Sequence Models

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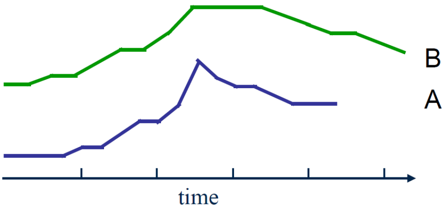

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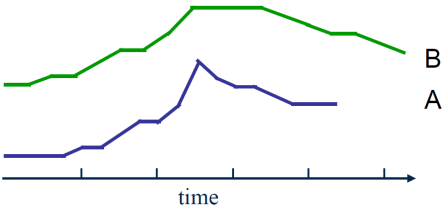

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$$D(A, B) = \frac{1}{N} \sum_{n=1}^{N} \|\mathbf{x}_n - \mathbf{y}_n\|$$

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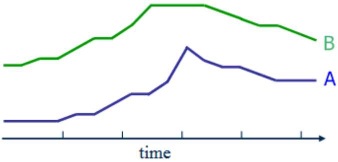

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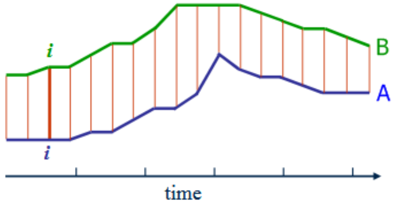

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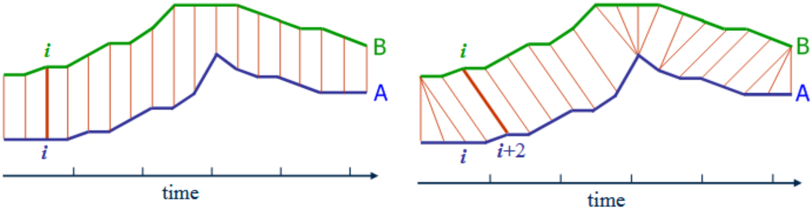

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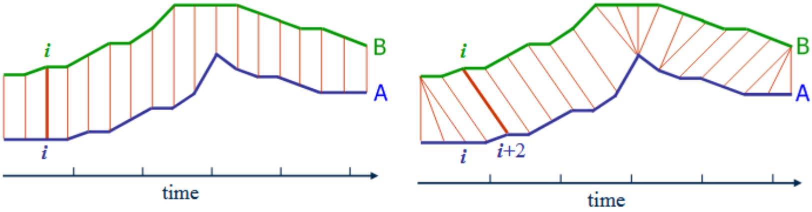

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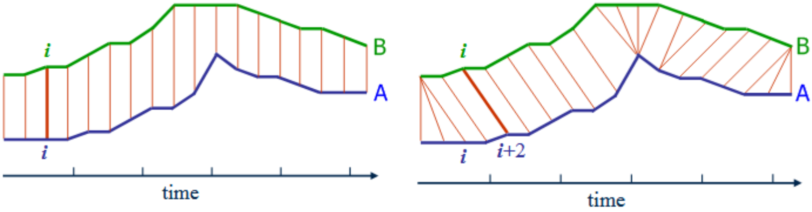

- Nonlinear (elastic) alignment produces a more intuitive similarity measure
- Allows similar shapes to match even if they are out of phase in time

Distance between Sequences

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- Let A and B be two varying-length sequences

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- Find the corresponding points along best alignment path

$$(p_1, q_1), (p_2, q_2), \cdots (p_K, q_K) \qquad 1 \leq p_k \leq M \;\&\; 1 \leq q_k \leq N$$

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- Need to follow constraints while identifying corresponding points

Distance Matrix

$$A = (4, 3, 7, 0, 2, 6, 5)$$
$$B = (3, 7, 1, 6, 1, 5, 4, 4)$$

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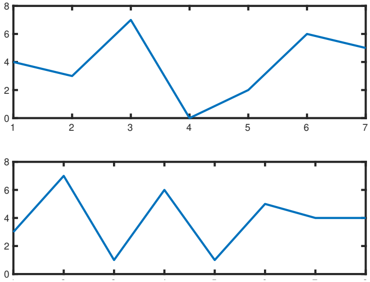

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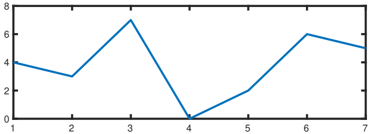

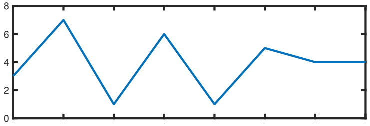

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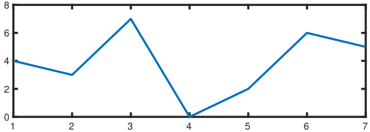

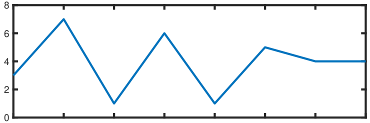

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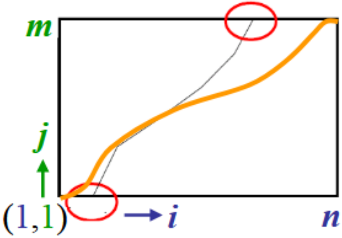
4	0	1	9	16	4	4	1
4	0	1	9	16	4	4	1
5	1	4	4	25	9	1	0
1	9	4	36	1	1	25	16
6	4	9	1	36	16	0	1
1	9	4	36	1	1	25	16
7	9	16	0	49	25	1	4
3	1	0	16	9	1	9	4
	4	3	7	0	2	6	5

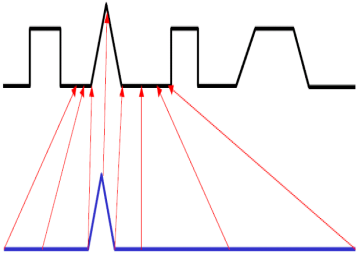
Boundary Condition

- The alignment path should start at bottom-left and end at top-right.

$$(p_1, q_1) = (1, 1) \qquad (p_K, q_K) = (M, N)$$

- Ensures that the alignment is not limited to a partial sequence



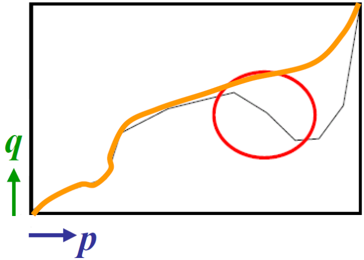


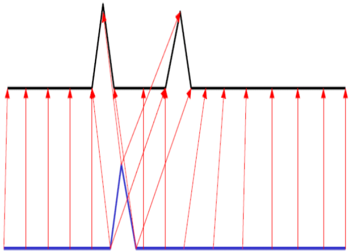
Monotonicity

- The alignment path should not go back in 'time' index.

$$p_{k-1} \leq p_k \qquad q_{k-1} \leq q_k$$

- Ensures that the features are not repeated in the alignment



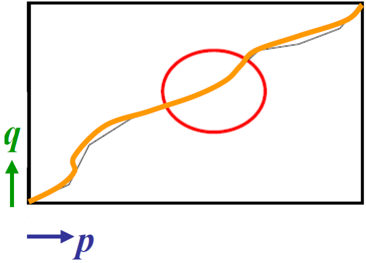


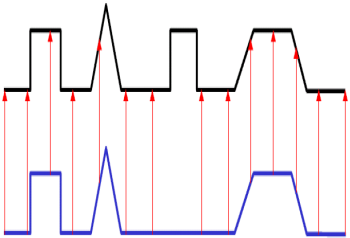
Continuity

- The alignment path should not jump in 'time' index.

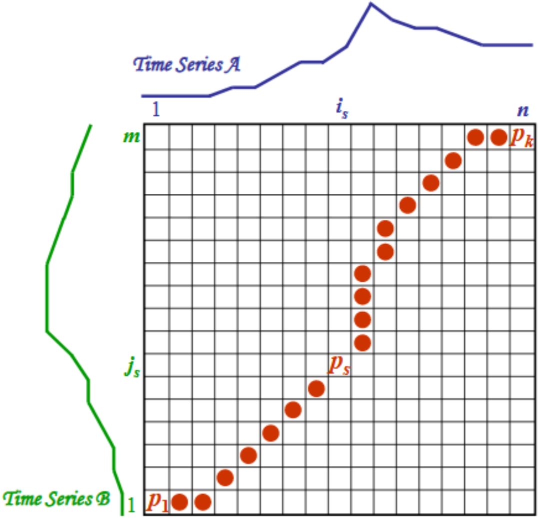
$$p_k - p_{k-1} \leq 1 \qquad q_k - q_{k-1} \leq 1$$

- Ensures that the alignment path does not omit important features





DTW Alignment Path



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8	4	0	1	9	16	4	4	1
7	4	0	1	9	16	4	4	1
6	5	1	4	4	25	9	1	0
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7	4	33 0	32 1	40 9	47 16	29 4	22 4	19(6,6) 1
6	5	33 1	31 4	31 4	41 25	25 9	18(5,5) 1	18(1,1) 0
5	1	32 9	27 4	51 36	16(3,4) 1	17(4,5) 1	28 25	19 16
4	6	23 4	23 9	15(2,3) 1	38 36	18 16	3(5,3) 0	4(6,4) 1
3	1	19 9	14(1,2) 4	37 36	2(3,2) 1	3(4,3) 1	28 25	44 16
2	7	10(1,1) 9	17(1,1) 16	1(2,2) 0	50(3,2) 49	51(4,1) 25	28(5,1) 1	32(6,2) 4
1	3	1 1	1(1,1) 0	17(2,1) 16	26(3,1) 9	27(4,1) 1	36(5,1) 9	40(6,1) 4
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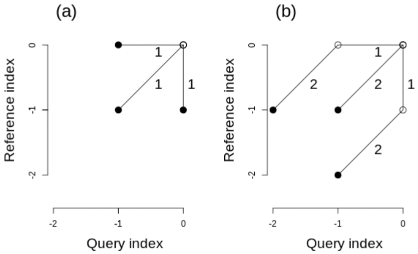
- $\mathbf{C}(M,N)$ indicates the similarity measure between the sequences
- Best possible alignment can be obtained by backtracking $P(i,j)$

$$q_T^* = (M,N) \qquad q_{t-1}^* = \mathbf{P}(q_t^*) \qquad t = T, T-1, \cdots 1$$

Summary of DTW

- DTW - quantifies distance between varying-length sequences
- Computational complexity: $O(NM)$
- DTW is not a distance metric
  - DTW is not commutative: $D(A, B) \neq D(B, A)$ for asymmetric steps
  - Does not satisfy triangular inequality
- DTW is a template-matching technique
  - Does not account for statistical variations
- Statistical models - Markov models and hidden Markov models

Symmetric vs Asymmetric Step Patterns



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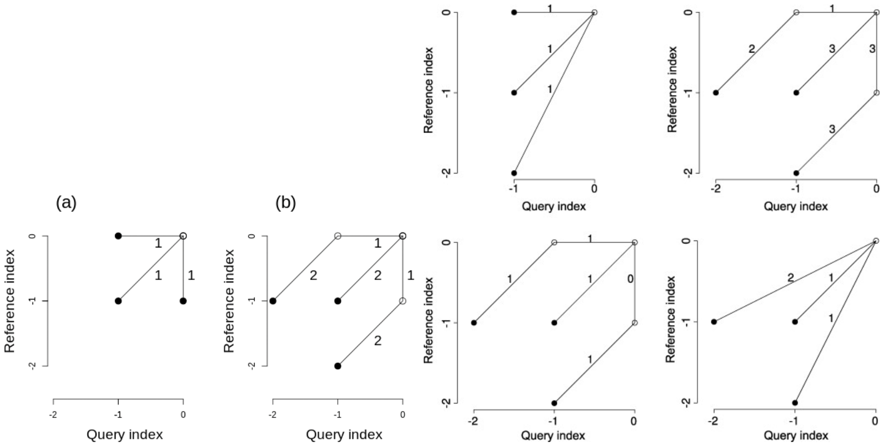

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  - Tomorrows weather depends only on today's weather, no matter what happened prior to that.

First Order Markov Model

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- Future depends only on present (and not on past)

$$p(q_n \ / \ q_{n-1}, \ q_{n-2}, \cdots, \ q_1) = p(q_n / q_{n-1})$$

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  - N - Number of distinct states (or symbols) of the model

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$$p(q_n \;/\; q_{n-1},\; q_{n-2}, \cdots,\; q_1) = p(q_n/q_{n-1})$$

q_ks can take any of the N discrete symbols, e.g. S, C or R

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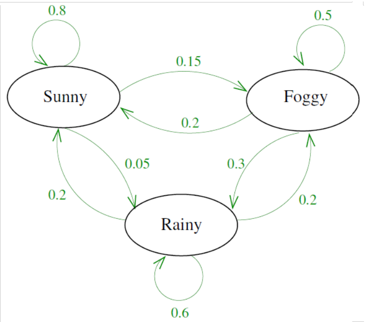

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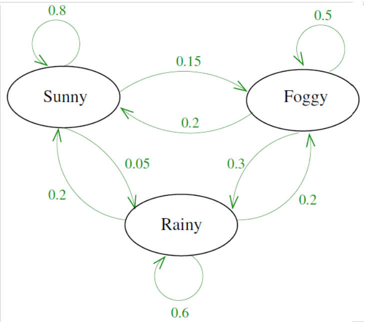

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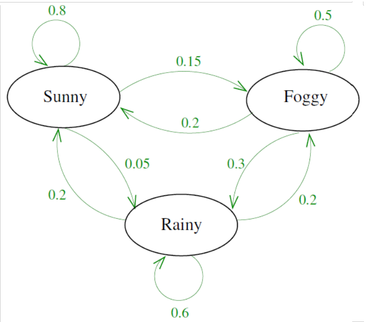

$$P[q_4 = R/q_3 = C,\ q_2 = S,\ q_1 = S] = P[q_4 = R/q_3 = C] = 0.3$$

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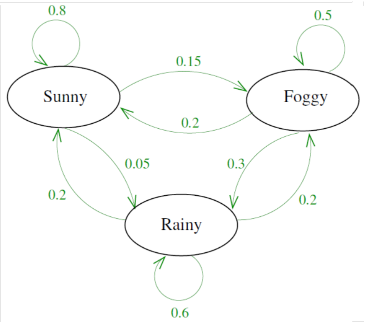

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$$\begin{aligned} P[q_2 = S, q_3 = R/q_1 = S] &= P[q_2 = S/q_1 = S]P[q_3 = R/q_2 = S] \\ &= 0.8 \times 0.05 = 0.04 \end{aligned}$$

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- Spoken dialogue systems, summerization etc.

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  - fifth an of futures the an incorporated a a the inflation most dollars quarter in is mass

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  - (I am sam)
  - (Sam I am)
  - (I do not like green eggs and ham)
  - Bigram probabilities:
    P(I/() = 2/3 P(Sam/()=1/3 P(am/I)=2/3 P()/Sam) = 1/2
    P(Sam/am) = 1/2 P(do/I)=1/3

Bigram Generated Sentences

- I must have taken into this way out of her by one hand, with which was one has been knocked off again
- outside new car parking lot of the agreement reached
- this would a record november

Summary of Markov Model

- First order Markov model can be completely described by
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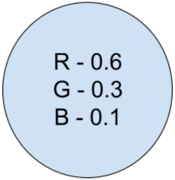
- State is deterministic in the case of Markov models
  - Each state produces a specific observation symbol
  - There is no difference between state and symbol
  - Given the observation symbol sequence, state sequence can be exactly determined

Basket Balls Experiment

	Deterministic State	Latent Variable
RV		

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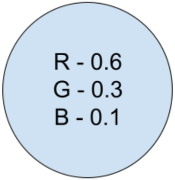
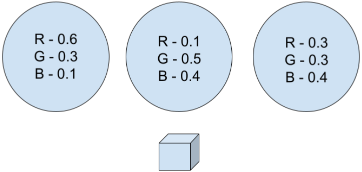
RV		

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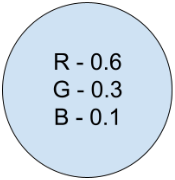
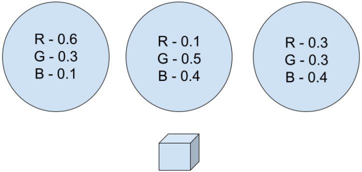
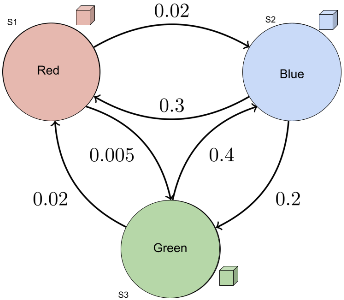
RV		
RP		

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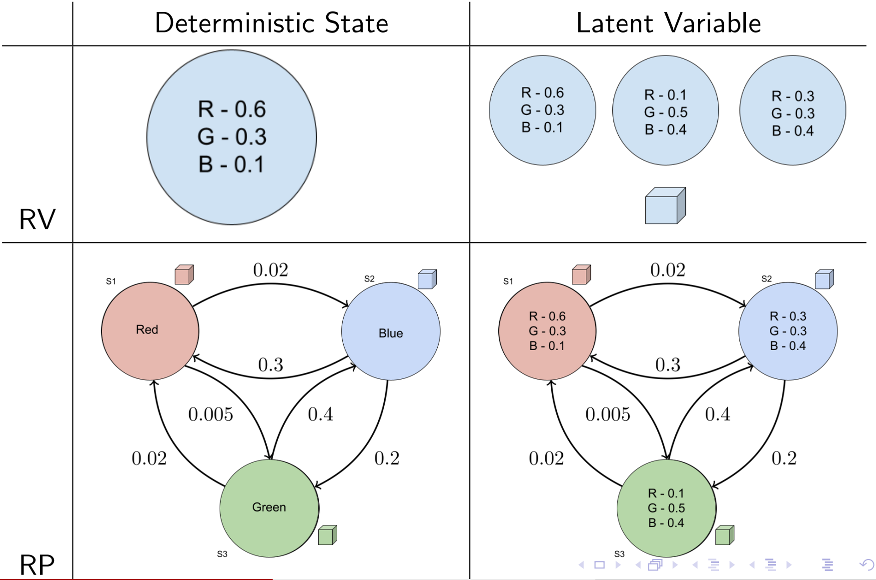

Hidden Markov Model

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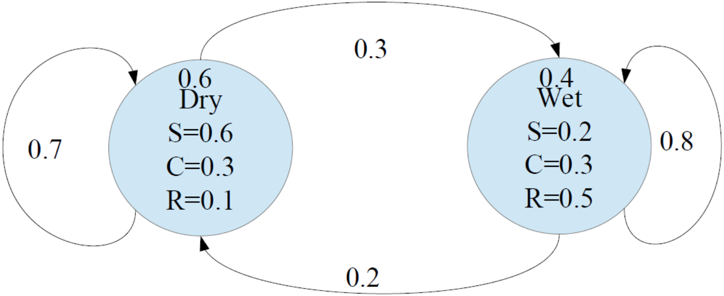

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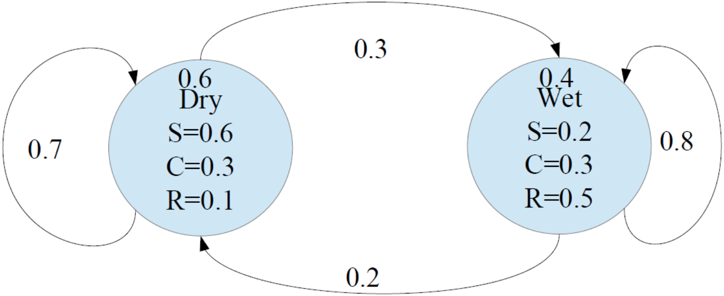

- State and symbol are different.

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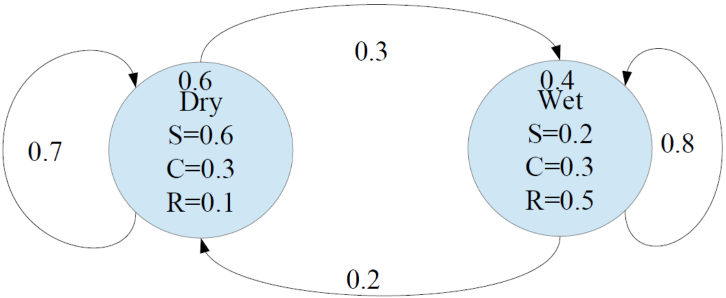

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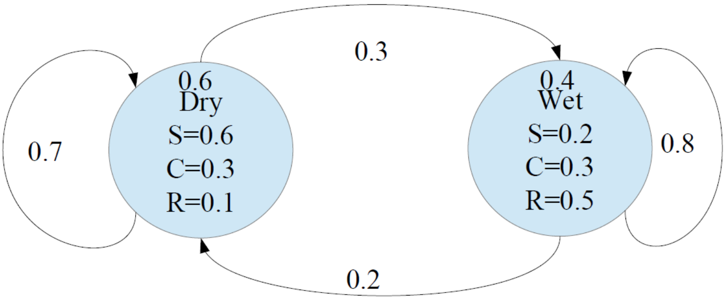

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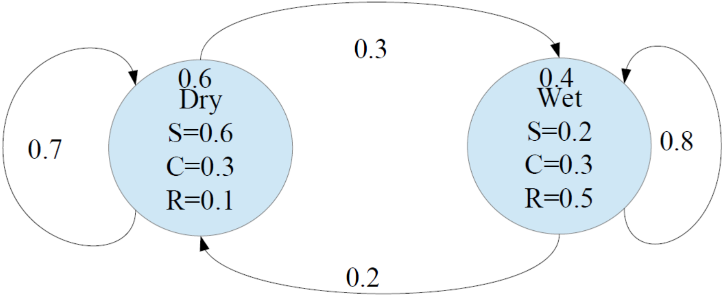

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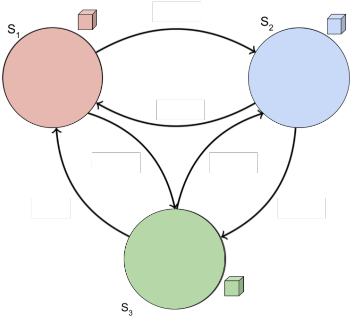
- Which state sequence produced the observation sequence?

HMM Parameters $\lambda = (\mathbf{A}, \mathbf{B}, \Pi)$



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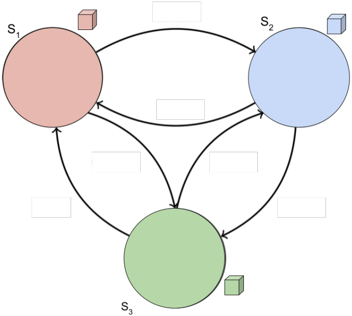

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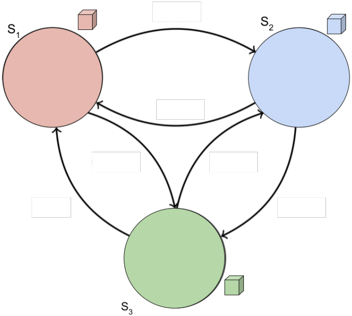

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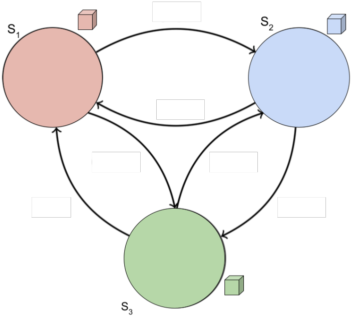

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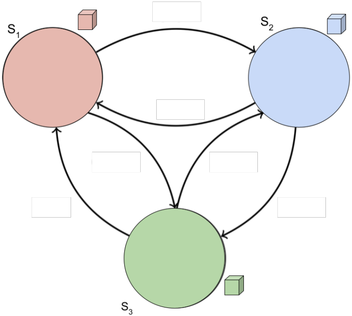

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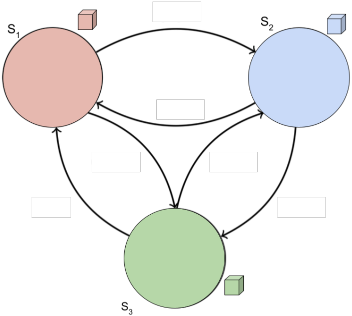

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- Emission Probability

$$\mathbf{B}_{N\times M} = [b_j(v_k)] = P[o_t = v_k / q_t = s_j]$$

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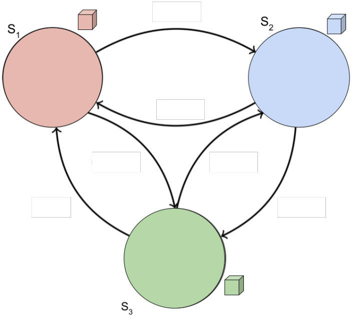

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- Observation symbol sequence

$$O = (o_1, o_2, \cdots, o_t, \cdots o_T)$$

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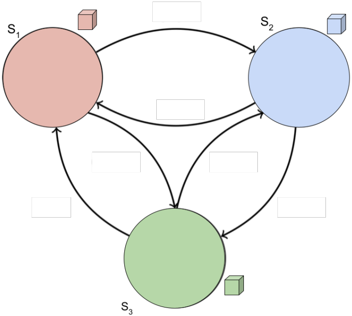

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$$P[O, Q/\lambda] = P[o_1, o_2, \cdots o_T, q_1, q_2, \cdots q_T/\lambda]$$

Evaluating $P[O/\lambda]$

- Probability of producing OSS O while passing through specific SS Q

$$
\begin{aligned}
P[O, Q/\lambda] &= P[o_1, o_2, \cdots o_T, q_1, q_2, \cdots q_T/\lambda] \\
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\end{aligned}
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\end{aligned}
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Evaluating $P[O/\lambda]$

- Probability of producing OSS O while passing through specific SS Q

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- Evaluating $P[O, Q/\lambda]$ requires $2T - 1$ multiplications
- Probability of generating OSS O given the model λ

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- A 5-state model with 100 observations requires around 10^{72} FPOs!

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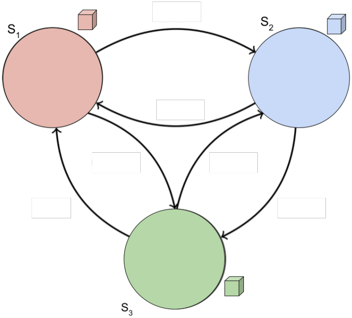
  - Baum-Welch algorithm can be used to estimate the model parameters.

HMM Parameters $\lambda = (\mathbf{A}, \mathbf{B}, \Pi)$ (Recap)



- N - Number of states

$$s_1, s_2, \cdots, s_i, \cdots s_N$$

- M - Number of symbols

$$v_1, v_2, \cdots, v_k, \cdots v_M$$

- Initial state probability

$$\Pi = [\pi_i] = P[q_1 = s_i] \qquad 1 \leq i \leq N$$

- State transition probability

$$\mathbf{A}_{N\times N} = [a_{ij}] = P[q_t = s_j / q_{t-1} = s_i]$$

- Emission Probability

$$\mathbf{B}_{N\times M} = [b_j(v_k)] = P[o_t = v_k / q_t = s_j]$$

- Observation symbol sequence

$$O = (o_1, o_2, \cdots, o_t, \cdots o_T)$$

- Sate sequence

$$Q = (q_1, q_2, \cdots, q_t, \cdots q_T)$$

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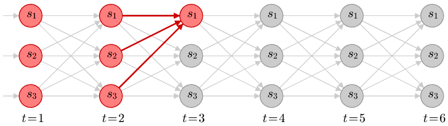

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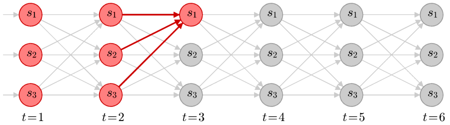

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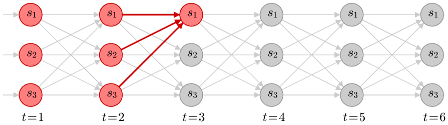

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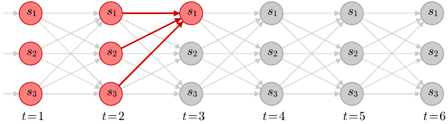

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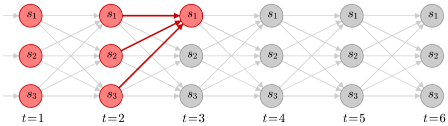

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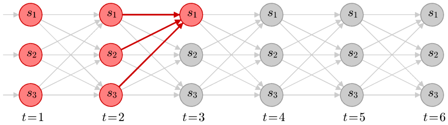

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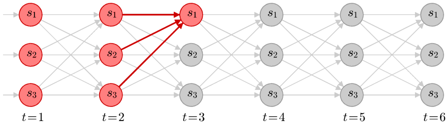

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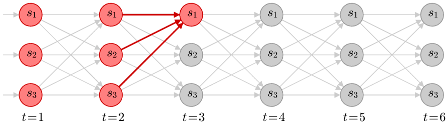

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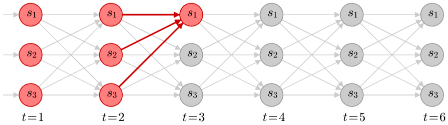

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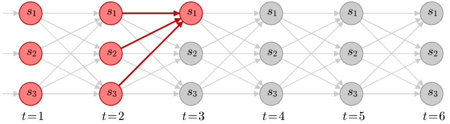

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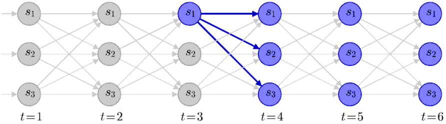

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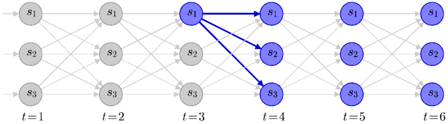

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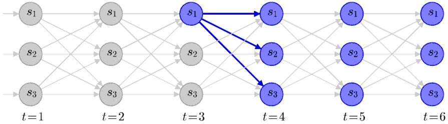

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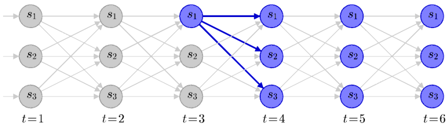

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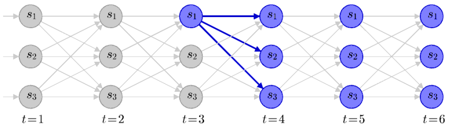

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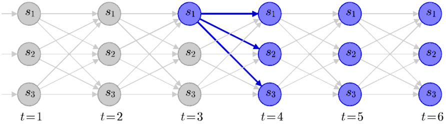

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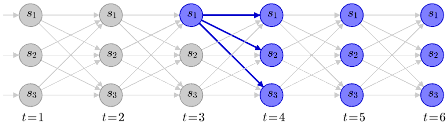

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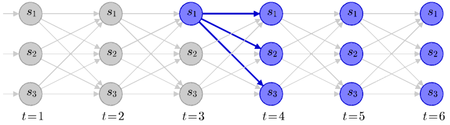

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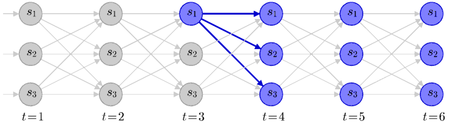

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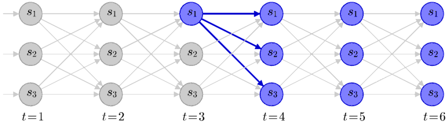

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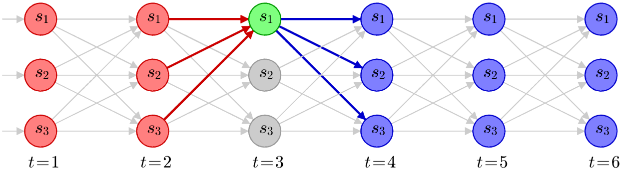

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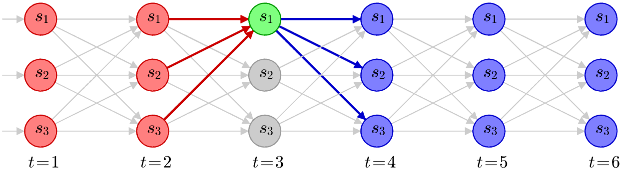

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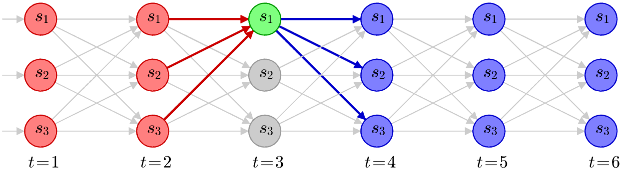

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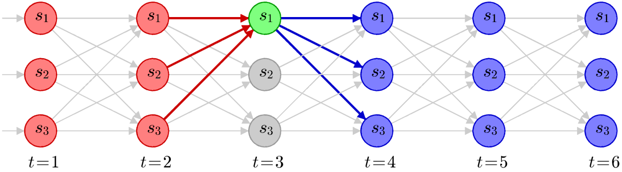
Probability of State



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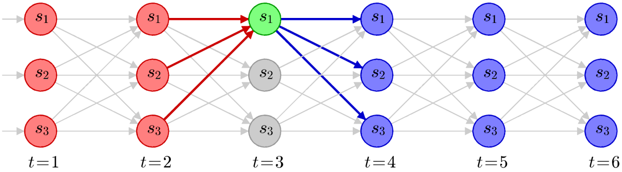

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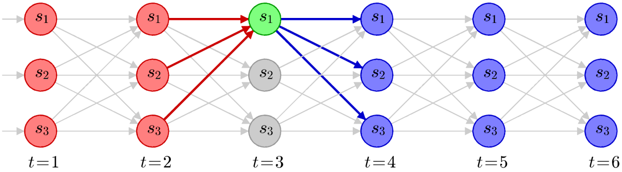

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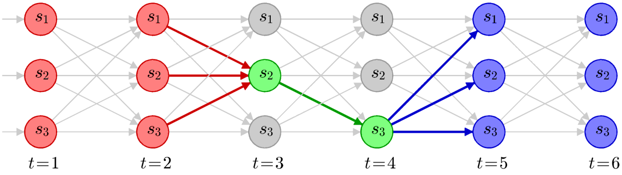
$$\begin{aligned}\gamma_t(i) &= P[q_t = s_i / O, \lambda] \\ &= \frac{P[q_t = s_i, O/\lambda]}{P[O/\lambda]} \\ &= \frac{\alpha_t(i)\beta_t(i)}{\sum_{i=1}^{N} \alpha_t(i)\beta_t(i)}\end{aligned}$$

Probability of State Transition



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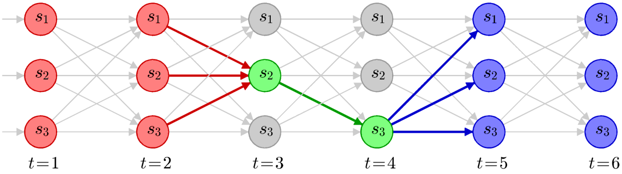
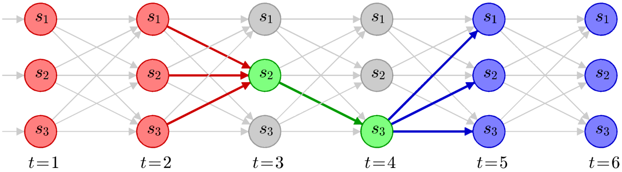


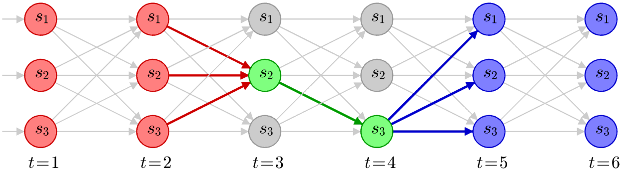
Probability of State Transition



- Probability of being in state s_i at time t and state s_j at time $t+1$, given OSS O and model λ

$$\xi_t(i,j) \quad = \quad P[q_t = s_i, q_{t+1} = s_j / O, \lambda]$$

Probability of State Transition



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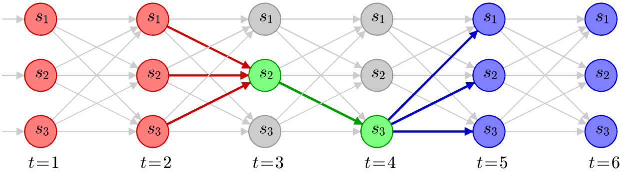
$$\begin{aligned}\xi_t(i,j) &= P[q_t = s_i, q_{t+1} = s_j / O, \lambda] \\ &= \frac{P[q_t = s_i, q_{t+1} = s_j, O/\lambda]}{P[O/\lambda]}\end{aligned}$$

Probability of State Transition



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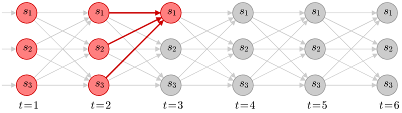
$$
\begin{aligned}
\xi_t(i,j) &= P[q_t = s_i, q_{t+1} = s_j / O, \lambda] \\
&= \frac{P[q_t = s_i, q_{t+1} = s_j, O/\lambda]}{P[O/\lambda]} \\
&= \frac{\alpha_t(i) a_{ij} b_j(o_{t+1}) \beta_{t+1}(j)}{\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_t(i) a_{ij} b_j(o_{t+1}) \beta_{t+1}(j)}
\end{aligned}
$$

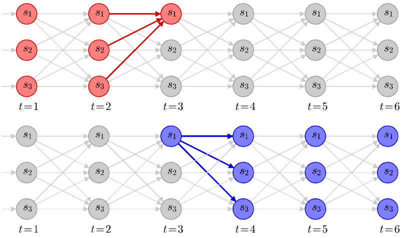
HMM Variables (Recap)



Forward variable:

$$\alpha_t(i) = P[o_1, o_2, \cdots o_t, q_t = s_i/\lambda]$$

HMM Variables (Recap)



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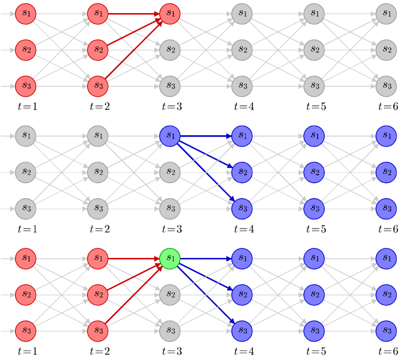
HMM Variables (Recap)



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State marginals:

$$\gamma_t(i) = P[q_t = s_i/O, \lambda]$$

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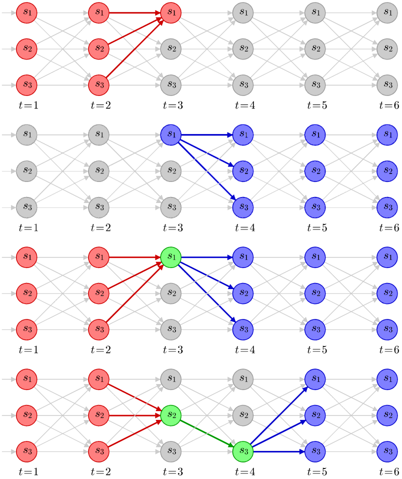

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State-transition marginals:

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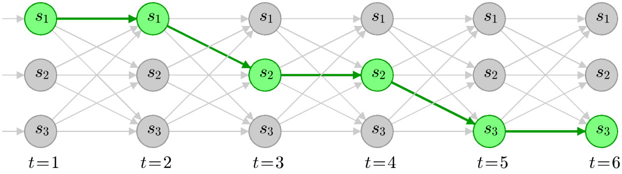
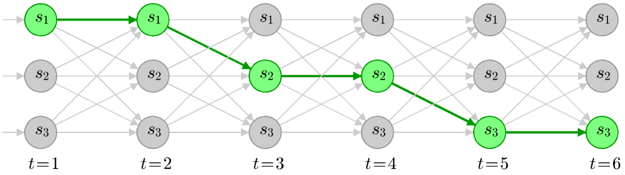
Optimal State Sequence



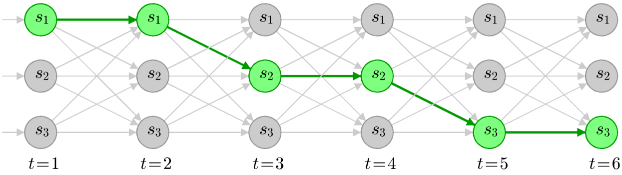
Optimal State Sequence



- Can local-best states result in global-best path?

$$q_t = \arg\max_{1 \le i \le N} \gamma_t(i) \qquad 1 \le t \le T$$

Optimal State Sequence



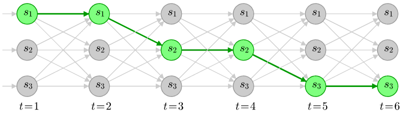
- Can local-best states result in global-best path?

$$q_t = \arg\max_{1 \leq i \leq N} \gamma_t(i) \qquad 1 \leq t \leq T$$

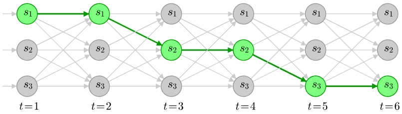
- Probability along the single best path ending in state S_i while generating the partial OSS $(o_1, o_2, \ldots o_t)$ given the

$$\delta_t(i) = \max_{q_1, q_2, \ldots q_{t-1}} P[o_1, o_2, \cdots o_t, q_1, q_2, \cdots q_t = s_i / \lambda]$$

Viterbi Algorithm

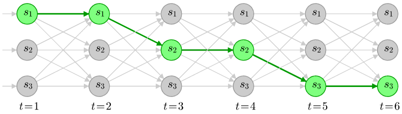


Viterbi Algorithm



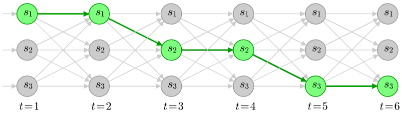
- Probability of producing a P-OSS $o_1, \cdots o_t$ along single best path ending in $q_t = S_i$.

Viterbi Algorithm



- Probability of producing a P-OSS $o_1, \cdots o_t$ along single best path ending in $q_t = S_i$.
- Initialization

Viterbi Algorithm



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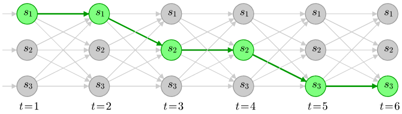
$$\begin{aligned} \delta_1(i) &= \pi_i b_i(o_1) \\ \psi_1(i) &= 0 \quad 1 \leq i \leq N \end{aligned}$$

Viterbi Algorithm



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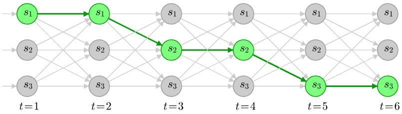

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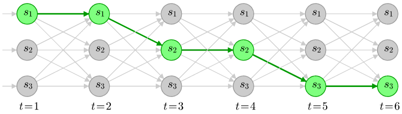
$$1 \leq j \leq N \quad 1 < t \leq T$$

Viterbi Algorithm



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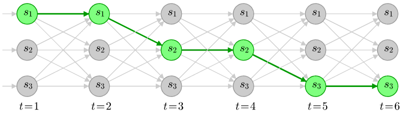

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$$P^* = \max_{1\leq i\leq N} \delta_T(i)$$
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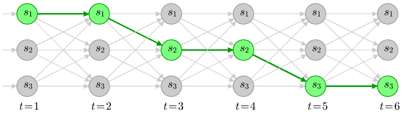
Viterbi Algorithm



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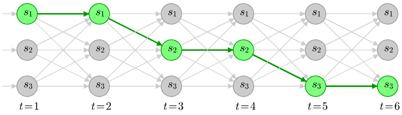
- Back-tracking

Viterbi Algorithm



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- Back-tracking

$$q_t^* = \psi_{t+1}(q_{t+1}^*)$$

$$t = T-1, T-2, \cdots 1$$

HMM Parameter Estimation

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- Given the OSS O, estimate the model parameters λ

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$$\mathcal{L}(\lambda) = \log P[O/\lambda]$$

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- Gradient ascent would be extremely slow
- Baum-Welch algorithm is used for iterative estimation

Parameter Estimation - Markov Model

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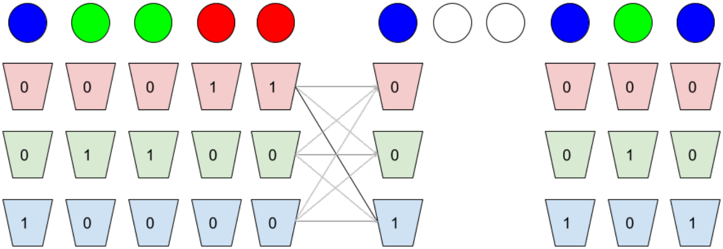
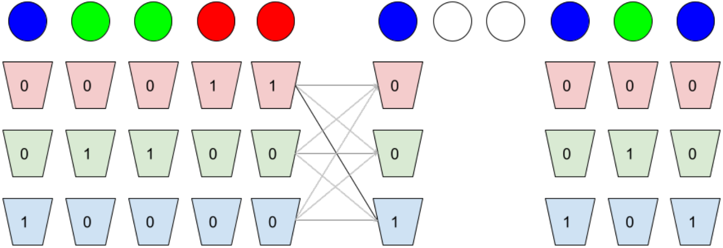
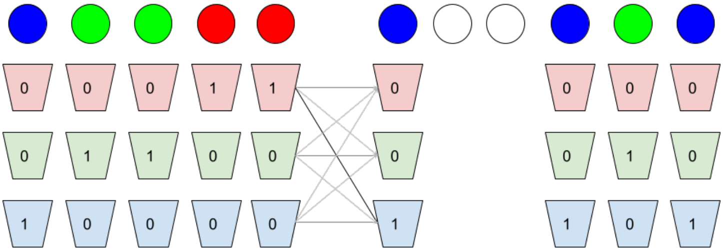


Parameter Estimation - Markov Model



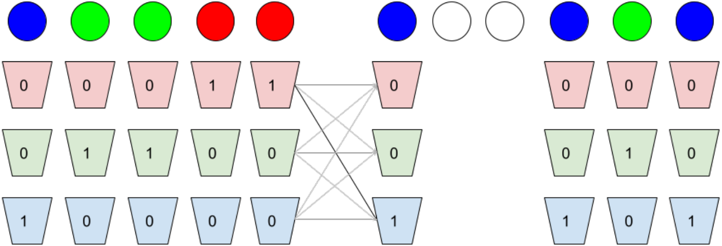
$$\hat{\pi}_R =$$

Parameter Estimation - Markov Model



$$\hat{\pi}_R = \text{Probability of starting from R} =$$

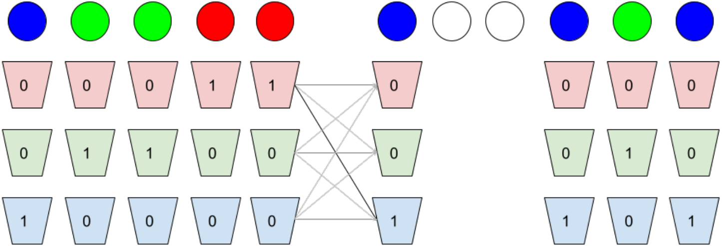
Parameter Estimation - Markov Model



$$\hat{\pi}_R = \text{Probability of starting from R} = \gamma_1(R)$$

$$\hat{a}_{RB} =$$

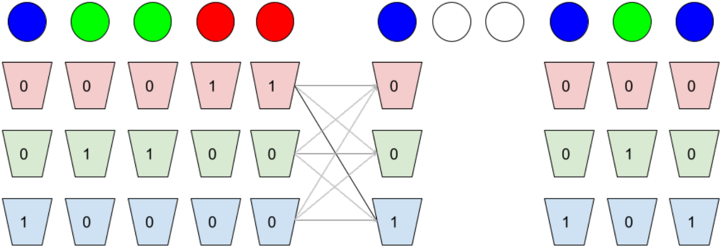
Parameter Estimation - Markov Model



$$\hat{\pi}_R \quad = \text{Probability of starting from R} \quad = \gamma_1(R)$$

$$\hat{a}_{RB} \quad = \frac{\text{Number of transitions from R to B}}{\text{Number of transitions from R}} \quad =$$

Parameter Estimation - Markov Model



$$\hat{\pi}_R = \text{Probability of starting from R} = \gamma_1(R)$$

$$\hat{a}_{RB} = \frac{\text{Number of transitions from R to B}}{\text{Number of transitions from R}} = \frac{\sum_{t=1}^{T-1} \xi_t(R,B)}{\sum_{t=1}^{T-1} \gamma_t(R)}$$

Parameter Estimation - HMM

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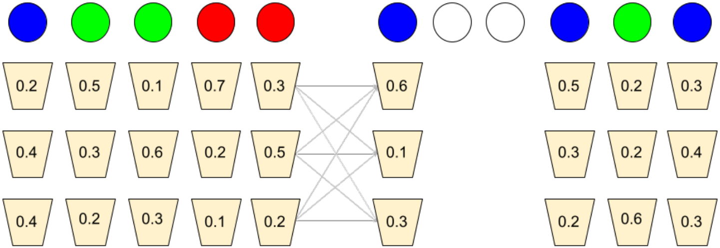
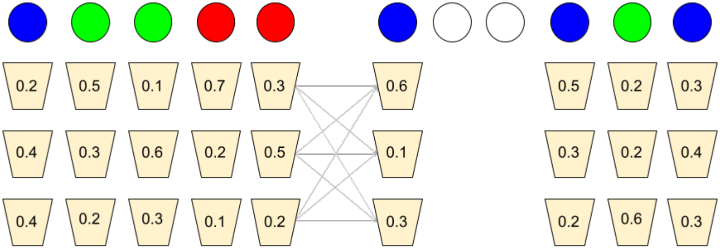
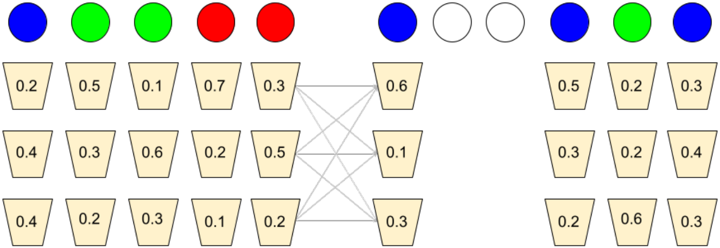


Parameter Estimation - HMM



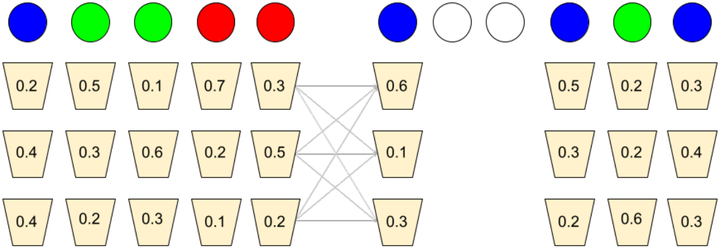
$$\hat{\pi}_i \qquad =$$

Parameter Estimation - HMM



$$\hat{\pi}_i \quad = \text{Probability of starting from } s_i \quad =$$

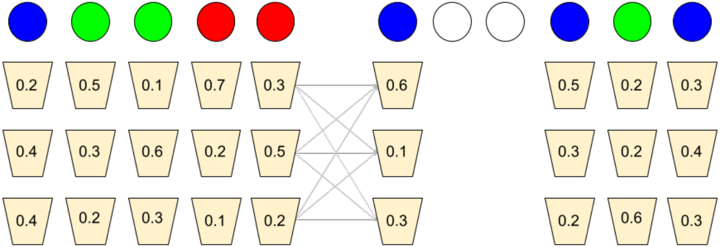
Parameter Estimation - HMM



$$\hat{\pi}_i \quad = \text{Probability of starting from } s_i \quad = \gamma_1(i)$$

$$\hat{a}_{ij} \quad =$$

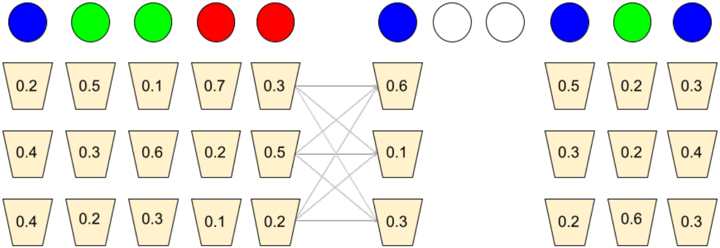
Parameter Estimation - HMM



$$\hat{\pi}_i \quad = \text{Probability of starting from } s_i \quad = \gamma_1(i)$$

$$\hat{a}_{ij} \quad = \frac{\text{Expected number of transitions from } s_i \text{ to } s_j}{\text{Expected number of transitions from } s_i} \quad =$$

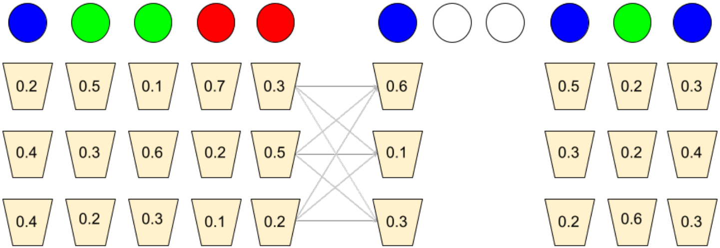
Parameter Estimation - HMM



$$\hat{\pi}_i = \text{Probability of starting from } s_i = \gamma_1(i)$$

$$\hat{a}_{ij} = \frac{\text{Expected number of transitions from } s_i \text{ to } s_j}{\text{Expected number of transitions from } s_i} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)}$$

Parameter Estimation - HMM



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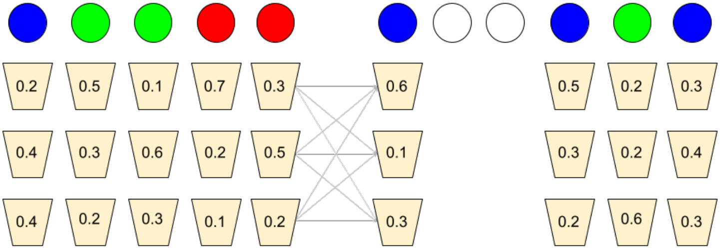
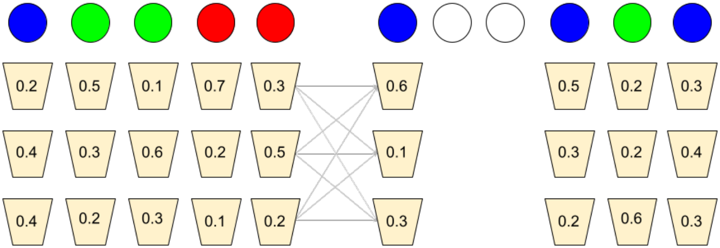


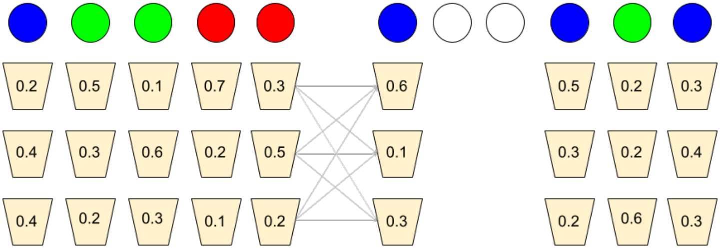
$$\hat{b}_j(v_k) \quad =$$

Parameter Estimation - HMM



$$\hat{b}_j(v_k) = \frac{\text{Expected number of times in } s_j \text{ observing } v_k}{\text{Expected number of times in } s_j} =$$

Parameter Estimation - HMM



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Baum-Welch Algorithm

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- Initialize HMM parameters $\lambda = (\mathbf{A}, \mathbf{B}, \pi)$ randomly such that

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$$\hat{\pi}_i = \gamma_1(i) \qquad \hat{a}_{ij} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)} \qquad \hat{b}_j(v_k) = \frac{\sum_{t=1,\ \text{s.t}\ o_t=v_k}^{T} \gamma_t(j)}{\sum_{t=1}^{T} \gamma_t(j)}$$

Continuous Process

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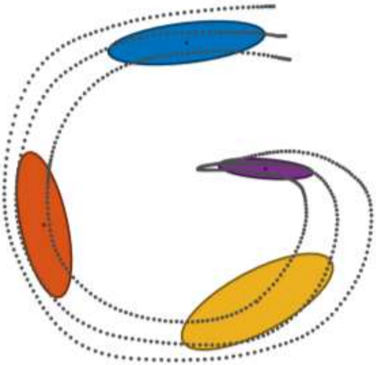

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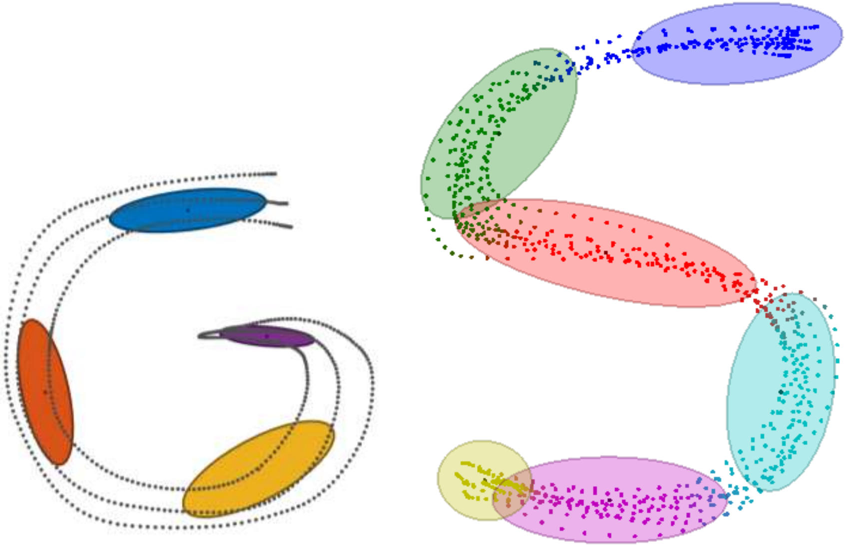

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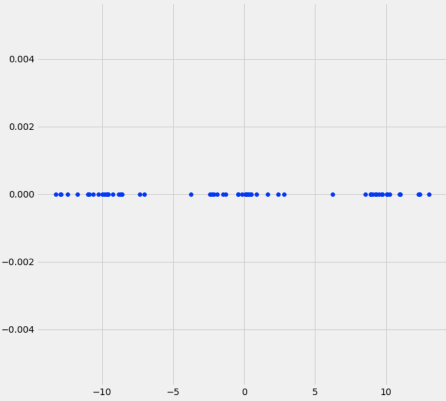
- In order to ensure that the pdf of the state integrates to 1

$$\sum_{k=1}^{M} w_{jk} = 1$$

When one Gaussian is not enough!



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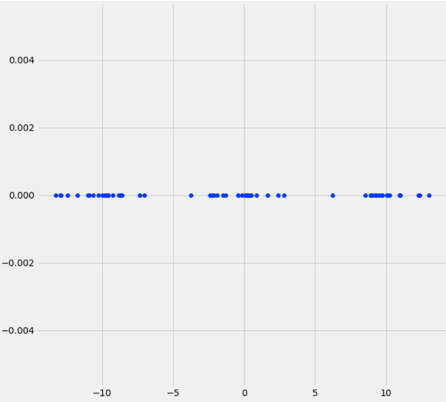
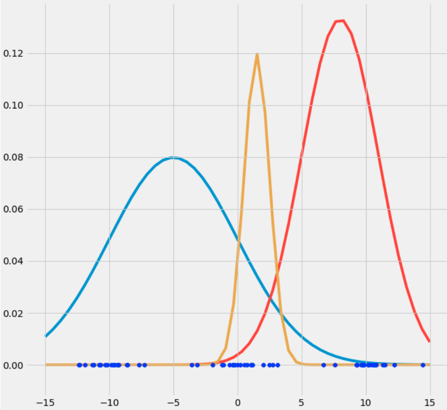
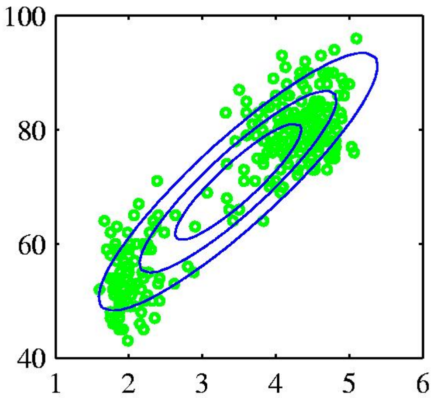




- Real-world datasets can have multimodal distribution
- Single Gaussian may not be a good approximation

2-D Illustration



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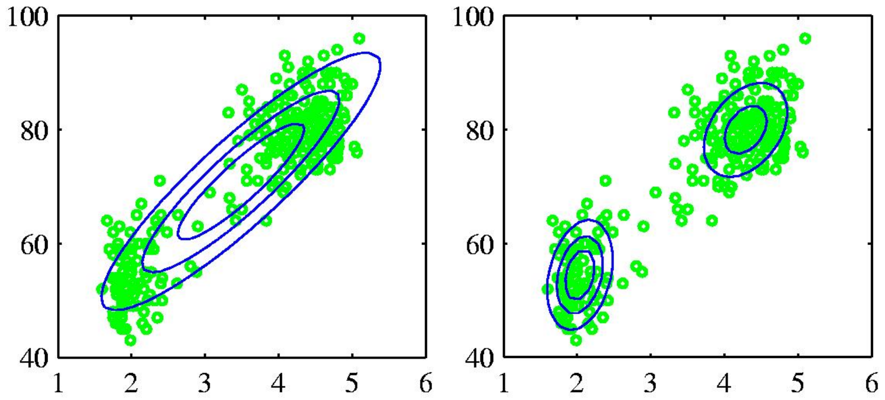
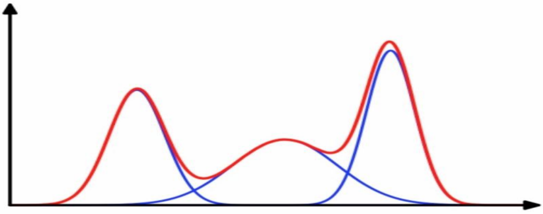


Gaussian Mixture Model (GMM)



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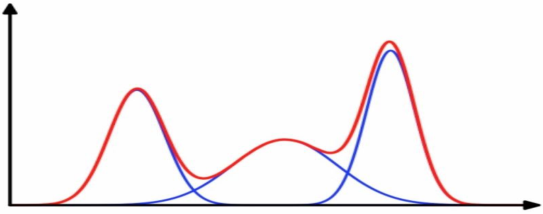

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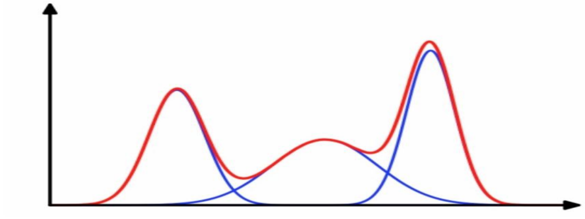

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$$p(x) = \sum_{m=1}^{M} \pi_m \mathcal{N}(x/\mu_m, \sigma_m^2)$$

- PDF should be positive and integrate to one

$$\pi_m \geq 0 \qquad \& \qquad \sum^{M} \pi_m = 1$$

Sampling from GMM

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- Given the observed data, we need to estimate the underlying pdf
  - Number of Gaussian components M is not known
  - M is typically determined empirically
  - Parameters of the GMM are estimated by maximizing the ML criterion

ML Formulation for GMM

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$$J(\theta) = \sum_{n=1}^{N} \log \left(\sum_{m=1}^{M} \pi_m \mathcal{N}(x_n/\mu_m, \sigma_m^2) \right) + \lambda \sum_{m=1}^{M} \pi_m = 1$$

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Interpretation of γ_{nk}

$$\gamma_{nk} = \frac{\pi_k \mathcal{N}(x_n/\mu_k, \sigma_k^2)}{\sum_{m=1}^{M} \pi_m \mathcal{N}(x_n/\mu_m, \sigma_m^2)}$$

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- After estimation, γ_{nk} can be interpreted as posterior probability

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- Are they in closed form? Can we directly compute them from data?

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- Compute log-likelihood with updated parameters
- Repeat E-step and M-step until convergence

Illustration of EM Iterations

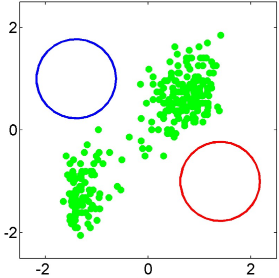


Illustration of EM Iterations

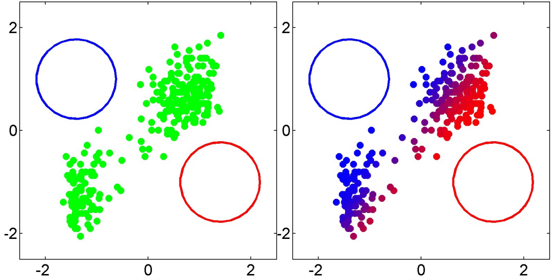


Illustration of EM Iterations

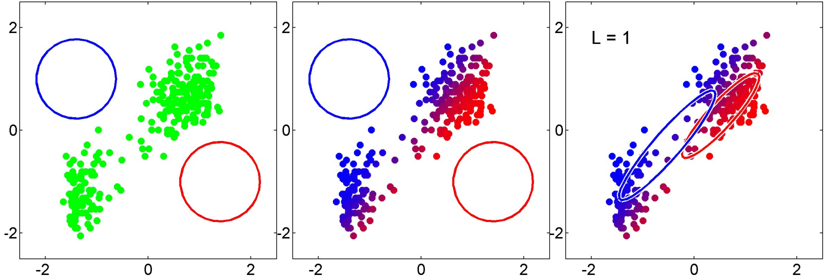
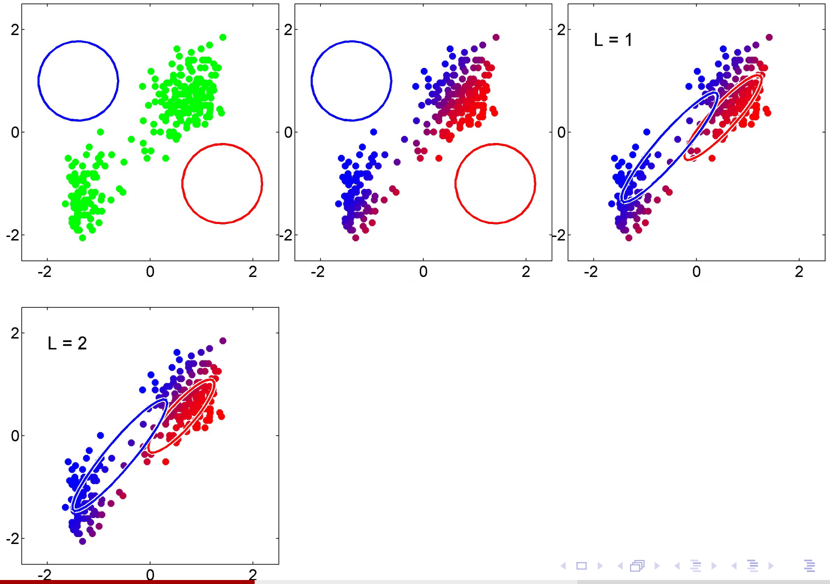


Illustration of EM Iterations



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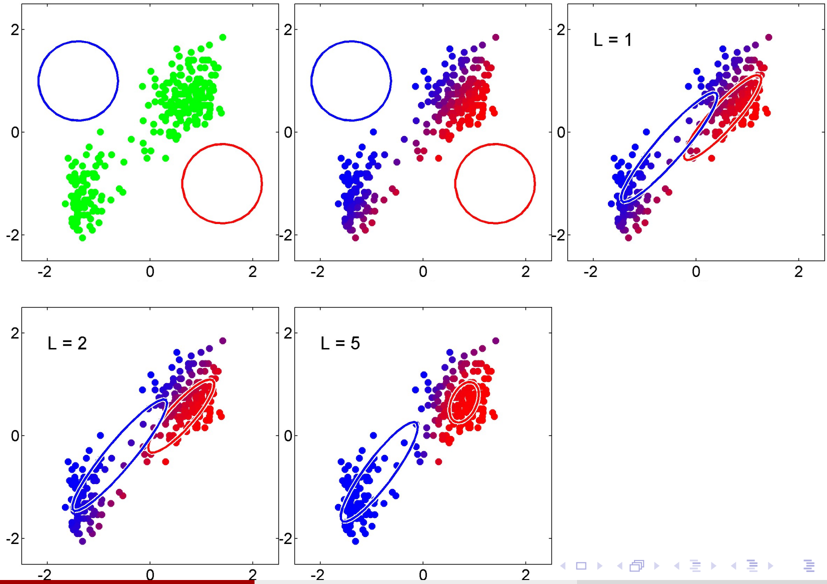

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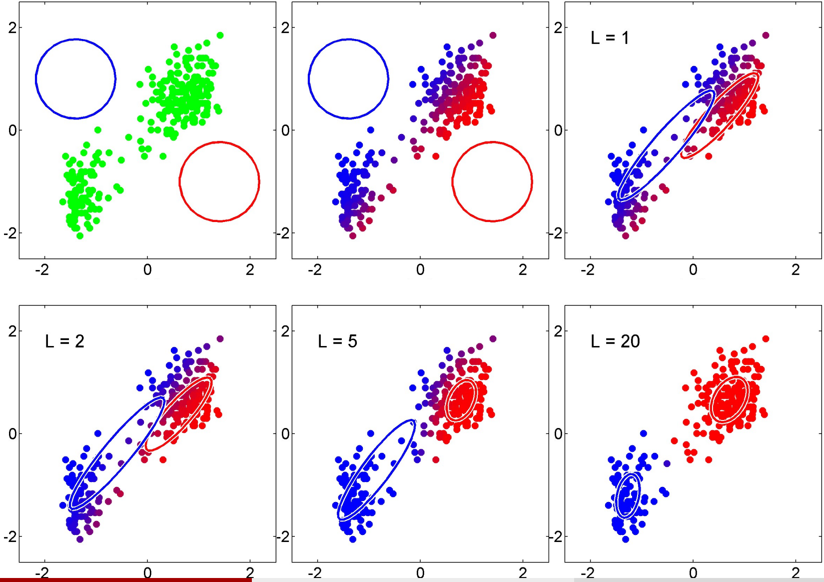

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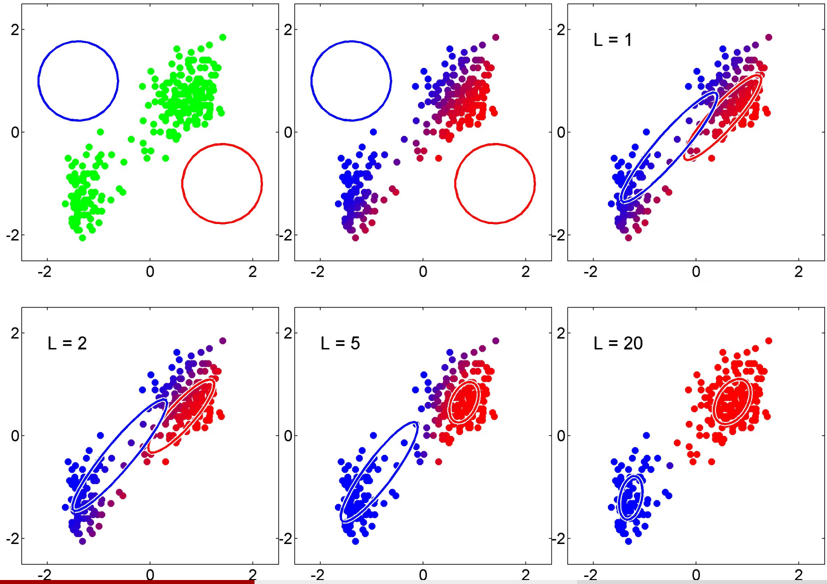
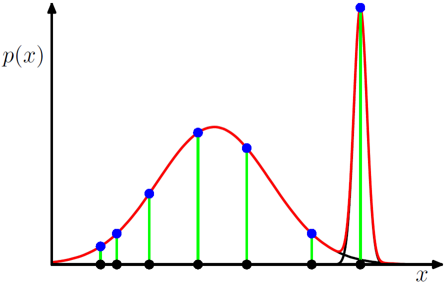


Singularities in Likelihood Function



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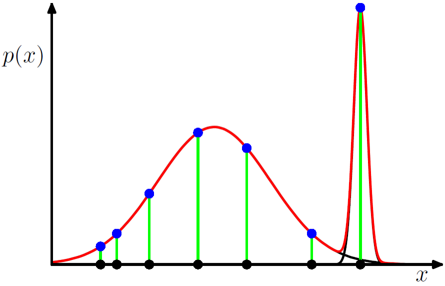
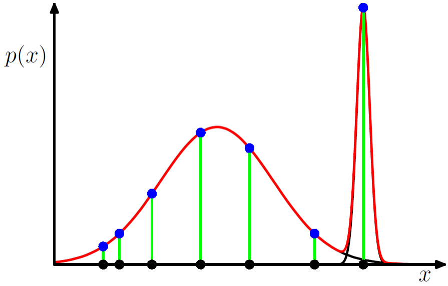


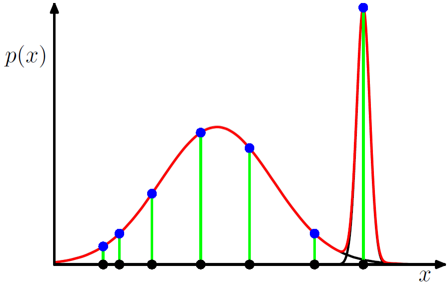
- One component has finite variance

Singularities in Likelihood Function



- One component has finite variance
- Some Gaussian components can collapse to specific data points

Singularities in Likelihood Function



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- Some Gaussian components can collapse to specific data points
- Shrinking variance can lead to ever increasing log-likelihood!

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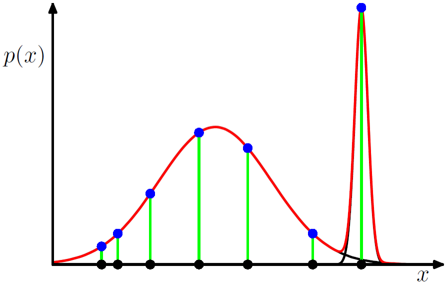

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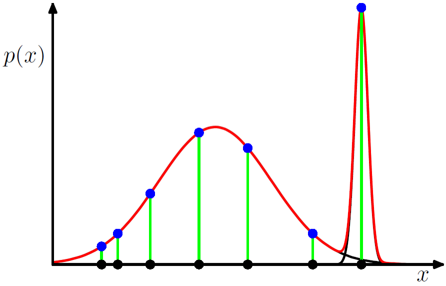
- ML approach can result in severe over-fitting (local maximum)

Singularities in Likelihood Function



- One component has finite variance
- Some Gaussian components can collapse to specific data points
- Shrinking variance can lead to ever increasing log-likelihood!
- ML approach can result in severe over-fitting (local maximum)
- Poor initialization, outliers, smaller dataset, more components