Drag Coefficient and Terminal Velocity of Spherical and Non-Spherical Particles

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Drag Coefficient and Terminal Velocity of Spherical and Nonspherical Particles

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SUMMARY

Explicit equations are developed for the drag coefficient and for the terminal velocity of falling spherical and nonspherical particles. The goodness of fit of these equations to the reported experimental data is evaluated and is compared with that of other recently proposed equations.

Accurate design charts for C_D and u_t are prepared and displayed for all particle sphericities.

INTRODUCTION

There are well over 30 equations in the literature relating the drag coefficient C_D to the Reynolds number Re of spheres falling at their terminal velocities. These correlations are of varying complexity, and contain as many as 18 arbitrary constants. Clift *et al.* [1], Khan and Richardson [2], and Haider [3] list many of these correlations. For nonspherical particles, however, no generalized expression for C_D *vs.* Re is available. This paper develops and presents such a correlation.

Usually it is the terminal velocity, rather than the drag coefficient, which is of ultimate interest. However, we have found only three equations in the literature which give u_t explicitly, again only for spheres, while none are available for nonspherical particles. This paper also develops such a correlation for particles of all sphericities.

DRAG COEFFICIENT FOR SPHERICAL PARTICLES

Consider the three most recently proposed drag equations, all of which happen to contain five arbitrary constants. First, Khan and Richardson [2] compiled experimental results of various researchers, and using nonlinear regression on 300 data points, proposed the following drag equation for $Re < 3 \times 10^5$:

$$C_D = (2.25\,Re^{-0.31} + 0.36\,Re^{0.06})^{3.45} \tag{1}$$

Flemmer and Banks [4] proposed, for $Re < 8.6 \times 10^4$,

$$C_D = \frac{24}{Re} 10^E \tag{2}$$

where

$$E = 0.261\,Re^{0.369} - 0.105\,Re^{0.431} - \frac{0.124}{1 + (\log_{10} Re)^2}$$

Turton and Levenspiel [5], using the equation form proposed by Clift and Gauvin [6] plus 408 previously reported experimental data points, presented the following correlation for $Re < 2.6 \times 10^5$:

$$C_D = \frac{24}{Re}(1 + 0.173\,Re^{0.657}) + \frac{0.413}{1 + 16300\,Re^{-1.09}} \tag{3}$$

After looking at the regression analysis for spherical and nonspherical particles and the results of Clift and Gauvin [6] and Turton and Levenspiel [5], the following four-parameter general drag correlation is proposed here:

$$C_D = \frac{24}{Re}(1 + A\,Re^B) + \frac{C}{1 + \dfrac{D}{Re}} \tag{4}$$

The values of A, B, C, and D in eqn. (4) were found by minimizing the sum of squares error Q, which for n data points is defined as

$$Q = \sum_{i=1}^{n} (\log_{10} C_{D,\mathrm{exp}} - \log_{10} C_{D,\mathrm{cal}})^2 \tag{5}$$

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This was done with nonlinear regression software [7] which uses the Gauss–Newton method. The experimental data used in finding the best values of the four parameters were the same 408 data points compiled by Turton and Levenspiel [5]. The final equation for predicting C_D of spheres is then found to be

$$C_D = \frac{24}{Re}(1 + 0.1806\,Re^{0.6459}) + \frac{0.4251}{1 + \dfrac{6880.95}{Re}} \tag{6}$$

The goodness of fit of eqns. (1), (2), (3), and (6) to the data are compared by the numbers in the last column of Table 1. Here the RMS deviation measures the average fractional displacement of the measured C_D-values from the correlation line. Mathematically,

$$\text{RMS} = \left(\frac{Q}{n}\right)^{1/2} = \left[\frac{\sum_{i=1}^{n}(\log_{10} C_{D,\text{exp}} - \log_{10} C_{D,\text{cal}})^2}{n}\right]^{1/2} \tag{7}$$

Finally, the lowest curve in Fig. 1 shows the fit of eqn. (6) to the experimental data.

TABLE 1

Comparison of fit to the data of the most recent drag correlations for spheres

Researchers	Equation No.	*Re* range	RMS deviation in calculated C_D
Khan and Richardson [2]	Eqn. (1)	$Re < 3 \times 10^5$	0.041
Flemmer and Banks [4]	Eqn. (2)	$Re < 8.6 \times 10^4$	0.066
Turton and Levenspiel [5]	Eqn. (3)	$Re < 2.6 \times 10^5$	0.025
This work	Eqn. (6)	$Re < 2.6 \times 10^5$	0.024

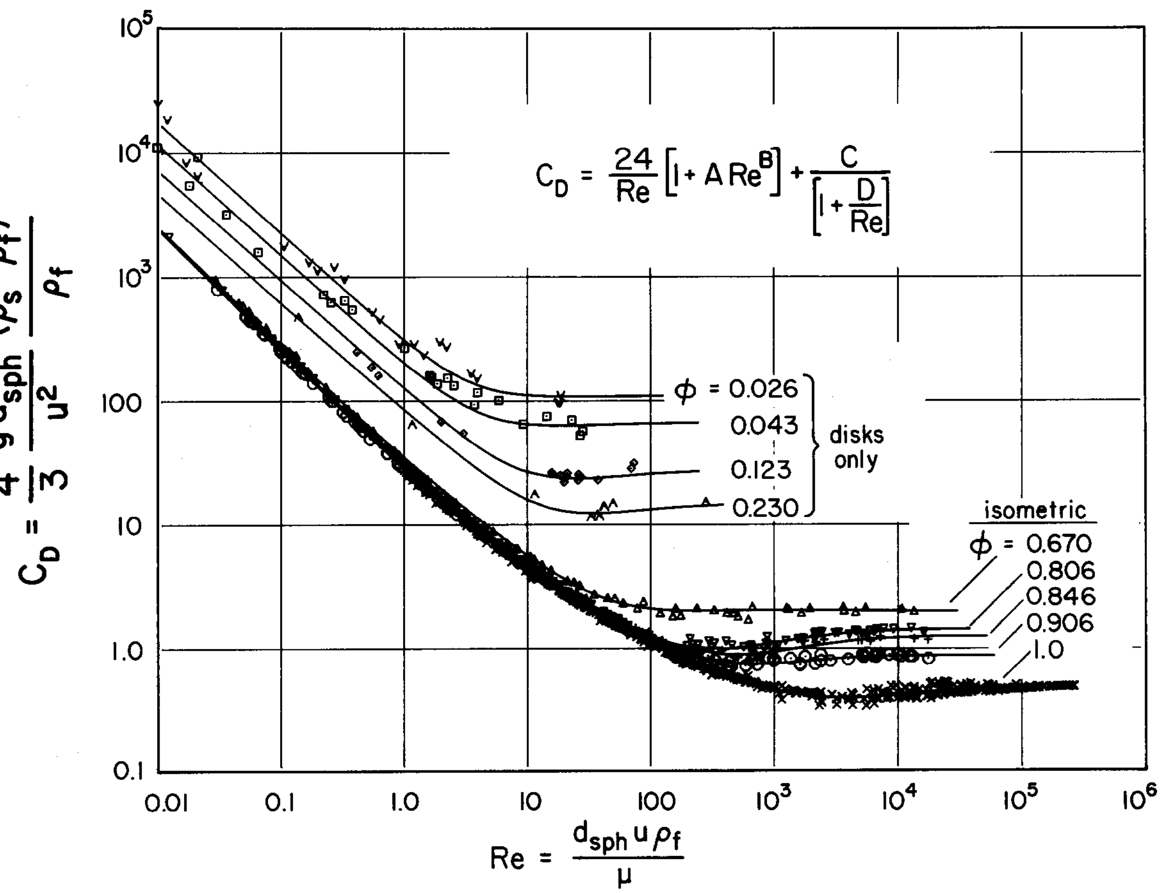

Fig. 1. Reported drag coefficients for spherical particles (408 data points) and nonspherical particles (506 data points).

DRAG COEFFICIENT FOR NONSPHERICAL PARTICLES

Here we correlate the reported experimental drag data for nonspherical particles by a direct extension of the four-parameter equation form of eqn. (4), but first we need a measure of particle shape and of particle size.

Wadell [8] introduced the concept of particle sphericity ϕ to account for particle shape, thus

$$\phi = \frac{s}{S} \tag{8}$$

where s is the surface of a sphere having the same volume as the particle and S is the actual surface area of the particle.

For close to isometrically shaped particles, those with no one very much longer or very much shorter dimension, the sphericity is a useful measure, most likely the best single parameter for describing the shape for falling particles.

As for particle size, a whole host of measures have been proposed. We have chosen to use the equivalent spherical diameter d_{sph}, which is the diameter of a sphere having the same volume as the particle. All correlations presented hereafter are based on this measure of particle size. For isotropically shaped particles,

$$d_{sph} \approx d_{screen} \tag{9}$$

Experimental drag data for eight different particle sphericities were compiled. For $\phi \geqslant 0.670$, the data were adapted from Pettyjohn and Christiansen [9], who studied the following isometric nonspherical shapes:

Cube octahedrons	($\phi = 0.906$)	136 data points
Octahedrons	($\phi = 0.846$)	80 data points
Cubes	($\phi = 0.806$)	136 data points
Tetrahedrons	($\phi = 0.670$)	67 data points

For lower sphericities, $\phi < 0.670$, the data used were for thin free-falling disks extracted from Schmiedel [10], Squires and Squires [11], and Willmarth *et al.* [12]. In this case, C_D, Re and ϕ were recalculated based upon d_{sph} using the information provided about the particles and the fluids used in the experiment.

The data were plotted for various ϕ-values and were observed to follow drag curves similar to that for spherical particles except that the drag curves lie higher and higher as ϕ drops from unity. In other words, particles experience higher drag as they become less spherical.

The data were fitted using the same equation form as used for spheres, namely eqn. (4). Once again, the best values of the four parameters were evaluated and are summarized in Table 2.

Figure 1 shows the experimental data and the best fitted drag curves as predicted by eqn. (4) with parameter estimates given in Table 2. Column 4 of Table 3 shows the goodness of fit of this equation to the data by listing its RMS deviation at the various ϕ-values.

TABLE 2

Best values of parameters to be used in eqn. (4) for predicting C_D for particles of various sphericities

ϕ	A	B	C	D
1.000	0.1806	0.6459	0.4251	6880.95
0.906	0.2155	0.6028	0.8203	1080.835
0.846	0.2559	0.5876	1.2191	1154.13
0.806	0.2734	0.5510	1.406	762.39
0.670	0.4531	0.4484	1.945	101.178
0.230	2.5	0.21	15	30
0.123	4.2	0.16	28	19
0.043	7	0.13	67	7
0.026	11	0.12	110	5

COMPREHENSIVE DRAG EQUATION FOR FALLING PARTICLES

Table 2 shows that the four parameters of eqn. (4) are functions of ϕ, and once this functionality is established, it would make it convenient to interpolate for C_D for sphericities other than the ones listed in Table 2. Keeping this in mind, values of the four parameters were plotted against ϕ and a reasonable order polynomial was fitted through the data using least-squares fit. The result is

$$A = \exp(2.3288 - 6.4581\phi + 2.4486\phi^2) \tag{10a}$$

$$B = 0.0964 + 0.5565\phi \tag{10b}$$

$$C = \exp(4.905 - 13.8944\phi + 18.4222\phi^2 - 10.2599\phi^3) \tag{10c}$$

$$D = \exp(1.4681 + 12.2584\phi - 20.7322\phi^2 + 15.8855\phi^3) \tag{10d}$$

Substituting the above relations into eqn. (4) yields

TABLE 3

Deviation of predicted C_D from experiment for different sphericities

Shapes	Number of data points	ϕ	RMS deviation of eqn. (4) with parameter values given in Table 2	RMS deviation of eqn. (11)	RMS deviation of eqn. (12)
Spheres	408	1.000	0.024	0.031	0.058
Isometric solids	136	0.906	0.017	0.024	0.040
	80	0.846	0.018	0.022	0.038
	136	0.806	0.024	0.030	0.035
	67	0.670	0.024	0.034	0.044
Disks, Non-isometric solids	10	0.230	0.049	0.067	0.160
	17	0.123	0.043	0.077	0.147
	30	0.043	0.107	0.095	0.112
	30	0.026	0.107	0.154	0.217

$$C_D = \frac{24}{Re}\left[1 + \exp(2.3288 - 6.4581\phi + 2.4486\phi^2)\, Re^{(0.0964 + 0.5565\phi)}\right] + \frac{Re \times \exp(4.905 - 13.8944\phi + 18.4222\phi^2 - 10.2599\phi^3)}{Re + \exp(1.4681 + 12.2584\phi - 20.7322\phi^2 + 15.8855\phi^3)} \tag{11}$$

Equation (11) predicts C_D quite accurately but it is rather tedious to use. As an approximation, a linear relation was developed for each of the parameters as a function of ϕ. When substituted into eqn. (4), this gives a much simpler expression,

$$C_D = \frac{24}{Re}\left[1 + [8.1716 \exp(-4.0655\phi)] \times Re^{(0.0964 + 0.5565\phi)}\right] + \frac{73.69\, Re \exp(-5.0748\phi)}{Re + 5.378 \exp(6.2122\phi)} \tag{12}$$

Table 3 also compares the goodness of fit of eqns. (11) and (12) to the experimental data. For isometric particles ($\phi \geqslant 0.67$), the fit is quite good, but is poorer for disks. Also, for spheres we may prefer to use the simpler eqn. (6) with its RMS error of 2.4% rather than eqns. (11) or (12) with their RMS errors of 3.1% and 5.8%, respectively.

Figure 2 presents a design chart for the drag coefficient C_D using eqn. (6) for spheres and the comprehensive drag equation, eqn. (11), for all other sphericities.

TERMINAL VELOCITY OF FALLING SOLIDS

To find the terminal velocity u_t of particles from any of the proposed C_D *vs.* Re expressions requires a tedious trial and error procedure since u_t is present in both variables. Thus, it would be useful to have an expression which explicitly gives u_t in terms of the system variables.

So far, three explicit equations have been reported in the literature to determine the terminal velocities of free-falling spheres, however none have been proposed for nonspherical particles. Here, we develop a simple, explicit, and reasonably accurate equation capable of predicting terminal velocities of free-falling spheres as well as nonspherical particles.

Before presenting anyone's results, we define two dimensionless quantities: a dimensionless terminal velocity u_* and a dimensionless particle diameter d_* as follows

$$u_* = \left(\frac{4}{3}\frac{Re}{C_D}\right)^{1/3} = u_t\left[\frac{\rho_f^2}{g\mu(\rho_s - \rho_f)}\right]^{1/3} \tag{13}$$

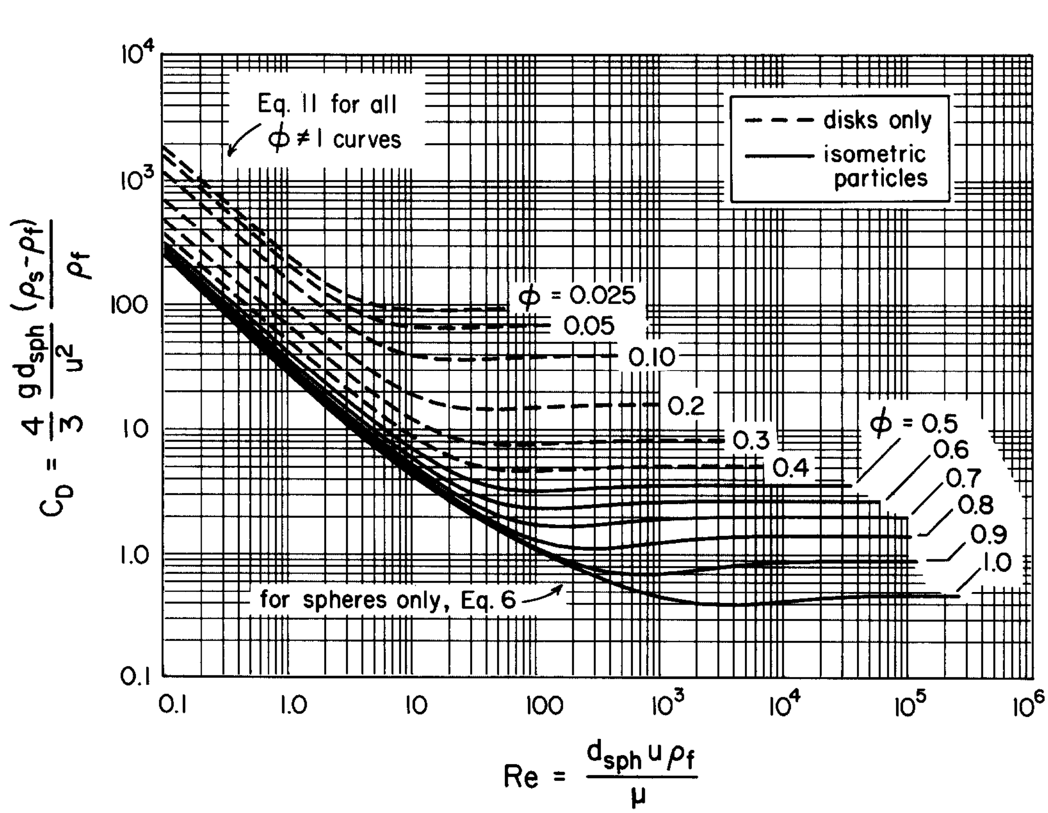

Fig. 2. Design chart for drag coefficients of single free-falling particles.

and

$$d_* = \left(\frac{3}{4} C_D Re^2\right)^{1/3} = d_{sph}\left[\frac{g\rho_f(\rho_s - \rho_f)}{\mu^2}\right]^{1/3} \quad (14)$$

Zigrang and Sylvester [13] presented an explicit equation for particle settling velocities in solid–liquid suspensions. In the limit as the volume fraction of solids in the system approaches zero, their equation gives the terminal velocity of a single sphere settling in a liquid as

$$u_* = \frac{[(14.51 + 1.83 d_*^{3/2})^{1/2} - 3.81]^2}{d_*} \quad (15)$$

Khan and Richardson [2], using nonlinear regression on their compiled experimental data, presented the following explicit correlation for u_t of spheres:

$$Re = (2.33 d_*^{0.018} - 1.53 d_*^{-0.016})^{13.3} \quad (16)$$

Recently, Turton and Clark [14], making use of the asymptotic expressions for u_t for very low and very high Re, came up with

$$u_* = \left[\left(\frac{18}{d_*^2}\right)^{0.824} + \left(\frac{0.321}{d_*}\right)^{0.412}\right]^{-1.214} \quad (17)$$

This correlation comes from the general two-parameter expression for u_*:

$$u_* = \left[\frac{1}{\left(\frac{18}{d_*^2}\right)^{K_2} + \left(\frac{3K_1}{4 d_*^{0.5}}\right)^{K_2}}\right]^{1/K_2} \quad (18)$$

In the present study, values of K_1 and K_2 for the five different sphericities ($0.67 \leqslant \phi \leqslant 1.00$) were found by minimizing the sum of squares error for u_*, defined similarly as in eqn. (5). Experimental data analysed are the same 408 points for spherical particles as previously used in the analysis of drag coefficients, plus all the data from Pettyjohn and Christiansen [9] for nonspherical isometric particles. Table 4 gives the best-fit values of K_1 and K_2 along with RMS deviation for various sphericities, and Fig. 3 shows the resulting fit of eqn. (18) to the data. For the sake of clarity, the ϕ = 0.846 line and data are omitted from this figure.

One may notice from Table 4 that the values of K_2 for different ϕ stay close to 1.

TABLE 4

Best values of K_1 and K_2 to be used in eqn. (18) for calculating the terminal velocity of particles of various sphericities

ϕ	K_1	K_2	RMS deviation of u_*
1.000	0.7554	0.8243	0.0245
0.906	0.9999	0.9677	0.0212
0.846	1.1272	0.9697	0.0257
0.806	1.2024	1.0222	0.0327
0.670	1.5469	0.9464	0.0275

Setting $K_2 = 1$ in eqn. (18) essentially amounts to adding low and high Re contributions in parallel. On doing this, eqn. (18) reduces to the simple one-parameter expression

$$u_* = \left(\frac{18}{d_*^{\,2}} + \frac{3K_1}{4d_*^{\,0.5}}\right)^{-1} \tag{19}$$

Once again, nonlinear regression was performed on eqn. (19). The resulting best-fit values for K_1 and the corresponding RMS deviation are shown in Table 5.

One notices from this table that K_1 increases regularly as ϕ decreases. This may be approximated by the linear relationship

$$K_1 = 3.1131 - 2.3252\,\phi \tag{20}$$

Substituting eqn. (20) into eqn. (19) yields the following simple general correlation for

TABLE 5

Best values of K_1 to be used in the simple eqn. (19) for calculating u_* of particles of different sphericities

ϕ	K_1	RMS deviation of u_*
1.000	0.8039	0.0403
0.906	1.0142	0.0218
0.846	1.1416	0.0261
0.806	1.1926	0.0329
0.670	1.5824	0.0288

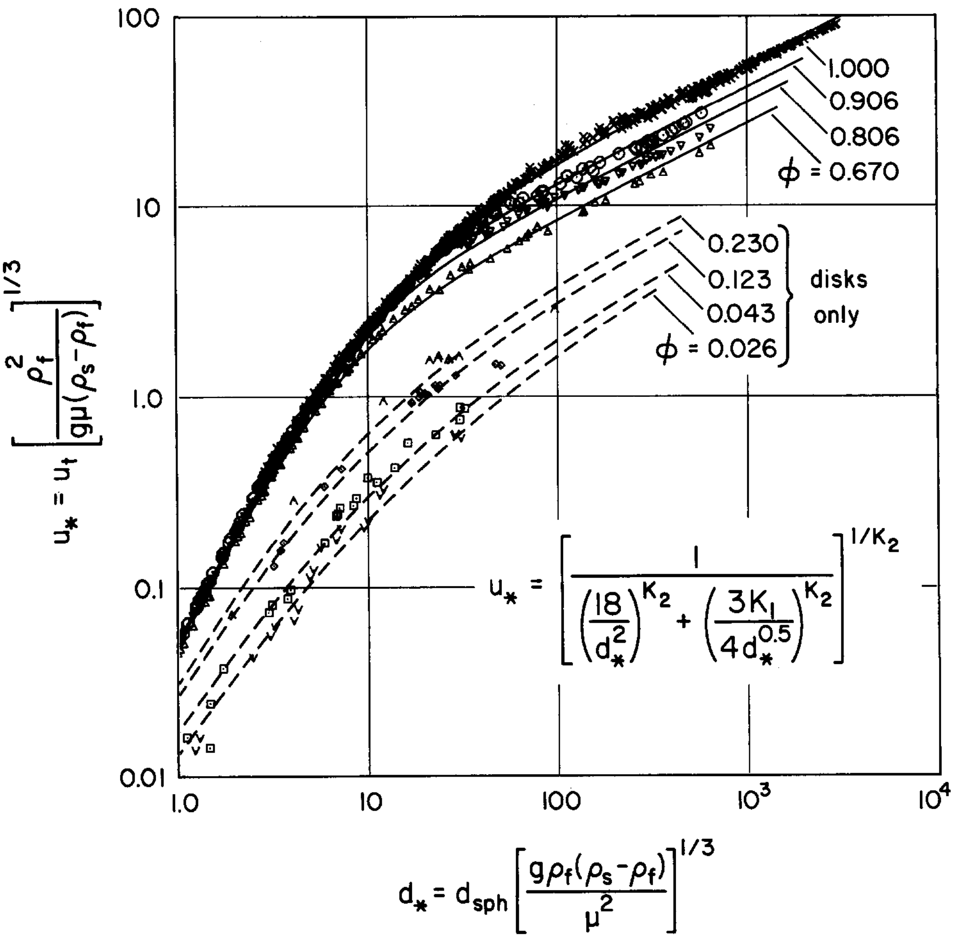

Fig. 3. Reported terminal velocities for both spherical and nonspherical particles.

predicting terminal velocities for isometric particles, given information on the particles and physical properties of the fluid,

$$u_* = \left[\frac{18}{d_*^2} + \frac{(2.3348 - 1.7439\,\phi)}{d_*^{0.5}}\right]^{-1} \qquad 0.5 \leqslant \phi \leqslant 1 \qquad (21)$$

Table 6 shows the goodness of fit of eqn. (21) to the data for different sphericities as well as the goodness of fit of all the recommended equations for spheres.

Finally, Fig. 4 displays the best-fit terminal velocity curves for different particle sphericities.

TABLE 6

Goodness of fit of various explicit equations for u_* to the reported data

Researchers	ϕ	Equation No.	RMS deviation of equation from data
Zigrang and Sylvester [13]	1.0	Eqn. (15)	0.041
Khan and Richardson [2]	1.0	Eqn. (16)	0.033
Turton and Clark [14]	1.0	Eqn. (17)	0.024
This work, for spheres	1.0	Eqn. (21)	0.041
This work, for nonspherical particles	0.906	Eqn. (21)	0.022
	0.846	Eqn. (21)	0.026
	0.806	Eqn. (21)	0.035
	0.670	Eqn. (21)	0.029

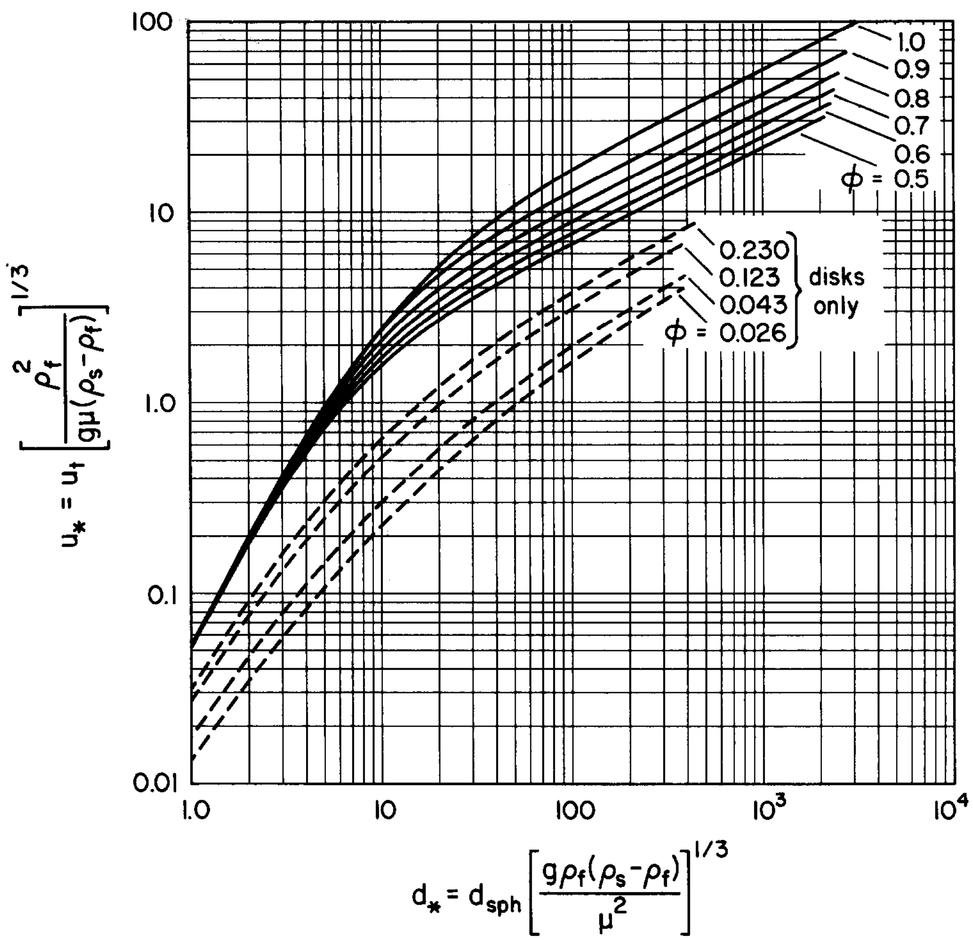

Fig. 4. Design chart for finding the terminal velocity of single free-falling particles.

FINAL RECOMMENDATIONS

Spherical Particles: 408 data points, $Re < 2.6 \times 10^5$

— For finding C_D use eqn. (6): RMS deviation = 2.4%

— For finding u_t use eqn. (17): RMS deviation = 2.4%

Nonspherical Particles: 419 isometric data points, $Re < 25\,000$; 87 disk data points, $Re < 500$

— For finding C_D with a cumbersome but accurate expression use eqn. (11): RMS deviation ≈ 3% for isometric particles.

— For finding C_D with a simpler expression use eqn. (12): RMS deviation ≈ 5% for isometric particles.

— For finding u_t for $\phi \neq 1$ use eqn. (21): RMS deviation ≈ 3% for isometric particles.

Design Charts:

— Figure 2, based on eqn. (6) for $\phi = 1$ and eqn. (11) for $\phi \neq 1$, relates C_D with Re.

— Figure 4 is for finding u_* and consequently u_t. It is based on eqn. (18) with K_1 and K_2 values given in Table 4 for spheres and on eqn. (21) for nonspherical isometric particles.

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LIST OF SYMBOLS

A, B, C, D	fitted constants in eqn. (4)
C_D	drag coefficient, $= \dfrac{4}{3} \dfrac{g d_{sph}}{u_t^2} \dfrac{(\rho_s - \rho_f)}{\rho_f}$
d_{scr}	screen size of particles, m
d_{sph}	equivalent spherical diameter, or diameter of sphere which has same volume as particle, m
d_*	dimensionless particle diameter, see eqn. (14)
g	acceleration due to gravity, = 9.81 m/s^2
K_1, K_2	fitted constants in eqn. (18).
Re	Reynolds number based on equivalent spherical diameter of particle, $= \dfrac{d_{sph} u_t \rho_f}{\mu}$
RMS	root-mean-square deviation of data points from correlation, on log–log plot, see eqn. (7)
u_t	terminal velocity of particle in fluid, m/s
u_*	dimensionless particle velocity, see eqn. (13)

Greek symbols

μ	viscosity of fluid, kg/m.s)
ρ_f	density of fluid, kg/m^3
ρ_s	density of particle, kg/m^3
ϕ	particle sphericity, see eqn. (8)

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