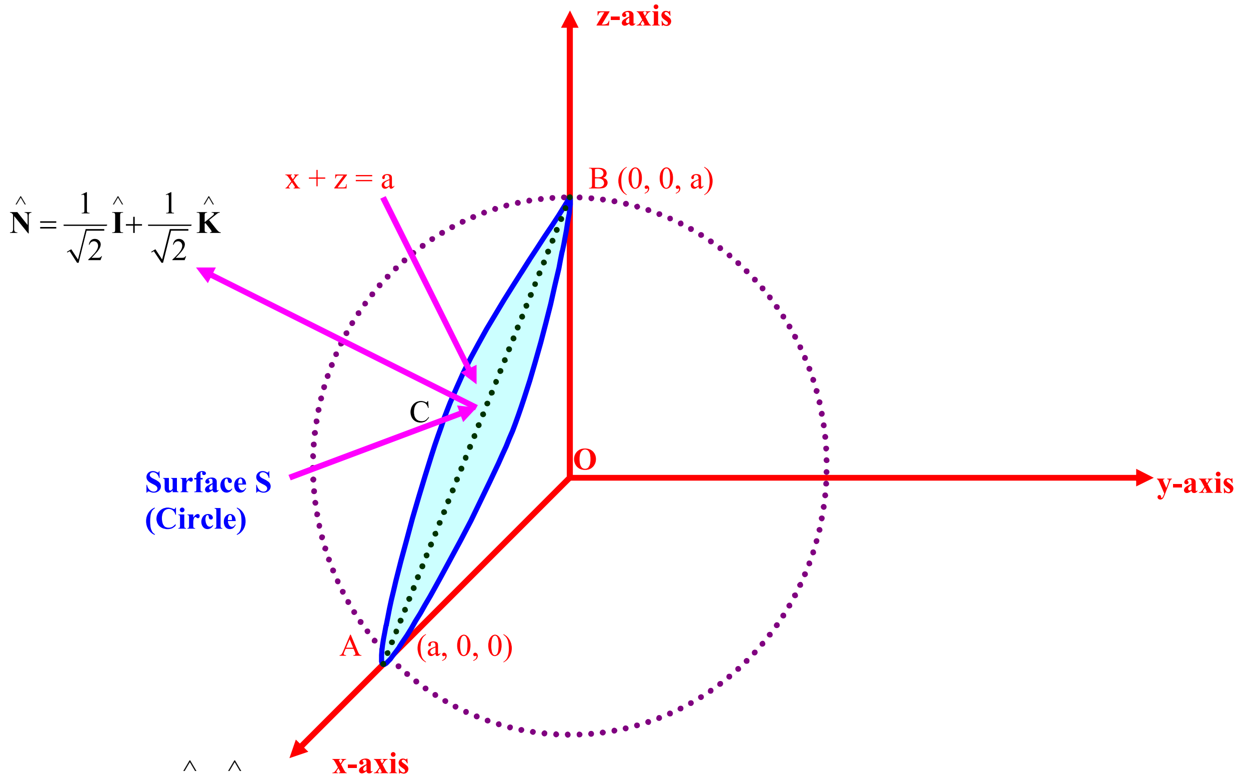
Q.No.2: **Apply Stoke's theorem** to evaluate $\int_C (ydx + zdy + xdz)$, where C is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$.

Sol.: The curve C is evidently a circle lying in the plane $x + z = a$, and having A(a, 0, 0), B(0, 0, a) as the extremities of the diameter (see figure).

$$\therefore \int_C (ydx + zdy + xdz) = \int_C \left(y\hat{\mathbf{I}} + z\hat{\mathbf{J}} + x\hat{\mathbf{K}}\right).d\mathbf{R} = \int_S \text{curl}\left(y\hat{\mathbf{I}} + z\hat{\mathbf{J}} + x\hat{\mathbf{K}}\right).\hat{\mathbf{N}}\,ds,$$

(by Stoke's theorem)

where S is the circle on AB as diameter. Now equation of the plane through A and B is $x + z = a \Rightarrow f = x + z - a = 0$.



$$\therefore \hat{N} = \frac{\nabla f}{|\nabla f|} = \frac{\hat{I}+\hat{K}}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}\hat{I} + \frac{1}{\sqrt{2}}\hat{K}.$$

Also
$$\text{curl}\left(y\hat{I} + z\hat{J} + x\hat{K}\right) = \begin{vmatrix} \hat{I} & \hat{J} & \hat{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{I}(-1) + \hat{J}(-1) + \hat{K}(-1) = -\left[\hat{I} + \hat{J} + \hat{K}\right].$$

$$\therefore \int_C (ydx + zdy + xdz) = \int_S -\left(\hat{I} + \hat{J} + \hat{K}\right).\left(\frac{1}{\sqrt{2}}\hat{I} + \frac{1}{\sqrt{2}}\hat{K}\right)ds$$

$$= -\frac{2}{\sqrt{2}}\int_S ds = -\frac{2}{\sqrt{2}}\pi\left(\frac{a}{\sqrt{2}}\right)^2 = -\frac{\pi a^2}{\sqrt{2}}. \text{ Ans.} \left[\therefore \int_S ds \text{ is the area of the circle, whose radius is } \frac{a}{\sqrt{2}}\right]$$

Q.No.3: Evaluate **by Stoke's theorem** $\oint_C (yz\,dx + zx\,dy + xy\,dz)$, where C is the curve

$$x^2 + y^2 = 1,\ z = y^2.$$

Sol.: Stoke's theorem states that, if S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\hat{\mathbf{I}} + f_2\hat{\mathbf{J}} + f_3\hat{\mathbf{K}}$ be any continuously differentiable vector point function, then

$$\int_C \mathbf{F}.d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds,$$

where $\hat{\mathbf{N}}$ is a unit vector external normal at any point of S. Then

$$\oint_C (yz\,dx + zx\,dy + xy\,dz) = \oint_C \left(yz\hat{\mathbf{I}} + zx\hat{\mathbf{J}} + xy\hat{\mathbf{K}}\right).\left(dx\hat{\mathbf{I}} + dy\hat{\mathbf{J}} + dz\hat{\mathbf{K}}\right)$$

$$= \oint_C \mathbf{F}.d\mathbf{R} = \oint_S \nabla\times\mathbf{F}.d\mathbf{S} \quad \text{(by Stoke's theorem)}$$

Now $\text{curl}\,\mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = (x-x)\hat{\mathbf{I}} + (y-y)\hat{\mathbf{J}} + (z-z)\hat{\mathbf{K}} = (0)\hat{\mathbf{I}} + (0)\hat{\mathbf{J}} + (0)\hat{\mathbf{K}} = \mathbf{0}$

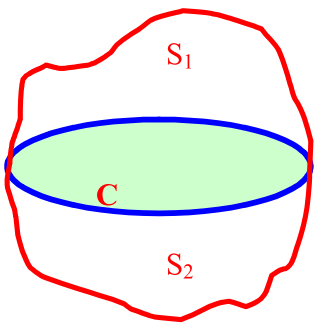
$$\oint_C (yz\,dx + zx\,dy + xy\,dz) = \oint_S \nabla\times\mathbf{F}.d\mathbf{S} = \oint_S \mathbf{0}.d\mathbf{S} = 0.\ \text{Ans.}$$

Q.No.4: If S be any closed surface, prove that $\int_S \text{curl}\,\mathbf{F}.d\mathbf{S} = 0$.

Sol.: Cut open the surface S by any plane and let S_1, S_2 denotes its upper and lower portions. Let C be the common curve bounding both these portions. Then

$$\int_S \text{curl}\mathbf{F}.d\mathbf{S} = \int_{S_1} \text{curl}\mathbf{F}.d\mathbf{S} + \int_{S_2} \text{curl}\mathbf{F}.d\mathbf{S},$$

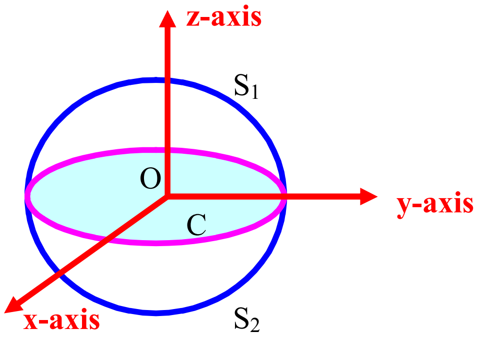
$$= \int_C \mathbf{F}.d\mathbf{R} - \int_C \mathbf{F}.d\mathbf{R} = 0. \qquad \text{(by Stoke's theorem)}$$



The second integral is negative because it is traversed in a direction opposite to that of the first.

Q.No.5: If S be the surface of the sphere $x^2 + y^2 + z^2 = 1$, prove that $\int_S \text{curl}\,\mathbf{F}\,.\,d\mathbf{S} = 0$.

Sol.:



Cut the surface S of the sphere $x^2 + y^2 + z^2 = 1$ by any plane and let S_1, S_2 denote its upper and lower portions. Let C be the common curve bounding both these portions.

Then, by Stoke's theorem which states that, if S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\,\hat{\mathbf{I}} + f_2\,\hat{\mathbf{J}} + f_3\,\hat{\mathbf{K}}$ be any continuously differentiable vector point function, then

$$\int_C \mathbf{F}.\,d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}\,.\,\hat{\mathbf{N}}\,ds\,,$$

where $\hat{\mathbf{N}}$ is a unit vector external normal at any point of S.

Now $\int_C \mathbf{F}\,.\,d\mathbf{R} = \int_{S_1} \text{curl}\,\mathbf{F}\,.\,d\mathbf{S}$ and $\int_C \mathbf{F}\,.\,d\mathbf{R} = -\int_{S_2} \text{curl}\,\mathbf{F}\,.\,d\mathbf{S}$

$$\therefore \int_S \text{curl}\,\mathbf{F}\,.\,d\mathbf{S} = \int_{S_1} \text{curl}\,\mathbf{F}\,.\,d\mathbf{S} + \int_{S_2} \text{curl}\,\mathbf{F}\,.\,d\mathbf{S} = \int_C \mathbf{F}\,.\,d\mathbf{R} - \int_C \mathbf{F}\,.\,d\mathbf{R} = 0\,.$$

The second integral is negative because the normal unit vectors of these two surfaces are in opposite direction, hence the direction of integral is also opposite.

This completes the proof.

Q.No.6: **Using Stoke's theorem** evaluate $\int_C \left[(x + y)dx + (2x - z)dy + (y + z)dz\right]$

where C is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).

Sol.: Here $\mathbf{F} = (x+y)\hat{\mathbf{I}} + (2x-z)\hat{\mathbf{J}} + (y+z)\hat{\mathbf{K}}$.

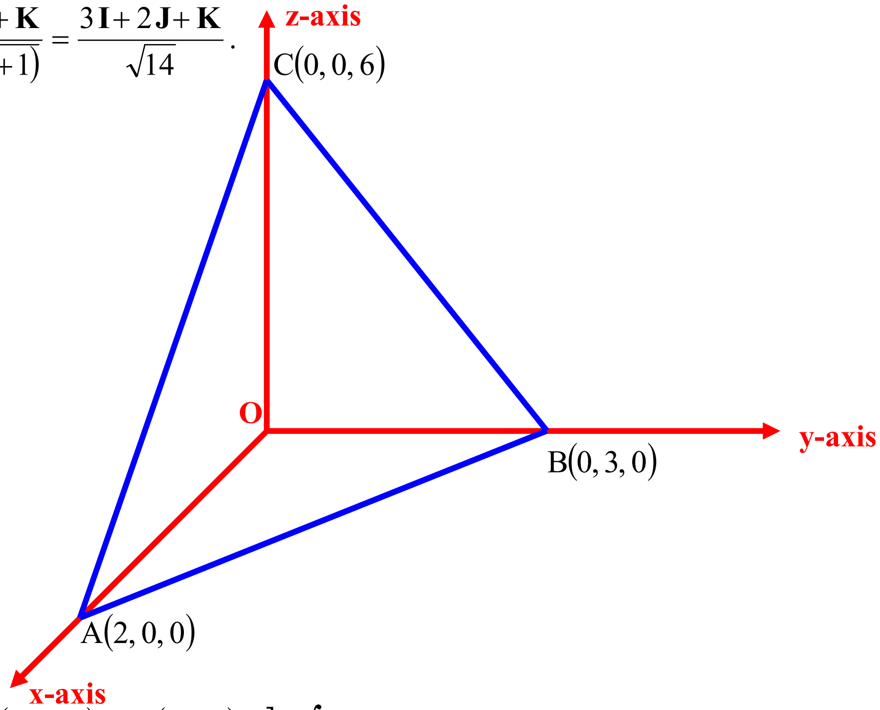
$$\therefore \text{ curl } \mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = (1+1)\hat{\mathbf{I}} + (0)\hat{\mathbf{J}} + (2-1)\hat{\mathbf{K}} = 2\hat{\mathbf{I}} + \hat{\mathbf{K}}.$$

Also, the equation of the plane through A, B, C is

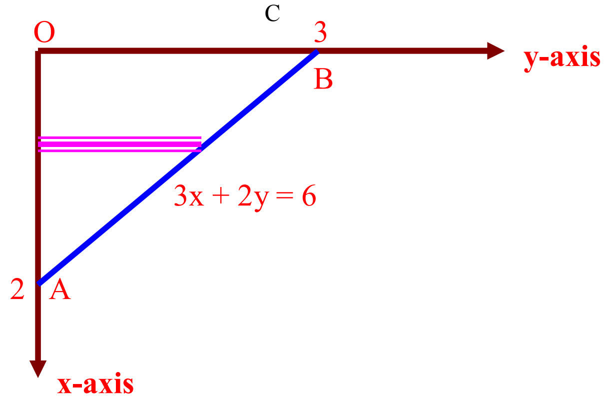
$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \Rightarrow 3x + 2y + z = 6.$$

Then the vector $\mathbf{N}$ normal to this plane is $\nabla(3x+2y+z-6) = 3\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + \hat{\mathbf{K}}$.

$$\therefore \hat{\mathbf{N}} = \frac{\mathbf{N}}{|\mathbf{N}|} = \frac{3\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + \hat{\mathbf{K}}}{\sqrt{(9+4+1)}} = \frac{3\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + \hat{\mathbf{K}}}{\sqrt{14}}.$$



Hence $\int_C [(x+y)dx + (2x-z)dy + (y+z)dz] = \int_C \mathbf{F}.d\mathbf{R}$



$= \int_S \text{curl}\, \mathbf{F}.\hat{\mathbf{N}}\, ds$, where S is the triangle ABC

$$= \int_S \left(2\hat{\mathbf{I}} + \hat{\mathbf{K}}\right).\left(\frac{3\hat{\mathbf{I}} + 2\hat{\mathbf{J}} + \hat{\mathbf{K}}}{\sqrt{14}}\right) ds = \frac{1}{\sqrt{14}}(6+1)\int_S ds \text{, where } ds = \frac{dxdy}{\left|\hat{\mathbf{N}}.\hat{\mathbf{K}}\right|} = \frac{dxdy}{\frac{1}{\sqrt{14}}}.$$

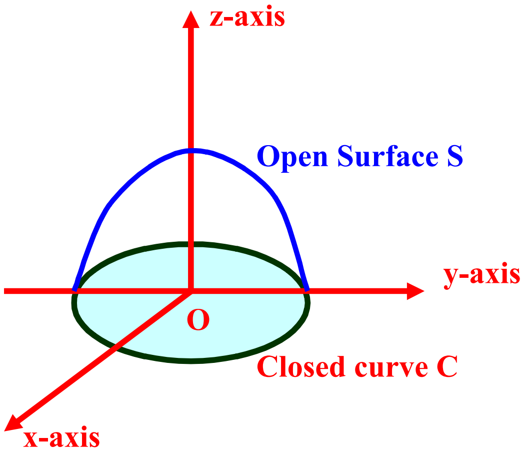
$$= \frac{7}{\sqrt{14}} \iint_S \frac{dxdy}{\frac{1}{\sqrt{14}}} = 7\int_0^2 \left(\int_0^{\frac{6-3x}{2}} dy\right) dx = 7\int_0^2 \left(\frac{6-3x}{2}\right) dx = 7\left[\frac{6}{2}x - \frac{3}{2}\frac{x^2}{2}\right]_0^2$$

$= 7[6-3] = 21$.Ans.

Q.No.7: Evaluate $\int \nabla \times \mathbf{V}.d\mathbf{S}$ over the surface of the paraboloid

$$z = 1 - x^2 - y^2,\ z \geq 0 \text{, where } \mathbf{V} = y\hat{\mathbf{I}} + z\hat{\mathbf{J}} + x\hat{\mathbf{K}}.$$

Sol.:



Stoke's theorem states that, if S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\hat{\mathbf{I}} + f_2\hat{\mathbf{J}} + f_3\hat{\mathbf{K}}$ be any continuously differentiable vector point function, then

$$\int_C \mathbf{F}.d\mathbf{R} = \int_S \text{curl}\, \mathbf{F}.\hat{\mathbf{N}}\, ds,$$

where $\hat{\mathbf{N}}$ is a unit vector external normal at any point of S.

Here given $\mathbf{V} = y\hat{\mathbf{I}} + z\hat{\mathbf{J}} + x\hat{\mathbf{K}}$, then

$$\int \nabla\times \mathbf{V}.d\mathbf{S} = \int_C \mathbf{V}.d\mathbf{R} = \int_C \left(y\hat{\mathbf{I}} + z\hat{\mathbf{J}} + x\hat{\mathbf{K}}\right).\left(dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}} + dz\,\hat{\mathbf{K}}\right)$$

$$= \int_C (ydx + zdy + xdz)$$

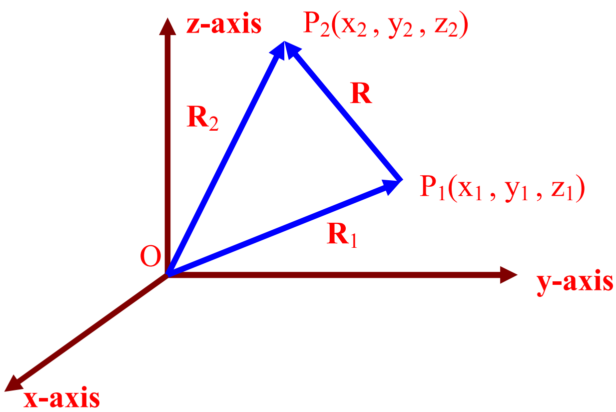
$$= \int_C ydx, \qquad \text{since on the closed curve C, } z = 0$$

$$= 4\times\int_0^1 \sqrt{1-x^2}\,dx = 4\times\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right]_0^1$$

$$= 4\times\frac{1}{2}.\frac{\pi}{2} = \pi. \text{ Ans.}$$

Q.No.8: Show that $\int_C \mathbf{R}.d\mathbf{R} = 0$ independently of the origin of **R.**

Sol.:



Take a position vector **R** such that it is origination from point $P_1(x_1, y_1, z_1)$ to the point $P_2(x_2, y_2, z_2)$ having position vectors $\mathbf{R}_1$ and $\mathbf{R}_2$, hence **R** is independent of origin.

$$\mathbf{R} = (x_2 - x_1)\hat{\mathbf{I}} + (y_2 - y_1)\hat{\mathbf{J}} + (z_2 - z_1)\hat{\mathbf{K}}$$

$$\Rightarrow d\mathbf{R} = d(x_2 - x_1)\hat{\mathbf{I}} + d(y_2 - y_1)\hat{\mathbf{J}} + d(z_2 - z_1)\hat{\mathbf{K}}.$$

Then, by Stoke's theorem which states that, if S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\hat{\mathbf{I}} + f_2\hat{\mathbf{J}} + f_3\hat{\mathbf{K}}$ be any continuously differentiable vector point function, then

$$\int_C \mathbf{F}.d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}.\hat{\mathbf{N}}\,ds,$$

where $\overset{\wedge}{\mathbf{N}}$ is a unit vector external normal at any point of S.

$$\int_C \mathbf{R}.d\mathbf{R} = \int_S \text{curl}\,\mathbf{R}\,.\,d\mathbf{S}$$

$$\text{Since curl}\,\mathbf{R} = \begin{vmatrix} \overset{\wedge}{\mathbf{I}} & \overset{\wedge}{\mathbf{J}} & \overset{\wedge}{\mathbf{K}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) \end{vmatrix} = (0-0)\overset{\wedge}{\mathbf{I}} - (0-0)\overset{\wedge}{\mathbf{J}} + (0-0)\overset{\wedge}{\mathbf{K}} = \mathbf{0}$$

$\int_C \mathbf{R}.d\mathbf{R} = \int_S \text{curl}\,\mathbf{R}\,.\,d\mathbf{S} = \int_S \mathbf{0}\,.\,d\mathbf{S} = 0$. This completes the proof.

Q.No.9: Prove that $\int_C \mathbf{A}\times\mathbf{R}.\,d\mathbf{R} = 2\mathbf{A}.\int dS$, $\mathbf{A}$ being any constant vector, and deduce that $\oint_C \mathbf{R}\times d\mathbf{R}$ is twice the vector area of the surface enclosed by C.

Sol.: Stoke's theorem states that, if S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\,\overset{\wedge}{\mathbf{I}} + f_2\,\overset{\wedge}{\mathbf{J}} + f_3\,\overset{\wedge}{\mathbf{K}}$ be any continuously differentiable vector point function, then

$$\int_C \mathbf{F}.\,d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}\,.\,\overset{\wedge}{\mathbf{N}}\,ds\,,$$

where $\overset{\wedge}{\mathbf{N}}$ is a unit vector external normal at any point of S. Then

$$\int_C (\mathbf{A}\times\mathbf{R}).\,d\mathbf{R} = \int_S \text{curl}\,(\mathbf{A}\times\mathbf{R}).\;d\mathbf{S} = \int_S \nabla\times(\mathbf{A}\times\mathbf{R}).\;d\mathbf{S}$$

$$= \int_S [\mathbf{A}(\nabla.\mathbf{R}) - (\mathbf{A}.\nabla)\mathbf{R}].\;d\mathbf{S} = \int_S [\mathbf{A}(3) - \mathbf{A}].\;d\mathbf{S} = \int_S 2\mathbf{A}\,.\;d\mathbf{S}$$

$$= 2\mathbf{A}\int_S d\mathbf{S}.$$

Consider an area in xy-plane enclosed by curve C.

Since $\mathbf{R} = x\,\overset{\wedge}{\mathbf{I}} + y\,\overset{\wedge}{\mathbf{J}} \Rightarrow d\mathbf{R} = dx\,\overset{\wedge}{\mathbf{I}} + dy\,\overset{\wedge}{\mathbf{J}}$, then

$$\mathbf{R}\times d\mathbf{R} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ x & y & 0 \\ dx & dy & 0 \end{vmatrix} = (0-0)\hat{\mathbf{I}} - (0-0)\hat{\mathbf{J}} + (xdy - ydx)\hat{\mathbf{K}} = (xdy - ydx)\hat{\mathbf{K}}$$

$$= \left[\left(-y\,\hat{\mathbf{I}} + x\,\hat{\mathbf{J}}\right).\left(dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}}\right)\right]\hat{\mathbf{K}} = [\mathbf{U}.d\mathbf{R}]\hat{\mathbf{K}}, \quad \text{where } \mathbf{U} = \left(-y\,\hat{\mathbf{I}} + x\,\hat{\mathbf{J}}\right).$$

Then, by Stoke's theorem and substituting the value of

$$\nabla\times\mathbf{U} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = (0-0)\hat{\mathbf{I}} - (0-0)\hat{\mathbf{J}} + (1+1)\hat{\mathbf{K}} = 2\hat{\mathbf{K}}.$$

Then, we get $\oint_C \mathbf{R}\times d\mathbf{R} = \int_C (\mathbf{U}.d\mathbf{R})\hat{\mathbf{K}} = \int_S (\nabla\times\mathbf{U}).\hat{\mathbf{K}}.d\mathbf{S} = \int_S 2\,\hat{\mathbf{K}}.\hat{\mathbf{K}}\,d\mathbf{S}$

$$= 2\int_S d\mathbf{S} = \text{Twice the vector area of surface enclosed by C}$$

This completes the proof.

Q.No.10: If ϕ is a scalar point function, use Stoke's theorem to prove that

$$\text{curl}(\text{grad}\,\phi) = 0.$$

Sol.: Stoke's theorem states that, if S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\,\hat{\mathbf{I}} + f_2\,\hat{\mathbf{J}} + f_3\,\hat{\mathbf{K}}$ be any continuously differentiable vector point function, then

$$\int_C \mathbf{F}.\,d\mathbf{R} = \int_S \text{curl}\,\mathbf{F}\,.\,\hat{\mathbf{N}}\,ds,$$

where $\hat{\mathbf{N}}$ is a unit vector external normal at any point of S. Then

$$\int_S \nabla\times\nabla\phi\,.\,d\mathbf{S} = \int_C \nabla\phi.\,d\mathbf{R} = \int_C \left(\hat{\mathbf{I}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{J}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{K}}\frac{\partial\phi}{\partial z}\right).\left(dx\,\hat{\mathbf{I}} + dy\,\hat{\mathbf{J}} + dz\,\hat{\mathbf{K}}\right)$$

$$= \int_C \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = \int_C d\phi = 0.$$

Since ϕ is a scalar point function and its integration is carried on a closed curve, whose initial and final limits will be same.

Hence

$$\int_S \nabla\times\nabla\phi.d\mathbf{S} = 0 \Rightarrow \nabla\times\nabla\phi = 0 \Rightarrow \text{curl}(\text{grad}\,\phi) = 0.$$

This completes the proof.

Q.No.11.: Evaluate $\iint_S (\nabla\times\mathbf{A}).\hat{\mathbf{N}}\,ds$ over the surface of intersection of the cylinders $x^2+y^2=a^2$, $x^2+z^2=a^2$ which is included in the first octant, given that $\mathbf{A} = 2yz\hat{\mathbf{I}} - (x+3y-2)\hat{\mathbf{J}} + (x^2+z)\hat{\mathbf{K}}$.

Sol.: By Stokes theorem the given surface integral can be converted to a line integral,

i.e. $$SI = \iint_S (\nabla\times\mathbf{A}).\hat{\mathbf{N}}\,ds = \oint_c \mathbf{A}.d\mathbf{R} = LI.$$

Here c is the curve consisting of the four curves

$c_1:\ x^2+z^2=a^2,\ y=0;$ $\quad c_2:\ x^2+y^2=a^2,\ z=0;$

$c_3: x=0,\ y=a,\ 0\le z\le a;\ c_4: x=0,\ z=a,\ 0\le y\le a.$

$$LI = \oint_c \mathbf{A}.d\mathbf{R} = \int_{c_1+c_2+c_3+c_4} = \int_{c_1} + \int_{c_2} + \int_{c_3} + \int_{c_4} = LI_1 + LI_2 + LI_2 + LI_3 + LI_4$$

On the curve c_1: $y=0,\ x^2+z^2=a^2$

$$LI_1 = \int_{c_1} \mathbf{A}.d\mathbf{R} = \int_{c_1} (x^2+z)dz = \int_a^0 \left[(a^2-z^2)+z\right]dz = -\frac{2}{3}a^3 - \frac{a^2}{2}$$

On the curve c_2: $z=0,\ x^2+y^2=a^2$

$$LI_2 = \int_{c_2} \mathbf{A}.d\mathbf{R} \int_{c_2} -(x+3y-2)dy = -\int_0^a \left(\sqrt{a^2-y^2}+3y-2\right)dy = -\frac{\pi a^2}{4} - \frac{3}{2}a^2 + 2a.$$

On the curve c_3: $x=0,\ y=a,\ 0\le z\le a$

$$LI_3 = \int_{c_3} \mathbf{A}.d\mathbf{R} = \int_0^a z dz = \frac{a^2}{2}$$

On the curve c_4: $x=0,\ z=a,\ 0\le y\le a$

$$LI_4 = \int_{c_4} \mathbf{A}.d\mathbf{R} = \int_a^0 (2-3y)dz = -2a + \frac{3a^2}{2}.$$

Thus $SI = \iint_S (\nabla \times A).\hat{\mathbf{N}}\, ds = LI = \left(\frac{-2a^3}{3} - \frac{a^2}{2}\right) + \left(-\frac{\pi a^2}{4} - \frac{3a^2}{2} + 2a\right) + \frac{a^2}{2} + \left(-2a + \frac{3a^2}{2}\right)$

$$SI = \frac{-a^2}{12}(3\pi + 8a)$$

Q.No.12.: Prove that $\oint_c f d\mathbf{R} = \iint_S d\mathbf{S} \times \nabla f$.

Sol.: Choose $\mathbf{A} = f\mathbf{C}$, where $\mathbf{C}$ is a constant vector, in the Stoke's theorem.

Then LHS $= \oint_c \mathbf{A}.d\mathbf{R} = \oint_c f\mathbf{C}.d\mathbf{R} = \oint_c \mathbf{C}.(fd\mathbf{R}) = \mathbf{C}.\oint_c fd\mathbf{R}.$

Now $\nabla \times \mathbf{A} = \nabla \times (f\mathbf{C}) = (\nabla f) \times \mathbf{C} + f(\nabla \times \mathbf{C}) = \nabla f \times \mathbf{C}$, since $\nabla \times \mathbf{C} = 0$.

So $(\nabla \times \mathbf{A}).\hat{\mathbf{N}} = (\nabla f \times \mathbf{C}).\hat{\mathbf{N}} = \mathbf{C}.\left(\hat{\mathbf{N}} \times \nabla f\right)$

RHS $= \iint_S (\nabla \times \mathbf{A}).\hat{\mathbf{N}}\, ds = \iint_S \mathbf{C}.\left(\hat{\mathbf{N}} \times \nabla f\right) ds = \mathbf{C}.\iint_S \left(\hat{\mathbf{N}} \times \nabla f\right) ds$

Thus $\mathbf{C}.\oint_c fd\mathbf{R} = \mathbf{C}.\iint_S \left(\hat{\mathbf{N}} \times \nabla f\right) ds$.

Since this is true for any arbitrary constant $\mathbf{C}$, thus, we get

$$\oint_c fd\mathbf{R} = \iint_S \left(\hat{\mathbf{N}} \times \nabla f\right) ds = \iint_S \hat{\mathbf{N}}\, ds \times \nabla f = \iint_S d\mathbf{S} \times \nabla f.$$

Hence the result.

Q.No.13.: If C is a simple closed curve in the xy-plane not enclosing the origin.

Show that $\int_C \mathbf{F}.d\mathbf{R} = 0$, where $\mathbf{F} = \dfrac{y\hat{\mathbf{I}} - x\hat{\mathbf{J}}}{x^2 + y^2}$.

Sol.: Since we know that $\oint \mathbf{F}.d\mathbf{R} = \iint (\nabla \times \mathbf{F}) d\mathbf{S}$.

Now $\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} & 0 \end{vmatrix}$

$$= \hat{\mathbf{I}}(0) + \hat{\mathbf{J}}(0) + \hat{\mathbf{K}}\left[-\frac{\left(x^2+y^2\right)1 - x.2x}{\left(x^2+y^2\right)^2} - \frac{\left(x^2+y^2\right)1 - 2y.y}{\left(x^2+y^2\right)^2} \right]$$

$$= \hat{\mathbf{I}}(0) + \hat{\mathbf{J}}(0) + \hat{\mathbf{K}}\left[-\frac{x^2+y^2-2x^2}{\left(x^2+y^2\right)^2} - \frac{x^2+y^2-2y^2}{\left(x^2+y^2\right)^2} \right]$$

$$= \hat{\mathbf{I}}(0) + \hat{\mathbf{J}}(0) + \hat{\mathbf{K}}\left[-\frac{y^2-x^2}{\left(x^2+y^2\right)^2} - \frac{x^2-y^2}{\left(x^2+y^2\right)^2} \right]$$

$$= \hat{\mathbf{I}}(0) + \hat{\mathbf{J}}(0) + \hat{\mathbf{K}}\left[\frac{x^2-y^2}{\left(x^2+y^2\right)^2} - \frac{x^2-y^2}{\left(x^2+y^2\right)^2} \right]$$

$$= \hat{\mathbf{I}}(0) + \hat{\mathbf{J}}(0) + \hat{\mathbf{K}}[0] = \mathbf{0}$$

$\therefore$ Required integral $\oint \mathbf{F}.d\mathbf{R} = \iint (\mathbf{0})d\mathbf{S} = 0$

This completes the proof.

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Home Assignments

Q.No.1.: Verify Stokes theorem in $\mathbf{A} = x^2\hat{\mathbf{I}} + xy\hat{\mathbf{J}}$ where S is square $0 \le x \le a$, $0 \le y \le a$ in the xy-plane.

Hint: c : square $0 \le x \le a$, $0 \le y \le a$, $z = 0$.

Ans.: $a^3/2$

Q.No.2.: **Verify Stokes theorem** in $\mathbf{A} = e^z\left(\hat{\mathbf{I}} + \sin y\,\hat{\mathbf{J}} + \cos y\,\hat{\mathbf{K}}\right)$ where $S : z = y^2$, $0 \le x \le 4$, $0 \le y \le 2$.

Ans.: $\pm 4\left(1 - e^4\right)$ from (0, 0, 0) to (4, 0, 0)

$\mp 4e^4$ from (4, 2, 4) to (0, 2, 4)

The integrals over the parabolas cancel each other.

Q.No.3.: **Verify Stokes theorem** in $\mathbf{A} = y^2\,\hat{\mathbf{I}} + z^2\,\hat{\mathbf{J}} + x^2\,\hat{\mathbf{K}}$ where S : portion of paraboloid $x^2 + y^2 = z$, $y \ge 0$, $z \le 1$.

Ans.: $\pm\frac{4}{3}$.

Q.No.4.: **Using Stoke's theorem** evaluate $\iint_S (\nabla \times \mathbf{A}).\hat{\mathbf{N}}\,ds$ where S is the surface of hemisphere $x^2 + y^2 + z^2 = 16$ above the xy-plane and $\mathbf{A} = \left(x^2 + y - 4\right)\hat{\mathbf{I}} + 3xy\,\hat{\mathbf{J}} + \left(2xz + z^2\right)\hat{\mathbf{K}}$.

Ans.: -16π.

Q.No.5.: If $\mathbf{A} = \left(y^2 + z^2 + x^2\right)\hat{\mathbf{I}} + \left(z^2 + x^2 - y^2\right)\hat{\mathbf{J}} + \left(x^2 + y^2 - z^2\right)\hat{\mathbf{K}}$ then **using Stoke's theorem** evaluate $\iint_S (\nabla \times \mathbf{A}).\hat{\mathbf{N}}\,ds$ taken over the surface $S = x^2 + y^2 - 2ax + az = 0$, $z \ge 0$.

Ans.: $2\pi a^2$.

Q.No.6.: **Using Stoke's theorem** evaluate $\iint_S \nabla \times \left(y\,\hat{\mathbf{I}} + z\,\hat{\mathbf{J}} + x\,\hat{\mathbf{K}}\right).\hat{\mathbf{N}}\,ds$ over the surface of the paraboloid $z = 1 - x^2 - y^2$, $z \ge 0$.

Ans.: π.

Q.No.7.: **Using Stoke's theorem** evaluate $\iint\limits_S \nabla\times\left(y\hat{\mathbf{I}}+2x\hat{\mathbf{J}}+z\hat{\mathbf{K}}\right).\hat{\mathbf{N}}\,ds$ where S is the paraboloid $z=1-x^2-y^2$, $z\geq 0$.

Ans.: π

Q.No.8.: What is the surface integral of the normal component of the curl of the vector function $(x+y)\hat{\mathbf{I}}+(y-x)\hat{\mathbf{J}}+z^3\hat{\mathbf{K}}$ over the upper half of the sphere $x^2+y^2+z^2=1$.

Ans.: -2π.

Q.No.9.: **Using Stoke's theorem** evaluate $\int\limits_c (ydx+zdy+xdz)$ where c is the curve given by $x^2+y^2+z^2-2ax-2ay=0$, x + y = 2a, beginning at the point (2a, 0, 0) and going at first below the z-plane.

Ans.: $-2\sqrt{2}\pi a^2$.

Q.No.10.: **Using Stoke's theorem** evaluate $\oint\limits_c (\sin zdx-\cos dy+\sin ydz)$ where c: rectangle $0\leq x\leq\pi$, $0\leq y\leq 1$, $z=3$.

Ans.: 2.

Q.No.11.: Evaluate $\oint\limits_c (ydx+xz^3dy-zy^2dz)$

(a) directly

(b) using Stokes theorem, given that c is the circle : $x^2+y^2=4$, $z=-3$.

Ans.: $=-28.4\pi$

Q.No.12.: Evaluate $\oint\limits_c (4adx-2xdy+2xdz)$ where c is the ellipse $x^2+y^2=1$, z = y + 1.

(a) directly

(b) using Stokes theorem

Ans.: -4π.

Q.No.13.: Find **(Using Stoke's theorem)** the **work done** by the force F in the displacement around the closed curve c where

$\mathbf{F} = 2xy^3z\hat{\mathbf{I}} + 3x^2y^2 \sin z\hat{\mathbf{J}} + x^2y^3 \cos z\hat{\mathbf{K}}$, c : intersection of paraboloid $z = x^2 + y^2$ and cylinder $(x-1)^2 + y^2 = 1$.

Ans.: 0

Q.No.14.: Find **(Using Stoke's theorem)** the **work done** by the force F in the displacement around the closed curve c where $\mathbf{F} = x^3\hat{\mathbf{I}} + e^{3y}\hat{\mathbf{J}} + e^{-3z}\hat{\mathbf{K}}$, c: $x^2 + 9y^2 = 9$, $z = x^2$.

Ans.: 0.

Q.No.15.: Show that if $\mathbf{F} = f_1\hat{\mathbf{I}} + f_2\hat{\mathbf{J}} + f_3\hat{\mathbf{K}}$ **F** and S satisfy the assumptions of Stoke's theorem and **F**= grad f, then $\int_C (f_1dx + f_2dy + f_3dz) = 0$, where C is the boundary of S.

Q.No.16.: If $\nabla \times \mathbf{A} = 0$, then prove that $\oint_c \mathbf{A}.d\mathbf{R} = 0$ for every closed curve c.

Q.No.17.: Prove that $\oint_c d\mathbf{R} \times \mathbf{B} = \iint_S \left(\hat{\mathbf{N}} \times \nabla\right) \times \mathbf{B} ds$.

Hint: Choose $\mathbf{A} = \mathbf{B} \times \mathbf{C}$, where **c** is a constant vector, and apply Stokes theorem. Note that $\left(\hat{\mathbf{N}} \times \nabla\right) \times \mathbf{B} = \nabla\left(\mathbf{B}.\hat{\mathbf{N}}\right) - \hat{\mathbf{N}}(\nabla.\mathbf{B})$.

Q.No.18.: Prove that. $\oint_c f\nabla g.d\mathbf{R} = \iint_S (\nabla f \times \nabla g).\hat{\mathbf{N}} ds$ and deduce that $\oint_c f\nabla f.d\mathbf{R} = 0$.

Hint: Take $\mathbf{A} = f\nabla g$ in Stokes theorem. Note that $\nabla \times \nabla g = 0$.

For deduction, take f = g and note that $\nabla f \times \nabla f = 0$.

10th Topic

Vector Calculus

Volume integral

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VOLUME INTEGRAL:

Consider a continuous vector function $\mathbf{F}(\mathbf{R})$ defined everywhere in the region E enclosed by surface S. Now divide this region E into finite, i.e. 'n' number of sub regions $E_1, E_2, \ldots\ldots\ldots, E_n$. Let δv_i be the volume of the sub-region E_i enclosing any point, whose position vector is $\mathbf{R}_i$.

Now consider the sum $\mathbf{V} = \sum_{i=1}^{n} \mathbf{F}(\mathbf{R}_i)\delta v_i$.

The limit of this sum, as $n \to \infty$, and consequently $\delta v_i \to 0$, is called the **volume integral or space integral** of $\mathbf{F}(\mathbf{R})$ over E and symbolically this can be written as $\int_E \mathbf{F} dv$.

Thus $\lim_{\substack{n \to \infty \\ \delta v_i \to 0}} \sum_{i=1}^{n} \mathbf{F}(\mathbf{R}_i)\delta v_i = \int_E \mathbf{F} dv$.

Now if $\mathbf{F}(\mathbf{R}) = f(x, y, z)\hat{\mathbf{I}} + \phi(x, y, z)\hat{\mathbf{J}} + \psi(x, y, z)\hat{\mathbf{K}}$ and $dv = dx\, dy\, dz$.

Then $\int_E \mathbf{F}dv = \hat{\mathbf{I}}\iiint_E f dx\,dy\,dz + \hat{\mathbf{J}}\iiint_E \phi dx\,dy\,dz + \hat{\mathbf{K}}\iiint_E \psi dx\,dy\,dz$.

Q.No.1.: Evaluate $\iiint_V f dV$ where f = 2x + y, V is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes x = 0, y = 0 and z = 0.

Sol.: This closed region is covered if x and z varies covering the area OAB and y varies from 0 to 2. Thus

$$\iiint_V (2x+y)dV = \int_{y=0}^{2}\int_{z=0}^{4}\left(\int_{x=0}^{\sqrt{4-z}}(2x+y)dx\right)dzdy = \int_0^2\int_0^4 \left(x^2+xy\right)\Big|_0^{\sqrt{4-z}} dzdy$$

$$= \int_0^2\left(\int_0^4\left[(4-z)+y\sqrt{4-z}\right]dz\right)dy = \int_0^2 4z - \frac{z^2}{2} - \frac{2}{3}y(4-z)^{3/2}\Big|_0^4$$

$$= \int_0^2\left(8+\frac{16}{3}y\right)dy = 8y + \frac{16}{6}y^2\Big|_0^2 = \frac{80}{3}.$$

Q.No.2.: If V is the region in the first octant bounded by $y^2 + z^2 = 9$ and the plane x = 2 and $\mathbf{F} = 2x^2y\hat{\mathbf{I}} - y^2\hat{\mathbf{J}} + 4xz^2\hat{\mathbf{K}}$. Then evaluate $\iiint_V (\nabla.\mathbf{F})dV$.

Sol.: $\nabla.\mathbf{F} = 4xy - 2y + 8xz$.

The volume V of the solid region is covered by covering the plane region OAB while x varies from 0 to 2.

$$\iiint_V (\nabla.\mathbf{F})dV = \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}}(4xy-2y+8xz)dzdydx$$

$$= \int_0^2\int_0^3 4xyz - 2yz + 4xz^2\Big|_0^{\sqrt{9-y^2}} dydx$$

$$= \int_0^2\int_0^3\left[(4xy-2y)\sqrt{9-y^2} + 4x\left(9-y^2\right)\right]dydx$$

$$= \int_0^2 (4x-2) - \frac{1}{3}\left(9-y^2\right)^{3/2} + 4x\left(9y - \frac{y^3}{3}\right)\Big|_0^3 dx$$

$$=\int_0^2 \left[9(4x-2)+72x\right]dx = 18x^2-18x+36x^2\Big|_0^2 = 180.$$

Q.No.3.: Evaluate $\iiint_V \nabla\times\mathbf{A}.dV$ where $\mathbf{A}=(x+2y)\hat{\mathbf{I}}-3z\hat{\mathbf{J}}+x\hat{\mathbf{K}}$ and V is the closed region in the first octant bounded by the plane 2x + 2y + z = 4.

Sol.: The solid region is covered by covering the plane region OAB in the xy-plane while z is varying from 0 to the plane 2x + 2y + z = 4.

Thus z varies from 0 to $4-2x-2y$,

y varies from 0 to $2-x$

and x varies from 0 to 2.

Here $\nabla\times\mathbf{A}=\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+y & -3z & x\end{vmatrix}$

$$\iiint_V \nabla\times\mathbf{A}.dV = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y}\left(3\hat{\mathbf{I}}-\hat{\mathbf{J}}+2\hat{\mathbf{K}}\right)dzdydx$$

$$=\left(3\hat{\mathbf{I}}-\hat{\mathbf{J}}+2\hat{\mathbf{K}}\right)\int_0^2\int_0^{2x}(4-2x-2y)dydz$$

$$=\left(3\hat{\mathbf{I}}-\hat{\mathbf{J}}+2\hat{\mathbf{K}}\right)\int_0^2\left(\left(2-x^2\right)-\frac{(2-x)^2}{2}\right)dx$$

$$=\left(3\hat{\mathbf{I}}-\hat{\mathbf{J}}+2\hat{\mathbf{K}}\right)\left[4x+\frac{x^3}{3}-2x^2\right]_0^2$$

$$=\frac{8}{3}\left(3\hat{\mathbf{I}}-\hat{\mathbf{J}}+2\hat{\mathbf{K}}\right).$$

Q.No.4.: Find the volume enclosed between the two surfaces $S_1: z=8-x^2-y^2$ and $S_2: z=x^2-3y^2$.

Sol.: Eliminating z from the given two surfaces S_1 and S_2, we get $8-x^2-y^2=z=x^2+3y^2$ i.e., $x^2+2y^2=4$.

Thus the given two surfaces S_1 and S_2 intersect on the elliptic cylinder $x^2 + 2y^2 = 4$.

So the solid region between S_1 and S_2 is covered when

z varies from $x^2 + 3y^2$ to $8 - x^2 - y^2$

y varies from $-\sqrt{\frac{4-x^2}{2}}$ to, $\sqrt{\frac{4-x^2}{2}}$

and x varies from -2 to 2.

So the required volume V enclosed between the two surface S_1 and S_2 is

$$V = \int_{-2}^{2} \int_{-\frac{\sqrt{(4-x)^2}}{2}}^{\frac{\sqrt{(4-x)^2}}{2}} \int_{x^2+3y^2}^{8-x^2-y^2} dzdydx = \int_{-2}^{2} \int_{-\frac{\sqrt{(4-x)^2}}{2}}^{\frac{\sqrt{(4-x)^2}}{2}} \left(8 - 2x^2 - 4y^2\right) dydx$$

$$= \int_{-2}^{2} \left[2\left(8 - 2x^2\right)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx$$

$$V = \frac{4\sqrt{2}}{3} \int_{-2}^{2} \left(4 - x^2\right)^{3/2} dx = 8\pi\sqrt{2}.$$ Cubic units. Ans.

Home Assignment

Q.No.1.: Evaluate $\iiint_V fdV$ where $f = 45x^2 y$ and V denotes the closed region bounded by the planes $4x + 2y + z = 8$, $x = 0$, $y = 0$, $z = 0$

Ans.: 128.

Q.No.2.: If $\mathbf{A} = \left(2x^2 - 3z\right)\hat{\mathbf{I}} - 2xy\,\hat{\mathbf{J}} - 4x\,\hat{\mathbf{K}}$ and V is the closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$, evaluate $\iiint_V (\nabla \times \mathbf{A})dV$.

Ans.: $\frac{8}{3}\left(\hat{\mathbf{J}} - \hat{\mathbf{K}}\right)$.

Q.No.3.: Evaluate $\iiint_V \mathbf{A}dV$ where $\mathbf{A} = x\,\hat{\mathbf{I}} + y\,\hat{\mathbf{J}} + 2z\,\hat{\mathbf{K}}$ and V is the volume enclosed by the planes $x = 0$, $y = 0$, $z = 0$ and the surface $z = x^2$.

Ans.: $\frac{ab^4}{4}\hat{\mathbf{I}}+\frac{a^2b^3}{3}\hat{\mathbf{J}}+\frac{4ab^5}{5}\hat{\mathbf{K}}$.

Q.No.4.: Evaluate $\iiint\limits_V \mathbf{B}dV$ where V is the region bounded by the surfaces x = 0, y = 0, y = 6, $z = x^2$, z = 4 and $\mathbf{B} = xz\hat{\mathbf{I}} - x\hat{\mathbf{J}} + y^2\hat{\mathbf{K}}$.

Ans.: $128\hat{\mathbf{I}} - 24\hat{\mathbf{J}} + 384\hat{\mathbf{K}}$.

Q.No.5.: If $\mathbf{A} = \left(x^3 - yz\right)\hat{\mathbf{I}} - 2x^3y\hat{\mathbf{J}} + 2\hat{\mathbf{K}}$, evaluate $\iiint\limits_V (\nabla.\mathbf{A})dV$ over the volume of a cube of side b.

Ans.: $\frac{1}{3}b^3$.

Q.No.6.: Evaluate $\iiint\limits_V (\nabla.\mathbf{B})dV$ over the solid region of the sphere $x^2+y^2+z^2=a^2$ when $\mathbf{B} = px\hat{\mathbf{I}} + qy\hat{\mathbf{J}} + rz\hat{\mathbf{K}}$ where p, q, r are constant.

Ans.: $\frac{4}{3}\pi a^2 (p+q+r)$.

Q.No.7.: Find the volume of the region common to the intersecting cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

Ans.: $\frac{16a^3}{3}$.

Q.No.8.: Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$ and above by the planes z = 2y.

Ans.: $\frac{\pi}{2}$.

Q.No.9.: Find the volume cut from the sphere $x^2+y^2+z^2=4a^2$ by the cylinder $x^2+y^2=a^2$.

Ans.: $\frac{4a^3\left(8-3\sqrt{3}\right)}{3}$.

Q.No.10.: Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = 2a^2$ and below by the paraboloid $az = x^2 + y^2$.

Ans.: $\dfrac{\left(8\sqrt{2} - 7\right)\pi a^3}{6}$.

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11th Topic

Vector Calculus

Gauss Divergence Theorem,

Green's Theorem, Harmonic function

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Gauss Divergence Theorem:
(Relation between surface and volume integrals)

Statement: If $\mathbf{F}$ is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F}.\hat{\mathbf{N}}\, ds = \int_E \operatorname{div}\mathbf{F}\, dv,$$

where $\hat{\mathbf{N}}$ is the unit vector external normal at any point of S.

Proof: If $\mathbf{F}(\mathbf{R}) = f(x, y, z)\hat{\mathbf{I}} + \phi(x, y, z)\hat{\mathbf{J}} + \psi(x, y, z)\hat{\mathbf{K}}$.

Then it is required to prove that

$$\iint_S (f dydz + \phi dzdx + \psi dxdy) = \iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}\right) dx\, dy\, dz \qquad (1)$$

Case I: Firstly consider such a surface S that a line parallel to z-axis cuts it in two points; say $P_1(x, y, z_1)$ and $P_2(x, y, z_2)$ $(z_1 \le z_2)$ as shown in figure.

If S projects into the area A_z on the xy-plane, then

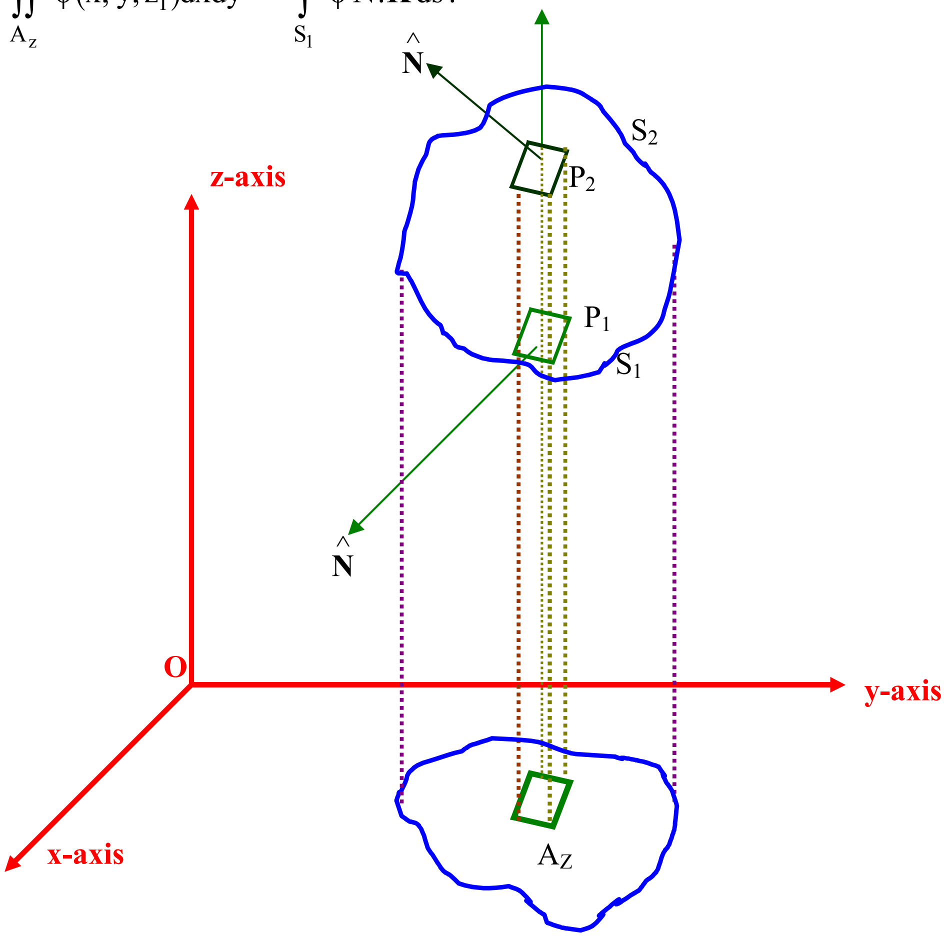
$$\iiint_E \frac{\partial \psi}{\partial z} dx\, dy\, dz = \iint_{A_z} \left(\int_{z_1}^{z_2} \frac{\partial \psi}{\partial z} dz \right) dxdy$$

$$= \iint_{A_z} [\psi(x, y, z_2) - \psi(x, y, z_1)] dxdy = \iint_{A_z} \psi(x, y, z_2) dxdy - \iint_{A_z} \psi(x, y, z_1) dxdy. \quad (2)$$

Let S_1, S_2 be the lower and upper parts of the surface S corresponding to the points P_1 and P_2 respectively and $\hat{N}$ be the unit external normal vector at any points of S. As the external normal at any point of S_2 makes an acute angle with the positive direction of z-axis and that at any point of S_1 an obtuse angle, therefore

$$\iint_{A_z} \psi(x, y, z_2) dxdy = \int_{S_2} \psi\, \hat{N}.\hat{K}\, ds. \quad (3)$$

$$\iint_{A_z} \psi(x, y, z_1) dxdy = -\int_{S_1} \psi\, \hat{N}.\hat{K}\, ds. \quad (4)$$



Using (3) and (4), (2) now becomes

$$\iiint_E \frac{\partial\psi}{\partial z} dx\, dy\, dz = \int_{S_2} \psi \hat{\mathbf{N}}.\hat{\mathbf{K}}\, ds + \int_{S_1} \psi \hat{\mathbf{N}}.\hat{\mathbf{K}}\, d = \int_S \psi \hat{\mathbf{N}}.\hat{\mathbf{K}}\, ds. \tag{5}$$

Similarly, we have

$$\iiint_E \frac{\partial f}{\partial x} dx\, dy\, dz = \int_{S_2} f \hat{\mathbf{N}}.\hat{\mathbf{I}}\, ds + \int_{S_1} f \hat{\mathbf{N}}.\hat{\mathbf{I}}\, d = \int_S f \hat{\mathbf{N}}.\hat{\mathbf{I}}\, ds. \tag{6}$$

$$\iiint_E \frac{\partial\phi}{\partial y} dx\, dy\, dz = \int_{S_2} f \hat{\mathbf{N}}.\hat{\mathbf{J}}\, ds + \int_{S_1} f \hat{\mathbf{N}}.\hat{\mathbf{J}}\, d = \int_S \phi \hat{\mathbf{N}}.\hat{\mathbf{J}}\, ds. \tag{7}$$

Addition of (5), (6) and (7) gives

$$\iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial z}\right) dx\, dy\, dz = \int_S \left(f\hat{\mathbf{I}} + \phi\hat{\mathbf{J}} + \psi\hat{\mathbf{K}}.\right)\hat{\mathbf{N}}\, ds,$$ which is same as (1).

Case II: Secondly, consider a general region E. Assume that it can be split up into a finite number of sub–regions each of which is met by a line parallel to any axis in only two points. Applying (1) to each of these sub-regions and adding the results, the volume integrals will combine to give the volume integral over the whole region E. Also the surface integrals over the common boundaries of two sub-regions cancel because each occurs twice and having corresponding normals in opposite directions whereas the remaining surface integrals combine to give the surface integral over the entire surface S.

Case III: Finally consider a region E bounded by two closed surfaces S_1, S_2 (S_1 being within S_2). Noting that the outward normal at points of S_1 is directed inwards (i. e. away from S_2) and introducing an additional surface cutting S_1, S_2 so that all parts of E are bounded by a single closed surface, the truth of the theorem follows as before. Thus theorem also holds for regions enclosed by several surfaces.

Hence, the theorem is completely established.

Green's Theorem:

Statement: If ϕ and ψ are scalar point functions possessing continuous derivatives of the first and second orders, then

$$\int_E \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dv = \int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) ds, \tag{i}$$

where, $\frac{\partial}{\partial n}$ denotes differentiation in the direction of the external normal to the bounding surface S enclosing the region E.

Proof: Applying Divergence theorem : $\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_E \nabla.\mathbf{F}dv$ to the function $\phi\nabla\psi$,

we get $\int_S (\phi\nabla\psi).\hat{\mathbf{N}}\,ds = \int_E \nabla.(\phi\nabla\psi)dv$

$$= \int_E \left(\nabla\phi.\nabla\psi + \phi\nabla^2\psi\right)dv \qquad [\because \nabla.(f\mathbf{G}) = \nabla f.\mathbf{G} + f\nabla.\mathbf{G}]$$

$$= \int_E \nabla\phi.\nabla\psi dv + \int_E \phi\nabla^2\psi dv. \qquad \text{(ii)}$$

Interchanging ϕ and ψ, (ii) gives

$$\int_S (\psi\nabla\phi).\hat{\mathbf{N}}\,ds = \int_E \nabla\psi.\nabla\phi dv + \int_E \psi\nabla^2\phi dv. \qquad \text{(iii)}$$

Subtracting (iii) from (ii), we get

$$\int_S (\phi\nabla\psi - \psi\nabla\phi).\hat{\mathbf{N}}\,ds = \int_E \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)dv.$$

But $\nabla\psi.\hat{\mathbf{N}} = \frac{\partial\psi}{\partial n}$ the directional derivatives of ψ along the external normal at any point of S.

Hence $\int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds = \int_E \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)dv.$

This completes the proof.

Harmonic function:

Definition: A scalar point function ϕ satisfying the Laplace's equation $\nabla^2\phi = 0$ at every point of a region E, is called a harmonic function in E.

If ϕ and ψ be both harmonic functions in E, then (i) gives

$\int_S \phi\frac{\partial\psi}{\partial n}ds = \int_S \psi\frac{\partial\phi}{\partial n}ds$, which is known as **Green's reciprocal theorem.**

Now let us discuss some problems based upon Gauss divergence theorem.

Verify Divergence theorem for the following problems:

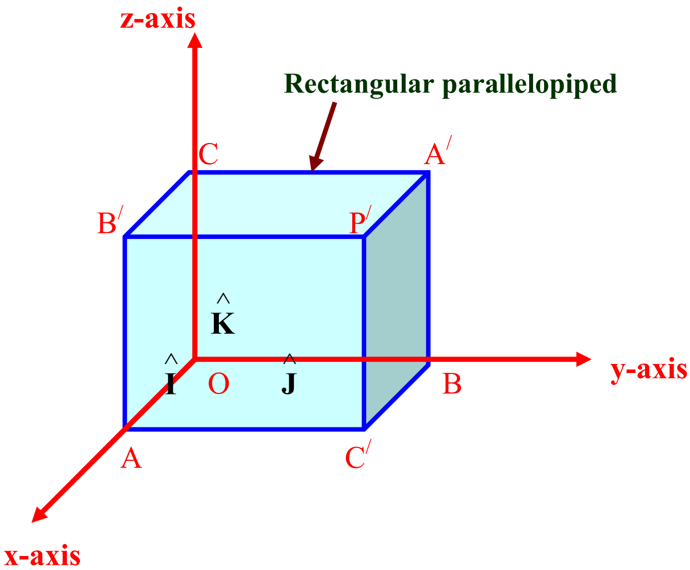
Q.No.1.: Verify Divergence theorem for $\mathbf{F} = \left(x^2 - yz\right)\hat{\mathbf{I}} + \left(y^2 - zx\right)\hat{\mathbf{J}} + \left(z^2 - xy\right)\hat{\mathbf{K}}$ taken over the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$.

Sol.: Gauss divergence theorem states that, if $\mathbf{F}$ is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_E \operatorname{div}\mathbf{F}\,dv,$$

where $\hat{\mathbf{N}}$ is the unit external normal vector at any point of S.

Now since $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}\left(x^2 - yz\right) + \frac{\partial}{\partial y}\left(y^2 - zx\right) + \frac{\partial}{\partial z}\left(z^2 - xy\right) = 2(x + y + z)$.



$$\therefore \int_R \operatorname{div}\mathbf{F}\,dv = 2\int_0^c \int_0^b \left[\int_0^a (x + y + z)dx\right] dy\,dz$$

$$= 2\int_o^c \left[\int_0^b \left(\frac{a^2}{2} + ya + za\right)dy\right]dz = 2\int_o^c \left(\frac{a^2}{2}b + \frac{ab^2}{2} + abz\right)dz$$

$$= 2\left(\frac{a^2 b}{2}c + \frac{ab^2}{2}c + ab\frac{c^2}{2}\right) = abc(a + b + c). \qquad (i)$$

Also $\int\limits_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int\limits_{S_1} \mathbf{F}.\hat{\mathbf{N}}\,ds + \int\limits_{S_2} \mathbf{F}.\hat{\mathbf{N}}\,ds + \ldots\ldots\ldots\ldots + \int\limits_{S_6} \mathbf{F}.\hat{\mathbf{N}}\,ds$,

where the surface S_1 is the face $OAC'B$, surface S_2 is the face $CB'P'A'$,

surface S_3 is the face $OBA'C$, surface S_4 is the face $AC'P'B'$,

surface S_5 is the face $OCB'A$ and surface S_6 is the face $BA'P'C'$.

Now $\int\limits_{S_1} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int\limits_{S_1} \mathbf{F}.\left(-\hat{\mathbf{K}}\right)ds = -\int\limits_0^b \int\limits_0^a (0 - xy)dxdy = \frac{a^2b^2}{4}$. $[\because z = 0]$

$\int\limits_{S_2} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int\limits_{S_2} \mathbf{F}.\left(\hat{\mathbf{K}}\right)ds = -\int\limits_0^b \int\limits_0^a \left(c^2 - xy\right)dxdy = abc^2 - \frac{a^2b^2}{4}$. $[\because z = c]$

Similarly, $\int\limits_{S_3} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int\limits_{S_3} \mathbf{F}.\left(-\hat{\mathbf{I}}\right)ds = -\int\limits_0^c \int\limits_0^b (0 - yz)dydz = \frac{b^2c^2}{4}$. $[\because x = 0]$

$\int\limits_{S_4} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int\limits_{S_4} \mathbf{F}.\left(\hat{\mathbf{I}}\right)ds = \int\limits_0^c \int\limits_0^b \left(a^2 - yz\right)dydz = a^2bc - \frac{b^2c^2}{4}$. $[\because x = a]$

$\int\limits_{S_5} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int\limits_{S_4} \mathbf{F}\left(-\hat{\mathbf{J}}\right) = -\int\limits_0^c \int\limits_0^b (0 - zx)dxdz = \frac{c^2a^2}{4}$. $[\because y = 0]$

$\int\limits_{S_6} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int\limits_{S_5} \mathbf{F}.\left(\hat{\mathbf{J}}\right)ds = \int\limits_0^c \int\limits_0^a \left(b^2 - zx\right)dxdz = ab^2c - \frac{c^2a^2}{4}$. $[\because y = b]$

Thus $\int\limits_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \frac{a^2b^2}{4} + \left(abc^2 - \frac{a^2b^2}{4}\right) + \frac{b^2c^2}{4} + \left(a^2bc - \frac{a^2c^2}{4}\right) + \frac{c^2a^2}{4} + \left(ab^2c - \frac{c^2a^2}{4}\right)$

$\Rightarrow \int\limits_S \mathbf{F}.\hat{\mathbf{N}}\,ds = abc(a + b + c)$. (ii)

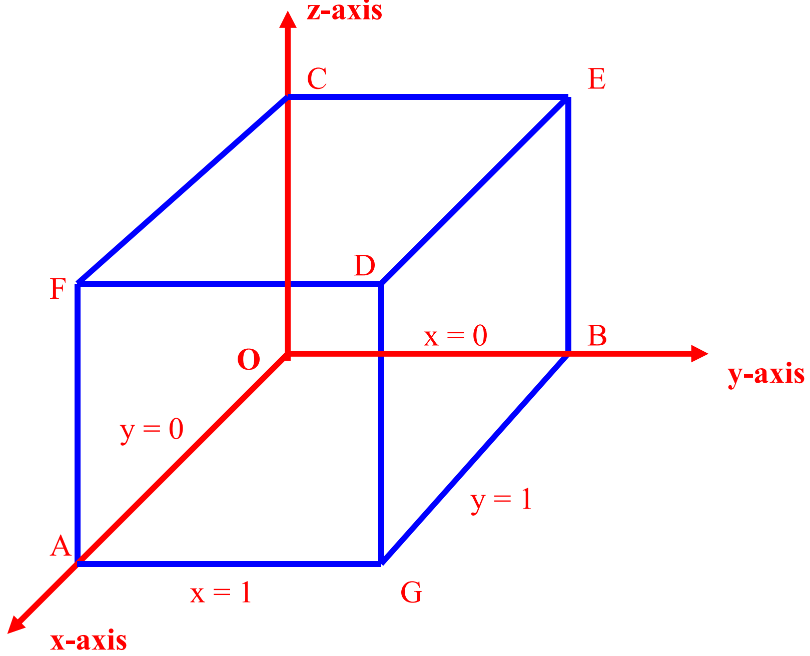
Hence, the divergence theorem is verified from equation of (i) and (ii).

Q.No.2: Verify divergence theorem for $\mathbf{F} = 4xz\hat{\mathbf{I}} - y^2\hat{\mathbf{J}} + yz\hat{\mathbf{K}}$, taken over the cube bounded by x = 0, x = 1; y = 0, y = 1; z = 0, z = 1.

Sol.: Gauss divergence theorem states that if **F** is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_E \text{div}\mathbf{F}dv,$$

where $\hat{\mathbf{N}}$ is the unit external normal vector at any point of S.



$$\text{R.H.S.} = \int_E \text{div}\,\mathbf{F}\,dv = \int_0^1\int_0^1\int_0^1 (4z - y)dz\,dy\,dx = \int_0^1\int_0^1 \left|2z^2 - yz\right|_0^1 dy\,dx$$

$$= \int_0^1\left[\int_0^1 (2-y)\,dy\right]dx = \int_0^1 \left(2y - \frac{y^2}{2}\right)_0^1 dx = \int_0^1 \frac{3}{2}dx = \frac{3}{2}[x]_0^1 = \frac{3}{2}$$

Again $\int_S \mathbf{F}.\mathbf{N}ds = \int_{S_1}\mathbf{F}.\mathbf{N}ds + \int_{S_2}\mathbf{F}.\mathbf{N}ds + \int_{S_3}\mathbf{F}.\mathbf{N}ds + \int_{S_4}\mathbf{F}.\mathbf{N}ds + \int_{S_5}\mathbf{F}.\mathbf{N}ds + \int_{S_6}\mathbf{F}.\mathbf{N}ds$

where S_1 is the surface of the cube where z = 0, S_2 is the surface where z = 1,

S_3 is the surface where x = 0, S_4 is the surface where x = 1,

S_5 is the surface where y = 0 and S_6 is the surface where y = 1

Then $\int_{S_1} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_{S_1} \mathbf{F}.\left(-\hat{\mathbf{K}}\right)ds = -\int_0^1\int_0^1 yz\,dx\,dy = 0$, $\qquad (\because z = 0)$

$$\int_{S_2} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_{S_2} \mathbf{F}.\left(\hat{\mathbf{K}}\right)ds = \int_0^1\int_0^1 yz\,dx\,dy = \int_0^1\int_0^1 y\,dx\,dy = \int_0^1 y\,dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2} \qquad (\because z = 1)$$

$$\int_{S_3} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_{S_3} \mathbf{F}.\left(-\hat{\mathbf{I}}\right)ds = -\int_0^1\int_0^1 4xz\,dy\,dz = 0 \;, \qquad (\because x = 0)$$

$$\int_{S_4} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_{S_4} \mathbf{F}.\left(\hat{\mathbf{I}}\right)ds = \int_0^1\int_0^1 4xz\,dy\,dz = 4\int_0^1\int_0^1 z\,dy\,dz = 4\int_0^1 z[y]_0^1\,dz = 4\left[\frac{z^2}{2}\right]_0^1 = 2 \quad (\because x = 1)$$

$$\int_{S_5} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_{S_5} \mathbf{F}.\left(-\hat{\mathbf{J}}\right)ds = \int_0^1\int_0^1 y^2\,dz\,dx = 0 \qquad (\because y = 0)$$

$$\int_{S_6} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_{S_6} \mathbf{F}.\left(\hat{\mathbf{J}}\right)ds = -\int_0^1\int_0^1 y^2\,dz\,dx = -\int_0^1\int_0^1 dz\,dx = -1 \;, \qquad (\because y = 1)$$

Adding all these surface integrals, we get

$$\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = 0 + \frac{1}{2} + 0 + 2 - 0 - 1 = \frac{3}{2}.$$

Hence $\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_E \mathrm{div}\mathbf{F}\,dv$.

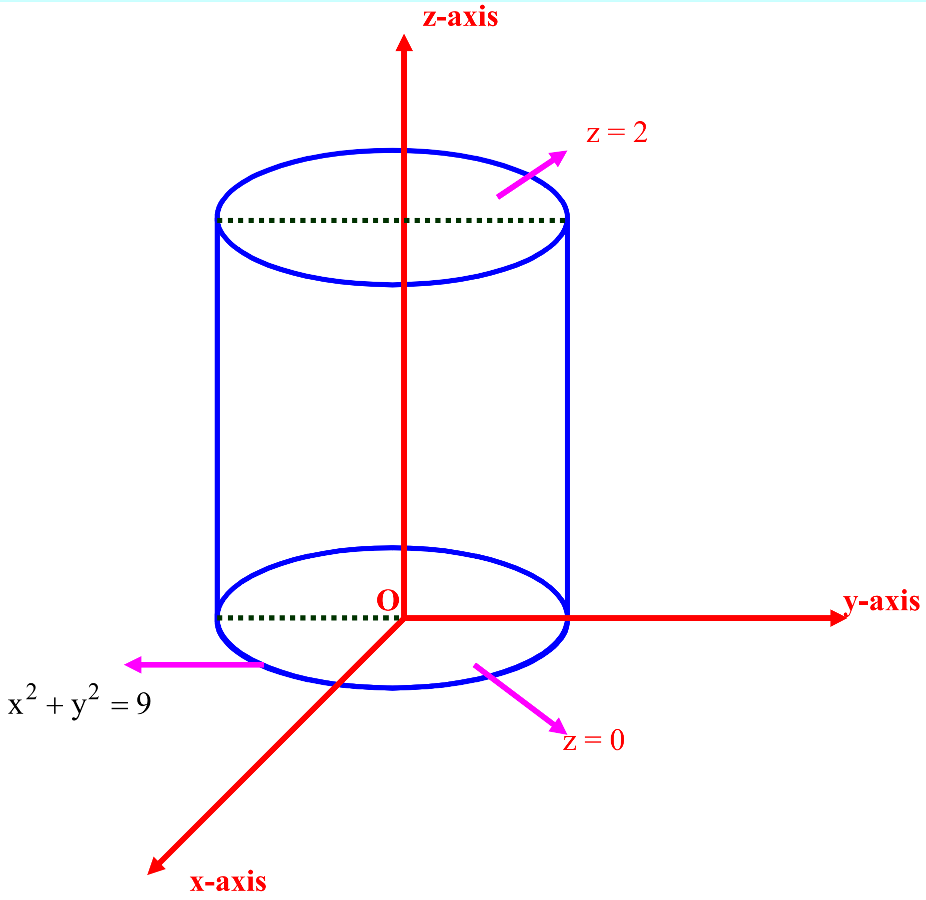
Hence, the Gauss divergence theorem is verified.

Q.No.3: Verify Gauss divergence theorem for the function $\mathbf{F} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z^2\hat{\mathbf{K}}$ over the cylindrical region bounded by $x^2 + y^2 = 9,\ z = 0$ and $z = 2$.

Sol.: Gauss divergence theorem states that if **F** is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_E \mathrm{div}\mathbf{F}\,dv,$$

where $\hat{\mathbf{N}}$ is the unit external normal vector at any point of S.



$$\text{R.H.S.} = \int_E \operatorname{div} \mathbf{F}\, dv = \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^2 (2+2z)\,dz\,dx\,dy = 2\int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \left| z + \frac{z^2}{2} \right|_0^2 dx\,dy$$

$$= 8\int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} dx\,dy = 16\int_{-3}^{3} \sqrt{9-y^2}\,dy = 32\int_0^3 \sqrt{9-y^2}\,dy$$

Put $y = 3\sin\theta$, and so $dy = 3\cos\theta\, d\theta$ and θ varies from 0 to $\frac{\pi}{2}$, we get

$$\therefore\ \text{R.H.S.} = 32\int_0^{\pi/2} 3\cos\theta.3\cos\theta d\theta = 32 \times 9 \int_0^{\pi/2} \cos^2\theta . d\theta = 288 \times \left(\frac{1}{2} \times \frac{\pi}{2} \right) = 72\pi .$$

Again $\int_S \mathbf{F}.\hat{\mathbf{N}}\, ds = \int_{S_1} \mathbf{F}.\hat{\mathbf{N}}\, ds + \int_{S_2} \mathbf{F}.\hat{\mathbf{N}}\, ds + \int_{S_3} \mathbf{F}.\hat{\mathbf{N}}\, ds ,$

where S_1 is the surface of the cylinder where z = 0, S_2 is the surface where z = 2, S_3 is the curved surface of the cylinder.

Then $\int\limits_{S_1} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int\limits_{S_1} \mathbf{F}.\left(-\hat{\mathbf{K}}\right)ds = -\int\limits_{-3}^{3}\int\limits_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\left(z^2\right)dx\,dy = 0\ \left(\because z = 0\right),$

$$\int\limits_{S_2} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int\limits_{S_2} \mathbf{F}.\left(\hat{\mathbf{K}}\right)ds = \int\limits_{-3}^{3}\int\limits_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\left(z^2\right)dx\,dy = 16\int\limits_{0}^{3}\int\limits_{0}^{\sqrt{9-y^2}} dx\,dy\ \left(\because z = 2\right)$$

$$= 16\int\limits_{0}^{3}\sqrt{9-y^2}\,dy.$$

Put $y = 3\sin\theta$, and so $dy = 3\cos\theta\,d\theta$ and θ varies from 0 to $\frac{\pi}{2}$, we get

$$\therefore\ \text{R.H.S.} = 16\int\limits_{0}^{\pi/2} 3\cos\theta.3\cos\theta d\theta = 16\times 9\int\limits_{0}^{\pi/2}\cos^2\theta.d\theta = 144\times\left(\frac{1}{2}\times\frac{\pi}{2}\right) = 36\pi.$$

To find $\hat{\mathbf{N}}$ in S_3, we note that

$$\nabla\left(x^2+y^2\right) = 2x\hat{\mathbf{I}} + 2y\hat{\mathbf{J}},\quad \therefore\ \hat{\mathbf{N}} = \frac{2x\hat{\mathbf{I}} + 2y\hat{\mathbf{J}}}{\sqrt{4x^2+4y^2}} = \frac{x\hat{\mathbf{I}} + y\hat{\mathbf{J}}}{3}\quad \left(\because\ x^2+y^2 = 9\right)$$

Also on S_3, i.e. $x^2+y^2 = 9$, $x = 3\cos\theta$, $y = 3\sin\theta$ and $ds = 3d\theta dz$.

$$\left[ds = \frac{dydz}{\left|\hat{\mathbf{N}}.\hat{\mathbf{I}}\right|} = \frac{dydz}{x/3} = 3.\frac{dydz}{x} = 3.\frac{3\cos\theta d\theta dz}{3\cos\theta} = 3d\theta dz\right].$$

To cover the whole surface S_3, z varies from 0 to 2 and θ varies from 0 to 2π.

$$\int\limits_{S_3} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int\limits_{S_3}\left(x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z^2\hat{\mathbf{K}}\right).\left(\frac{x\hat{\mathbf{I}} + y\hat{\mathbf{J}}}{3}\right)ds = \int\limits_{S_3}\left(\frac{x^2+y^2}{3}\right)3d\theta dz = 9\int\limits_{0}^{2}\int\limits_{0}^{2\pi} d\theta dz$$

$$= 9\int\limits_{0}^{2}\int\limits_{0}^{2\pi} d\theta dz = 9\int\limits_{0}^{2} 2\pi dz = 18\pi.(2) = 36\pi.$$

Adding all the surface integrals, we get $\int\limits_{S} \mathbf{F}.\hat{\mathbf{N}}\,ds = 0 + 36\pi + 36\pi = 72\pi$.

Hence, $\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_E \operatorname{div}\mathbf{F}\,dv$.

Hence, Gauss divergence theorem is verified.

Q.No.4.: Verify the divergence theorem for $\mathbf{A} = 2x^2y\hat{\mathbf{I}} - y^2\hat{\mathbf{J}} + 4xz^2\hat{\mathbf{K}}$ taken over the region in the first octant boundary by the cylinder $y^2 + z^2 = 9$ and the plane $x = 2$.

Sol.: Here $\nabla.\mathbf{A} = 4xy - 2y + 8xz$

RHS $= \iiint_V \nabla.\mathbf{A}dV$

The solid region covered when z varies from 0 to $\sqrt{9-y^2}$, y varies from 0 to 3 and x varies from 0 to 2 (height of cylinder) so

$$\text{RHS} = \int_0^2 \int_0^3 \int_0^{\sqrt{9-y^2}} (4xy - 2y + 8xz)\,dzdydx$$

$$= \int_0^2 \int_0^3 \left[(4xy - 2y)\left(\sqrt{9-y^2}\right) + 4x\left(9-y^2\right)\right]dydx$$

$$= \int_0^2 \left.\frac{(2-4x)}{2}\frac{\left(9-y^2\right)^{3/2}}{3/2}\right|_0^3 dx + \int_0^2 \left. 36yx - 4x\frac{y^3}{3}\right|_0^3 dx$$

$$= \int_0^2 -18(1-2x)\,dx + \int_0^2 (108x - 36)\,dx$$

RHS = 180.

The entire surface S consists of five surfaces S_1, S_2, S_3, S_4, So

$$\text{LHS} = \iint_S \mathbf{A}.\hat{\mathbf{N}}\,dS = \iint_{S_1} + \iint_{S_2} + \ldots\ldots.. + \iint_{S_5} = SI_1 + SI_2 + \ldots\ldots\ldots + S_5$$

On S_1: OAB: $x = 0$, $\hat{\mathbf{N}} = -\hat{\mathbf{I}}$, $\mathbf{A}.\hat{\mathbf{N}} = 0$ so

$$SI_1 = \iint_{S_1} \mathbf{A}.\hat{\mathbf{N}}\,dS = 0.$$

On S_2: CED: $x = 2$, $\hat{\mathbf{N}} = \hat{\mathbf{I}}$, $\mathbf{A}.\hat{\mathbf{N}} = 8y$ so

$$SI_2 = \iint_{S_2} \mathbf{A}.\hat{\mathbf{N}}\,dS = \iint_{S_2} 8ydydz. = \int_0^3 \int_0^{\sqrt{9-z^2}} 8ydydz = \int_0^3 4\left(9-z^2\right)dz = 72.$$

On S_3: OBDE: $y = 0$, $\hat{\mathbf{N}} = -\hat{\mathbf{J}}$, $\mathbf{A}.\hat{\mathbf{N}} = 0$ so $SI_3 = \iint_{S_3} \mathbf{A}.\hat{\mathbf{N}}\,dS = 0$.

On S_4: OACE: $z = 0$, $\hat{\mathbf{N}} = -\hat{\mathbf{K}}$, $\mathbf{A}.\hat{\mathbf{N}} = 0$ so $SI_4 = \iint_{S_4} \mathbf{A}.\hat{\mathbf{N}}\,dS = 0$.

On S_5: the curved surface ABDC of the cylinder: $y^2 + z^2 = 9$

Unit normal $\hat{\mathbf{N}}$ to S_5 :

$$\frac{\nabla\left(y^2+z^2\right)}{\left|\nabla\left(y^2+z^2\right)\right|} = \frac{2y\hat{\mathbf{J}} + 2z\hat{\mathbf{K}}}{\sqrt{4y^2+4z^2}}$$

$$\hat{\mathbf{N}} = \frac{y\hat{\mathbf{J}} + z\hat{\mathbf{K}}}{3}$$

So that $\mathbf{A}.\hat{\mathbf{N}} = \dfrac{-y^3 + 4xz^3}{3}$

$$\hat{\mathbf{N}}.\hat{\mathbf{K}} = \frac{y\hat{\mathbf{J}} + z\hat{\mathbf{K}}}{3}.\hat{\mathbf{K}} = \frac{z}{3} = \frac{\sqrt{9-y^2}}{3}$$

Projecting the surface S_5 on the yx-plane

$$SI_5 = \iint_{S_5} \mathbf{A}.\hat{\mathbf{N}}\,dS = \iint \frac{\left(4xz^3 - y^3\right)}{3}.\frac{dxdy}{n.k} = \iint_R \frac{\left(4xz^3 - y^3\right)}{3\dfrac{\sqrt{9-y^2}}{3}}dxdy$$

$$= \int_{x=0}^{2} \int_{y=0}^{3} \left[4x\left(9-y^2\right) - y^3\left(9-y^2\right)^{-1/2}\right]dydx = \int_0^2 72xdx + 18\int_0^2 dx = 144 - 36 = 108$$

LHS = 0 + 72 + 0 + 0 + 108 = 180.

Hence the divergence theorem is verified.

Use divergence theorem in the following problems:

Q.No.1: Using divergence theorem, prove that

$$\textbf{(i)} \int_S \mathbf{R}\,.\,d\mathbf{S} = 3V\,, \quad \textbf{(ii)} \int_E \nabla r^2\,.\,d\mathbf{S} = 6V\,,$$

where S is any closed surface enclosing a volume V and $r^2 = x^2 + y^2 + z^2$.

Sol.: Applying Gauss Divergence theorem, which states that if $\mathbf{F}$ is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_E \mathrm{div}\mathbf{F}dv\,,$$

where $\hat{\mathbf{N}}$ is the unit external normal vector at any point of S.

(i) Here $\mathbf{F} = \mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}$.

$$\mathrm{div}\mathbf{R} = \nabla.\mathbf{R} = \nabla.\left[x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}\right] = \left[\hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}\right].\left[x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}\right] = 1 + 1 + 1 = 3.$$

Hence, by Gauss Divergence theorem, we have

$$\int_S \mathbf{R}\,.\,\hat{\mathbf{N}}\,ds = \int_V \mathrm{div}\,\mathbf{R}\,dv = \int_V 3\,dv = 3\int_V dv = 3V.$$

(ii) Here $\mathbf{F} = \nabla r^2 = \left(\frac{\partial}{\partial x}\hat{\mathbf{I}} + \frac{\partial}{\partial y}\hat{\mathbf{J}} + \frac{\partial}{\partial z}\hat{\mathbf{K}}\right)\left(x^2 + y^2 + z^2\right) = 2\left(x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}\right)$.

$$\mathrm{div}\nabla r^2 = \nabla.\nabla r^2 = \nabla.2\left[x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}\right] = \left[\hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}\right].2\left[x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}\right]$$
$$= 2(1 + 1 + 1) = 6$$

Hence, by Gauss Divergence theorem, we have

$$\int_S \nabla r^2\,.\,d\mathbf{S} = \int_S \nabla r^2\,.\,\hat{\mathbf{N}}\,ds = \int_V \mathrm{div}\left(\nabla r^2\right)dv = \int_V 6\,dv = 6\int_V dv = 6\mathbf{V}.$$

Hence, proved the result**.**

Q.No.2: If S is any closed surface enclosing a volume V and $\mathbf{F} = ax\hat{\mathbf{I}} + by\hat{\mathbf{J}} + cz\hat{\mathbf{K}}$,

prove by using Gauss Divergence theorem that $\int_S \mathbf{F}\,.\,\hat{\mathbf{N}}\,ds = (a + b + c)V$.

Sol.: Applying Gauss Divergence theorem, which states that if **F** is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_E \operatorname{div}\mathbf{F}\,dv,$$

where $\hat{\mathbf{N}}$ is the unit external normal vector at any point of S.

Here $\mathbf{F} = ax\,\hat{\mathbf{I}} + by\,\hat{\mathbf{J}} + cz\,\hat{\mathbf{K}}$.

$$\therefore \operatorname{div}\mathbf{F} = \nabla.\mathbf{F} = \nabla.\left[ax\,\hat{\mathbf{I}} + by\,\hat{\mathbf{J}} + cz\,\hat{\mathbf{K}}\right] = \left[\hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}\right].\left[ax\,\hat{\mathbf{I}} + by\,\hat{\mathbf{J}} + cz\,\hat{\mathbf{K}}\right]$$

$$= a + b + c.$$

Hence, by Gauss Divergence theorem, we have

$$\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_V \operatorname{div}\mathbf{F}\,dv = \int_V (a+b+c)\,dv = (a+b+c)\int_V dv = (a+b+c)V.$$

Hence proved**.**

Q.No.3: For any closed surface S, prove by using Gauss Divergence theorem that

$$\int_S \left[x(y-z)\hat{\mathbf{I}} + y(z-x)\hat{\mathbf{J}} + z(x-y)\hat{\mathbf{K}}\right].\hat{\mathbf{N}}\,ds = 0.$$

Sol.: Applying Gauss Divergence theorem, which states that if **F** is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_E \operatorname{div}\mathbf{F}\,dv,$$

where $\hat{\mathbf{N}}$ is the unit external normal vector at any point of S.

Here $\mathbf{F} = x(y-z)\hat{\mathbf{I}} + y(z-x)\hat{\mathbf{J}} + z(x-y)\hat{\mathbf{K}}$.

$$\operatorname{div}\mathbf{F} = \nabla.\mathbf{F} = \nabla.\left[x(y-z)\hat{\mathbf{I}} + y(z-x)\hat{\mathbf{J}} + z(x-y)\hat{\mathbf{K}}\right]$$

$$= \left[\hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}\right].\left[x(y-z)\hat{\mathbf{I}} + y(z-x)\hat{\mathbf{J}} + z(x-y)\hat{\mathbf{K}}\right]$$

$$= (y-z) + (z-x) + (x-y) = 0\ .$$

Hence, by Gauss Divergence theorem, we have

$$\int_S \mathbf{F}\,.\,\overset{\wedge}{\mathbf{N}}\,ds = \int_E \text{div}\,\mathbf{F}\,dv = \int_E (0)\,dv = 0.$$

Hence, proved the result.

Q.No.4: Using divergence theorem, evaluate $\iint (xdydz + ydzdx + zdxdy)$ over the surface of the sphere of radius a.

Sol.: Applying Gauss Divergence theorem, which states that if **F** is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F}.\overset{\wedge}{\mathbf{N}}\,ds = \int_E \text{div}\mathbf{F}dv,$$

where $\overset{\wedge}{\mathbf{N}}$ is the unit external normal vector at any point of S.

Here $\mathbf{F} = x\overset{\wedge}{\mathbf{I}} + y\overset{\wedge}{\mathbf{J}} + z\overset{\wedge}{\mathbf{K}}$.

$$\text{div}\mathbf{F} = \nabla.\mathbf{F} = \nabla.\left[x\overset{\wedge}{\mathbf{I}} + y\overset{\wedge}{\mathbf{J}} + z\overset{\wedge}{\mathbf{K}}\right] = \left[\overset{\wedge}{\mathbf{I}}\frac{\partial}{\partial x} + \overset{\wedge}{\mathbf{J}}\frac{\partial}{\partial y} + \overset{\wedge}{\mathbf{K}}\frac{\partial}{\partial z}\right].\left[x\overset{\wedge}{\mathbf{I}} + y\overset{\wedge}{\mathbf{J}} + z\overset{\wedge}{\mathbf{K}}\right] = 1 + 1 + 1 = 3.$$

Hence, by Gauss Divergence theorem, we have

$$\iint (xdydz + ydzdx + zdxdy) = \int_S \mathbf{F}\,.\,\overset{\wedge}{\mathbf{N}}\,ds$$

$$= \int_V \text{div}\,\mathbf{F}\,dv = \int_V 3\,dv = 3\int_V dv = 3 \times \left(\frac{4}{3}\pi a^3\right) = 4\pi a^3.\ \text{Ans.}$$

Here $\int_V dv$ is the volume of the sphere of radius a.

Q.No.5: Using divergence theorem, evaluate $\int_S \mathbf{R}\,.\,\overset{\wedge}{\mathbf{N}}\,ds$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 9$.

Sol.: Applying Gauss Divergence theorem, which states that if **F** is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F}.\overset{\wedge}{\mathbf{N}}\,ds = \int_E \text{div}\mathbf{F}dv,$$

where $\overset{\wedge}{\mathbf{N}}$ is the unit external normal vector at any point of S.

Here $\mathbf{F} = \mathbf{R} = x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}$.

$$\operatorname{div}\mathbf{R} = \nabla.\mathbf{R} = \nabla.\left[x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}\right] = \left[\hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}\right].\left[x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}\right] = 1 + 1 + 1 = 3.$$

Hence, by Gauss Divergence theorem, we have

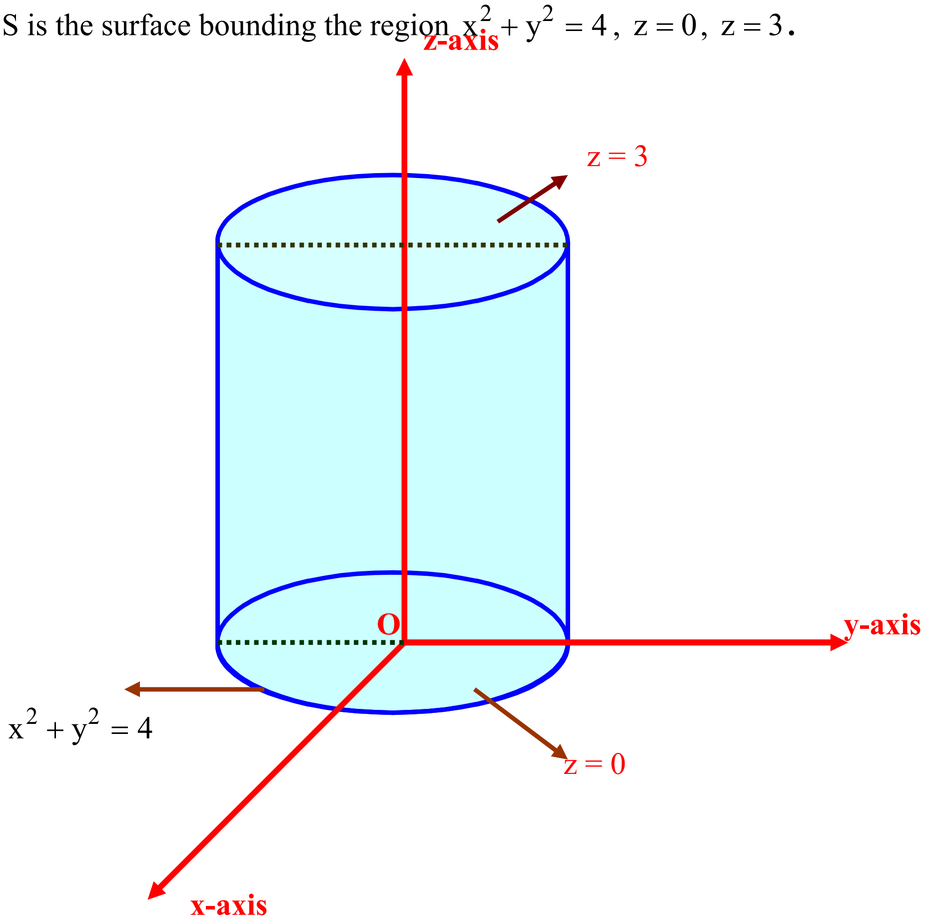
$$\int_S \mathbf{R}.\hat{\mathbf{N}}\,ds = \int_V \operatorname{div}\mathbf{R}\,dv = \int_V 3\,dv = 3\int_V dv = 3 \times \left(\frac{4}{3}\pi 3^3\right) = 108\pi. \text{ Ans.}$$

Here $\int_V dv$ is the volume of the sphere $x^2 + y^2 + z^2 = 9$, whose radius is 3.

Q.No.6.: Using divergence theorem, evaluate $\int_S \mathbf{F}.d\mathbf{S}$ where $\mathbf{F} = 4x\hat{\mathbf{I}} - 2y^2\hat{\mathbf{J}} + z^2\hat{\mathbf{K}}$ and

S is the surface bounding the region $x^2 + y^2 = 4$, $z = 0$, $z = 3$.

Sol.:



By divergence theorem, we get

$$\int_S \mathbf{F}.d\mathbf{S} = \int_V \text{div}\,\mathbf{F}\,dv = \int_V \left[\frac{\partial}{\partial x}(4x)+\frac{\partial}{\partial y}\left(-2y^2\right)+\frac{\partial}{\partial z}\left(-z^2\right)\right]dv$$

$$= \iiint_V (4-4y+2z)dx\,dy\,dz = \int_{-2}^{2}\int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}}\int_0^3 (4-4y+2z)dzdydx$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \left|4z-4yz+z^2\right|_0^3 dydx = \int_{-2}^{2}\int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} (12-12y+9)dydx$$

$$= \int_{-2}^{2}\left[\left|21y-6y^2\right|_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}}\right]dx = 42\int_{-2}^{2}\sqrt{(4-x^2)}dx = 84\int_0^2\sqrt{(4-x^2)}dx$$

$$= 84\left|\frac{x\sqrt{(4-x^2)}}{2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right|_0^2 = 84\left[\left(0+\frac{2\pi}{2}\right)-(0+0)\right] = 84\pi.\ \text{Ans.}$$

Q.No.7: Using divergence theorem, evaluate $\int_S \left(yz\,\hat{\mathbf{I}}+zx\,\hat{\mathbf{J}}+xy\,\hat{\mathbf{K}}\right).d\mathbf{S}$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ in first octant .

Sol.: The surface of the region V**:** OABC is piecewise (see the figure) and is comprised of four surfaces (i) S_1 - circular quadrant OBC in the yz-plane,

(ii) S_2 - circular quadrant OCA in the zx- plane,

(iii) S_3 - circular quadrant OAB in the xy- plane, and

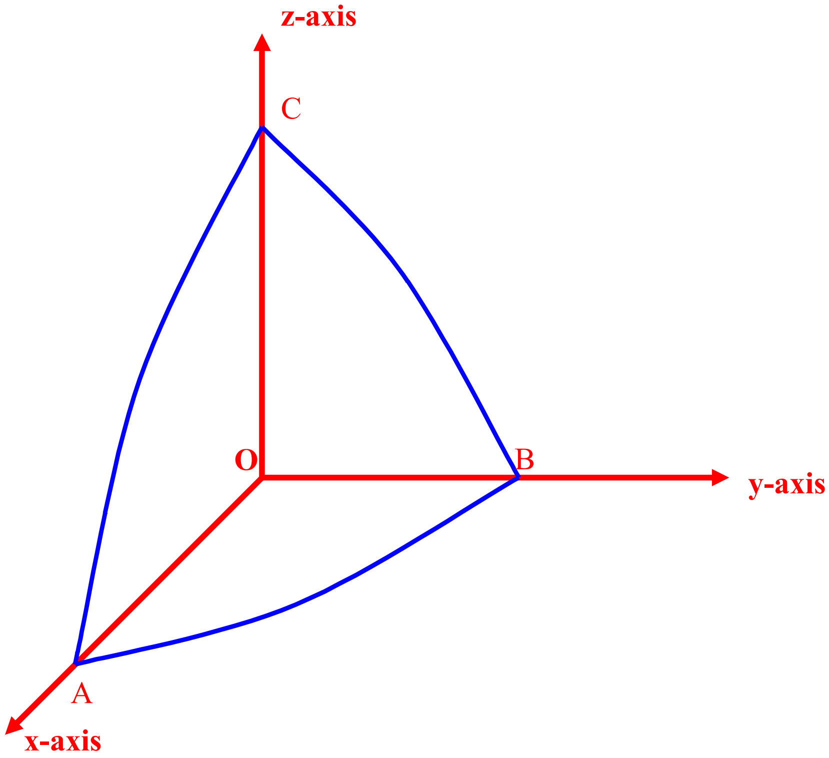
(iv) S - surface ABC of the sphere in the first octant.

Also $\mathbf{F} = yz\,\hat{\mathbf{I}}+zx\,\hat{\mathbf{J}}+xy\,\hat{\mathbf{K}}$.

By Gauss divergence theorem, we get

$$\int_V \text{div}\mathbf{F}dv = \int_{S_1}\mathbf{F}.d\mathbf{S}+\int_{S_2}\mathbf{F}.d\mathbf{S}+\int_{S_3}\mathbf{F}.d\mathbf{S}+\int_S\mathbf{F}.d\mathbf{S}. \qquad \text{(i)}$$

Now $\text{div}\mathbf{F} = \frac{\partial}{\partial x}(yz)+\frac{\partial}{\partial y}(zx)+\frac{\partial}{\partial z}(xy) = 0$.



For the surface S_1, $x = 0$,

$$\therefore \int_{S_1} \mathbf{F}.d\mathbf{S} = \int_0^a \int_0^{\sqrt{(a^2-y^2)}} \left(yz\,\hat{\mathbf{I}}\right).\left[\left(-\hat{\mathbf{I}}\right)dydz\right] = -\int_0^a \int_0^{\sqrt{(a^2-y^2)}} yz\,dy\,dz = -\int_0^a \left[\frac{z^2}{2}\right]_0^{\sqrt{(a^2-y^2)}} ydy$$

$$= -\frac{1}{2}\int_0^a \left(a^2 - y^2\right)ydy = -\frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_0^a = -\frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = -\frac{a^4}{8}.$$

Similarly, $\int_{S_2} \mathbf{F}.d\mathbf{S} = \int_{S_3} \mathbf{F}.d\mathbf{S} = -\frac{a^4}{8}$.

Thus, (i) becomes $0 = \left(-\frac{a^4}{8} - \frac{a^4}{8} - \frac{a^4}{8}\right) + \int_S \mathbf{F}.d\mathbf{S}$ whence $\int_S \mathbf{F}.d\mathbf{S} = \frac{3a^4}{8}$. **Ans.**

Q.No.8.: Apply Divergence theorem to evaluate $\int \left(\ell x^2 + my^2 + nz^2\right)ds$ taken over the sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$; ℓ, m, n being the direction cosine of the external normal to the sphere.

Sol.: The parametric equations of the sphere are $x = a + \rho\sin\theta\cos\phi$, $y = b + \rho\sin\theta\sin\phi$, $z = c + \rho\cos\theta$ and to cover the whole sphere , r varies from 0 to ρ, θ varies from 0 to π and ϕ varies from 0 to 2π.

$$\therefore \int_S \left(\ell x^2 + my^2 + nz^2\right)ds = \int_S \left(x^2 \hat{\mathbf{I}} + y^2 \hat{\mathbf{J}} + z^2 \hat{\mathrm{K}}\right).\hat{\mathbf{N}}\,ds$$

$$= \int_V \operatorname{div}\left(x^2 \hat{\mathbf{I}} + y^2 \hat{\mathbf{J}} + z^2 \hat{\mathbf{K}}\right)dv = 2\int_V (x + y + z)dv$$

$$= 2\int_0^{2\pi}\int_0^{\pi}\int_0^{\rho} [(a + b + c) + \rho(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)] \times r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 2(a + b + c)\frac{\rho^3}{3}\left|-\cos\theta\right|_o^{\pi}.2\pi = \frac{8\pi}{3}(a + b + c)\rho^3.$$ **Ans.**

Q.No.9: Use divergence theorem to evaluate $\int_S \mathbf{F}.d\mathbf{S}$, where $\mathbf{F} = x^3 \hat{\mathbf{I}} + y^3 \hat{\mathbf{J}} + z^3 \hat{\mathbf{K}}$, and

S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Sol.: Applying Gauss Divergence theorem, which states that if **F** is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_E \operatorname{div}\mathbf{F}dv,$$

where $\hat{\mathbf{N}}$ is the unit external normal vector at any point of S.

Here $\mathbf{F} = x^3 \hat{\mathbf{I}} + y^3 \hat{\mathbf{J}} + z^3 \hat{\mathbf{K}}$.

$$\operatorname{div}\mathbf{F} = \nabla.\mathbf{F} = \nabla.\left[x^3 \hat{\mathbf{I}} + y^3 \hat{\mathbf{J}} + z^3 \hat{\mathbf{K}}\right] = \left[\hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}\right].\left[x^3 \hat{\mathbf{I}} + y^3 \hat{\mathbf{J}} + z^3 \hat{\mathbf{K}}\right]$$

$$= 3\left(x^2 + y^2 + z^2\right).$$

Hence, by Gauss Divergence theorem, we have

$$\int_S \mathbf{F}.d\mathbf{S} = \int_V \operatorname{div}\mathbf{F}\,dv = 3\int_V \left(x^2 + y^2 + z^2\right)dv = 3\iiint_V \left(x^2 + y^2 + z^2\right)dxdydz.$$

Now change the rectangular co-ordinates (x, y, z) to spherical polar co-ordinates $(r,\ \theta,\ \phi)$.

Put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $J = \dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

$\Rightarrow dx\,dy\,dz = J\ dr\ d\theta\ d\phi = \left(r^2\sin\theta\right) dr\ d\theta\ d\phi$.

$$\int_S \mathbf{F}\,.\,d\mathbf{S} = 3\iiint_V \left(x^2+y^2+z^2\right)dx\,dy\,dz = 3\times 8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^a r^2\left(r^2\sin\theta\right)dr\,d\theta\,d\phi$$

$$= 24\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(\int_0^a r^4 dr\right)\sin\theta\,d\theta\,d\phi = \frac{24a^5}{5}\int_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}}\sin\theta d\theta\right)d\phi = \frac{24a^5}{5}\int_0^{\frac{\pi}{2}}d\phi = \frac{12}{5}\pi a^5.\ \text{Ans.}$$

Q.No.10: Using divergence theorem, evaluate $\int_S \mathbf{F}.d\mathbf{S}$

where $\mathbf{F} = y^2z^2\,\hat{\mathbf{I}} + z^2x^2\,\hat{\mathbf{J}} + x^2y^2\,\hat{\mathbf{K}}$ and S is the upper part of the sphere $x^2+y^2+z^2 = a^2$ above XOY plane**.**

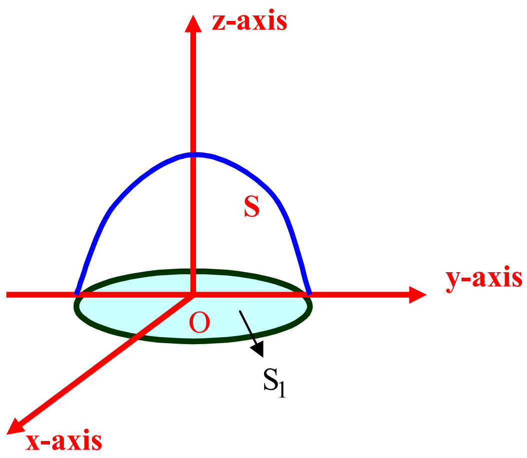
Sol.: Applying Gauss Divergence theorem, which states that if **F** is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F}.\hat{\mathbf{N}}ds = \int_E \text{div}\mathbf{F}dv,$$

where $\hat{\mathbf{N}}$ is the unit external normal vector at any point of S.

Here $\int_{S+S_1} \mathbf{F}\,.\,\hat{\mathbf{N}}ds = \int_E \text{div}\,\mathbf{F}\,dv \Rightarrow \int_S \mathbf{F}\,.\,\hat{\mathbf{N}}ds + \int_{S_1}\mathbf{F}\,.\,\hat{\mathbf{N}}ds = \int_E \text{div}\,\mathbf{F}\,dv$,

where S is the upper part of the sphere and S_1 is the lower part of the sphere i.e. circle.



Here $\mathbf{F} = y^2z^2\hat{\mathbf{I}} + z^2x^2\hat{\mathbf{J}} + x^2y^2\hat{\mathbf{K}}$.

$$\operatorname{div}\mathbf{F} = \nabla.\mathbf{F} = \nabla.\left[y^2z^2\hat{\mathbf{I}} + z^2x^2\hat{\mathbf{J}} + x^2y^2\hat{\mathbf{K}}\right]$$

$$= \left[\hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}\right].\left[y^2z^2\hat{\mathbf{I}} + z^2x^2\hat{\mathbf{J}} + x^2y^2\hat{\mathbf{K}}\right]$$

$$= 0 \ .$$

$$\Rightarrow \int_S \mathbf{F}.\hat{\mathbf{N}}\,ds + \int_{S_1} \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_E \operatorname{div}\mathbf{F}\,dv = 0$$

$$\Rightarrow \int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = -\int_{S_1} \mathbf{F}.\hat{\mathbf{N}}\,ds$$

$$\Rightarrow \int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = -\int_{S_1} \mathbf{F}.\hat{\mathbf{N}}\,ds = -\int_{S_1} \mathbf{F}.\left(-\hat{\mathbf{K}}\right)ds = \int_{S_1} \mathbf{F}.\hat{\mathbf{K}}\,ds = 4\int_0^a \int_0^{\sqrt{a^2-y^2}} x^2y^2\,dxdy$$

$$= 4\int_0^a \frac{\left(\sqrt{a^2-y^2}\right)^3}{3} y^2dy.$$

Put $y = a\sin\theta$, and so $dy = a\cos\theta\, d\theta$ and θ varies from 0 to $\frac{\pi}{2}$, we get

$$= \frac{4}{3}\int_0^{\pi/2} \left(a^3\cos^3\theta\right)\left(a\sin\theta\right)^2 a\cos\theta d\theta$$

$$= \frac{4a^6}{3}\int_0^{\pi/2} \cos^4\theta\sin^2\theta d\theta = \frac{4a^6}{3}\times\left(\frac{3.1.1}{6.4.2}\times\frac{\pi}{2}\right) = \frac{\pi a^6}{24}. \text{ Ans.}$$

Q.No.11: By transforming to triple integral, evaluate $\iint_S \left(x^3dydz + x^2ydzdx + x^2zdxdy\right)$,

where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ and the circular discs $z = 0$ and $z = b$.

Sol.: Applying Gauss Divergence theorem, which states that if **F** is a continuously differentiable vector function in the region E bounded by the closed surface S, then

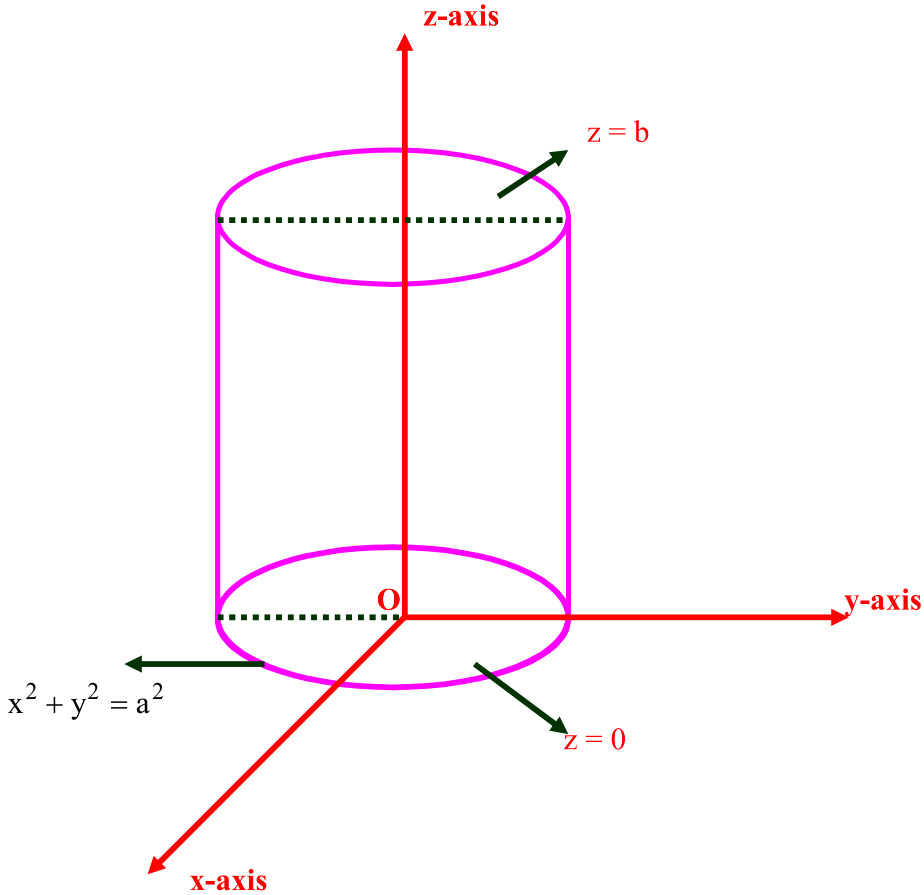
$$\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_E \operatorname{div}\mathbf{F}dv,$$

where $\hat{\mathbf{N}}$ is the unit external normal vector at any point of S.

Here $\mathbf{F} = x^3\hat{\mathbf{I}} + x^2y\hat{\mathbf{J}} + x^2z\hat{\mathbf{K}}$.

$$\operatorname{div}\mathbf{F} = \nabla.\mathbf{F} = \nabla.\left[x^3\hat{\mathbf{I}} + x^2y\hat{\mathbf{J}} + x^2z\hat{\mathbf{K}}\right] = \left[\hat{\mathbf{I}}\frac{\partial}{\partial x} + \hat{\mathbf{J}}\frac{\partial}{\partial y} + \hat{\mathbf{K}}\frac{\partial}{\partial z}\right].\left[x^3\hat{\mathbf{I}} + x^2y\hat{\mathbf{J}} + x^2z\hat{\mathbf{K}}\right]$$

$$= 3x^2 + x^2 + x^2 = 5x^2 .$$



Hence, by Gauss Divergence theorem, we have

$$\iint\left(x^3 dydz + x^2 ydzdx + x^2 zdxdy\right) = \int_S \mathbf{F}\,.\,\mathbf{N}ds = \int_V \operatorname{div}\mathbf{F}\,dv = \int_V 5x^2\,dv$$

$$= 5\int_0^b \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} x^2\,dx\,dy\,dz = 5\times 4\int_0^b \int_0^a \int_0^{\sqrt{a^2-y^2}} x^2\,dx\,dy\,dz$$

$$= 20\int_0^b \int_0^a \left|\frac{x^3}{3}\right|_0^{\sqrt{a^2-y^2}} dy\,dz = \frac{20}{3}\int_0^b \int_0^a \left(a^2 - y^2\right)dy\,dz .$$

Put $y = a\sin\theta$, and so $dy = a\cos\theta\, d\theta$ and θ varies from 0 to $\frac{\pi}{2}$, we get

$$\iint\left(x^3 dydz + x^2 ydzdx + x^2 zdxdy\right) = \frac{20}{3}\int_0^b\int_0^a \left(a^2 - y^2\right)dy\,dz = \frac{20}{3}\int_0^b\left(\int_0^{\frac{\pi}{2}}\left(a^4\cos^4\theta\right)d\theta\right)dz$$

$$= \frac{20\,a^4}{3}\int_0^b\left(\frac{3.1}{4.2}\times\frac{\pi}{2}\right)dz = \frac{5\,\pi a^4}{4}\int_0^b dz = \frac{5}{4}\pi a^4 b \text{ . Ans.}$$

Q.No.12: Using divergence theorem, evaluate $\int_S (\nabla\times\mathbf{F}).d\mathbf{S}$, where S is the surface of the paraboloid $x^2 + y^2 + z = 4$ above the xy-plane, and

$\mathbf{F} = \left(x^2 + y - 4\right)\hat{\mathbf{I}} + 3xy\hat{\mathbf{J}} + \left(2xz + z^2\right)\hat{\mathbf{K}}$.

Sol.: Here S is not a closed surface. The surface $x^2 + y^2 + z = 4$ meets the xy-plane in a circle C given by $x^2 + y^2 = 4,\ \ z = 0$.

Let S_1 be the plane region bounded by the circle C,

S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface.

Then, by Gauss Divergence theorem, which states that if $\mathbf{F}$ is a continuously differentiable vector function in the region E bounded by the closed surface S', then

$$\int_S \mathbf{F}.\hat{\mathbf{N}}ds = \int_E \text{div}\mathbf{F}dv,$$

where $\hat{\mathbf{N}}$ is the unit external normal vector at any point of S.

We have $\int_{S'}(\nabla\times\mathbf{F}).\hat{\mathbf{N}}ds = \int_E \text{div curl}\,\mathbf{F}\,dv = 0,\quad (\because \text{div curl}\,\mathbf{F} = 0)$

$$\Rightarrow \int_{S'}(\nabla\times\mathbf{F}).\hat{\mathbf{N}}ds = \int_S(\nabla\times\mathbf{F}).\hat{\mathbf{N}}ds + \int_{S_1}(\nabla\times\mathbf{F}).\hat{\mathbf{N}}ds = 0$$

$$\Rightarrow \int_S(\nabla\times\mathbf{F}).\hat{\mathbf{N}}ds = -\int_{S_1}(\nabla\times\mathbf{F}).\hat{\mathbf{N}}ds = -\int_{S_1}(\nabla\times\mathbf{F}).\left(-\hat{\mathbf{K}}\right)ds \qquad \left(\because \text{On } S_1, \hat{\mathbf{N}} = -\hat{\mathbf{K}}\right)$$

$$= \int_{S_1} (\nabla \times \mathbf{F}).\hat{\mathbf{K}}\, ds.$$

Now $\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y - 4 & 3xy & 2xz + z^2 \end{vmatrix} = \hat{\mathbf{I}}(0) + \hat{\mathbf{J}}(2z) + \hat{\mathbf{K}}(3y-1)$.

Then $(\nabla \times \mathbf{F}).\hat{\mathbf{K}} = (3y-1)$.

$$\Rightarrow \int_S (\nabla \times \mathbf{F}).\hat{\mathbf{N}}\, ds = \int_{S_1} (\nabla \times \mathbf{F}).\hat{\mathbf{K}}\, ds = \int_{S_1} (3y-1) ds = \iint_{S_1} (3y-1) dx dy.$$

Put $x = r\sin\theta,\ y = r\cos\theta,\quad dxdy = r dr d\theta$

$$= \int_0^{2\pi}\int_0^{2} (3r\cos\theta - 1) r dr d\theta = \int_0^{2\pi}\int_0^{2} \left(3r^2\cos\theta - r\right) dr d\theta$$

$$= \int_0^{2\pi} \left(3\frac{r^3}{3}\cos\theta - \frac{r^2}{2}\right)_0^2 d\theta = \int_0^{2\pi} (8\cos\theta - 2) d\theta$$

$$= (-8\sin\theta - 2\theta)_0^{2\pi} = -4\pi. \text{ Ans.}$$

Q.No.13: If $\mathbf{F} = \text{grad}\,\phi$ and $\nabla^2\phi = -4\pi\rho$, prove that $\int_S \mathbf{F}.\mathbf{N} ds = -4\pi\rho \int_V dV$ where the symbols have their usual meanings**.**

Sol.: Applying Gauss Divergence theorem, which states that if $\mathbf{F}$ is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F}.\hat{\mathbf{N}} ds = \int_E \text{div}\mathbf{F} dv,$$

where $\hat{\mathbf{N}}$ is the unit external normal vector at any point of S.

Here we have given $\mathbf{F} = \text{grad}\,\phi$ and $\nabla^2\phi = -4\pi\rho$, therefore use Divergence theorem to the function $\text{grad}\,\phi$, we have

$$\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_V \operatorname{div}\mathbf{F}\,dV = \int_V \nabla.\mathbf{F}\,dV = \int_V \nabla.\nabla\phi\,dV = \int_V \nabla^2\phi\,dV = \int_V (-4\pi\rho)dV = -4\pi\rho\int_V dV$$

This completes the proof.

Q.No.14.: By Gauss's divergence theorem find the volume V of a region bounded by a surface S.

Sol.: By Gauss's divergence theorem

$$\iiint_V \nabla.\mathbf{A}dV = \iint_S \mathbf{A}.\hat{\mathbf{N}}\,dS \qquad \text{(i)}$$

Choose $\mathbf{A} = x\hat{\mathbf{I}}$, so that $\nabla.\mathbf{A} = 1$, with this (i) reduces to

$$V = \text{Volume} = \iiint 1.dV = \iint_S x\left(\hat{\mathbf{I}}.\hat{\mathbf{N}}\right)dS = \iint_S xdydz$$

Similarly by taking $\mathbf{A} = y\hat{\mathbf{J}}$ and $\mathbf{A} = z\hat{\mathbf{K}}$, we get

$$V = \iint_S ydzdx \quad \text{and} \quad V = \iint_S zdxdy$$

$$\Rightarrow V = \frac{1}{3}\iint_S (xdydz + ydzdx + zdxdy).$$

Q.No.15.: By Gauss's divergence theorem, evaluate $\iint_S e^x dydz - ye^x dzdx + 3zdxdy$

where S is the surface of cylinder $x^2 + y^2 = c^2$, $0 \le z \le h$.

Sol.: Here $A_1 = e^x$, $A_2 = -ye^x$, $A_3 = 3z$, so that

$\nabla.A = \dfrac{\partial A_1}{\partial x} + \dfrac{\partial A_2}{\partial y} + \dfrac{\partial A_3}{\partial z} = e^x - e^x + 3 = 3$ using divergence theorem, given surface integral

$$= \iiint \nabla.\mathbf{A}dV = 3\iiint dxdydz = 3\int_{z=0}^{h}\int_{-c}^{c}\int_{-\sqrt{c^2-x^2}}^{\sqrt{c^2-x^2}} dydxdz = 3.2.2.\int_0^h\int_0^c\int_0^{\sqrt{c^2-x^2}} dydxdz$$

$$= 12h\int_0^c \sqrt{c^2-x^2}dx \quad a = 3\pi hc^2.$$

Q.No.16.: By Gauss's divergence theorem, evaluate $\iint\limits_S \mathbf{A}.\overset{\wedge}{\mathbf{N}}\,dS$ where

$2xy\overset{\wedge}{\mathbf{I}}+yz^2\overset{\wedge}{\mathbf{J}}+xz\overset{\wedge}{\mathbf{K}}$, and Sis the surface of the region boundary by x = 0, y = 0, z = 0, y = 3 and x + 2z = 6.

Sol.: $\nabla.\mathbf{A}=2y+z^2+x$

By Gauss's divergence theorem

$$SI=\iint\limits_S \mathbf{A}.\overset{\wedge}{\mathbf{N}}\,dS=\iiint\limits_V .\mathbf{A}.dV=\iiint\limits_V\left(2y+z^2+x\right)dV=\int\limits_{y=0}^{3}\int\limits_{x=0}^{6}\int\limits_{z=0}^{\frac{6-x}{2}}\left(2y+z^2+x\right)dzdxdy$$

$$=\int\limits_0^3\int\limits_0^6 (2y+x)z+\frac{z^3}{3}\Bigg|_0^{\frac{6-x}{2}}=\int\limits_0^3\int\limits_0^6\left[y(6-x)+\frac{6x-x^2}{2}+\frac{1}{24}(6-x)^3\right]dx$$

$$=\int\limits_0^3 y\left(6x-\frac{x^2}{2}\right)+\frac{1}{2}\left(\frac{6x^2}{2}-\frac{x^3}{3}\right)-\frac{1}{24}\left(\frac{6-x}{4}\right)\Bigg|_0^6=\int\limits_0^3\left[18y+216\left(\frac{1}{12}+\frac{1}{16}\right)\right]dy=\frac{351}{2}$$

Q.No.17.: By Gauss's divergence theorem, evaluate $\iint\limits_S \frac{\mathbf{R}}{r^2}.\overset{\wedge}{\mathbf{N}}\,dS$.

Sol.: Take $\mathbf{A}=\frac{\mathbf{R}}{r^2}=\frac{x\overset{\wedge}{\mathbf{I}}+y\overset{\wedge}{\mathbf{J}}+z\overset{\wedge}{\mathbf{K}}}{x^2+y^2+z^2}$ so that

$$\nabla.\mathbf{A}=\nabla.\left(\frac{\mathbf{R}}{r^2}\right)=\frac{\left(r^2-2x^2\right)}{r^4}+\frac{\left(r^2-2y^2\right)}{r^4}+\frac{\left(r^2-2z^2\right)}{r^4}=\frac{\left(3r^2-2r^2\right)}{r^4}=\frac{1}{r^2}$$

Applying Gauss's divergence theorem, we get

$$\iint \frac{\mathbf{r}}{r^2}.\overset{\wedge}{\mathbf{N}}\,dS=\iint\limits_S \mathbf{A}.\overset{\wedge}{\mathbf{N}}\,dS=\iiint\limits_V \nabla.\mathbf{A}dV=\iiint\limits_V \frac{1}{r^2}dV.$$

Q.No.18.: By Gauss's divergence theorem, evaluate $\iint\limits_S r^5.\overset{\wedge}{\mathbf{N}}\,dS$.

Sol.: Put $f=r^5$ so that $\nabla f=\nabla r^5=5r^3\mathbf{R}$

$$\iint\limits_S r^5\,\overset{\wedge}{\mathbf{N}}\,dS=\iint f\,\overset{\wedge}{\mathbf{N}}\,dS=\iiint\limits_V \nabla f dV=\iiint\limits_V 5r^3\mathbf{R}dV.$$

Q.No.19.: By Gauss's divergence theorem, evaluate $\iint\limits_S \mathbf{B}.\overset{\wedge}{\mathbf{N}}\,dS$ when $\mathbf{B}=\nabla\times\mathbf{A}$ and S is any closed surface.

Sol.: By Gauss's divergence theorem, we get

$$\iint_S \mathbf{B}.\hat{\mathbf{N}}\,dS = \iiint_V \nabla.\mathbf{B}dV = \iiint_V \nabla.(\nabla\times\mathbf{A})dV = 0$$

Since $\nabla.(\nabla\times\mathbf{A}) = 0$ for any $\mathbf{A}$.

Q.No.20.: Prove that $\iiint \nabla f dV = \iint_S f\,\hat{\mathbf{N}}\,dS$.

Sol.: Choose $\mathbf{A} = f\mathbf{C}$ and $\mathbf{C}$ is a constant vector

So that $\nabla.\mathbf{A} = V.(f\mathbf{C}) = \mathbf{C}.\nabla f + f\nabla.\mathbf{C} = \mathbf{C}.\nabla f$

Since $\nabla.\mathbf{C} = 0$

Also $\mathbf{A}.\hat{\mathbf{N}} = (f\mathbf{C}).\hat{\mathbf{N}} = \left(f\,\hat{\mathbf{N}}\right).\mathbf{C} = \mathbf{C}.\left(f\,\hat{\mathbf{N}}\right)$.

Applying Gauss divergence theorem

$$\iiint_V \nabla.\mathbf{A}dV = \iiint_V \mathbf{C}.\nabla f dV = \iint_S \mathbf{A}.\hat{\mathbf{N}}\,dS = \iint_S \mathbf{C}.\left(f\,\hat{\mathbf{N}}\right)dS$$

$$\Rightarrow \mathbf{C}.\iiint_V \nabla f dV = \mathbf{C}.\iint_S f\,\hat{\mathbf{N}}\,dS$$

Since **c** is arbitrary constant vector, the result follows.

Q.No.21.: Prove that $\iint_S \mathbf{R}\times d\mathbf{S} = \mathbf{0}$ for any closed surface S.

Sol.: We know that

$$\iint_S \hat{\mathbf{N}}\times\mathbf{B}dS = \iiint_V \nabla\times\mathbf{B}dV \qquad \text{(i)}$$

Consider $\iint_S \mathbf{R}\times d\mathbf{S} = \iint_S \mathbf{R}\times\hat{\mathbf{N}}\,dS = -\iint \hat{\mathbf{N}}\times\mathbf{R}dS$

Choose $\mathbf{B} = -\mathbf{R}$ in the result (i), Note that

$$\nabla\times\mathbf{B} = \nabla\times(-\mathbf{R}) = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & -y & -z \end{vmatrix}$$

Thus $\iint \mathbf{R}\times d\mathbf{S} = -\iint \hat{\mathbf{N}}\times\mathbf{R}dS = -\iiint\limits_V \nabla\times\mathbf{R}dV = 0$.

Using divergence theorem, let us evaluate flux of some vector fields:

Q.No.1.: Compute the **flux** of the vector field

$$\mathbf{A} = \left(\frac{x^2y}{1+y^2}+6yz^2\right)\hat{\mathbf{I}}+2x\ \text{arc}\ \tan y\ \hat{\mathbf{J}}-\frac{2xz(1+y)+1+y^2}{1+y^2}\hat{\mathbf{K}}$$ through the outer side of that part of the surface of the paraboloid of revolution $z = 1-x^2-y^2$ located above the xy-plane.

Sol.: The flux of **A** through a surface S is given by the surface integral

$$\text{Flux} = \iint\limits_S \mathbf{A}.\hat{\mathbf{N}}dS \qquad \text{(i)}$$

Since the given surface of S_1 is the surface of the paraboloid of revolution $z = 1-x^2-y^2$, which is not closed surface, so we close this surface from below with the circular portion S_2 of the xy-plane that is bounded by the circle $x^2+y^2=1$, $z = 0$.

Let V be the volume of the resulting solid bounded above by S_1 and below by S_2.

Now the flux (i) is calculated, using divergence theorem for the closed region V. Thus

$$\text{Flux} = \iint\limits_S \mathbf{A}.\hat{\mathbf{N}}dS = \iiint\limits_V (\nabla.\mathbf{A})dV = 0$$

Since $\nabla.\mathbf{A} = \dfrac{2xy}{1+y^2}+\dfrac{2x}{1+y^2}\dfrac{-2x(1+y)}{1+y^2} = 0$

Flux across S = $S_1 + S_2$ is additive. So

$$\text{Flux} = \iint\limits_S = \iint\limits_{S_1+S_2} = \iint\limits_{S_1} + \iint\limits_{S_2} = 0$$

Thus $\iint\limits_{S_1} \mathbf{A}.\hat{\mathbf{N}}dS = -\iint\limits_{S_2} \mathbf{A}.\hat{\mathbf{N}}dS$

i.e., flux across the required surface $S_1 = -\text{flux}$ across the circular region S_2.

On S_2 : $z = 0$, $x^2+y^2 \le 1$, $\hat{\mathbf{N}} = -\hat{\mathbf{K}}$ so that

$$\mathbf{A}.\hat{\mathbf{N}} = \left(\frac{x^2 y}{1+y^2}\hat{\mathbf{I}} + 2x \ \text{arc} \ \tan y\,\hat{\mathbf{J}} - \hat{\mathbf{K}} \right).\hat{\mathbf{K}} = 1$$

$$\iint_{S_2} \mathbf{A}.\hat{\mathbf{N}}\,dS = -\iint_{S_2} dS = S_2 = \text{area of the circular region}$$

$$= \pi r^2 = \pi.1^2 = \pi.$$

Thus the require flux of $\mathbf{A}$ across the outer side of that part of the surface S_1 of the paraboloid of revolution $z = 1 - x^2 - y^2$ is π.

Green's Formulas:

Green's first formula (identity)

Q.No.1.: Prove that $\iint_S f\frac{\partial g}{\partial n}dS = \iiint_V \left(f\nabla^2 g + \nabla f.\nabla g\right)dV$.

Sol.: Choose $\mathbf{A} = f\nabla g$ in the divergence theorem then

$$\nabla.\mathbf{A} = \nabla.(f\nabla g) = f\nabla.\nabla g + \nabla f.\nabla f = f\nabla^2 g + \nabla f.\nabla g$$

$$\mathbf{A}.\hat{\mathbf{N}} = \hat{\mathbf{N}}.f\nabla g = f\,\hat{\mathbf{N}}.\nabla g = f\nabla g.\hat{\mathbf{N}} = f\frac{\partial g}{\partial n}$$

From divergence theorem

$$\iint_S \mathbf{A}.\hat{\mathbf{N}}\,dS = \iint_S f\frac{\partial g}{\partial n}dS = \iiint_V \nabla.\mathbf{A}dV = \iiint_V \left(f\nabla^2 g + \nabla f.\nabla g\right)dV$$

Since $\frac{\partial g}{\partial n}dS = \nabla g.\hat{\mathbf{N}}\,dS = \nabla g.d\mathbf{S}$

Green's first identity can be written as

$$\iint_S f\nabla g.d\mathbf{S} = \iiint_V \left(f\nabla^2 g + \nabla f.\nabla g\right)dV$$

Green's second formula (identity) or Symmetric theorem:

Q.No.2.: Show that $\iiint \left(f\nabla^2 g - g\nabla^2 f\right)dV = \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)dS$.

Sol.: From Green's first identity, we have

$$\iint_S f\frac{\partial g}{\partial n}dS = \iiint_V \left(f\nabla^2 g + \nabla f.\nabla g\right)dV \qquad \text{(i)}$$

Interchanging f and g, we obtain

$$\iint_S g\frac{\partial f}{\partial n}dS = \iiint_V \left(g\nabla^2 f + \nabla g.\nabla f\right)dV \qquad \text{(ii)}$$

Subtracting (ii) from (i), we get

$$\iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)dS = \iiint_V \left(f\nabla^2 g - g\nabla^2 f\right)dV$$

Note: $\iint_S \left[f\left(\nabla g.\hat{\mathbf{N}}\right) - g\nabla f.\hat{\mathbf{N}}\right].dS = \iint_S \left(f\nabla g - g\nabla f\right).\hat{\mathbf{N}}\,dS = \iint_S \left(f\nabla g - g\nabla f\right)d\mathbf{S}$

Green's first identity can be written as

$$\iint_S \left(f\nabla g - g\nabla f.\right)d\mathbf{S} = \iiint_V \left(f\nabla^2 g - g\nabla^2 f\right)dV$$

Home Assignment

Q.No.1.: Verify Gauss divergence theorem for $\mathbf{A} = 4x\hat{\mathbf{I}} - 2y^2\hat{\mathbf{J}} + z^2\hat{\mathbf{K}}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$, $z = 3$.

Ans.: Common value: 84π.

Q.No.2.: Verify Gauss divergence theorem for $\mathbf{A} = \left(x^3 - yz\right)\hat{\mathbf{I}} - 2x^2y\hat{\mathbf{J}} + z\hat{\mathbf{K}}$ taken over the entire surface of the cube $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$.

Ans.: Common value: $\frac{a^2}{3} + a^3$.

Q.No.3.: Verify Gauss divergence theorem for $\mathbf{A} = ax\hat{\mathbf{I}} + by\hat{\mathbf{J}} + cz\hat{\mathbf{K}}$, theorem taken over the entire surface of the sphere of radius d and centered at origin.

Ans.: Common value: $\frac{4\pi}{3}d^3\left(a + b + c\right)$

Q.No.4.: Verify Gauss divergence theorem for $\mathbf{A} = 2xy\hat{\mathbf{I}} + yz^2\hat{\mathbf{J}} + xz\hat{\mathbf{K}}$ and S is the total surface of the rectangular parallelopiped bounded by the coordinate planes and $x = 1$, $y = 2$, $z = 3$.

Ans.: Common value: 33.

Q.No.5.: **Verify** Gauss divergence theorem for $\mathbf{A} = 2xz\,\hat{\mathbf{I}} + yz\,\hat{\mathbf{J}} + z^2\,\hat{\mathbf{K}}$ over the upper half of the sphere $x^2 + y^2 + z^2 = a^2$.

Ans.: Common value: $\dfrac{5\pi a^2}{4}$.

Q.No.6.: **Verify** Gauss divergence theorem for $\mathbf{A} = \left(x^2 - yz\right)\hat{\mathbf{I}} + \left(y^2 - zx\right)\hat{\mathbf{J}} + \left(z^2 - xy\right)\hat{\mathbf{K}}$ taken over the rectangular parallelopiped bounded by the coordinate planes and $x = a,\ y = b$ and $z = c$.

Ans.: Common value: $abc(a + b + c)$.

Q.No.7.: **Verify** Gauss divergence theorem for $\mathbf{A} = x^2\,\hat{\mathbf{I}} + y^2\,\hat{\mathbf{J}} + z^2\,\hat{\mathbf{K}}$ taken over the surface of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Ans.: Common value: 0.

Q.No.8.: **Verify** Gauss divergence theorem for $\mathbf{A} = x\,\hat{\mathbf{I}} + y\,\hat{\mathbf{J}}$ taken over the half of the unit sphere $x^2 + y^2 + z^2 = 1$

Ans.: Common value: $\dfrac{4\pi}{3}$.

Q.No.9.: **Verify** Gauss divergence theorem for $\mathbf{A} = x^3\,\hat{\mathbf{I}} + x^2y\,\hat{\mathbf{J}} + x^2z\,\hat{\mathbf{K}}$ taken over the closed region of the cylinder $x^2 + y^2 = a^2$, bounded by the planes z = 0 and $z = b$.

Ans.: Common value: $\dfrac{5\pi b a^4}{4}$.

Q.No.10.: Using divergence theorem, evaluate the surface integral

$$\iint_S yz\,dydz + zx\,dzdx + xy\,dxdy \text{ when } S:\ x^2 + y^2 + z^2 = 4.$$

Ans.: 0.

Q.No.11.: Using divergence theorem, evaluate the surface integral

$$\iint_S x^2 dydz + x^2 y\,dzdx + x^2 dxdy \text{ where } S: \text{closed surface consisting of the}$$

circular cylinder $x^2+y^2=a^2$, $(0\le z\le b)$ and the circular disk $z=0$ and $z=b$, $(x^2+y^2\le a^2)$

Ans.: $\dfrac{5\pi a^4 b}{4}$.

Q.No.12.: Using divergence theorem, evaluate the surface integral

$$\iint_S \sin x dydz+(2-\cos x)ydzdx \ \text{ where S : parallelopiped } 0\le x\le 3,\ 0\le y\le 2,\ 0\le z\le 1.$$

Ans.: 12.

Q.No.13.: Using divergence theorem, evaluate the surface integral

$$\iint_S \left(ax^2+by^2+cz^2\right)dS \ \text{ where S : sphere of the unit radius centered at origin.}$$

Ans.: $\dfrac{4\pi(a+b+c)}{3}$.

Q.No.14.: Using divergence theorem, evaluate the surface integral

$$\iint_S \left(x^2-yz\right)dzdy-2x^2ydzdx+zdxdy \ \text{ where S : cube of the side b and three of whose edges are along the axis.}$$

Ans.: $\dfrac{b^2\left(b^2+3\right)}{3}$.

Q.No.15.: Using divergence theorem, evaluate the surface integral

$$\iint_S 9xdydz+y\cosh^2 xdzdx-z\sinh^2 xdxdy \ \text{ where S : ellipsoid } 4x^2+y^2+9z^2=36.$$

Ans.: 480π.

Q.No.16.: Using divergence theorem, evaluate the surface integral

$$\iint_S \sin xdydz+ydzdx+zdxdy \ \text{ where S : surface of } 0\le x\le\frac{\pi}{2},\ x\le y\le z,\ 0\le z\le 1.$$

Ans.: $\dfrac{3}{2}-\dfrac{\pi^2}{4}$.

Q.No.17.: Using divergence theorem, evaluate the surface integral

$$\iint_S \mathbf{R}.\hat{\mathbf{N}}\,dS \quad \text{where S : sphere of radius 2 with center at origin.}$$

Ans.: 32π.

Q.No.18.: Using divergence theorem, evaluate the surface integral

$$\iint_S \mathbf{R}.\hat{\mathbf{N}}\,dS \text{ where S : surface of cube bounded by planes } x=-1,\ y=-1, z=-1$$

,

$x = 1,\ y = 1, z = 1.$

Ans.: 24.

Q.No.19.: Using divergence theorem, evaluate the surface integral

$$\iint_S \mathbf{F}.\hat{\mathbf{N}}\,dS \text{ where } \mathbf{F} = 2xy\hat{\mathbf{I}} + yz^2\hat{\mathbf{J}} + xz\hat{\mathbf{K}} \text{ and S : surface of parallelopiped}$$

$0 \le x \le 2,\ 0 \le y \le 1,\ 0 \le z \le 3.$

Ans.: 30.

Q.No.20.: If S is any closed surface enclosing a volume V and $\mathbf{A} = ax\hat{\mathbf{I}} + by\hat{\mathbf{J}} + cz\hat{\mathbf{K}}$, then evaluate $\iint_S \mathbf{A}.\hat{\mathbf{N}}\,dS$.

Ans.: $(a+b+c)V$.

Q.No.21.: If $\hat{\mathbf{N}}$ is the unit outward drawn normal to any closed surface of area S, then evaluate $\iiint \nabla.\hat{\mathbf{N}}\,dV$.

Ans.: S.

Q.No.22.: Prove that $\iint_S \hat{\mathbf{N}}\,dS = \mathbf{0}$ for any closed surface S.

Hint: Choose f = 1.

Q.No.23.: Prove that $\iint_S \hat{\mathbf{N}} \times \mathbf{B}dS = \iiint_V \nabla \times \mathbf{B}dV$.

Hint: Take $\mathbf{A} = \mathbf{B} \times \mathbf{C}$ in divergence theorem, with **C** any arbitrary constant vector.

Q.No.24.: Evaluate $\iiint\limits_V \nabla \times \mathbf{B}dV$ where V is the region bounded by a closed surface S and **B** is always normal to S.

Hint: Normal $\hat{\mathbf{N}}$ to S and $\mathbf{B}$ are parallel, so $\hat{\mathbf{N}} \times \mathbf{B} = 0$.

Ans.: 0

Q.No.25.: Prove that $\iint\limits_S \frac{\partial f}{\partial n} dS = \iiint\limits_V \nabla^2 f dV$. Further if f is harmonic (solution of Laplace's equation) in a domain D, then evaluate $\iint \frac{\partial f}{\partial n} dS$.

Hint: Take $\mathbf{A} = \nabla \mathbf{f}$ in divergence theorem and note that $\mathbf{A}.\hat{\mathbf{N}} = \nabla f.\hat{\mathbf{N}} = \frac{\partial f}{\partial n}$.

Ans.: 0

Q.No.26.: If f and g are harmonic in V then evaluate $\iint\limits_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) dS$.

Hint: Use Green's second identity and note that $\nabla^2 f = 0$ and $\nabla^2 g = 0$

Ans.: 0.

Q.No.27.: Let S be a closed surface and let **R** be the position vector of any point (x, y, z) measured from an origin 0. Then prove that

$$\iint\limits_S \frac{\hat{\mathbf{N}}.\mathbf{R}}{r^3} dS = 0 \quad \text{if 0 lies outside S}$$

$$= 4\pi \quad \text{if 0 lies inside S}$$

Hint: If 0 lies outside S, note that $r \neq 0$ and $\nabla.\frac{\mathbf{R}}{r^3} = 0$. Now use divergence theorem if 0 lies inside S, enclose 0 by a small sphere S* of radius a then from the above result

$$\iint\limits_{S+S^*} \frac{\hat{\mathbf{N}}.\mathbf{R}}{r^3} dS = 0$$

Note that $\iint\limits_{S^*} \frac{\hat{\mathbf{N}}.\mathbf{R}}{r^3} dS = -4\pi$

Q.No.28.: Prove that $\iint\limits_{S} \nabla\left(x^2+y^2+z^2\right).\hat{\mathbf{N}}\,dS = 6V$ where S is any closed surface enclosing a volume V.

Q.No.29.: If $\mathbf{A}=\left(x^2+y-4\right)\hat{\mathbf{I}}+3xy\,\hat{\mathbf{J}}+\left(2xz+z^2\right)\hat{\mathbf{K}}$ and S is the surface of paraboloid $x^2+y^2+z=4$ above the xy-plane, evaluate $\iint\limits_{S} (\nabla\times\mathbf{A}).\hat{\mathbf{N}}\,dS$.

Ans.: -4π.

Q.No.30.: Compute the **flux** of the vector $\mathbf{A}=4x\,\hat{\mathbf{I}}-y\,\hat{\mathbf{J}}+z\,\hat{\mathbf{K}}$ through the surface of a torus.

Hint: Volume of torus with R_1 and R_2 as the inner and outer radii of the torus is $\frac{\pi^2}{4}\left(R_2-R_1\right)^2\left(R_2-R_1\right)$.

Ans.: Flux $=\pi^2\left(R_2-R_1\right)^2\left(R_2-R_1\right)$.

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12^{th} Topic

(Last Lecture)

Vector Calculus

Irrotational fields, Solenoidal fields

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IRROTATIONAL FIELDS:

Definition: An irrotational field $\mathbf{F}$ is characterised by any one of the following conditions:

(i) $\nabla \times \mathbf{F} = \mathbf{0}$.

(ii) Circulation $\int \mathbf{F}.d\mathbf{R}$ along every closed curve is zero.

(iii) $\mathbf{F} = \nabla\phi$, if the domain is simply connected.

Proof: (ii) If $\nabla \times \mathbf{F} = \mathbf{0}$, then, by Stoke's theorem,

$$\int_C \mathbf{F}.d\mathbf{R} = \int_S \nabla \times \mathbf{F}.d\mathbf{S} = 0$$

i.e. the circulation along every closed curve is zero.

(iii) Again since $\nabla \times \nabla\phi = \mathbf{0}$.

But in an irrotational field, $\nabla \times \mathbf{F} = \mathbf{0}$.

$\Rightarrow \mathbf{F} = \nabla\phi$.

Thus, the vector $\mathbf{F}$ can always be expressed as the gradient of a scalar function ϕ provided the domain is simply connected.

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Such a scalar function ϕ is called the **potential**.

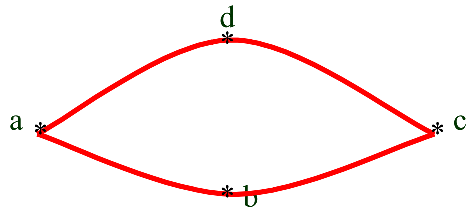
In a rotational field, $\mathbf{F}$ cannot be expressed as the gradient of a scalar potential.

Remarks:

(1).: In an irrotational field, the line integral of $\mathbf{F}$ between two points is independent of the path of the integration and is equal to the potential difference between these points.

Proof: If (a b c d) by any closed contour in an irrotational field $\mathbf{F}$, then

$$\int_{abcd} \mathbf{F}.d\mathbf{R} = \int_{abc} \mathbf{F}.d\mathbf{R} + \int_{cda} \mathbf{F}.d\mathbf{R} = 0$$



$$\Rightarrow \int_{abc} \mathbf{F}.d\mathbf{R} = \int_{adc} \mathbf{F}.d\mathbf{R}.$$

Thus, the value of the line integral is independent of the path joining the end points.

2nd Part: Further, substituting $\mathbf{F} = \nabla\phi$, we get

$$\int_a^c \mathbf{F}.d\mathbf{R} = \int_a^c \nabla\phi.d\mathbf{R} = \int_a^c \left(\hat{\mathbf{I}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{J}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{K}}\frac{\partial\phi}{\partial z}\right).\left(\hat{\mathbf{I}}\,dx + \hat{\mathbf{J}}dy + dz\,\hat{\mathbf{K}}\right)$$

$$= \int_a^c \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = \int_a^c d\phi = \phi_c - \phi_a.$$

(2).: If $\mathbf{F}$ is a vector acting on a particle, then $\oint \mathbf{F}.d\mathbf{R}$ represent the work done in moving the particle around a closed path.

When $\oint \mathbf{F}.d\mathbf{R} = 0$, the field is said to be conservative, i.e. no work is done in displacement from a point a to another point in the field and back to point a and the mechanical energy is conserved.

Thus every irrotational field is conservative.

SOLENOIDAL FIELDS:

Definition: A solenoidal field $\mathbf{F}$ is characterised by anyone of the following conditions.

(i) $\nabla.\mathbf{F} = 0$.

(ii) flux $\int \mathbf{F}.\hat{\mathbf{N}}ds$ across every closed surface is zero.

(iii) $\mathbf{F} = \nabla \times \mathbf{V}$.

Proof: (ii) If $\nabla.\mathbf{F} = 0$, then, the Divergence theorem,

$$\int_S \mathbf{F}.\hat{\mathbf{N}}\,ds = \int_V \nabla.\mathbf{F}\,dv = 0$$

i.e. the flux across every closed surface is zero.

(iii) Again since $\nabla.\nabla\times\mathbf{V} = 0$.

But in a Solenoidal field, $\nabla.\mathbf{F} = 0$.

$\Rightarrow \mathbf{F} = \nabla\times\mathbf{V}$.

Thus, the vector $\mathbf{F}$ can always be expressed as the curl of a vector function $\mathbf{V}$.

Q.No.1.: A vector field is given by $\mathbf{F} = \left(x^2 - y^2 + x\right)\hat{\mathbf{I}} - (2xy + y)\hat{\mathbf{J}}$.

Show that the field is **irrotational** and find its **scalar potential**.

Hence, evaluate the **line integral** from (2, 1) to (1, 2).

Sol.: (i): Since $\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 + x & -(2xy + y) & 0 \end{vmatrix} = \mathbf{0}$.

$\Rightarrow$ This vector field is irrotational.

(ii): Since the vector field is irrotational $\Rightarrow$ $\mathbf{F}$ can be expressed as the gradient of a scalar potential i. e. $\mathbf{F} = \nabla\phi \Rightarrow \left(x^2 - y^2 + x\right)\hat{\mathbf{I}} - (2xy + y)\hat{\mathbf{J}} = \nabla\phi = \dfrac{\partial\phi}{\partial x}\hat{\mathbf{I}} + \dfrac{\partial\phi}{\partial y}\hat{\mathbf{J}}$

$$\Rightarrow \frac{\partial\phi}{\partial x} = x^2 - y^2 + x \qquad \text{(i)}$$

and $$\frac{\partial\phi}{\partial y} = -(2xy + y). \qquad \text{(ii)}$$

Integrating (i) w. r. t. x, keeping y constant, we get

$$\phi = \frac{x^3}{3} - y^2x + \frac{x^2}{2} + f(y). \qquad \text{(iii)}$$

Similarly, integrating (ii) w. r. t. y, keeping x constant, we get

$$\phi = -xy^2 - \frac{y^2}{2} + g(x). \qquad \text{(iv)}$$

Equating (iii) and (iv), we get

$$\frac{x^3}{3} - y^2x + \frac{x^2}{2} + f(y) = -xy^2 - \frac{y^2}{2} + g(x)$$

$$\therefore \; f(y) = -\frac{y^2}{2} \text{ and } g(x) = \frac{x^3}{3} + \frac{x^2}{2}.$$

Hence, $\phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2}$. Ans.

(iii): Since the vector field **F** is irrotational, then

$\int$ **F**.d**R** from (2, 1) to (1, 2)$=\phi_{1,\,2} - \phi_{2,\,1}$

$$= \left(\frac{1}{3} - 1\times 4 + \frac{1}{2} - \frac{4}{2}\right) - \left(\frac{8}{3} - 2\times 1 + \frac{4}{2} - \frac{1}{2}\right) = -7\frac{1}{3}. \text{ Ans.}$$

Q.No.2.: A fluid motion is given by $\mathbf{V} = (y+z)\hat{\mathbf{I}} + (z+x)\hat{\mathbf{J}} + (x+y)\hat{\mathbf{K}}$.

(a) Is the motion **irrotational**? If so, find the **velocity potential**.

(b) Is the motion possible for incompressible fluid?

Sol.: (a): Since $\nabla\times\mathbf{V} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = \hat{\mathbf{I}}(1-1) - \hat{\mathbf{J}}(1-1) + \hat{\mathbf{K}}(1-1) = \mathbf{0}$.

$\Rightarrow$ The motion is irrotational and if ϕ is the velocity potential then $\mathbf{V} = \nabla\phi$.

i. e., $(y+z)\hat{\mathbf{I}} + (z+x)\hat{\mathbf{J}} + (x+y)\hat{\mathbf{K}} = \frac{\partial\phi}{\partial x}\hat{\mathbf{I}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{J}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{K}}$

$$\Rightarrow \frac{\partial\phi}{\partial x} = y+z,\; \frac{\partial\phi}{\partial y} = z+x,\; \frac{\partial\phi}{\partial z} = x+y.$$

Integrating these, we get $\phi = (y+z)x + f_1(y, z)$ (i)

$\phi = (z+x)y + f_2(z, x)$ (ii)

$\phi = (x+y)z + f_3(x, y)$ (iii)

Equality of (i), (ii) and (iii), requires that

$f_1(y, z) = yz,\; f_2(z, x) = zx,\; f_3(x, y) = xy$.

Hence $\phi = yz + zx + xy$. Ans.

(b) The fluid motion is possible, if $\mathbf{V}$ satisfies the equation of continuity, which for an incompressible fluid is $\nabla.\mathbf{V}=0$.

Here $\nabla.\mathbf{V}=\frac{\partial}{\partial x}(y+z)+\frac{\partial}{\partial y}(z+x)+\frac{\partial}{\partial z}(x+y)=0$.

Hence, the fluid motion is possible**.**

Q.No.3.: Show that $r^{\alpha}\mathbf{R}$ is an irrotational vector for any value of α, but is solenoidal only if $\alpha+3=0$, where $\mathbf{R}=x\hat{\mathbf{I}}+y\hat{\mathbf{J}}+z\hat{\mathbf{K}}$ and r is the magnitude of $\mathbf{R}$**.**

Sol.: Given $\mathbf{R}=x\hat{\mathbf{I}}+y\hat{\mathbf{J}}+z\hat{\mathbf{K}}$.

$\therefore r=\sqrt{x^2+y^2+z^2} \Rightarrow r^{\alpha}=\left(x^2+y^2+z^2\right)^{\alpha/2}$

and $r^{\alpha}\mathbf{R}=\left(x^2+y^2+z^2\right)^{\alpha/2}\left(x\hat{\mathbf{I}}+y\hat{\mathbf{J}}+z\hat{\mathbf{K}}\right)=\mathbf{A}$ [vector, say]

1st part: Given $\mathbf{A}$ i.e. $\alpha^2\mathbf{R}$ will be a irrotational vector, then $\nabla\times\mathbf{A}=\mathbf{0}$.

$$\text{Now } \nabla\times\mathbf{A}=\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xr^{\alpha} & yr^{\alpha} & zr^{\alpha}\end{vmatrix}$$

$$=\left(\frac{\partial}{\partial y}zr^{\alpha}-\frac{\partial}{\partial z}yr^{\alpha}\right)\hat{\mathbf{I}}-\left(\frac{\partial}{\partial x}zr^{\alpha}-\frac{\partial}{\partial z}xr^{\alpha}\right)\hat{\mathbf{J}}+\left(\frac{\partial}{\partial x}yr^{\alpha}-\frac{\partial}{\partial y}xr^{\alpha}\right)\hat{\mathbf{K}}. \quad (1)$$

Now, $\frac{\partial}{\partial y}zr^{\alpha}-\frac{\partial}{\partial z}yr^{\alpha}=z\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{\alpha/2}-y\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{\alpha/2}$

$$=z.\frac{\alpha}{2}\left(x^2+y^2+z^2\right)^{\frac{\alpha}{2}-1}.2y-y.\frac{\alpha}{2}\left(x^2+y^2+z^2\right)^{\frac{\alpha}{2}-1}.2z$$

$$=\left(yz\alpha-yz\alpha\right)\left(x^2+y^2+z^2\right)^{\frac{\alpha}{2}-1}=0.$$

Similarly, $\frac{\partial}{\partial x}zr^{\alpha}-\frac{\partial}{\partial z}xr^{\alpha}=\frac{\partial}{\partial x}yr^{2}-\frac{\partial}{\partial y}xr^{2}=0$.

$\therefore \nabla\times\mathbf{A}=0\hat{\mathbf{I}}-0\hat{\mathbf{J}}+0\hat{\mathbf{K}}=\mathbf{0}$**.**

$\Rightarrow r^{\alpha}\mathbf{R}$ is an irrotational vector for any value of α because it is independent of value of α.

2nd Part: Now, $r^{\alpha}\mathbf{R}$ will be a solenoidal vector if $\nabla.\mathbf{A} = 0$, where $\mathbf{A} = r^{\alpha}\mathbf{R}$

i. e. $\left(\frac{\partial}{\partial x}\hat{\mathbf{I}} + \frac{\partial}{\partial y}\hat{\mathbf{J}} + \frac{\partial}{\partial z}\hat{\mathbf{K}}\right)\left[\left(x^2+y^2+z^2\right)^{\alpha/2}\left(x\hat{\mathbf{I}} + y\hat{\mathbf{J}} + z\hat{\mathbf{K}}\right)\right] = 0$

$$\Rightarrow \frac{\partial}{\partial x}\left[\left(x^2+y^2+z^2\right)^{\alpha/2}.x\right] + \frac{\partial}{\partial y}\left[\left(x^2+y^2+z^2\right)^{\alpha/2}.y\right] + \frac{\partial}{\partial z}\left[\left(x^2+y^2+z^2\right)^{\alpha/2}.z\right] = 0$$

$$\Rightarrow \left[x.\frac{\alpha}{2}\left(x^2+y^2+z^2\right)^{\frac{\alpha}{2}-1}(2x) + \left(x^2+y^2+z^2\right)^{\alpha/2}\right]$$

$$+\left[y.\frac{\alpha}{2}\left(x^2+y^2+z^2\right)^{\frac{\alpha}{2}-1}(2y) + \left(x^2+y^2+z^2\right)^{\alpha/2}\right]$$

$$+\left[z.\frac{\alpha}{2}\left(x^2+y^2+z^2\right)^{\frac{\alpha}{2}-1}(2z) + \left(x^2+y^2+z^2\right)^{\alpha/2}\right] = 0$$

$$\Rightarrow \left(x^2+y^2+z^2\right)^{\frac{\alpha}{2}-1}\alpha\left(x^2+y^2+z^2\right) + 3\left(x^2+y^2+z^2\right)^{\alpha/2} = 0$$

$$\Rightarrow \left(x^2+y^2+z^2\right)^{\alpha/2}\alpha + 3\left(x^2+y^2+z^2\right)^{\alpha/2} = 0$$

$\Rightarrow \alpha + 3 = 0$. $\Rightarrow r^{\alpha}\mathbf{R}$ will be a solenoidal vector if $\alpha + 3 = 0$.

This completes the proof.

Q.No.4.: If ϕ is a solution of the Laplace's equation, prove that $\nabla\phi$ is both solenoidal and irrotational.

Sol.: Since ϕ is a solution of Laplace's equation (given).

$\therefore \nabla^2\phi = 0$ (Laplace's equation) $\Rightarrow \nabla.(\nabla\phi) = 0$.

A vector is said to be solenoidal if it satisfies the conditions $\nabla.(\text{vector}) = 0$.

Hence, $\nabla\phi$ is solenoidal. (1)

Now, $\nabla \times (\nabla \phi) = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$

$$= \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right) \hat{\mathbf{I}} - \left(\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z} \right) \hat{\mathbf{J}} + \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right) \hat{\mathbf{K}} = \mathbf{0}.$$

$\Rightarrow \nabla \times (\nabla \phi) = \mathbf{0}$,

which is the condition for a vector to be irrotational.

Hence, $\nabla \phi$ is irrotational. (2)

From (1) and (2), it is proved that $\nabla \phi$ is both solenoidal and irrotational.

Q.No.5.: If $\mathbf{A}$ and $\mathbf{B}$ are irrotational, prove that $\mathbf{A} \times \mathbf{B}$ is solenoidal.

Sol.: Given that $\mathbf{A}$ and $\mathbf{B}$ are irrotational.

$\therefore$ we have, $\nabla \times \mathbf{A} = \mathbf{0}$ and $\nabla \times \mathbf{B} = \mathbf{0}$. (1)

To prove: $\mathbf{A} \times \mathbf{B}$ is solenoidal.

i.e. to prove $\nabla.(\mathbf{A} \times \mathbf{B}) = 0$.

L.H.S. $= \nabla.(\mathbf{A} \times \mathbf{B})$

$= \mathbf{B}.(\nabla \times \mathbf{A}) - \mathbf{A}.(\nabla \times \mathbf{B})$ $\quad [\because \nabla.(\mathbf{F} \times \mathbf{G}) = \mathbf{G}.(\nabla \times \mathbf{F}) - \mathbf{F}.(\nabla \times \mathbf{G})]$

$= \mathbf{B}.(0) - \mathbf{A}(0) = 0 - 0 = 0$ =R.H.S. [by (1)]

Hence, $\mathbf{A} \times \mathbf{B}$ is solenoidal.

Q.No.6.: Show that the vector field defined by $\mathbf{F} = (x^2 + xy^2)\hat{\mathbf{I}} + (y^2 + x^2 y)\hat{\mathbf{J}}$ is conservative and find the scalar potential. Hence evaluate $\int \mathbf{F}.d\mathbf{R}$ from (1, 2) to (0, 1).

Sol.: (a) Given $\mathbf{F} = (x^2 + xy^2)\hat{\mathbf{I}} + (y^2 + x^2 y)\hat{\mathbf{J}}$.

Since we know that every irrotational field is conservative, so we have to show that $\mathbf{F}$ is irrotational field.

i.e. To show: $\nabla \times \mathbf{F} = \mathbf{0}$.

$$\therefore \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2 y & 0 \end{vmatrix}$$

$$= \hat{\mathbf{I}}\left[0 - \frac{\partial}{\partial z}\left(y^2 + x^2 y\right)\right] - \hat{\mathbf{J}}\left[0 - \frac{\partial}{\partial z}\left(x^2 + xy^2\right)\right] + \hat{\mathbf{K}}\left[\frac{\partial}{\partial x}\left(y^2 + x^2 y\right) - \frac{\partial}{\partial y}\left(x^2 + xy^2\right)\right]$$

$$= \hat{\mathbf{I}}(0-0) - \hat{\mathbf{J}}(0-0) + \hat{\mathbf{K}}(2x.y - x.2y) = \hat{\mathbf{I}}(0-0) - \hat{\mathbf{J}}(0-0) + \hat{\mathbf{K}}(0) = \mathbf{0}.$$

$\Rightarrow \mathbf{F}$ is irrotational.

Hence, $\mathbf{F}$ is conservative.

(b) Scalar potential:

Let ϕ be the scalar potential, then $\mathbf{F}$ can be expressed as a gradient of the scalar potential. i.e. $\mathbf{F} = \nabla\phi$

$$\Rightarrow \left(x^2 + xy^2\right)\hat{\mathbf{I}} + \left(y^2 + x^2 y\right)\hat{\mathbf{J}} = \frac{\partial\phi}{\partial x}\hat{\mathbf{I}} + \frac{\partial\phi}{\partial x}\hat{\mathbf{J}}$$

$$\Rightarrow \frac{d\phi}{dx} = x^2 + xy^2 \qquad (1)$$

$$\text{and } \frac{d\phi}{dy} = y^2 + x^2 y. \qquad (2)$$

Integrating (1) w. r. t. x keeping y as a constant, we get

$$\int \frac{d\phi}{dx} = \int \left(x^2 + xy^2\right)dx \Rightarrow \phi = \frac{x^3}{3} + \frac{x^2y^2}{2} + f(y). \qquad (3)$$

Integrating (2) w. r. t. y keeping x as a constant, we get

$$\int d\phi = \int \left(y^2 + x^2 y\right)dy \Rightarrow \phi = \frac{y^3}{3} + \frac{x^2y^2}{2} + f(x). \qquad (4)$$

Now, equating (3) and (4), we get

$$\frac{x^3}{3} + \frac{x^2y^2}{2} + f(y) = \frac{y^3}{3} + \frac{x^2y^2}{2} + f(x)$$

$$\Rightarrow \frac{x^3}{3} + f(y) = \frac{y^3}{3} + f(x).$$

Equality of this result requires that

$$f(x)=\frac{x^3}{3} \text{ and } f(y)=\frac{y^3}{3}. \tag{5}$$

Put (5) in (3), we get

$$\phi=\frac{x^3}{3}+\frac{x^2y^2}{2}+\frac{y^3}{3}. \quad \text{Hence } \phi=\frac{x^3+y^3}{3}+\frac{x^2y^2}{2}. \text{ Ans.}$$

(c) Since **F** is irrotational.

$\therefore \int$ **F**.d**R** from (1, 2) to (0, 1) $=\phi_{(0,1)}-\phi_{(1,2)}$

$$=\left(\frac{0}{3}+\frac{0.1^2}{2}+\frac{1}{3}\right)-\left(\frac{1}{3}+\frac{1^2.2^2}{2}+\frac{2^3}{3}\right)$$

$$=0+0+\frac{1}{3}-\frac{1}{3}-\frac{4}{2}-\frac{8}{3}=\frac{-12-16}{6}=\frac{-28}{6}=\frac{-14}{3}=-4\frac{2}{3}\textbf{.} \text{ Ans.}$$

Q.No.7.: Find the work done by the variable force $\mathbf{F}=2y\,\hat{\mathbf{I}}+xy\,\hat{\mathbf{J}}$ on a particle when it is displaced from the origin to the point $\mathbf{R}=4\,\hat{\mathbf{I}}+2\,\hat{\mathbf{J}}$ along the parabola $y^2=x$.

Sol.: Given $\mathbf{F}=2y\,\hat{\mathbf{I}}+xy\,\hat{\mathbf{J}}$. Now $d\mathbf{R}=dx\,\hat{\mathbf{I}}+dy\,\hat{\mathbf{J}}$

$\therefore$ work done $=\int$ **F**.d**R** from (0, 0) to (4, 2) along $y^2=x$

$$=\int(2ydx+xydy)$$

$$=\int_{y=0}^{y=2}\left(2y.2ydy+y^2.ydy\right) \qquad \left[\because y^2=x \Rightarrow 2ydy=dx\right]$$

$$=\int_0^2\left(4y^2+y^3\right)dy=\left[\frac{4}{3}y^3+\frac{1}{4}y^4\right]_0^2=\left[\frac{4}{3}.8+\frac{1}{4}.16\right]=\frac{44}{3}=14\frac{2}{3}\textbf{.} \text{ Ans.}$$

Q.No.8.: Show that the vector field given by $\mathbf{A}=3x^2y\,\hat{\mathbf{I}}+\left(x^3-2yz^2\right)\hat{\mathbf{J}}+\left(3z^2-2y^2z\right)\hat{\mathbf{K}}$ is irrotational but not solenoidal. Also find $\phi(x,y,z)$ such that $\nabla\phi=\mathbf{A}$.

Sol.: (a) Given $\mathbf{A}=3x^2y\,\hat{\mathbf{I}}+\left(x^3-2yz^2\right)\hat{\mathbf{J}}+\left(3z^2-2y^2z\right)\hat{\mathbf{K}}$.

Now, $\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial y} \\ 3x^2y & \left(x^3 - 2yz^2\right) & \left(3z^2 - 2y^2z\right) \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}\left(3z^2 - 2y^2z\right) - \frac{\partial}{\partial z}\left(x^3 - 2yz^2\right)\right]\hat{\mathbf{I}} - \left[\frac{\partial}{\partial x}\left(3z^2 - 2y^2z\right) - \frac{\partial}{\partial z}\left(3x^2y\right)\right]\hat{\mathbf{J}}$$

$$+ \left[\frac{\partial}{\partial x}\left(x^3 - 2yz^2\right) - \frac{\partial}{\partial y}\left(3x^2y\right)\right]\hat{\mathbf{K}}$$

$$= (-4yz + 4yz)\hat{\mathbf{I}} - (0-0)\hat{\mathbf{J}} + \left(3x^2 - 3x^2\right)\hat{\mathbf{K}}$$

$$= 0\hat{\mathbf{I}} - 0\hat{\mathbf{J}} + 0\hat{\mathbf{K}} = \mathbf{0}.$$

$\Rightarrow$ The vector field $\mathbf{A}$ is irrotational**.**

Again, $\nabla.\mathbf{A} = \dfrac{\partial}{\partial x}\left(3x^2y\right) + \dfrac{\partial}{\partial y}\left(x^3 - 2yz^2\right) + \dfrac{\partial}{\partial z}\left(3x^2 - 2y^2z\right)$

$$= (6xy) + \left(0 - 2z^2\right) + \left(6z - 2y^2\right) \neq 0.$$

$\Rightarrow$ The vector field $\mathbf{A}$ is not solenoidal**.**

(b) We have to find out $\phi(x, y, z)$ such that $\nabla\phi = \mathbf{A}$.

Now, $\nabla\phi = \mathbf{A}$.

$$\Rightarrow \frac{\partial\phi}{\partial x}\hat{\mathbf{I}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{J}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{K}} = 3x^2y\,\hat{\mathbf{I}} + \left(x^3 - 2yz^2\right)\hat{\mathbf{J}} + \left(3z^2 - 2y^2z\right)\hat{\mathbf{K}}.$$

Comparing coefficients of $\hat{\mathbf{I}}, \hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$, we get

$$\frac{\partial\phi}{\partial x} = 3x^2y. \tag{i}$$

$$\frac{\partial\phi}{\partial y} = \left(x^3 - 2yz^2\right). \tag{ii}$$

$$\frac{\partial\phi}{\partial z} = \left(3z^2 - 2y^2z\right). \tag{iii}$$

Integrating both sides of (i) w. r. t. to x keeping y and z constant, we get

$\phi = x^3 y + f(y, z)$. (iv)

Similarly, integrating both sides of (ii) w. r. t. to y keeping z and x constant, we get

$\phi = x^3 y - y^2 z^2 + g(z, x)$. (v)

Similarly, integrating both sides of (iii) w. r. t. to z keeping x and y constant, we get

$\phi = z^3 - y^2 z^2 + h(x, y)$. (vi)

Equality of (iv), (v) and (vi), requires that

$f(y,z) = -y^2 z^2 + z^3,\ g(z,x) = z^3,\ h(x,y) = x^3 y$.

Hence, $\phi = x^3 y - y^2 z^2 + z^3$. Ans.

Q.No.9.: Show that the following vectors are irrotational and find the scalar potential in each case: **(i)** $(x^2 - yz)\hat{\mathbf{I}} + (y^2 - zx)\hat{\mathbf{J}} + (z^2 - xy)\hat{\mathbf{K}}$

(ii) $2xy\,\hat{\mathbf{I}} + (x^2 + 2yz)\hat{\mathbf{J}} + (y^2 + 1)\hat{\mathbf{K}}$

(iii) $(6xy + z^3)\hat{\mathbf{I}} + (3x^2 - z)\hat{\mathbf{J}} + (3xz^2 - y)\hat{\mathbf{K}}$.

Sol.: (i) Given $\mathbf{A} = (x^2 - yz)\hat{\mathbf{I}} + (y^2 - zx)\hat{\mathbf{J}} + (z^2 - xy)\hat{\mathbf{K}}$.

Since $\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - yz) & (y^2 - zx) & (z^2 - xy) \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(z^2 - xy) - \frac{\partial}{\partial z}(y^2 - zx)\right]\hat{\mathbf{I}} - \left[\frac{\partial}{\partial x}(z^2 - xy) - \frac{\partial}{\partial z}(x^2 - yz)\right]\hat{\mathbf{J}}$$

$$- \left[\frac{\partial}{\partial x}(y^2 - zx) - \frac{\partial}{\partial y}(x^2 - yz)\right]\hat{\mathbf{K}}$$

$$= (-x + x)\hat{\mathbf{I}} - (-y + y)\hat{\mathbf{J}} + (-z + z)\hat{\mathbf{K}} = 0.\hat{\mathbf{I}} - 0.\hat{\mathbf{J}} + 0.\hat{\mathbf{K}} = \mathbf{0}.$$

$\Rightarrow \mathbf{A}$ is irrotational vector.

Let $\phi(x,\ y,\ z)$ be the scalar potential.

$\therefore \nabla\phi = \mathbf{A}$.

$$\Rightarrow \frac{\partial \phi}{\partial x}\hat{\mathbf{I}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{J}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{K}} = \left(x^2 - yz\right)\hat{\mathbf{I}} + \left(y^2 - zx\right)\hat{\mathbf{J}} + \left(z^2 - xy\right)\hat{\mathbf{K}}.$$

Comparing the coefficient of $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$, $\hat{\mathbf{K}}$, we get

$$\frac{\partial \phi}{\partial x} = \left(x^2 - yz\right). \qquad \text{(i)}$$

$$\frac{\partial \phi}{\partial y} = \left(y^2 - zx\right). \qquad \text{(ii)}$$

$$\frac{\partial \phi}{\partial z} = \left(z^2 - xy\right). \qquad \text{(iii)}$$

Integrating both sides of (i) w. r. t. x keeping y and z constant, we get

$$\phi = \frac{1}{3}x^3 - xyz + f(y, z). \qquad \text{(iv)}$$

Similarly, integrating both sides of (ii) w. r. t. y keeping z and x constant, we get

$$\phi = \frac{1}{3}y^3 - xyz + g(z, x). \qquad \text{(v)}$$

Similarly, integrating both sides of (iii) w. r. t. z keeping x and y constant, we get

$$\phi = \frac{1}{3}z^3 - xyz + h(x, y). \qquad \text{(vi)}$$

Equality of (iv), (v) and (vi), requires that

$$f(y,z) = \frac{1}{3}\left(y^3 + z^3\right), \; g(z,x) = \frac{1}{3}\left(z^3 + x^3\right), \; h(x,y) = \frac{1}{3}\left(x^3 + y^3\right)$$

Hence, $\phi = \frac{1}{3}\left(x^3 + y^3 + z^3\right) - xyz$. **Ans.**

(ii) Given $\mathbf{A} = 2xy\,\hat{\mathbf{I}} + \left(x^2 + 2yz\right)\hat{\mathbf{J}} + \left(y^2 + 1\right)\hat{\mathbf{K}}$.

Since $\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & \left(x^2 + 2yz\right) & \left(y^2 + 1\right) \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}\left(y^2 + 1\right) - \frac{\partial}{\partial z}\left(x^2 + 2yz\right)\right]\hat{\mathbf{I}} - \left[\frac{\partial}{\partial x}\left(y^2 + 1\right) - \frac{\partial}{\partial z}(2xy)\right]\hat{\mathbf{J}}$$

$$-\left[\frac{\partial}{\partial x}\left(x^2+2yz\right)-\frac{\partial}{\partial y}(2xy)\right]\hat{\mathbf{K}}$$

$$=(2y-2y)\hat{\mathbf{I}}-(0-0)\hat{\mathbf{J}}+(2x-2x)\hat{\mathbf{K}}=0.\hat{\mathbf{I}}-0.\hat{\mathbf{J}}+0.\hat{\mathbf{K}}=\mathbf{0}.$$

$\Rightarrow \mathbf{A}$ is irrotational vector .

Let $\phi(x,\ y,\ z)$ be the scalar potential.

$\therefore \nabla.\phi = \mathbf{A}$.

$$\Rightarrow \frac{\partial\phi}{\partial x}\hat{\mathbf{I}}+\frac{\partial\phi}{\partial y}\hat{\mathbf{J}}+\frac{\partial\phi}{\partial z}\hat{\mathbf{K}}=2xy\,\hat{\mathbf{I}}+\left(x^2+2yz\right)\hat{\mathbf{J}}+\left(y^2+1\right)\hat{\mathbf{K}}.$$

Comparing the coefficient of $\hat{\mathbf{I}},\ \hat{\mathbf{J}},\ \hat{\mathbf{K}}$, we get

$$\frac{\partial\phi}{\partial x}=2xy. \qquad \text{(i)}$$

$$\frac{\partial\phi}{\partial y}=\left(x^2+2yz\right). \qquad \text{(ii)}$$

$$\frac{\partial\phi}{\partial z}=\left(y^2+1\right). \qquad \text{(iii)}$$

Integrating both sides of (i) w. r. t. x keeping y and z constant, we get

$$\phi = x^2y+f(y,z). \qquad \text{(iv)}$$

Similarly, integrating both sides of (ii) w. r. t. y keeping z and x constant, we get

$$\phi = x^2y+y^2x+g(z,x). \qquad \text{(v)}$$

Similarly, integrating both sides of (iii) w. r. t. z keeping x and y constant, we get

$$\phi = y^2z+z+h(x,y). \qquad \text{(vi)}$$

Equality of (iv), (v) and (vi), requires that

$f(y,z)=y^2z+z,\ g(z,x)=z,\ h(x,y)=x^2y$.

Hence, $\phi = x^2y+y^2z+z$. Ans**.**

(iii) Let $\mathbf{A}=\left(6xy+z^3\right)\hat{\mathbf{I}}+\left(3x^2-z\right)\hat{\mathbf{J}}+\left(3xz^2-y\right)\hat{\mathbf{K}}$**.**

Since $\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (6xy + z^3) & (3x^2 - z) & (3xz^2 - y) \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z)\right]\hat{\mathbf{I}} - \left[\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(6xy + z^3)\right]\hat{\mathbf{J}}$$

$$- \left[\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3)\right]\hat{\mathbf{K}}$$

$$= (-1 + 1)\hat{\mathbf{I}} - (3z^2 - 3z^2)\hat{\mathbf{J}} + (6x - 6x)\hat{\mathbf{K}} = 0.\hat{\mathbf{I}} - 0.\hat{\mathbf{J}} + 0.\hat{\mathbf{K}} = \mathbf{0}.$$

$\Rightarrow \mathbf{A}$ is irrotational vector **.**

Let $\phi(x,\ y,\ z)$ be the scalar potential.

$\therefore \nabla\phi = \mathbf{A}$.

$$\Rightarrow \frac{\partial \phi}{\partial x}\hat{\mathbf{I}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{J}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{K}} = (6xy + z^3)\hat{\mathbf{I}} + (3x^2 - z)\hat{\mathbf{J}} + (3xz^2 - y)\hat{\mathbf{K}}.$$

Comparing the coefficient of $\hat{\mathbf{I}},\ \hat{\mathbf{J}},\ \hat{\mathbf{K}}$, we get

$$\frac{\partial \phi}{\partial x} = (6xy + z^3). \tag{i}$$

$$\frac{\partial \phi}{\partial y} = (3x^2 - z). \tag{ii}$$

$$\frac{\partial \phi}{\partial z} = (3xz^2 - y). \tag{iii}$$

Integrating both sides of (i) w. r. t. x keeping y and z constant, we get

$$\phi = 3x^2y + xz^3 + f(y, z). \tag{iv}$$

Similarly, integrating both sides of (ii) w. r. t. y keeping z and x constant, we get

$$\phi = 3x^2y - yz + g(z, x). \tag{v}$$

Similarly, integrating both sides of (iii) w. r. t. z keeping x and y constant, we get

$$\phi = xz^3 - yz + h(x, y). \tag{vi}$$

Equality of (iv), (v) and (vi), requires that

$f(y,z) = -yz,\ g(z,x) = xz^2,\ h(x,y) = 3x^2y$.

Hence, $\phi = 3x^2y + xz^3 - yz$. Ans.

Q.No.10.: Fluid motion is given by $\mathbf{V} = ax\hat{\mathbf{I}} + ay\hat{\mathbf{J}} - 2az\hat{\mathbf{K}}$

(i) Is it possible to find out the velocity potential? If so, find it.

(ii) Is the motion possible for an incompressible fluid?

Sol.:(i) Given, $\mathbf{V} = ax\hat{\mathbf{I}} + ay\hat{\mathbf{J}} - 2az\hat{\mathbf{K}}$.

$$\nabla \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax & ay & -2az \end{vmatrix}$$

$$= \hat{\mathbf{I}}\left[\frac{\partial}{\partial y}(-2az) - \frac{\partial}{\partial z}(ay)\right] - \hat{\mathbf{J}}\left[\frac{\partial}{\partial x}(-2az) - \frac{\partial}{\partial z}(ax)\right] + \hat{\mathbf{K}}\left[\frac{\partial}{\partial y}(ay) - \frac{\partial}{\partial y}(ax)\right]$$

$$= \hat{\mathbf{I}}.0 - \hat{\mathbf{J}}.0 + \hat{\mathbf{K}}.0 = \mathbf{0}.$$

Since, $\nabla \times \mathbf{v} = \mathbf{0}$.

Hence, $\mathbf{V}$ can be expressed as the velocity potential with gradient.

Now, $\mathbf{V} = \nabla\phi$.

$$\Rightarrow ax\hat{\mathbf{I}} + ay\hat{\mathbf{J}} - 2az\hat{\mathbf{K}} = \frac{\partial \phi}{\partial x}\hat{\mathbf{I}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{J}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{K}}.$$

Comparing the coefficient of $\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}$, we get

$$\frac{\partial \phi}{\partial x} = ax,\ \frac{\partial \phi}{\partial y} = ay,\ \frac{\partial \phi}{\partial z} = -2az.$$

Integrating the above equation, we get

$$\phi = \frac{ax^2}{2} + f(y,z). \quad \text{(i)}$$

$$\phi = \frac{ay^2}{2} + g(x,z). \quad \text{(ii)}$$

$$\phi = -az^2 + h(x,y). \quad \text{(iii)}$$

Equality of (i), (ii) and (iii), requires that

$$f(y,z)=\frac{ay^2}{2}-az^2,\ g(z,x)=\frac{ax^2}{2}-az^2,\ h(x,y)=\frac{ax^2}{2}+\frac{ay^2}{2}$$

Hence $\phi=\frac{ax^2}{2}+\frac{ay^2}{2}-az^2 \Rightarrow \phi=\frac{a}{2}\left(x^2+y^2-2z^2\right)$. Ans.

(ii) For incompressible liquid $\nabla.\mathbf{V}=\left(\frac{\partial\phi}{\partial x}\hat{\mathbf{I}}+\frac{\partial\phi}{\partial y}\hat{\mathbf{J}}+\frac{\partial\phi}{\partial z}\hat{\mathbf{K}}\right).\left(ax\,\hat{\mathbf{I}}+ay\,\hat{\mathbf{J}}-2a\,\hat{\mathbf{K}}\right)$

$$=a+a-2a=0.$$

Hence, the given fluid is incompressible.

Q.No.11.: Find the constant a, b, c, so that

$$\mathbf{F}=(x+2y+az)\hat{\mathbf{I}}+(bx-3y-z)\hat{\mathbf{J}}+(4x+cy+2z)\hat{\mathbf{K}}\text{, is irrotational.}$$

Sol.: For irrotational field, we have $\nabla\times\mathbf{F}=\mathbf{0}$

$$\Rightarrow\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z)\end{vmatrix}=\mathbf{0}$$

$$\Rightarrow\left[\frac{\partial}{\partial y}(4x+cy+2z)-\frac{\partial}{\partial z}(bx-3y-z)\right]\hat{\mathbf{I}}-\left[\frac{\partial}{\partial x}(4x+cy+2z)-\frac{\partial}{\partial z}(x+2y+az)\right]\hat{\mathbf{J}}$$

$$-\left[\frac{\partial}{\partial x}(bx-3y-z)-\frac{\partial}{\partial y}(x+2y+az)\right]\hat{\mathbf{K}}=\mathbf{0}$$

$$\Rightarrow(c+1)\hat{\mathbf{I}}+(a-4)\hat{\mathbf{J}}+(b-2)\hat{\mathbf{K}}=\mathbf{0}.$$

Comparing the both sides of the equation, we get

$a=4$, $b=2$, $c=-1$. Ans.

Q.No.12.: Find the constant 'a' so that **V** is a conservative vector field, where

$$\mathbf{V}=\left(axy-z^3\right)\hat{\mathbf{I}}+(a-2)x^2\hat{\mathbf{J}}+(1-a)xz^2\hat{\mathbf{K}}.$$

Calculate its scalar potential from $(1,-4,2)$ to $(1,2,-3)$ in the field.

Sol.: The field **V** is conservative if $\nabla\times\mathbf{V}=\mathbf{0}$

$$\Rightarrow \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = \mathbf{0}$$

$$\Rightarrow \hat{\mathbf{I}}\left[\left(\frac{\partial}{\partial y}(1-a)xz^2 - \frac{\partial}{\partial z}(a-2)x^2\right)\right] - \hat{\mathbf{J}}\left[\left(\frac{\partial}{\partial x}(1-a)xz^2 - \frac{\partial}{\partial z}\left(axy - z^3\right)\right)\right]$$

$$+\hat{\mathbf{K}}\left[\frac{\partial}{\partial x}(a-2)x^2 - \frac{\partial}{\partial y}\left(axy - z^2\right)\right] = \mathbf{0}$$

$$\Rightarrow 0.\hat{\mathbf{I}} - \hat{\mathbf{J}}\left[(1-a)z^2 + 3z^2\right] + \hat{\mathbf{K}}\left[2x(a-2) - ax\right] = \mathbf{0}.$$

Comparing both sides of the equation, we get

$4z^2 = az^2$ and $ax = 4x$

$\Rightarrow a = 4$. Ans.

If $\mathbf{V} = \nabla\phi$, then, we have

$$\left(axy - z^3\right)\hat{\mathbf{I}} + (a-2)x^2\,\hat{\mathbf{J}} + (1-a)xz^2\,\hat{\mathbf{K}} = \frac{\partial\phi}{\partial x}\hat{\mathbf{I}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{J}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{K}}$$

Putting a = 4, we get

$$\left(4xy - z^3\right)\hat{\mathbf{I}} + 2x^2\,\hat{\mathbf{J}} - 3xz^2\,\hat{\mathbf{K}} = \frac{\partial\phi}{\partial x}\hat{\mathbf{I}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{J}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{K}}$$

Comparing both sides of the equation, we get

$$\frac{\partial\phi}{\partial x} = 4xy - z^3, \quad \frac{\partial\phi}{\partial y} = 2x^2, \quad \frac{\partial\phi}{\partial z} = -3xz^2$$

Integrating these equations, we get

$$\phi = 2x^2y - xz^3 + f(y,\ z). \qquad \text{(i)}$$

$$\phi = 2x^2y + g(x,\ z). \qquad \text{(ii)}$$

$$\phi = -xz^3 + h(x,\ y)\ . \qquad \text{(iii)}$$

Equality of (i), (ii) and (iii), requires that

$f(y,z) = 0,\ g(z,x) = -xz^3,\ h(x,y) = 2x^2y$.

Hence, $\phi = 2x^2y - xz^3$. Ans.

Work done $= \int_{(1,\,-4,\,2)}^{(1,\,2,\,-3)} \mathbf{F}.d\mathbf{R} = \phi_{(1,\,2,\,-3)} - \phi_{(1,\,-4,\,2)} = \left[2(1)^2(2)-1(-3)^3\right]-\left[2(1)^2(-4)-1(2)^3\right]$

$= 4+27-(-8-8) = 31+16$.

Work done = 47. **Ans.**

Q.No.13.: Prove that $\mathbf{F} = \left(y^2\cos x + z^2\right)\hat{\mathbf{I}} + \left(2y\sin x - 4\right)\hat{\mathbf{J}} + \left(3xz^2+2\right)\hat{\mathbf{K}}$ is

(a). conservative field.

(b). Find scalar potential of **F**.

(c). Find work done in moving an object in this field from $P_1(0,\ 1,\ -1)$ to

$P_2 = \left(\frac{\pi}{2},\ -1,\ 1\right)$.

Sol.: (a). F is conservative of $\nabla\times\mathbf{F} = 0$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^2 & 2y\sin x - 4 & 3xz^2+2 \end{vmatrix}$$

$= \hat{\mathbf{I}}(0-0) - \hat{\mathbf{J}}\left(3x^2-3z^2\right) + \hat{\mathbf{K}}\left(2y\cos x - 2y\cos x\right) = 0$.

Hence **F** is conservative.

(b). Let f be the scalar potential such that $\mathbf{F} = \nabla f$

then comparing the components of $\hat{\mathbf{I}},\ \hat{\mathbf{J}},\ \hat{\mathbf{K}}$, we get

$$\frac{\partial f}{\partial x} = y^2\cos x + z^3 \qquad \text{(i)}$$

$$\frac{\partial f}{\partial y} = 2y\sin x - 4 \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial z} = 3xz^2 + 2 \qquad \text{(iii)}$$

Integrating (i) partially w.r.t. x, we get

$$f = y^2\sin x + xz^3 + g(y,z) \qquad \text{(iv)}$$

Differentiating (iv) partially w.r.t. y and using (ii), we get

$$2y\sin x + 0 + \frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} = 2y\sin x - 4$$

Integrating w.r.t. y, we get

$$g(y,z) = -4y + c_1(z) \qquad \text{(v)}$$

Substituting (v) in (iv), we get

$$f = y^2 \sin x + xz^3 - 4y + c_1(z) \qquad \text{(vi)}$$

Differentiating (vi) partially w.r.t. z and using (iii), we get

$$0 + 3xz^2 - 0 + \frac{dc_1}{dz} = \frac{\partial f}{\partial z} = xz^2 + 2$$

Integrating w.r.t. z, we get

$$c_1(z) = z^2 + c \qquad \text{(vii)}$$

Substituting (vii) in (vi), we get

$f(x, y, z) = y^2 \sin x + xz^3 - 4y + z^2 + c$.

(c). Work done

$$= f(P_2) - f(P_1) = f\left(\frac{\pi}{2}, -1,\ 2\right) - f\left(0,\ 1,\ -1\right) = 12 + 4\pi.$$

Q.No.14.: Show that $\left(z - e^{-x}\sin y\right)dx + \left(1 + e^{-x}\cos y\right)dy + \left(x - 8z\right)dz$ is an exact differential of a function f and find f.

Sol.: Let $\mathbf{A} = A_1\hat{\mathbf{I}} + A_2\hat{\mathbf{J}} + A_3\hat{\mathbf{K}} = \left(z - e^{-x}\sin y\right)\hat{\mathbf{I}} + \left(1 + e^{-x}\cos y\right)\hat{\mathbf{J}} + \left(x - 8z\right)\hat{\mathbf{K}}$

Then

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z - e^{-x}\sin y & 1 + e^{-x}\cos y & x - 8z \end{vmatrix}$$

$$= \hat{\mathbf{I}}(0-0) - \hat{\mathbf{J}}(1-1) + \hat{\mathbf{K}}\left(-e^{-x}\cos y - \left(-e^{-x}\cos y\right)\right) = 0$$

Since $\nabla \times \mathbf{A} = 0$

$A_1 dx + A_2 dy + A_3 dz$ will be an exact differential

$$\left(z - e^{-x}\sin y\right)dx + \left(1 + e^{-x}\cos y\right)dy + \left(x - 8z\right)dz = df$$

Regrouping

$$\left(zdx + xdz\right) - 8zdz + dy + \left(e^{-x}\cos ydy - e^{-x}\sin ydx\right) = df$$

$$\because df = d\left(xz\right) - d\left(4z^2\right) + dy + d\left(e^{-x}\sin y\right)$$

$$\therefore f = xz - 4z^2 + y + e^{-x}\sin y.$$

Home Assignments

Q.No.1.: Determine whether the force field $\mathbf{F} = 2xz\hat{\mathbf{I}} + \left(x^2 - y\right)\hat{\mathbf{J}} + \left(2z - x^2\right)\hat{\mathbf{K}}$ is conservative or not.

Ans.: $\nabla \times \mathbf{F} \neq 0$ so non-conservative.

Q.No.2.: (a). Prove that $\mathbf{F} = \left(4xy - 3x^2z^2\right)\hat{\mathbf{I}} + 2x^2\hat{\mathbf{J}} - 2x^3z\hat{\mathbf{K}}$ is a conservative field.

(b). Find its scalar potential f.

(c). Also find the work done in moving an object in this field from (1, 1, 1) to (0, 0, 0).

Ans.: (a). $\nabla \times \mathbf{F} = 0$, so conservative.

(b). Scalar Potential $f = 2x^2y - x^3z^2 + c$.

(c). Work done = $f(1,\ 1,\ 1) - f(0,\ 0,\ 0)$

Q.No.3.: If $\mathbf{A} = \left(2xy + z^2\right)\hat{\mathbf{I}} + x^2\hat{\mathbf{J}} - 3xz^2\hat{\mathbf{K}}$:

(a). Prove that the line integral $\int_c \mathbf{A}.d\mathbf{R}$ is independent of the curve c joining two given points $P_1\left(1, -2,\ 1\right)$ and $P_2(3, 1, 4)$.

(b). Show that there exists a scalar function f such that $\mathbf{A} = \nabla f$ and find f.

(c). Also find the work done in moving a an object from P_1 to P_2.

Ans.: (a). $\nabla \times \mathbf{A} = 0$, **A** is conservative, so line integral is independent of path.

(b). $x^2y + xz^3 +$ constant

(c). Work done : 202.

Q.No.4.: Find b such that the force field $\mathbf{A} = \left(e^xz - bx\right)\hat{\mathbf{I}} + \left(1 - bx^2\right)\hat{\mathbf{J}} + \left(e^x + bz\right)\hat{\mathbf{K}}$ is conservative. Find the scalar potential f of **A** when **A** is conservative.

Ans.: $b = 0,\ f = y + ze^{x} + c$.

Q.No.5.: Find the scalar potential f of $\mathbf{F} = (z + \sin y)\hat{\mathbf{I}} + (-z + x\cos y)\hat{\mathbf{J}} + (x - y)\hat{\mathbf{K}}$.

Ans.: $f = xz + x\sin y - yz + c$.

Q.No.6.: If $\int_{P_1}^{P_2} \mathbf{A}.d\mathbf{R}$ is independent of the path joining any two given points P_1 and P_2 in a given region then $\oint_c \mathbf{A}.d\mathbf{R} = 0$ for all closed paths in the region passing through P_1 and P_2.

Hint: $P_1BP_2DP_1$ be any closed curve c

$$\oint_c = \int_{P_1BP_2DP_1} = \int_{P_1BP_2} + \int_{P_2DP_1} = \int_{P_1BP_2} - \int_{P_1DP_2}.$$

Best of Luck

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