Econ 1078-002 Midterm Exam 2
April 4th, 2018

ANSWER KEY

- Do not open the exam booklet until instructed to do so. Close your exam booklet promptly when I say time is up.
- Read the questions closely, and answer them completely. **Answer the questions you find easiest first!!!!**
- The exam is for 100 points. **There are 10 points of extra credit questions at the back**, so a score of 110/100 is possible.
- Show your work to get partial credit (i.e., convince me you understand what's asked of you and what you're doing).
- Please write your answers clearly on the exam booklet, and draw a box around your final answer. There are useful formulas on the last page.
- Please raise your hand if a question is unclear or you need scratch paper. Take as much scratch paper as you need.
- Assume all values are in $\mathbb{R}$. When there are multiple real solutions, provide all of them unless explicitly directed otherwise. If there is no real solution, state this with a brief explanation of why (1 sentence or less). You are not required to provide complex solutions.
- Deep breaths. It's going to be ok. Do your best, and have fun!

Name: ______________________________

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: ______________________________

Date: ______________________________

1 Mechanical questions (88 points)

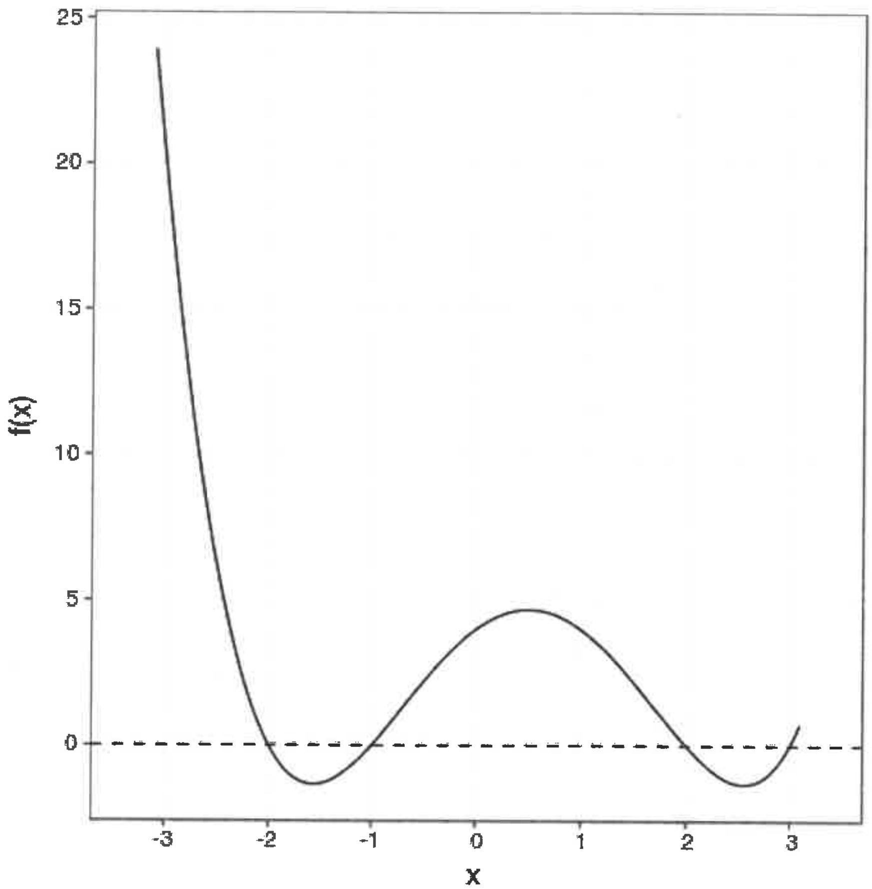
1. (13 points) Let $f(x) = p^x(1-p)^{1-x}$. Show that $\ln(f(x)) = \ln(0.5)$ when $p = 0.5$.

When $p = 0.5$,

$$
\begin{aligned}
\ln(f(x)) &= \ln\left(p^x(1-p)^{1-x}\right) \\
&= \ln(p^x) + \ln\left((1-p)^{1-x}\right) \\
&= x\ln(p) + (1-x)\ln(1-p) \\
&= x\ln(0.5) + (1-x)\ln(0.5) \\
&= \cancel{x\ln(0.5)} + \ln(0.5) - \cancel{x\ln(0.5)} \\
&= \ln(0.5).
\end{aligned}
$$

T.S.

2. (13 points) Use the method of linear factors to find a function $f(x)$ which produces the graph below, given that $f(1) = 4$.



roots at : $-2, -1, 2, 3$ $\Rightarrow$ $f(x) = b\,(x+2)(x+1)(x-2)(x-3)$

$f(1) = 4$, and $f(1) = b\,(1+2)(1+1)(1-2)(1-3) = b(3)(2)(-1)(-2) = 12b$

$\Rightarrow 4 = 12b$

$\Rightarrow b = 4/12 = 1/3$

So $\boxed{f(x) = \frac{1}{3}(x+2)(x+1)(x-2)(x-3)}$

3. (13 points) Find all integer solutions to $e^{-x^6+8x^3} = (e^{-7})^{-1}$ (hint: you don't need to check $\{-7, +7\}$, and you are solving for x)

$e^{-x^6+8x^3} = (e^{-7})^{-1} = e^7$. Since bases are same,

$\Rightarrow \quad -x^6 + 8x^3 = 7$

Factors of $7 = \{-1, +1, -7, +7\}$

$\underline{-1}$: $-(-1)^6 + 8(-1)^3 = -1 - 8 = -9 \neq 7$ ✗

$\underline{+1}$: $-(1)^6 + 8(1)^3 = -1 + 8 = 7$ ✓

$\boxed{+1 \text{ is the integer solution}}$

4. (13 points) Suppose $Y = 2, X = 3$. Use equation (1) to plot $\epsilon(\beta)$ and $\epsilon(\beta)^2$ with β on the horizontal axis. Label all axis intercepts and places where $\epsilon(\beta)$ and $\epsilon(\beta)^2$ intersect.

$$Y = X\beta - \epsilon \tag{1}$$

$\varepsilon = Y - X\beta$. With the given numbers,

$\varepsilon(\beta) = 2 - 3\beta \leftarrow$ linear in β

$a > 0 \Rightarrow$ cup (min, no max)

$\varepsilon(\beta)^2 = (2-3\beta)^2 = (2-3\beta)(2-3\beta) = \underbrace{4}_{c} \underbrace{- 12\beta}_{b} + \underbrace{9}_{a}\beta^2 \leftarrow$ quadratic in β

$\underline{\varepsilon(\beta):}$ $\varepsilon(0) = 2$

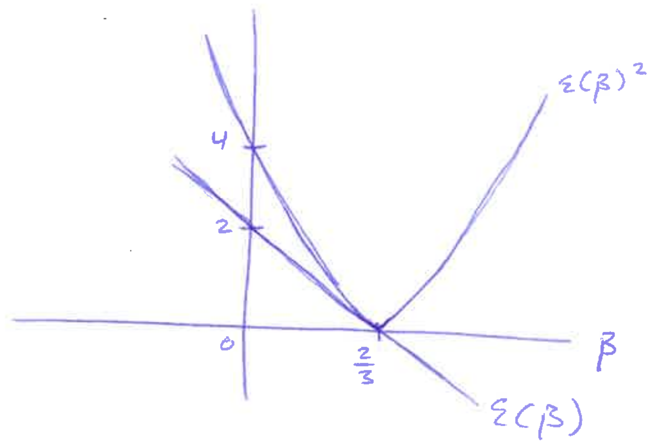
$\beta : \varepsilon(\beta) = 0$

$\Rightarrow \beta = 2/3$

$\underline{\varepsilon(\beta)^2:}$ $\varepsilon(0) = 4$

β : $\varepsilon(\beta)^2$ is minimized

$\Rightarrow \beta = -\frac{b}{2a} = \frac{12}{18} = \frac{2}{3}$



5. (10 points) Use the equation below to determine whether $M(q) = L(q+1) - L(q)$ is increasing, decreasing, or constant as q increases:

$$pq = rq + L(q) + qM(q)$$

Assume that $p > r$, $L(q)$ is always nonnegative, and that $L(q)/q$ is increasing as q increases. (Hint: if $f(x)$ is increasing, $-f(x)$ is decreasing.)

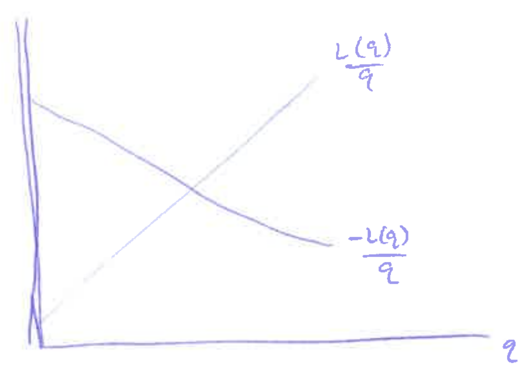
$$\frac{p\cancel{q}}{\cancel{q}} = \frac{r\cancel{q}}{\cancel{q}} + \frac{L(q)}{q} + \frac{\cancel{q}M(q)}{\cancel{q}}$$

$$\Rightarrow p = r + \frac{L(q)}{q} + M(q)$$

$$\Rightarrow \underbrace{p - r}_{\text{constant}} - \underbrace{\frac{L(q)}{q}}_{\text{increasing}} = M(q)$$

$$M(q) = \text{constant} - \text{increasing}$$
$$= \text{constant} + \text{decreasing}$$

$$\Rightarrow \boxed{M(q) \text{ is decreasing}}$$



6. (13 points) Prove that $x-2$ is a factor of $P(x) = x^6 - 2x^5 + x^4 - 2x^3 + x^2 - 2x$.

Proof:

$$P(2) = 2^6 - 2(2)^5 + 2^4 - 2(2)^3 + 2^2 - 2(2)$$
$$= (2^6 - 2^6) + (2^4 - 2^4) + (2^2 - 2^2)$$
$$= 0$$

So $x-2$ is a factor of $P(x)$.

T.S.

7. (13 points) (This question has two parts)

(a) (8 points) Solve for x: $\exp(\exp(-r(ax+b))) = 3$, where $a, b, r > 0$. Label your solution x^*.

$$\ln\left(e^{e^{-r(ax+b)}}\right) = \ln(3)$$

$$\Rightarrow \ln\left(e^{-r(ax+b)}\right) = \ln(\ln(3))$$

$$\Rightarrow -r(ax+b) = \ln(\ln(3))$$

$$\Rightarrow ax+b = -\frac{\ln(\ln(3))}{r}$$

$$\Rightarrow x = -\frac{\ln(\ln(3))}{ar} - \frac{b}{a}$$

$$\boxed{x^* = -\frac{1}{a}\left(\frac{\ln(\ln(3))}{r} + b\right)}$$

(b) (5 points) Is $x^* > 0$? (Hint: Derive a necessary condition, $x > 0 \implies$ (*blank*), and see if it is true.)

$$x^* > 0 \Rightarrow -\frac{1}{a}\left(\frac{\ln(\ln(3))}{r} + b\right) > 0$$

$$\Rightarrow \frac{\ln(\ln(3))}{r} + b < 0$$

The above statement is false, because $r > 0$, $b > 0$, and $\ln(\ln(3)) > 0$

$$\Rightarrow \boxed{x^* < 0}$$

2 Word problem (12 points)

8. (12 points) The stock of fish in a fishery grows according to the equation $y(x) = rx(1-\frac{x}{K}) - Ex$, where x is the level of the fish stock (i.e., how many fish are in the fishery), E is the level of harvest effort (bigger E means more fish are harvested), and y is the growth rate of the fish population as a function of x and E. The "steady state" level of the fish stock is $\tilde{x} : y(\tilde{x}) = 0$. This condition is used to derive the function $\tilde{x}(E) = \frac{K}{r}(r-E)$. This industry's profit as a function of harvest effort is $\pi(E) = pE\tilde{x}(E) - cE$, where p is the price per unit effort and c is the cost per unit effort. Assume that $p > c > 0$.

(a) (3 points) Solve for $E_0 : \pi(E_0) = 0$.

$$\pi(E) = pE\tilde{x}(E) - cE$$
$$= pE\frac{k}{r}(r-E) - cE$$
$$\pi(E) = -\frac{pk}{r}E^2 + (pk-c)E$$

quadratic => 2 roots

$$= E\left(-\frac{pk}{r}E + (pk-c)\right)$$

$E_0 : \pi(E_0) = 0 \quad \Rightarrow \quad E_0\left(-\frac{pk}{r}E_0 + (pk-c)\right) = 0$

One solution : $\boxed{E_0 = 0}$ → E_{01}

other solution : $\boxed{E_0 = \frac{r(pk-c)}{pk}}$ → E_{02}

(b) (3 points) Solve for E^* which optimizes $\pi(E)$. Is $\pi(E)$ a cup or a cap?

By symmetry of quadratic functions,

$$E^* = \frac{E_{01} + E_{02}}{2} = \frac{1}{2}\left(0 + \frac{r}{pk}(pk-c)\right)$$

$$\Rightarrow \boxed{E^* = \frac{r}{2pk}(pk-c)}$$

$\pi(E) = -\frac{pk}{r}E^2 + (pk-c)E$. $-\frac{pk}{r} < 0$, so $\boxed{\pi(E) \text{ is a cap.}}$

(c) (3 points) Suppose $p = 10, r = 4, K = 20$. Plot $E_0(c)$ and $E^*(c)$ with c on the horizontal axis (plot over the interval $[0, 12]$ for c, and stop assuming that $p > c$). Label your axis intercepts and any points where the functions intersect.

$E^* = \frac{r}{2pk}(pk - c)$. With the given numbers,

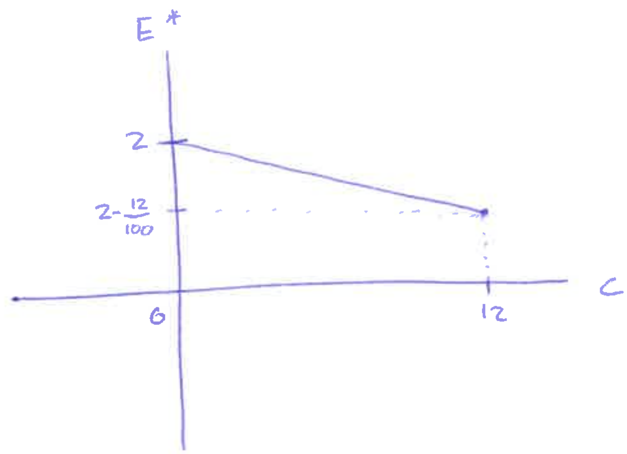
$E^* = \frac{4}{2(10)(20)}((10)(20) - c) = \frac{\cancel{4}^{1}}{\cancel{400}_{100}}(200 - c) = \frac{200}{100} - \frac{c}{100}$

$\Rightarrow E^*(c) = 2 - \frac{c}{100}$

$E^*(0) = 2$

$E^*(12) = 2 - \frac{12}{100}$

$E^*(c)$ is linear in c



(d) (3 points) Determine which is larger, $y(\tilde{x}(E^*))$ or $y(\tilde{x}(E_0))$. You may use numbers from part (c) if necessary.

The problem tells us that $\tilde{x} : y(\tilde{x}) = 0$. So, $\forall E$, $y(\tilde{x}) = 0$, i.e.

$$\boxed{y(\tilde{x}(E^*)) = y(\tilde{x}(E_0)) = 0}.$$

3 (Extra credit) Multiple choice questions (10 points)

Circle the correct answer.

9. (2.5 points) $\ln(AK^{\alpha}L^{1-\alpha}) =$

   (a) $A + K^{\ln(\alpha)} + L^{\ln(1-\alpha)}$

   (b) $A\ln(K^{\alpha}L^{1-\alpha})$

   (c) $A + \alpha\ln(K) + (1-\alpha)\ln(L)$

   (d) $\ln(A) + \alpha\ln(K) + (1-\alpha)\ln(L)$

   (e) None of the above

10. (2.5 points) How many roots does a polynomial of degree 7 have?

    (a) 6, including any complex roots

    (b) 7, including any complex roots

    (c) 8, the eighth root is complex

    (d) $7^7 - 1$, the -1 accounts for the complex root

    (e) None of the above

11. (2.5 points) Which of the following functions is an exponential function of x? (Assume $a > 0$.)

    1. $f(x) = ax^2$
    2. $f(x) = 2a^x$
    3. $f(x) = xa^2$
    4. All of the above
    5. None of the above

exponential function:
$f(x) = Aa^x, \quad a > 0$

12. (2.5 points) A function $g(x)$ is "homogeneous of degree m" if $g(tx) = t^m g(x)$. If $f(tK, tL) = tf(K, L)$, then $f(K, L)$ is

    (a) homogeneous of degree 0

    (b) homogeneous of degree 1

    (c) homogeneous of degree m

    (d) All of the above

    (e) None of the above

END

4 Formulas you may find useful

- Quadratic formula: $x_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Quadratic optimization rule: $x^* = -\frac{b}{2a}$
- $\exp(1) \equiv e \approx 2.718$
- $10\left(\frac{7}{10}\right)^2 - 14\left(\frac{7}{10}\right) + 5 = 0.1$
- Powers of 2: $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, $2^6 = 64$