ANSWER KEY

Econ 1078-002 Midterm Exam 1

February 21, 2018

- Do not open the exam booklet until instructed to do so. Close your exam booklet promptly when I say time is up.
- Read the questions closely, and answer them completely. Answer the questions you find easiest first!!!!
- Only answer one of the word problems - choose whichever you like.
- The exam is for 36 points - pay attention to the point values.
- Show your work to get partial credit (i.e., convince me you understand what's asked of you and what you're doing).
- Please write your answers clearly on the exam booklet, and draw a box around your final answer.
- Take as much scratch paper as you need. If you want me to review your scratch paper, write your name on it and include it in your completed exam.
- Please raise your hand if a question is unclear.
- Deep breaths. It's going to be ok. Do your best, and have fun!

Name: ______________________________

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: ______________________________

Date: ______________________________

Assume all values are in $\mathbb{R}$. When asked to solve an inequality, provide an inequality on the variable(s) which makes the statement true. When there are multiple real solutions, provide all of them unless explicitly directed otherwise. If there is no real solution, state this with a brief explanation why (1 sentence or less). You are not required to provide complex solutions.

1 Multiple choice questions (8 points)

1. (2 points) Consider two sets, $A \subset \Omega$ and $B \subset \Omega$, where Ω is the universal set. If neither A nor B is empty, which of the following statements **must** be true?
   (a) $A \cap B = \emptyset$
   (b) $A \setminus B = B \setminus A$
   (c) $(A \cap B) \subseteq A$
   (d) All of the above
   (e) None of the above ✓

2. (2 points) Which of the following statements says **only** "Breathing is a sufficient condition for living"?
   (a) breathing + living = necessity
   (b) breathing $\implies$ living ✓
   (c) living $\implies$ breathing
   (d) All of the above
   (e) None of the above

3. (2 points) How many solutions can a system of two linear equations with two unknowns have?
   1. 0
   2. 1
   3. Infinitely many
   4. All of the above ✓
   5. None of the above

4. (2 points) Which of the following sets is the set of solutions to the equation $\frac{x^2-16}{x(x-4)} = 0$? (The "set of solutions to an equation" is a set in which every element is a solution to the given equation.)
   (a) $\{-4, 0, +4\}$
   (b) $\{-4, +4\}$
   (c) $\{-4\}$ ✓
   (d) All of the above
   (e) None of the above

2 Mechanical questions (23 points)

1. (2 points) Solve for x:

$$10x^2 + 9x = 0.$$

$\Rightarrow x(10x+9) = 0$

$\Rightarrow \boxed{x=0}$ or $\boxed{x = -\frac{9}{10}}$

1 1

$10x + 9 = 0$
$\Rightarrow x = -\frac{9}{10}$

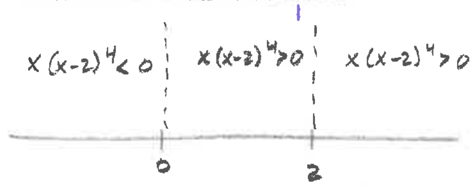
2. (3 points) Find all values of x for which the inequality is true:

$$x(x-2)^4 > 0.$$

$x(x-2)^4 > 0$

$\Rightarrow \boxed{x > 0 \text{ AND } x \neq 2}$

2 1



- $(x-2)^4$ is always nonnegative because even power.
- $(x-2)^4 = 0$ if $x = 2$; we don't want that.
- $a \cdot b > 0$ if both a & b have the same sign.

3. (4 points) Solve for (x, y):

$$5x + by = 4 \quad \text{— (1)}$$
$$dx + 4y = -7 \quad \text{— (2)}$$

What restriction(s) do you need on the parameters b, d to ensure that a unique solution exists?

Solving

From (1),
$x = \frac{4-by}{5}$

Plugging into (2),

$d\left(\frac{4-by}{5}\right) + 4y = -7$

$\Rightarrow 4d - bdy + 20y = -35$

$\Rightarrow y(20 - bd) = -35 - 4d$

$\Rightarrow \boxed{y^* = \frac{-35-4d}{20-bd}}$ — 1.5

Plugging back into our eqn. for x,

$\boxed{x^* = \frac{4 - b\left(\frac{-35-4d}{20-bd}\right)}{5}}$ — 1.5

Existence:

To make sure the solutions are not undefined, we check what we need to avoid illegal operations.

Here, if $bd = 20$, we will be dividing by 0. To avoid this,

$\boxed{bd \neq 20}$ — 1

4. (4 points) Label the set of possible solutions to the equation $(x-c)(x+5)=0$ as A, and label the set of solutions to the equation $(x-5)(x-c)(x-1)=0$ as B. (c is a parameter, x is what you are solving for.)

   (a) What is the set $A \cup B$?

   (b) What is the set $A \setminus B$?

$A = \{c, -5\}$

$B = \{5, c, 1\}$

(a) $\boxed{A \cup B = \{-5, 5, c, 1\}}$ -2

(b) $\boxed{A \setminus B = \{-5\}}$ -2

5. (2 points) Given that $Y = K^{\alpha}L^{1-\alpha}$, verify that $\frac{Y}{K} = \left(\frac{L}{K}\right)^{1-\alpha}$

$$\frac{Y}{K} = \frac{K^{\alpha}L^{1-\alpha}}{K}$$

$$= K^{\alpha}L^{1-\alpha}K^{-1}$$

$$= L^{1-\alpha}K^{\alpha-1}$$ -1 for reaching here

$$= \frac{L^{1-\alpha}}{K^{-(\alpha-1)}}$$

$$= \frac{L^{1-\alpha}}{K^{1-\alpha}}$$

$$\Rightarrow \boxed{\frac{Y}{K} = \left(\frac{L}{K}\right)^{1-\alpha}}$$

$a \cdot b = 0$
$\Rightarrow a = 0, b = 0,$
or both

6. (4 points) Solve for x:

$$\overbrace{rx}^{a}\overbrace{\left(1 - \frac{x}{K}\right)}^{b} = 0.$$

Derive a necessary condition for all solutions to be nonnegative.

Solving

$a = 0$: $rx = 0 \Rightarrow \boxed{x_1^* = 0}$ −1.5

$b = 0$: $\left(1 - \frac{x}{K}\right) = 0$

$\Rightarrow \left(\frac{K}{K} - \frac{x}{K}\right) = 0$

$\Rightarrow \boxed{x_2^* = K}$ −1.5

Necessary condition

We want our solutions to be nonnegative, that is, $x^* \geq 0$ for both solutions.

- $0 \geq 0$ is already true, so we're fine there
- $K \geq 0$ is something we don't know from the problem, so we need to assume it.

$\boxed{x_1^*, x_2^* \geq 0 \Rightarrow K \geq 0}$ −1

7. (4 points) Solve for x:

$$|5x - 4| = a.$$

Provide a restriction on a which will make this equation have only 1 solution.

Solving

positive

$5x - 4 = a$

$\Rightarrow 5x = 4 + a$

$\Rightarrow \boxed{x_1^* = \frac{4+a}{5}}$

1.5

negative

$5x - 4 = -a$

$\Rightarrow 5x = -a + 4$

$\Rightarrow \boxed{x_2^* = \frac{4-a}{5}}$

1.5

Restriction

To get only one solution by restricting a, we should ask ourselves, "for what value of a is $\frac{4+a}{5} = \frac{4-a}{5}$?"

We can solve this equation for a, giving $a = 0$. "If $a = 0$, $\frac{4+a}{5} = \frac{4-a}{5}$".

$\boxed{a = 0}$ −1

→ Geometrically, this says that there is only 1 way that $5x - 4$ can be 0 units away from 0.

3 Word problems (5 points)

Answer **one** of the following word problems. You may choose either.

1. (4 points) Your cousin is considering investing in one of two accounts. The first will take his money and provide 5% interest per year. The second will take his money, offer 8% interest on the total amount per year, but charge a management fee of 10% of the initial amount in the first year. Both accounts compound the interest each year. Your cousin is starting with $x > 0$ dollars and is going to invest for 10 years.

   (a) (2 points) Write formulas for the amounts he'll receive at the end of 10 years from each account.

$$\boxed{A = x\left(1+\frac{5}{100}\right)^{10}} \quad -1$$

$$B = x\underbrace{\left(1-\frac{10}{100}\right)}_{\text{mgmt. fee}}\left(1+\frac{8}{100}\right)^{10} \Rightarrow \boxed{B = 0.9x\left(1+\frac{8}{100}\right)^{10}} \quad -1$$

   (b) (1 point) Using the formulas you derived, which account would you advise your cousin to invest in to maximize his return?

We ask, "Is $A > B$?"

$$A > B \Rightarrow x\left(1+\frac{5}{100}\right)^{10} > 0.9x\left(1+\frac{8}{100}\right)^{10}$$

$$\Rightarrow x(1.05)^{10} > 0.9x(1.08)^{10}$$

$$\Rightarrow \frac{\cancel{x}}{0.9\cancel{x}} > \frac{(1.08)^{10}}{(1.05)^{10}}$$

$$\Rightarrow \frac{1}{0.9} > \frac{2.16}{1.63}$$

$$\Rightarrow 1.\overline{11} > 1.33 \quad \Rightarrow\!\Leftarrow$$

Since $A > B$ implies a false statement, it is false.

So $B > A$,

$\boxed{\text{Invest in account } B}$ -1

   (c) (2 points) Write a proof showing that your recommendation does not depend on the amount your cousin starts with, x.

We claim that $B > A$ $\overset{\text{"for all"}}{\forall}$ $x > 0$.

<u>Proof:</u> Let $x > 0$. $B > A \Rightarrow 0.9x(1.08)^{10} > x(1.05)^{10}$

$\Rightarrow 1.33 > 1.\overline{11}$, (see algebra in part (b))

Since $B > A$ implies a statement that is true regardless of the specific x chosen (as long as $x > 0$), the recommendation does not depend on which $x > 0$ we choose.

<u>T.S.</u>

2. (4 points) The macroeconomy of Elon Musk's Mars Colony can be represented by two equations: the national income accounting equation, and the statistical consumption equation. Let C denote consumption, Y denote output, I denote investment, G denote government spending, and NX denote net export flows with Earth. Letting a and b be two parameters, the statistical consumption equation is

$$C = a + bY.$$

The national income accounting equation states that output is the sum of consumption, investment, government spending, and net export flows.

(a) (1 point) Write a formula for the national income accounting equation based on the problem statement

$$\boxed{Y = C + I + G + NX}$$

−1

(b) (2 points) Suppose that $a = 5$, $b = 0.9$, $I = 20$, $NX = 2$, and $G = 10$. President Musk thinks that raising NX by 4 will increase Y by 1 or less. Is he right?

Solving for Y^*

$Y = C + I + G + NX$

$C = a + bY$

$\Rightarrow Y = a + bY + I + G + NX$

$\Rightarrow Y(1-b) = a + I + G + NX$

$\Rightarrow Y^* = \dfrac{a+I+G+NX}{1-b}$.

$Y_{new} = \dfrac{a+I+G+(NX+4)}{1-b}$.

$$Y_{new} - Y^* = \frac{a+I+G+NX+4}{1-b} - \frac{a+I+G+NX}{1-b}$$

$$= \frac{\cancel{a}+\cancel{I}+\cancel{G}+\cancel{NX}+4-\cancel{a}-\cancel{I}-\cancel{G}-\cancel{NX}}{1-b}$$

$$= \frac{4}{1-b}$$ ∴ Plugging in $b = 0.9$,

$$Y_{new} - Y^* = \frac{4}{1-0.9} = \frac{4}{0.1} = 40.$$

$\boxed{\text{No, Musk is wrong that } Y_{new} - Y^* \leq 1.}$

−2

(c) (2 points) Given $b = 0.9$, derive a necessary condition on the change in NX so that it increases Y by 1 or less.

smiley face

Let the change in NX be ☺.

$$Y_{new} = \frac{a+I+G+NX+☺}{1-b}$$

From the algebra above,

$$Y_{new} - Y^* = \frac{☺}{1-b}$$

If $b = 0.9$,

$$Y_{new} - Y^* = \frac{☺}{0.1}$$

We want to know,

"for what value of ☺ is $Y_{new} - Y^* \leq 1$?"

$\Rightarrow Y_{new} - Y^* \leq 1 \Rightarrow \dfrac{☺}{0.1} \leq 1$

$\Rightarrow \boxed{☺ \leq 0.1}$

−2

"The change in NX must be less than or equal to 0.1 for $Y_{new} - Y^* \leq 1$ when $b = 0.9$."

4 Formulas that may be useful

- Compounded growth: $K = P\left(1 + \frac{r}{100}\right)^t$
- Quadratic identity 1: $(a+b)^2 = a^2 + 2ab + b^2$
- Quadratic identity 2: $(a-b)^2 = a^2 - 2ab + b^2$
- Quadratic identity 3: $a^2 - b^2 = (a+b)(a-b)$