10/1/19

EE5103: Concentration Inequalities

Today:
- Review basic inequalities
- Law of large numbers (proof)
- Demo on LLN and CLT
- Hoeffding's inequality

Review: Markov Inequality: For a non-negative RV X and for any $\varepsilon > 0$

$$\boxed{P(X \geq \varepsilon) \leq \frac{E[X]}{\varepsilon}}$$

Chebyshev Inequality: for any RV X with bounded variance,

$$\boxed{P(|X - E[X]| \geq \varepsilon) \leq \frac{Var(X)}{\varepsilon^2}}$$

Chernoff bound: For any RV X with bounded variance and for any $t > 0$

$$\boxed{P(e^{tX} \geq e^{t\varepsilon}) \leq \frac{E[e^{tX}]}{e^{t\varepsilon}}}$$

Law of large numbers: (LLN):

$$P\left(\lim_{n\to\infty} \left|\frac{1}{n} S_n - \mu\right| \geq \varepsilon \right) = 0$$

$S_n = \sum_{i=1}^{n} X_i$ are i.i.d RVs with mean μ and bounded variance σ^2

Applying Chebyshev inequality,

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \leq \frac{Var\left(\frac{S_n}{n}\right)}{\varepsilon^2}$$

$$Var(S_n) = \sum_{i=1}^{n} Var(X_i) = n \cdot \sigma^2$$

$$Var\left(\frac{S_n}{n}\right) = \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) = \frac{\sigma^2}{n}$$

$$\therefore P\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \leq \frac{\sigma^2}{n\varepsilon^2}$$

$$\Rightarrow \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2}$$

$$= 0. \quad \blacksquare$$

- Hoeffding's inequality:

Motivation: $S_n = \sum_{i=1}^{n} X_i$ $\quad X_i$ are i.i.d

- Applying Chebyshev inequality to $\frac{S_n}{n}$ gives us

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \leq \frac{\sigma^2}{n\varepsilon^2}$$ where $Var(X_i) = \sigma^2$, $E[X_i] = \mu$.

- How about the tail probability

$$P\left(\frac{S_n}{n} - \mu \geq \varepsilon\right) ?$$

We know $P\left(\frac{S_n}{n} - \mu \geq \varepsilon\right) = P\left(e^{\lambda\left(\frac{S_n}{n} - \mu\right)} \geq e^{\lambda\varepsilon}\right)$

$$\overset{\text{Apply M.I.}}{\leq} \frac{E\left[e^{\lambda\left(\frac{S_n}{n} - \mu\right)}\right]}{e^{\lambda\varepsilon}}$$

$$= e^{-\lambda\varepsilon} E\left[e^{\lambda \cdot \left[\frac{1}{n}\sum_{i=1}^{n} X_i - E[X_i]\right]}\right]$$

$$= e^{-\lambda\varepsilon} \prod_{i=1}^{n} E\left[e^{\lambda \cdot \frac{[X_i - E[X_i]]}{n}}\right]$$

Question: How can we come up with a tight bound for the moment generating function of $(X - E[X])$.

Hoeffding's lemma: If X is a random variable with $E[X] = 0$ and $a \leq X \leq b$, then for any $s > 0$

$$E[e^{sX}] \leq e^{\frac{s^2(b-a)^2}{8}}.$$