28/1/19

EE5603: Concentration Inequalities

Today:
- Review
- Complete proof of McDiarmid's inequality
- Efron-Stein inequality

- Review:
  - The McDiarmid's inequality looks a lot like the Hoeffding's inequality
  - Can we express the difference $V = g(X^n) - E(g(X^n))$ as a sum of V_i where V_i are such that:

(i) $V = \sum_{i=1}^{n} V_i$

(ii) V_i depend only on X^i

(iii) given X^{i-1}, there exist functions U_i, L_i such that

$$\boxed{U_i - L_i \le c_i}, \qquad L_i \le V_i \le U_i$$

Recall:

$$U_i = \sup_{x' \in \mathcal{X}} \left\{ E[g(X^n) \mid X^{i-1}, x'] - E[g(X^n) \mid X^{i-1}] \right\}$$

$$L_i = \inf_{x \in \mathcal{X}} \left\{ E[g(X^n) \mid X^{i-1}, x] - E[g(X^n) \mid X^{i-1}] \right\}$$

$$U_i - L_i = \sup_{x' \in \mathcal{X}} \left\{ E[g(X^n) \mid X^{i-1}, x'] - E[g(X^n) \mid X^{i-1}] \right\} - \inf_{x \in \mathcal{X}} \left\{ E[g(X^n) \mid X^{i-1}, x] - E[g(X^n) \mid X^{i-1}] \right\}$$

$$= \sup_{x \in \mathcal{X}} \sup_{x' \in \mathcal{X}} \left\{ E[g(X^n) \mid X^{i-1}, x] - E[g(X^n) \mid X^{i-1}, x'] \right\}$$

(from defn of inf & sup)

$$= \sup_{x \in \mathcal{X}} \sup_{x' \in \mathcal{X}} \int \left[g(x^n \mid x^{i-1}, x) - g(x^n \mid x^{i-1}, x') \right] dP_{x_{i+1}^n}$$

$$\leq \sup_{x \in \mathcal{X}} \sup_{x' \in \mathcal{X}} \int \left| g(x^n \mid x^{i-1}, x) - g(x^n \mid x^{i-1}, x') \right| dP_{x_{i+1}^n}$$

(since $\int f-g \leq \int |f-g|$)

$$\leq c_i$$

(from bounded difference property)

$\therefore \quad U_i - L_i \leq c_i \quad \& \quad L_i \leq V_i \leq U_i \leq L_i + c_i$

Observation: McDiarmid's inequality is a powerful result since we only require (i) independence of RVs $X_1 \dots X_n$,

(ii) $g(X^n)$ to satisfy bounded differences property

Note that we did not impose any restrictions on the distributions of X_i

- <u>Efron-Stein inequality</u>: Let $X_1 \dots X_n$ be independent RVs. Let $f: \mathcal{X}^n \to \mathbb{R}$ be a square integrable function. Let $Z = f(X_1 \dots X_n)$,

$$\mathrm{Var}(Z) \leq \sum_{i=1}^{n} E\left(Z - E^{i}(Z)\right)^2 \stackrel{\text{def}}{=} v.$$

$$E_i(Z) = E\left[f(x^n) \mid x^i\right].$$

To dos:

- $\Delta_i = E_i(z) - E_{i-1}(z)$
- $Z - E[z] = \sum_{i=1}^{n} \Delta_i$
- If $E^i(z) = \int f(x_1 \ldots x_{i-1}, x_i, x_{i+1} \ldots x_n) \, dP(x_i)$

$$E_i[E^i(z)] = E_{i-1}(z)$$