11/1/19 EES 303: Concentration Inequalities

Today:
- Review
- Hoeffding's Inequality
  - Lemma
  - Proof
- Motivate McDiarmid's inequality

Review:
- $S_n = \sum_{i=1}^{n} X_i$ where $X_i$ are iid RVs with $E[X_i] = \mu$, $Var(X_i) = \sigma^2 < \infty$
- Chebyshev inequality: $P\left( \left| \frac{S_n}{n} - \mu \right| \geq \varepsilon \right) \leq \frac{\sigma^2}{n\varepsilon^2}$
- Chernoff bound: $P\left( \frac{S_n}{n} - \mu \geq \varepsilon \right) \leq e^{-\lambda\varepsilon} \cdot \prod_{i=1}^{n} E\left[e^{\lambda \cdot \frac{(X_i - E[X_i])}{n}}\right]$

Motivation: Want to bound the MGF of zero mean RVs.

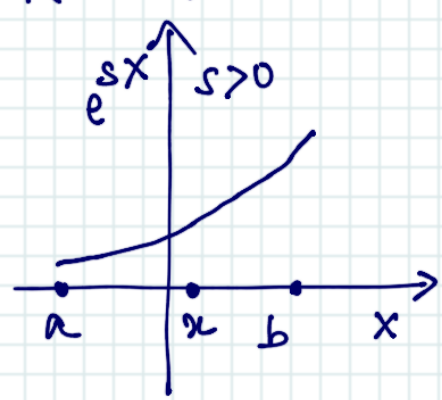
Hoeffding's inequality:
- Hoeffding's Lemma: If $X$ is a random variable with $E[X] = 0$ and $a \leq X \leq b$, then $E[e^{sX}] \leq \underline{\underline{e^{\frac{s^2(b-a)^2}{8}}}}$.

Proof:
- Note that $e^{sx}$ is a convex function

$\Rightarrow$ we can apply the Jensen's inequality i.e.

$$f(\underbrace{\theta \cdot a + (1-\theta) \cdot b}) \leq \theta \cdot f(a) + (1-\theta) \cdot f(b) \quad 0 \leq \theta \leq 1$$

let $f(x) = e^{sx}$, $\theta = \frac{b-x}{b-a}$




$$e^{(sx)} \leq \frac{b-x}{b-a} \cdot f(a) + \frac{x-a}{b-a} \cdot f(b)$$

(from Jensen's)

let $\theta = \frac{b-x}{b-a}$, $a \leq x \leq b$

Applying expectation operator,

$$E[e^{sX}] \leq E\left[\left(\frac{b-x}{b-a}\right) f(a)\right] + E\left[\left(\frac{x-a}{b-a}\right) f(b)\right]$$ (Expectation doesn't change inequality)

$$= \left(\frac{b}{b-a}\right) \cdot f(a) + \left(\frac{-a}{b-a}\right) \cdot f(b) \quad (\because E[X]=0)$$

Notation: $p = \frac{-a}{(b-a)}$ ; $h = s.(b-a)$.

Show that RHS $\left(\frac{b}{b-a}\right) \cdot f(a) + \left(\frac{-a}{b-a}\right) f(b)$

$$= e^{L(h)}, \text{ where}$$

$$L(h) = -hp + \log\left((1-p) + p \cdot e^{h}\right)$$

$$L(0) = 0$$

$$L'(0) = -p + \frac{1}{(1-p) + p \cdot e^{h}} \cdot p \cdot e^{h} \Big|_{h=0} = 0$$

$$L''(0) = p \cdot (1-p)$$

$$\boxed{L''(0) \leq \frac{1}{4}}$$ (H.W.)

$$L(h) = L(0) + \frac{(h-0)}{1!} \cdot L'(0) + \frac{h^2}{2} L''(0) + \cdots$$

$$= \frac{h^2}{2} \cdot L''(0) + \cdots$$

$$L(h) \leq h^2/8 + h.o.t.$$

$$h^2/8 = \frac{s^2 (b-a)^2}{8}$$

$$E\left[e^{sX}\right] \leq E\left[e^{L(h)}\right] \leq e^{\frac{s^2(b-a)^2}{8}}$$

Hoeffding's inequality:

If $S_n = \sum_{i=1}^{n} X_i$ where $X_i$'s are independent RVs with $a_i \leq X_i \leq b_i$,

then
$$P\{S_n - ES_n \geq t\} \leq \exp\left[\frac{-2t^2}{\sum_{i=1}^{n}(b_i-a_i)^2}\right],$$

$$P\{S_n - ES_n \leq -t\} \leq \exp\left[-\frac{2t^2}{\sum_{i=1}^{n}(b_i-a_i)^2}\right].$$

Proof: Let's consider the following tail bound:

$$P\{S_n - ES_n \geq t\} \leq e^{-st} \cdot \prod_{i=1}^{n} E\left[\exp s(X_i - E[X_i])\right]$$

(due to chernoff bound)

$$\leq e^{-st} \prod_{i=1}^{n} e^{\frac{s^2 \cdot (b_i - a_i^2)}{8}}$$

$$= \exp\underbrace{\left[-st + \sum_{i=1}^{n} \frac{s^2(b_i-a_i)^2}{8}\right]}_{g(s)}$$

Since we are interest in the tightest bound possible, find $s$ that minimizes $g(s)$. where $g(s) = -st + \sum_{i=1}^{n} \frac{s^2 \cdot (b_i - a_i)^2}{8}$

Setting $g'(s) = 0$ gives $s = \frac{4t}{\sum_{i=1}^{n}(b_i-a_i)^2}$ | Verify $g''(s) \geq 0$ at this $s$.

Plugging this $s$ into our expression gives us the result.