Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3−scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.com/gadepall/EE1310/master/filter/codes/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?

Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.

Solution:

```
{
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   "execution_count": 1,
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    "import␣soundfile␣as␣sf\n",
    "from␣scipy␣import␣signal\n",
    "\n",
    "#read␣.wav␣file␣\n",
    "input_signal,fs␣=␣sf.read('filter_codes_Sound_Noise.wav')␣\n",
    "\n",
    "#sampling␣frequency␣of␣Input␣signal\n",
    "sampl_freq=fs\n",
    "\n",
    "#order␣of␣the␣filter\n",
    "order=4␣␣␣\n",
    "\n",
    "#cutoff␣frquency␣4kHz\n",
    "cutoff_freq=4000.0␣␣\n",
    "\n",
    "#digital␣frequency\n",
    "Wn=2*cutoff_freq/sampl_freq␣␣\n",
    "\n",
    "#␣b␣and␣a␣are␣numerator␣and␣denominator␣polynomials␣respectively\n",
    "b,␣a␣=␣signal.butter(order,Wn,␣'low')␣\n",
    "\n",
```

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```
    "#filter the input signal with butterworth filter\n",
    "output_signal = signal.filtfilt(b, a, input_signal)\n",
    "#output_signal = signal.lfilter(b, a, input_signal)\n",
    "\n",
    "#write the output signal into .wav file\n",
    "sf.write('Sound_With_ReducedNoise.wav', output_signal, fs) "
   ]
  },
  {
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   "version": "3.9.12"
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   }
  }
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}
```


2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?

Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), \quad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.

Solution: The following code yields Fig. 3.2.

```
wget https://github.com/gadepall/EE1310/raw/master/filter/codes/xnyn.py
```

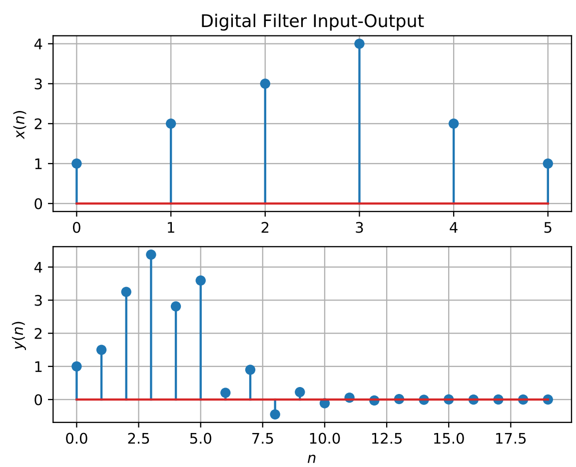

Fig. 3.2

3.3 Repeat the above exercise using a C code.

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1. **Solution:**

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{2}{z^4} + \frac{1}{z^5} \tag{4.7}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.8}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.9}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.10}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.12}$$

is

$$U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.13}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.14}$$

and from (4.12),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.15}$$

$$= \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.16}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-az^{-1}} \quad |z| > |a| \tag{4.17}$$

Solution: On applying Z-transform:

$$A(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.18}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{z^n}, \quad |z| > |a| \tag{4.19}$$

$$= \frac{1}{1-\frac{a}{z}} \tag{4.20}$$

Using the formula for infinite geometric progression with $r = \frac{a}{z}$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.21}$$

Plot $|H\left(e^{\jmath\omega}\right)|$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discrete Time Fourier Transform* (DTFT) of $h(n)$.
Solution: The following code plots Fig. 4.6.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/dtft.
    ipynb
```

$$H\left(z\right) = \frac{1+z^2}{z^2+\frac{z}{2}} \tag{4.22}$$

$$H(e^{\jmath\omega}) = \frac{1+e^{2\jmath\omega}}{e^{2\jmath\omega}+\frac{e^{\jmath\omega}}{2}} \tag{4.23}$$

$$H(e^{\jmath\omega}) = \frac{2e^{\jmath\omega}\left(e^{\jmath\omega}+e^{-\jmath\omega}\right)}{2e^{2\jmath\omega}+e^{\jmath\omega}} \tag{4.24}$$

$$H(e^{\jmath\omega}) = \frac{4cos(w)}{1+2e^{\jmath\omega}} \tag{4.25}$$

$$H\left(e^{\jmath\omega}\right) = \frac{cos(w)}{1+2cos(w)+2jsin(w)} \tag{4.26}$$

$$|H\left(e^{\jmath\omega}\right)| = \frac{|cos(w)|}{\sqrt{5+4cos(w)}} \tag{4.27}$$

Therefore, the period is 2π

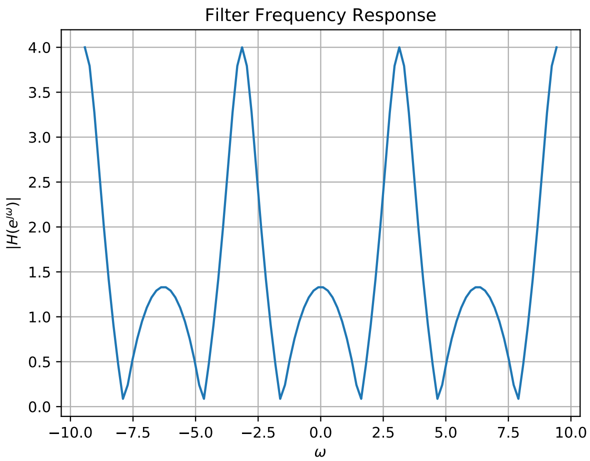

Fig. 4.6: $|H(e^{\jmath\omega})|$

4.7 Express $h(n)$ in terms of $H(e^{\jmath\omega})$.

We know that,

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)e^{-\jmath nz} \tag{4.28}$$

Now,

$$\int_{-\pi}^{\pi} H(z)e^{\jmath kz}\,dz \tag{4.29}$$

$$= \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} h(n)e^{-\jmath nz}e^{\jmath kz}\,dz \tag{4.30}$$

If n = k, the above integral evaluates to:

$$2\pi h(k) = 2\pi h(n) \tag{4.31}$$

If n ≠ k, the above integral evaluates to:

$$\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} h(n)e^{\jmath z(k-n)}\,dz \tag{4.32}$$

$$= \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{\jmath z(k-n)}\,dz \tag{4.33}$$

$$= \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} cos(z(k-n)) + \jmath sin(z(k-n))\,dz \tag{4.34}$$

Taking z(k-n) = t,

$$\sum_{n=-\infty}^{\infty} h(n) \int_{-\pi(k-n)}^{\pi(k-n)} cos(t) + \jmath sin(t)\,dt \tag{4.35}$$

$$= 0 \tag{4.36}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.10).
Solution:

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{z^{-1}}{2}} \tag{5.2}$$

Let

$$z^{-1} = t \tag{5.3}$$

Hence,

$$1 + t^2 = (2t - 4)(1 + \frac{t}{2}) + 5 \tag{5.4}$$

$$H(z) = (2z^{-1} - 4) + \frac{5}{1 + \frac{1}{2z}} \tag{5.5}$$

$$= \frac{2}{z} - 4 + 5 - \frac{5}{2z} + \frac{5}{4z^2} + ... \tag{5.6}$$

$$= 1 - \frac{1}{2z} + \frac{5}{4z^2} - \frac{5}{8z^3} + \frac{5}{16z^4} \tag{5.7}$$

Hence, by comparing with the coefficients with the definition of z-transform

$$h(0) = 1 \tag{5.8}$$

$$h(1) = -\frac{1}{2} \tag{5.9}$$

$$h(2) = \frac{5}{4} \tag{5.10}$$

$$h(3) = -\frac{5}{8} \tag{5.11}$$

$$h(4) = \frac{5}{16} \tag{5.12}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.13}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: From (4.10),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.14}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.15}$$

For $\left|\frac{1}{2z}\right| < 1$ or $|z| > \frac{1}{2}$ using (4.17) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots Fig. 5.3.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/hn.py
```

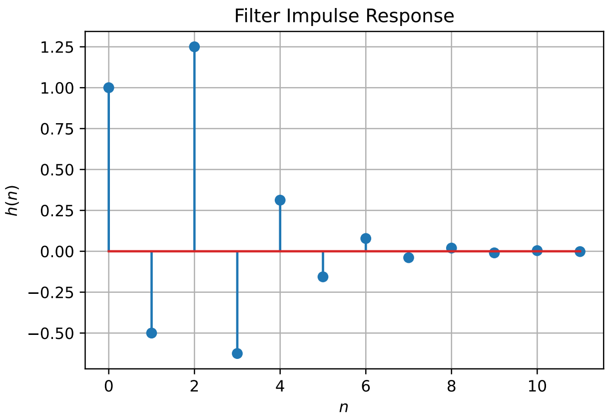

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

h(n) is bounded as it tends to 0 as n tends to 12

5.4 Convergent? Justify using the ratio test.
Solution:

$$h(n) = (\frac{-1}{2})^{n}u(n) + (\frac{-1}{2})^{n-2}u(n-2) \tag{5.16}$$

As n $\rightarrow \infty$,

$$h(n)/h(n-1) = \frac{(\frac{-1}{2})^{n} + (\frac{-1}{2})^{n-2}}{(\frac{-1}{2})^{n-1} + (\frac{-1}{2})^{n-3}} = -\frac{1}{2} \tag{5.17}$$

As $\left|\frac{h(n)}{h(n-1)}\right|$<1, the series is convergent due to which it is bounded.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.18}$$

Is the system defined by (3.2) stable for the impulse response in (5.13)?
Solution:

$$\sum_{n=0}^{n=11} h(n) = 1-0.5+1.25-0.625+0.3125-0.15625+0.07812 \tag{5.19}$$

5.6 Verify the above result using a python code.

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.20}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/hndef
    .ipynb
```

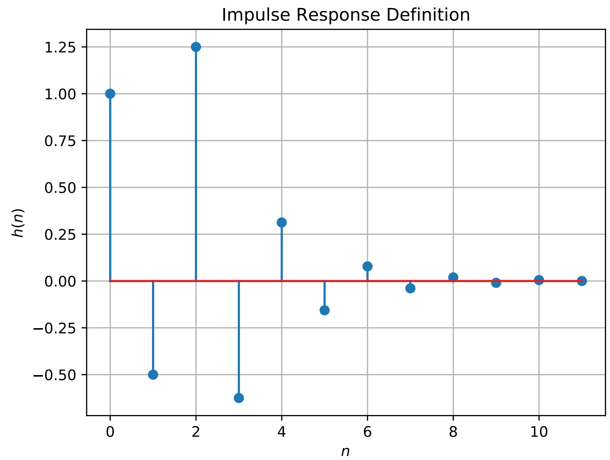

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{5.21}$$

Comment. The operation in (5.21) is known as *convolution*.
Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/
    ynconv.ipynb
```

5.9 Express the above convolution using a Teoplitz matrix.

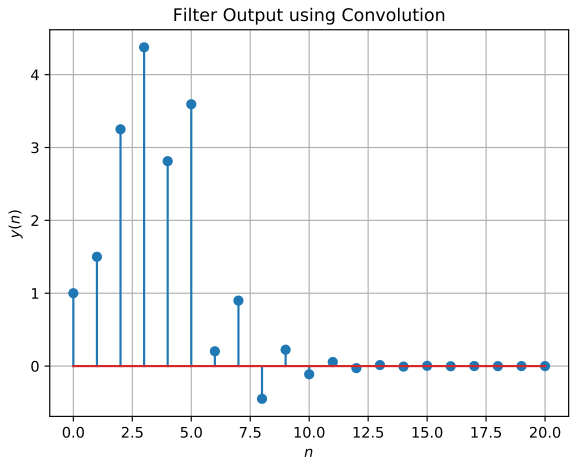

Fig. 5.8: $y(n)$ from the definition of convolution

5.10 Show that

$$y(n)=\sum_{n=-\infty}^{\infty}x(n-k)h(k) \tag{5.22}$$

Solution: We know that,

$$y(n)=\sum_{n=-\infty}^{\infty}x(k)h(n-k) \tag{5.23}$$

Taking k = n-k

$$y(n)=\sum_{n=-\infty}^{\infty}x(n-k)h(k) \tag{5.24}$$

5.11 Oppenheimer question 3.10(e):

$$x[n]=u[n+10]-u[n+5] \tag{5.25}$$

$$=\begin{cases}1 & -10<=n<-6\\ 0 & otherwise\end{cases} \tag{5.26}$$

x[n] is finite length and has only positive powers of z, hence, its ROC is $|z|<\infty$

6 DFT and FFT

6.1 Compute

$$X(k)\triangleq\sum_{n=0}^{N-1}x(n)e^{-\jmath 2\pi kn/N},\quad k=0,1,\dots,N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

6.2 Compute

$$Y(k)=X(k)H(k) \tag{6.2}$$

6.3 Compute

$$y(n)=\frac{1}{N}\sum_{k=0}^{N-1}Y(k)\cdot e^{\jmath 2\pi kn/N},\quad n=0,1,\dots,N-1 \tag{6.3}$$

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/yndft.
    ipynb
```

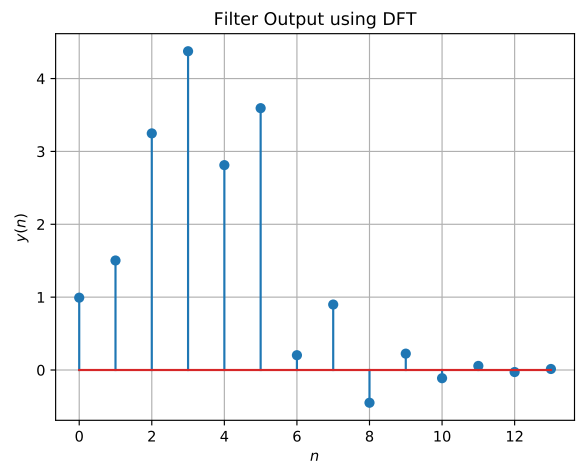

Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k),H(k)$ and $y(n)$ through FFT and IFFT.

6.5 Wherever possible, express all the above equations as matrix equations.

6.6 Verify the above equations by generating the DFT matrix in python.

7 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

7.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M}a(m)\,y(n-m)=\sum_{k=0}^{N}b(k)\,x(n-k) \tag{7.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

7.2 Repeat all the exercises in the previous sections for the above a and b.

7.3 What is the sampling frequency of the input signal?

Solution: Sampling frequency(fs)=44.1kHZ.

7.4 What is type, order and cutoff-frequency of the above butterworth filter

Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

7.5 Modifying the code with different input parameters and to get the best possible output.