Econ 1078-002 Word Problems

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This document lists several types of word problems we have covered over the semester. This does not cover every variation we have considered.

An investment problem

1. Your cousin is considering investing in one of two accounts. The first will take his money and provide 5% interest per year. The second will take his money, give him a bonus 20% of his initial investment, but then only offer 4% interest on the total amount per year. Both accounts compound the interest each year. Your cousin is starting with $x > 0$ dollars and is going to invest for 10 years.
   (a) Write formulas for the amounts he'll receive at the end of 10 years from each account.
   (b) Using the formulas you derived, which account would you advise your cousin to invest in to maximize his return?
   (c) How would your advice change if your cousin were investing for 20 years? Briefly (1 sentence or less, short math statements ok) explain why.

A depreciation problem

2. You have an asset valued at \$500 when $t = 0$, and want to calculate its value after 3 and 7 years under different depreciation assumptions.
   (a) Suppose the asset is completely depreciated after 10 years (straight line depreciation). Write a formula $V_1(t)$ for its value in year t.
   (b) Suppose the asset continuously depreciates (exponential depreciation) and is worth 250 in year 5. Write a formula $V_2(t)$ for its value in year t.
   (c) Compare $V_1(3)$ with $V_2(3)$. Which is larger?
   (d) Compare $V_1(7)$ with $V_2(7)$. Which is larger?
   (e) If you were selling the truck at the depreciated value at 7 years, which assumption would you use to maximize your value? (Ignore any tax effects.)
   (f) Plot V_1 and V_2, with t on the horizontal axis. Label all axis intercepts and the intersection of V_1 and V_2.

A linear solving-for-parameters problem

3. An "inverse demand curve" is a function $p(q)$ (price as a function of quantity), with a linear inverse demand curve being $p(q) = aq + b$, $a, b \in \mathbb{R}$.

(a) Suppose the company you work for knows that they face a linear inverse demand curve, and they just raised their price for widgets from \$4 per unit (where they sold 5 units per month) to \$6 dollars (where they sold 2 units per month). Formally, the observed quantity-price pairs are

$$(q_1, p_1) = (5, 4) \text{ and } (q_2, p_2) = (2, 6)$$

Use the observed quantity-price pairs to calculate the slope of the inverse demand curve. (Hint: use the definition of a linear inverse demand curve to derive an equation for the slope parameter as a function of two observed (q, p) values.)

(b) Use the formula you derived in part 1 to calculate the price at which demand for your product is exactly 0.

A budget constraint problem

4. Let X be the amount of good x which a consumer purchases, and Y be the amount of good y the consumer purchases. The prices of goods x and y are P_X and P_Y. The consumer's total income is I. Their budget constraint is an inequality relation stating that the total cost of purchases must be less than or equal to their income, i.e. $P_X X + P_Y Y \leq I$. Suppose the consumer spends all of their income on x and y. Write a proof to show that doubling the prices of both goods and the consumer's income results in no change in the amount of x purchased.

A fishery problem

5. The growth rate of a certain fish population is given by $y = rx(1 - \frac{x}{K})$, where $x \geq 0$ is a measure of the size of the fish population, $r > 0$ is a growth parameter, and $K > 0$ is the carrying capacity of the habitat. When fishers start harvesting fish, the growth rate y is modified by adding a negative Ex term, where $E \geq 0$ is the number of fishers. Thus, Ex represents the total number of fish caught by all the fishers in this particular industry.

   (a) Assume that r, K, and E are positive real numbers satisfying $r > E$, and plot the function $y(x)$

   (b) The "steady state" level of the fish stock (this just means $\tilde{x} : y(\tilde{x}) = 0$) as a function of the harvest effort is $\tilde{x} = \frac{K}{r}(r - E)$. Plot $\tilde{x}(E)$. Label the horizontal and vertical axis intercepts.

   (c) This industry's profit as a function of harvest effort is $\pi(E) = pE\tilde{x}(E) - cE$, where p is the price per unit effort c is the cost per unit effort. Assume that $p > c > 0$. Plot $\pi(E)$.

   (d) Solve for $E_0 : \pi(E_0) = 0$.

   (e) Solve for $E_{max} : \pi(E_{max}) > \pi(E)\ \forall E \neq E_{max}$.

   (f) Calculate $\tilde{x}(E_0)$ and $\tilde{x}(E_{max})$. Which is larger? Do you need any additional assumptions beyond what is given in the problem for your conclusion to hold?

   (g) Calculate $y(\tilde{x}(E_0))$ and $y(\tilde{x}(E_{max}))$. Which is larger? Do you need any additional assumptions beyond what is given in the problem for your conclusion to hold?

A monopoly problem

6. You're a monopolist trying to maximize profits. Your total revenue function is $P(Q)Q = (a - bQ)Q,\ (a, b > 0)$, and your total cost function is $C(Q) = \alpha Q + \beta Q^2,\ (\alpha, \beta > 0)$. Your profit function is total revenues minus total costs, call it $\pi(Q)$.

   (a) Write the profit function. Is it a cup or a cap? How do you know?

   (b) Plot the profit function with π on the vertical axis and Q on the horizontal axis. Label the intercepts on the plot. Assume $a > \alpha$.

(c) What value of Q maximizes profits? Call this Q^*.

(d) Calculate $\pi^* = \pi(Q^*)$. (Note that π^* should be a function of a.)

(e) Is $M(\pi^*(a)) \equiv \pi^*(a+1) - \pi^*(a)$ a linear function of a?

A duopoly problem

7. Two firms, 1 and 2, are competing in the market for widgets. x_1 is firm 1's widget production, and x_2 is firm 2's widget production. The inverse demand function (the price) is

$$P(x_1 + x_2) = A - B(x_1 + x_2),$$

where A and B are positive constants. The firms have identical cost functions,

$$c(x_i) = \alpha x_i,$$

where α is a positive constant such that $A > \alpha$.

(a) (1 point) Suppose x_2 is just a constant. Solve Firm 1's profit maximization problem for their optimal production level,

$$\max_{x_1} P(x_1 + x_2)x_1 - c(x_1).$$

(Your answer should be a function $x_1^*(x_2)$.)

(b) (1 point) Suppose x_1 is just a constant. Solve Firm 2's profit maximization problem for their optimal production level,

$$\max_{x_2} P(x_1 + x_2)x_2 - c(x_2).$$

(Your answer should be a function $x_2^*(x_1)$.)

(c) (2 points) Plot $x_1^*(x_2)$ and $x_2^*(x_1)$ with x_1 on the horizontal axis and x_2 on the vertical axis. **Use algebra** to solve the system of equations $x_1^*(x_2)$ and $x_2^*(x_1)$ for x_1^* and x_2^*. **Label your solution on the graph.**

(d) (1 point) Now suppose Firm 1 buys Firm 2, creating a monopoly in the market for widgets. Solve the new firm's profit maximization problem,

$$\max_{x} P(x)x - c(x).$$

(Your answer should be an equation for x. Call the solution $\hat{x}$.)

(e) (1 point) Use your solutions from 3. and 4. to prove the following results:

1. the monopoly produces less total output than the duopoly (i.e., that $\hat{x} < x_1^* + x_2^*$), and
2. the price is higher under the monopoly than under the duopoly (i.e., that $P(\hat{x}) > P(x_1^* + x_2^*)$).

A least-squares problem

8. Suppose you are an economist interested in a linear model, $Y = X\beta + \epsilon$, where (X, Y) are observed data points; Y is the variable you want to predict; X is a variable you think can explain Y; β is the parameter you want to estimate which describes the relationship between X and Y; and ϵ is a "statistical error" which accounts for things other than X which affect Y.[1] To estimate β you are going to try to minimize the squared error, $\epsilon(\beta)^2$.

[1] For example, maybe you're a health economist studying the effect of cigarette taxes on smoking rates. Y could be a measure of the number of people in an area who smoke, and X could be the amount of a cigarette tax in the area.

(a) You observe a data point, (X, Y). Write the squared error function, $\epsilon(\beta)^2$, for the model $Y = X\beta + \epsilon$. Can the squared error be minimized, i.e. is it a cup or a cap?

(b) Find the β which minimizes $\epsilon(\beta)^2$

(c) Suppose $Y = 1$ and $X = 0.5$. Plot $\epsilon(\beta)$ and $\epsilon(\beta)^2$ on the same graph. Your graph should have β on the horizontal axis, and the value of the functions on the vertical axis.

(d) Now suppose you get two data points, (X_1, Y_1) and (X_2, Y_2), giving you two equations, $Y_1 = X_1\beta + \epsilon_1$ and $Y_2 = X_2\beta + \epsilon_2$ (notice that β is common to both equations). Find the β which minimizes the "sum of squared errors" function, $\epsilon_1(\beta)^2 + \epsilon_2(\beta)^2$.

(e) Suppose $Y_1 = 1$, $X_1 = 1$, $Y_2 = 2$, $X_2 = 3$. Plot $\epsilon_1(\beta)^2 + \epsilon_2(\beta)^2$ with β on the horizontal axis. (Hint: Use these numbers to calculate β^* from your formula in part (b), then calculate $\epsilon_1(\beta^*)^2 + \epsilon_2(\beta^*)^2$.)

(f) Use the numbers from the previous part to calculate $\hat{Y_1} = X_1\beta^*$ and $\hat{Y_2} = X_2\beta^*$ ($\hat{Y_1}$ and $\hat{Y_2}$ should be numbers). Are $\hat{Y_1} = Y_1$ and $\hat{Y_2} = Y_2$?

(g) Suppose now you have n equations of the form $Y_i = X_i\beta + \epsilon_i$, $i = 1, \dots, n$. Write the sum of squared errors function, $\sum_{i=1}^{n} \epsilon_i^2$, using summation notation.

(h) Solve for the β^* which minimizes $\sum_{i=1}^{n} \epsilon_i^2$.

A satellite problem

9. A number of firms are competing to provide satellite services. Let $\pi > 0$ be the return generated by a satellite in period t, $\beta \in (0, 1)$ be the discount factor, and $F > 0$ be the cost of launching a satellite. $\phi(S_t) \in (0, 1)$ is the probability the satellite survives to the next period. The value of owning a satellite in period t is

$$V(S_t) = \pi + \phi(S_t)\beta V(S_{t+1}) \tag{1}$$

In each period, firms will launch satellites until profits are zero, giving us the equilibrium condition

$$\beta V(S_{t+1}) - F = 0 \tag{2}$$

(a) Use equations 1 and 2 to derive an equation for $V(S_t)$ which does not contain $V(S_{t+1})$.

(b) Use the equation you derived in part (a) to write $V(S_{t+1})$ as a function of π, F, and $\phi(S_{t+1})$ only. Combine this equation for $V(S_{t+1})$ with equation 2 to derive an equation for $\phi(S_{t+1})$ as a function of β, π, and F only.

(c) Use the restriction $0 < \phi(S_{t+1}) < 1$ with the equation you derived in part (b) to derive inequalities which relate $\beta\pi$ and F.

(d) Now, suppose that you're told $\phi(S_t) = e^{-\alpha S_t}$, where $\alpha > 0$. Use this fact with the equation you derived in part (b) to find a formula for S_{t+1} as a function of β, π, F, and α.

(e) Is $S_{t+1} < 0$? How do you know (1-2 sentences)?
(Hint: compare the formula you derived in part (d) to the restrictions you derived in part (c). What are the properties of the function you derived in part (d)?)

A zombie treatment problem

10. You are an analyst for a pharmaceutical company, and have been asked to determine whether it would be more profitable over the next quarter to invest in a treatment or a cure for a zombie plague. The number of cured individuals is H, the number of individuals being treated (but still infected) is I, and the total number is $N = H + I$. The average price received for a treatment is p_I and the average price received for a cure is p_H. The cost of producing a unit of treatment is c_I, and the cost of producing a unit of cure is c_H. The company's profits are

$$\pi(H, I) = p_I I - c_I I^2 + p_H - c_H H^2. \tag{3}$$

(a) Write the company's profits as a function of N and only H or I. (Hint: use the equation for the total number of individuals with the profit function.)

(b) Solve for the profit-maximizing choice of H or I. Label your solution H^* or I^*.

(c) Use the solved value of H^* or I^* to solve for the other. (Hint: use the equation for the total number of individuals.)

(d) Use your solved values to derive necessary conditions on the price, cost, and population parameters, p_H, p_I, c_H, c_I, N, for:

1. $H^* > I^*$ (the company wants to cure more people than it wants to treat);
2. $H^* = N$ (the company wants to cure everyone)
3. $H^* = 0$ (the company wants to cure no one)
4. $H^* > 0, I^* > 0$ (the company wants to treat some and cure some)