Open access to orbit and runaway space debris growth

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Abstract

As Earth's orbital space fills with satellites and debris, debris-producing collisions between orbiting bodies become more likely. Runaway space debris growth, known as Kessler Syndrome, may render Earth's orbits unusable for centuries. We present a dynamic physico-economic model of Earth orbit use under rational expectations with endogenous collision risk and Kessler Syndrome. When satellites can be destroyed in collisions with debris and other satellites, there is a manifold of open-access equilibria rather than a unique equilibrium. When debris can collide to produce more debris, this manifold allows Kessler Syndrome to occur along equilibrium paths. We show open access is increasingly and inefficiently likely to cause Kessler Syndrome as satellites become more profitable, highlighting the need for orbital-use management.

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1 Introduction

Earth's orbits are humanity's largest commons yet, and increasingly necessary for services like GPS and satellite imaging that power the modern world. As humans launch more satellites, the risk of collisions between orbiting objects increases. Such collisions can destroy satellites and produce orbital debris, further increasing the risk of future collisions, threatening active satellites and the future of human use of outer space. The worst-case scenario is runaway debris growth, known as Kessler Syndrome, wherein the production of debris due to collisions between orbiting bodies becomes self-sustaining and irreversible. In such a scenario valuable regions of orbital space may become unusable and impassable for decades, centuries, or longer. While a social planner may wish to avoid such a scenario, the current legal and institutional regime is one of open access where anyone with a rocket can place a satellite in orbit.

How will open access to orbit affect orbital debris accumulation, satellite collision risk, and the occurrence of Kessler Syndrome? How do short-run economic dynamics transition to long-run outcomes in orbit? These questions have been explored very little in economics, and have not yet been addressed in the physics and engineering or law and policy literatures.[1] In this paper, we build the first fully-coupled dynamic economic model of satellite launching and conduct theoretical and numerical analysis to explore the consequences of open access to orbit.

We offer four main findings about open-access orbit use. First, we show the equilibrium collision probability is determined by the excess rate of return on a satellite. This induces a manifold of open-access equilibria in the short run as long as satellites can be destroyed by debris or other satellites. Second, when collisions between debris produce more debris, the manifold of open-access equilibria makes it possible for Kessler Syndrome to occur along equilibrium paths. Third, we show open-access steady-state approach paths will almost surely be non-monotonic unless the number of launches available in a period is constrained. Environmental processes (e.g. sunspot activity) or technological innovations (e.g. new final-stage boosters) which shift the equilibrium manifold can therefore lead to short-run "rebound" effects where collision risk and debris temporarily rise above equilibrium levels. Finally, we show conditions under which open-access firms will cause Kessler Syndrome if the excess return on a satellite becomes high enough. This is inefficient, as a social planner facing the same parameters would avoid Kessler Syndrome.

To put our results in context and motivate the main features we model, we offer some physical and institutional detail on orbit use. Satellites produce debris over their lifecycle. Launching satellites produces orbital debris (spent rocket stages, separation bolts), satellites

[1] Wienzierl (2018) highlights several issues in the development of a space economy, from space debris to coordination problems and market design, which economists have the tools to address.

can produce some debris while in orbit (paint chips, lost tools, etc.), and satellites which are not deorbited or shifted to disposal orbits at the end of their life become debris. Satellites struck by debris can shatter into thousands of hazardous debris fragments.[2] Compounding the problem, collisions between debris objects can generate even more hazardous high-velocity debris. Debris accumulation can cause a cascading series of collisions between orbital objects, creating an expanding field of debris which can render an orbital region unusable and impassable for decades, centuries, or longer. Engineers and physicists call this phenomenon "collisional cascading" or "Kessler Syndrome" (Kessler and Cour-Palais, 1978). Kessler Syndrome can cause large economic losses, directly from damage to active satellites and indirectly from limiting access to space (Bradley and Wein, 2009; Schaub et al., 2015). Existing estimates of debris growth indicate that the risk of Kessler Syndrome is highest in low-Earth orbit (LEO), where it threatens imaging and future telecommunications satellites and can reduce access to higher orbits (Kessler et al., 2010; Lewis, 2020). In the worst-case scenario Kessler Syndrome could completely block human access to space, marking an end to services like GPS and satellite imaging. In 2019, the cost of a temporary GPS outage to the US private sector alone was estimated to be on the order of $1 billion per day, and as much as 50% higher if it occurred during the critical planting season (O'Connor et al., 2019). The global costs of long-run disruption of satellite services today are likely to be even higher. Some estimates show debris production due to collisions in LEO has already crossed the self-sustaining growth threshold (Liou and Johnson, 2008).

Existing legal frameworks for orbit use such as the Outer Space Treaty complicate the process of establishing orbital property rights and hinder debris cleanup efforts.[3] We offer a new perspective on how these institutions interact with orbital mechanics to make Kessler Syndrome possible, and a model environment in which alternative management institutions can be explored quantitatively.[4]

[2]Objects in orbit move at velocities higher than 5 km/s, so debris as small as 10 cm in diameter can be hazardous to active satellites. Debris as low as 900 km above the Earth's surface can take centuries to deorbit naturally, while debris at 36000 km can take even longer (Weeden, 2010). Currently, there are more than 2,000 operating satellites in orbit, up to 600,000 pieces of debris large enough to cause satellite loss, and millions of smaller particles that can degrade satellite performance (Ailor et al., 2010). Approximately 49 percent of currently-active satellites are in LEO, a share which is projected to grow in the near future (Union of Concerned Scientists, 2017).

[3]For example, Article 2 of the OST forbids national appropriation or claims of sovereignty over outer space, which is often interpreted as prohibiting national authorities from unilaterally establishing orbital property rights (Gorove, 1969).

[4]Ostrom et al. (1999) provides some insight into why decentralized orbit management may face challenges. Orbit users are a diverse and international group, ranging from national militaries and intelligence agencies to corporations, universities, and wealthy individuals. Excluding potential users from orbital regions without creating additional debris is difficult, and the relevant conflict resolution mechanisms can be unclear. National regulatory regimes must also contend with "launch leakage", which has happened at least once already (Selk, 2017). Weeden and Chow (2012) discuss some of these issues in more detail.

As Stavins (2011) notes, management of the commons is among the central issues of economics. While open access problems have been well-studied in terrestrial settings such as fisheries, forests, climate, oil fields, traffic, and invasive species management (Gordon, 1954; Scott, 1955; Nordhaus, 1982; Libecap and Wiggins, 1985; Bohn and Deacon, 2000; Duranton and Turner, 2011; Huang and Smith, 2014), open access to orbital resources is not as well understood. Though results from these other settings provide some helpful intuition, open access and the orbital mechanics governing collision risk and debris production create unique physico-economic feedback loops. We extend the literature on how open access and a lack of property rights affects resource use and management to a new, increasingly-relevant context.

Commons problems, particularly in biophysical commons, often involve dynamic externalities (Haveman, 1973; Brown Jr, 1974; Levhari and Mirman, 1980; Reinganum and Stokey, 1985; Farzin, 1996; Sherstyuk et al., 2016). Such externalities also occur in arms races, growth, industrial development, and other settings with strategic or dynamic interactions (Simaan and Cruz, 1975; Boldrin, 1992; Henderson, 1997; Beaudry, Galizia, and Portier, 2020). Though results from other settings again offer helpful intuition, the unique physical properties of orbits create a novel dynamic externality which illuminates the natures of congestion and pollution more generally. We add to the literature on decentralized and optimal responses to interacting dynamic externalities by showing how profit-maximizing responses to congestion can limit pollution production, and biophysical conditions under which profit maximization leads to runaway pollutant accumulation due to a fold bifurcation. The bifurcation emerges due to fragment-generating collisions between debris objects, but is modulated by the degree to which debris reduces satellite profitability in the short run.

To our knowledge, this paper is the first to bridge the short-term and long-term physical and economic dynamics of perfectly-competitive open-access and socially-optimal orbit use in a physically-general environment which allows for Kessler Syndrome to be caused over multiple periods. While Sandler and Schulze (1981) account for collision risk when studying geostationary belt position allocation, debris accumulation and general orbital regions are not considered. Our analysis generalizes those presented in Macauley (2015); Adilov, Alexander, and Cunningham (2015, 2018), Grzelka and Wagner (2019), and Rouillon (2020) in the physical dimensions by allowing non-stationary dynamics and runaway debris growth, and makes the sources of external effects clear by explicitly considering the economics of general forms of couplings between physical state variables. While our model is most similar to that of Rouillon (2020), we generalize the analysis by considering a planner who owns both the current stock of satellites in orbit as well as the rights to launch to that orbit in perpetuity, and by allowing debris to collide with other debris (which is especially relevant for analyzing long-run orbit use (Lewis, 2020)). Unlike Rao (2018), Grzelka and Wagner (2019), Rao, Burgess, and Kaffine

(2020), and Béal, Deschamps, and Moulin (2020), we focus primarily on economic dynamics rather than policy instrument design.

Though prior work has established that open-access launch rates exceed the socially-optimal launch rate and result in excess collision risk and debris production (Adilov, Alexander, and Cunningham, 2015; Rouillon, 2020), our framework yields the novel insight that there can be a manifold of open-access equilibria in the short run due to couplings between satellites and debris in the collision risk function. In certain cases, short-run open-access equilibria can cause Kessler Syndrome, in which case a steady state will never be reached. Our results contrast with those in Adilov, Alexander, and Cunningham (2018), where open access is found to never cause Kessler Syndrome. The contrast is due to our differing definitions of Kessler Syndrome and degrees of physical generality. Adilov, Alexander, and Cunningham (2018) consider a definition of Kessler Syndrome where satellites are destroyed with probability one ("unusable orbits") and disallow collisions between debris objects. We define Kessler Syndrome as states where the debris stock diverges to infinity ("runaway debris growth") and allow collisions between debris objects. Both features are critical for understanding dynamics of orbit use, as collisions between debris are becoming increasingly likely and are expected to dominate the long-run dynamics of the orbital environment (Davenport, 2020; Lewis, 2020). Once runaway debris growth occurs in our model, the collision probability will eventually reach one (runaway debris growth eventually renders orbits unusable). Finally, the physical generality we consider allows us to derive conditions under which exogenous parameter changes (such as sunspot activity or launch cost reductions) under open access will cause short-run rebound effects with excess debris and collision risk. While physical processes like sunspot activity are beyond human control, they must be accounted for in designing orbital-use management strategies. Processes like changes in launch costs or launch technologies, on the other hand, offer potential direct policy levers for managing orbit use more effectively.

The organization of this paper is as follows. In section 2 we describe the physical model we use to study the orbital environment. In section 3 we describe the economic primitives we use to study orbit use. In this section we also define the open-access equilibrium and characterize the open-access launch rate. In section 4 we derive the dynamic properties of open-access equilibria. In section 5, we pose the planner's problem, decompose the marginal external cost of a satellite, and identify cases where Kessler Syndrome is inefficient. We illustrate key results with numerical simulations. We conclude in section 7.

2 Physical model

Consider a spherical shell around the Earth, say the region between 600-650km above mean sea level. Active satellites and debris move through the orbital shell ("orbit") in elliptical paths. Influences from the Earth and other celestial bodies perturb their motion and may cause their paths to intersect, upon which the colliding bodies shatter into more debris. The stock of active satellites in the orbit is periodically replenished by new launches, which may bring with them more debris. While active satellites expend fuel to counteract the perturbations they face, debris do not and eventually fall back to Earth and burn up in the atmosphere. If the supply of satellites is high enough for long enough, the production of debris from collisions may become self-sustaining.[5]

The number of active satellites in orbit in period $t+1$ (S_{t+1}) is the number of launches in the previous period (X_t) plus the number of satellites which survived the previous period ($S_t(1-L(S_t,D_t))$). The amount of debris in orbit in $t+1$ (D_{t+1}) is the amount from the previous period which did not decay ($D_t(1-\delta)$), plus the number of new fragments created in collisions ($G(S_t,D_t)$), plus the amount of debris in the shell created by new launches (mX_t).[6] The laws of motion for the satellite and debris stocks formalize this:

$$S_{t+1} = S_t(1-L(S_t,D_t)) + X_t \tag{1}$$

$$D_{t+1} = D_t(1-\delta) + G(S_t,D_t) + mX_t. \tag{2}$$

$L(S_t,D_t)$ is the proportion of orbiting active satellites which are destroyed in collisions, and $G(S_t,D_t)$ is the number of new fragments created by collisions between orbiting bodies. Since we assume active satellites are identical, $L(S_t,D_t)$ is also the probability an individual satellite is destroyed in a collision. We assume that the collision probability is twice continuously differentiable, nonnegative, increasing in each argument, and bounded below by 0 and above by 1.[7] $\delta > 0$ is the rate of orbital decay for debris, and $m \geq 0$ is the amount of launch

[5]Paths which span shells are possible and useful for some applications, but highly elliptical orbits (such as Molniya orbits) are the exception rather than the rule. A Molniya orbit has a low perigee over the Southern Hemisphere and a high apogee over the Northern Hemisphere. Molniya orbits require less power to cover regions in the Northern Hemisphere (e.g. former Soviet Union countries) than geosynchronous orbits, due to the low incidence angles of rays from the Northern Hemisphere to geosynchronous positions. This "shell of interest" approach is frequently used in debris modeling, e.g. Rossi et al. (1998) and Bradley and Wein (2009), though higher fidelity models use large numbers of small regions to track individual objects, e.g Liou et al. (2004), Liou and Johnson (2008), and Liou and Johnson (2009). We abstract from the composition of orbital stocks, and assume that all satellites and debris are identical.

[6]Satellites' orbits also decay and satellites eventually cease being productive, features we abstract from in the main model. We extend the model to include this in section B.2 of the Appendix. This abstraction makes some results a little clearer but does not qualitatively affect them. Rouillon (2020) includes these features in the main model and derives qualitatively similar results, albeit in a continuous-time setting.

[7]Satellite operators try to avoid collisions by maneuvering their satellites when possible; the collision probability in this model should be thought of as the probability of collisions which could not be avoided, with easily avoided

debris created by new satellites.

We assume the new-fragment function G is twice continuously differentiable, nonnegative, strictly increasing in each argument, zero when there are no objects in orbit ($G(0,0) = 0$), and unbounded above ($\lim_{S\to\infty} G(S,D) = \lim_{D\to\infty} G(S,D) = \infty$). We also assume the effect of the first satellite or debris fragment on new fragment formation is negligible, i.e. $G_S(0,D) = G_D(S,0) = 0$ (letting subscripts denote partial derivatives). To derive results about the occurrence of Kessler Syndrome, we also assume that the growth in new fragments due to debris alone will eventually be greater than δ, i.e. the fragment autocatalysis rate (defined below) will eventually be positive.[8]

Definition 1. *(Fragment autocatalysis rate) The fragment autocatalysis rate is the net rate of fragment growth due to marginal debris objects. Formally,*

$$G_D(S,D) - \delta.$$

Positive fragment autocatalysis rates are a necessary condition for existence of unstable fixed points of long-run orbit use under open access (described precisely in Proposition 3), and are associated with Kessler Syndrome. Figure 3 illustrates this connection.

Definition 2. *(Kessler Syndrome) The Kessler region for a launch policy X is the set of states (satellite and debris levels) such that under X the debris stock will grow to infinity, i.e.*

$$\kappa_X \equiv \{(S,D) : \lim_{t\to\infty} D_{t+1} = \infty \mid S_0 = S, D_0 = D, X_t = X(S_t,D_t)\}$$

Kessler Syndrome occurs when the satellite and debris stocks enter the Kessler region, i.e. $(S_t,D_t) \in \kappa_X$.

This definition of Kessler Syndrome can accommodate open access, optimal management, or any other type of orbital management institution.[9] We define the *unavoidable Kessler region*, where Kessler syndrome will occur even if there are no new launches, as the Kessler

collisions optimized away. Collisions which could have been avoided but were not due to human error are included in L. Implicitly we are assuming operators are imperfect cost-minimizers when maneuvering satellites. Even when operators can maneuver their satellites and are aware of impending collisions, coordination can be challenging and plagued by technical glitches (Brodkin, 2017).

[8] This assumption is consistent with long-run engineering studies of the evolution of the orbital environment, e.g. Liou (2006a); Lewis (2020).

[9] In Kessler and Cour-Palais (1978), collisional cascading is defined as one of the "possible consequences of continuing unrestrained launch activities". "Unrestrained launch activities" is not defined precisely, though that paper and other engineering studies of the debris environment typically assume either continuation of historical trends indefinitely or continuation with total launch cessation at an arbitrary date (Rossi et al., 1998; Liou, 2006b; Bradley and Wein, 2009). Open-access launch behavior is consistent with "unrestrained launch activities", current legal institutions around orbit use, and economic behavior. In Kessler et al. (2010), "runaway debris growth" is defined in terms of a launch policy which holds the stock of "intact objects" (active satellites and unfragmented debris objects) constant. Such a policy is less consistent with rational economic behavior—eventually the required rate of replacement should

region under a never-launch policy, i.e.

$$\kappa_0 \equiv \{(S,D) : \lim_{t\to\infty} D_{t+1} = \infty \mid S_0 = S, D_0 = D, X_t \equiv 0\}.$$

The unavoidable Kessler region is a subset of the Kessler region for any launch policy constrained to have nonnegative numbers of launches ($\kappa_0 \subseteq \kappa_X \; \forall X : X_t \geq 0 \; \forall t$), e.g. a policy without a satellite removal technology.

Using the definition of the Kessler region for a launch policy, we define the *Kessler threshold* of a launch policy, D_X^κ. The Kessler threshold of a launch policy is the maximum sustainable debris level given that policy:

$$D_X^\kappa \equiv \sup\{D : (S,D) \notin \kappa_X\}. \tag{3}$$

We characterize D_X^κ analytically for open access, and compute examples of the Kessler regions under open access and optimal management.

For orbits where the fragment autocatalysis rate is negative for all satellite and debris levels, Kessler Syndrome is impossible. Our assumption that the new fragments function G is monotonically increasing in both arguments, and the lack of debris removal technologies, imply that Kessler Syndrome is an absorbing state. Intuitively, the Kessler threshold is increasing in the decay rate, making lower-altitude orbits (where drag from the Earth's atmosphere makes debris decay faster than higher orbits) physically less vulnerable to Kessler Syndrome than higher-altitude orbits.

2.1 Three types of physical couplings

We use "physical coupling" to refer to ways that variables in the laws of motion for satellite and debris stocks are connected to each other. These couplings give structure to the laws of motion and drive the physical dynamics of orbit use. There are five economically relevant physical couplings between objects in orbit: the collision probability satellite coupling and collision probability debris coupling (the first type); the new fragment formation satellite coupling and new fragment formation debris coupling (the second type); and the launch debris coupling (the third type). If all five couplings were turned off, there would be no problems of excess satellite congestion or debris accumulation. The strength of a coupling between X and $f(\cdot)$ is the absolute value of $\frac{\partial f}{\partial X}$, with $\frac{\partial f}{\partial X} = 0$ when there is no coupling. Principles of orbital mechanics imply any non-zero coupling is positive.

make launchers prefer to invest their funds elsewhere. In any case, our definition encompasses all such definitions and makes the dependence on the launch policy explicit.

The collision probability couplings The collision probability couplings refer to the arguments of the collision probability function, L. When these couplings are turned off, the collision probability is exogenous: $L(\cdot) = L$. If the collision probability is exogenous, there is no congestion externality even when the probability varies over time. Even if open access causes Kessler Syndrome due to other couplings, the open access launch rate is efficient in the sense that the planner would produce the same outcome.

If the collision probability is coupled with the satellite stock ($L(\cdot) = L(S)$), then there can be a "steady state" congestion externality. Despite any other couplings and their effects on debris growth, the only source of inefficiency is that open access results in too many satellites in orbit relative to the optimal plan. This describes a world where debris is harmless but active satellites aren't perfectly coordinated to avoid collisions with each other. If the collision probability is coupled with the debris stock ($L(\cdot) = L(D)$) but not the satellite stock, there can be a "dynamic" congestion externality. Due to debris accumulation and the coupling, there can be persistent consequences for specific launch histories. This describes a world where active satellites are perfectly coordinated to avoid collisions with each other, but debris are hazardous and not always successfully avoided. Such a function might be a good representation of the situation in the geostationary belt, where active satellites are coordinated to have virtually no motion relative to each other but debris objects are not. The "fully coupled" case links collision probability to both the satellite and debris stocks ($L(\cdot) = L(S,D)$). In this world, active satellites face collision risk from debris and each other. Adilov, Alexander, and Cunningham (2015, 2018) focus on a collision probability function with only a debris coupling. Rouillon (2020) focuses on the steady state and abstracts from the details of the collision probability couplings. The existence of both collision probability couplings allows us to define the marginal rate of technical substitution between satellites and debris in producing collision probability, $\frac{L_D}{L_S}$, which we show in Proposition 3 to be important in determining to the stability of open-access steady states.

The launch debris coupling The launch debris coupling refers to the amount of launch debris created by a new launch, i.e. the parameter m in the debris law of motion $\left(\frac{\partial D_{t+1}}{\partial X_t} = m\right)$. When collision probability is coupled with the debris stock, the launch debris coupling reduces the equilibrium and optimal number of launches per period, since launches themselves produce debris. Thus, the launch debris coupling can "stabilize" open access orbit use by forcing operators to internalize some of the persistent effects of their launches, shown in Proposition 3.

The new fragment formation couplings The new fragment formation couplings refer to the arguments of the new fragment function, $G(\cdot)$. If the new fragment function is coupled to the satellite stock, $G(\cdot) = G(S)$ (a world where only active satellites fragment upon collision), then there can be persistent consequences of excess satellite levels. When the new fragment function is coupled to the debris stock, $G(\cdot) = G(D)$ (a world where only debris fragment upon collision), there can be multiple open-access steady states and Kessler Syndrome becomes possible. This multiplicity will exist independent of the collision probability and launch debris couplings. When the collision probability is only coupled to the satellite stock or uncoupled ($L(\cdot) = L(S)$ or $L(\cdot) = L$), the multiplicity is only in debris levels. When the collision probability is fully coupled ($L(\cdot) = L(S,D)$) and the new-fragment function is coupled with only the debris stock, multiple open-access steady states in both satellites and debris can exist: one with low debris and high satellites, and one with high debris and low satellites. In both cases, only the low-debris equilibrium is stable. Adilov, Alexander, and Cunningham (2015, 2018) focus on new-fragment formation functions coupled only to the satellite stock.

For simulations and figures, we use the following fully-coupled functional forms:

$$L(S,D) = 1 - e^{-\alpha_{SS}S - \alpha_{SD}D} \tag{4}$$

$$G(S,D) = \beta_{SS}(1 - e^{-\alpha_{SS}S})S + \beta_{SD}(1 - e^{-\alpha_{SD}D})S + \beta_{DD}(1 - e^{-\alpha_{DD}D})D, \tag{5}$$

where $\alpha_{SS}, \alpha_{SD}, \alpha_{DD}, \beta_{SS}, \beta_{SD}, \beta_{DD}$ are all positive physical parameters. We derive these forms and give some physical intuition about the parameters in Appendix A.

3 Economic model

Active satellites provide services to individuals, firms, governments, research agencies, and other entities. They tend to be information services like mobile broadband, images of the Earth, and positioning/timing. To focus on the dynamics of collisions and debris, we ignore such differentiation. Adilov, Alexander, and Cunningham (2015) account for product differentiation in a two-period setting.

Satellites are identical, infinitely lived unless destroyed in a collision, and produce a single unit of output per period. The net payoff from this output is $\pi > 0$, and each satellite costs $F > \pi$ to plan, build, and launch.[10] The market for satellite output is perfectly competitive, so the price per unit of output (the private marginal benefit) is the same as its social marginal benefit. Costs and payoffs are constant over time.[11]

[10]This is a simplification to focus on the margin of launch decisions. Operational costs like managing receiver stations on the ground or monitoring the satellite to perform stationkeeping are incurred each period, and may push $\pi \leq 0$. We assume that the firms have correctly forecasted $\pi > 0$ before deciding to launch. This also allows us to abstract from the margins of decisions to sell or deorbit satellites, or to purchase already-orbiting satellites. Assuming $F > \pi$ allows us to ignore the uninteresting case where satellites are always profitable to launch even if they only survive for a single period.

[11]We relax these assumptions in Appendix B, modeling the effects of finite satellite lifetimes in Section B.2 and of

We consider two scenarios: open access by profit-maximizing firms which can own up to one satellite at a time—or equivalently that all firms are choosing whether or not to launch identical constellations of satellites at any given time, changing the units of S_t from individual satellites to constellations—and launching controlled by a fleet planner who owns all satellites.[12] We focus on the open-access problem in this section and the following, deferring discussion of the social planner's problem until section 5. The value of a satellite, $Q(S_t, D_t, X_t)$, is the sum of present payoffs and the expected discounted value of its remaining lifetime payoffs:

$$Q(S_t, D_t, X_t) = \pi + \beta(1 - L(S_t, D_t))Q(S_{t+1}, D_{t+1}, X_{t+1}), \tag{6}$$

where $X_t = \int_0^\infty x_{it} di$ is the aggregate launch rate based on each potential launcher's entry decision $x_{it} \in \{0, 1\}$, and $\beta = (1+r)^{-1}$. Firms cannot choose to deorbit satellites.[13] The maximum number of launches possible in one period is $\bar{X}$.[14]

A firm which does not own a satellite in period t decides whether to pay F to plan, build, and launch a satellite which will reach orbit and start generating payoffs in period $t+1$, or to wait and decide again in period $t+1$.[15] All potential launchers are risk-neutral profit maximiz-

time-varying rates of return in Section B.3.

[12] Our assumption that each firm is identified with a single satellite is equivalent to assuming that either the marginal satellite in an equilibrium is launched by such a firm, or that firms may own multiple satellites but ignore their inframarginal satellites in launching. In either case we could obtain the same equilibrium condition given by equation 9 below. Rouillon (2020) uses the latter interpretation and arrives at an equivalent equilibrium condition (the condition in Rouillon (2020) is the continuous-time limit of ours with finite satellite lifetimes). We maintain our stricter interpretation to avoid the complications of solving the dynamic satellite inventory management problem facing a constellation owner outside the steady state, with the attendant assumptions about when and how they choose to refresh their constellation. See Adilov et al. (2019) for a model of GEO satellite inventory management. Versions of those orbital warehousing issues are also relevant to LEO constellation management, with additional dynamic and strategic complications through the collision risk and debris growth functions. The constellation management problem is eminently important to study, but given the complications we leave it to future research. As long as there exists a marginal potential launcher who is considering the profitability of a single satellite, our results hold.

[13] This assumption does not matter in our setting, where satellite payoffs and launch costs are common to all firms and constant over time: if it is ever profitable to launch in this setting, it is never profitable to deorbit. Generating nontrivial dynamics from the option to deorbit requires heterogeneous and time-varying payoffs or costs. We abstract from this to keep the dynamical analysis manageable but this is an interesting avenue for future research.

[14] The launch constraint does not bind in our model except in scenarios where we illustrate the effects of a binding launch constraint.

[15] There is a difference between the timescale of physical interactions in orbit and the timescale of launch decisions. The former occur continuously, while the latter do not. Historically, the median time to build for a geostationary satellite has been 3.5 years (TelAstra, Inc., 2017). While LEO satellite take less time to build, launches to LEO are still constrained by the availability of launch windows. Globally, launch windows are typically available in clusters no closer than two weeks apart, and launches may be postponed for days or weeks at the last minute due to inclement weather (Union of Concerned Scientists, 2017). We include the one-period lag from the decision to launch to the beginning of a satellite's productive life to capture this gap in timescales.

ers, and the value of potential launcher i at the beginning of period t is

$$V_i(S_t, D_t, X_t) = \max_{x_{it} \in \{0,1\}} \{(1 - x_{it})\beta V_i(S_{t+1}, D_{t+1}, X_{t+1}) + x_{it}[\beta Q(S_{t+1}, D_{t+1}, X_{t+1}) - F]\} \quad (7)$$

$$\text{s.t. } S_{t+1} = S_t(1 - L(S_t, D_t)) + X_t$$

$$D_{t+1} = D_t(1 - \delta) + G(S_t, D_t) + mX_t$$

While firms may choose to launch or not, all satellites earn *ex-ante* identical payoffs.

3.1 Open-access equilibrium

An open-access equilibrium is a state and launch rate such that all firms are indifferent between launching a satellite or not. We define this formally below.

Definition 3. *(Open-access equilibrium) An open-access equilibrium is a nonnegative vector (S_t, D_t, X_t) such that the marginal potential launcher i in period t is indifferent between launching ($x_{it} = 1$) or not launching ($x_{it} = 0$) a satellite, subject to the laws of motion of the satellite and debris stocks (equations 1 and 2). An interior open-access equilibrium will exist as long as $\pi < (1 + r)F$ and $\pi > rF$.*

Definition 3 implies that in an open-access equilibrium firms will launch satellites until launching another satellite provides zero profits, i.e.

$$X_t \in [0, \bar{X}] : \; V_i(S_t, D_t, X_t) = 0 \quad (8)$$

$$\implies \beta Q(S_{t+1}, D_{t+1}, X_{t+1}) = F. \quad (9)$$

Equation 9 is the zero-profit condition which defines the open-access equilibrium. The resulting value of a satellite is the period profit plus the expected value of its survival under open access (substituting equation 9 into equation 6),

$$Q(S_t, D_t, X_t) = \pi + (1 - L(S_t, D_t))F. \quad (10)$$

The open access condition and the satellite value can be rewritten to show that in an interior equilibrium the flow of benefits generated by a satellite is equated with the flow of opportunity costs and expected collision costs (the marginal private costs of satellite ownership):

$$F = \beta Q(S_{t+1}, D_{t+1}, X_{t+1}) \quad (11)$$

$$\implies \pi = rF + L(S_{t+1}, D_{t+1})F. \quad (12)$$

Using equation 12, we define $\rho(X_t, S_t, D_t) = \pi - rF - L(S_{t+1}, D_{t+1})F$ as the collision-risk-adjusted return on a satellite.[16] Given the upper bound on launches, the open-access launch rate is then

$$\hat{X}_t \in [0, \bar{X}] \text{ if } \rho(\hat{X}_t, S_t, D_t) = 0, \tag{13}$$

with $\hat{X}_t = 0$ if $\rho(0, S_t, D_t) < 0$ and $\hat{X}_t = \bar{X}$ if $\rho(\bar{X}, S_t, D_t) > 0$.

Given that $L(S,D) \in [0,1]\ \forall(S,D)$ since L is a probability, equation 12 provides existence conditions for an open-access equilibrium ($\pi < (1+r)F$ and $\pi > rF$). $\pi < (1+r)F$ requires a satellite to produce less profit in a single period than the total cost of launching it inclusive of opportunity costs. If $\pi > (1+r)F$, then there can exist no nonnegative value of X_t which makes the marginal potential launcher is indifferent—the marginal potential launcher will strictly prefer launching over not launching. $\pi > rF$ requires a satellite to cover at least the opportunity cost of funds. If $\pi < rF$, they will strictly prefer not launching over launching. We define the one-period rate of return on a satellite as $\pi/F \equiv r_s > 0$ and the excess return on a satellite as $r_s - r$. The existence conditions then become $r_s - r < 1$ and $r_s - r > 0$.

By dividing equation 12 by F on both sides, we see that when the collision probability function is fully coupled with satellites and debris, there exists a manifold of interior open-access equilibria. This manifold is the isoquant where the collision probability is equal to the excess return on a satellite. Increases in the excess return on a satellite therefore increase the equilibrium collision probability.

Proposition 1 (Equilibrium collision probability)**.** *Given that an interior open-access equilibrium exists and the collision probability function is coupled with both the satellite and debris stocks, there are multiple vectors (S_t, D_t, X_t) such that*

$$L(S_{t+1}, D_{t+1}) = r_s - r.$$

Increases in the excess return on a satellite cause X_t, S_{t+1}, D_{t+1}, and $L(S_{t+1}, D_{t+1})$ to increase.

In cases where the economic parameters π, F, β are time-varying, Proposition 1 implies that the equilibrium collision probability in period t under open access will be determined by the values of the economic parameters in period $t-1$, i.e. the period when the newest satellites in orbit were launched.

Increases in the excess return on a satellite shift the equilibrium collision probability isoquant outwards. Since every point on the excess-return isoquant of the collision probability

[16]The expected collisions cost term in equation 12, $L(S_{t+1}, D_{t+1})F$, is analogous to a default premium required by satellite operators to compensate them for the probability that their asset will "default" through destruction. In this way, satellites can be thought of as risky assets similar to corporate bonds, e.g. Fons (1987).

function is an open-access equilibrium, which equilibrium is reached in any period will depend on the initial conditions and physical dynamics. This continuum is an implication of the physics of orbit use—as long as the collision probability function is coupled with the satellite and debris stocks, the same collision probability level can be realized by different combinations of satellite and debris levels in the next period. Figure 1 illustrates these results.

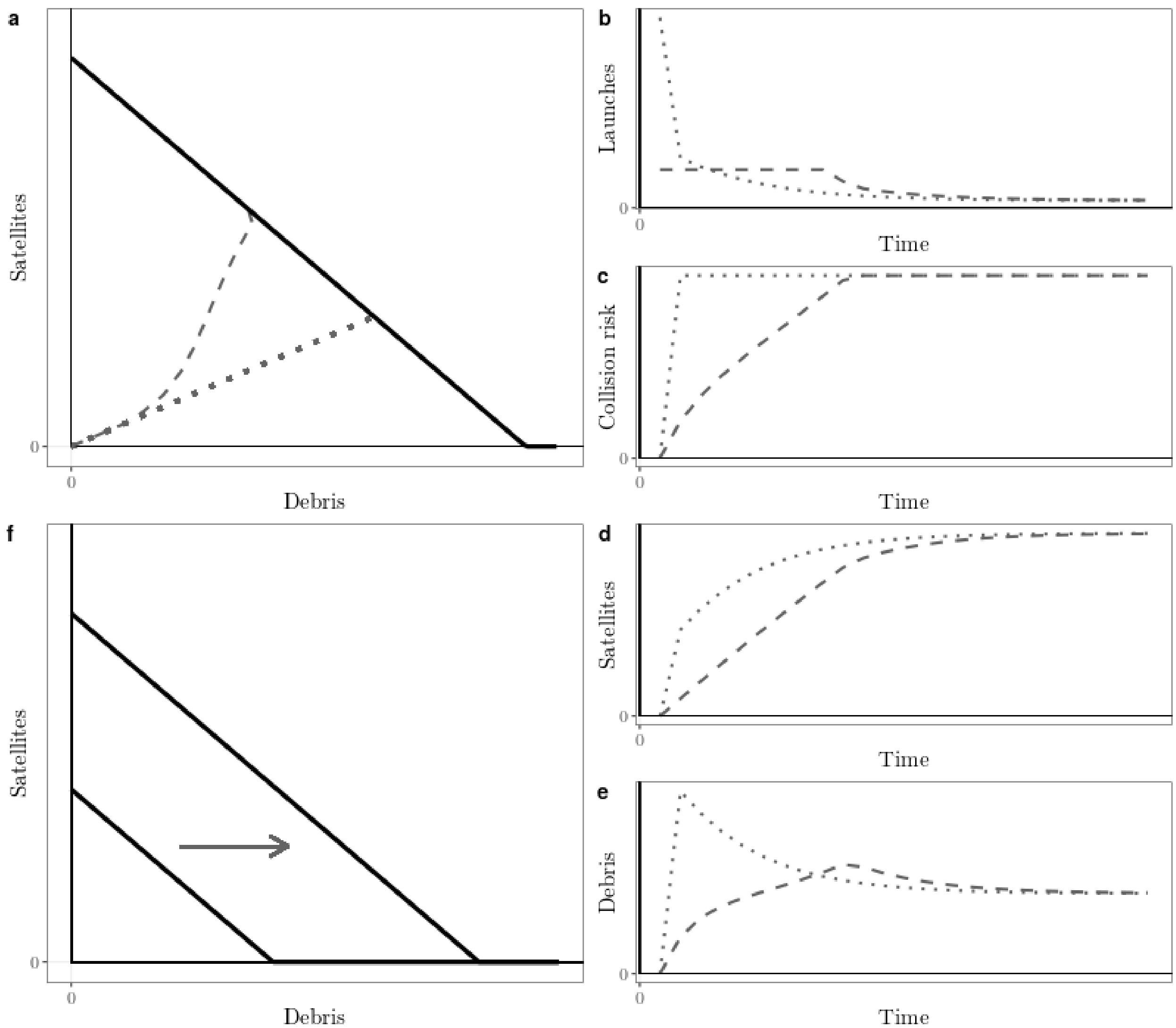

Figure 1: Panel a shows two approach paths from the same initial condition: one which reaches the equilibrium isoquant immediately (dotted straight line), the other with a binding launch constraint ($\bar{X}$) which takes multiple periods to reach the equilibrium isoquant (dashed curving line). Panels b, c, d, e show the time evolution of the launch rate, collision risk, satellite stock, and debris stock from panel a. Panel f shows the equilibrium isoquant shifting outwards when the excess return on a satellite increases.

4 Dynamic properties of open access

In addition to a continuum of open-access equilibria, there can multiple open-access steady states. Removing period t subscripts and letting $'$ superscripts denote period $t+1$ variables, open-access steady states are defined by the collision probability equaling the excess return on a satellite ($L(S,D) = L(S',D') = r_s - r$) along with the usual stationarity conditions on the state variables:

$$(S,D) : L(S,D) = r_s - r, \tag{14}$$

$$\begin{aligned} S' = S &\implies S = (1 - L(S,D))S + X \\ &\implies X = L(S,D)S, \end{aligned} \tag{15}$$

$$\begin{aligned} D' = D &\implies D = (1-\delta)D + G(S,D) + mX \\ &\implies \delta D = G(S,D) + mX. \end{aligned} \tag{16}$$

The following lemma simplifies our analysis in the rest of the section, allowing us to reduce the system of equations 14, 15, and 16 to a single equation.

Lemma 1 (Reduction). *Given a fully-coupled collision probability function, the steady states of an open-access equilibrium are defined by solutions to*

$$\mathscr{Y}(D) = -\delta D + G(\hat{S},D) + m(r_s - r)\hat{S} = 0, \tag{17}$$

where $\hat{S} = S(r_s - r, D) \geq 0$ *is such that, for a given* D*,*

$$L(\hat{S},D) = r_s - r$$

when such an S *exists and* 0 *otherwise.*

Intuitively, the open-access condition links the steady-state satellite and debris stocks and reduces the system's degrees of freedom. Using Lemma 1, we establish the existence of multiple steady states when the collision probability function is fully coupled and the new-fragment-debris coupling exists.

Proposition 2 (Multiplicity). *Given a positive excess return on a satellite and a fully-coupled collision probability function, multiple open-access steady states can exist if the new-fragment-debris coupling exists.*

This multiplicity is possible in part because open access makes the equilibrium satellite stock a decreasing function of the debris stock. Increases in the steady-state debris stock initially cause the number of new fragments created to fall as open-access launchers respond by launching fewer satellites. Eventually, the fragment-reducing effect of fewer satellites is dominated by the fragment-producing effect of additional debris through the new fragment debris coupling. The number of open-access steady states will depend on the shapes of L and G. For example, if L is strictly increasing and G is strictly convex, there can be up to two open-access solutions to equation 17. Figure 2 illustrates Proposition 2 in the space of equation 17 and the accompanying phase diagram of the satellite-debris system.

Proposition 3 connects the stability of the fixed points to the fragment autocatalysis rate and the marginal rate of technical substitution of satellites for debris in producing collision probability.

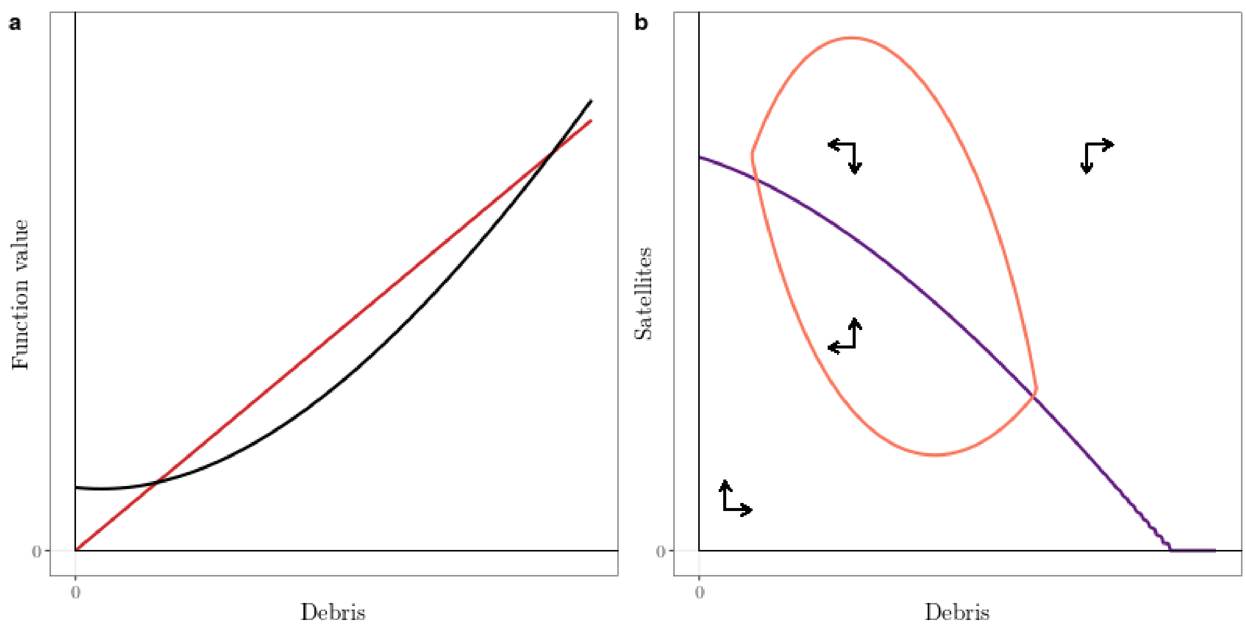

Figure 2: Panel a shows equation 17 for a parameterization with two steady states. The red line shows δD, while the black line shows $G(\hat{S}, D) + m(r_s - r)\hat{S}$. Panel b shows the phase diagram of the same parameterization, with the purple line showing the satellite nullcline and the pink line showing the debris nullcline. Only the lower-debris steady state is stable.

Proposition 3 (Local stability)**.** *Given a positive excess return on a satellite, new-fragment-satellite coupling, and launch-debris coupling, open-access steady states will be locally stable if and only if the fragment autocatalysis rate is small enough, or the marginal rate of technical substitution of satellites for debris in collision probability is high enough. When G is strictly convex in both arguments and two steady states exist, the higher-debris steady state is unstable.*

The key equation from the proof is

$$\frac{\partial \mathscr{Y}}{\partial D}(D^*) = (G_D(S^*, D^*) - \delta) - \frac{L_D(S^*, D^*)}{L_S(S^*, D^*)}(G_S(S^*, D^*) + m(r_s - r)), \tag{18}$$

where $S^* \equiv S(r_s - r, D^*)$. The stability of an open-access steady state depends on two factors, shown on the right hand side of equation 18. The first term is the fragment autocatalysis rate, $G_D - \delta$, which can ensure stability if it is close enough to negative infinity. This has a clear physical intuition: higher decay rates indicate regions with greater natural renewability, while stronger new fragment debris couplings indicate stronger positive feedbacks between debris stocks.

The second term is the equilibrium effect of new satellites on debris growth. This term is increasing in the marginal rate of technical substitution (MRTS) of satellites for debris in collision probability, as a larger MRTS indicates the collision probability function is more sensitive to debris fragments than to satellites. A larger MRTS incentivizes profit-maximizing satellite launchers to be more responsive to debris stocks than they would be with a smaller MRTS, thus stabilizing the steady state. The effect of the MRTS is scaled by the strength of the new-fragment-satellite coupling, the launch-debris coupling, and the excess return on a satellite. The new fragment satellite and launch debris couplings are both byproducts of putting satellites into orbit, and force satellite operators to account for the risks they face in orbit or impose on others.

When there are only two solutions to equation 17, say if L is strictly increasing and G is strictly convex, then the higher-debris solution is a repelling fixed point. The debris level associated with this fixed point is then the Kessler threshold defined earlier in equation 3, D_X^κ. D_X^κ was defined as the maximum sustainable debris level for this reason—if the larger fixed point is repelling, any debris level beyond it will eventually diverge to infinity. This is formalized in Lemma 2.

Lemma 2 (Open-access Kessler threshold)**.** *When G is strictly convex in both arguments and two steady states exist, let*

$$\underline{D} = \min\{D : \mathscr{Y}(D) = 0\}$$

be the stable steady state, and

$$\bar{D} = \max\{D : \mathscr{Y}(D) = 0\}$$

be the unstable steady state. Then

$$\bar{D} = D_X^\kappa.$$

The connection between the Kessler threshold and the unstable steady state allows us to characterize the Kessler region as the complement of the stable steady state's basin of attraction. Intuitively, paths from an initial condition must be governed by the attracting or repelling behavior induced by one of the steady states. So initial conditions which do not lead to the stable steady state must be governed by the unstable one, which will drive the debris stock to infinity and the satellite stock to zero.

Given the potential for an unstable steady state, it is natural to wonder whether open access can lead to Kessler Syndrome. There are two cases where this can occur:

1. the initial condition is inside the Kessler region;
2. the fragment autocatalysis rate or rate of excess return is high enough no stable steady state exists.

The first case can occur due to non-economic factors, such as changes in environmental parameters (e.g. sunspots) or military activity in space (e.g. anti-satellite missile tests which create debris). The second can occur due to economic or non-economic factors, with examples of the former including increases in the profitability of a satellite. Figure 3 illustrates this example. In both cases Kessler Syndrome is unless the initial condition is in the unavoidable Kessler region. As long as Kessler Syndrome is possible, open access is "more likely" to cause Kessler Syndrome as the excess return on a satellite increases (in the sense that the Kessler basin is expanding in the excess return on a satellite), and certain to cause Kessler Syndrome if the excess return exceeds the maximum sustainable level. We define the maximum sustainable excess return below and then state the result.

Definition 4. *The maximum sustainable excess return, R^{max}, is the maximum rate of excess return on a satellite such that an open-access steady state exists. R^{max} is characterized by*

$$
\begin{aligned}
&-\delta D_X^{\kappa} + G(S_X^{\kappa}, D_X^{\kappa}) + mR^{max} S_X^{\kappa} = 0, \\
&G_D(S_X^{\kappa}, D_X^{\kappa}) - \frac{L_D(S_X^{\kappa}, D_X^{\kappa})}{L_S(S_X^{\kappa}, D_X^{\kappa})}(G_S(S_X^{\kappa}, D_X^{\kappa}) + mR^{max}) = \delta,
\end{aligned}
$$

where $S_X^{\kappa} \equiv \hat{S}(D_X^{\kappa})$, $\hat{S}(D)$ is as defined in Lemma 1, and D_X^{κ} is the open-access Kessler threshold.

Proposition 4 (Open-access Kessler Syndrome)**.** *When the rate of excess return is positive and the new fragment function is fully coupled and strictly convex in both arguments,*

1. *open access will cause Kessler Syndrome if the excess return on a satellite exceeds the maximum sustainable excess return R^{max};*
2. *the open-access Kessler region will expand as the excess return approaches R^{max}.*

Intuitively, as the rate of excess return on a satellite increases the two steady states are squeezed closer to each other.[17] When the excess rate of return is equal to the maximum

[17] This is similar to a sub-critical fold bifurcation in the excess rate of return, though the unstable steady state isn't annihilated when the two steady states collide. Note that making the per-period payoff a decreasing function of the total satellite stock would not remove the bifurcation; it would require restrictions on the new fragment function, e.g. removing the new fragment debris coupling.

sustainable level, the two steady states collide. When the rate of excess return exceeds the maximum sustainable excess return ($r_s - r > R^{max}$), Kessler Syndrome is the only long-run possibility under open access. If $R^{max} < 1$, then Kessler Syndrome will occur before open access drives the collision probability to one. If $R^{max} \geq 1$, then Kessler Syndrome under open access is inconsistent with the existence of an open-access steady state; since existence of an open-access equilibrium requires the excess rate of return to be less than one, the maximum sustainable level will not be crossed.

Figure 3 illustrates Proposition 4. The debris level associated with the unstable steady state is D^{κ}_{X}, the maximum sustainable debris level. As the rate of excess return on a satellite increases, the basin of attraction for the stable open-access steady state (the yellow shaded region) shrinks and D^{κ}_{X} moves inward while the stable open-access steady state debris level moves outward, squeezing the two fixed points closer to each other. Eventually, the stable steady state crosses the boundary where the fragment autocatalysis rate becomes positive (the red dashed line). Once this occurs, all initial conditions will lead to Kessler Syndrome.

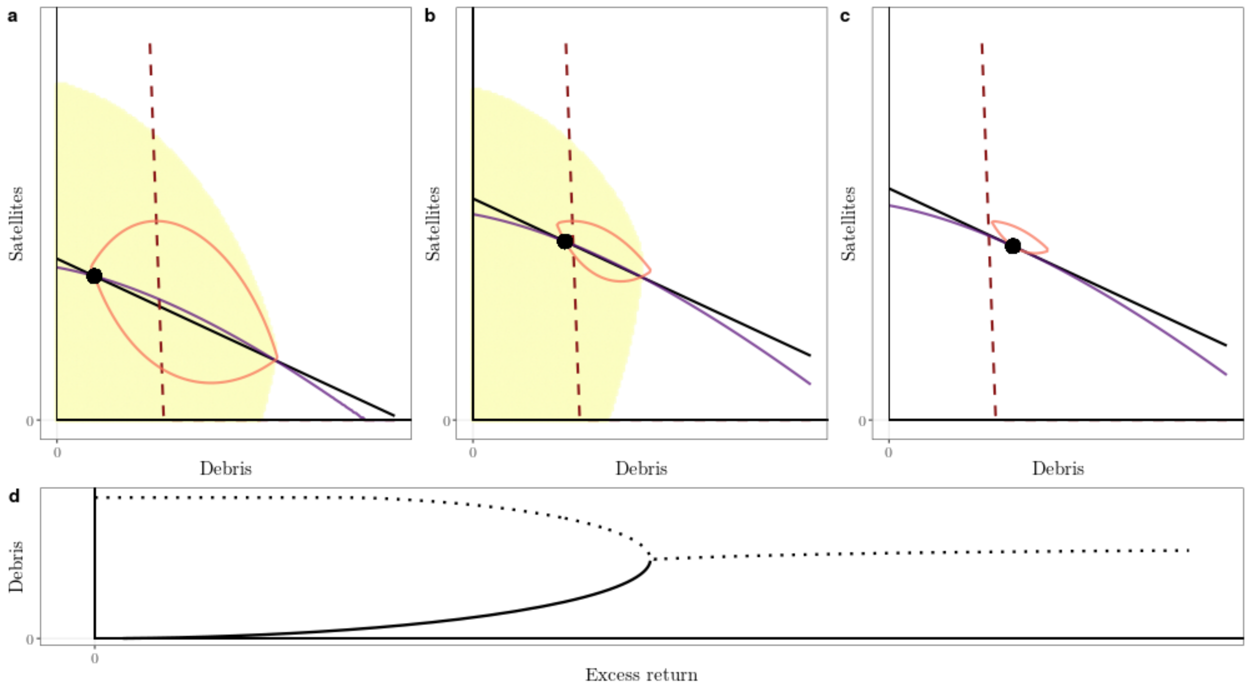

Figure 3: An illustration of how increases in the rate of excess return on a satellite can cause Kessler Syndrome. In panels a-c, the shaded yellow region shows the basin of attraction of the stable open-access steady state, and the dashed red line shows the boundary where the fragment autocatalysis rate become positive. The solid black line shows the equilibrium isoquant, and the purple and pink lines show the satellite and debris nullclines. The black dot shows the stable (in panels a and b) or only (in panel c) open-access steady state. From panel a to c the rate of excess return steadily increases, shrinking the basin of attraction of the stable open-access steady state or equivalently increasing the size of the open-access Kessler region. When the stable open-access steady state crosses the fragment autocatalysis rate boundary, it loses stability and its basin of attraction vanishes. Panel d shows the stable (solid line) and unstable (dotted line) open-access steady-state debris levels colliding as the rate of excess return on a satellite increases.

Provided there is a stable open-access steady state to approach, it is important to understand the open-access approach paths. If open access monotonically approaches stable steady states, then parameter changes which shift the steady state may not cause costly spikes in the debris stock. Such parameter changes include technological advances (e.g. better-shielded satellites), policy guidelines (e.g. encouragement to use frangibolts instead of exploding bolts for booster separation), environmental processes (e.g. sunspot activity which causes δ to vary), or economic changes (e.g. increases in the excess return on a satellite).

Proposition 5 shows it is unlikely monotonic approach paths occur when the launch rate is unconstrained. Initial conditions which result in smooth approach paths to a stable steady state form a set of Lebesgue measure zero. Almost all initial conditions instead cause the state variables to overshoot the steady state.

Proposition 5 (Overshooting)**.** *Given a new fragment formation function which is strictly con-*

vex in both arguments, a launch constraint which does not bind, and a non-atomic probability distribution over states with positive launch rates, open-access paths will almost surely overshoot the stable open-access steady state.

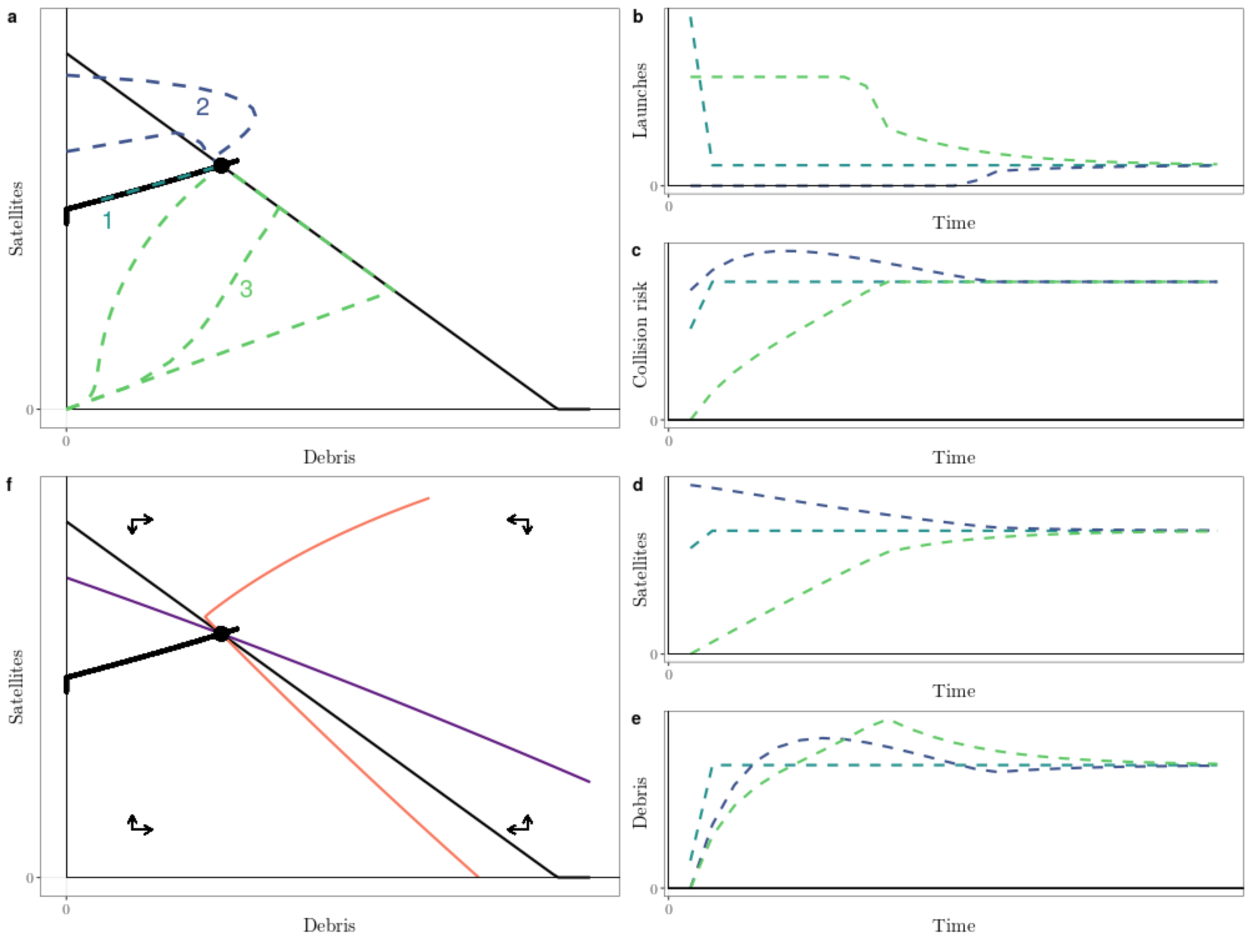

Figure 4: In panels a and f, the thin black line shows the equilibrium isoquant. The thick black point in panels a and f on the equilibrium isoquant is the stable open-access steady state. The thick black line in panels a and f connecting the steady state to the y-axis is the manifold of points which can converge to the stable steady state in one step (the "one-step set"). Panel a illustrates Proposition 5 with sample paths. The teal dashed line along the thick black line shows a path converging to the steady state in one step. The green dashed lines show paths which overshoot in debris, all of which converge to the steady state along the equilibrium isoquant. The purple dashed lines show paths which overshoot at least in satellites. These paths follow the equilibrium isoquant to the steady state whenever possible, though the physical dynamics force two of the five paths shown to approach the steady state from outside the action region. Panels b-e show the launch rate, collision risk, and satellite and debris stocks for sample paths 1-3 over time. Panel f shows the equilibrium isoquant (thin black), the one-step set (thick black), and the satellite (purple) and debris nullclines (pink).

Proposition 5 highlights two properties of open access, illustrated in figure 4. First, open access firms attempt to immediately equilibriate the collision probability by moving to the isoquant where it is equal to the excess return, even if the point on the isoquant that can be reached in one period is not a steady state. This is analogous to a driver speeding to reach their destination and not decelerating as they approach. This same property implies that, when the physical dynamics permit, open-access firms will approach the steady state along the equilibrium isoquant.

Second, the overshooting becomes more severe as the initial condition and physical dynamics bring firms to points on the equilibrium isoquant which are farther from the manifold leading to the open-access steady state. In the driving analogy, this is the effect of starting location on the uninternalized acceleration: when the driver is farther away, they must accelerate more and go faster to reach their destination in the same amount of time. In cases where the fragment autocatalysis rate becomes positive, this leads to overshooting in both state variables.

This suggests some scope for orbit use stabilization policies which impose a cap on the number of launches per period, since slowing the rate at which firms approach the equilibrium isoquant can reduce the amount of overshooting. This can be seen in figure 4. For example, an unconstrained path from the $(0,0)$ initial condition (the straight line from $(0,0)$) immediately reaches the equilibrium isoquant by overshooting the steady-state debris level. The overshooting is reduced when the launch rate is constrained (e.g. path 3, as shown in figure 4b-e), with the most tightly-constrained (green) path showing no overshooting.

Proposition 5 suggests parameter changes which lower the open-access steady-state debris level may still cause short-run rebound effects. Sunspot activity is an environmental process which can cause such a shift. Sunspots induce differential heating of the Earth's atmosphere, causing it to expand and contract over time and changing the radiation pressure exerted on satellites. These changes increase the debris decay rate, and force satellites to expend more fuel to remain in their intended orbit. Figure 5 shows an example of debris rebound effects under periodic decay rate variation similar to sunspot activity.

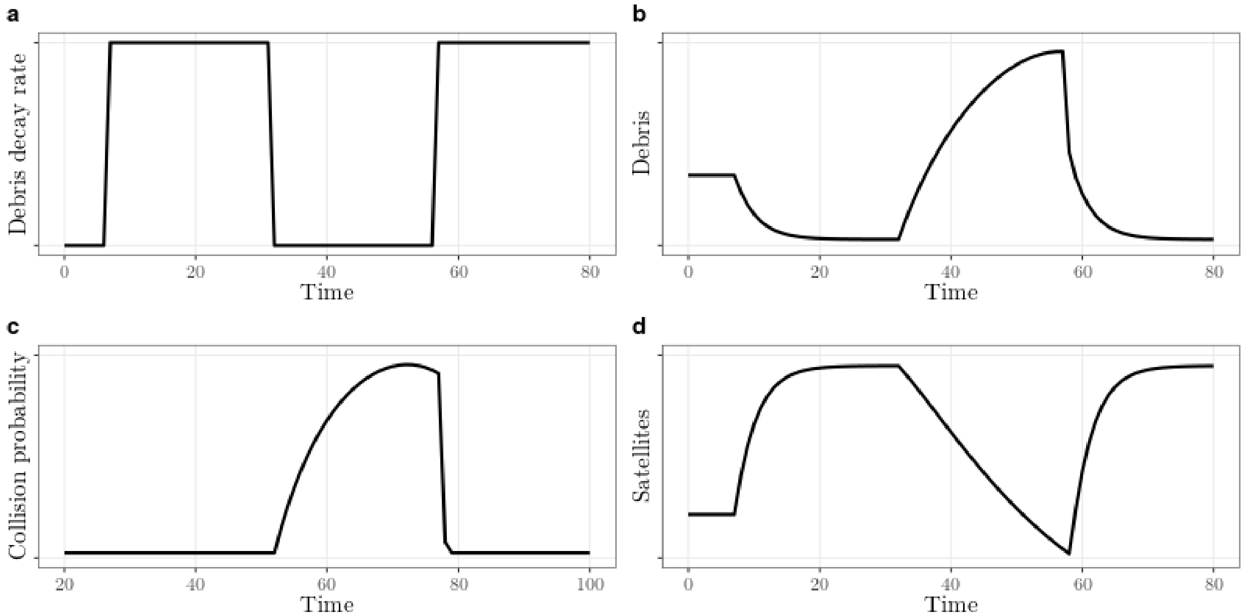

Figure 5: Panel a shows the debris decay rate varying over time. Panels b-d show equilibrium debris stock, equilibrium collision probability, and equilibrium satellite stock responding to the changes in the decay rate. The initial increase in the decay rate reduces the debris stock and allows more satellites to be sustained. When the decay rate falls, the debris stock grows rapidly and the satellite stock falls. This growth in debris causes the collision probability to rise above the equilibrium level, which stops launch activity. The growth in debris is halted only by the decay rate once again increasing. Once the collision probability is back in equilibrium, new satellites are again launched to take advantage of the higher decay rate.

5 The planner's problem

The fleet planner has exclusive rights in perpetuity to the orbit and all satellites in it. The planner therefore controls launches to maximize the net present value of the satellite fleet, equating the per-period marginal benefit of another satellite with its per-period social marginal cost. The per-period social marginal cost is the sum of the opportunity cost and collision risk ($rF + L(S_{t+1}, D_{t+1})F$), plus the effect of the marginal satellite on the fleet through future collisions and debris growth ($\xi(S_{t+1}, D_{t+1})$, described below). Firms which do not own exclusive orbital rights internalize only the opportunity cost of funds and the risk to their satellite, ignoring the costs the satellite imposes on the rest of the fleet.

The fleet planner solves

$$W(S_t, D_t) = \max_{X_t \in [0, \bar{X}]} \{\pi S_t - FX_t + \beta W(S_{t+1}, D_{t+1})\} \tag{19}$$

$$\text{s.t.} \quad S_{t+1} = S_t(1 - L(S_t, D_t)) + X_t \tag{20}$$

$$D_{t+1} = D_t(1 - \delta) + G(S_t, D_t) + mX_t. \tag{21}$$

The planner's optimal interior launch rate equates the flow of marginal benefits and costs, such that

$$\pi = rF + L(S_{t+1}, D_{t+1})F + \xi(S_{t+1}, D_{t+1}) \tag{22}$$

$$\implies L(S_{t+1}, D_{t+1}) = r_s - r - \mu(S_{t+1}, D_{t+1}). \tag{23}$$

where $\xi(S_{t+1}, D_{t+1})$ is defined as the marginal external cost of another launch in t and $\mu(S_{t+1}, D_{t+1}) = \xi(S_{t+1}, D_{t+1})/F$ is the rate of external cost per dollar spent on a launch in t. We derive equation 22 and $\xi(S,D)$ in the general (non-interior, non-stationary) case in the Appendix.

Similar to the open-access case, we define $\rho^*(X_t, S_t, D_t) = \pi - rF - L(S_{t+1}, D_{t+1})F - \xi(S_{t+1}, D_{t+1})$ as the risk-and-externality-adjusted excess return on a satellite.[18] Given the upper bound on launches, the planner's launch policy is then

$$X_t^* \in [0, \bar{X}] \text{ if } \rho^*(X_t^*, S_t, D_t) = 0, \tag{24}$$

with $X_t^* = 0$ if $\rho^*(0, S_t, D_t) < 0$ and $X_t^* = \bar{X}$ if $\rho^*(\bar{X}, S_t, D_t) > 0$.

While the full expression is lengthy, analyzing the interior steady state reveals the core intuition for the marginal external cost. Suppressing function arguments, the marginal external cost in an interior steady state is

$$\begin{aligned}\xi(S,D) = \; & \underbrace{L_S SF}_{\substack{\text{“congestion” channel:} \\ \text{marginal cost of satellites colliding} \\ \text{with each other due to crowding}}} + \underbrace{\beta\left(G_S + m(L + SL_S)\right) L_D SF + (1-\beta) m L_D SF}_{\substack{\text{“pollution hazard” channel:} \\ \text{marginal cost of satellites colliding} \\ \text{with new fragments and launch debris}}} \\ & + \underbrace{(1 - \delta + G_D)\left(\beta(\pi - (L + L_S S)F) - (1-\beta)F\right)}_{\substack{\text{“pollution persistence” channel:} \\ \text{marginal cost of persistent debris} \\ \text{and debris growth}}}\end{aligned} \tag{25}$$

The first term of $\xi(S_{t+1}, D_{t+1})$, the congestion channel, represents the cost of additional satellite collision probability due to satellite crowding. As long as the collision probability is coupled with the satellite stock and new satellites weakly increase the probability of satellite-destroying collisions ($L_S \geq 0$), this term is nonnegative.

[18] Continuing the intuition of satellites as risky assets, the planner can be viewed as internalizing not only the risk the asset stops producing payoffs ($L(S_{t+1}, D_{t+1})F$) but also the effect of each additional asset acquired on their portfolio as a whole ($\xi(S_{t+1}, D_{t+1})$).

The second term, the pollution hazard channel, is the cost of additional collisions with debris. There are two components of this channel, reflecting debris costs incurred over time and immediately. The first component, $\left(G_S + m(L+SL_S)\right) L_D SF$, represents the marginal cost of colliding with fragments generated by collisions involving other satellites (including fragments from collisions between satellites and launch debris). This component reflects the long-run risk to the fleet created by the marginal satellite, as it may be destroyed in a collision and generate additional fragments. The second component, $mL_D SF$, represents the marginal cost of the new satellite's launch debris—an immediate hazard to the fleet. As the discount factor approaches one only the long-run debris cost matters, while as it approaches zero only the immediate debris cost matters. As the number of launch debris fragments from the marginal satellite goes to zero, this channel reduces to the discounted marginal cost of collisions from fragments of satellite-debris collisions ($\beta G_S L_D SF$, a long-run cost) only. As long as the new fragment satellite and collision probability debris couplings are weakly positive, i.e. having more satellites in orbit weakly increases the number of fragments produced in collisions and having more objects in orbit weakly increases the probability of a collision, this term is nonnegative.

The third term, the pollution persistence channel, is the cost of debris which does not decay and new fragments produced in collisions between debris objects. If all debris in orbit decayed at the end of each period and the debris coupling in the new fragment function was inactive, this channel would disappear. The cost associated with this channel is the forgone discounted payoff from a satellite net of collision risk, congestion costs, and the opportunity cost of the funds used to deploy the asset. As long as the excess rate of return weakly exceeds the collision risk and marginal rate of congestion costs ($r_s - r \geq L(S,D) + L_S(S,D)S$), this term is nonnegative.

Proposition 6 establishes that the marginal external cost in an interior steady state is nonnegative as long as the excess rate of return on a satellite weakly exceeds the collision risk and marginal rate of congestion costs.

Proposition 6 (Negative externality)**.** *If the excess rate of return on a satellite weakly exceeds the collision risk and marginal rate of congestion costs in an optimal interior steady state, then the marginal external cost is weakly positive and the optimal collision risk is weakly lower than the open-access collision risk, i.e.*

$$r_s - r \geq L(S,D) + L_S(S,D)S \implies \xi(S,D) \geq 0, \tag{26}$$

and

$$\xi(S,D) \geq 0 \implies L(S^*,D^*) \leq L(\hat{S},\hat{D}). \tag{27}$$

As long as the marginal external cost is positive, the planner will maintain a lower collision risk in an interior steady state than open-access firms would. Kessler Syndrome therefore seems less likely under the planner's management than under open access. While full analysis of the dynamics of the optimal launch plan is beyond our scope here, we calculate numerical examples under illustrative parameterizations (shown in figure 6) to develop some intuition for how the optimal launch plan manages the risk of Kessler Syndrome.

In the first set of scenarios (shown in the top row of figure 6, panels a-c) we increase the excess return on a satellite by progressively lowering the cost to build and launch a satellite. This corresponds to technological changes such as reusable rockets or more efficient satellite manufacturing processes. In the second set of scenarios (shown in the middle row of figure 6, panels d-f), we increase the discount rate while holding the payoff generated and costs incurred by a satellite constant. This corresponds to improved outside investment options and a lower excess return. In the third set of scenarios (shown in the bottom row of figure 6, panels g-i), we increase the discount rate while decreasing the launch costs, such that the excess return is held constant. This corresponds to improved outside investment options and technological improvements in satellite launching.

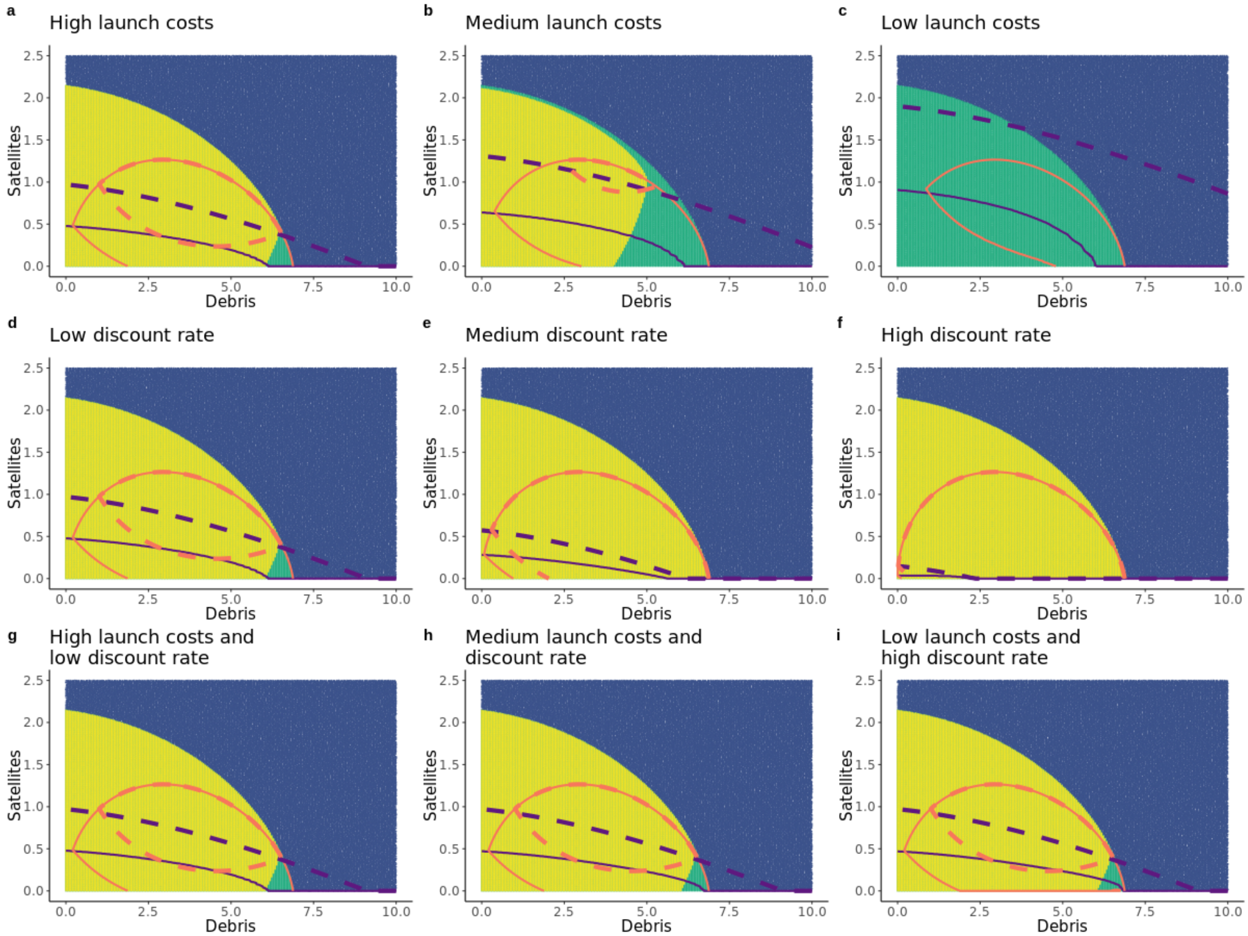

Figure 6: The blue shading marks the unavoidable Kessler region, the green shading marks the basin of attraction to the planner's stable steady state, and the yellow shading marks the basin of attraction to the open-access stable steady state. The solid orange and purple lines show the debris and satellite nullclines under the planner's launch policy, while the dashed lines show the nullclines under the open-access launch policy. From panels a to c, the excess rate of return on a satellite is increasing due to decreases in launch costs, holding the discount rate constant. From panels d to f, the discount rate rises, holding launch costs constant (implying lower excess return). From panels g to i, the launch costs fall and discount rate rises such that the excess return is held constant.

We emphasize three features in figure 6, both of which underscore the centrality of the excess return on a satellite in determining orbit-use patterns. First, when the planner has a stable basin, it appears to saturate the complement of the unavoidable Kessler region. Though the planner does respond to increases in the excess return on a satellite by launching more satellites and tolerating more debris, the increase is smaller than what occurs under open access and can be such that the planner maintains the same stable basin. Thus, increases in the excess return on a satellite may not only make Kessler Syndrome more likely under open access, but also less efficient in that there are more states where open access causes Kessler Syndrome while the planner avoids it.

Second, as the discount rate increases both the planner and open-access firms launch fewer satellites. At high discount rates the planner may prefer never launching over causing Kessler Syndrome. The contrast with biological resources like fisheries is noteworthy. Whereas high discount rates may incentivize a profit-maximizing fishery manager to drive a population to extinction, profit-maximizing orbit managers and open-access firms do not appear to face incentives to render an orbit unusable due to time preference. This is driven by the difference in how investment relates to biological and artificial population levels. Biological populations can reproduce without human investment, allowing a user to profit today at the expense of tomorrow by investing and harvesting many members of the species. Artificial populations require upfront investment and only generate profits over time—investing more today may reduce tomorrow's profits due to collisions and debris, but won't increase the profits today. Increases in the discount rate only serve to lengthen the payback period required before the investment is profitable, making debris risks more salient.[19]

Third, the discount rate affects the optimal launch policy independent of the excess return on a satellite. While higher discount rates (holding the excess return constant) may not change the optimal steady state, it does change the optimal launch policy by making the planner willing to launch more at higher debris levels.

Together, these figures illustrate how the economic dynamics of orbit use differ substantially from previously-studied natural resources. Most importantly, they show that the inefficiencies in open-access orbit use are of two types: inefficient long-run stable satellite and debris levels, and inefficient acceptance of long-run instability. While the existence of Kessler Syndrome is not required for open-access orbit use to be inefficient, open-access firms also appear to cause Kessler Syndrome when the planner would avoid it. Further research is needed to understand the dynamic properties of optimal orbit use.

6 Discussion

Resource collapse is not an unusual possibility in natural resource contexts (e.g. Clark (1973); Costello, Gaines, and Lynham (2008); Gars and Spiro (2018); Costello and Grainger (2018); Berry, Fenichel, and Robinson (2019)), but the way it manifests in orbit use is unique. In particular, the role of the discount rate in orbit use is quite different from its role in biological resource use. Rather than higher discount rates inducing resource collapse, all else equal higher discount rates will reduce the excess return on a satellite, thus reducing open-access

[19] We assume satellites are financed such that the launcher accrues profits immediately and pays an annuity for the life of the asset. But the intuition is the same: higher discount rates imply costlier financing, making satellite investments less attractive.

and optimal orbit use and potentially staving off resource collapse. This result may seem counterintuitive—shouldn't higher discount rates induce less consideration for outcomes in the future?—but is a natural consequence of rent dissipation. Higher discount rates reduce the excess return on a satellite (equivalently, lengthen the payback period), reducing the incentive to launch.

It is important to note that open-access rent dissipation in orbit is fundamentally the same as in other settings. Launchers and operators do not benefit from self-restraint because they have no reason to expect others to follow suit. The excess launching generated by this behavior is similar to overexploitation in other common resource contexts. However, the potential for multiple combinations of satellites and debris to produce identical risk levels (due to the presence of both collision risk couplings, L_S and L_D) creates a multiplicity of equilibria in the short run. While multiplicity occurs in other natural resource settings due to nonlinearities inherent in the resource dynamics (May, 1974; Clark and Munro, 1975), it is typically a multiplicity of steady states.

As in biological resource contexts, once resource collapse in orbit has been initiated it will continue unless the population levels are altered. While in fisheries and other biological contexts this often amounts to introducing new members of the harvested species, the reverse is true for orbit use: to prevent or stop Kessler Syndrome, the satellite or debris populations (or both) must be reduced. This is driven by the difference between biological reproduction and "reproduction" of debris. Since any mass on orbit is a source of new debris, lowering the rate at which debris "reproduces" requires reducing the mass on orbit. Combined with equation 3, framing the result in terms of "debris reproduction" clarifies the role of the new fragment formation function, G, in determining whether Kessler Syndrome is possible and why it might occur. If the new fragment function is not coupled with the debris stock ($G_D \equiv 0$), then equation 3 is always negative and Kessler Syndrome is impossible. Intuitively, $G_D \equiv 0$ means new fragments can only form due to active satellites. Thus, as the debris stock grows, collision risk also increases, eventually leading firms to stop launching satellites. If $G_D \equiv 0$ then the debris stock will shrink as the number of active satellites dwindles, until firms can begin launching again and the stable steady state is reached. But if $G_D > 0$, debris can reproduce without active satellites. Since multiple combinations of satellites and debris can produce the level of collision risk required for rent dissipation, firms may stop launching only after there is enough mass on orbit for "debris reproduction" without additional human input. As noted above, the new fragment debris coupling is considered a key factor in long-run debris population dynamics, though these studies ignore the economic incentives to launch satellites (Liou, 2006a; Lewis, 2020). Given that near-collisions between debris objects are already occurring, it seems only a matter of time before this coupling acquires greater empirical significance (Davenport, 2020).

7 Conclusion

In this paper we present a dynamic physico-economic model of orbit use under rational expectations with endogenous collision risk and Kessler Syndrome. We show how both economic and physical parameters drive equilibrium short- and long-run orbit-use patterns, derive the marginal external cost of a satellite, explore the multiplicity, instability, and convergence dynamics of open-access steady states, and examine the relationships between open-access orbit use, optimal orbit use, and Kessler Syndrome. We highlight three messages regarding orbital-use management.

First, under open access too many firms will launch satellites because they won't internalize the risks they impose on other orbit users. Though profit maximizing satellite owners have incentives to reduce launches as the risk of a collision grows, they do not respond to debris growth or collision risk optimally. This inefficiency is independent of whether Kessler Syndrome is possible or not. Further, changes in economic parameters are unlikely to make Kessler Syndrome optimal, though further research quantifying coupling strengths, economic parameters, and their relationships is needed. Second, rebound effects can result in higher debris levels when the rate of debris decay increases. This suggests sunspot activity may drive debris and collision risk cycles in LEO even if satellite operators are forward-looking and rational. Third, under open access Kessler Syndrome is more likely as the excess return on a satellite rises, even if firms will respond to orbital congestion by launching fewer satellites. As launch costs fall and new commercial satellite applications become viable, LEO is thus increasingly and inefficiently likely to experience Kessler Syndrome. While it may seem paradoxical that the very changes which make orbit use profitable can also increase the risk of resource collapse, such dynamics occur frequently in bioeconomic commons problems. These results, along with the long-run consequences of debris accumulation occurring today, underscore the urgent need for reforms to orbital-use management institutions.

Economists tend to focus on property rights, corrective taxes, or other market-based mechanisms to solve externality problems. While these mechanisms can ensure efficient orbit use and prevent Kessler Syndrome, more study is needed to understand how orbital management policies should be designed in light of the unique physical features of this resource. Whether they are enforced by states, structured as self-enforcing agreements between private actors, or some combination of the two, orbits are a global commons and will require global policy solutions.

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Appendices

A Proofs and derivations

A.1 Proofs

Proposition 1 (Equilibrium collision probability)**.** *Given that an interior open-access equilibrium exists and the collision probability function is coupled with both the satellite and debris stocks, there are multiple vectors* (S_t, D_t, X_t) *such that*

$$L(S_{t+1}, D_{t+1}) = r_s - r.$$

Increases in the excess return on a satellite cause X_t*,* S_{t+1}*,* D_{t+1}*, and* $L(S_{t+1}, D_{t+1})$ *to increase.*

Proof. Taking equation 12 and dividing by F, the equilibrium collision probability in $t+1$ can be written as

$$L(S_{t+1}, D_{t+1}) = r_s - r, \tag{28}$$

where $r_s - r$ is the excess return on a satellite. By inspection, increases in the excess return will also increase the equilibrium collision probability.

Since L is continuous and increasing in both arguments, there is a continuum of points (S, D) such that equation 12 holds—the level set of L equal to $r_s - r$.

To see the effect of increases in the excess return on the launch rate in t and the debris stock in $t+1$, we apply the Implicit Function Theorem to equation 12:

$$\begin{aligned}
\frac{\partial X_t}{\partial (r_s - r)} &= \frac{1}{L_S(S_{t+1}, D_{t+1}) + mL_D(S_{t+1}, D_{t+1})} > 0\\
\frac{\partial D_{t+1}}{\partial (r_s - r)} &= \frac{1}{L_D(S_{t+1}, D_{t+1})} > 0\\
\frac{\partial S_{t+1}}{\partial (r_s - r)} &= \frac{1}{L_S(S_{t+1}, D_{t+1})} > 0,
\end{aligned}$$

where variable-subscripts indicate partial derivatives, i.e. $L_S \equiv \frac{\partial L}{\partial S}$, and $L_S, L_D > 0$ by assumption. $\square$

Lemma 1 (Reduction)**.** *Given a fully-coupled collision probability function, the steady states of an open-access equilibrium are defined by solutions to*

$$\mathscr{Y}(D) = -\delta D + G(\hat{S}, D) + m(r_s - r)\hat{S} = 0, \tag{17}$$

where $\hat{S} = S(r_s - r, D) \geq 0$ *is such that, for a given* D*,*

$$L(\hat{S}, D) = r_s - r$$

when such an S *exists and* 0 *otherwise.*

Proof. The open-access steady states are defined by equations 14, 15, and 16. Equation 14 implicitly determines the number of satellites as a function of the amount of debris, the excess return on a satellite, and the collision rate function,

$$L(S, D) = r_s - r \implies S = S(r_s - r, D). \tag{29}$$

Since L is monotone increasing in each argument, $S(r_s - r, D)$ is monotone decreasing in D. Since S must be nonnegative, there exists a nonnegative $D^S : S(r_s - r, D) = 0 \; \forall D \geq D^S$. So we have

$$\hat{S} = \begin{cases} S(r_s - r, D) \text{ if } D \in [0, D^S) \\ 0 \text{ if } D \geq D^S \end{cases} \tag{30}$$

Using $\hat{S}$ we can reduce equations 14, 15, and 16 to a single equation in debris,

$$\mathscr{Y}(D) = -\delta D + G(\hat{S}, D) + m(r_s - r)\hat{S},$$

with

$$\{\hat{D} \geq 0 : \delta \hat{D} = G(\hat{S}, \hat{D}) + m(r_s - r)\hat{S}\} \tag{31}$$

being the open-access steady states.

□

Proposition 2 (Multiplicity)**.** *Given a positive excess return on a satellite and a fully-coupled collision probability function, multiple open-access steady states can exist if the new-fragment-debris coupling exists.*

Proof. Using the reduction to equation 17 from Lemma 1, we focus our attention on solutions to

$$\mathscr{Y}(D) = -\delta D + G(\hat{S}, D) + m(r_s - r)\hat{S}.$$

δD is monotonically increasing in D with $\delta D = 0$ when $D = 0$, and $m(r_s - r)\hat{S}$ is monotonically decreasing in D with $\hat{S} > 0$ when $D = 0$, but $\hat{G} \equiv G(\hat{S}, D)$ is nonmonotone in D. To see this, note

$$\frac{d\hat{G}}{dD}(\hat{S},D) = \underbrace{\frac{\partial G}{\partial S}}_{\geq 0}\underbrace{\frac{\partial \hat{S}}{\partial D}}_{\leq 0} + \underbrace{\frac{\partial G}{\partial D}}_{\geq 0}, \text{ with} \tag{32}$$

$$\frac{d\hat{G}}{dD}(\hat{S},0) = \frac{\partial G}{\partial S}\frac{\partial \hat{S}}{\partial D} < 0 \text{ and} \tag{33}$$

$$\frac{d\hat{G}}{dD}(0,D^S) = \frac{\partial G}{\partial D} > 0, \tag{34}$$

where $\frac{\partial \hat{S}}{\partial D} = -\frac{L_D}{L_S} \leq 0$ by application of the Implicit Function Theorem on equation 12.

Let $\hat{D}$ be a solution to equation 17. If $G_D > 0$, then $\hat{G}$ is nonmonotone in D and the existence or uniqueness of $\hat{D}$ cannot be guaranteed. If G_D is large enough, $\hat{D}$ will not exist; if G_D is not too small, multiple $\hat{D}$ will exist. If $G_D = 0$, then the existence of $\hat{D}$ also ensures its uniqueness. If G_D is strictly convex in both arguments, at most two $\hat{D}$ can exist. □

Proposition 3 (Local stability)**.** *Given a positive excess return on a satellite, new-fragment-satellite coupling, and launch-debris coupling, open-access steady states will be locally stable if and only if the fragment autocatalysis rate is small enough, or the marginal rate of technical substitution of satellites for debris in collision probability is high enough. When G is strictly convex in both arguments and two steady states exist, the higher-debris steady state is unstable.*

Proof. We use the reduction from Lemma 1 to simplify the proof. The open access steady states are solutions to equation 17, and the sign of $\frac{\partial \mathscr{Y}}{\partial D}$ at the solutions allows us to classify the stability of the system. Applying the Implicit Function Theorem to equation 14 to calculate S_D, then differentiating $\mathscr{Y}$ in the neighborhood of an arbitrary solution D^*,

$$\frac{\partial \mathscr{Y}}{\partial D}(D^*) = (G_D(S^*,D^*) - \delta) - \frac{L_D(S^*,D^*)}{L_S(S^*,D^*)}(G_S(S^*,D^*) + m(r_s - r)), \tag{35}$$

where $S^* \equiv S(r_s - r, D^*)$. $(G_D(S^*,D^*) - \delta)$ is the fragment autocatalysis rate and $\frac{L_D(S^*,D^*)}{L_S(S^*,D^*)}$ is the MRTS of satellites for debris in collision probability. Both $G_S(S^*,D^*)$ and $m(r_s - r)$ are positive by assumption. So $\frac{\partial \mathscr{Y}}{\partial D}(D^*) < 0$ can hold if and only if the fragment autocatalysis rate is small enough, or the MRTS of satellites for debris in collision probability is large enough. □

Lemma 2 (Open-access Kessler threshold)**.** *When G is strictly convex in both arguments and two steady states exist, let*

$$\underline{D} = \min\{D : \mathscr{Y}(D) = 0\}$$

be the stable steady state, and

$$\bar{D} = \max\{D : \mathscr{Y}(D) = 0\}$$

be the unstable steady state. Then

$$\bar{D} = D^{\kappa}_{X}.$$

Proof. When G is stictly convex in both arguments and two steady states exist, the reduction

$$\mathscr{Y}(D) = -\delta D + G(\hat{S}, D) + m(r_s - r)\hat{S} = 0$$

has two solutions. The curve $G(\hat{S}, D) + m(r_s - r)\hat{S}$ is above δD when $D = 0$ and again as $D \to \infty$, as shown in Proposition 2 and illustrated in panel a of figure 2. So the first solution,

$$\underline{D} = \min\{D : \mathscr{Y}(D) = 0\},$$

must occur when $G(\hat{S}, D) + m(r_s - r)\hat{S}$ approaches δD from above, and the second,

$$\bar{D} = \max\{D : \mathscr{Y}(D) = 0\},$$

when $G(\hat{S}, D) + m(r_s - r)\hat{S}$ approaches δD from below.

This implies that at the first solution, $\underline{D}$,

$$\frac{\partial \mathscr{Y}}{\partial D}(\underline{D}) = (G_D(\underline{S}, \underline{D}) - \delta) - \frac{L_D(\underline{S}, \underline{D})}{L_S(\underline{S}, \underline{D})}(G_S(\underline{S}, \underline{D}) + m(r_s - r)) < 0 \tag{36}$$

and at the second solution, $\bar{D}$,

$$\frac{\partial \mathscr{Y}}{\partial D}(\bar{D}) = (G_D(\bar{S}, \bar{D}) - \delta) - \frac{L_D(\bar{S}, \bar{D})}{L_S(\bar{S}, \bar{D})}(G_S(\bar{S}, \bar{D}) + m(r_s - r)) > 0 \tag{37}$$

where $\underline{S} = \hat{S}(\underline{D})$ and $\bar{S} = \hat{S}(\bar{D})$.

Since $\bar{D}$ is an unstable fixed point of $\mathscr{Y}(D) = 0$, initial conditions $D > \bar{D}$ must diverge from $\bar{D}$. But $\bar{D}$ is the largest fixed point of $\mathscr{Y}(D) = 0$, so such trajectories must diverge to $+\infty$. The Kessler region and threshold are defined as

$$\kappa_X \equiv \{(S, D) : \lim_{t \to \infty} D_{t+1} = \infty \mid S_0 = S, D_0 = D, X_t = X(S_t, D_t)\}$$
$$D^{\kappa}_{X} \equiv \sup\{D : (S, D) \notin \kappa_X\}.$$

So from the definition of D^{κ}_{X} and the properties of $\bar{D}$, we have that $\bar{D} = D^{\kappa}_{X}$.

$\square$

Proposition 4 (Open-access Kessler Syndrome)**.** *When the rate of excess return is positive and the new fragment function is fully coupled and strictly convex in both arguments,*

1. *open access will cause Kessler Syndrome if the excess return on a satellite exceeds the maximum sustainable excess return R^{max};*
2. *the open-access Kessler region will expand as the excess return approaches R^{max}.*

Proof. We begin with the single-equation reduction from Lemma 1:

$$\mathscr{Y}(D) = -\delta D + G(\hat{S}, D) + m(r_s - r)\hat{S}. \tag{38}$$

If the new fragment formation function G is strictly convex in both arguments, there are up to two solutions to the above equation.[20] From Lemma 2 we have that the smaller (attracting) solution is $\underline{D}$ and the larger (repelling) solution is D_X^κ. The basin of attraction for $\underline{D}$ in the D dimension is thus at most $[0, D_X^\kappa)$. We say "at most" because, as figure 3 illustrates, in two dimensions the basin depends on both S and D.

Next, consider how open-access steady-state debris levels change as the excess return on a satellite changes. We suppress function arguments to reduce notation; all functions are evaluated at an arbitrary open-access steady state. Applying the Implicit Function Theorem to equation 17, and applying it again to equation 14 to calculate $\frac{\partial S}{\partial(r_s - r)}$, we get:

$$\frac{\partial D}{\partial(r_s - r)} = -\frac{\frac{G_S}{L_S} + m(S(r_s - r, D) + \frac{r_s - r}{L_S})}{(G_D - \delta) - \frac{L_D}{L_S}(G_S + m(r_s - r))} \lesseqgtr 0$$

Given our assumptions, the numerator is always positive. From Proposition 3, an open-access steady state is locally stable if and only if the denominator is negative. So if the steady state is stable, increases in the excess rate of return on a satellite will cause the debris level to increase. If the steady state is unstable, increases in the excess rate of return on a satellite will cause the debris level to decrease. So we have

$$\frac{\partial \underline{D}}{\partial(r_s - r)} > 0, \quad \frac{\partial D_X^\kappa}{\partial(r_s - r)} < 0,$$

so as $r_s - r$ increases, $\underline{D}$ and D_X^κ are squeezed closer together. As this happens, the basin of attraction for $\underline{D}$ must shrink under any non-atomic measure applied to $[0, D_X^\kappa)$. Since the open-access Kessler region is the complement of the stable steady state's basin of attraction, as the basin shrinks, the open-access Kessler region must expand.

Eventually, increases in $r_s - r$ will result in $\underline{D} = D_X^\kappa = D^*$. This is because increases in $r_s - r$ shift the curve $G(\hat{S}, D) + m(r_s - r)\hat{S}$ upwards through both the coefficient on $\hat{S}$ and the

[20] Strict convexity of G is a sufficient but not necessary condition for this result. However, it is a reasonable physical assumption and simplifies the proof.

value of $\hat{S}$ itself—recall from Proposition 1 that increases in $r_s - r$ shift the equilibrium isoquant outwards, and from Lemma 1 that $\hat{S}$ is the equilibrium satellite stock given D.

Thus there exists a threshold value R^{max} for $r_s - r$ such that

1. there is a single solution to $\mathscr{Y}(D) = 0$, and
2. for $r_s - r > R^{max}$ there is no solution to $\mathscr{Y}(D) = 0$.

Defining this threshold as R^{max} and letting that single solution be D^* (with $S^* \equiv \hat{S}(D^*)$), the first condition can be formalized as

$$R^{max} : \mathscr{Y}(D^*; R^{max}) = -\delta D^* + G(S^*, D^*) + mR^{max} S^* = 0. \tag{39}$$

The second condition implies that R^{max} is such that $G(\hat{S}, D) + mR^{max}\hat{S}$ is tangent to δD at D^*, so that when $r_s - r > R^{max}$, $G(\hat{S}, D) + mR^{max}\hat{S}$ and δD do not intersect. The second condition can therefore be formalized as

$$R^{max} : G_D(S^*, D^*) - \frac{L_D(S^*, D^*)}{L_S(S^*, D^*)}(G_S(S^*, D^*) + mR^{\max}) = \delta. \tag{40}$$

The tangency condition in equation 40 combined with the properties of the curve $G(S^*, D^*) + mR^{max}S^*$ imply that $G(S^*, D^*) + mR^{max}S^*$ approaches and departs the point of tangency from above. For any $\mu > 0$, $\mathscr{Y}_D(D^*; R^{max} - \mu) < 0$ and $\mathscr{Y}_D(D^*; R^{max} - \mu) > 0$. So trajectories under R^{max} which approach D^* will be driven away from D^* if they receive any small positive perturbation. Consequently, the steady state induced by R^{max} must be an unstable one, i.e. $D^* = D_X^{\kappa}$.

□

Proposition 5 (Overshooting)**.** *Given a new fragment formation function which is strictly convex in both arguments, a launch constraint which does not bind, and a non-atomic probability distribution over states with positive launch rates, open-access paths will almost surely overshoot the stable open-access steady state.*

Proof. We first define the following sets and functions, where $S, D \geq 0$ is assumed:

- The action region: the set of states with positive open-access launch rates,

$$A \equiv \left\{(S, D) : r_s - r - L(S', D') \geq 0\right\}, \tag{41}$$

  where

$$\begin{aligned} S' &= S(1 - L(S, D)) + X \\ D' &= D(1 - \delta) + G(S, D) + mX, \end{aligned}$$

- The equilibrium isoquant:

$$E \equiv \left\{(S,D) : r_s - r - L(S,D) = 0\right\}. \tag{42}$$

- The stable open-access steady state: following the reduction to equation 17 from Lemma 1,

$$E_s \equiv \Bigg\{(\hat{S},D) : \mathscr{Y}(D) = -\delta D + G(\hat{S},D) + m(r_s - r)\hat{S} = 0,$$
$$\hat{S} : L(\hat{S},D) = r_s - r,\ \ \mathscr{Y}'(D) < 0\Bigg\}. \tag{43}$$

- The physical dynamics: the effect of orbital mechanics on the satellite and debris stocks in a period,
$$P_{SD}(S,D) \equiv \big(S(1-L(S,D)), D(1-\delta)+G(S,D)\big). \tag{44}$$

- The one-step set: the set of states from which one period's physical dynamics, followed by launching, will read an open-access steady state,

$$A_{P1} \equiv \{(S,D) : P_{SD}(S,D) + (X, mX) \in E_s,\ \ X \in (0,\bar{X}]\}. \tag{45}$$

- The one-step ray: the set of states from which one period of launching will reach an open-access steady state,

$$A_1 \equiv \left\{(S,D) : (S+X, D+mX) \in E_s,\ \ X \in (0,\bar{X}]\right\}, \tag{46}$$

  where m is the same as in the debris law of motion. The one-step ray can be viewed as part of a decomposition of the satellite and debris laws of motion: after a period's physical dynamics have been applied, launches to the stable steady state occur from the one-step ray.

Since the launch constraint does not bind, any point in A will reach a point in E. Since E_s contains at most one element given the strict convexity of G while E is a manifold, $E_s \subset E$. Given that L is increasing in both arguments, points in $E \setminus E_s$ must therefore have either larger S and smaller D than E_s, or vice versa. Consequently, reaching points in $E \setminus E_s$ constitutes overshooting E_s in one state variable and undershooting in the other.

By definition, $A_1 \subseteq A$. Since E_s contains at most one element, A_1 is a single line segment, so $A_1 \subset A$. The Lebesgue measure on A of A_1 is therefore zero.

From the definition of P_{SD}, it is bijective on $\mathbb{R}^2_+$. To show this, we separate the physical dynamics into two functions,

$$P_S(S,D) = S(1 - L(S,D))$$
$$P_D(S,D = D(1-\delta) + G(S,D).$$

P_D is a sum of strictly monotone increasing functions, so is strictly monotone increasing as well. Strictly monotone functions are bijections, so P_D is a bijection. So for two arbitrary pairs (S_1,D_1) and (S_2,D_2) we have

$$P_{SD}(S_1,D_1) = P_{SD}(S_2,D_2) \iff P_S(S_1,D_1) = P_S(S_2,D_2) \,\&P_D(S_1,D_1) = P_D(S_2,D_2).$$

But since P_D is a bijection, $P_D(S_1,D_1) = P_D(S_2,D_2) \implies (S_1,D_1) = (S_2,D_2)$, implying P_{SD} is also a bijection.

Since P_{SD} is a bijection, the Lebesgue measure of the pre-image of A_1 under P_{SD},

$$P_{SD}^{-1}(A_1) \equiv \left\{(S,D) : P_{SD}(S,D) \in A_1\right\},$$

is the same as the Lebesgue measure of A_1. Since $P_{SD}^{-1}(A_1) = A_{P1}$, the Lebesgue measure on A of A_{P1} is also zero. Lebesgue measure is isomorphic to any non-atomic probability measure, so the one-step set is measure zero under any non-atomic probability measure. This gives the desired result: initial conditions with positive open-access launch rates drawn from a non-atomic probability measure over states will almost surely overshoot the stable open-access steady state.

□

Proposition 6 (Negative externality). *If the excess rate of return on a satellite weakly exceeds the collision risk and marginal rate of congestion costs in an optimal interior steady state, then the marginal external cost is weakly positive and the optimal collision risk is weakly lower than the open-access collision risk, i.e.*

$$r_s - r \geq L(S,D) + L_S(S,D)S \implies \xi(S,D) \geq 0, \tag{26}$$

and

$$\xi(S,D) \geq 0 \implies L(S^*,D^*) \leq L(\hat{S},\hat{D}). \tag{27}$$

Proof. Let $r_s - r \geq L(S,D) + L_S(S,D)S$ at an interior optimal steady state (S,D). Suppressing function arguments, the marginal external cost is

$$\xi = L_S SF + \beta \left(G_S + m(L + SL_S)\right) L_D SF + (1-\beta) m L_D SF$$
$$+ (1 - \delta + G_D) \left(\beta(\pi - (L + L_S S)F) - (1-\beta)F\right) \tag{47}$$

All terms in this equation are weakly positive for all S and D, and the first two terms on the right-hand side add non-negative components separately or in a convex combination weighted by β. To see that $r_s - r \geq L(S,D) + L_S(S,D)S$ is equivalent to the third term being weakly positive, note that

$$\beta(\pi - (L + L_S S)F) - (1-\beta)F \geq 0 \tag{48}$$
$$\iff \frac{\pi - (L + L_S S)F}{F} \geq \frac{1-\beta}{\beta} \tag{49}$$
$$\iff r_s - (L + L_S S) \geq r \tag{50}$$
$$\iff r_s - r \geq L + L_S S. \tag{51}$$

Since all the components of $\xi(S,D)$ are weakly positive, $\xi(S,D) \geq 0$ at an optimal interior steady state.

Next, consider the open-access interior steady-state launch rate, $\hat{X}$, and the optimal interior steady-state launch rate, X^*. An interior open-access steady state, $(\hat{S}, \hat{D})$, is such that

$$\pi = rF + L(\hat{S}, \hat{D})F \tag{52}$$

and an interior optimal steady state, (S^*, D^*), is such that

$$\pi = rF + L(S^*, D^*)F + \xi(S^*, D^*). \tag{53}$$

Since ξ is weakly positive at an optimal interior steady state, it must be that $L(S^*, D^*) \leq L(\hat{S}, \hat{D})$.

$\square$

A.2 Derivations

A.2.1 Optimal launch policy and marginal external cost

The sequence version of the fleet planner's problem is

$$\max_{\{X_t,S_{t+1},D_{t+1}\}_{t=0}^{\infty}} S_t Q(S_t,D_t,X_t)+\beta\sum_{\tau=t}^{\infty}\beta^{\tau-t-1}X_\tau(\beta Q(S_{\tau+1},D_{\tau+1},X_{\tau+1})-F) \tag{54}$$

$$\text{s.t. } Q(S_t,D_t,X_t)=\pi+\beta(1-L(S_t,D_t))Q(S_{t+1},D_{t+1},X_{t+1}) \tag{55}$$

$$S_{t+1}\leq S_t(1-L(S_t,D_t))+X_t \tag{56}$$

$$D_{t+1}\geq D_t(1-\delta)+G(S_t,D_t)+mX_t \tag{57}$$

$$X_t\in[0,\bar{X}]\ \ \forall t \tag{58}$$

$$S_{t+1}\geq 0, D_{t+1}\geq 0 \tag{59}$$

$$S_0=s_0, D_0=d_0 \tag{60}$$

The planner's Lagrangian is

$$\mathscr{L}(X,S,D,\lambda,\gamma)=\sum_{t=0}^{\infty}\beta^t\Bigg\{\pi S_t-FX_t+\lambda_{S_t}\big(S_t(1-L(S_t,D_t))+X_t-S_{t+1}\big) \tag{61}$$

$$+\lambda_{D_t}\big(D_{t+1}-D_t(1-\delta)-G(S_t,D_t)-mX_t\big) \tag{62}$$

$$+\gamma_{X_t}X_t+\gamma_{\bar{X}_t}(\bar{X}-X_t)+\gamma_{S_t}S_{t+1}+\gamma_{D_t}D_{t+1}\Bigg\} \tag{63}$$

The first-order necessary conditions for an optimal launch path are, $\forall t$ up to T,

$$\mathscr{L}_{X_t}=-F+\lambda_{S_t}-m\lambda_{D_t}+\gamma_{X_t}-\gamma_{\bar{X}_t}=0 \tag{64}$$

$$\mathscr{L}_{S_{t+1}}=\beta\{\pi+\lambda_{S_{t+1}}(1-L(S_{t+1},D_{t+1})-S_{t+1}L_S(S_{t+1},D_{t+1})) \tag{65}$$

$$-\lambda_{D_{t+1}}G_S(S_{t+1},D_{t+1})\}+\gamma_{S_t}-\lambda_{S_t}=0 \tag{66}$$

$$\mathscr{L}_{D_{t+1}}=\beta\{\lambda_{D_{t+1}}(\delta-1-G_D(S_{t+1},D_{t+1}))-\lambda_{S_{t+1}}S_{t+1}L_D(S_{t+1},D_{t+1})\}+\lambda_{D_t}+\gamma_{D_t}=0 \tag{67}$$

$$\mathscr{L}_{S_{T+1}}=\gamma_{S_T}-\lambda_{S_T}=0 \tag{68}$$

$$\mathscr{L}_{D_{T+1}}=\lambda_{D_T}+\gamma_{D_T}=0 \tag{69}$$

with complementary slackness and transversality conditions

$$\lambda_{St}\left(S_t(1-L_t+X_t-S_{t+1}\right)=0 \tag{70}$$

$$\lambda_{Dt}\left(D_{t+1}-D_t(1-\delta)-G_t-mX_t\right)=0 \tag{71}$$

$$\gamma_{Xt}X_t=0, \tag{72}$$

$$\gamma_{\bar{X}t}(\bar{X}-X_t)=0, \tag{73}$$

$$\gamma_{St}S_{t+1}=0, \tag{74}$$

$$\gamma_{Dt}D_{t+1}=0 \tag{75}$$

$$\lim_{T\to\infty}\beta^T\lambda_{ST}S_{T+1}=0 \tag{76}$$

$$\lim_{T\to\infty}-\beta^T\lambda_{DT}D_{T+1}=0. \tag{77}$$

In what follows we drop time subscripts to reduce notational clutter. Period t values are shown with no subscript, period $t+1$ values are marked with a $'$ after the variable, and period $t-1$ values are marked with a $'$ before the variable e.g. $S_{t-1}\equiv {}'S$, $S_t\equiv S$, $S_{t+1}\equiv S'$. By (64),

$$\lambda_S=\frac{F+\beta m\lambda_D-\gamma_X+\gamma_{\bar{X}}}{\beta}. \tag{78}$$

In the next period, this becomes

$$\lambda_S'=\frac{F+\beta m\lambda_D'-\gamma_X'+\gamma_{\bar{X}}'}{\beta}. \tag{79}$$

By (66) and (67),

$$\lambda_S=\pi+\frac{\gamma_S}{\beta}+\beta\{\lambda_S'(1-L(S',D')-S'L_S(S',D'))-\lambda_D'G_S(S',D')\} \tag{80}$$

$$\lambda_D=\beta\{\lambda_D'(1+G_D(S',D')-\delta)+\lambda_S'S'L_D(S',D')\}-\frac{\gamma_D}{\beta}. \tag{81}$$

Using (79),

$$\lambda_S=\pi+\frac{\gamma_S}{\beta}-F(L(S',D')+S'L_S(S',D')-1)-\beta\lambda_DG_S(S',D') \tag{82}$$

$$-\beta m\lambda_D(L(S',D')+S'L_S(S',D')-1)+(L(S',D')+S'L_S(S',D')-1)(\gamma_X'-\gamma_{\bar{X}}') \tag{83}$$

$$\lambda_D=FS'L_D(S',D')+\beta W_D(S',D')(1+G_D(S',D')-\delta)+\beta m\lambda_DS'L_D(S',D') \tag{84}$$

$$-\left(\frac{\gamma_D}{\beta}+S'L_D(S',D')(\gamma_X'-\gamma_{\bar{X}}'\right) \tag{85}$$

Define,

$$\alpha'_1 = \pi + (1 - L(S', D') - S' L_S(S', D'))F \tag{86}$$

$$\alpha'_2 = S' L_D(S', D')F \tag{87}$$

$$\Gamma'_1 = G_S(S', D') - m(1 - L(S', D') - S' L_S(S', D')) \tag{88}$$

$$\Gamma'_2 = 1 - \delta + G_D(S', D') + mS' L_D(S', D') \tag{89}$$

$$\kappa'_1 = \frac{\gamma_S}{\beta} - (\gamma_{X'} - \gamma_{\bar{X}'})(1 - L(S', D') - S' L_S(S', D')) \tag{90}$$

$$\kappa'_2 = \frac{\gamma_D}{\beta} + S' L_D(S', D')(\gamma_{X'} - \gamma_{\bar{X}'}) \tag{91}$$

so that

$$\lambda_S = \alpha'_1 - \beta \lambda'_D \Gamma'_1 + \kappa'_1 \tag{92}$$

$$\lambda_D = \alpha'_2 + \beta \lambda'_S \Gamma'_2 - \kappa'_2 \tag{93}$$

Then,

$$\lambda'_D = \frac{\lambda_D - \alpha'_2 + \kappa'_2}{\beta \Gamma'_2} \tag{94}$$

Substitute (78) and (94) in (92) to get the following expression for $W_D(S, D)$

$$\frac{\beta\{\Gamma'_1(\alpha'_2 - \kappa'_2) + \Gamma'_2(\alpha'_1 + \kappa'_1)\} + \Gamma'_2(\gamma_X - \gamma_{\bar{X}} - F)}{\beta(\Gamma'_1 + m\Gamma'_2)} \tag{95}$$

Iterate 95 to period $t+1$ and substitute into 94 to obtain

$$\lambda'_D = \frac{\beta\{\Gamma''_1(\alpha''_2 - \kappa''_2) + \Gamma''_2(\alpha''_1 + \kappa''_1)\} + \Gamma''_2(\gamma'_X - \gamma'_{\bar{X}} - F)}{\beta(\Gamma''_1 + m\Gamma''_2)} \tag{96}$$

Use (95) and (96) in (93) to get

$$\alpha'_1 = m(\alpha'_2 - \kappa'_2) - \kappa'_1 + \frac{1}{\beta}(\gamma_{\bar{X}} - \gamma_X + F) + \frac{\Gamma'_1 + m\Gamma'_2}{\Gamma''_1 + m\Gamma''_2}\left(\Gamma''_1 \beta(\alpha''_2 - \kappa''_2) + \Gamma''_2(\beta(\alpha''_1 + \kappa''_1) - F + \gamma'_X - \gamma'_{\bar{X}})\right) \tag{97}$$

Evaluate (97) in the previous time period as:

$$\alpha_1 = m(\alpha_2 - \kappa_2) - \kappa_1 + \frac{1}{\beta}({}'\gamma_{\bar{X}} - {}'\gamma_X + F) + \frac{\Gamma_1 + m\Gamma_2}{\Gamma'_1 + m\Gamma'_2}\left(\Gamma'_1 \beta(\alpha'_2 - \kappa'_2) + \Gamma'_2(\beta(\alpha'_1 + \kappa'_1) - F + \gamma_X - \gamma_{\bar{X}})\right) \tag{98}$$

Subtract $F(\frac{1}{\beta}+L')$ from both sides and add $F(L+SL_S)$ to both sides to obtain

$$\pi - rF - FL(S',D') = F(L(S,D)+SL_S(S,D)-L(S',D')) + m(\alpha_2-\kappa_2) - \kappa_1 + \frac{1}{\beta}('\gamma_{\bar{X}} -' \gamma_X) + \frac{\Gamma_1+m\Gamma_2}{\Gamma_1'+m\Gamma_2'} \left(\Gamma_1'\beta(\alpha_2'-\kappa_2') + \Gamma_2'(\beta(\alpha_1'+\kappa_1') - F + \gamma_X - \gamma_{\bar{X}})\right) \tag{99}$$

$$\implies \xi(S',D') = L_S(S,D)SF + \big(L(S,D)-L(S',D')\big)F + \frac{\Gamma_1+m\Gamma_2}{\Gamma_1'+m\Gamma_2'}\Gamma_2'(\beta\alpha_1'-F) + \beta\frac{\Gamma_1+m\Gamma_2}{\Gamma_1'+m\Gamma_2'}\Gamma_1'\alpha_2' + m\alpha_2 + \frac{\Gamma_1+m\Gamma_2}{\Gamma_1'+m\Gamma_2'}\left(\Gamma_2'(\beta\kappa_1'+\gamma_X-\gamma_{\bar{X}}) - \beta\Gamma_1'\kappa_2'\right) - (m\kappa_2+\kappa_1) + \frac{1}{\beta}(\gamma_{\bar{X}}-\gamma_X). \tag{100}$$

Along an interior launch path, the MEC $\xi(S',D')$ reduces to

$$\xi(S',D') = L_S(S,D)SF + \big(L(S,D)-L(S',D')\big)F + \frac{\Gamma_1+m\Gamma_2}{\Gamma_1'+m\Gamma_2'}\Gamma_2'(\beta\alpha_1'-F) + \beta\frac{\Gamma_1+m\Gamma_2}{\Gamma_1'+m\Gamma_2'}\Gamma_1'\alpha_2' + m\alpha_2, \tag{101}$$

and in an interior steady state the MEC further reduces to

$$\xi(S,D) = L_S(S,D)SF + \Gamma_2(\beta\alpha_1 - F) + \beta(\Gamma_1+m)\alpha_2. \tag{102}$$

A.2.2 Forms for the collision probability function and new fragment formation function

For numerical simulations, we model the probability that objects of type j are struck by objects of type k as

$$p_{jk}(k_t) = 1 - e^{-\alpha_{jk}k_t}, \tag{103}$$

where $\alpha_{jk} > 0$ is a physical parameter reflecting the relative mean sizes, speeds, and inclinations of the object types. The probability a satellite is destroyed is the sum of the probabilities it is struck by debris and by other satellites, adjusted for the probability it is struck by both. For satellite-satellite and satellite-debris collisions, equation 103 gives us

$$L(S,D) = p_{SS}(S) + p_{SD}(D) - p_{SS}(S)p_{SD}(D) \tag{104}$$

$$= (1-e^{-\alpha_{SS}S}) + (1-e^{-\alpha_{SD}D}) - (1-e^{-\alpha_{SS}S})(1-e^{-\alpha_{SD}D})$$

$$\implies L(S,D) = 1 - e^{-\alpha_{SS}S-\alpha_{SD}D}. \tag{105}$$

We write the new fragment formation function as

$$G(S,D) = F_{SD}p_{SD}(D) + F_{SS}p_{SS}(S) + F_{DD}p_{DD}(D), \tag{106}$$

where F_{jk} models the number of fragments produced in a collision between objects of type j and k. Letting $F_{SS} = \beta_{SS}S$, $F_{SD} = \beta_{SD}S$, and $F_{DD} = \beta_{DD}D$ where $\beta_{jk} > 0$ is a physical parameter reflecting the physical compositions and masses of the colliding objects, and using the forms in equation 103, we obtain

$$G(S,D) = \beta_S S(1 - e^{-\alpha_{SS}S})S + \beta_{SD}(1 - e^{-\alpha_{SD}D})S + \beta_{DD}(1 - e^{-\alpha_{DD}D})D. \tag{107}$$

The form in equation 105 is convenient as it allows us to solve explicitly for the open access launch rate and is easy to manipulate. Similar forms have been used in engineering studies of the orbital debris environment (Bradley and Wein, 2009; Letizia et al., 2017; Letizia, Lemmens, and Krag, 2018).

To derive equation 105, we map the setting of satellites and debris colliding in an orbital shell to a setting where balls are randomly dropped in bins, i.e. satellites and debris are mapped to balls and their positions in the orbital shell are mapped to bins. The probability of a specific satellite being struck by another object is then equivalent to the probability that a randomly-dropped ball ends up in a bin containing the specific ball we are focusing on. This is a version of the "pigeonhole principle", used in Béal, Deschamps, and Moulin (2020) to derive a similar form for satellite-satellite collisions.

Suppose we have b equally-sized bins and $n+1$ balls in total, where $b \geq n+1$. Without loss of generality, we label the ball we are interested in as i. We will first place i into an arbitrary bin, and then drop the remaining N balls into the b bins with equal probability over bins. The probability a ball is dropped into a given bin is $\frac{1}{b}$, and the probability a ball is not dropped into a given bin is then $\frac{b-1}{b} = 1 - \frac{1}{b}$. As we drop the remaining n balls, the probability that none of the balls is dropped in the same bin containing j is

$$Pr(\text{no collision with } i) = \left(1 - \frac{1}{b}\right)^n \tag{108}$$

Consequently, the probability that any of the n balls are dropped into i's bin is

$$Pr(\text{collision with } i) = 1 - \left(1 - \frac{1}{b}\right)^n. \tag{109}$$

Now suppose we are interested in the probability that members of a collection of j balls, $1 \leq j < b$, end up in a bin with one of the remaining $n+1-j$ balls. The probability that any of the remaining balls end up in a bin with any of the j balls we are interested in is then

$$Pr(\text{collision with } i) = 1 - \left(1 - \frac{j}{b}\right)^{n+1-j}. \tag{110}$$

As the number of bins and balls grow large ($\lim_{b,n\to\infty}$), we obtain

$$Pr(\text{collision with } i) = 1 - e^{-j}. \tag{111}$$

Though neither the number of objects in orbits nor the possible positions they could occupy is infinite, the negative natural exponential form is likely a reasonable approximation. If we suppose that we have two types of balls j and k of different sizes and bins the size of the smallest type of ball, we get that the probability a ball of type k is dropped into in a bin with a ball of type j as

$$Pr(k\text{–}j \text{ collision}) = 1 - \left(1 - \frac{\alpha_{jk}k}{b}\right)^{n+1-k} \tag{112}$$

$$\implies \lim_{b,n\to\infty} Pr(k\text{–}j \text{ collision}) = 1 - e^{-\alpha_{jk}k}, \tag{113}$$

which is the form in equation 103, where α_{jk} is a nonnegative parameter indexing the relative sizes of objects j and k. In the orbital context, α_{jk} reflects not only the sizes of the objects but also their relative speeds and inclinations. From here we obtain the form of L by applying standard rules of probability to satellite-satellite and satellite-debris collisions. Equation 107 follows from the form of L.

B Physical and economic extensions

B.1 Spectrum congestion or price effects

Satellite applications generally require transmissions to and from the Earth. These transmissions may be the satellite's main output or incidental to its operation. In both cases, satellite operators must secure spectrum use rights from the appropriate national authorities for their broadcast and receiving locations. How will spectrum management affect collisions and debris growth?

Spectrum congestion degrades signal quality, making the per-period output of a satellite decreasing in the number of satellites in orbit, i.e. $\pi = \pi(S), \pi'(S) < 0$. If satellites are launched only when they have appropriate spectrum rights and spectrum use is optimally managed, then firms will be forced to account for their marginal effects on spectrum congestion in their decision to launch or not. The equilibrium condition becomes

$$L(S_{t+1}, D_{t+1}) = r_s(S_{t+1}) + r_s'(S_{t+1}) - r, \tag{114}$$

where $r_s'(S_{t+1}) = \pi'(S_{t+1})/F < 0$ by assumption. The equilibrium set would no longer be a collision probability isoquant, although it would still be a manifold in the state space. Even if it isn't managed optimally, spectrum congestion will reduce the equilibrium collision probability by reducing the rate of excess return on a satellite.[21]

Open access orbit use will still be inefficient. Although spectrum congestion can reduce the chance of Kessler Syndrome, efficient spectrum management will not incorporate the marginal external cost of collisions and debris growth, $\xi(S_{t+1}, D_{t+1})$. Collision and debris growth management policies could be implemented through spectrum pricing. The rest of the analysis in this paper still goes through when spectrum congestion is considered, though some proofs become more complicated since the equilibrium set is no longer a collision probability isoquant.

Mathematically, this modification would also apply to price effects induced by additional firms entering a particular orbit. Such effects may be relevant for orbits where the dominant satellite application does not face terrestrial competition, such as satellite imaging in sun-synchronous orbits. In orbits where the price of the service provided by satellites is pinned down by terrestrial applications, such as LEO or GEO internet or television service provided to urban areas, either the baseline model where r_s is constant or the interpretation of $r_s'(S)$ as spectrum congestion is appropriate. Both spectrum and price effects may be relevant for applications which require substantial spectrum use and face little terrestrial competition, such as satellite telecom service to remote areas. We put the spectrum congestion interpretation first since all satellites require some spectrum and could interfere with each other regardless of application, whereas price effects are application-specific. Price effects, however, can be orbit-agnostic in the sense that satellite systems in different physical orbits may have price effects on each other if they compete in the same market.

B.2 The effects of limited satellite lifespans

Satellites do not produce payoffs forever until destroyed in a collision. Over 1967-2015, planned satellite lifetimes ranged from 3 months to 20 years, with longer lifetimes being more representative of larger and more expensive GEO satellites.[22] How would finite lifetimes affect

[21] If spectrum use were also under open access then the marginal congestion effect ($r_s'(S_{t+1})$) would not be in the equilibrium condition, but the equilibrium set would still not be a collision probability isoquant.

[22] These numbers are taken from the Union of Concerned Scientists' publicly available data on satellites. The data are available at https://www.ucsusa.org/nuclear-weapons/space-weapons/satellite-database.

the collision probability and debris growth problems?

Suppose satellite lifetimes are finite and exogenously distributed with mean μ^{-1}. Satellites live at least one time period in this setting, so that $\mu^{-1} > 1$. The probability that a satellite will exogenously "die" in any given period is then μ. The value of a single satellite becomes

$$Q(S_t, D_t, X_t) = \pi + \beta(1-\mu)(1-L(S_t, D_t))Q(S_{t+1}, D_{t+1}, X_{t+1}), \tag{115}$$

and the equilibrium condition becomes

$$\beta Q(S_{t+1}, D_{t+1}, X_{t+1}) = F \tag{116}$$

$$\implies L(S_{t+1}, D_{t+1}) = \frac{r_s - r - \mu}{1-\mu}, \tag{117}$$

which is lower than the equilibrium collision probability when satellites are infinitely lived ($\mu = 0$). Intuitively, the fact that the satellite will stop generating payoffs at some point reduces its expected present value, and with it the incentive to launch. All else equal, shorter lifetimes reduce the equilibrium collision probability. The rest of the analysis in this paper goes through with minor modifications. Note that equation 117 is the non-stationary discrete-time analog of the physico-economic equilibrium condition in Definition 2 of Rouillon (2020).

Satellites built for GEO tend to be longer lived than satellites launched for LEO. If shorter lifetimes tend to reduce satellite costs, then the downward shift in the collision probability isoquant from the shorter lifetimes will be balanced against the upward shift caused by higher rates of return. The net effect may be higher or lower equilibrium rates of collisions.

The distinction between exogenous and endogenous lifetimes is relevant here. The above analysis hinges on satellite lifetimes being exogenously set. This is not the case in reality. Satellite lifetimes are determined by cost minimization concerns, technological constraints, expectations of component failures, and expectations of future technological change. Incorporating all these features to realistically model the choice of satellite life along with launch decisions and their effects on orbital stock dynamics is beyond our scope in this paper, though it is an interesting area for future research. The assumption that the end-of-life is random simplifies the analysis, but does not change any conclusions over imposing a pre-specified end date in this model.[23]

[23] This simplification could matter in a model where firms own multiple satellites and have to plan replacements.

B.3 The effects of changes in satellite payoffs and launch costs over time

Though it simplifies the analysis, the rate of return on a satellite is not constant over time. How would changes over time in these economic parameters affect our conclusions? For simplicity, suppose that costs, payoffs, and discount rate vary exogenously and are known in advance.[24] The open access equilibrium condition is then

$$\pi_{t+1} = (1+r_t)F_t - (1 - L(S_{t+1}, D_{t+1}))F_{t+1} \tag{118}$$

Equation 118 can be rewritten as

$$L(S_{t+1}, D_{t+1}) = 1 + \frac{\pi_{t+1} - (1+r_t)F_t}{F_{t+1}} \tag{119}$$

$$\implies L(S_{t+1}, D_{t+1}) = 1 + r_{s,t+1} - (1+r_t)\frac{F_t}{F_{t+1}} \tag{120}$$

$$\implies L(S_{t+1}, D_{t+1}) = \underbrace{\left(r_{s,t+1} - r\frac{F_t}{F_{t+1}}\right)}_{\substack{\text{excess return from} \\ \text{satellite ownership}}} + \underbrace{\left(1 - \frac{F_t}{F_{t+1}}\right)}_{\substack{\text{capital gains} \\ \text{from changes in} \\ \text{satellite launch cost}}} . \tag{121}$$

If $\pi_{t+1} > (1+r)F_t$, then the one period return on a satellite is greater than the gross return on the launch cost from the safe asset, and the collision probability will be 1. Ignoring that corner case, the equilibrium collision probability is decreasing in the current cost of launching a satellite, but increasing in the future cost of launching a satellite. All else equal, the collision probability in $t+1$ will be lower when F_{t+1} increases. This highlights the role of the launch cost under open access: if firms enter until zero profits in each period, future increases in the cost deter firms from entering in the future, increasing the value of satellites already in orbit. Alternately, the equilibrium collision risk can be decomposed into two components: one representing the excess returns of satellite ownership, and the other representing the capital gains from changes in the cost of launching a satellite under open access.

When the costs and payoffs are time-varying, the equilibrium set is still a collision probability isoquant, though the isoquant selected may vary over time. These changes do not affect the physical dynamics or the Kessler threshold, though they may affect how close the selected equilibrium is to the threshold. If the parameters vary so that the ratio $\frac{\pi_{t+1}-(1+r)F_t}{F_{t+1}}$ is stationary, then the equilibrium set will stay on the same isoquant.

[24] Uncertainty over costs, payoffs, and discount rates doesn't change the qualitative results, though it introduces expectations over the changes. Endogeneity in the changes, for example due to investment in R&D or marketing, may have more significant consequences which are beyond the scope of this paper.

B.4 Open access equilibrium between multiple shells

We have so far focused on a single orbital shell in isolation. While we can expand the "shell of interest" to cover a large region, in practice this is unlikely to yield accurate results over heterogeneous regions and use cases. Consider two orbital shells, H and L, with access costs $F^H > F^L$ and equal payoffs π. What would open access imply for the collision risks between the shells? Letting H superscripts index the higher orbit and L superscripts index the lower one, the new laws of motion are

$$\begin{aligned}
S^H_{t+1} &= S^H_t(1 - L^H(S^H_t, D^H_t)) + X^H_t \\
D^H_{t+1} &= D^H_t(1 - \delta^{HE} - \delta^{HL}) + G(S^H_t, D^H_t) + \mu_L X^H_t + \delta^{LH} D^L_t \\
S^L_{t+1} &= S^L_t(1 - L^L(S^L_t, D^L_t)) + X^L_t \\
D^L_{t+1} &= D^L_t(1 - \delta^{LE} - \delta^{LH}) + G(S^L_t, D^L_t) + \mu_L X^L_t + \delta^{HL} D^H_t
\end{aligned}$$

with the debris transport matrix

$$\mathbf{D} = \begin{array}{r} \\ H \\ L \\ E \end{array} \begin{array}{c} \begin{array}{ccc} H & L & E \end{array} \\ \left\| \begin{array}{rrr} 1 - \delta^{HL} - \delta^{HE} & \delta^{HL} & \delta^{HE} \\ \delta^{LH} & 1 - \delta^{LH} - \delta^{LE} & \delta^{LE} \\ 0 & 0 & 1 \end{array} \right\| \end{array},$$

where E represents the Earth. All transport coefficients are bounded in $[0,1]$ and sum to 1 across rows, with row labels indexing the source of debris and column labels indexing the destination.

The Bellman equations for the launch decision are now

$$Q^H_t = \pi + \beta(1 - L^H_t)Q^H_{t+1} \tag{122}$$

$$Q^L_t = \pi + \beta(1 - L^L_t)Q^L_{t+1} \tag{123}$$

$$V_{it} = \max_{x_{it} \in \{0,L,H\}} \{\mathbb{1}(x_{it} = 0)\beta V_{t+1} + \mathbb{1}(x_{it} = H)[\beta Q^H_{t+1} - F^H] \tag{124}$$

$$+ \mathbb{1}(x_{it} = L)[\beta Q^L_{t+1} - F^L]\}, \tag{125}$$

where we suppress function arguments and use time subscripts for brevity (though these are still infinite-horizon Bellman equations). In an open-access equilibrium, the payoffs from owning a satellite in either shell should be zero, i.e.

$$
\begin{aligned}
(X_t^H, X_t^L) : V_{it} = 0 &\implies \beta Q_{t+1}^H - F^H = 0,\ \beta Q_{t+1}^L - F^L = 0 && (126)\\
&\implies \beta[\pi + (1 - L_{t+1}^H)F^H] - F^H = 0,\ \beta[\pi + (1 - L_{t+1}^L)F^L] - F^L = 0 && (127)\\
&\implies \pi = \left(\frac{1-\beta}{\beta}\right)F^H + L_{t+1}^H F^H,\ \pi = \left(\frac{1-\beta}{\beta}\right)F^L + L_{t+1}^L F^L && (128)\\
&\implies rF^H + L_{t+1}^H F^H = rF^L + L_{t+1}^L F^L && (129)\\
&\implies \frac{F^L}{F^H} = \frac{r + L_{t+1}^H}{r + L_{t+1}^L} && (130)
\end{aligned}
$$

In this case the cheaper orbit will have higher equilibrium collision risk, e.g. if the lower orbit is cheaper to access then the higher orbit will have lower equilibrium collision risk:

$$F^L < F^H \implies L_{t+1}^H < L_{t+1}^L. \tag{131}$$

If the payoffs earned by satellites in either shell are not identical, but rather π^H and π^L with $\pi^H = \pi^L + \gamma$, then we have

$$\frac{F^L}{F^H} = \frac{r + L_{t+1}^H - \frac{\gamma}{F^H}}{r + L_{t+1}^L} \tag{132}$$

If the lower orbit is cheaper to access, $F^L < F^H \implies L_{t+1}^H - \frac{\gamma}{F^H} < L_{t+1}^L$. This gives us two possibilities:

- If the higher orbit generates greater payoffs, either orbit may be riskier in equilibrium depending on how big the return premium to the higher orbit is: $\gamma > 0 \implies L_{t+1}^H \lesseqgtr L_{t+1}^L$.
- If the lower orbit generates greater payoffs, then the higher orbit will have lower equilibrium collision risk: $\gamma < 0 \implies L_{t+1}^H < L_{t+1}^L$.

The physical and economic dynamics in the multi-shell setting are likely richer than those of the single-shell setting.