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NUMERICAL METHODS TO STUDY FLOWS IN CONCENTRIC CIRCULAR PIPES USING RADIAL MESHES

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ABSTRACT

This project presents numerical methods for studying flows in concentric pipes with circular cross-sectional geometries. The focus is on the use of the finite volume method (FVM) to model and analyze fluid flow phenomena in these pipes, which can be approximated by the Navier-Stokes equation as,

$$\frac{\partial(\rho w)}{\partial t} = \nabla \cdot (\mu \nabla w) + \frac{dp}{dz} \tag{1}$$

where, $\phi = w, \Gamma = \mu, S_\phi = -\frac{dp}{dz}$

The FVM is used to solve the above equation in a discretized form. Then, we implement a TDMA for a radial mesh by invoking proper boundary conditions, such as no-slip and stationary walls, i.e.,w(r=R) = 0. We also implement unsteady diffusion algorithms to compute changes at every time step. The findings of this study can contribute to the design and optimization of fluid transport systems in various industrial applications.

NOMENCLATURE

- ρ Density of the fluid
- μ Dynamic viscosity
- w Flow velocity (scalar)
- p Pressure (scalar)
- z Length along longitudinal direction (scalar)
- f Weighing parameter
- D Dimensionality (2 in our case)

INTRODUCTION

This project explores the problem of radial meshes on concentric circular pipes. The situation poses two challenges. The solver development faces difficulties in certain aspects, such as invoking cyclicity in the tridiagonal matrix algorithm (TDMA). Traditional TDMA is solved on a tridiagonal matrix, but the matrix cannot be tri-diagonalized using simple methods in a radial mesh. Hence, we invoke the use of the Sherman- Morrison formula. This solves the issue of cyclicity. Secondly, the convergence in the unsteady scenario could be better, which shall be further explained. We implement an unsteady state solver with all three schemes for $f = 1, 0, 0.5$ and compare the results.

GOVERNING EQUATIONS

Our primary equation will be,

$$\frac{\partial(\rho w)}{\partial t} = \nabla \cdot (\mu \nabla w) + \frac{dp}{dz} \tag{2}$$

We analyze this expression for two cases, namely steady and unsteady.

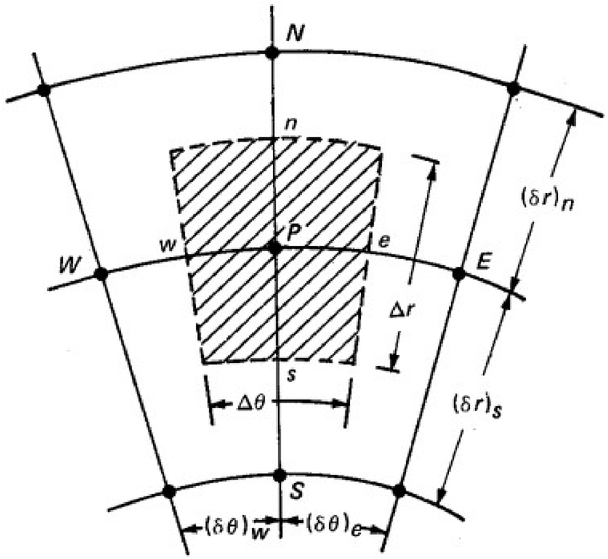

FIGURE 1. Control volume in polar cooordinates

Steady state

At a steady state,

$$\nabla \cdot (\mu \nabla w) + \frac{dp}{dz} = 0 \tag{3}$$

Taking integration over a control volume,

$$\int_{cv} \nabla \cdot (\mu \nabla w) dv + \int_{cv} (\frac{dp}{dz})\, dv = 0 \tag{4}$$

Using the Gauss Divergence theorem,

$$\int_{cs} (\mu \nabla w) \cdot d\mathbf{A} + \int_{cv} (\frac{dp}{dz})\, dv = 0 \tag{5}$$

Summing over the faces and assuming a constant pressure gradient,

$$\sum_{f=e,w,n,s} (\mu \nabla w)_f \cdot \mathbf{A}_f + (\frac{dp}{dz})\, \Delta v = 0 \tag{6}$$

We discretize this equation on a radial setting, as shown in Figure 1.

$$a_P w_P = \sum_{nb} a_{nb} w_{n\ b} + b \tag{7}$$

where, $a_E = \frac{\mu \Delta r}{r_e \delta \theta_e}, a_W = \frac{\mu \Delta r}{r_w \delta \theta_w}, a_N = \frac{\mu r_n \Delta \theta}{\delta \theta_n}, a_S = \frac{\mu r_s \Delta \theta}{\delta \theta_s}$, $a_P = \sum_{nb} a_{nb}, b = (\frac{dp}{dz}) \Delta v$

But this discrete equation is cyclic; hence we implement the Sherman - Morrison algorithm with a traditional TDMA to solve it.

Algorithm

1. Initialize the coefficient matrices, $a_P, a_W, a_E, a_N, a_S, s$,and set the control parameters such as $\mu, \frac{dp}{dz}, d\theta, dr$.
2. Setup the boundary conditions, here we set $w(r = R_{in}) = 0$ and $w(r = R_{out}) = 0$, i.e., a no-slip conditions on both boundaries. Update the coefficient matrices as well.
3. Perform radial TDMA using the Sherman- Morrison formula, sweeping radially and performing TDMA in the anti-clockwise sense.
4. Iterate till convergence, the ϕ matrix stores the converged results.

Unsteady state

For the unsteady state,

$$\frac{\partial(\rho w)}{\partial t} = \nabla \cdot (\mu \nabla w) + \frac{dp}{dz} \tag{8}$$

Taking integration over a control volume and a time step,

$$\int_{\Delta t} \int_{cv} \frac{\partial(\rho w)}{\partial t} dv \cdot dt = \int_{\Delta t} \int_{cv} \nabla \cdot (\mu \nabla w) dv \cdot dt + \int_{\Delta t} \int_{cv} (\frac{dp}{dz})\, dv \cdot dt \tag{9}$$

Using the Gauss Divergence theorem,

$$\int_{\Delta t} \int_{cv} \frac{\partial(\rho w)}{\partial t} dv \cdot dt = \int_{\Delta t} \sum_f (\mu \nabla w)_f \cdot \mathbf{A}_f \cdot dt + \int_{\Delta t} \int_{cv} (\frac{dp}{dz})\, dv \cdot dt \tag{10}$$

Applying the weighing parameters and assuming a constant pressure gradient,

$$a_P^1 w_P^1 = \sum_{nb} a_{nb} (f w_{nb}^1 + (1-f) w_{nb}^0) + w_P^0 (a_P^0 - (1-f) \sum_{nb} a_{nb}) + b \tag{11}$$

For $f = 0$,

$$a_P^1 w_P^1 = \sum_{nb} a_{nb} w_{nb}^0 + w_P^0 (a_P^0 - \sum_{nb} a_{nb}) + b \tag{12}$$

For $f = 0.5$,

$$a_P^1 w_P^1 = \sum_{nb} a_{nb} (\frac{w_{nb}^0 + w_{nb}^1}{2}) + w_P^0 (a_P^0 - \frac{\sum_{nb} a_{nb}}{2}) + b \tag{13}$$

For $f = 1$,

$$a_P^1 w_P^1 = \sum_{nb} a_{nb} w_{nb}^1 + w_P^0 a_P^0 + b \quad (14)$$

where, $a_P^0 = \frac{\rho \Delta v}{\Delta t}, a_P^1 = a_P^0 + f \sum_{nb} a_{nb}, b = (\frac{dP}{dz})\Delta v$

Algorithm

1. Initialize the coefficient matrices, $a_P, a_W, a_E, a_N, a_S, s$,and set the control parameters such as $\mu, \frac{dp}{dz}, d\theta, dr$.
2. Setup the boundary conditions, here we set $w(r = R_{in}) = 0$ and $w(r = R_{out}) = 0$, i.e., a no-slip conditions on both boundaries. Update the coefficient matrices as well.
3. Perform radial TDMA using the Sherman- Morrison formula, sweeping radially and performing TDMA in the anti-clockwise sense.
4. Iterate till convergence of both time and space, the $\phi_{timenew}$ matrix stores the converged results.

RESULTS AND DISCUSSIONS
Steady State

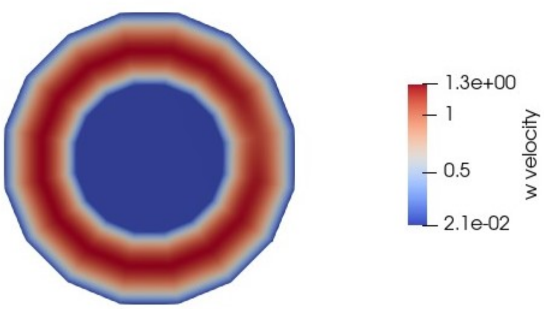

Figure 2: Pipe with inner radius 1m and outer radius 2m, 100 radial and 16 angular divisions

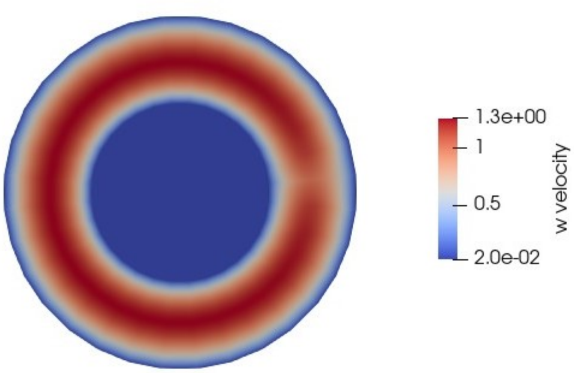

Figure 3: Pipe with inner radius 1m and outer radius 2m, 100 radial and 32 angular divisions

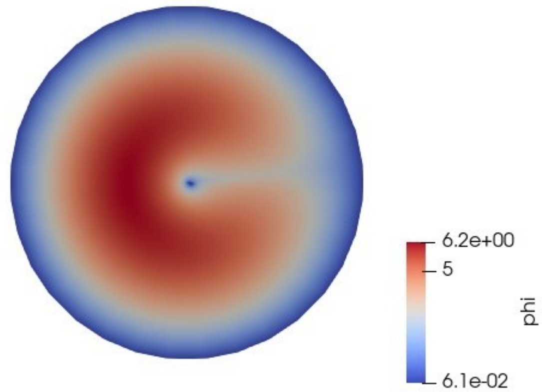

Figure 4: Pipe with no inner radius and outer radius 2m, 100 radial and 32 angular divisions

An investigation of a trivial problem was carried out. The problem specifies a concentric pipe with no-slip boundary conditions implemented. We perform finite volume methods on the radial mesh, as described in Figure 1. In Figure 2, the mesh was discretized into 100 radial and 16 angular meshes. We observe the solution to be smooth, and as expected, the maximum velocity occurs somewhere at the center of the radial direction, i.e., close to $\frac{R_{in}+R_{out}}{2}$.In Figure 3, we increase the angular mesh size to 32, and we start to notice discontinuities in the $\theta = 0$ direction. This is because of the convergence issues associated with the Sherman-Morrison implementation. We also notice the velocity profiles in both figures are the same, which is expected.In Figure 4, we implement a border case of $R_{in} = 0$. The solver converges the scenario better than expected, and we obtain a smooth velocity profile.We also plot variations of the maximum pipe velocity, with the dynamic viscosity of the fluid, μ, and also with the pressure gradient, $\frac{dP}{dz}$ in figures 5 and 6.

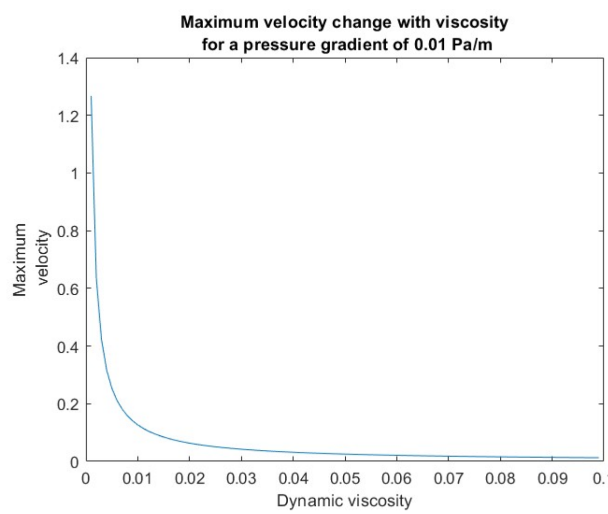

Figure 5 : Maximum velocity with dynamic viscosity for $\frac{dP}{dz} = 0.01 Pa/m$

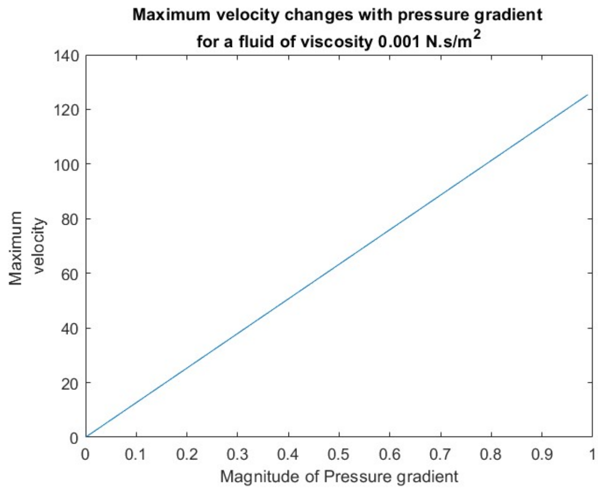

Figure 6 : Maximum velocity with pressure gradient for $\mu = 0.001 N.s/m^2$

Unsteady State

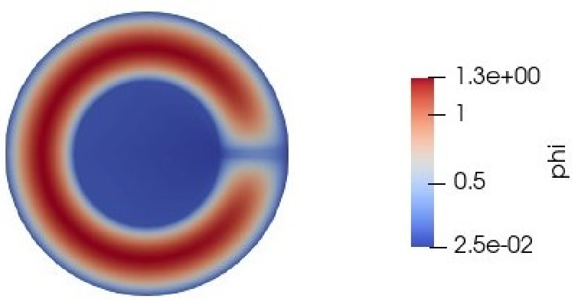

Figure 7: Unsteady diffusion for an inner radius of 1m and outer radius of 2m using an explicit scheme

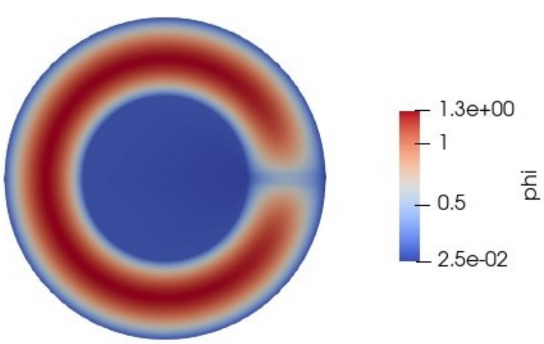

Figure 8: Unsteady diffusion for an inner radius of 1m and outer radius of 2m using a Crank-Nicolson scheme

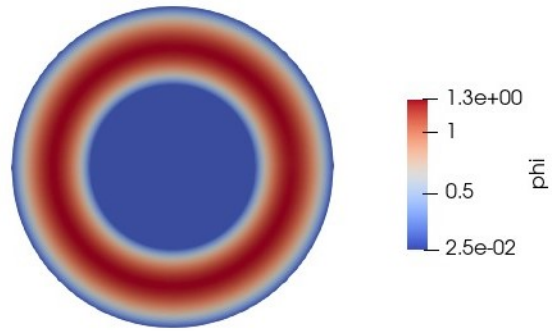

Figure 9: Unsteady diffusion for an inner radius of 1m and outer radius of 2m using an implicit scheme

In the unsteady scenario, we choose a time step of $\Delta t = 0.02$ secs, $\rho = 1$ and a grid size of 100 radial and 32 angular meshes. This gives the solver enough convergence time. In Figures 7,8, and 9, we compare the accuracy of the final result obtained. We notice the magnitudes of the velocity match in all cases, thereby confirming the validity of the solver.

The following are the total time steps taken for convergence for each of the schemes,

Grid size	Scheme used	Total time steps taken
100x32	Explicit	100867
100x32	Crank-Nicolson	100965
100x32	Implicit	102820

These may sound deviant but remember that we maintain similar Δt throughout. This value of Δt also satisfies the Von-Neumann stablity criteria, as $\Delta t \leq \frac{\rho \Delta r^2}{2D\mu}$.

CONCLUSION

Thus, solvers for the steady and unsteady case of radial pipe flows have been implemented. Our results show validation with experimental data, and some things we also realize, such as keeping a grid size of 100x32 and increasing smoothness, but the convergence of the Sherman-Morrison formula is affected. We could extend the solver to run more iterations to solve the convergence issue, but keeping the computational effort in mind, we keep that aside. We also observe the times taken by different time-stepping schemes to converge.

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