Some Notes on Fibonacci Sequence

Appi's Coolest Brother

Nov 12, 2016

Definitions

The Fibonacci Sequence

A **sequence** of numbers is simply an ordered list (finite or infinite) of numbers. So $\pi, 2, 2.5$ is a sequence, as is $1, 0, 1, 0, 1, 0, ...$ You get the idea. We define the **Fibonacci sequences** to be the sequence of counting numbers (f_n) formed by the following rules:

$$f_0 = 1 \tag{1}$$

$$f_1 = 1 \tag{2}$$

$$f_{n+2} = f_{n+1} + f_n \tag{3}$$

So we can see that the sequence goes $1, 1, 2, 3, 5, 8, 13, 21, ...$ and so on.

Growth Rates

If we have the sequence $2, 4, 8, 16, ...$ we can see that each number is twice the previous one. Motivated by this observation, we can define the **growth rate** to be the ratio of a number in the sequence to its predecessor. For sequences like the powers of 2, this happens to correspond to the *actual* ratios exactly, in that $2 = \frac{8}{4} = \frac{16}{8} = \frac{32}{16}....$ However, such a ratio is not always the case. Clearly, the Fibonacci numbers don't have a nice, exact ratio for each pair of numbers. However, we claim that as we get further and further down the sequence, the ratio eventually stabilizes. Formally, we can say

$$R = \lim_{n\to\infty} \frac{f_{n+1}}{f_n} \tag{4}$$

Don't worry if this notation seems confusing. All we're saying is that as we get further down the sequence, we're interested in the ratio of a Fibonacci number to the one that came before it.

Notice that for some sequences, it doesn't make sense to try to find this number! The sequence $1, 2, 1, 2, 1, 2, ...$ has two candidate for such a ratio, namely $2, \frac{1}{2}$. But which one is it? It turns out that no matter how far we go, since these ratios keep oscillating, it's not meaningful to try to find the ratio. Formally, we say that the *sequence diverges*.

The Theorem

Claim: $R = \lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2}$, AKA the Golden Ratio.

Proof: If such a limit exists, then call it R. We'll do some manipulation first:

$$R = \lim_{n\to\infty} \frac{f_{n+1}}{f_n} \tag{5}$$

$$= \lim_{n\to\infty} \frac{f_n + f_{n-1}}{f_n} \qquad \text{since } f_{n+1} = f_n + f_{n-1} \tag{6}$$

$$= \lim_{n\to\infty} (\frac{f_{n-1}}{f_n} + 1) \tag{7}$$

$$= \frac{1}{R} + 1 \tag{8}$$

This last step might seem a bit dubious. But notice that $\frac{f_{n-1}}{f_n} = (\frac{f_n}{f_{n-1}})^{-1}$. Next, notice that as n goes to ∞, so do $n+1, n-1$, so we don't really care about the fact that we're comparing f_{n-1} to f_n instead of f_{n+1} to f_n. So, to "expand out" our step more:

$$\lim_{n\to\infty} \frac{f_{n-1}}{f_n} = \lim_{n\to\infty} \frac{1}{\frac{f_n}{f_{n-1}}} \tag{9}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{f_{n+1}}{f_n}} \tag{10}$$

$$\frac{1}{R} \tag{11}$$

$$\tag{12}$$

Now, returning to the fact that $R = \frac{1}{R} + 1$ we can do some simple algebra to finish.

$$R = 1 + \frac{1}{R} \tag{13}$$

$$R^2 = R + 1 \tag{14}$$

$$R^2 - R - 1 = 0 \tag{15}$$

Now, we simply use the quadratic formula to find that $R = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$. Finally, we toss out $\frac{1-\sqrt{5}}{2}$ as a candidate for the answer since this is a negative number, and all Fibonacci numbers are positive. So our final answer is:

$$R = \frac{1+\sqrt{5}}{2} \tag{16}$$