INDIAN INSTITUTE OF SPACE SCIENCE AND TECHNOLOGY
THIRUVANANTHAPURAM

Assignment #1

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Contents

Metric Space,Normed Space,Vector Space

1. Show that the set X of all integers with metric defined by d(m,n) = |m-n| is a complete metric space.

A sequence x_1,x_2,x_3... is called a Caushy's sequence if
$\forall \epsilon > 0 \qquad \exists N| \; \forall m, n > N \; d(x_m, x_n) < \epsilon$
Take ϵ=$\frac{1}{2}$.
Then $\exists N : \; \forall m, n > N \; d(x_m, x_n) < \frac{1}{2}$
Since X is the set of integers and the difference cannot exceed $\frac{1}{2}$, then it should be 0; ie if a metric d is defined then $d(x_m, x_n)$ should be zero if ϵ=$\frac{1}{2}$.

$$\implies d(x_m, x_n) = |m - n| = 0$$

$$\implies x_m = x_n \forall m, n > N$$

Thus the sequence $\{x_n\} \longrightarrow 0$ when n>N. Hence all cauchy's sequences converges,so (X, d) is a complete metric space.

2. Show that d(x,y) = $\sqrt{|x-y|}$ defines a metric on the set of all real numbers

$$d(x, y) = \sqrt{|x-y|} \forall x, y \in \mathbb{R}$$

To check that whether d is a metric, we need to verify all the axioms of a metric.
a. Since |x-y| > 0 $\forall x, y \in \mathbb{R}$ d(x,y)>0 always.
and d(x,y)=0 only when |x-y|=0 $\implies$ x=y.

b.

$$\forall x, y \in \mathbb{R} |x-y| = |y-x|$$

$$\implies d(x, y) = \sqrt{|x-y|} = \sqrt{|y-x|} = d(y, x)$$

c.Triangular inequality
it states that $\forall x, y, z \in \mathbb{R}$

$$\sqrt{|x-z|} \le \sqrt{|x-y|} + \sqrt{|y-z|}$$

we know that,

$$|x-z| = |x-y+y-z| \le |x-y| + |y-z|$$

Thus it follows from the properties of square roots and the above inequality

$$\sqrt{|x-z|} \le \sqrt{|x-y|} + \sqrt{|y-z|}$$

Hence all the axioms are satisfied. So d(x,y) = $\sqrt{|x-y|}$ is a metric in $\mathbb{R}$.

3. Show that the closure $\overline{Y}$ of a subspace Y of a normed space X is again a vector space.

Inorder to prove that $\overline{Y}$ is a vector space it is sufficient to establish that

$$\alpha x + \beta y \in \overline{Y} \forall x, y \in \overline{Y}$$

and α&β are from the underlying field *F*.
We know that $0 \in \overline{Y}$ since $Y \subset \overline{Y}$. Since x,y $\in \overline{Y}$ there exists $x_i, y_i \in X$ such that $x_i \longrightarrow x$ and $y_i \longrightarrow y$.Since multiplication and addition are continuous,

$$\alpha x_i + \beta y_i \longrightarrow \alpha x + \beta y$$

Therefore,$\alpha x + \beta y \in \overline{Y}$

4. Show that in an inner product space, x⊥y iff $||x+\alpha y|| \geq ||x|| \forall \alpha \in \mathbb{R}$

we know that,

$$||x+\alpha y||^2 = \langle x+\alpha y, x+\alpha y\rangle$$

$$= \langle x,x\rangle + \alpha\langle x,y\rangle + \alpha\langle y,x\rangle + \alpha^2\langle y,y\rangle$$

Assuming the underlying field to be $\mathbb{R}$, the inner product becomes symmetric, and we obtain

$$||x+\alpha y||^2 = \langle x,x\rangle + 2*\alpha\langle x,y\rangle + \alpha^2\langle y,y\rangle$$

If x⊥y then $\langle x,y\rangle$=0. Thus

$$||x+\alpha y||^2 = ||x||^2 + \alpha^2||y||^2$$

$$\implies ||x+\alpha y||^2 \geq ||x||^2$$

$$\implies ||x+\alpha y|| \geq ||x||$$

since, $\alpha^2||y||^2$ is always a positive value. This will violate only when the following two conditions occur simultaneously.
I) x is not perpendicular to y
II) $2*\alpha\langle x,y\rangle \geq -\alpha^2||y||^2$
Thus only if part is also verified.

5. Find $\langle u,v\rangle$, where v=$(1+2i,3-i)^T$, $u=(-2+i,4)^T$

$$\langle u,v\rangle = \langle(-2+i,4),(1+2i,3-i)\rangle$$

for complex numbers $\langle(x1,x2)(y1,y2)\rangle = x1*\overline{y1} + x2*\overline{y2}$

$$= (-2+i)(1-2i) + 4(3+i)$$

$$= -2+4i+i+2+12+4i$$

$$= 9i+12$$

6. Which of the following subsets of $\mathbb{R}^3$ constitute a subspace of $\mathbb{R}^3$? [x=$(\eta_1,\eta_2,\eta_3)^T$]

(a) **All x with** $\eta_1=\eta_2$ **and** $\eta_3=0$**.**
(b) **All x with** $\eta_1=\eta_2+1$
a)
Let Z={All x with $\eta_1=\eta_2$ and $\eta_3=0$}.
Consider X=$(x,x,0)$,Y=$(y,y,0) \in Z$

$$X+Y = (x+y,x+y,0) \in Z$$

$$\alpha X = (\alpha x,\alpha x,0) \in Z$$

Thus Z is closed under addition and scalar multiplication,hence it is a subspace of $\mathbb{R}^3$.
b)
Let Z={All x with $\eta_1=\eta_2+1$}.
Consider X=$(x+1,x,p)$,Y=$(y+1,y,q) \in Z$ where p,q$\in\mathbb{R}$.

$$X+Y = (x+y+2,x+y,p+q)) \notin Z$$

because $\eta_1 \neq \eta_2+1$ is violated here. Hence Z is not closed under addition. So it is not a subspace of $\mathbb{R}^3$.

7. Show that the norm ||x|| is the distance from x to 0

Every normed space is a metric space or norm induces a metric on a vector space. Thus in a metric space with an induced norm

$$d(x,y) = ||x-y||$$

We know that d(x,y) is,

$$d: X * X \longrightarrow \mathbb{K}$$

and norm is,

$$||\cdot||: X \longrightarrow \mathbb{K}$$

where $\mathbb{K}$ is the underlying field.
if ||x|| is a metric in a metric space then we have,

$$d(x,y) = ||x||$$

$$\implies y = 0$$

It implies that we are calculating the distance from origin. Hence ||x|| is the distance from 0.

8. If in an inner product space $\langle x,u\rangle = \langle x,v\rangle$ for all x , show that u=v.

Since $\langle x,u\rangle = \langle x,v\rangle$,

$$\langle x,u\rangle - \langle x,v\rangle = \langle x,u-v\rangle = 0$$

But we know that inner product is zero only when one of the two vectors is zero.(orthogonality case can be avoided, since x is neither orthogonal to u,nor to v, hence it cannot be orthogonal to a linear combination of u & v.)
Here x cannot be zero ∀ x.

$$\implies u - v = 0$$

$$\implies u = v$$

9. Prove that $||T_1T_2|| \leq ||T_1|| * ||T_2||; ||T^n|| \leq ||T||^n$

This property is called submultiplicative property and is only valid for matrix norms.
An induced matrix norm ||T|| is defined as

$$||T|| = \max_{x\neq 0} \frac{||Tx||}{||x||}$$

Thus $||T_1T_2||$ is

$$||T_1T_2|| = \max_{x\neq 0} \frac{||T_1T_2x||}{||x||}$$

$$= \max_{x\neq 0} \frac{||T_1T_2x||}{||T_2x||}\frac{||T_2x||}{||x||}$$

Putting $T_2x = y$ in the first part

$$\leq \max_{y\neq 0} \frac{||T_1y||}{||y||} * \max_{x\neq 0} \frac{||T_2x||}{||x||}$$

$$\leq ||T_1|| * ||T_2||$$

b)

$$||T^n|| \leq ||T|| * ||T\ldots T||$$

$$\leq ||T|| * ||T|| * ||T\ldots T||$$

$$\leq ||T|| * ||T|| * \ldots * ||T||$$

$$\leq ||T||^n$$

10. For a real inner product space prove that $\langle x, y\rangle = \frac{1}{4}(||x+y||^2 - ||x-y||^2)$

We know that,

$$||x+y||^2 = \langle x+y, x+y\rangle$$

$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$

but for real IPS $\langle x,y\rangle = \langle y,x\rangle$.also $\langle x,x\rangle = ||x||^2$,Then,

$$||x+y||^2 = ||x||^2 + 2 * \langle x,y\rangle + ||y||^2$$

Similarly,

$$||x-y||^2 = ||x||^2 - 2 * \langle x,y\rangle + ||y||^2$$

$$||x+y||^2 - ||x-y||^2 = 4 * \langle x,y\rangle$$

Thus,

$$\langle x,y\rangle = \frac{1}{4}(||x+y||^2 - ||x-y||^2)$$

11. Define T: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by T(x,y)=(x,0). Is T a linear operator ?

An operator is said to be linear if
a)D(T) and R(T) are vector spaces over the same field $\mathbb{K}$.
b)T(x+y) = T(x)+T(y)
$T(\alpha x) = \alpha T(x)$

In this case,D(T) = $\mathbb{R}^2$ is a vector space.
R(T) = (x,0) where x∈ $\mathbb{R}$ is also a vector space. Let X,Y ∈ $\mathbb{R}^2$

$$T(X+Y) = T((x_1,x_2)+(y_1,y_2))$$

$$= T(x_1+y_1, x_2+y_2)$$

$$= (x_1+y_1, 0) = (x_1,0)+(y_1,0)$$

$$= T(X)+T(Y)$$

Checking for the other condition,

$$T(\alpha X) = T(\alpha x_1, \alpha x_2)$$

$$= (\alpha x_1, 0) = \alpha(x_1, 0)$$

$$= \alpha T(X)$$

All the conditions are satisfied. Hence T is a linear operator.

12. Show that a discrete metric space is complete.

Discrete metric ρ on a set X is defined by,

$$\rho(x,y) = \begin{cases} 1 & \text{if} x \neq y \\ 0 & \text{if} x = y \end{cases}$$

for any x,y ∈ X. Here (X,ρ) is a disrete metric space.
A sequence x_1,x_2,x_3... is called a Caushy's sequence if
$\forall \epsilon > 0 \qquad \exists N| \; \forall m,n > N \; d(x_m,x_n) < \epsilon$

Take $\epsilon = \frac{1}{2}$.
Then $\exists N : \ \forall m, n > N \ d(x_m, x_n) < \frac{1}{2}$
But possible values of d are {0,1}. Since distance cannot exceed $\frac{1}{2}$, $d(x_m, x_n)$ should be zero.

$$\implies d(x_m, x_n) = 0 \forall m, n > N$$

$$\implies x_n = x_m$$

Thus $\{x_n\} \longrightarrow x \ \forall n > N$. This means every Caushy's sequence converges in X. So discrete metric space is complete.

13. Descibe Weistrass appoximation theorem.

Theorem: if f is a continuous real valued function on *[a,b]* and if any $\epsilon > 0$ is given, then there exists a polynomial p on *[a,b]* such that

$$|f(x) - p(x)| < \epsilon$$

for all x in [a,b]. In words, any continuous function on a closed and bounded interval can be uniformly approximated on that interval by polynomials to any degree of accuracy[1].

Because polynomials are among the simplest functions, and because computers can directly evaluate polynomials, this theorem has both practical and theoretical relevance, especially in polynomial interpolation[2].As a consequence of the Weierstrass approximation theorem, one can show that the space $C[a, b]$ is separable: the polynomial functions are dense, and each polynomial function can be uniformly approximated by one with rational coefficients; there are only countably many polynomials with rational coefficients.

References

[1] Weisstein, Eric W. "Weierstrass Approximation Theorem." From MathWorld–A Wolfram Web Resource. *http://mathworld.wolfram.com/WeierstrassApproximationTheorem.html*

[2] Stone-Weierstrass theorem, *http://en.wikipedia.org/wiki/Stone-Weierstrass_theorem*