MASTER OF COMPUTER APPLICATIONS

SEMESTER 1

DISCRETE MATHEMATICS AND GRAPH THEORY

Unit 9

Mathematical Logic

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1. INTRODUCTION

Mathematical logic serves as the cornerstone for systematically evaluating reasoning and formalizing logical structures, essential in diverse disciplines like mathematics, computer science, philosophy, and linguistics. At its core are statements—declarative sentences symbolized using variables, constants, and logical symbols—which can be either true or false. The language of logic employs connectives such as conjunction (∧), disjunction (∨), negation (¬), implication (→), and biconditional (↔), allowing the construction of complex statements from simpler ones. These connectives serve as fundamental tools to amalgamate individual statements, forming compound statements whose truth values are determined by the logical relationships established. Normal forms like Conjunctive Normal Form (CNF) and Disjunctive Normal Form (DNF) provide standardized representations for logical expressions, simplifying complex statements and aiding in logical analysis. Well-formed formulas (WFFs) are syntactically correct expressions adhering to the formal language's syntax rules, ensuring meaningfulness within the logical system. Implication, denoted by the symbol →, signifies conditional statements where the truth of one statement implies the truth of another, crucial in establishing logical relationships and deriving conclusions from given premises. Tautologies, on the other hand, are statements that hold true under all interpretations, regardless of the truth values of their constituent parts. Recognizing tautologies assists in identifying logical truths and understanding the nature of valid arguments, reinforcing the foundational role of mathematical logic in systematically analyzing and manipulating logical statements to derive precise deductions and inferences across various domains of knowledge.

1.1. Objectives:

At studying this unit, you should be able to:

- *Remember the Statements and Notations, Connectives, Normal Forms, Well-formed Formulas, Implication, Tautology.*

2. DEFINITIONS AND EXAMPLES

2.1. Statement

A statement is a declarative sentence that is either true or false, but not both.

A statement is a declarative sentence which is either true or false but not both.

A proposition is also called as statement.

Example:

1. Karnataka is in England (false), India is a country (true), 26=69 (false), these are some example of statement or proposition
2. The following are not propositions: “Do your assignment?” (Command), “What is your Name”? (Question), “This sentence is Equal” (Neither T or F), C is Odd number (C value not determined)

Note: Based on the number of statements or proposition it is mainly categorised as primary statement and compound Proposition.

3. CONNECTIVES

Any word or Expression used to connect 2 or more statements then such words are called connectives.

The word or phrases like not, and, or, if then, if and only if[iff], etc., are called logical statement.

Note: Based on the number of statements or proposition it is mainly categorised as primary or simple statement and compound Proposition.

3.1. Simple statement

The proposition which is free from logical connectives are called **simple Statement.**

3.2. Compound statement

Any statement obtained by combining two or more statement by logical connective.

3.3. Negation [NOT]

If $\boldsymbol{p}$ be any proposition, then negation of $\boldsymbol{p}$ is read as $\boldsymbol{not\ p}$ and it is denoted by $\neg\boldsymbol{p}$ or $\sim\boldsymbol{p}$.

If the truth value of $\boldsymbol{p}$ is **1** then truth value of $\boldsymbol{not\ p}$ is **0** & vice versa.

Truth table:

p	$\neg p$	O/P
F	T	1
T	F	0

Example: Find the negation of the proposition "Michael's PC runs Linux".

The negation is: "Michael's PC does not run Linux".

3.4. Conjunction [AND]

Let p and q be simple proposition, then conjunction of p & q is read as $p\ and\ q$ and is denoted by $p \wedge q$.

The truth value of $p \wedge q$ is 1 if both p & q values are 1, otherwise value of $p \wedge q$ is 0.

Truth table:

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$	**O/P**
F	**F**	**F**	**0**
F	**T**	**F**	**0**
T	**F**	**F**	**0**
T	**T**	**T**	**1**

3.5. Disjunction [OR]

Let p and q be simple proposition, then disjunction of p & q is read as $p\ or\ q$ and is denoted by $p \vee q$.

The truth value of $p \vee q$ is 1 if any one of p and q is 1 but 0 if both p & q are 0.

Truth table:

p	q	$p \vee q$	O/P
F	F	F	0
F	T	T	1
T	F	T	1
T	T	T	1

3.6. Exclusive Disjunction [Either or but not both]

Let p and q be simple proposition, then exclusive disjunction of p & q is read as $either\ p\ or\ q\ but\ not\ both$ and is denoted by $p \veebar q$.

The truth value of $p \veebar q$ is 1 if any one of value of p & q is different but 0 if both p & q are same.

Truth table:

p	q	$p \veebar q$	O/P
F	F	F	0
F	T	T	1
T	F	T	1
T	T	F	0

3.7. Conditional [If, then]

Let p and q be simple proposition, then conditional of p & q is read as $If\ p,\ then\ q$ and is denoted by $p \rightarrow q$.

The truth value of $p \rightarrow q$ is 0 if p is 1 & q is 0 and otherwise it is 1.

Truth table:

p	q	$p \to q$	O/P
F	F	T	1
F	T	T	1
T	F	F	0
T	T	T	1

3.8. Biconditional [Iff]

Let p and q be simple proposition, then biconditional of p & q is read as $p\ iff\ q$ [p if and only if q] and is denoted by $p \leftrightarrow q$.

The truth value of $p \leftrightarrow q$ is 1 only when both p & q are same otherwise 0.

Truth table:

p	q	$p \leftrightarrow q$	O/P
F	F	T	1
F	T	F	0
T	F	F	0
T	T	T	1

Note: The biconditional statement is also read as $If\ p\ then\ q\ and\ if\ q\ then\ p$.

3.9. Duality

The dual of a statement formula is obtained by replacing ∨ by ∧ , ∧ by ∨ , T by F F by T. A dual is obtained by replacing T (tautology) by (contradiction), F and, by T.

The principle of duality in mathematical logic is a fundamental concept that describes a relationship between logical expressions. It asserts that any theorem or property in propositional logic remains valid if we obtain a new theorem or property by interchanging certain elements while preserving the logical structure. This interchange involves replacing certain operations or elements with their dual counterparts.

In the context of propositional logic, duality primarily involves two logical operations: conjunction (∧) and disjunction (∨), along with the constants true and false, also known as 1 and 0 respectively. The principle of duality is based on the following transformations:

Interchanging ∨ and ∧: When you take any theorem or logical expression involving conjunction (∧) and disjunction (∨), you can obtain a new valid theorem by replacing all occurrences of ∧ with ∨ and vice versa. For example:

Original expression: P ∧ (Q ∨ R)

Dual expression: P ∨ (Q ∧ R)

Interchanging 1 and 0: This principle also extends to the constants true (1) and false (0). Swapping these constants while preserving the logical structure results in an equivalent expression. For instance:

Original expression: P ∨ 1

Dual expression: P ∧ 0

Preservation of Properties: The principle of duality doesn't just interchange symbols but also preserves certain logical properties. For example, if an expression is a tautology (always true), its dual expression is also a tautology. Similarly, if an expression is satisfiable (can be true under some interpretation), its dual is also satisfiable.

The principle of duality allows for simplification and the derivation of new theorems or logical expressions by transforming existing ones. It is a powerful tool in logic that helps in proving theorems, simplifying expressions, and verifying properties by exploiting the relationships between logical operations.

Example:

Write the dual of $(p \vee q) \vee \{(\neg p \vee F_0) \wedge (q \vee T_0)\}$

Solution

$(p \wedge q) \wedge \{(\neg p \wedge T_0) \vee (q \wedge F_0)\}$.

Problem

(P → Q) ∧ (Q → R) ↔ (P → R)

Solution

(¬P ∨ Q) ∧ (¬Q ∨ R) ↔ (¬P ∨ R)

Problem

(P ∨ (Q ∧ ¬R)) ↔ ((P ∨ Q) ∧ (P ∨ ¬R))

Solution

(¬P ∧ (Q ∨ R)) ↔ ((¬P ∧ ¬Q) ∨ (¬P ∧ R))

Problem: Prove the following statement and its dual are equal with the help of truth table

u=P∧(Q∨R) and u^d=P∨(Q∧R)

P	Q	R	Q∨R	P∧(Q∨R)
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

u^d=P∨(Q∧R)

P	Q	R	Q∧R	P∨(Q∧R)
T	T	T	T	T

T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

4. THE LAWS OF LOGIC OR EQUIVALENCE FORMULAS

Let $p,\ q$ & r be any proposition, T_0 denotes tautology and F_0 denotes contradiction

Law of double negation:

$\neg\neg p \Leftrightarrow p$

Idempotent law:

$p \vee p \Leftrightarrow p$

$p \wedge p \Leftrightarrow p$

Identity law:

$p \vee F_0 \Leftrightarrow p$

$p \wedge T_0 \Leftrightarrow p$

Inverse Law:

$p \vee \neg p \Leftrightarrow T_0$

$p \wedge \neg p \Leftrightarrow F_0$

Domination Laws:

$p \vee T_0 \Leftrightarrow T_0$

$p \wedge F_0 \Leftrightarrow F_0$

Commutative law:

$p \vee q \Leftrightarrow q \vee p$

$p \wedge q \Leftrightarrow q \wedge p$

Associate law:

$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

Distributive law:

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

DE Morgan's law:

$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

Absorption law:

$[p \vee (p \wedge q)] \Leftrightarrow p$

$[p \wedge (p \vee q)] \Leftrightarrow p$

Law of negation of conditional:

$\neg(p \rightarrow q) \Leftrightarrow p \wedge {\sim}q$

$p \rightarrow q \Leftrightarrow {\sim}p \vee q$

Example: Simplify or Prove the following logical statement without using truth tables $p \vee q \wedge \{\neg[\neg p \wedge q]\}$

Solution

by using DeMorgan's law

$\Leftrightarrow (p \vee q) \wedge \{\neg\neg p \vee \neg q\}$

By using law of double negation

$\Leftrightarrow (p \vee q) \wedge \{p \vee \neg q\}$

by Distributive law

$\Leftrightarrow p \vee (q \wedge \neg q)$

By Inverse law

$\Leftrightarrow p \vee F_0$

By Identity law

$\Leftrightarrow p$

5. RULES OF INFERENCE

Consider a set of Proposition $p_1, p_2, \dots, p_n$ and q then the compound proposition $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q[\Rightarrow q]$ is called an argument

Here $p_1, p_2, \dots, p_n$ are called premises q is called conclusion.

Note: An argument is generally written as follows

$$p_1$$

$$p_2$$

$$\vdots$$

$$p_n$$

$\therefore q$

The argument is valid when all $p_1, p_2, \ldots, p_n$ are true and likewise q is true.

i.e., The conclusion is true only in the case of valid argument

We use the following rules known as the rules of inference to check the validity of an argument

Rules of conjunctive simplification

$$\frac{p \wedge q}{\therefore p} \quad \text{or} \quad \frac{p \wedge q}{\therefore q}$$

i.e., $p \wedge q \Rightarrow p$ or $p \wedge q \Rightarrow q$

Rule of Disjunctive Amplification

$$\frac{p}{\therefore p \vee q} \quad \text{or} \quad \frac{q}{\therefore p \vee q}$$

i.e., $p \Rightarrow p \vee q$ or $q \Rightarrow p \vee q$

Rule of Syllogism

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

i.e., $[(p \to q) \wedge (q \to r)] \Rightarrow p \to r$

Modus Pones [Rule of Detachment / Method of Affirming]

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

i.e., $[(p \to q) \wedge p] \Rightarrow q$

Modus Tollens [Method of denying]

$p \to q$

$\sim q$

$\therefore \sim p$

i.e., $[(p \to q) \wedge \sim q] \Rightarrow \sim p$

Rule of Disjunctive syllogism

$p \vee q$

$\sim p$

$\therefore q$

i.e., $[(p \vee q) \wedge \sim p] \Rightarrow q$

Rule of Contradiction

$\sim p \to F_0$

$\therefore p$

Thus, the validity is established with aid of above argument but some cases we also use the laws of logic, logical equivalent or tautology if its not possible to prove using about when we use truth table

Example: Test the validity of the following arguments

1. *Andrea can program in C++ and she can program in Java. Therefore Andrea can Program in C++*

   Solution

   p: Andrea can Program in C++

q: Andrea can Program in Java

Thus, the argument reads as follows

$p \wedge q$

$\therefore p$

i.e., $p \wedge q \Rightarrow p$ by rule of conjunctive simplification, we conclude that the argument is valid.

2. If a person is poor, then he is unhappy

If a person is unhappy, he dies young .

$\therefore$ Poor person dies young.

Solution

p: A person is poor.

q: Person is unhappy.

r: Person dies young.

Thus, the argument reads as

$p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

i.e., $[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow p \rightarrow r$ by rule of syllogism, we conclude the argument is valid.

6. VARIABLES AND QUANTIFIERS

In Mathematical discussions, declarative sentance such as those given below

Example:

1) x+3=6

2) x^2<10
3) x divides 4
4) x = √2, These sentences are not propositions unless symbol x is specified.

These sentences of these kind are called open sentences or **open statements** And symbol x which is unspecified is called **free variable**

Let us consider open sentence (1) and set of Real number "R".

This sentence becomes a proposition if x is replaced by and=y element of R

Here we say R is Universe (universe of discourse) for variable x

For Example: if x is replaced as 3,

The sentence x+3=6 becomes true Proposition

And if x =5 it becomes False Proposition

Note:

- Open statement containing variable x are denoted by p(x), q(x) etc..
- If U is universe for variable x in an open statement p(x) and if a ∈ U, then proposition got by replacing a by a in p(x) is denoted by p(a)
- open statement p(x) becomes proposition only when x is replaced by chosen element of the universe

6.1. Quantifier

For all of you, there exists information about quantifiers below. We often quantify a variable for a statement, or predicate, by claiming a statement holds for all values of the quantity or we say there exists a quantity for which the statement holds (at least one).

Thus we can write this in shorthand as follows:

∀x ∈ A, P(x), which claims: for all x in the set A, the statement P(x) is true.

$\exists x \in A, P(x)$, which claims: there exists at least one x in the set A such that the statement P(x) is true.

There are many equivalent way to express these quantifiers in English. Here are a few examples:

6.1.1.1. Universal Quantifier:

Here are a few ways to say $\forall x \in N$:

"For all natural numbers x, . . .", "For all x ∈ N, . . .", "For any x ∈ N, . . .", "For every x ∈ N, . . ."

6.1.1.2. Existential Quantifier:

Here are a few ways to say $\exists x \in N$:

"There exists a natural number x such that . . .", "There exists x ∈ N such that . . .", "For at least one x ∈ N

. . .", "For some x ∈ N . . ."

6.1.1.3. Nested (or Compound) Quantifiers:

If there is more than one quantity it is fairly common to see more than one quantifier in a statement. When we see nested quantifiers we must take special care in the order they appear (as this can effect the meaning).

RULE 1 : If we are using the same quantifier, then the ordering doesn't matter.

Examples

For all x ∈ R and for all y ∈ R, x + y = 4.', is the same as

'For all y ∈ R and for all x ∈ R, x + y = 4.', which is the same as

'For all x, y ∈ R, x + y = 4.' (Note: You should be able to tell that this is a false statement.)

RULE 2 : If we are using mixed quantifiers, then the ordering DOES matter.

Examples

- 'For all x ∈ R, there exists y ∈ R such that x + y = 4.'

This statement says that the following in this exact order:

1. The variable x can set as ANY real number.
2. After x is set, we can find at AT LEAST ONE y based on x such that x + y = 4.

In other words, the variable y is nested 'inside' in a sense, so we are saying that y can be found after we

already select x. (Note: This is a true statement.)

6.2. Truth values of quantified statement

The following Rules are employed for determining the truth value of a quantified statement

Rule-1: The statement '$\forall x \in S, p(x)$' is true only when p(x) is true for each $x \in S$.

Rule-2: The statement '$\forall x \in S, p(x)$' is False only when p(x) is False for every $x \in S$.

As a consequence of rule 1 and 2 indicated above, we obtain the following rules of inference

Rule-3: If an open statement **p(x) is known to be true** for all x in a universe S and if $a \in S$, then p(a) is true [known as rule of universal specification)

Rule-4: If an open statement **p(x) is proved to be true** for any arbitrary x chosen from set S, then quantified statement "$\forall x \in S, p(x)$' is true [known as rule of universal generalization]

Example:

1. $\forall x, [p(x) \wedge q(x)] \quad \Leftrightarrow \quad (\forall x, p(x)) \wedge (\forall x, q(x))$
2. $\exists x, [p(x) \vee q(x)] \quad \Leftrightarrow \quad (\exists x, p(x)) \vee (\exists x, q(x))$
3. $\exists x, [p(x) \rightarrow q(x)] \quad \Leftrightarrow \quad \exists x, [\sim p(x) \vee q(x)]$
4. $\forall x, \sim p(x) \quad \Leftrightarrow \quad$ for no x, p(x)

Negating a Quantified Statement

The negation of "all A are B" is "at least one A is not B".

The negation of "no A are B" is "at least one A is B".

The negation of “at least one A is B” is “no A are B”.

The negation of “at least one A is not B” is “all A are B”.

6.3. Free and bound variables

Two essentially different ways in which we use individual variables in first-order formulae:

1. Free variables: used to denote unknown or unspecified objects, as in $(5<x)\vee(x2+x-2=0)$.
2. Bound variables: used to quantify, as in $\exists x((5<x)\vee(x2+x-2=0))$ and $\forall x((5<x)\vee(x2+x-2=0))$.

Note that the same variable can be both free and bound in a formula, Example:

x in the formula $x>0\wedge \exists\, x\, (5<x)$. A formula with no bound variables is an open formula.

A formula with no free variables is a closed formula, or a sentence.

7. WELL-FORMED FORMULA

A Well-Formed Formula (WFF) is an expression that combines variables, parentheses, and connective symbols.

Definition : A propositional variable is a symbol representing any proposition. Note that a propositional variable is not a proposition but can be replaced by a proposition. Any statement involving a propositional variable and logical connectives is a well-formed formula.

Three rules make up the simple syntax of propositional logic.

1) A WFF is a capital letter by itself.

2) " May be used as a prefix to any WFF. (A WFF will also be the end result.)

3) Any two WFFs can be combined with a "•", "", "", or "" in between them, with the resulting value enclosed in parentheses. (This will also be a WFF.).

Formulas formed using syntax rules and examples.

WFF	EXPLNATION
A	by rule 1
~A	by rule 2, since A is a WFF
(~A.B)	by rule 3, joining ~A and B
((~A • B) ∧~~C)	by rule 3, joining (~A • B) and ~~C
~((~A • B) ∧~~C)	by rule 2, since ((~A • B) ∧~~C) is WFF

Unformed formulas include

Non-WFF	EXPLNATION
A~	the ~ belongs on the left side of the negated proposition
(A •)	there's no WFF on the right side of the •
(A • ∧ B)	cannot be formed by the rules of syntax

Example : Obtain truth value for $\alpha = (P \rightarrow Q) \wedge (Q \rightarrow P)$.

Solution : The truth table for the given well formed formula

P	Q	(P→Q)	(Q→P)	α
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

7.1. Tautology [T_0]

A compound proposition which is true [1] for all possible truth values of its components [Primitive] is called **tautology.**

Example:

P	~P	P∨~P
T	F	**T**
F	T	**T**

P ∨ ¬ P always takes value T for all possible truth value of P, it is a tautology.

7.2. Contradiction / Absurdity [F_0]

A compound proposition which is false [0] for all possible truth value of its components is called **contradiction.**

Example:

P	~P	P^~P
T	F	**F**
F	T	**F**

P ∨ ¬ P always takes value F for all possible truth value of P, it is a contradiction

7.3. Contingency

A compound proposition that can be true or false i.e., neither tautology nor contradiction depending upon the ruth values of its compound is called **contingency.**

7.4. Logical Implication

Informally statement, a statement p logically implies a statement q if the truth or p guarantees the truth of q This happens exactly when $p \to q$ is a tautology. Note that we are not concerned about what happens if p is false. This is because of the truth table for implies: $p \to q$ is true (by definition) when p is false. We use the notation $p \Rightarrow q$

7.5. Tautological Implications

A statement formula A is said to tautologically imply a statement B if and only if A → B is a tautology. In this case we write A ⇒ B, which is read as A implies B.

Note: ⇒ is not a connective, A ⇒ B is not a statement formula.

7.6. Logical Equivalence

Let u & v be two compound proposition which are said to be logically equivalent whenever u & v have same truth value.

In other words, its biconditional is a tautology. It is denoted by $u \Leftrightarrow v$.

7.6.1. The Laws of logic [logical equivalence]

Let $p,\ q$ & r be any proposition, T_0 denotes tautology and F_0 denotes Contradiction.

Law of double negation:

$\neg\neg p \Leftrightarrow p$

Idempotent law:

$p \vee p \Leftrightarrow p$

$p \wedge p \Leftrightarrow p$

Identity law:

$p \vee F_0 \Leftrightarrow p$

$p \wedge T_0 \Leftrightarrow p$

Inverse Law:

$p \vee \neg p \Leftrightarrow T_0$

$p \wedge \neg p \Leftrightarrow F_0$

Domination Laws:

$p \vee T_0 \Leftrightarrow T_0$

$p \wedge F_0 \Leftrightarrow F_0$

Commutative law:

$p \vee q \Leftrightarrow q \vee p$

$p \wedge q \Leftrightarrow q \wedge p$

Associate law:

$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

Distributive law:

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

DeMorgan's law:

$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

Absorption law:

$[p \vee (p \wedge q)] \Leftrightarrow p$

$[p \wedge (p \vee q)] \Leftrightarrow p$

Law of negation of conditional:

$\neg(p \rightarrow q) \Leftrightarrow p \wedge \sim q$

$p \rightarrow q \Leftrightarrow \sim p \vee q$

8. NORMAL FORM OF A WELL FORMED FORMULA

One of the main problem in logic is to determine whether the given statement is a tautology or a contradiction. One method to determine it is the method of truth tables. Other method is to reduce the statement form to, called normal form.
If P & Q are two propositional variables we get various well formed formula. The number of distinct truth values for formulas in P and Q is 2^4.

In general we have 2^n possible combinations of truth values of statements replacing the n variables

Here we give a method of reducing a given formula to an equivalent form called a 'normal form'. We use 'sum' for disjunction, 'product' for conjunction and 'literal' either for P or for ¬ P , where P is any propositional variable.

8.1. Elementary Product

A product of the variables and their negations in a formula is called an elementary product. If p and q are any two atomic variables, then $p, \sim p \wedge q, \neg q \wedge p, \ldots$, are some examples of elementary products.

8.2. Elementary Sum

A sum of the variables and their negations in a formula is called an elementary sum. If P and Q are any two atomic variables, then $q, \sim p \vee \neg q, p \vee q, \ldots$, are some examples of elementary sums.

8.3. Normal Forms

We can convert any proposition in two normal forms −

1. Conjunctive Normal Form (CNF)

2. Disjunctive Normal Form (DNF)

8.4. Conjunctive Normal Form

A compound statement is in conjunctive normal form if it is obtained by operating AND among variables (negation of variables included) connected with ORs.

Example: $(p \vee q) \wedge (p \vee r)$

8.5. Disjunctive Normal Form

A compound statement is in disjunctive normal form if it is obtained by operating OR among variables (negation of variables included) connected with ANDs.

Example: $(q \wedge r) \vee (\neg q \wedge r)$

Examples

1. Obtain Disjunctive normal form of $p \wedge (p \to q)$

Solution

Consider $p \wedge (p \to q) \Leftrightarrow p \wedge (\neg p \vee q)$ (Law of negation of conditional)

$\Leftrightarrow (p \wedge \neg p) \vee (p \wedge q)$ (Distributive law)

Which is a sum of elementary product

Thus, the required Disjunctive normal form of $p \wedge (p \to q)$ is $(p \wedge \neg p) \vee (p \wedge q)$

2. Obtain Conjunctive normal form of $\neg(p \to q) \vee (r \to p)$

Solution

Consider$\neg(p \to q) \vee (r \to p) \Leftrightarrow \neg(\neg p \vee q) \vee (\neg r \vee p)$ (Law of negation of conditional)

$\Leftrightarrow (p \wedge \neg q) \vee (\neg r \vee p)$ (Law of Double negation)

$\Leftrightarrow (p \vee \neg r \vee p) \wedge (\neg q \vee \neg r \vee p)$ (Distributive Law)

Which is a product of elementary sums

Thus, the required Conjunctive normal form of $\neg(p \to q) \vee (r \to p)$

is$(p \vee \neg r \vee p) \wedge (\neg q \vee \neg r \vee p)$

9. SELF-ASSESSMENT QUESTIONS

1. P.T $[(p \vee q) \wedge (p \vee \neg q)] \vee q \Leftrightarrow p \vee q$
2. Write negation of the following statement.
   i. $p = 2 + 3 > 1$.
   ii. It will rain tomorrow, or it will snow tomorrow.
   iii. p: It is cold.
3. Find the truth vale of each WFF if p & r is 1 and q is 0.
   i. $\sim p \wedge \sim q$.
   ii. $(\sim p \vee q) \wedge r$.
   iii. $p \vee q \vee r$.
   iv. $\sim(p \vee q) \wedge r$.
4. Indicate how many rows are needed is the truth table for the compound proposition $(p \vee \neg q) \leftrightarrow [(\neg r \wedge s) \rightarrow t]$. Find the truth value of WFF if p & r are true and q, s, & t are false.

10. SUMMARY

In this chapter we studied about Statements and Notations, Connectives, Normal Forms, Well-formed Formulas, Implication, Tautology.

11. ANSWERS TO SELF ASSESSMENT QUESTIONS

1. $LHS \Leftrightarrow [(p \vee q) \wedge (p \vee \neg q)] \vee q$

   By distributive law

   $LHS \Leftrightarrow \{p \vee [q \wedge \neg q]\} \vee q$

   By Inverse law

   $LHS \Leftrightarrow \{p \vee F_0\} \vee q$

   By Identity law

   $LHS \Leftrightarrow p \vee q \Leftrightarrow RHS$

2. $\sim p = 2 + 3 \leq 1.$

   It will not rain tomorrow, and it will not snow tomorrow.

   $\sim p$: *It is not cold*

3. Given $p = 1,\ q = 0,\ r = 1.$

   Thus $\sim p = 0, \sim q = 1,\ \sim r = 0.$

   i. $\sim p \wedge \sim q = 0 \wedge 1 = 0.$
   ii. $(\sim p \vee q) \wedge r = (0 \vee 0) \wedge 1 = 0 \wedge 1 = 0.$
   iii. $p \vee q \vee r = 1 \vee 0 \vee 1 = 1 \vee 1 = 1.$
   iv. $\sim(p \vee q) \wedge r = \sim(1 \vee 0) \wedge r = \sim(1) \wedge 1 = 0 \wedge 1 = 0.$

4. The give compound statement contains $n = 5$ propositions.

   $\therefore$ the number rows required in the truth table is $2^n = 2^5 = 32.$

   Consider

   $(p \vee \neg q) \leftrightarrow [(\neg r \wedge s) \rightarrow t] =$

   $= (1 \vee 1) \leftrightarrow [(0 \wedge 0) \rightarrow 0]$

   $= 1 \leftrightarrow [0 \rightarrow 0]$

$= 1 \leftrightarrow 1$

$= 1$

12. REFERENCES

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