Homework 8
Colorado CSCI 5454

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People I studied with for this homework: Gautam G, Praddunya

Problem 1

Solution:

π for S and T, where the graph contains a directed edge from S to T and A self-edge to itself to T.

let's take an example of the below graph:

The Normalized Adjacency Matrix WG is - $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

The above matrix $\pi(t) \approxeq 1$. Since there are no edges out of t, the probability of a vertex landing on t remains the same once it arrives t.

As we know that π is a stationary distribution if πW = π Let's assume the initial distribution to be $\begin{pmatrix} 1/2 & 1/2 \end{pmatrix}$

$\pi * W = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix} * \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$= \begin{pmatrix} 0 & 1 \end{pmatrix}$

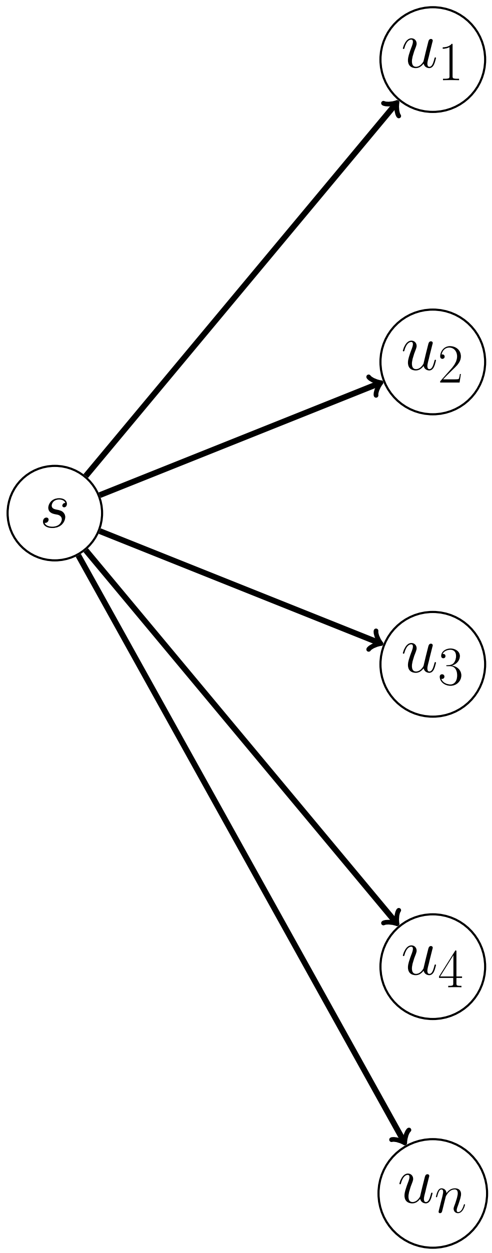
It will carry forward the same for the other examples, as we have the self-loop for the sink t. So, for normalized Adjacency Matrix has the value 1 in that position, when we multiply with W, it will have that value carry to that position. So π is the stationary distribution as it won't go to a different vertex, as it has no other vertex than itself.

From this, we can say that the probability is 1. and π is the stationary distribution.

Problem 2

Solution:

- Let's consider the start graph which we have already used previously



- Now the normalized adjacency matrix for the above graph is written below

$$
\begin{pmatrix}
 & u_1 & u_2 & u_3 & . & . & . & . & u_n \\
u_1 & 0 & 1/n & 1/n & . & . & . & . & 1/n \\
u_2 & 1 & 0 & 0 & . & . & . & . & 0 \\
u_3 & 1 & 0 & 0 & . & . & . & . & 0 \\
u_4 & 1 & 0 & 0 & . & . & . & . & 0 \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
u_n & 1 & 0 & 0 & . & . & . & . & 0
\end{pmatrix}
$$

- consider initial stationary distribution as $\begin{pmatrix} 1/2 & 1/2n & 1/2n & 1/2n & . & . & . & . & 1/2n \end{pmatrix}$
- if we perform $\pi W_G =$

$$
\begin{pmatrix} 1/2 & 1/2n & 1/2n & 1/2n & . & . & . & . & 1/2n \end{pmatrix} *
\begin{pmatrix}
 & u_1 & u_2 & u_3 & . & . & . & . & u_n \\
u_1 & 0 & 1/n & 1/n & . & . & . & . & 1/n \\
u_2 & 1 & 0 & 0 & . & . & . & . & 0 \\
u_3 & 1 & 0 & 0 & . & . & . & . & 0 \\
u_4 & 1 & 0 & 0 & . & . & . & . & 0 \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
u_n & 1 & 0 & 0 & . & . & . & . & 0
\end{pmatrix}
$$

$= \begin{pmatrix} 1/2 & 1/2n & 1/2n & 1/2n & . & . & . & . & 1/2n \end{pmatrix}$

- As after matrix multiplication we get π. So we can say that $\pi(W)$ is a stationary distribution.
- This can be claimed with certainty since the only other option, after reaching a leaf node, is to return to the central node and then leap to the leaf node. The central node won't have a 1/2 probability of choosing any of the leaf nodes in this scenario.
- So, from above our consideration for$\pi(w) = 0.5$.

Problem 3

lets say adjacency matrix:

$$
A_G[i,j] = \begin{cases} 1 & \text{if } i,j \in E \\ 0 & \text{otherwise} \end{cases}
$$

- As given $A_G[i,j] = A_G[j,i]$ as it's an undirected and an unweighted graph, it means $A_G[i,j] = A_G[j,i] = 1$.

- So, as $A_G[i,j] = A_G[j,i]$ for all the vertices, we can say that A_G will be equal to it's transpose. It means that $A_G = A_G^T$.

- So, as $A_G[i,j] = A_G[j,i]$ for all the vertices, we can say that A_G will be equal to it's transpose. So, $A_G = A_G^T$.

  we can say that:

$$B = A_G * A_G$$

$$=> B = A_G * A_G^T$$

- Now, we need the sum of all the elements at $[i,i]$ position. That means that we are finding the dot product of the i^{th} row and the i^{th} column.

- Now as we know that the adjacency matrix is equal to its transpose, we can say that the i^{th} row in the matrix and the i^{th} column in the transpose will be equal to each other.

- So basically when we consider the dot product of i^{th} row and the i^{th} column, we will be multiplying 1 with 1 and 0's with 0's (as we multiply with the same vector).

- So, we can say that:

$$B[i,i] = \text{sum of 1's in the } i^{th} \text{ row in the original adjacency matrix}$$

$$\sum_{i=1}^{n} B[i,i] = \text{sum of 1's in all the rows}$$

- Now we know that we have m edges. For suppose for an edge between i and j, we can say that it will have a 1 in the matrix in 2 places.

- They are $A_G[i,j]$ and $A_G[j,i]$. As we know that $i \neq j$, we can say that for m edges, there will be $2m$ 1's. So, the sum of all the 1's in the matrix will be,

$$\sum_{i=1}^{n} B[i,i] = \text{sum of 1's in all the rows}$$

$$= 2m * 1$$

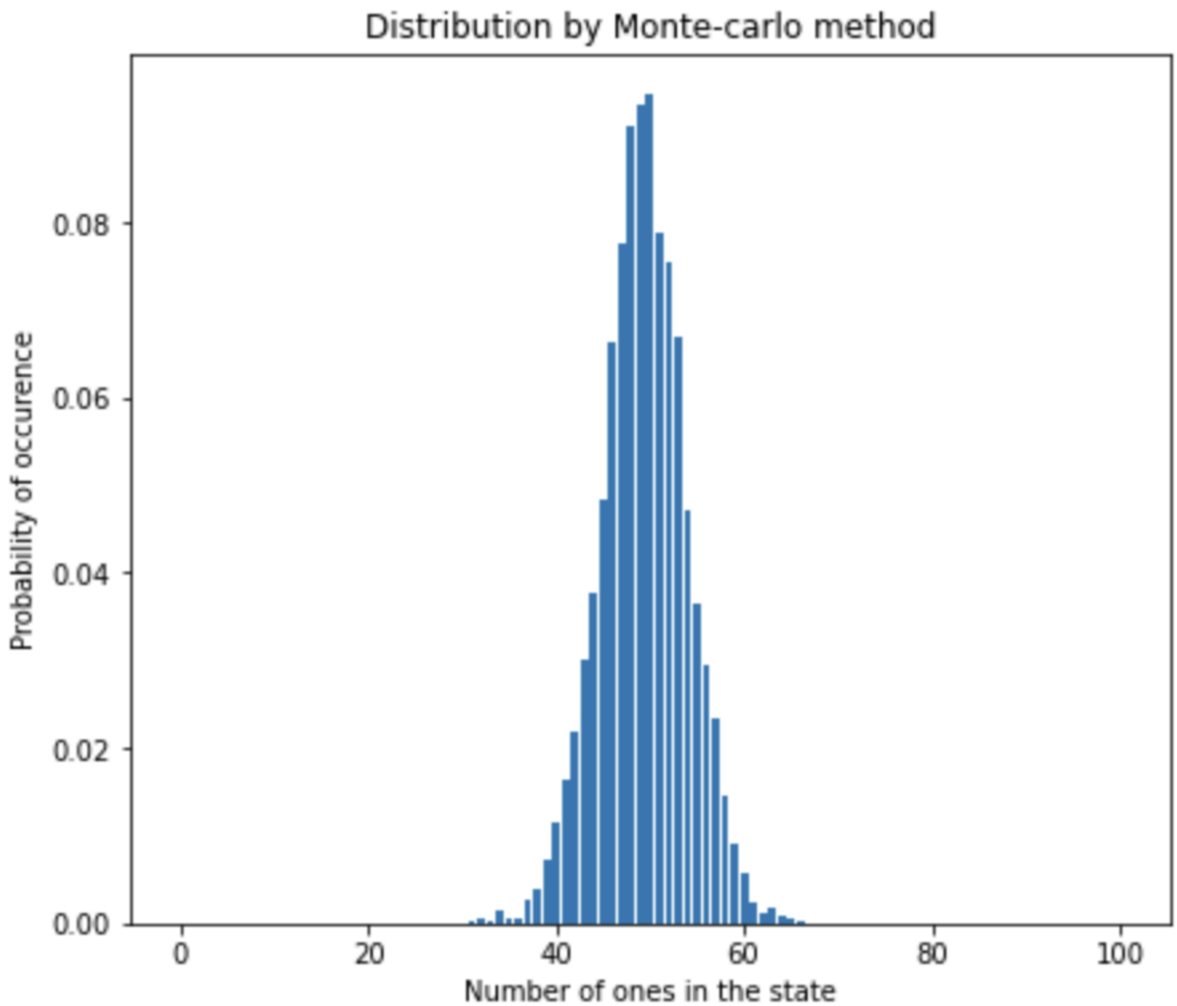
$$\sum_{i=1}^{n} B[i,i] = 2m$$

Problem 4

Part a

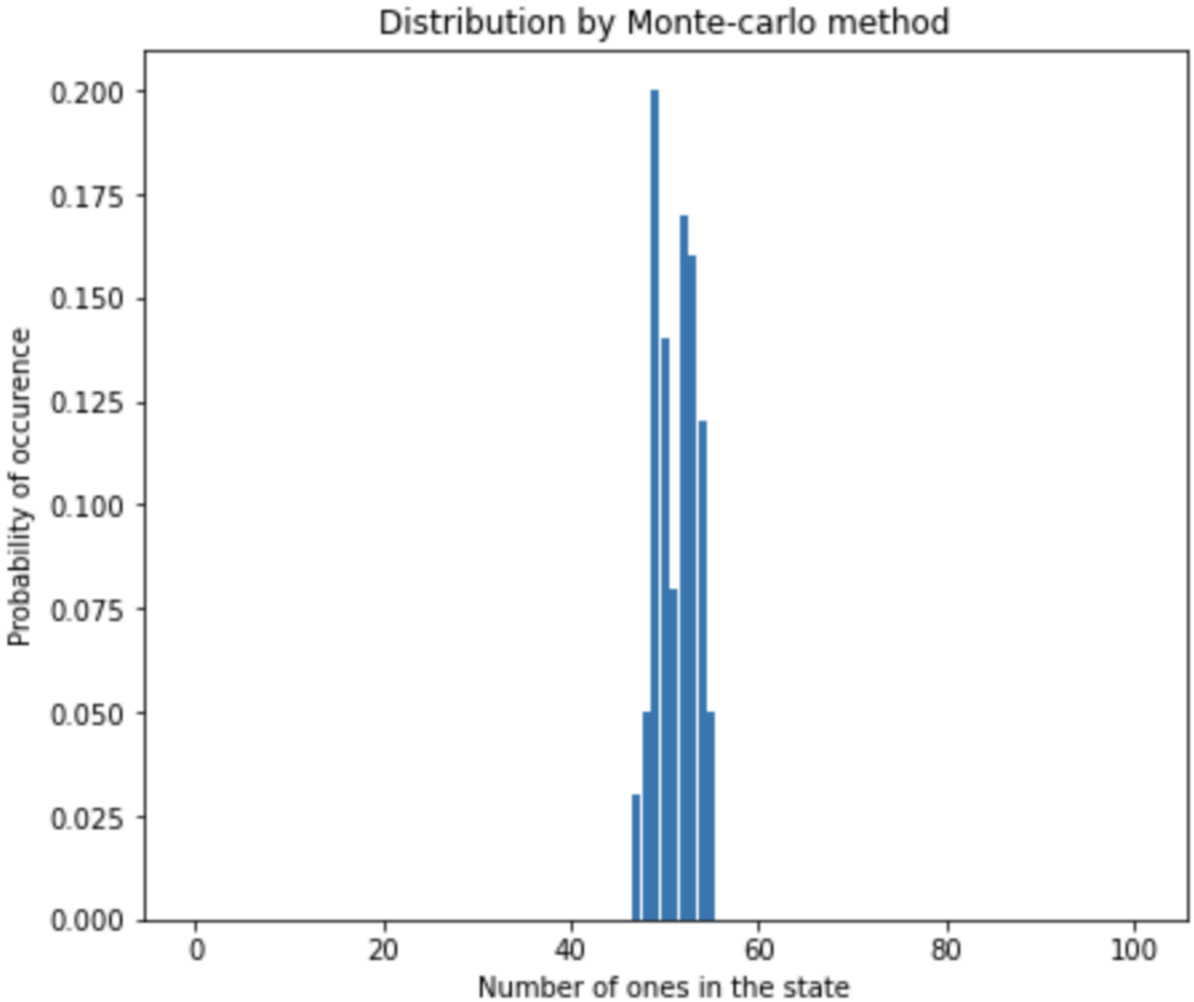
- The state of the system can be described by a vector $x \in \{0, 1\}^n$, i.e. a list of n bits. So the number of states of the system is 2^n.
- the graph in the Metropolis-Hastings algorithm. For this, we represent a state as a vertex and connect a state to all the states where they differ in their state at exactly one position. This is how we find neighbors of a state.
- So to find the degree of a vertex, we see that for a state, its neighbor can differ at one of the positions in its bits and there are n positions. So maximum degree r is n.

Part c



the histogram for distribution of Monte-carlo method is attached for T = 10000.

Part d



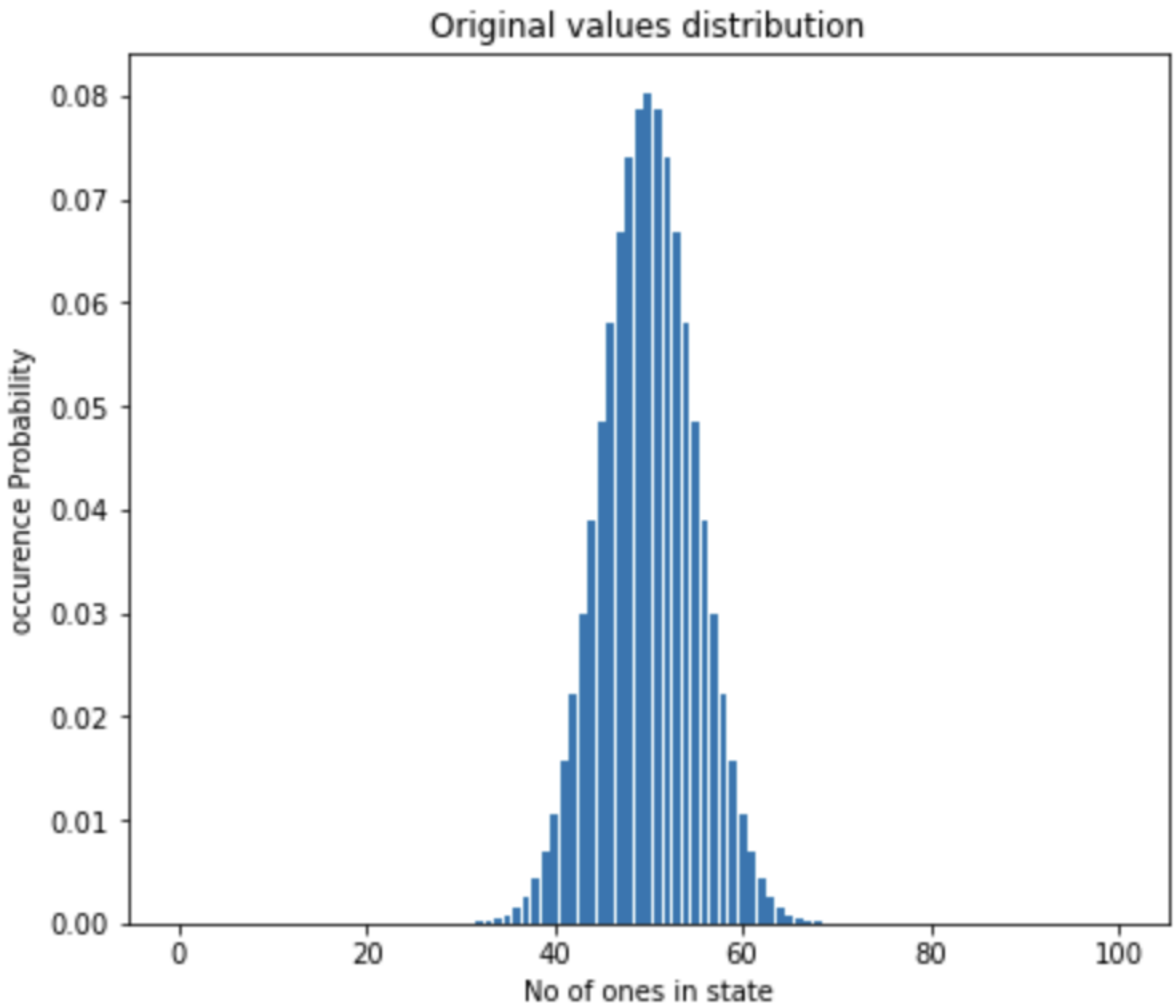
The histogram attached for T = 100 is right skewed. we know that original distribution is around 50. From above T=10000 is closer to the original distribution but for T=100 it's on the right skew.

part e

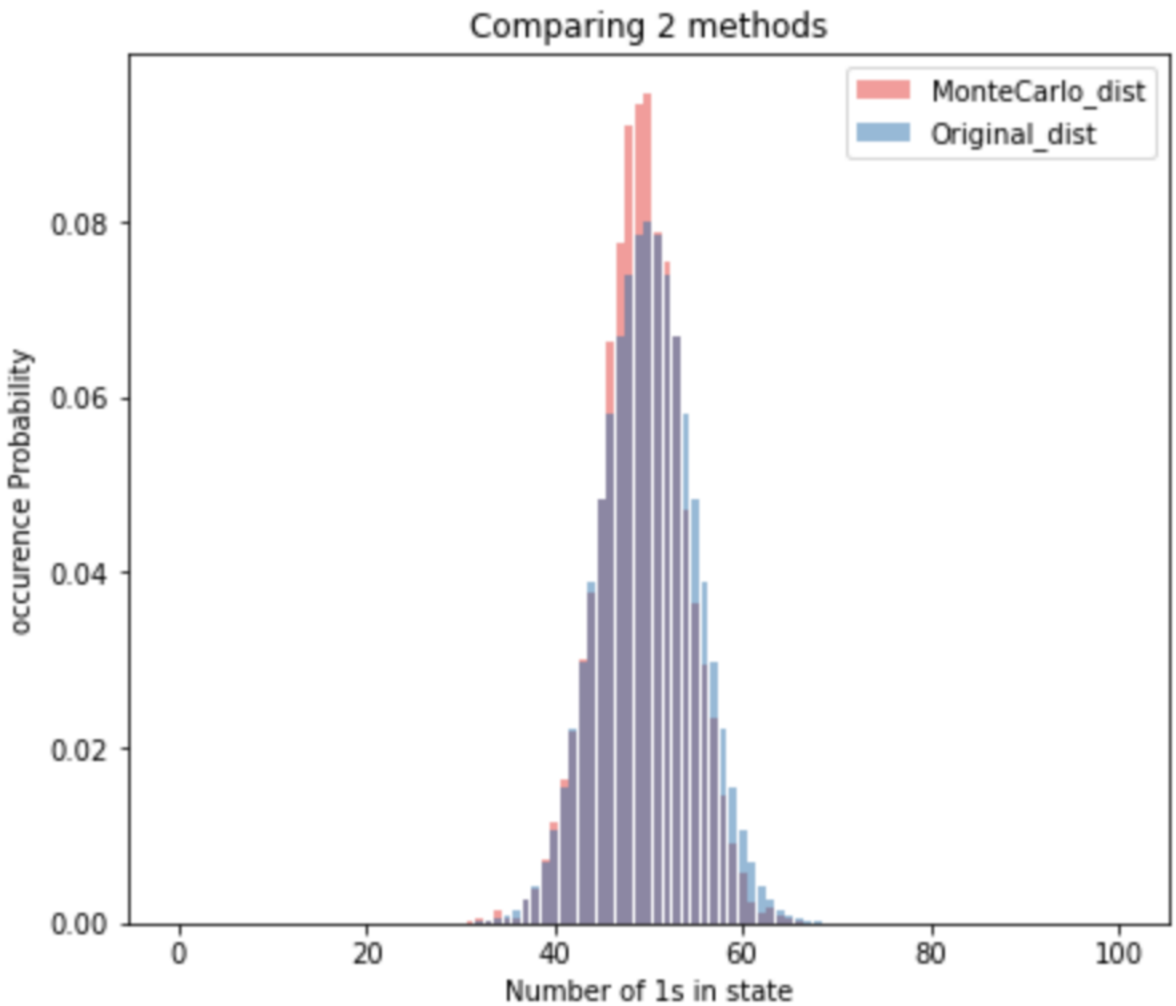
When we substitute w(x) with the (x/n) then we get the below equation. we need to calculate the original probability for the number of ones in the state.

So we have to choose x ones from n, these many states are possible which have the number of 1's as x. So the probability of this occurring is proportional to $(nx)H2(x/n)$ To get the original probability we need to scale this.

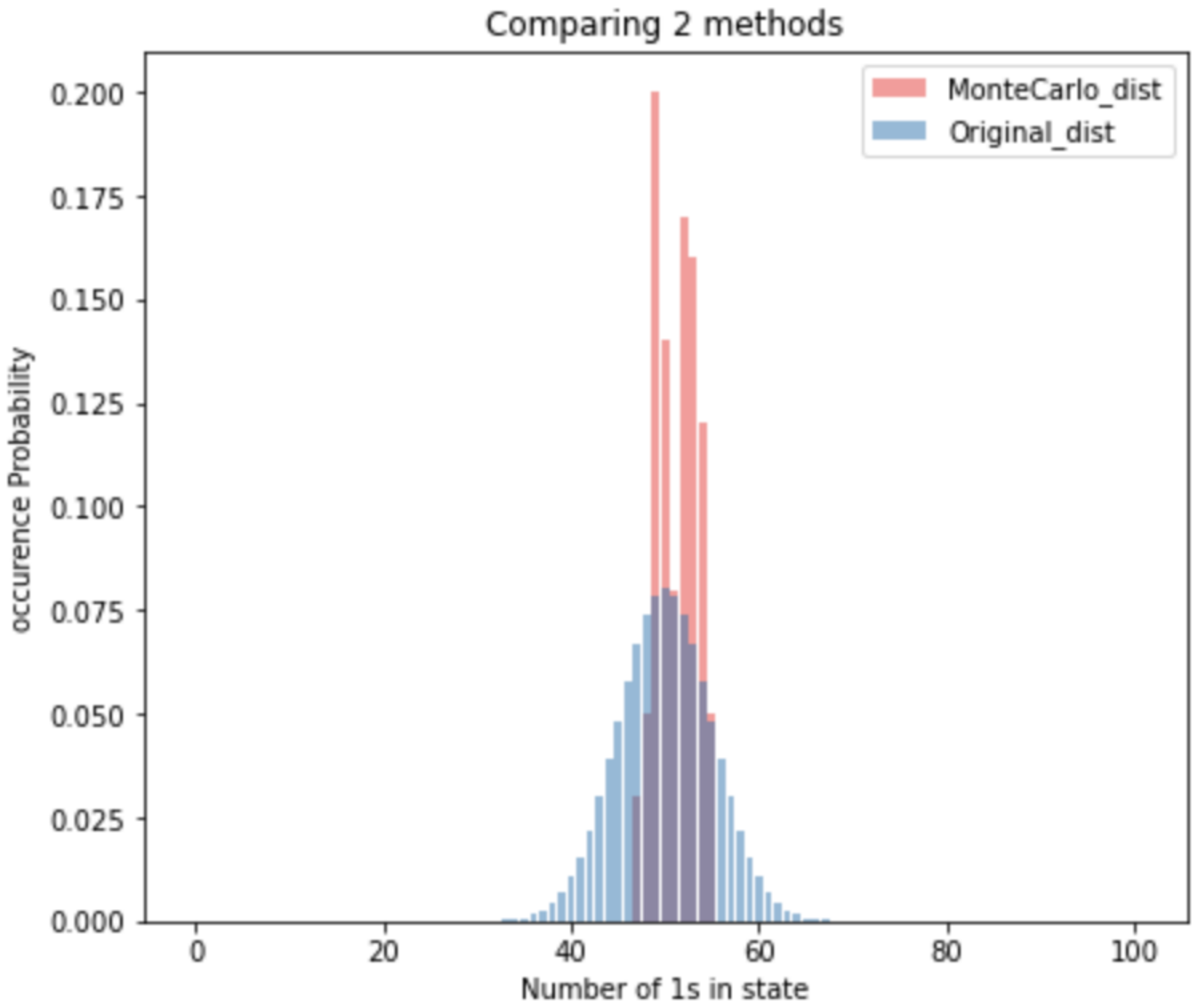
$$Pr(X = x) = \binom{n}{x} * H_2(x/n) / \sum_{x=0}^{n} \binom{n}{x} * H_2(x/n) \tag{1}$$



Below is the comparison with the original graph can be found below.



Comparision between two histograms for Original values vs MonteCarlo distribution when



T=100

Part b

The complete code will be there at the end of the pdf with all the distribution and graph codes