4/1/19 EE5603: Concentration Inequalities

Today
- Recall motivation + measure theoretic prob.
- Basic inequalities with proofs and examples
  - Markov
  - Chebyshev
  - Chernoff
  - LLN

Basic intro to measure theoretic probability:

- $(\Omega, \mathcal{F}, P)$ the probability triplet

Ω: set of possible outcomes ω.

$\mathcal{F}$: σ-algebra defined on Ω that satisfies the following axioms

Axioms: A.1. $\Omega \in \mathcal{F}$

A.2: If $A \in \mathcal{F}$, $A^c \in \mathcal{F}$

A.3: If $A \in \mathcal{F}$, $B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

From A.2 & A.3 we can show that if $A \in \mathcal{F}$, $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Pf: $A^c \in \mathcal{F}$, $B^c \in \mathcal{F}$ (from A.2)

$\Rightarrow A^c \cup B^c \in \mathcal{F}$ (from A.3)

$\Rightarrow (A^c \cup B^c)^c \in \mathcal{F}$ (from A.2)

we know $A \cap B = (A^c \cup B^c)^c$

$\therefore A \cap B \in \mathcal{F}$.

P: A probability measure defined on $\mathcal{F}$ that satisfies the following axioms

P.1: $P(A) \geq 0$ for all $A \in \mathcal{F}$

P.2: $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ for disjoint sets A_1, A_2

P.3: $P(\Omega) = 1$.

A random variable X maps Ω to $\mathbb{R}$ and is $\mathcal{F}$-measurable.

For any ε, $\{\omega : X(\omega) \leq \varepsilon\} \in \mathcal{F}$.

Recall defn of a.s. convergence: $P(\lim_{n \to \infty} X_n = X) = 1$

Can be interpreted as $P(\omega : \lim_{n \to \infty} X_n(\omega) = X(\omega)) = 1$

Ex: $\Omega = \{a, b, c, d\}$

σ-algebra $\mathcal{F} = \{\Omega, \phi, \{a, b\}, \{c, d\}\}$

- $F_X(x) = P\{\omega : X(\omega) \leq x\}$

$$= P(X \leq x).$$

- $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

- Review / prove basic inequalities:

- Markov inequality: For a non-negative RV X, and for any $\varepsilon > 0$

$$P(X \geq \varepsilon) \leq \frac{E[X]}{\varepsilon}.$$

$$E[X] = \int_0^\infty P(X \geq t)\,dt$$

$$= \int_0^\infty \int_t^\infty f_X(x)\,dx\;dt$$

$$\vdots ?$$

$$= \int_0^\infty t \cdot f_X(t)\,dt$$

<u>Discrete case:</u> $X = n \quad n \in \mathbb{N},\; p(n)$

$$E[X] = \sum_{n=1}^{\infty} n \cdot p(n) = 1 \cdot p(1) + 2\,p(2) \cdots$$

$$\Updownarrow$$

$$= \sum_{m=0}^{\infty} p(X \geq m)$$

$$= \cancel{1 \cdot p(0)} + \overbrace{1 \cdot p(1) + 1 \cdot p(2) \cdots}^{p(X \geq 1)}$$

$$+ \overbrace{1\,p(2) + 1 \cdot p(3) \cdots}^{p(X \geq 2)}$$

$$+ \quad 1 \cdot p(3) + \cdots$$

$$E[X] = \int_{-\infty}^{\infty} x\,f_X(x)\,dx = \int_0^\infty x \cdot f_X(x)\,dx \quad (\because X \geq 0)$$

$$= \int_0^\varepsilon x \cdot f_x(x) \cdot dx + \int_\varepsilon^\infty x \cdot f_X(x)\,dx$$

$$\geq \int_\varepsilon^\infty x \cdot f_x(x)\,dx$$

$$\geq \int_\varepsilon^\infty \varepsilon \cdot f_X(x)\,dx \quad \left(\text{since } x \text{ is non negative}\right)$$

$$= \varepsilon \int_{\varepsilon}^{\infty} f_X(x)\, dx$$

$$= \varepsilon \cdot P(X \geq \varepsilon)$$

i.e. $E[X] \geq \varepsilon \cdot P(X \geq \varepsilon)$

or $P(X \geq \varepsilon) \leq \frac{E[X]}{\varepsilon}$; $\varepsilon > 0$

Chebyshev inequality! For any random variable X, and for any $\varepsilon > 0$, $P[\,|X - E[X]| \geq \varepsilon\,] \leq \frac{Var(X)}{\varepsilon^2}$

If $\phi(x)$ is strictly monotonic and increasing,

$$P(X \geq \varepsilon) = P(\phi(X) \geq \phi(\varepsilon)).$$

Using this result and letting $Y = |X - E[X]|$

$$P(Y \geq \varepsilon) = P(\phi(Y) \geq \phi(\varepsilon)) \quad \text{where } \phi(x) = x^2$$

$$= P[\,(X - E[X])^2 \geq \varepsilon^2\,]$$

clearly $\phi(Y)$ is non-negative and the MI is valid.

$$\Rightarrow \quad P(Y \geq \varepsilon) \leq \frac{E[(X - E[X])^2]}{\varepsilon^2}$$