25/1/19 EE5808: Concentration Inequalities

Today:
- Review
- Complete proof of McDiarmid's inequality

Review: Bounded differences property: If $f: \mathcal{X}^n \to \mathbb{R}$, there exists a non negative c_i for all i between 1 and n such that

$$\sup_{x_1 \ldots x_n, x_i' \in \mathcal{X}} \left| f(x_1 \ldots x_i, \ldots x_n) - f(x_1 \ldots x_i', \ldots x_n) \right| \leq c_i$$

McDiarmid's inequality: If $X^n = (X_1 \ldots X_n)$ is a set of n indep. RVs, $g: \mathcal{X}^n \to \mathbb{R}$ is a function that has the bounded differences property, then for any $t > 0$

$$P(g(X^n) - E[g(X^n)] \geq t) \leq \exp\left(\frac{-2t^2}{\sum_{i=1}^n c_i^2}\right),$$

$$P(g(X^n) - E[g(X^n)] \leq -t) \leq \exp\left(\frac{-2t^2}{\sum_{i=1}^n c_i^2}\right).$$

Proof outline: Find V_i such that:

(i) $V = g(X^n) - E[g(X^n)] = \sum_{i=1}^n V_i$

(ii) V_i depends only on X^i

(iii) Given $\underline{X^{i-1}}$, there exist U_i, L_i such that $L_i \leq V_i \leq U_i$ and $\underline{U_i - L_i} \leq c_i$

If $V_i = E[g(X^n) | X^i] - E[g(X^n) | X^{i-1}]$,

$$\sum_{i=1}^n V_i = E[g(X^n) | X^n] - \cancel{E[g(X^n) | X^{n-1}]} + \cancel{E[g(X^n) | X^{n-1}]} - E[g(X^n) | X^{n-1}] + \ldots + \cancel{E[g(X^n) | X^1]} - E[g(X^n)]$$

$$= \underbrace{E[g(X^n) \mid X^n]} - E[g(X^n)]$$

$$= \quad g(X^n) \quad - E[g(X^n)]$$

$$= V$$

- $E[g(X^n) \mid X^i] = \int g(x^n) . f_{X_{i+1}}(x_{i+1}) \cdots f_{X_n}(x_n) . \, dx_{i+1} \cdots dx_n$

It follows that V_i satisfies property (ii)

- $L_i = \inf_{x \in \mathcal{X}} [E[g(X^n) \mid X^{i-1}, x] - E[g(X^n) \mid X^{i-1}]]$

$U_i = \sup_{x' \in \mathcal{X}} [E[g(X^n) \mid X^{i-1}, x'] - E[g(X^n) \mid X^{i-1}]]$

From the defn of inf, sup & V_i,

$$L_i \leq V_i \leq U_i$$

Recall:
$V_i = E[g(X^n) \mid X^i] - E[g(X^n) \mid X^{i-1}]$

- Show that $U_i - L_i \leq c_i$