POPL

Assignment-I

ITISH AGARWAL
(18CS30021)

Q1.

(a) $(\lambda x.((xz)(\lambda y.(xy))))$

(b) $(\lambda x.(xz))\ (\lambda y.(w(\lambda w.(((wy)z)x))))$

(c) $(\lambda x.((xy)(\lambda x.(y\,x))))$

Q2. The underlined variables are free variables:

(a) $\lambda x.\ x\underline{z}\lambda y.\ xy$

(b) $(\lambda x.\ x\underline{z})\ \lambda y.\ \underline{w}\ \lambda w.\, w\ y\ \underline{z}\ \underline{x}$

(c) $\lambda x.\ x\underline{y}\lambda x\ \underline{y}\ x$

Q3. PTO

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Q3. (a) NOT (NOT TRUE) = TRUE

where,

$$NOT = \lambda x. ((x\ FALSE)\ TRUE)$$

$$TRUE = \lambda x. \lambda y. x$$

$$FALSE = \lambda x. \lambda y. y$$

$$\begin{aligned}
NOT\ TRUE &= (\lambda x. ((x\ FALSE)\ TRUE))\ (\lambda x. \lambda y. x) \\
&= (((\lambda x. \lambda y. x)\ FALSE)\ TRUE) \\
&= (\lambda y. FALSE)\ TRUE \\
&= FALSE
\end{aligned}$$

$\therefore$

$$\begin{aligned}
NOT\ (NOT\ TRUE) &= NOT\ FALSE \\
&= (\lambda x. ((x\ FALSE)\ TRUE))\ (\lambda y. \lambda y. y) \\
&= (((\lambda x. \lambda y. y)\ FALSE)\ TRUE) \\
&= (\lambda y. y)\ TRUE \\
&= TRUE
\end{aligned}$$

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Hence,

$$NOT\ (NOT\ TRUE) = TRUE$$

(b) OR FALSE TRUE = TRUE

given

$OR = \lambda x.\lambda y.((x\ TRUE)y)$

$TRUE = \lambda x.\lambda y.x$

$FALSE = \lambda x.\lambda y.y$

OR FALSE TRUE

$= (\lambda x.\lambda y.((x\ TRUE)y)\ (\lambda x.\lambda y.y)(\lambda x.\lambda y.x)$

$= (\lambda x.\lambda y.((x\ TRUE)y)\ (\lambda a.\lambda b.b)\ (\lambda n.\lambda m.n)$

$= (\lambda y.(((\lambda a.\lambda b.b)TRUE)y))(\lambda n.\lambda m.n)$

$= (\lambda y.(((\lambda b.b)y))\ (\lambda n.\lambda m.n)$

$= (\lambda y.y)\ (\lambda n.\lambda m.n)$

$= \lambda n.\lambda m.n$

$= TRUE$

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(c) SUCC 2 = 3

$LHS = (\lambda z.\lambda f.\lambda y.f(zfy))(\lambda f.\lambda y.f(fy))$

$= (\lambda z.\lambda f.\lambda y.f(zfy))(\lambda a.\lambda b.a(ab))$

$= \lambda f.\lambda y.f((\lambda a.\lambda b.a(ab))fy)$

$= \lambda f.\lambda y.f((\lambda b.f(fb))y)$

$= \lambda f.\lambda y.f(f(f(y)))$

$= \lambda f.\lambda y.f(f(fy)))$ (Proved)

$= 3$

(d) $(Y\ FACT)\ 2 = 2$

Given $Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$

$FACT = \lambda f.\lambda n.\text{IF } n=0 \text{ THEN } 1 \text{ ELSE } n*(f(n-1))$

Now,

$Y\ FACT = FACT\ (Y\ FACT)$

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$\therefore\ (FACT\ (Y\ FACT))\ 2$

$= ((\lambda f.\lambda n.\text{IF } n=0 \text{ THEN } 1 \text{ ELSE } n*(f(n-1)))\ (Y\ FACT))\ 2$

$= (\lambda n.\text{IF } n=0 \text{ THEN } 1 \text{ ELSE } n*((Y FACT)(n-1)))\ 2$

$= \text{IF } 2=0 \text{ THEN } 1 \text{ ELSE } 2*((Y FACT)(1))$

$= 2*((Y FACT)(1))$ (As $2 \neq 0$)

~~Now~~

Now,

$(Y FACT)\ 1 = (FACT\ (Y\ FACT))\ 1$

$= ((\lambda n.\text{IF } n=0 \text{ THEN } 1 \text{ ELSE } n*((Y\ FACT)(n-1)))\ 1)$

PTO

$= (\text{IF } 1=0 \text{ THEN } 1 \text{ ELSE } 1^* (\text{YFACT } 0))$

$= (1^* (\text{YFACT}) \ 0)$ (As $1 \neq 0$)

NOW,

$(\text{Y FACT}) 0 = \text{FACT } (\text{Y FACT}) \ 0$

$= (\lambda n. \text{ IF } n=0 \text{ THEN } 1 \text{ ELSE } n^* (\text{Y FACT}(n-1)) \ 0)$

$= (\text{IF } 0=0 \text{ THEN } 1 \text{ ELSE } 0^* (\text{YFACT}(-1)))$

$= 1$ (As $0=0$)

Hence,

$(\text{Y FACT}) \ 2 = 2^* 1^* 1$

$= 2$ (Proved)

$= \text{RHS}$

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(e) PTO

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(e) $mul = \lambda n.\lambda m.\lambda x.(n(mx))$

we have,

$mul\ \bar{3}\ \bar{3}$

$= (\lambda n.\lambda m.\lambda x\ (n(mx)))\ \bar{3}\ \bar{3}$

$= (\lambda m.\lambda x\ (\bar{3}\ (mx)))\ \bar{3} \qquad (\bar{3} = \lambda g.\lambda y.(g(g(gy))))$

$= \lambda x.(\bar{3}\ (\bar{3}\ x))$

$= \lambda x.(\bar{3}\ ((\lambda g.\lambda y.g(g(gy)))\ x)$

$= \lambda x.(\bar{3}\ (\lambda y.(x(x(xy))))) \qquad (\bar{3} = \lambda f.\lambda z.(f(f(fz))))$

$= \lambda x.((\lambda f.\lambda z.\ f(f(fz)))\ (\lambda y.\ x(x(xy))))$

$= \lambda x.(\lambda z.(\lambda y.\ x(x(xy)))((\lambda y.\ x(x(xy)))((\lambda y.\ x(x(xy)))z))$

$= \lambda x.(\lambda z.(\lambda y.\ x(x(xy)))((\lambda y.x(x(xy)))(x(x(xz)))))$

$= \lambda x.(\lambda z.(\lambda y.\ x(x(xy)))(x(x(x(x(x(xz)))))))$

$= \lambda x.\lambda z\ (x(x(x(x(x(x(x(x(xz)))))))))$

$= \bar{9}$

(g) PTO

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(PART (f) done at the end)

(g) IF FALSE THEN x ELSE $y = y$

Given IF a THEN b ELSE $c = \lambda a.\lambda b.\lambda c\ abc$

TRUE $= \lambda x.\lambda y.x$

FALSE $= \lambda x.\lambda y.y$

$= (\lambda a.\lambda b.\lambda c\ abc)$ FALSE xy

$= (\lambda b.\lambda c.\ (\lambda m.\lambda n.n)\ bc)\ xy$

$= (\lambda c.(\lambda m.\lambda n.n)\ x c)\ y$

$= ((\lambda m.\lambda n.n)\ x y)$

$= (\lambda n.n)\ y$

$= y$

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(h)(i) We have, $\text{add} = \lambda n.\lambda m.\lambda f.\lambda x\ nf(mfx)$

LHS

$= \text{add}\ ab$

$= (\lambda n.\lambda m.\lambda f.\lambda x\ nf(mfx))\ a\ b$

$= (\lambda m.\lambda f.\lambda x\ a f(mfx))\ b$

$= \lambda f.\lambda x\ a f(bfx)$

Also,

$af = (\lambda g.\lambda y.g^a y)f = \lambda y.f^a y$

$bf = (\lambda g.\lambda z.g^b z)\ f = \lambda z.f^b z$

$\therefore\ b f x = f^b x$

$\therefore\ \lambda f. \lambda x\ (\lambda y\ f^a y)\ f^b x$

$= \lambda f. \lambda x\ f^a f^b x$

$= \lambda f. \lambda x\ f^{a+b} x \leftrightarrow (1)$

RHS

$=$ add b a

$= (\lambda n. \lambda m. \lambda f. \lambda x\ n f\ (m f x))\ b\ a$

$= \lambda f. \lambda x\ b f\ (a f x)$

$= \lambda f. \lambda x\ (\lambda y. f^b y)\ (f^a x)$

$= \lambda f. \lambda x\ f^b f^a x = \lambda f. \lambda x\ f^{b+a} x$

$= \lambda f. \lambda x\ f^{a+b} x$

$=$ eqn (1)

$=$ LHS

$\therefore$ Hence proved

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(ii) We have,

mul ab = mul ba,

$$mul = \lambda n.\lambda m.\lambda x\,(n(mx))$$

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<u>LHS</u>

$$= mul\ a\ b$$

$$= \lambda n.\lambda m.\lambda x\,(n(mx))\ a\ b$$

$$= \lambda m.\lambda x\,(a\,(mx))\ b$$

$$= \lambda x.\,(a\,(bx))$$

Now,

$$bx = (\lambda f.\lambda y.f^b y)\,x = \lambda y.\,x^b y$$

$$\therefore = \lambda x.\,(a\,(\lambda y.\,x^b y))$$

$$= \lambda x.\,((\lambda f.\lambda z.f^a z)\,(\lambda y.\,x^b y))$$

$$= \lambda x.\,(\lambda z.(\lambda y.\,x^b y)^a z)$$

$$= \lambda x.\,(\lambda z.\,((\lambda y.\,x^b y)(\lambda y.\,x^b y))\ \ldots\ a \text{ times} \ldots z)$$

$$= \lambda x.\,(\lambda z.\,x^b (\lambda y.\,x^b y) \ldots (a-1) \text{ times} \ldots z)$$

$$= \lambda x.(\lambda z.\,x^b\,x^{(a-1)b}\,z)$$

$$= \lambda x.\lambda z.\,x^{ab}\,z \longleftrightarrow (1)$$

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RHS

$= \text{mul} \; b \; a$

$= (\lambda n. \lambda m. \lambda x (n (m x))) \; b \; a$

$= \lambda x. (b (a x))$

$= \lambda x. ((\lambda f. \lambda z. f^b z) (\lambda y. x^a y))$

$= \lambda x. (\lambda z. (\lambda y. x^a y)^b z)$

$= \lambda x. (\lambda z. (\lambda y. x^a y) (\lambda y. x^a y) \ldots$ b times $\ldots z)$

$= \lambda x. (\lambda z. x (\lambda y. x^a y) \ldots$ (b-1) times $\ldots z)$

$= \lambda x. (\lambda z \; x^a x^a \ldots$ b times $\ldots z)$

$= \lambda x. \lambda z. x^{ab} z$

$=$ eq (1)

$=$ LHS

Hence proved

(f) Solve: add $\overline{8}$ $\overline{1}$

add $= \lambda n. \lambda m. \lambda f. \lambda x. n f (m f x)$

Now,

$(\text{add} \; \overline{8}) \; \overline{1} =$

PTO

$= (\lambda n. \lambda m. \lambda f. \lambda x. nf(mfx)) \ \bar{8} \ \bar{1}$

$\Big\downarrow$ β-reduction

$= \lambda f. \lambda x. \bar{8} \quad f (\bar{1} \ f x)$

$\Big\downarrow$ Def^n of church ~~num~~ numerals

$= \lambda f. \lambda x \ (\lambda f. \lambda x. f^8 x) f ((\lambda f. \lambda x. fx))$

$\Big\downarrow$ β-reduction

$= \lambda f. \lambda x. (\lambda f. \lambda x. f^8 x) \ f (f x)$

$\Big\downarrow$ β-reduction

$= \lambda f. \lambda x f^8 (f x)$

$= \lambda f. \lambda x. f^9. x$

$= \bar{9}$

$=$ RHS

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