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EE22MTECH02003

Test - 2

(A) Given, $X \sim p(x/\theta) = \theta^2 x \exp(-\theta x) u(x)$

Assuming independent & identically distributions, likelihood function is —

$$p(X, \theta) = \prod_{n=1}^{N} p(x_n/\theta)$$

$$= \prod_{n=1}^{N} \theta^2 x_n \exp(-\theta x_n) u(x_n)$$

Now, to maximize likelihood :-

$$J(\theta) = \log p(X, \theta)$$

$$= \log \sum_{n=1}^{N} \theta^2 x_n \exp(-\theta x_n) u(x_n)$$

$$= \sum_{n=1}^{N} \left[\log(\theta^2 x_n) - \theta x_n\right]$$

Taking derivative :-

$$\frac{\partial J(\theta)}{\partial \theta} = \sum_{n=1}^{N} \left(\frac{1}{\theta^2 x_n} 2\theta x_n - x_n\right)$$

$$= \sum_{n=1}^{N} \left(\frac{2}{\theta} - x_n\right) = \frac{2N}{\theta} - \sum_{n=1}^{N} x_n$$

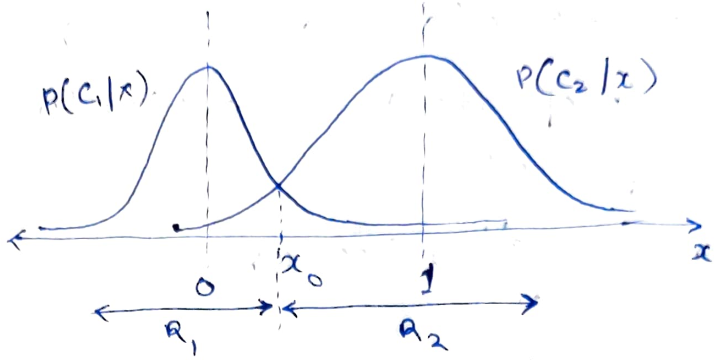
Equating it to zero to get maximized sol^n :-

$$\Rightarrow \frac{1}{\theta} = \frac{1}{2N}\left(\sum_{n=1}^{N} x_n\right)$$

$$\Rightarrow \boxed{\hat{\theta} = \frac{2N}{\sum_{n=1}^{N} x_n}}$$

ML Estimate of θ

(2) Given, class conditional densities for two classes :- $\mathcal{N}(0, \sigma^2)$ & $\mathcal{N}(1, \sigma^2)$



Now, assigning x to C_j that minimizes
$$\sum_K L_{Kj}\, p[C_K \mid x]$$

Given, Loss matrix : $L = \begin{bmatrix} 0 & L_{12} \\ L_{21} & 0 \end{bmatrix}$

from the plot, decision boundary is where weighted joint probabilities are same i.e.

$$\Rightarrow L_{12}\, p(x, C_1) = L_{21}\, p(x, C_2)$$

$$\Rightarrow L_{12}\, p(C_1)\, \mathcal{N}(0, \sigma^2) = L_{21}\, p(C_2)\, \mathcal{N}(1, \sigma^2)$$

$$\Rightarrow \log L_{12}\, p(C_1) - \frac{x_0^2}{2\sigma^2} = \log L_{21}\, p(C_2) - \frac{(x-1)^2}{2\sigma^2}$$

$$\Rightarrow \frac{(x_0-1)^2}{2\sigma^2} - \frac{x_0^2}{2\sigma^2} = \log \frac{L_{21}\, p(C_2)}{L_{12}\, p(C_1)}$$

$$\Rightarrow \frac{x_0^2 + 1 - 2x_0 - x_0^2}{2\sigma^2} = \log \frac{L_{21}\, p(C_2)}{L_{12}\, p(C_1)}$$

$$\Rightarrow \frac{1}{2\sigma^2} - \frac{x_0}{\sigma^2} = \log \frac{L_{21}\, p(C_2)}{L_{12}\, p(C_1)}$$

$$\Rightarrow \boxed{x_0 = \frac{1}{2} - \sigma^2 \log \frac{L_{21}\, p(C_2)}{L_{12}\, p(C_1)}}$$

1 Given, $0 \leq a \leq b$.

$\therefore \quad \sqrt{a}(\sqrt{b} - \sqrt{a}) \geq 0$

$\sqrt{ab} - a \geq 0$

$\therefore \quad \boxed{a \leq \sqrt{ab}}$ ———— (a)

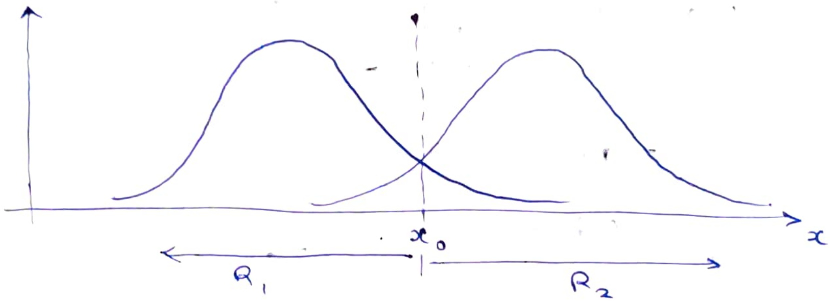
Now, we know,

$$p(\text{mistake}) = \int_{R_1} p(x, C_2)\,dx + \int_{R_2} p(x, C_1)\,dx$$

Now, we have to choose region R_1 such that :- $p(x, C_1) > p(x, C_2)$ ———— (b)

And choose region R_2 such that :-

$p(x, C_1) < p(x, C_2)$...



$\therefore$ (say for R_1) :-

$$\frac{p(x, C_1)}{P(x)} > \frac{p(x, C_2)}{P(x)}$$

Decision Rule will be to assign class C_1 if :- $p[C_1/x] > p[C_2/x]$

using (a) & (b) we can write :—

$$\int_{R_1} p(x, C_2)\,dx \leq \int_{R_1} \sqrt{p(x, C_1)\,p(x, C_2)}\,dx$$

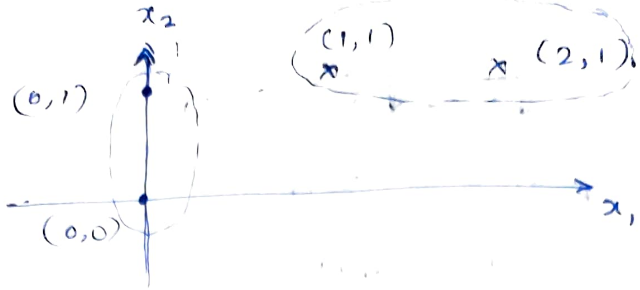
And same will be the case for R_2

Hence,

$$\boxed{p(\text{mistake}) \leq \int \sqrt{p(x, C_1)\,p(x, C_2)}\,dx}$$

3) Given,

(a)



$\therefore$ $$\underline{X} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}_{5\times 2} \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}_{5\times 2}$$

from Least squares :-

$$w_* = (X^T X)^{-1} X^T T \quad \Leftarrow \text{pseudo inverse sol}^n$$

$$= \left[\begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\right]^{-1}\begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\underline{w}_* = \begin{pmatrix} -0.5 & 0.5 \\ 0.75 & 0.25 \end{pmatrix}_{2\times 2}$$

$\therefore$ Linear discriminant function :-

$$\underline{y} = w_*^T x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0.5 & 0.25 \\ 0 & 1 \end{pmatrix}$$

(b) we know,

$$\bar{w}_K = \underbrace{\bar{w}_0}_{=0} + \sum_{k=1}^{K} x_k t_k$$

So norm is bounded by:-

$$\| w_K \| = \left\| \sum_{k=1}^{K} \bar{x}_k t_k \right\| \le \sum_{k=1}^{K} \| \bar{x}_k \|$$

let α be the maximum norm in data the :-

$$\alpha = \max_n \| x_n \|$$

$$\therefore \quad \| \bar{\omega}_K \| \leq K \alpha$$

Now,

$$\bar{\omega}^T_* \bar{\omega}_K = \sum_{K=1}^{K} \omega^T_* \bar{x}_K t_K$$

$$\beta = \min \bar{\omega}_* \bar{x}_n t_n$$

Now,

$$\bar{\omega}^T_* \omega_K \geq K\beta$$

$$\| \bar{\omega}_K \| \geq \frac{1}{\| \omega^* \|} K^2 \beta^2$$

~~Lower bound~~

range of norm $\bar{\omega}_K$ is :-

$$K^2 \beta' \leq \| \omega_K \| < K \alpha$$

$\therefore$ there must be a point where

LB = UB

Hence convergence reached.