MATH 3190 Homework 5

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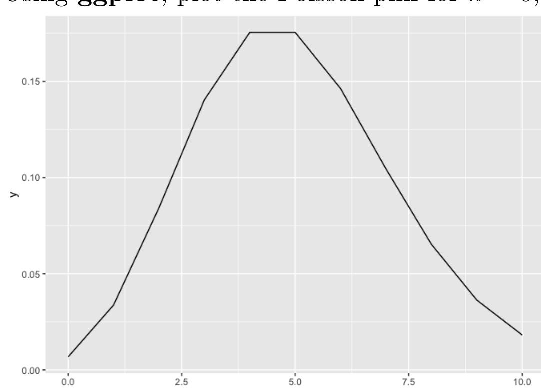
Here you will practice what you learned in the maximum likelihood estimation. Please turn this in as an RMarkdown document. You can either add your solution in Latex or you can write it by hand and input a scanned version or picture into the R Markdown. 'Turn it in' by uploading to your GitHub repository.

1. (20 points) Suppose $\mathbf{x} = (x_1, \ldots, x_N)^T$ follow a Poisson distribution with a parameter $\lambda > 0$ and p.m.f. given by

$$P(x = k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

Answer the following questions:

(a) Using **ggplot**, plot the Poisson pmf for $k = 0, 1, \ldots, 10$ when $\lambda = 5$.



(b) Assuming $\mathbf{x}$ is observed, give the likelihood $L(\lambda|\mathbf{x})$ and log-likelihood $l(\lambda|\mathbf{x})$ functions.

i. Likelihood Function

$$L(\lambda|x_1, \ldots, x_{10}) = \prod_{i=1}^{10} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}.$$

ii. Log-Likelihood function

$$l(\lambda|x_1, \dots, x_n) = \ln(\prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}).$$

Simplifying:

$$l(\lambda|x_1, \dots, x_n) = \sum_{i=1}^{n} \ln(\frac{\lambda^{x_i} e^{-\lambda}}{x_i!}).$$

$$l(\lambda|x_1, \dots, x_n) = \sum_{i=1}^{n} (x_i \ln(\lambda) + \ln(e^{-\lambda}) - \ln(x_i!)).$$

$$l(\lambda|x_1, \dots, x_n) = \sum_{i=1}^{n} (x_i \ln(\lambda) - \lambda - \ln(x_i!)).$$

$$l(\lambda|x_1, \dots, x_n) = -n\lambda + \ln(\lambda) \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \ln(x_i!).$$

(c) Find the Maximum Likelihood Estimator (MLE) $\hat{\lambda}$ for λ.

i. Calculate the derivative of log likelihood function with respect to λ.

$$\frac{d}{d\lambda}(l(\lambda|x_1, \dots, x_n)) = \frac{d}{d\lambda}(-n\lambda + \ln(\lambda) \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \ln(x_i!)).$$

$$\frac{d}{d\lambda}(l(\lambda|x_1, \dots, x_n)) = -n + \frac{1}{\lambda} \sum_{i=1}^{n} x_i.$$

ii. Let's set the derivative to 0 to get the MLE.

$$-n + \frac{1}{\lambda} \sum_{i=1}^{n} x_i = 0$$

$$\lambda = \frac{1}{n} \sum_{i=1}^{n} x_i$$

(d) Show that your estimator is in fact a maximum: i.e., check the boundary values of the log-likelihood, and check that the second derivative of the log-likelihood is negative everywhere.

i. Checking if the second derivative is negative everywhere.

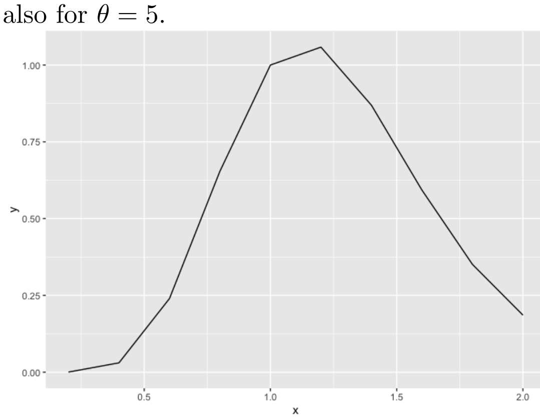
$$\frac{d^2}{d\lambda^2}(l(\lambda|x_1, \dots, x_n)) = \frac{-1}{\lambda^2} \sum_{i=1}^{n} x_i.$$

ii. For $\lambda > 0$, the MLE will be negative everywhere

2. (20 points) Suppose $\mathbf{x} = (x_1, \ldots, x_N)^T$ are *iid* random variables with p.d.f. given by

$$f(x|\theta) = \theta x^{\theta-1}, \ 0 \leq x \leq 1, \ 0 < \theta < \infty.$$

(a) Using **ggplot**, plot the pdf for an individual x_i given $\theta = 0.5$ and also for $\theta = 5$.



(b) Give the likelihood $L(\theta|\mathbf{x})$ and log-likelihood $l(\theta|\mathbf{x})$ functions.

i. Likelihood Function

$$L(\theta|x_1, \ldots, x_n) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$$

ii. Log-Likelihood function

$$l(\lambda|x_1, \ldots, x_n) = \ln(\prod_{i=1}^{n} \theta x_i^{\theta-1})$$

Simplifying:

$$l(\lambda|x_1, \ldots, x_n) = \sum_{i=1}^{n} \ln(\theta x_i^{\theta-1})$$

$$l(\lambda|x_1, \ldots, x_n) = n\ln(\theta) + (\theta - 1)\sum_{i=1}^{n} \ln(x_i)$$

(c) Find the Maximum Likelihood Estimator (MLE) $\hat{\theta}$ for θ.

i. Calculate the derivative of log likelihood function with respect to θ.

$$\frac{d}{d\theta}(l(\theta|x_1, \ldots, x_n)) = \frac{d}{d\theta}(n\ln(\theta) + (\theta - 1)\sum_{i=1}^{n} \ln(x_i)).$$

$$\frac{d}{d\theta}(l(\theta|x_1,\ldots,x_n)) = \frac{n}{\theta} + \sum_{i=1}^{n}\ln(x_i)$$

ii. Let's set the derivative to 0

$$\frac{n}{\theta} + \sum_{i=1}^{n}\ln(x_i) = 0$$

$$\frac{n}{\theta} = -\sum_{i=1}^{n}\ln(x_i)$$

$$\theta = \frac{-n}{\sum_{i=1}^{n}\ln(x_i)}$$

(d) Show that your estimator is in fact a maximum: i.e., check the boundary values of the log-likelihood, and check that the second derivative of the log-likelihood is negative everywhere.

i. Checking whether the second derivative is negative everywhere.

$$\frac{d^2}{d\theta^2}(l(\theta|x_1,\ldots,x_n)) = \frac{-n}{\theta^2}$$

ii. For any $\theta > 0$, the MLE is negative everywhere

iii. As you can see from the graph that the boundary conditions are satisfied since the function goes to 0, so the function we got will be the MLE.

3. (20 points) Suppose $\mathbf{x} = (x_1, \ldots, x_N)^T$ are iid random variables from a $Normal(0, \sigma^2)$ distribution. The pdf is given by

$$f(x|\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} e^{-\frac{x^2}{2\sigma^2}},\ -\infty < x < \infty,\ \sigma^2 > 0.$$

Find the Maximum Likelihood Estimator (MLE) $\hat{\sigma}^2$ for σ^2. Is it what you thought it would be? Why or why not?

(a) Likelihood function

$$L(\sigma^2|x) = \prod_{i=1}^{n}\left(\frac{1}{2\pi\sigma^2}\right)^{1/2} e^{-\frac{x_i^2}{2\sigma^2}}$$

(b) Log Likelihood function

$$l(\sigma^2|x) = \ln(\prod_{i=1}^{n}\left(\frac{1}{2\pi\sigma^2}\right)^{1/2} e^{-\frac{x_i^2}{2\sigma^2}})$$

$$l(\sigma^2|x) = \sum_{i=1}^{n}(\ln((\frac{1}{2\pi\sigma^2})^{1/2}) + \ln(e^{-\frac{x_i^2}{2\sigma^2}}))$$

$$l(\sigma^2|x) = \sum_{i=1}^{n}(\frac{-1}{2}\ln(2\pi\sigma^2) - \frac{x_i^2}{2\sigma^2})$$

$$l(\sigma^2|x) = \sum_{i=1}^{n}(\frac{-1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma^2) - \frac{x_i^2}{2\sigma^2})$$

$$l(\sigma^2|x) = \sum_{i=1}^{n}(\frac{-1}{2}\ln(2\pi) - \ln(\sigma) - \frac{x_i^2}{2\sigma^2})$$

$$l(\sigma^2|x) = (\frac{-n}{2}\ln(2\pi) - n\ln(\sigma) - \sum_{i=1}^{n}\frac{x_i^2}{2\sigma^2})$$

(c) Now let's take the derivative with respect to σ

$$\frac{-n}{\sigma} - \sum_{i=1}^{n}\frac{x_i^2}{\sigma^3} = 0$$

$$n = \sum_{i=1}^{n}\frac{x_i^2}{\sigma^2}$$

$$\sigma^2 = \sum_{i=1}^{n}\frac{x_i^2}{n}$$

(d) Yes, the MLE is what I thought it would be apart from the $(n-1)$ fact, when we calculate the variance of a distribution it is usually $(n-1)$ in the denominator and not n. Thus, this MLE, is fractionally smaller than the variance.