EE2801: DSP Lab
(Jan – Apr 2021)

Lecture 3

Contents

- Continuation of downsampling and upsampling
- Decimation
- Interpolation
- Practical Implementation of decimation and interpolation
- Cascading decimator and interpolator
- Generalizing decimation and interpolation to any factor M or L
- Reference

Downsampler(Compressors)

- Time domain relation between input and output

$$\boldsymbol{y[n] = x[Mn]}$$

- It is a Linear Time Varying (LTV) system.
- Frequency domain relation between input and output

$$Y\left(e^{j\omega}\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j\frac{(\omega-2\pi k)}{M}}\right)$$

- For M=2, $Y\left(e^{j\omega}\right) = \frac{1}{2}\sum_{k=0}^{1} X\left(e^{j\frac{(\omega-2\pi k)}{2}}\right)$
- For M=3, $Y\left(e^{j\omega}\right) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{j\frac{(\omega-2\pi k)}{3}}\right)$

Downsampler(contd....)

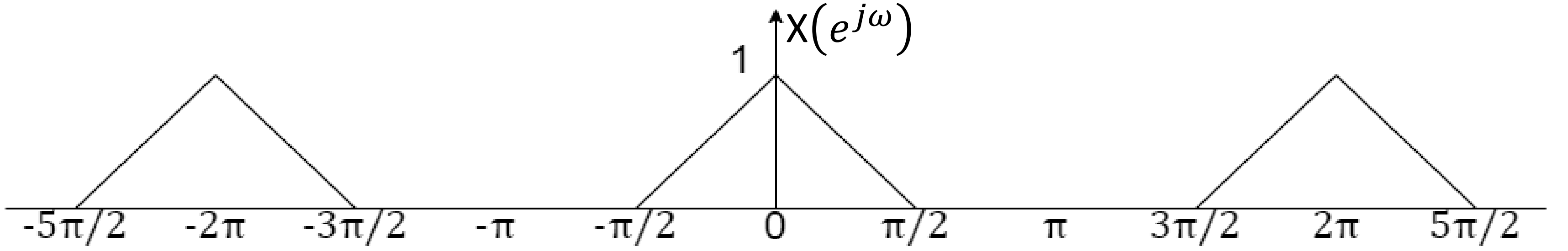


fig 1: Input to downsampler

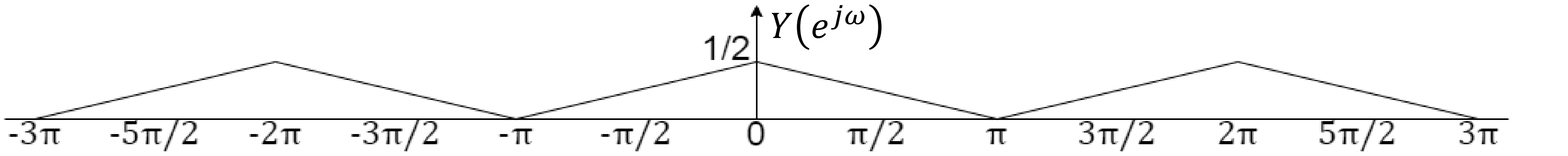


fig 2: downsampling with factor 2

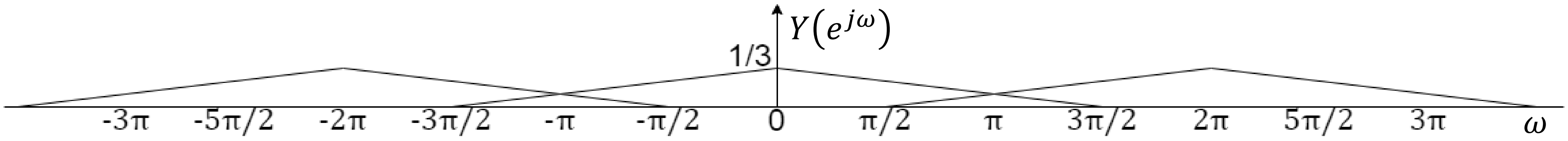


fig 3: downsampling with factor 3

Aliasing

Downsampler(contd….)

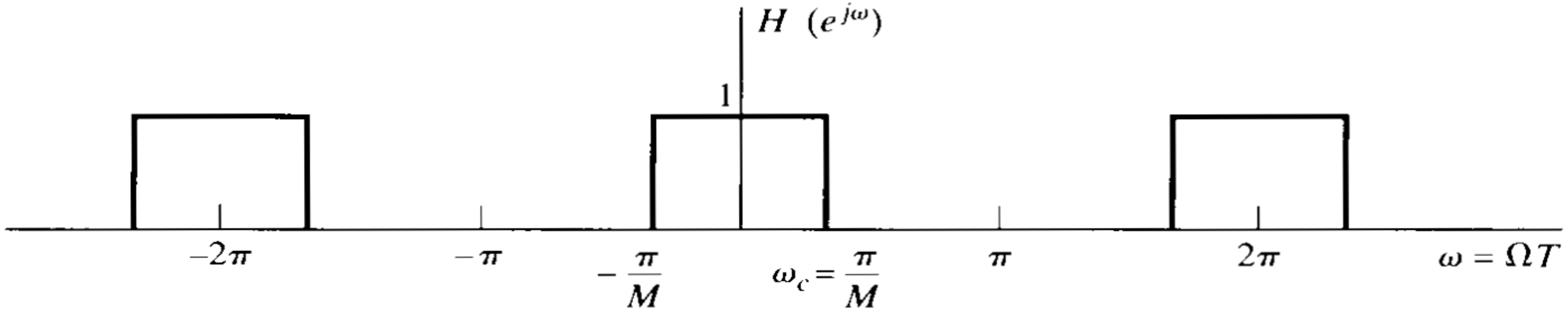


fig 4 : Low pass filter with cutoff freq π/M

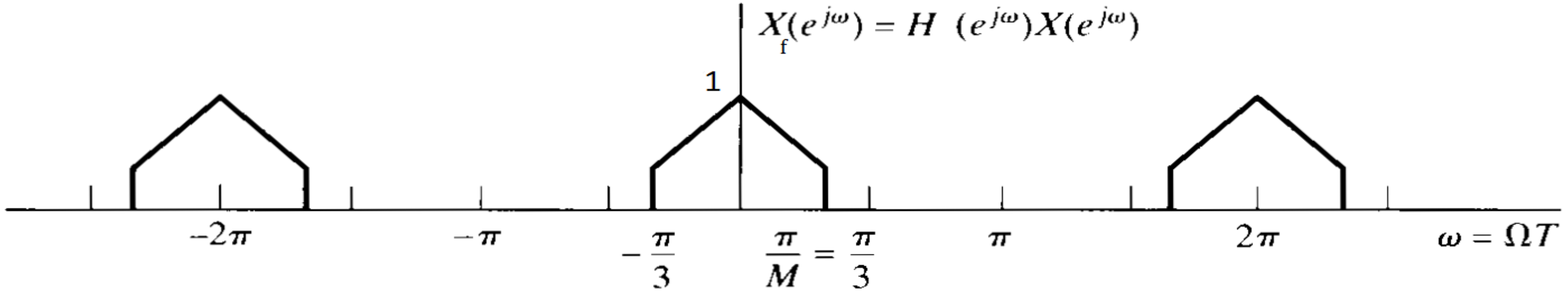


fig 5 : output of LPF

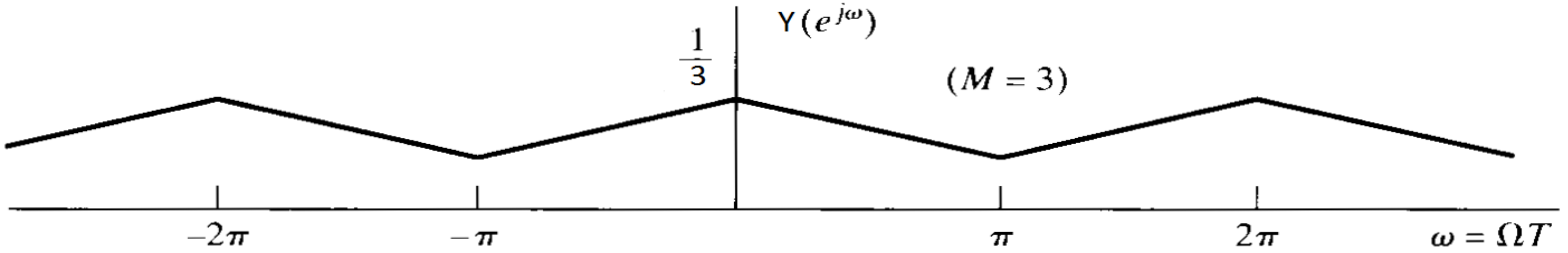


fig 6 : Downsampling with factor 3

This is known as decimation

Downsampler(contd….)

Observations

- BW of input signal is π.
- No aliasing when M=2
- Aliasing occurs when M=3
- Because signal BW < $\frac{2\pi}{M}$ when M=2 and signal BW > $\frac{2\pi}{M}$ when M=3

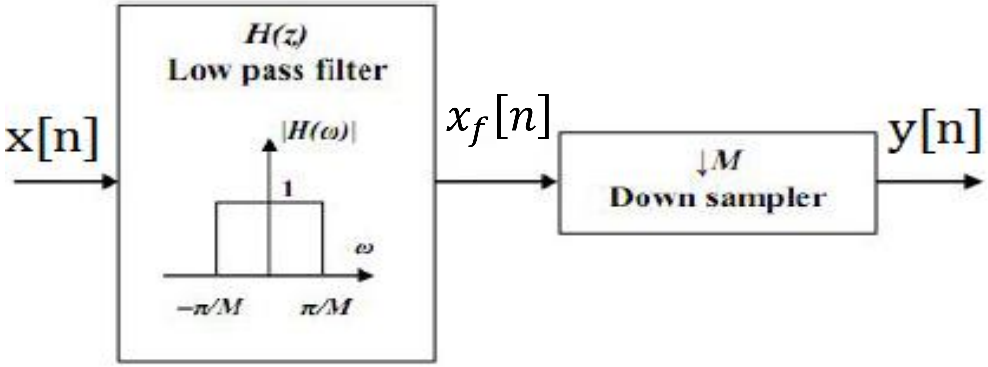
Exercise

- BW of input signal is 2π.
- M=2
- Aliasing will be there or not?
- Yes, because signal BW is more than $\frac{2\pi}{M}$

Key observation : Signal BW must be less than $\frac{2\pi}{M}$, where M is the downsampling factor.

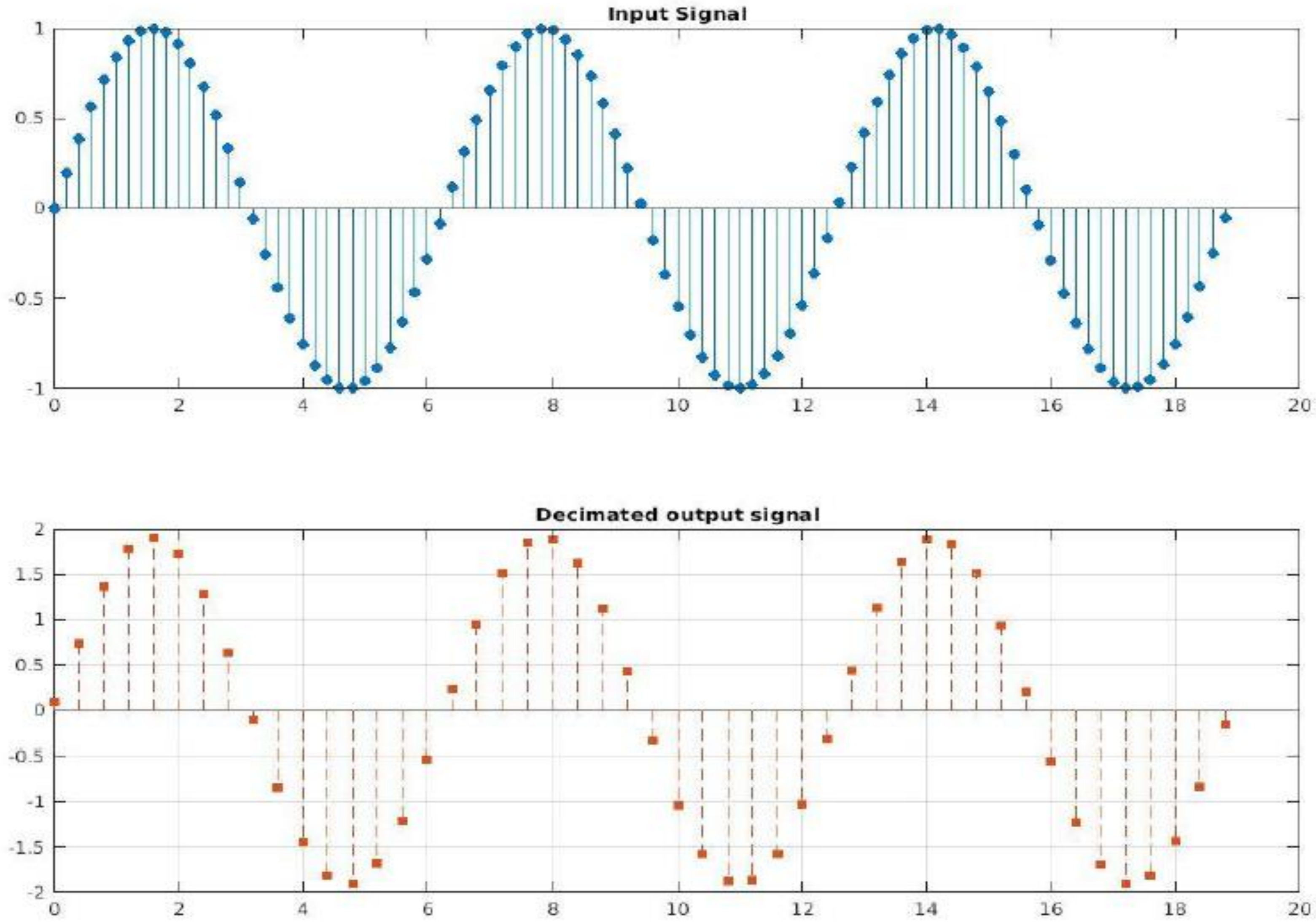
Decimation

- LPF followed by downsampler is known as decimator.
- The job of LPF is to prevent aliasing. Hence it is known as anti-aliasing filter.
- Cutoff frequency is π/M.
- When M = 2, then the LPF is also known as Half Band Filter(HBF) with cutoff frequency $\pi/2$.



Decimation

Decimation Example:



Upsampler(Expander)

- Time domain relation between input and output

$$\boldsymbol{y}[\boldsymbol{n}] = \begin{cases} \boldsymbol{x}[\boldsymbol{n}/\boldsymbol{L}], & \boldsymbol{if\ n\ is\ a\ multiple\ of\ L} \\ \boldsymbol{0}, & \boldsymbol{otherwise} \end{cases}$$

- It is a Linear Time Varying (LTV) system.
- Frequency domain relation between input and output

$$Y\left(e^{j\omega}\right) = X\left(e^{j\omega L}\right)$$

Upsampler(contd….)

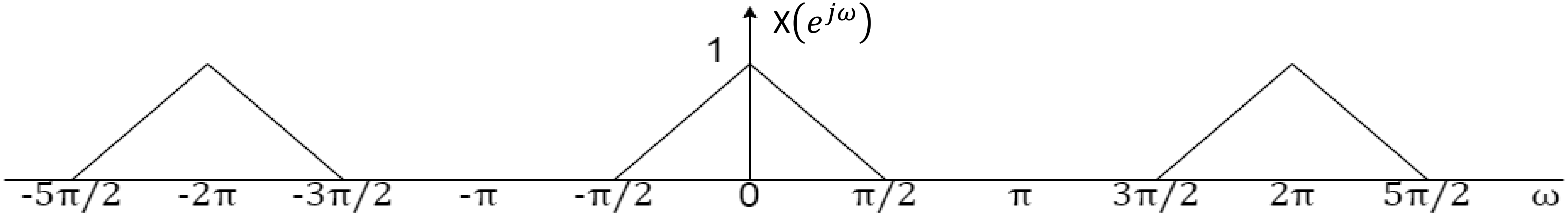


fig 1: Input to upsampler

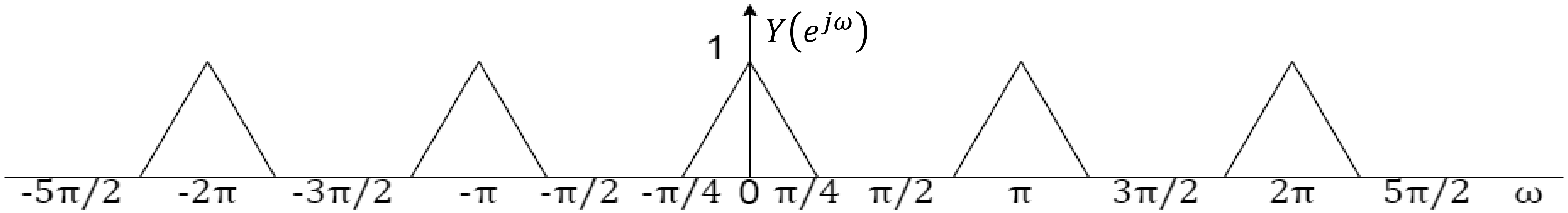


fig 2: Upsampling with L = 2

Unwanted image

Upsampler(contd….)

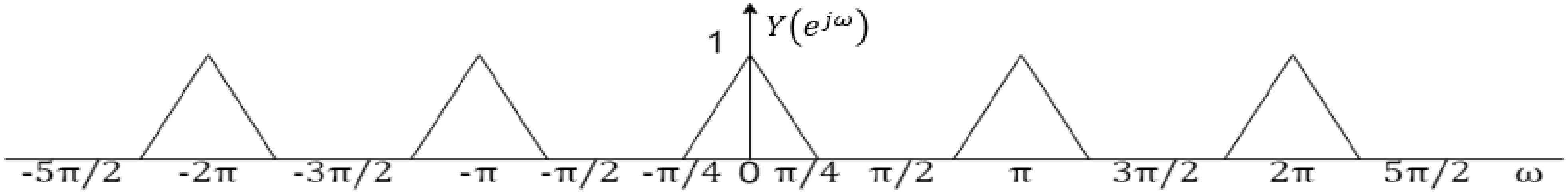


fig 2: Upsampling with L = 2

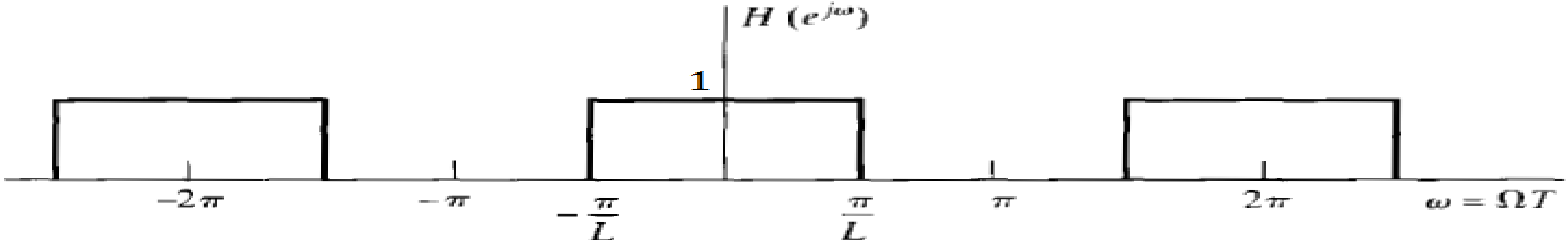


fig 3 : LPF with cutoff frequency π/L

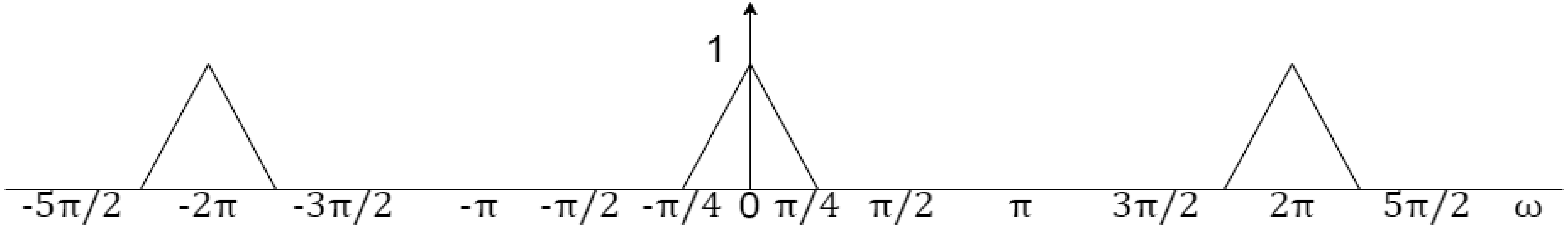
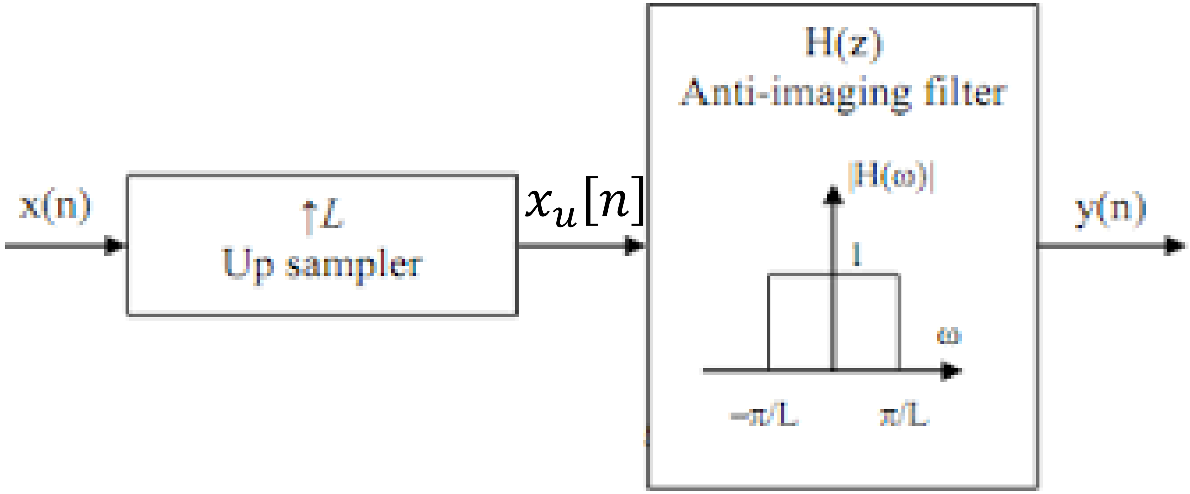


fig 4: output of LPF

This is known as interpolation

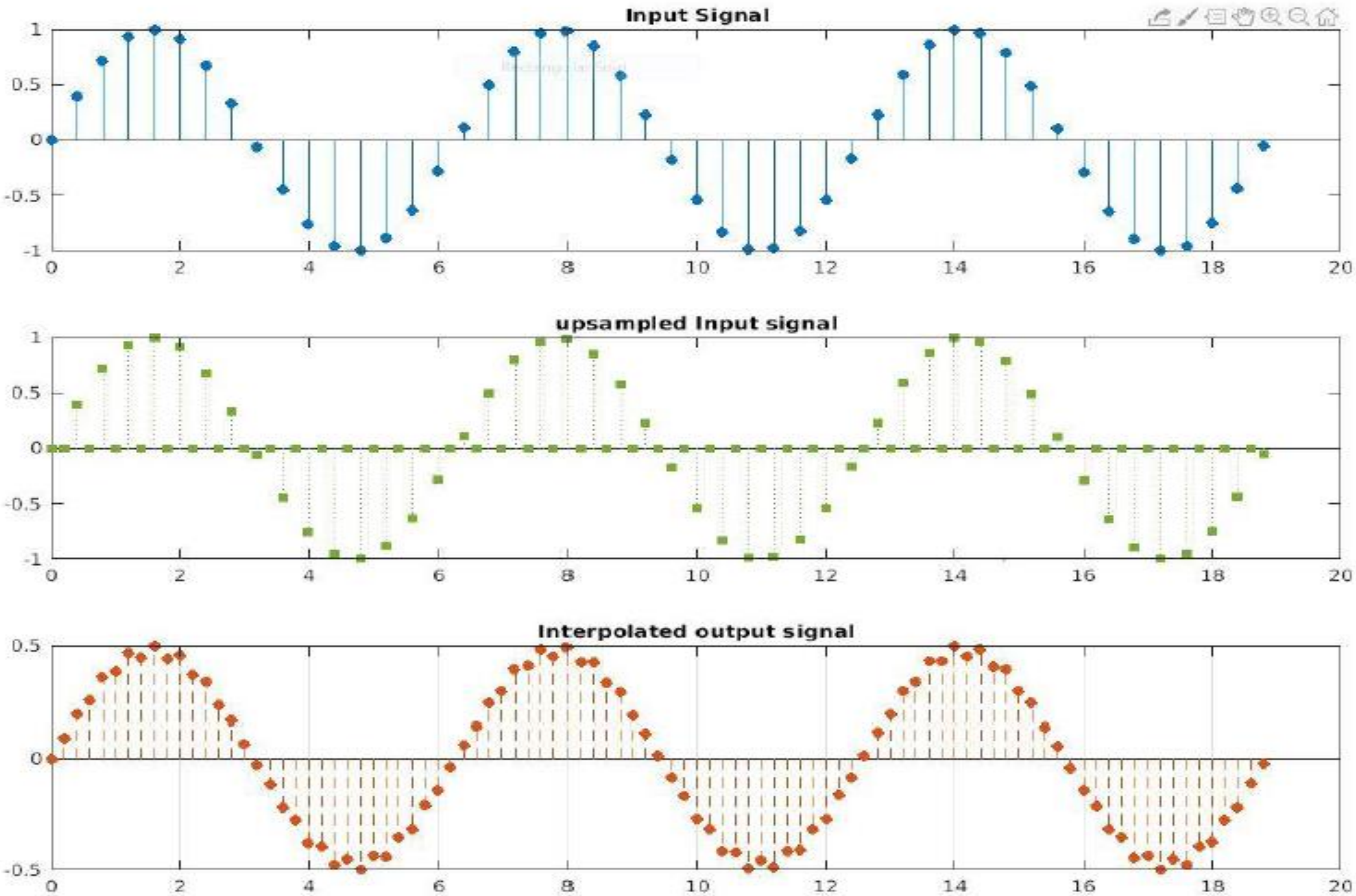
Interpolation

- Upsampler followed by LPF is known as interpolator.
- The job of LPF is to remove unwanted image of X$\left(e^{j\omega}\right)$. Hence it is known as anti-imaging filter.
- Cutoff frequency is π/L.
- When L = 2, then the LPF is also known as Half Band Filter(HBF) with cutoff frequency $\pi/2$.

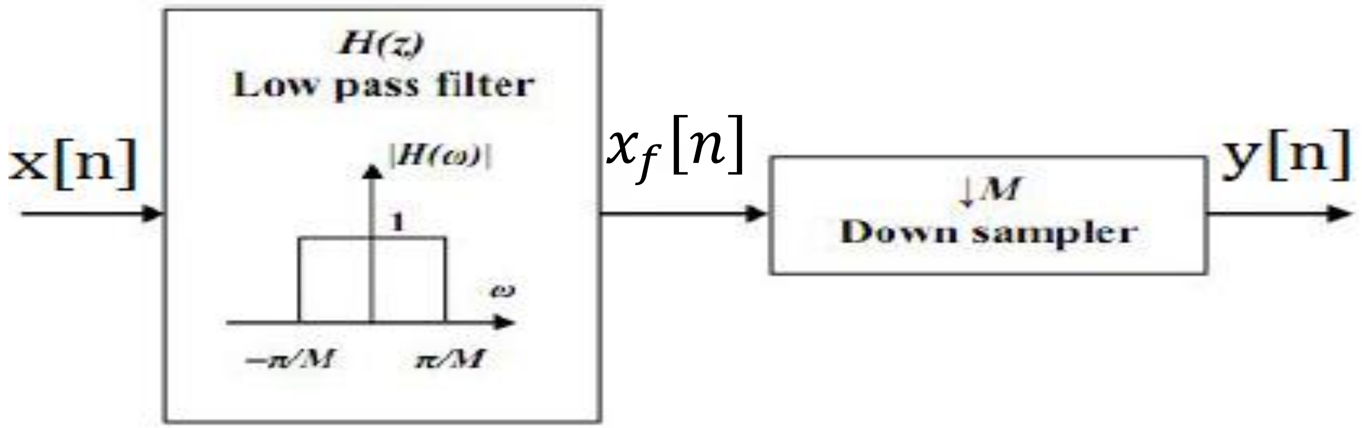


Interpolation

Interpolation Example:



Practical Implementation of decimation



- Length of x[n] is l_x
- Length of $x_f[n]$ is l_{xf}
- Length of $y[n]$ is l_y
- Impulse response of filter is h[n]
- Length of h[n] is l_h

Problem is

$x_f[n] = x[n] * h[n]$

Let $l_x = 36, l_h = 51$

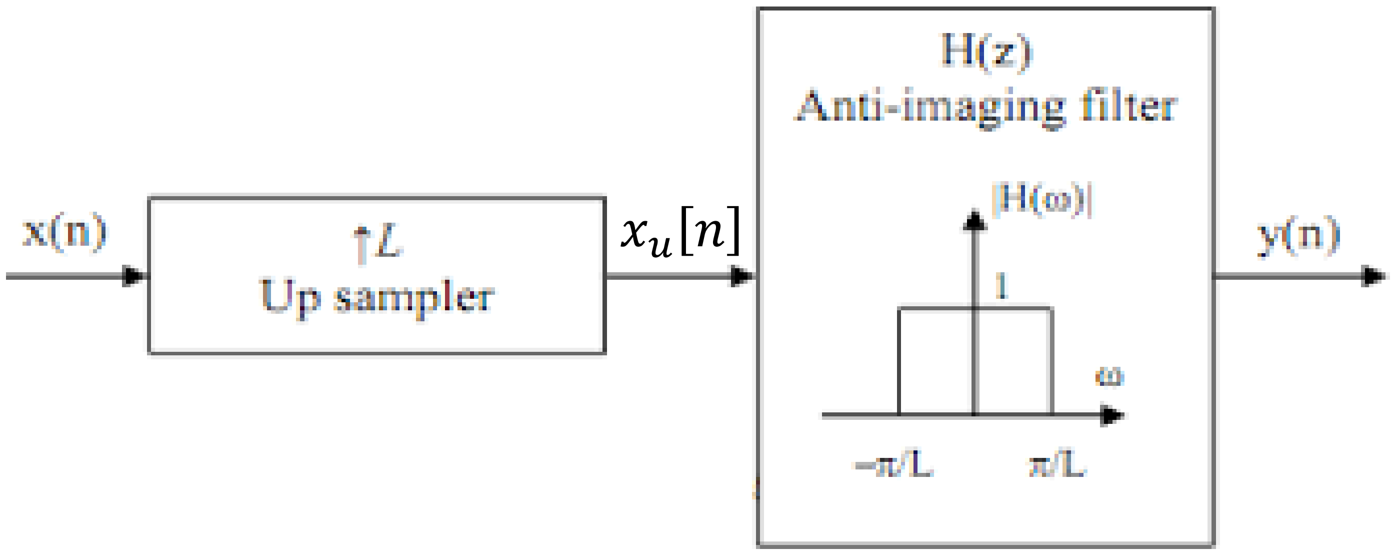
then $l_{xf} = 86, l_y = 43$

Solution is

Discard first and last $(l_h - 1)/2$ samples from $x_f[n]$, i.e. take only middle l_x sample of $x_f[n]$ and then do downsampling.

So now $l_y = 18$

Practical Implementation of interpolation



- Length of x[n] is l_x
- Length of $x_u[n]$ is l_{xu}
- Length of $y[n]$ is l_y

- Impulse response of filter is h[n]
- Length of h[n] is l_h

Problem is

$$y[n] = x_u[n] * h[n]$$

Let $l_x = 18$, $l_h = 51$

then $l_{xu} = 36$, $l_y = 86$

Solution is

Discard first and last $(l_h - 1)/2$ samples from $y[n]$, i.e. take only middle l_{xu} sample of $y[n]$.

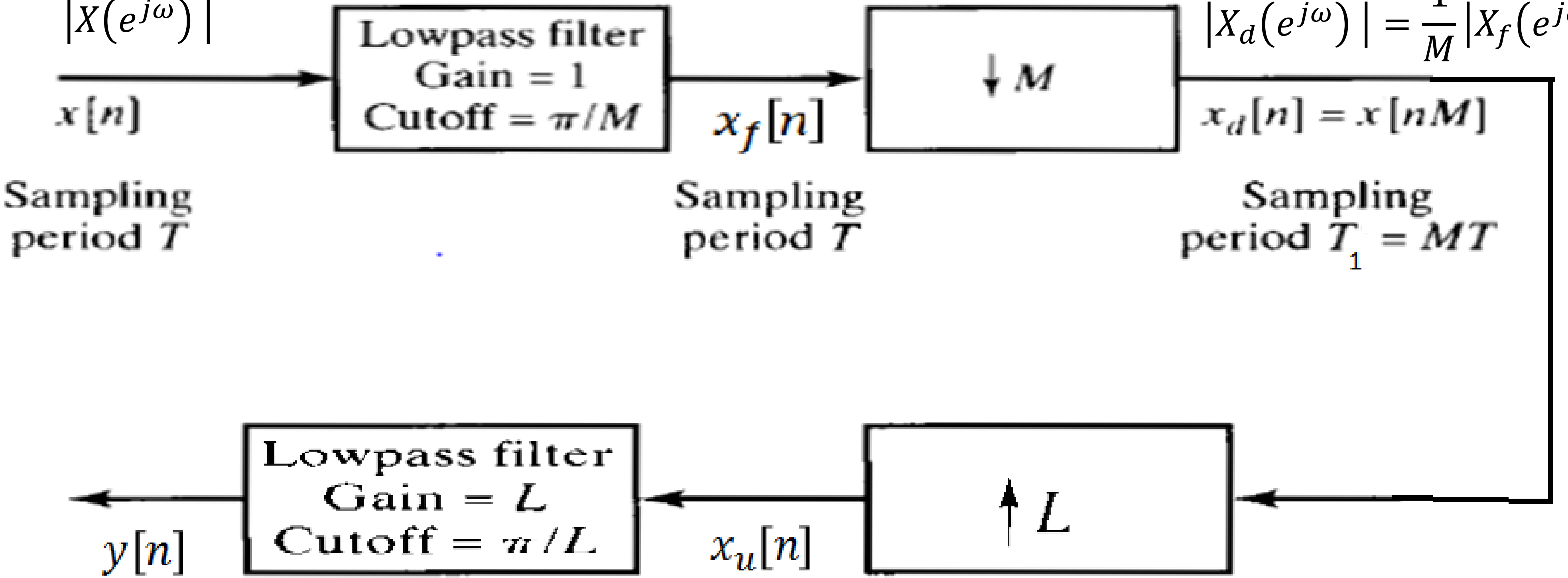
So now $l_y = 36$

Cascading of decimator and interpolator

$\left|X_f(e^{j\omega})\right| = \left|X(e^{j\omega})\right|$

$\left|X(e^{j\omega})\right|$

$\left|X_d(e^{j\omega})\right| = \frac{1}{M}\left|X_f(e^{j\omega})\right| = \frac{1}{M}\left|X(e^{j\omega})\right|$



Sampling period

$$T_2 = \frac{T_1}{L} = \frac{MT}{L}$$

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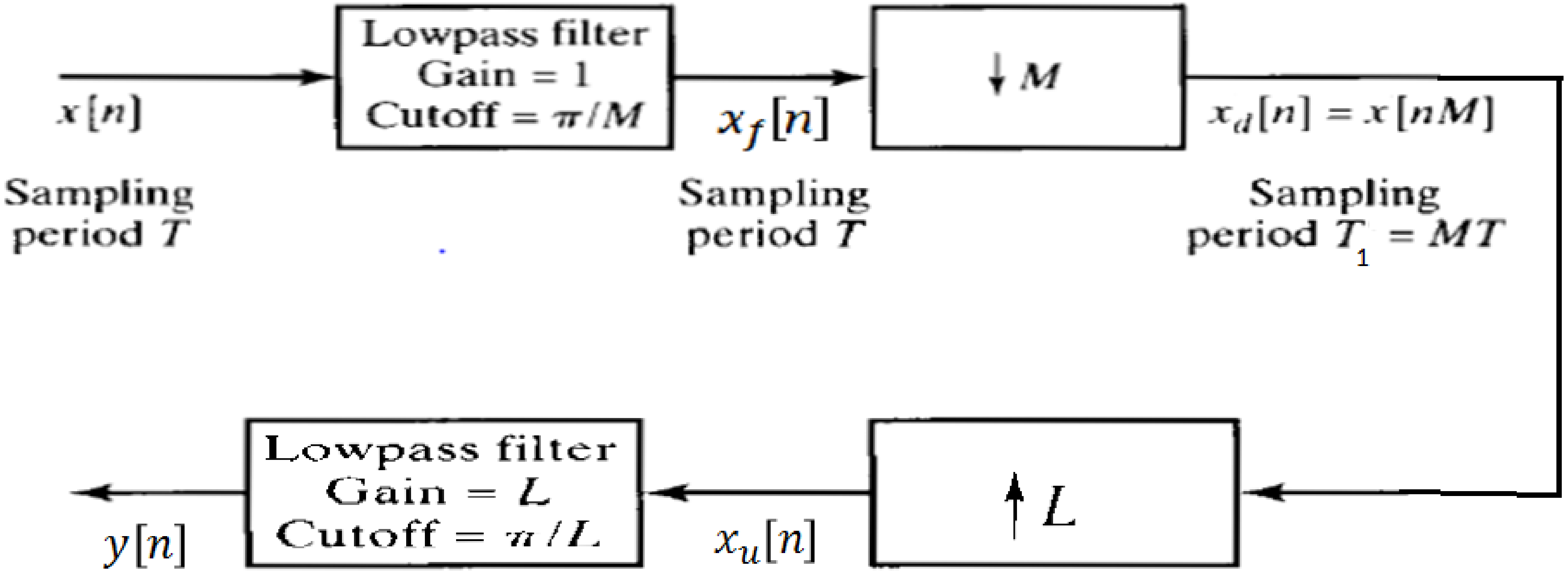
$\left|Y(e^{j\omega})\right| = \left|X_u(e^{j\omega})\right| = \frac{L}{M}\left|X(e^{j\omega})\right|$

$\left|X_u(e^{j\omega})\right| = \left|X_d(e^{j\omega})\right| = \frac{1}{M}\left|X(e^{j\omega})\right|$

Generalizing decimation and interpolation to any factor M or L



Sampling period

$$T_2 = \frac{T_1}{L} = \frac{MT}{L}$$

Sampling period

$$T_2 = \frac{T_1}{L} = \frac{MT}{L}$$

Note: 1. We will generalize here for M = L

2. Sampling frequency of $x[n]$ and $h[n]$ should be same.

Generalizing decimation and interpolation to any factor M or L

- **Ex-1:** Let decimation and interpolation factors are M = L = 3, what are the filter gain and cutoff frequency of anti-aliasing filter and anti-imaging filter?

  Ans: For anti-aliasing filter, gain = 1, cutoff frequency $\omega_c = \pi/3$.

  For anti-imaging filter, gain = 3, cutoff frequency $\omega_c = \pi/3$.

- **Ex-2:** Let decimation and interpolation factors are M = L = 4, what are the filter gain and cutoff frequency of anti-aliasing filter and anti-imaging filter?

  Ans: For anti-aliasing filter, gain = 1, cutoff frequency $\omega_c = \pi/4$.

  For anti-imaging filter, gain = 4, cutoff frequency $\omega_c = \pi/4$.

Reference

- Discrete Time Signal Processing by Alan V. Oppenheim and Ronald W. Schafer - link