• $X \sim N(0,1) \Rightarrow X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

i) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = A$

$$Y = AX = \begin{bmatrix} 2X_1 \\ 2X_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$$\Sigma = AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \quad |\Sigma| = 16$$

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$f_y(y_1, y_2) = \frac{e^{-\frac{(y-\mu)^T \Sigma^{-1} (y-\mu)}{2}}}{\sqrt{(2\pi)^k |\Sigma|}}$$

$$(y)^T \Sigma^{-1} (y) = [y_1 \; y_2] \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{y_1}{4} & \frac{y_2}{4} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \left[\frac{y_1^2 + y_2^2}{4} \right]$$

$$f_y(y_1, y_2) = \frac{e^{-\frac{(y_1^2 + y_2^2)}{8}}}{(2\pi)(4)} = \frac{e^{-\frac{(y_1^2+y_2^2)}{8}}}{8\pi}$$

$$\frac{\partial f_y}{\partial y_1} = \frac{e^{-\frac{(y_1^2 + y_2^2)}{8}}}{8\pi} \left(-\frac{y_1}{4} \right)$$

$$\frac{\partial f_Y}{\partial y_2} = \frac{e^{-\frac{(y_1^2+y_2^2)}{8}}\left(\frac{-y_2}{4}\right)}{8\pi}$$

Along $y=x$, taking directional derivative,

$$\left[\left(\frac{\partial f_Y}{\partial y_1}\right)\hat{i} + \left(\frac{\partial f_Y}{\partial y_2}\right)\hat{j}\right]\cdot\left[\frac{x}{\sqrt{2}}\hat{i} + \frac{y}{\sqrt{2}}\hat{j}\right]$$

$$= \left[\frac{-e^{-\frac{y_1^2}{4}}y_1^2}{32\pi\sqrt{2}} - \frac{e^{-\frac{y_1^2}{4}}y_2^2}{32\pi\sqrt{2}}\right]$$

$$= -\left[\frac{e^{-\frac{y_1^2}{4}}y_1^2}{16\pi\sqrt{2}}\right]$$

Along $y=-x$, taking directional derivative,

$$\left[\left(\frac{\partial f_Y}{\partial y_1}\right)\hat{i} + \left(\frac{\partial f_Y}{\partial y_2}\right)\hat{j}\right]\cdot\left[\frac{x}{\sqrt{2}}\hat{i} - \frac{y}{\sqrt{2}}\hat{j}\right]$$

$$= \left[\frac{-e^{-\frac{y_1^2}{4}}y_1^2}{32\pi\sqrt{2}} - \frac{e^{-\frac{y_1^2}{4}}y_1^2}{32\pi\sqrt{2}}\right]$$

$$= -\left[\frac{e^{-\frac{y_1^2}{4}}y_1^2}{16\pi\sqrt{2}}\right]$$

As the directional derivative is equal along $y=x$ and $y=-x$, we see that the shape of the equi-contours of joint density is circle. (As variance along x-axis = Variance along y-axis)

ii) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$\Sigma = AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}, \quad |\Sigma| = 9$$

$$\Sigma^{-1} = \frac{1}{9} \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

$$f_y(y_1, y_2) = \frac{e^{-\frac{(y-\mu)^T \Sigma^{-1} (y-\mu)}{2}}}{\sqrt{(2\pi)^k |\Sigma|}}$$

$$(y)^T \Sigma^{-1} (y) = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} \frac{5}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{5y_1 - 4y_2}{9} & \frac{-4y_1 + 5y_2}{9} \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5y_1^2 - 8y_1 y_2 + 5y_2^2}{9} \end{bmatrix}$$

$$f_y(y_1, y_2) = \frac{e^{-\left(\frac{5y_1^2 - 8y_1 y_2 + 5y_2^2}{2 \times 9}\right)}}{2\pi(3)}$$

$$\frac{\partial f_y}{\partial y_1} = \frac{e^{-\left(\frac{5y_1^2 - 8y_1 y_2 + 5y_2^2}{2 \times 9}\right)}}{2\pi(3)} \begin{bmatrix} \frac{-10y_1 + 8y_2}{2 \times 9 \times 6\pi} \end{bmatrix} [6\pi]$$

$$\frac{\partial f_y}{\partial y_2} = \frac{e^{-\left(\frac{5y_1^2 - 8y_1 y_2 + 5y_2^2}{2 \times 9}\right)}}{2\pi(3)} \begin{bmatrix} \frac{8y_1 - 10y_2}{2 \times 9 \times 6\pi} \end{bmatrix} [6\pi]$$

Along $y=x$, taking directional derivative,

$$\left[\left(\frac{\partial f_y}{\partial y_1}\right)\hat{i}+\left(\frac{\partial f_y}{\partial y_2}\right)\hat{j}\right]\cdot\left[\frac{x}{\sqrt{2}}\hat{i}+\frac{x}{\sqrt{2}}\hat{j}\right]$$

$$=\left[\left(\frac{e^{-\frac{(5y_1^2-8y_1y_2+5y_2^2)}{2\times 9}}}{(2\pi)(3)}\left[\frac{8y_2-10y_1}{2\times 9\times 6\pi}\right]\frac{(y_1)}{\sqrt{2}}\right)(6\pi)\right.$$

$$\left.+\left(\frac{e^{-\frac{(5y_1^2-8y_1y_2+5y_2^2)}{2\times 9}}}{(2\pi)(3)}\left[\frac{-10y_2+8y_1}{9\times 2\times 6\pi}\right]\left(\frac{y_1}{\sqrt{2}}\right)\right)(6\pi)\right]$$

$$=-\frac{4y_1^2\, e^{-\frac{(5y_1^2-8y_1^2+5y_1^2)}{18}}}{6\pi(18)(\sqrt{2})}=-\frac{4y_1^2 e^{-\frac{2y_1^2}{18}}}{108\pi\sqrt{2}}=-\frac{4y_1^2 e^{-\frac{y_1^2}{9}}}{108\pi\sqrt{2}}$$

Along $y=-x$,

$$\left[\left(\frac{\partial f_y}{\partial y_1}\right)\hat{i}+\left(\frac{\partial f_y}{\partial y_2}\right)\hat{j}\right]\cdot\left[\frac{x}{\sqrt{2}}\hat{i}-\frac{x}{\sqrt{2}}\hat{j}\right]$$

$$=\left[\left(\frac{e^{-\frac{(5y_1^2-8y_1y_2+5y_2^2)}{18}}}{6\pi}\left[\frac{8y_2-10y_1}{18\sqrt{2}}\right]y_1\right)\right.$$

$$\left.-\left(\frac{e^{-\frac{(5y_1^2-8y_1y_2+5y_2^2)}{18}}}{6\pi}\left[\frac{-10y_2+8y_1}{18\sqrt{2}}\right]y_1\right)\right].$$

$$=-\frac{y_1^2\, e^{-\frac{(5y_1^2+5y_1^2+18y_1^2)}{18}}(2)}{6\pi\sqrt{2}}=-\frac{36y_1^2 e^{-\frac{18y_1^2}{18}}}{108\pi\sqrt{2}}$$

$\therefore$ Rate of fall is greater along $y=-x$ than $y=x$ initially, hence the contour is aligned along $y=x$

iii) $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$\Sigma = AA^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}, \quad |\Sigma| = 9$$

$$\Sigma^{-1} = \frac{1}{9} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$f_y(y_1, y_2) = \frac{e^{-\frac{(y-\mu)^T \Sigma^{-1} (y-\mu)}{2}}}{\sqrt{(2\pi)^k |\Sigma|}}$$

$$y^T \Sigma^{-1} y = [y_1 \quad y_2] \begin{bmatrix} \frac{5}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5y_1 + 4y_2}{9} & \frac{4y_1 + 5y_2}{9} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5y_1^2 + 8y_1 y_2 + 5y_2^2}{9} \end{bmatrix}$$

$$f_y(y_1, y_2) = \frac{e^{-\frac{(5y_1^2 + 8y_1 y_2 + 5y_2^2)}{2 \times 9}}}{2\pi(3)}$$

$$\frac{\partial f_Y}{\partial y_1} = \frac{e^{-\frac{(5y_1^2+8y_1y_2+5y_2^2)}{18}}}{(2\pi)(3)} \left[\frac{-10y_1-8y_2}{18}\right]$$

$$\frac{\partial f_Y}{\partial y_2} = \frac{e^{-\frac{(5y_1^2+8y_1y_2+5y_2^2)}{18}}}{(2\pi)(3)} \left[\frac{-8y_1-10y_2}{18}\right]$$

Along $y=x$, taking directional derivative,

$$\left[\left(\frac{\partial f_Y}{\partial y_1}\right)\hat{i} + \left(\frac{\partial f_Y}{\partial y_2}\right)\hat{j}\right] \cdot \left[\frac{x}{\sqrt{2}}\hat{i} + \frac{x}{\sqrt{2}}\hat{j}\right]$$

$$= \left[\left(\frac{e^{-\frac{(5y_1^2+8y_1y_2+5y_2^2)}{18}}}{6\pi}\left[\frac{-10y_1-8y_2}{18}\right]\left(\frac{y_1}{\sqrt{2}}\right)\right) + \left(\frac{e^{-\frac{(5y_1^2+8y_1y_2+5y_2^2)}{18}}}{6\pi}\left[\frac{-8y_1-10y_2}{18}\right]\left(\frac{y_1}{\sqrt{2}}\right)\right)\right]$$

$$= \frac{-\sqrt{2}y_1^2}{6\pi}\left[e^{-y_1^2}\right]$$

Along $y=-x$, taking directional derivative,

$$\left[\left(\frac{\partial f_y}{\partial y_1}\right)\hat{i}+\left(\frac{\partial f_y}{\partial y_2}\right)\hat{j}\right]\cdot\left[\frac{x}{\sqrt{2}}\hat{i}-\frac{x}{\sqrt{2}}\hat{j}\right]$$

$$=\left[\left(\frac{e^{-\frac{(5y_1^2+8y_1y_2+5y_2^2)}{18}}}{6\pi}\left[\frac{-10y_1-8y_2}{18}\right]\left(\frac{y_1}{\sqrt{2}}\right)\right)\right.$$

$$\left.-\left(\frac{e^{-\frac{(5y_1^2+8y_1y_2+5y_2^2)}{18}}}{6\pi}\left[\frac{-8y_1-10y_2}{18}\right]\left(\frac{y_1}{\sqrt{2}}\right)\right)\right]$$

$$=\frac{-\sqrt{2}\,y_1^2 e^{-\frac{y_1^2}{9}}}{(6\pi)(9)}$$

$\therefore$ Rate of fall is greater along $y=x$ than $y=-x$ initially, hence the contour is aligned along $y=-x$