Econ 1078-002 Final Exam
May 6th, 2018

ANSWER KEY

- Do not open the exam booklet until instructed to do so. Close your exam booklet promptly when I say time is up.
- Read the questions closely, and answer them completely. **Answer the questions you find easiest first!!!!**
- The exam is for 130 points. **There are 10 points of extra credit questions at the back**, so a score of 140/130 is possible.
- Show your work to get partial credit (i.e., convince me you understand what's asked of you and what you're doing).
- Please write your answers clearly on the exam booklet, and draw a box around your final answer. There are useful formulas on the last page.
- Please raise your hand if a question is unclear or you need scratch paper. Take as much scratch paper as you need.
- Assume all values are in $\mathbb{R}$. When there are multiple real solutions, provide all of them unless explicitly directed otherwise. If there is no real solution, state this with a brief explanation of why (1 sentence or less). You are not required to provide complex solutions.
- Deep breaths. It's going to be ok. Do your best, and have fun!

Name: ______________________________

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: ______________________________

Date: ______________________________

1 Mechanical questions (114 points)

1. (13 points) Let $L(S, D) = e^{-aS-bD}$, and assume $a, b \neq 0$. Show that $L(\theta S, \theta D) = L(S, D)^{\theta}$.

$$L(\theta S, \theta D) = e^{-a\theta S - b\theta D} = e^{(-aS-bD)\theta} = \left(e^{-aS-bD}\right)^{\theta} = L(S,D)^{\theta}.$$

T.S.

2. (13 points)

$$f(x) = \ln(x-2).$$

Find the domain, range, and inverse of $f(x)$, and label the inverse as $g(y)$. Compute $f(g(x))$ for all integers in $[3, 10]$, and plot $f(g(x))$ over the real interval $[3, 10]$.

Domain : $(2, \infty)$

Range : $(-\infty, \infty)$

Inverse : $y = \ln(x-2)$

$\Rightarrow \exp(y) = x - 2$

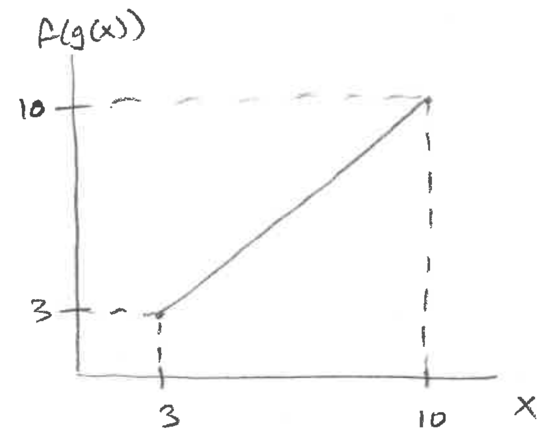
$\Rightarrow x = \exp(y) + 2$

$\boxed{g(y) = \exp(y) + 2}$

Since $f(\cdot)$ & $g(\cdot)$ are inverses of each other, $f(g(x)) = x \ \forall x$ where $f(\cdot)$ & $g(\cdot)$ are defined.

Since $[3, 10]$ is in the domain & range of $f(\cdot)$,

$\boxed{f(g(x)) = x \ \forall\, x \in [3, 10]}$



3. (13 points) Let $R(x) = px$, $C(x) = mx^2$, and $\beta \in (0, 1)$. We define the function $\pi(x)$,

$$\pi(x) = \sum_{t=0}^{\infty} \beta^t [R(x) - C(x)].$$

(a) Use a summation formula to write $\pi(x)$ without a $\sum$ symbol.

(b) Let $p = 2$, $m = 0.1$, and $\beta = 0.99$. Calculate the value of x which maximizes $\pi(x)$. (Your final answer should be a number.)

(a) $$\pi(x) = \boxed{\left(\frac{1}{1-\beta}\right)\left[R(x) - C(x)\right]} = \frac{R(x) - C(x)}{1-\beta}$$

(b) (Protip: Plug numbers in at the end.)

$$\pi(x) = \frac{px - mx^2}{1-\beta} = \underbrace{\left(\frac{p}{1-\beta}\right)}_{b} x - \underbrace{\left(\frac{m}{1-\beta}\right)}_{a} x^2 .$$ Using quadratic optimization rule,

$$x^* = \cancel{-}\frac{\left(\frac{p}{\cancel{1-\beta}}\right)}{2\left(\frac{m}{\cancel{1-\beta}}\right)} \Rightarrow x^* = \frac{p}{2m} = \frac{2}{2(0.1)} \Rightarrow \boxed{x^* = 10}$$

4. (13 points) Let

$$\sum_{i=1}^{n} r_i^2 = \sum_{i=1}^{n} (Y_i - X_i\beta)^2.$$

Find the value of β which minimizes $\sum_{i=1}^{n} r_i^2$.

$$\sum_{i=1}^{n} r_i^2 = \sum_{i=1}^{n} (Y_i - X_i\beta)(Y_i - X_i\beta)$$

$$= \sum_{i=1}^{n} (Y_i^2 - 2X_iY_i\beta + X_i^2\beta^2)$$

$$= \sum_{i=1}^{n} Y_i^2 - \sum_{i=1}^{n} 2X_iY_i\beta + \sum_{i=1}^{n} X_i^2\beta^2$$

$$= \underbrace{\sum_{i=1}^{n} Y_i^2}_{c} \underbrace{- 2\left(\sum_{i=1}^{n} X_iY_i\right)\beta}_{b} + \underbrace{\left(\sum_{i=1}^{n} X_i^2\right)}_{a}\beta^2$$

Using quadratic optimization rule,

$$\beta^* = \cancel{-}\frac{(\cancel{-}\cancel{2}\sum_{i=1}^{n} X_iY_i)}{\cancel{2}\sum_{i=1}^{n} X_i^2} \Rightarrow \boxed{\beta^* = \frac{\sum_{i=1}^{n} X_iY_i}{\sum_{i=1}^{n} X_i^2}}$$

5. (10 points) Use the equation below to plot $M(q) = L(q+1) - L(q)$ with q on the horizontal axis:

$$pq = rq + L(q) + qM(q)$$

Assume that $p > r$, ~~$L(q)$ is always nonnegative~~, that $L(q)/q$ is decreasing as q increases, and that $L(0)/0 \equiv 0$. (Hint: if $f(x)$ is decreasing, $-f(x)$ is increasing.)

Manipulate the original equation

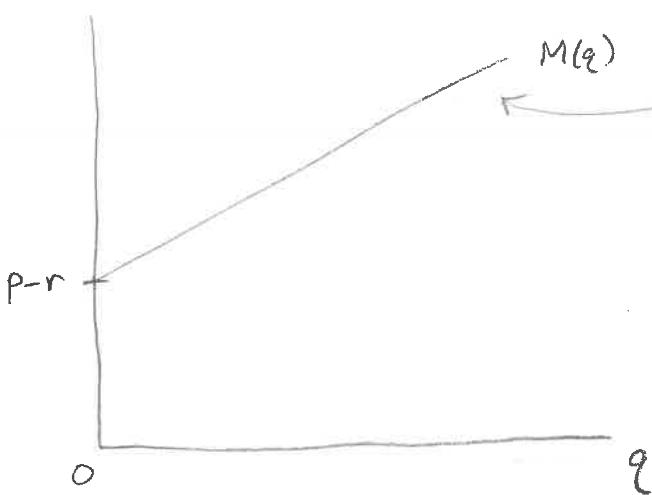
$$pq = rq + L(q) + qM(q)$$
$$\Rightarrow pq - rq - L(q) = qM(q)$$
$$\Rightarrow p - r - \frac{L(q)}{q} = M(q)$$

Examine the components

- $p-r$ is constant in q
- $\frac{L(q)}{q}$ is decreasing in q $\Longleftrightarrow$ $-\frac{L(q)}{q}$ is increasing in q
- So $M(q) = p - r - \frac{L(q)}{q} = \underset{(p-r)}{(\text{constant})} + \underset{(L(q)/q)}{(\text{increasing fn. of } q)}$
- If $q=0$, $M(0) = p - r - \frac{L(0)}{0} = p - r$ since we're given that $\frac{L(0)}{0} \equiv 0$.

How is this possible?

✓ $\frac{x}{0}$ is undefined in general, & written as ∞ or $-\infty$ when $x \neq 0$. But if $x=0$, we have $\frac{0}{0}$. This is also undefined, but not in the same way (it isn't growing without bound). (In Math Tools 2, you'll learn to use limits & derivatives to examine this case) $\frac{L(0)}{0} \equiv 0$ suggests that $L(q) \to 0$ as $q \to 0$ faster than a linear function of q would.



There may be some curvature here, but we don't have enough information to know. All we know is that $M(q)$ is increasing in q, & starts at $p-r$.

6. (13 points) The function $P(x_1, \ldots, x_n)$ is defined as

$$P(x_1, \ldots, x_n) = \frac{1}{1 + \sum_{i=1}^{n} a_i \exp(-x_i)}.$$

Suppose $x_1 = x_2 = \cdots = x_k = 1$, $x_{k+1} = x_{k+2} = \cdots = x_n = 0$, and $a_1 = a_2 = \cdots = a_n = 1$. Calculate $P(x_1, x_2, \ldots, x_n)$ when $n = 538$ and $k = 38$. (Your final answer should be a number or an expression containing only numbers and operations.)

Let's focus on the sum, $\sum_{i=1}^{n} a_i \exp(-x_i)$. We are given that $X_i = 1 \ \forall\, i \in [1, k]$, $X_i = 0 \ \forall\, i \in [k+1, n]$, $a_i = 1 \ \forall\, i \in [1, n]$. Rewriting the sum to reflect this,

$$\sum_{i=1}^{n} a_i \exp(-x_i) = \sum_{i=1}^{k} a_i \exp(-x_i) + \sum_{i=k+1}^{n} a_i \exp(-x_i)$$

$$= \sum_{i=1}^{k} (1) \exp(-1) + \sum_{i=k+1}^{n} (1) \exp(0)$$

$$= \sum_{i=1}^{k} e^{-1} + \sum_{i=k+1}^{n} 1$$. Applying sum-over-constant rules,

$$= k e^{-1} + (n + \cancel{1} - k - \cancel{1}) \cdot 1$$

$$= k e^{-1} + (n - k)$$. Plugging back into $P(x_1, \ldots, x_n)$,

$$P(x_1, \ldots, x_n) = \frac{1}{1 + [k e^{-1} + (n-k)]}$$. Using $n = 538$ & $k = 38$,

$$\boxed{P(x_1, \ldots, x_n) = \frac{1}{1 + 38e^{-1} + 500}} = \frac{1}{501 + 38e^{-1}} \approx 0.0019$$

7. (13 points) Prove that

$$P(n): \sum_{i=1}^{n} \frac{1}{i \cdot (i+1)} = \frac{n}{n+1}.$$

(We've used induction to do this, but you may use any method you like to prove the statement.)

Base case $(n=1)$ $\quad \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1+1} \Rightarrow \frac{1}{1(1+1)} = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{2}$ ✓

Induction step $\quad n+1^{th}$ term on LHS: $\frac{1}{(n+1)(n+2)}$. Assuming $P(n)$ is true, we want to add $\frac{1}{(n+1)(n+2)}$ to both sides of $P(n)$ & get to $P(n+1)$. $P(n+1)$ says " $\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n+1}{n+2}$ ".

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$\Rightarrow \sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{\cancel{(n+1)}(n+1)}{\cancel{(n+1)}(n+2)} = \frac{n+1}{n+2}.$$

Since adding the $n+1^{th}$ term to $P(n)$ gave us $P(n+1)$, $P(n) \Rightarrow P(n+1)$. Since we verified that $P(1)$ is true, $P(n)$ is true $\forall$ integer $n \geq 0$.

T. S.

8. (13 points) Plot the following system of equations:

$$(xy)^{1/2} = 4 \tag{1}$$

$$x + y \leq 8 \tag{2}$$

over the first quadrant (where $x > 0, y > 0$), with x on the horizontal axis and y on the vertical axis. Label all axis intercepts and points of intersection, and shade the region where inequality 2 holds. (Hint: after plotting the equality case of 2, look for where 1 and 2 intersect to get started on drawing 2. Substitute the equality case of 2 into 1 to find one coordinate of this intersection. Alternatively, you can plot both 1 and 2 by evaluating some points where they are true.)

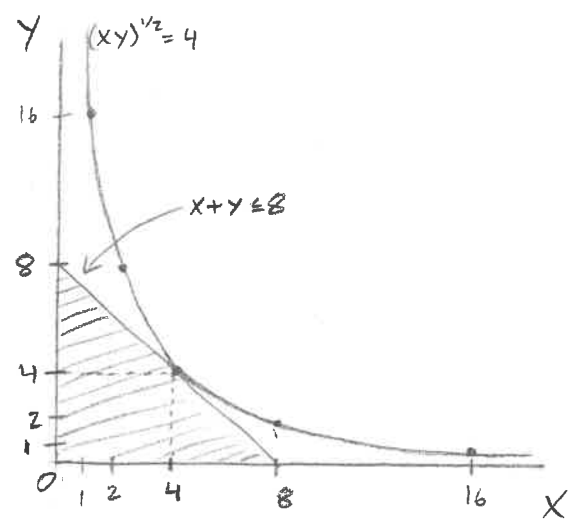
Plotting (2) : $X + Y \leq 8 \Rightarrow Y \leq 8 - X$

$X = 0 \Rightarrow Y \leq 8$,

$Y = 0 \Rightarrow X \leq 8$.

Plotting (1) : $(XY)^{1/2} = 4$

$\Rightarrow XY = 16 \Rightarrow Y = \frac{16}{X}$



Looking for the intersection : $Y = \frac{16}{X}$, $Y = 8 - X$

$\Rightarrow \frac{16}{X} = 8 - X \Rightarrow 16 = (8-X)X \Rightarrow -X^2 + 8X - 16 = 0$

$X^* = \frac{-8 \pm (8^2 - 4(-1)(-16))^{1/2}}{2(-1)} = \frac{-8 \pm (0)^{1/2}}{-2} = 4$

Evaluating points where (1) is true:

(1) $\Leftrightarrow Y = \frac{16}{X}$.

Now we can pick some values for x & calculate what y must be. We only need enough points to get the shape of the curve, since we know the location from the intersection with (1).

X	Y
1	16
2	8
4	4
6	$\frac{16}{6} \approx 2.67$
8	2
16	1

9. (13 points) Given that $u(c) = -\frac{e^{-ac}}{a}$, where $a, c > 0$, solve for x such that

$$u(c) = xu(1/a).$$

Plugging in the definition of $u(\cdot)$,

$$u(c) = x\, u\left(\tfrac{1}{a}\right)$$

$$\Rightarrow \cancel{-}\frac{e^{-ac}}{\cancel{a}} = x\left(\cancel{-}\frac{e^{-\cancel{a}(\frac{1}{\cancel{a}})}}{\cancel{a}}\right)$$

$$\Rightarrow e^{-ac} = x e^{-1}$$ (Multiply by e^1 on both sides to isolate x)

$$\Rightarrow \boxed{e^{-ac} \cdot e^{1} = x}$$ (This is sufficient)

$$\Rightarrow \boxed{x = e^{1-ac}}$$ (Simplifying further)

2 Word problem (16 points)

10. (12 points) Two firms, X and Y, are competing in the electricity market (a duopoly situation). x is firm X's electricity output level, y is firm Y's electricity output level. The inverse demand function (the price) is

$$P(x+y) = A - B(x+y),$$

where A and B are positive constants and $A > \alpha > 0$ and $A > B > 0$. The ways X and Y react to each other's electricity output in order to maximize profits are described by their *reaction curves*, $x(y)$ for X reacting to Y and $y(x)$ for Y reacting to X.

(a) (3 points) Suppose that

$$x(y) = \frac{1}{2B}(A - By - \alpha)$$
$$y(x) = \frac{1}{2B}(A - Bx - \alpha).$$

Plot the reaction curves with x on the horizontal axis and y on the vertical axis. Label the axes and all axis intercepts.

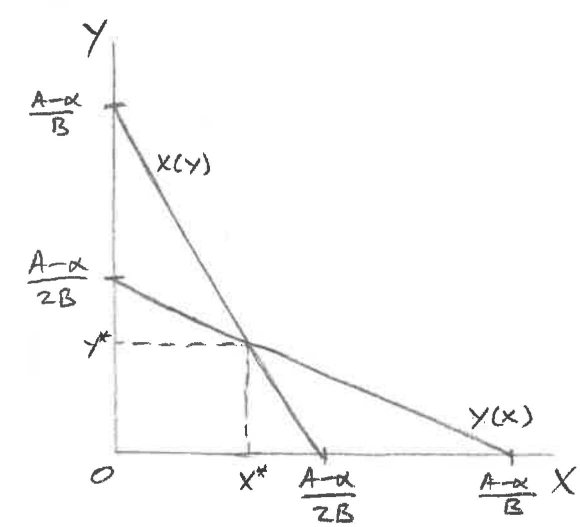
Shape & location of $y(x)$:

Shape : y is linear in x.

location: If $x=0$, $y = \frac{A-\alpha}{2B}$.

If $y=0$, $0 = \frac{1}{2B}(A-Bx-\alpha)$

$\Rightarrow A-Bx-\alpha=0 \Rightarrow x = \frac{A-\alpha}{B}$. (Note: $\frac{A-\alpha}{2B} < \frac{A-\alpha}{B}$)



Shape & location of $x(y)$:

Shape: x is linear in y.

location: If $y=0$, $x = \frac{A-\alpha}{2B}$.

If $x=0$, $y = \frac{A-\alpha}{2B}$.

(Note: Since $x(y)$ & $y(x)$ are symmetric, we can just use the analysis from $y(x)$.)

(b) (3 points) Solve the system of reaction curves from part (a) for their intersection. Label your solutions as x^* and y^*.

$$\left.\begin{aligned} y &= \frac{1}{2B}(A - Bx - \alpha) \\ x &= \frac{1}{2B}(A - By - \alpha) \end{aligned}\right\}$$ This is a linear system w/ 2 equations & 2 unknowns (x & y).

Using substitution,

$$y = \frac{1}{2B}\left(A - B\left(\frac{1}{2B}(A - By - \alpha)\right) - \alpha\right)$$

$$\Rightarrow y = \frac{A}{2B} - \frac{\cancel{B}}{2\cancel{B}}\left(\frac{A}{2B} - \frac{\cancel{B}}{2\cancel{B}}y - \frac{\alpha}{2B}\right) - \frac{\alpha}{2B}$$

$$\Rightarrow y = \frac{A}{2B} - \frac{A}{4B} + \frac{1}{4}y + \frac{\alpha}{4B} - \frac{\alpha}{2B}$$

$$\Rightarrow y - \frac{1}{4}y = \left(\frac{2A}{4B} - \frac{A}{4B}\right) + \left(\frac{\alpha}{4B} - \frac{2\alpha}{4B}\right)$$

$$\Rightarrow \frac{3}{4}y = \frac{A}{4B} - \frac{\alpha}{4B}$$

$$\Rightarrow y = \frac{\cancel{4}}{3}\cdot\frac{A}{\cancel{4}B} - \frac{\cancel{4}}{3}\cdot\frac{\alpha}{\cancel{4}B} \Rightarrow \boxed{y^* = \frac{A-\alpha}{3B}}$$

Plugging into $x(y)$,

$$x = \frac{1}{2B}\left(A - B\left(\frac{A-\alpha}{3B}\right) - \alpha\right)$$

$$= \frac{A}{2B} - \frac{1}{2}\left(\frac{A-\alpha}{3B}\right) - \frac{\alpha}{2B}$$

$$= \frac{A-\alpha}{2B} - \frac{A-\alpha}{6B}$$

$$= \frac{3(A-\alpha) - (A-\alpha)}{6B}$$

$$= \frac{\cancel{2}(A-\alpha)}{3\cancel{6}B} \Rightarrow \boxed{x^* = \frac{A-\alpha}{3B}}$$

Note: Since $x(y)$ & $y(x)$ are symmetric, we could have also just said "$y^* = x^* \Rightarrow x^* = \frac{A-\alpha}{3B}$" once we got y^*.

(c) (3 points) Suppose firms X and Y merge to form firm M. Find the level of electricity output m that maximizes firm M's profits,

$$\pi(m) = P(m)m - \alpha m.$$

Label your solution as m^*.

Using the definition $P(x+y) = A - B(x+y)$, $P(m) = A - Bm$.

So $$\pi(m) = (A - Bm)m - \alpha m$$
$$= Am - Bm^2 - \alpha m$$
$$= \underbrace{-B}_{a}m^2 + \underbrace{(A-\alpha)}_{b}m$$. Applying quadratic optimization rule,

$$m^* = +\frac{(A-\alpha)}{2(+B)} \Rightarrow \boxed{m^* = \frac{A-\alpha}{2B}}$$

(d) (3 points) This question has two parts.

(i) Determine whether $x^* + y^*$ is larger than m^* or not.

(ii) Determine whether $P(x^* + y^*)$ is larger than $P(m^*)$ or not. (Hint: you can use your answer from part (i) along with a property of $P(\cdot)$ to answer this without any calculations.)

(i) $x^*+y^* = \frac{A-\alpha}{3B} + \frac{A-\alpha}{3B} = \frac{2}{3}\left(\frac{A-\alpha}{B}\right)$. $m^* = \frac{1}{2}\left(\frac{A-\alpha}{B}\right)$.

Is $x^*+y^* > m^*$? Let's see:

$x^*+y^* > m^* \Rightarrow \frac{2}{3}\left(\cancel{\frac{A-\alpha}{B}}\right) > \frac{1}{2}\left(\cancel{\frac{A-\alpha}{B}}\right) \Rightarrow \frac{2}{3} > \frac{1}{2}$.

Since $\frac{2}{3} > \frac{1}{2}$ is true, $x^*+y^* > m^*$ is also true.

$\boxed{\text{Yes, } x^*+y^* > m^*.}$

(ii) With calculations: $P(x^*+y^*) = A - B(x^*+y^*) = A - \frac{2}{3}\cancel{B}\left(\frac{A-\alpha}{\cancel{B}}\right)$
$= A - \frac{2}{3}(A-\alpha)$

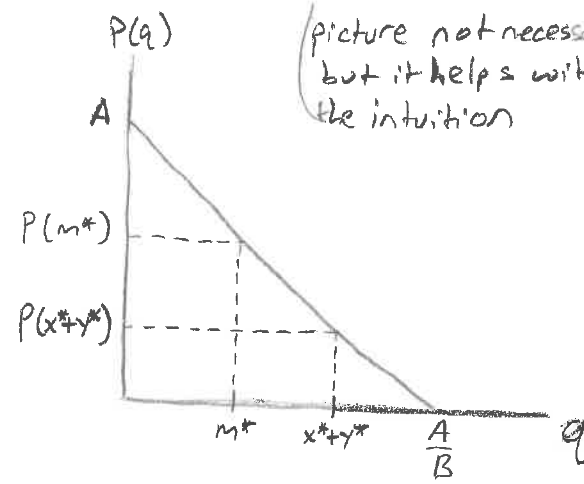
$P(m^*) = A - Bm^* = A - \cancel{B}\frac{1}{2}\left(\frac{A-\alpha}{\cancel{B}}\right) = A - \frac{1}{2}(A-\alpha)$.

$P(x^*+y^*) - P(m^*) = \cancel{A} - \frac{2}{3}(A-\alpha) - \cancel{A} + \frac{1}{2}(A-\alpha) = A-\alpha\left(\frac{1}{2}-\frac{2}{3}\right) < 0$,

so $P(x^*+y^*) < P(m^*)$ must be true.

Without calculations: $P(x) = A - Bx$ is a decreasing function of its argument. $x^*+y^* > m^*$ from (i). So, $P(x^*+y^*) < P(m^*)$ must be true.

$\boxed{\text{No, } P(x^*+y^*) < P(m^*).}$

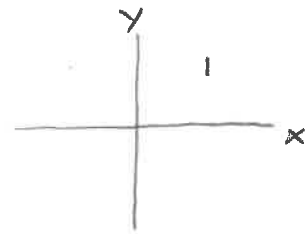


(picture not necessary but it helps with the intuition)

3 (Extra credit) Multiple choice questions (10 points)

Circle the correct answer.

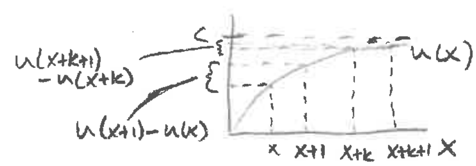
11. (2.5 points) Quadrant 1 is



   (a) where x and y are both positive
   (b) where x is positive and y is negative
   (c) where x is negative and y is positive
   (d) where x and y are both negative
   (e) None of the above

12. (2.5 points) Your teacher asserts that $u(x)$, defined over the integers, is both strictly increasing over $[0, \infty)$ and always less than some finite constant c. Which of the following statements is a sufficient condition for your teacher's statement?

   (a) This is impossible. If $u(x)$ is strictly increasing, it must eventually grow to $+\infty$ and cannot be always less than c.
   (b) $u(x)$ grows at a decreasing rate, so that $u(x+1) - u(x) > u(x+k+1) - u(x+k)$ for all $k > 1$.
   (c) This is impossible. If $u(x)$ is always less than some constant c, it must eventually stop growing and cannot be strictly increasing.
   (d) $u(x)$ grows over $[0, x)$, becomes equal to c at $\bar{x} < \infty$, and is then constant at c after.
   (e) None of the above



13. (2.5 points) Select the next term in the sequence below:

$$\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \ldots, \frac{1}{n\cdot(n+1)},$$

   (a) $\frac{1}{n\cdot(n+2)}$
   (b) $n+2$
   (c) $\frac{1}{(n+1)\cdot(n+2)}$
   (d) $\frac{1}{n+2}$
   (e) None of the above

14. (2.5 points) If $f(kx, ky) = f(x, y)^k$ and $f(1, 3) = 0.5$, then $f(3, 9) =$

   (a) 0.5
   (b) 3(0.5)
   (c) $0.5^0.5^0.5$
   (d) 0.5^3
   (e) None of the above

$f(3,9) = f(3\cdot 1, 3\cdot 3) = f(1,3)^3$

END

4 Formulas you may find useful

- Quadratic formula:

$$x_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Quadratic optimization rule:

$$x^* = -\frac{b}{2a}$$

- A summation formula:

$$\sum_{i=k}^{n} c = c(n+1-k)$$

- An infinite geometric sum formula:

$$\sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta}$$

  when $\beta \in (0,1)$

- Another infinite geometric sum formula:

$$\sum_{t=1}^{\infty} \beta^t = \frac{\beta}{1-\beta}$$

  when $\beta \in (0,1)$

- $\exp(1) \equiv e \approx 2.718$, and $exp(-1) \equiv e^{-1} \approx 0.368$
- Powers of 2: $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32, 2^6 = 64$, $2^7 = 128$, $2^8 = 256$, $2^9 = 512$, $2^{10} = 1024$