18/1/19

EE5603: Concentration Inequalities

Today:
- Review
- Complete proof of Bennett's inequality
- Demo: making sense of it all
- McDiarmid's inequality

Bennett's inequality: Let $X_1, \dots X_n$ be independent RVs with finite variance and $X_i \leq b$ for $b > 0$ almost surely for $i \leq n$. Let $v = \sum_{i=1}^{n} E X_i^2$.

If $S = \sum_{i=1}^{n} X_i - EX_i$, then for any $\lambda > 0$

$$\log E[e^{\lambda S}] \leq n \cdot \log\left(1 + \frac{v}{nb^2} \phi(\lambda b)\right) \leq \frac{v}{b^2} \phi(\lambda b).$$

where $\phi(u) = e^u - u - 1$, $u \in \mathbb{R}$.

Proof:
- Let us assume $b = 1$. — ①
- Observe that $u^{-2} \cdot \phi(u)$ is a non-decreasing function — ②
- From ① & ② we can write

$$(\lambda \cdot X_i)^{-2} \cdot \phi(\lambda X_i) \leq \lambda^{-2} \cdot \phi(\lambda).$$

$$\Rightarrow \quad \phi(\lambda X_i) \leq X_i^2 \phi(\lambda)$$

$$e^{\lambda X_i} - \lambda X_i - 1 \leq X_i^2 \cdot \phi(\lambda)$$

(Apply $E[\cdot]$).

$$E[e^{\lambda X_i} - \lambda X_i - 1] \leq E[X_i^2] \phi(\lambda).$$

(Rearranging)

$$E[e^{\lambda X_i}] \leq E[\lambda X_i] + 1 + E[X_i^2].\phi(\lambda)$$

(Applying $\log(.)$)

$$\log E[e^{\lambda X_i}] \leq \log\left[E[\lambda X_i] + 1 + E[X_i^2]\phi(\lambda)\right]$$

$$\log E[e^{\lambda X_i}] - \lambda E[X_i] \leq \log(1 + E[\lambda X_i] + E[X_i^2].\phi(\lambda))$$

(Summing over i)

$$\underbrace{\sum_{i=1}^{n} \log\left[\frac{E[e^{\lambda X_i}]}{e^{E[\lambda E_i]}}\right]}_{} \leq \sum_{i=1}^{n}\left[\log(1 + E[\lambda X_i] + E[X_i^2].\phi(\lambda)) - \lambda E[X_i]\right]$$

$$\psi_S(\lambda) \leq \sum_{i=1}^{n} \text{———} \;\; " \;\; \text{———} \quad -(2b)$$

Observe that $\log(\,)$ is a concave function i.e.

$$f(\lambda x + (1-\lambda).y) \geq \lambda.f(x) + (1-\lambda).f(y) \quad -(3)$$

$$\sum_{i=1}^{n} \log\left[1 + \lambda.E[X_i] + E[X_i^2].\phi(\lambda)\right] =$$

$$n.\sum_{i=1}^{n} \frac{1}{n}.\log\left[1 + \lambda E[X_i] + E[X_i^2].\phi(\lambda)\right]$$

From (3),

$$n \cdot \sum_{i=1}^{n} \frac{1}{n} \log \left[\underbrace{1 + \lambda \cdot E[X_i] + E[X_i^2]\,\phi(\lambda)} \right] \leq$$

$$n \log \cdot \left(\frac{1}{n} \sum_{i=1}^{n} \left(1 + \lambda \cdot E[X_i] + E[X_i^2] \cdot \phi(\lambda) \right) \right)$$

$$= n \cdot \log \left(1 + \sum_{i=1}^{n} \frac{\lambda \cdot E[X_i]}{n} + \frac{v}{n} \cdot \phi(\lambda) \right) \quad — 4$$

Applying 4 in 2 b gives us

$$\psi_S(\lambda) \leq n \cdot \left[\log \left(1 + \boxed{\sum_{i=1}^{n} \frac{\lambda \cdot E[X_i]}{n}} + \frac{v}{n} \cdot \phi(\lambda) \right) - \frac{1}{n} \sum_{i=1}^{n} \lambda E[X_i] \right]$$

observe $\log(1+x) \leq x \quad x \geq 0$. This helps us show

$$n \cdot \left[\log \left(1 + \sum_{i=1}^{n} \frac{\lambda \cdot E[X_i]}{n} + \frac{v}{n} \phi(\lambda) \right) - \frac{1}{n} \sum_{i=1}^{n} \lambda E[X_i] \right]$$

$$\leq n \left[\cancel{\sum_{i=1}^{n} \frac{\lambda \cdot E[X_i]}{n}} + \frac{v}{n} \phi(\lambda) - \cancel{\frac{\lambda}{n} \sum_{i=1}^{n} \lambda E[X_i]} \right]$$

$$= \boxed{v \cdot} \phi(\lambda)$$

Assuming $\frac{1}{n} \sum_{i=1}^{n} \lambda E[X_i] \geq 0$