Sensing and Planning for Autonomous Vehicles

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Linear Dynamical Systems

$$\mathbf{x}_t = \mathbf{F}_t \mathbf{x}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \mathbf{w}_t,$$

- $\mathbf{x}_t$ is the state vector containing the terms of interest for the system (e.g., position, velocity, heading) at time t

- $\mathbf{B}_t$ is the control input matrix which applies the effect of each control input parameter in the vector $\mathbf{u}_t$ on the state vector (e.g., applies the effect of the throttle setting on the system velocity and position)

- $\mathbf{F}_t$ is the state transition matrix which applies the effect of each system state parameter at time $t-1$ on the system state at time t (e.g., the position and velocity at time $t-1$ both affect the position at time t)

- $\mathbf{w}_t$ is the vector containing the process noise terms for each parameter in the state vector. The process noise is assumed to be drawn from a zero mean multivariate normal distribution with covariance given by the covariance matrix $\mathbf{Q}_t$.

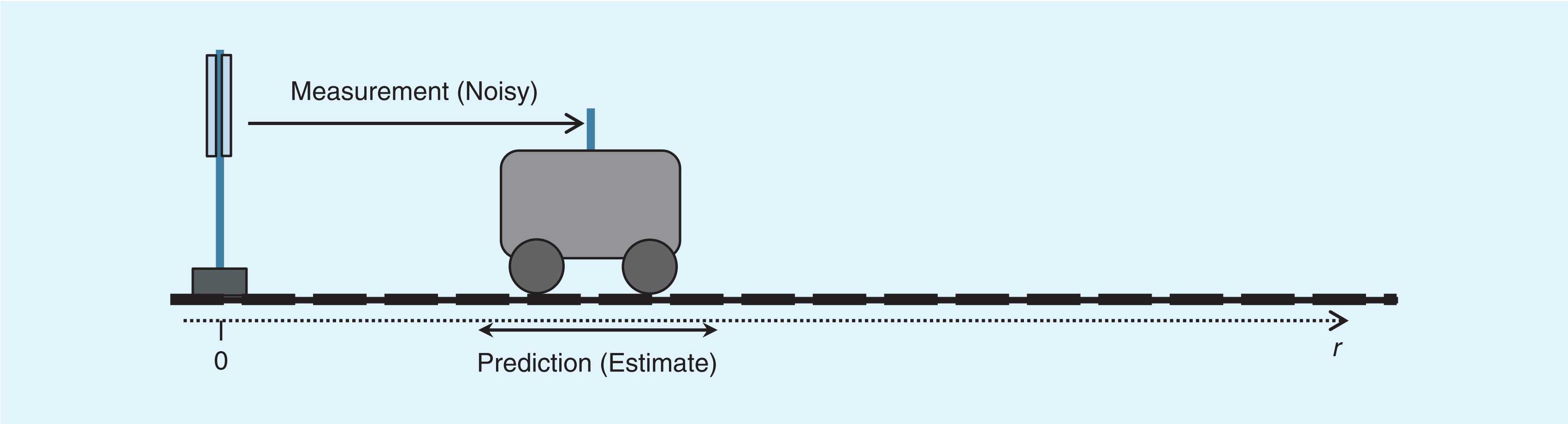
- $\mathbf{u}_t$ is the vector containing any control inputs (steering angle, throttle setting, braking force)

Measurement

$$\mathbf{z}_t = \mathbf{H}_t \mathbf{x}_t + \mathbf{v}_t\,,$$

- $\mathbf{z}_t$ is the vector of measurements
- $\mathbf{H}_t$ is the transformation matrix that maps the state vector parameters into the measurement domain
- $\mathbf{v}_t$ is the vector containing the measurement noise terms for each observation in the measurement vector. Like the process noise, the measurement noise is assumed to be zero mean Gaussian white noise with covariance $\mathbf{R}_t$.

Example



$$\mathbf{x}_t = \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix}.$$

$$x_t = x_{t-1} + (\dot{x}_{t-1} \times \Delta t) + \frac{f_t (\Delta t)^2}{2m}$$

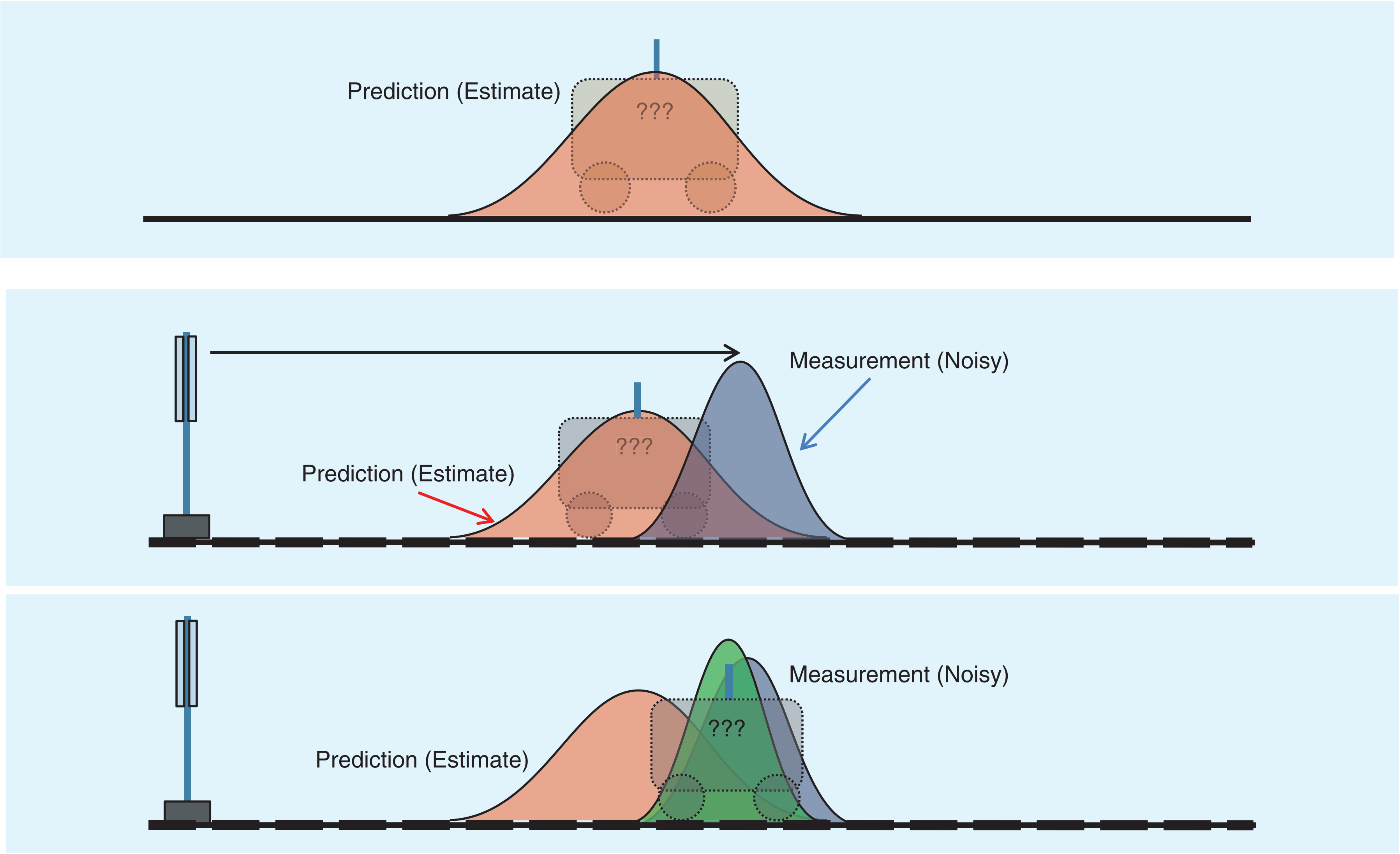
$$\dot{x}_t = \dot{x}_{t-1} + \frac{f_t \Delta t}{m}.$$

$$\begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \dot{x}_{t-1} \end{bmatrix} + \begin{bmatrix} \frac{(\Delta t)^2}{2} \\ \Delta t \end{bmatrix} \frac{f_t}{m}.$$

$$\mathbf{u}_t = \frac{f_t}{m}.$$

$$\mathbf{F}_t = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \text{and } \mathbf{B}_t = \begin{bmatrix} \frac{(\Delta t)^2}{2} \\ \Delta t \end{bmatrix}.$$

Position and Measurement



Scalar Kalman Filter

System $x_j = a x_{j-1} + b u_j + w_j$

Measurement $z_j = h x_j + v_j$

Prediction $\hat{x}_j^- = a \hat{x}_{j-1} + b u_j$

A priori covariance $p_j^- = a^2 p_{j-1} + Q$

Kalman Gain $k_j = \dfrac{h p_j^-}{h^2 p_j^- + R}$

Measurement Update $\hat{x}_j = \hat{x}_j^- + k_j \left(z_j - h \hat{x}_j^- \right)$

A posteriori covariance $p_j = p_j^- \left(1 - h k_j \right)$

Kalman Filter

System $$\mathbf{x}_t = \mathbf{F}_t \mathbf{x}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \mathbf{w}_t ,$$

Measurement $$\mathbf{z}_t = \mathbf{H}_t \mathbf{x}_t + \boldsymbol{\nu}_t ,$$

Prediction $$\hat{\mathbf{x}}_{t|t-1} = \mathbf{F}_t \hat{\mathbf{x}}_{t-1|t-1} + \mathbf{B}_t \mathbf{u}_t$$

A priori covariance $$\mathrm{P}_{t|t-1} = \mathrm{F}_t \mathrm{P}_{t-1|t-1} \mathbf{F}_t^{\mathrm{T}} + \mathrm{Q}_t ,$$

Kalman Gain $$\mathbf{K}_t = \mathbf{P}_{t|t-1} \mathbf{H}_t^{\mathbf{T}} (\mathbf{H}_t \mathbf{P}_{t|t-1} \mathbf{H}_t^{\mathbf{T}} + \mathbf{R}_t)^{-1} .$$

Measurement Update $$\hat{\mathbf{x}}_{t|t} = \hat{\mathrm{x}}_{t|t-1} + \mathrm{K}_t (\mathrm{z}_t - \mathbf{H}_t \hat{\mathbf{x}}_{t|t-1})$$

A posteriori covariance $$\mathbf{P}_{t|t} = \mathbf{P}_{t|t-1} - \mathbf{K}_t \mathbf{H}_t \mathbf{P}_{t|t-1} ,$$

Scalar Value

$$x_j = a x_{j-1} + b u_j$$

$$\mathbf{x}_t = \mathbf{F}_t \mathbf{x}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \mathbf{w}_t ,$$

$$x_j = a x_{j-1} + b u_j + w_j$$

$$z_j = h x_j + v_j$$

$$\mathbf{z}_t = \mathbf{H}_t \mathbf{x}_t + \boldsymbol{\nu}_t ,$$

$$e_j^- = x_j - \hat{x}_j^-$$

$$e_j = x_j - \hat{x}_j$$

$$\hat{x}_j^- = a \hat{x}_{j-1} + b u_j$$

$$p_j^- = E\left\{\left(e_j^-\right)^2\right\}$$

$$\text{Residual} = z_j - \hat{z}_j = z_j - h \hat{x}_j^-$$

$$p_j = E\left\{\left(e_j\right)^2\right\}$$

$$\hat{x}_j = \hat{x}_j^- + k(\text{Residual}) = \hat{x}_j^- + k\left(z_j - h \hat{x}_j^-\right)$$

Prediction covariance derivation

$$
\begin{aligned}
p_j^- &= E\left\{\left(x_j - \hat{x}_j^-\right)^2\right\} \\
&= E\left\{\left(a x_{j-1} + b u_j + w_j - \left(a \hat{x}_{j-1} + b u_j\right)\right)^2\right\} \\
&= E\left\{\left(a\left(x_{j-1} - \hat{x}_{j-1}\right) + w_j\right)^2\right\} \\
&= E\left\{a^2\left(x_{j-1} - \hat{x}_{j-1}\right)^2 + 2aw_j\left(x_{j-1} - \hat{x}_{j-1}\right) + w_j^2\right\}
\end{aligned}
$$

$$
E\left\{2aw_j\left(x_{j-1} - \hat{x}_{j-1}\right)\right\} = 2aE\left\{w_j\, e_{j-1}\right\} = 0
$$

$$
\begin{aligned}
p_j^- &= a^2 E\left\{\left(x_{j-1} - \hat{x}_{j-1}\right)^2\right\} + E\left\{w_j^2\right\} \\
&= a^2 p_{j-1} + Q
\end{aligned}
$$

Prediction

$$\mathbf{x}_t = \mathbf{F}_t \mathbf{x}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \mathbf{w}_t,$$

$$\hat{\mathbf{x}}_{t|t-1} = \mathbf{F}_t \hat{\mathbf{x}}_{t-1|t-1} + \mathbf{B}_t \mathbf{u}_t$$

$$\mathbf{P}_{t|t-1} = \mathbf{F}_t \mathbf{P}_{t-1|t-1} \mathbf{F}_t^{\mathrm{T}} + \mathbf{Q}_t,$$

$$P_{t|t-1} = \mathrm{E}[(x_t - \hat{x}_{t|t-1})(x_t - \hat{x}_{t|t-1})^T],$$

$$E[(x_{t-1} - \hat{x}_{t-1|t-1}) w_t^T] = E[w_t (x_{t-1} - \hat{x}_{t-1|t-1})^T] = 0$$

$$\Rightarrow P_{t|t-1} = F\mathrm{E}[(x_{t-1} - \hat{x}_{t-1|t-1})(x_{t-1} - \hat{x}_{t-1|t-1})^T] F^T + \mathrm{E}[w_t w_t^T]$$

$$\Rightarrow P_{t|t-1} = FP_{t-1|t-1} F^T + Q_t.$$

$$x_t - \hat{x}_{t|t-1} = F(x_{t-1} - \hat{x}_{t|t-1}) + w_t$$

$$\Rightarrow P_{t|t-1} = \mathrm{E}[(F(x_{t-1} - \hat{x}_{t-1|t-1}) + w_t) \times (F(x_{t-1} - \hat{x}_{t-1|t-1}) + w_t)^T]$$

$$= F\mathrm{E}[(x_{t-1} - \hat{x}_{t-1|t-1}) \times (x_{t-1} - \hat{x}_{t-1|t-1})^T] \times F^{\mathrm{T}} + F\mathrm{E}[(x_{t-1} - \hat{x}_{t-1|t-1}) w_t^T] + \mathrm{E}[w_t x_{t-1} - \hat{x}_{t-1|t-1}{}^T] F^T + \mathrm{E}[w_t w_t^T].$$

Measurement Derivation

$$p_j = E\left\{\left(x_j - \hat{x}_j\right)^2\right\}$$

$$p_j = E\left\{\left(x_j - \hat{x}_j^- - k\left(z_j - h\,\hat{x}_j^-\right)\right)^2\right\}$$

$$\frac{\partial p_j}{\partial k} = 0 = \frac{\partial E\left\{\left(x_j - \hat{x}_j^- - k\left(z_j - h\,\hat{x}_j^-\right)\right)^2\right\}}{\partial k}$$

$$= 2E\left\{\left(x_j - \hat{x}_j^- - k\left(z_j - h\,\hat{x}_j^-\right)\right)\left(z_j - h\,\hat{x}_j^-\right)\right\}$$

$$= 2E\left\{x_j z_j - \hat{x}_j^- z_j - k z_j^2 + kh\,\hat{x}_j^- z_j - h\,x_j \hat{x}_j^- + h\left(\hat{x}_j^-\right)^2 + kh\,z_j \hat{x}_j^- - kh^2\left(\hat{x}_j^-\right)^2\right\}$$

$$= 2E\left\{x_j z_j - \hat{x}_j^- z_j - h\,x_j \hat{x}_j^- + h\left(\hat{x}_j^-\right)^2\right\} - 2kE\left\{z_j^2 - 2h\,\hat{x}_j^- z_j + h^2\left(\hat{x}_j^-\right)^2\right\}$$

$$k = \frac{E\left\{x_j z_j - \hat{x}_j^- z_j - h\,x_j \hat{x}_j^- + h\left(\hat{x}_j^-\right)^2\right\}}{E\left\{z_j^2 - 2h\hat{x}_j^- z_j + h^2\left(\hat{x}_j^-\right)^2\right\}}$$

$$\text{numerator} = E\left\{x_j z_j - \hat{x}_j^- z_j - h\,x_j \hat{x}_j^- + h\left(\hat{x}_j^-\right)^2\right\}$$

$$= E\left\{x_j\left(h\,x_j + v_j\right) - \hat{x}_j^-\left(h\,x_j + v_j\right) - h\,x_j \hat{x}_j^- + h\left(\hat{x}_j^-\right)^2\right\}$$

$$= E\left\{h x_j^2 + x_j v_j - h\hat{x}_j^- x_j - \hat{x}_j^- v_j - h\,x_j \hat{x}_j^- + h\left(\hat{x}_j^-\right)^2\right\}$$

$$= E\left\{h x_j^2 - 2h\hat{x}_j^- x_j + h\left(\hat{x}_j^-\right)^2 + \left(x_j - \hat{x}_j^-\right)v_j\right\}$$

$$E\left\{\left(x_j - \hat{x}_j^-\right)v_j\right\} = E\left\{e_j^- v_j\right\} = 0$$

$$\text{numerator} = E\left\{h x_j^2 - 2h\hat{x}_j^- x_j + h\left(\hat{x}_j^-\right)^2\right\}$$

$$= hE\left\{\left(x_j - \hat{x}_j^-\right)^2\right\} = hE\left\{\left(e_j^-\right)^2\right\}$$

$$= h\,p_j^-$$

Measurement derivation

$$\text{denominator} = E\left\{ \left(h\,x_j + v_j \right)^2 - 2h\hat{x}_j^- \left(h\,x_j + v_j \right) + h^2 \left(\hat{x}_j^- \right)^2 \right\}$$

$$= E\left\{ h^2 x_j^2 + 2hx_j v_j + v_j^2 - 2h^2 \hat{x}_j^- x_j - 2h\hat{x}_j^- v_j + h^2 \left(\hat{x}_j^- \right)^2 \right\}$$

$$= E\left\{ h^2 x_j^2 - 2h^2 \hat{x}_j^- x_j + h^2 \left(\hat{x}_j^- \right)^2 + v_j^2 + 2h\left(x_j - \hat{x}_j^- \right) v_j \right\}$$

$$\text{denominator} = E\left\{ h^2 x_j^2 - 2h^2 \hat{x}_j^- x_j + h^2 \left(\hat{x}_j^- \right)^2 + v_j^2 \right\}$$

$$= h^2 E\left\{ x_j^2 - 2\hat{x}_j^- x_j + \left(\hat{x}_j^- \right)^2 \right\} + E\left\{ v_j^2 \right\}$$

$$= h^2 p_j^- + R$$

$$k = \frac{h\,p_j^-}{h^2 p_j^- + R}$$

Covariance update

$$p_j = E\left\{\left(x_j - \hat{x}_j\right)^2\right\}$$
$$= E\left\{\left(x_j - \left(\hat{x}_j^- - hk\hat{x}_j^- + kz_j\right)\right)^2\right\}$$
$$= E\left\{\left(x_j - \left(\hat{x}_j^- - hk\hat{x}_j^- + k\left(hx_j + v_j\right)\right)\right)^2\right\}$$
$$= E\left\{\left(\left(x_j - \hat{x}_j^-\right)\left(1 - hk\right) - kv_j\right)^2\right\}$$
$$= E\left\{\left(x_j - \hat{x}_j^-\right)^2\left(1 - hk\right)^2 - 2kv_j\left(x_j - \hat{x}_j^-\right)\left(1 - hk\right) + k^2v_j^2\right\}$$

$$E\left\{2kv_j\left(x_j - \hat{x}_j^-\right)\left(1 - hk\right)\right\} = 2k\left(1 - hk\right)E\left\{v_j\left(x_j - \hat{x}_j^-\right)\right\}$$
$$= 2k\left(1 - hk\right)E\left\{v_j e_j^-\right\}$$
$$= 0$$

$$p_j = E\left\{\left(x_j - \hat{x}_j^-\right)^2\left(1 - hk\right)^2\right\} + E\left\{k^2v_j^2\right\}$$
$$= \left(1 - hk\right)^2 E\left\{\left(x_j - \hat{x}_j^-\right)^2\right\} + k^2E\left\{v_j^2\right\}$$
$$= \left(1 - hk\right)^2 p_j^- + k^2R$$

$$R = \frac{p_j^-\left(h - h^2k\right)}{k}$$

$$p_j = \left(1 - hk\right)^2 p_j^- + k^2\frac{p_j^-\left(h - h^2k\right)}{k}$$
$$= p_j^-\left(1 - 2hk + h^2k^2 + kh - h^2k^2\right)$$
$$= p_j^-\left(1 - hk\right)$$

Measurement Update

$$\hat{\mathbf{x}}_{t|t} = \hat{\mathbf{x}}_{t|t-1} + \mathbf{K}_t(\mathbf{z}_t - \mathbf{H}_t\,\hat{\mathbf{x}}_{t|t-1})$$

$$\mathbf{P}_{t|t} = \mathbf{P}_{t|t-1} - \mathbf{K}_t\mathbf{H}_t\mathbf{P}_{t|t-1},$$

$$\mathbf{K}_t = \mathbf{P}_{t|t-1}\,\mathbf{H}_t^{\mathbf{T}}(\mathbf{H}_t\,\mathbf{P}_{t|t-1}\mathbf{H}_t^{\mathbf{T}} + \mathbf{R}_t)^{-1}.$$

Kalman Filter

System $$\mathbf{x}_t = \mathbf{F}_t \mathbf{x}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \mathbf{w}_t ,$$

Measurement $$\mathbf{z}_t = \mathbf{H}_t \mathbf{x}_t + \boldsymbol{\nu}_t ,$$

Prediction $$\hat{\mathbf{x}}_{t|t-1} = \mathbf{F}_t \hat{\mathbf{x}}_{t-1|t-1} + \mathbf{B}_t \mathbf{u}_t$$

A priori covariance $$\mathrm{P}_{t|t-1} = \mathrm{F}_t \mathrm{P}_{t-1|t-1} \mathrm{F}_t^{\mathrm{T}} + \mathrm{Q}_t ,$$

Kalman Gain $$\mathbf{K}_t = \mathbf{P}_{t|t-1} \mathbf{H}_t^{\mathrm{T}} (\mathbf{H}_t \mathbf{P}_{t|t-1} \mathbf{H}_t^{\mathrm{T}} + \mathbf{R}_t)^{-1} .$$

Measurement Update $$\hat{\mathbf{x}}_{t|t} = \hat{\mathrm{x}}_{t|t-1} + \mathrm{K}_t (\mathrm{z}_t - \mathbf{H}_t \hat{\mathbf{x}}_{t|t-1})$$

A posteriori covariance $$\mathbf{P}_{t|t} = \mathbf{P}_{t|t-1} - \mathbf{K}_t \mathbf{H}_t \mathbf{P}_{t|t-1} ,$$

Scalar Kalman Filter

System $$x_j = a x_{j-1} + b u_j + w_j$$

Measurement $$z_j = h x_j + v_j$$

Prediction $$\hat{x}_j^- = a \hat{x}_{j-1} + b u_j$$

A priori covariance $$p_j^- = a^2 p_{j-1} + Q$$

Kalman Gain $$k_j = \frac{h p_j^-}{h^2 p_j^- + R}$$

Measurement Update $$\hat{x}_j = \hat{x}_j^- + k_j \left(z_j - h \hat{x}_j^- \right)$$

A posteriori covariance $$p_j = p_j^- \left(1 - h k_j \right)$$