MATH 3190 Homework 4

Maximum Likelihood Estimators

Due 3/28/2022

Here you will practice what you learned in the maximum likelihood estimation. Please turn this in as an RMarkdown document. You can either add your solution in Latex or you can write it by hand and input a scanned version or picture into the R Markdown. 'Turn it in' by uploading to your GitHub repository.

1. (20 points) Suppose $\mathbf{x} = (x_1, \dots, x_N)^T$ follow a Poisson distribution with a parameter $\lambda > 0$ and p.m.f. given by

$$P(x = k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

Answer the following questions:

(a) Using **ggplot**, plot the Poisson pmf for $k = 0, 1, \dots, 10$ when $\lambda = 5$.

(b) Asummining $\mathbf{x}$ is observed, give the likelihood $L(\lambda|\mathbf{x})$ and log-likelihood $l(\lambda|\mathbf{x})$ functions.

i. Likelihood Function

$$L(\lambda|x_1, \dots, x_{10}) = \prod_{i=1}^{10} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}.$$

ii. Log-Likelihood function

$$l(\lambda|x_1, \dots, x_n) = \ln(\prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}).$$

Simplifying:

$$l(\lambda|x_1, \dots, x_n) = \prod_{i=1}^{n} \ln(\frac{\lambda^{x_i} e^{-\lambda}}{x_i!}).$$

$$l(\lambda|x_1, \dots, x_n) = \prod_{i=1}^{n} (x_i \ln(\lambda) + \ln(e^{-\lambda}) - \ln(x_i!)).$$

$$l(\lambda|x_1, \dots, x_n) = \prod_{i=1}^{n} (x_i \ln(\lambda) - \lambda - \ln(x_i!)).$$

$$l(\lambda|x_1, \dots, x_n) = -n\lambda + \ln(\lambda) \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \ln(x_i!).$$

(c) Find the Maximum Likelihood Estimator (MLE) $\hat{\lambda}$ for λ.

i. Calculate the derivative of log likelihood function with respect to λ.

$$\frac{d}{d\lambda}(l(\lambda|x_1,\ldots,x_n)) = \frac{d}{d\lambda}(-n\lambda + \ln(\lambda)\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \ln(x_i!)).$$

$$\frac{d}{d\lambda}(l(\lambda|x_1,\ldots,x_n)) = -10 + \frac{1}{\lambda}\sum_{i=1}^{n} x_i.$$

ii. Let's set the derivative to 0 to get the MLE.

$$-10 + \frac{1}{\lambda}\sum_{i=1}^{n} x_i = 0$$

$$\lambda = \frac{1}{n}\sum_{i=1}^{n} x_i$$

(d) Show that your estimator is in fact a maximum: i.e., check the boundary values of the log-likelihood, and check that the second derivative of the log-likelihood is zero everywhere.

i. Checking if the second derivative is zero everywhere.

$$\frac{d^2}{d\lambda^2}(l(\lambda|x_1,\ldots,x_n)) = \frac{-1}{\lambda^2}\sum_{i=1}^{n} x_i.$$

2. (20 points) Suppose $\mathbf{x} = (x_1,\ldots,x_N)^T$ are *iid* random variables with p.d.f. given by

$$f(x|\theta) = \theta x^{\theta-1},\ 0 \leq x \leq 1,\ 0 < \theta < \infty.$$

(a) Using **ggplot**, plot the pdf for an individual x_i given $\theta = 0.5$ and also for $\theta = 5$.

(b) Give the likelihood $L(\theta|\mathbf{x})$ and log-likelihood $l(\theta|\mathbf{x})$ functions.

i. Likelihood Function

$$L(\theta|x_1,\ldots,x_n) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$$

ii. Log-Likelihood function

$$l(\lambda|x_1,\ldots,x_n) = \ln(\prod_{i=1}^{n} \theta x_i^{\theta-1})$$

Simplifying:

$$l(\lambda|x_1,\ldots,x_n) = \sum_{i=1}^{n} \ln(\theta x_i^{\theta-1})$$

$$l(\lambda|x_1,\ldots,x_n) = n\ln(\theta) + (\theta-1)\sum_{i=1}^{n} \ln(x_i)$$

(c) Find the Maximum Likelihood Estimator (MLE) $\hat{\theta}$ for θ.

i. Calculate the derivative of log likelihood function with respect to θ.

$$\frac{d}{d\theta}(l(\theta|x_1,\ldots,x_n)) = \frac{d}{d\theta}(n\ln(\theta) + (\theta-1)\sum_{i=1}^{n} \ln(x_i)).$$

$$\frac{d}{d\theta}(l(\theta|x_1,\ldots,x_n)) = \frac{n}{\theta} + \sum_{i=1}^{n} \ln(x_i)$$

ii. Let's set the derivative to 0

$$\frac{n}{\theta} + \sum_{i=1}^{n} \ln(x_i) = 0$$

$$\frac{n}{\theta} = -\sum_{i=1}^{n} \ln(x_i)$$

$$\theta = \frac{-n}{\sum_{i=1}^{n} \ln(x_i)}$$

(d) Show that your estimator is in fact a maximum: i.e., check the boundary values of the log-likelihood, and check that the second derivative of the log-likelihood is zero everywhere.

3. (20 points) Suppose $\mathbf{x} = (x_1, \ldots, x_N)^T$ are *iid* random variables from a $Normal(0, \sigma^2)$ distribution. The pdf is given by

$$f(x|\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} e^{-\frac{x^2}{2\sigma^2}}, \ -\infty < x < \infty, \ \sigma^2 > 0.$$

Find the Maximum Likelihood Estimator (MLE) $\hat{\sigma}^2$ for σ^2. Is it what you thought it would be? Why or why not?

(a) Likelihood function

$$L(\sigma^2|x) = \prod_{i=1}^{n} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} e^{-\frac{x_i^2}{2\sigma^2}}$$

(b) Log Likelihood function

$$l(\sigma^2|x) = \ln(\prod_{i=1}^{n} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} e^{-\frac{x_i^2}{2\sigma^2}})$$

$$l(\sigma^2|x) = \sum_{i=1}^{n}(\ln((\frac{1}{2\pi\sigma^2})^{1/2}) + \ln(e^{-\frac{x_i^2}{2\sigma^2}}))$$

$$l(\sigma^2|x) = \sum_{i=1}^{n}(\frac{-1}{2}\ln(2\pi\sigma^2) - \frac{x_i^2}{2\sigma^2})$$

$$l(\sigma^2|x) = \sum_{i=1}^{n}(\frac{-1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma^2) - \frac{x_i^2}{2\sigma^2})$$

$$l(\sigma^2|x) = \sum_{i=1}^{n}(\frac{-1}{2}\ln(2\pi) - \ln(\sigma) - \frac{x_i^2}{2\sigma^2})$$

$$l(\sigma^2|x) = (\frac{-n}{2}\ln(2\pi) - n\ln(\sigma) - \sum_{i=1}^{n}\frac{x_i^2}{2\sigma^2})$$

(c) Now let's take the derivative with respect to σ

$$\frac{-n}{\sigma} - \sum_{i=1}^{n} \frac{x_i^2}{\sigma^3} = 0$$

$$n = \sum_{i=1}^{n} \frac{x_i^2}{\sigma^2}$$

$$\sigma^2 = \sum_{i=1}^{n} \frac{x_i^2}{n}$$

(d) Yes, the MLE is what I thought it would be since the maximum likelihood estimated for σ^2 is the variance of the measurements