24/1/19 EE5608: Concentration Inequalities

Today: • Motivation, review of sup & inf
• McDiarmid's inequality
– Bounded difference property
– Statement & proof

• Motivation: To come up with tail bounds on functions of independent RVs.

• First analyze functions that satisfy the $\boxed{\text{bounded differences property}}$.

Review: • If $A \subset \mathbb{R}$ is a set of real numbers, then $M \in \mathbb{R}$ is called an upper bound of A if $x \leq M$ for every $x \in A$.

• The least upper bound of A is called the supremum of A.

• If $A \subset \mathbb{R}$ is a set of real numbers, then $m \in \mathbb{R}$ is called a lower bound of A if $x \geq m$ for every $x \in A$.

• The greatest lower bound of A is called the infimum of A

• If $\sup A \in A$, then maximum of $A = \sup A$.
• If $\inf A \in A$, then minimum of $A = \inf A$.

Alternatively, $M = \sup A$ if and only if
• M is an upper bound of A
• For every $M' < M$, there exists an $x \in A$ such that $x > M'$.

- $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$

  $\inf A = 0, \quad \sup A = 1 = \max A$

  Minimum does not exist

- These notions carry over to functions when defined on the range of values taken by the function

  If $f : A \to \mathbb{R}$, $\inf_A f = \inf \{ f(x) : x \in A \}$,

  $\sup_A f = \sup \{ f(x) : x \in A \}$

- Ex: $f(x) = \begin{cases} x & 0 \leq x < 1 \\ 0 & x = 1 \end{cases}$

  $\inf_{[0,1]} f = \min_{[0,1]} f = 0$

  $\sup_{[0,1]} f = 1.$

<u>Bounded differences property</u>: Let X be a set and $f : X^n \to \mathbb{R}$. If there exist non negative c_i for all i $(1 \leq i \leq n)$, such that

$$\sup_{x_1, \dots x_n, x_i' \in X} \left| f(x_1, \dots x_i, \dots x_n) - f(x_1, \dots x_i', \dots x_n) \right| \leq c_i$$

<u>McDiarmid's inequality</u>: Let $X^n = (X_1 \dots X_n)$ be a collection of n independent RVs and $g : \mathcal{X}^n \to \mathbb{R}$ that has the bounded differences property, then for any $t > 0$

$$P(g(X^n) - E[g(X^n)] \geq t) \leq \exp\left[-\frac{2t^2}{\sum_{i=1}^n c_i^2}\right]$$

$$P(g(X^n) - E[g(X^n)] \leq -t) \leq \exp\left[-\frac{2t^2}{\sum_{i=1}^n c_i^2}\right]$$

<u>Proof outline:</u>

- Note that this bound looks a lot like the Hoeffding's inequality.
- Let $V = g(X^n) - E[g(X^n)]$
- If $V = \sum_{i=1}^n V_i$ such that each element of the sum V_i is bounded and the length of the bounded interval is c_i,

We can resort to our previous approach to prove the Hoeffding's inequality.

$$P(g(X^n) - E[g(X^n)] \geq t) = P(V \geq t)$$

$$= P(e^{sV} \geq e^{st})$$

$$\leq e^{-st} \cdot E[e^{sV}]$$

$$= e^{-st} \; E\left[e^{s \cdot \sum_{i=1}^n V_i}\right]$$

- V_i should also have the property that it depends only on X^i.

$$= e^{-st}\, E\left[E\left[e^{s\sum_{i=1}^{n} V_i} \,\middle|\, X^{n-1} \right]\right]$$

$$= e^{-st}\, E\left[e^{s\cdot\sum_{i=1}^{n-1} V_i} \cdot \underbrace{E\left[e^{sV_n} \,\middle|\, X^{n-1} \right]} \right]$$

- V_i should be such that given X^{i-1}, there exist L_i and U_i such that $L_i \le V_i \le U_i$ and $U_i - L_i \le c_i$

$$\le e^{-st} \cdot e^{\frac{s^2 \cdot c_n^2}{8}}\, E\left[e^{s\sum_{i=1}^{n-1} V_i} \right]$$

(due to Hoeffding's lemma)

$$\vdots$$

$$= e^{-st} \cdot e^{\frac{s^2 \sum_{i=1}^{n} c_i^2}{8}}$$