MASTER OF COMPUTER APPLICATIONS

SEMESTER 1

DISCRETE MATHEMATICS AND GRAPH THEORY

Unit 10

Introduction to Boolean Algebra

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1. INTRODUCTION

Boolean algebra is the branch of algebra that deals with logical operations and binary variables. It uses only the binary numbers i.e. 0 and 1. It is also called as Binary Algebra or logical Algebra. Boolean algebra was invented by George Boole in 1854. Boolean Algebra is used to analyze and simplify the digital (logic) circuits. It has been fundamental in the development of digital electronics and is provided for in all modern programming languages. Boolean Algebra is a set of rules used to simplify the given logic expression without changing its functionality.

1.1. Objectives

The objective of this topic is to make the students to

- *Understand basic Operations.*
- *Apply various properties of Boolean algebra..*

2. DEFINITIONS AND EXAMPLES

2.1. Boolean Variable:

It is a variable that represents the logical quantities such as 0 or 1. This variables are also called literals.

2.2. Boolean Algebra Operations:

The basic operations of Boolean algebra are as follows:

- Conjunction or AND operation **(∧)**
- Disjunction or OR operation **(∨)**
- Negation or Not operation **(¬)**

Suppose A and B are two Boolean variables then we can define the three operations as;

- A conjunction B or A AND B, satisfies A ∧ B = True, if A = B = True or else A ∧ B = False.
- A disjunction B or A OR B, satisfies A ∨ B = False, if A = B = False, else A ∨ B = True.
- Negation A or ¬A satisfies ¬A = False, if A = True and ¬A = True if A = False

Below is the table defining the symbols for all three basic operations.

Operator	Symbol	Precedence
NOT	$` \; or \; \neg$	Highest
AND	$\cdot \; or \; \wedge$	Middle
OR	$+ \; or \; \vee$	Lowest

2.3. Boolean Expression :

A Boolean expression is **a logical statement that is either TRUE or FALSE** and always produces a Boolean value.

2.4. Truth Table :

Truth table is a table that contains all possible values of logical variables/statements in a Boolean expression.

No. of possible combinations = 2^n, where n=number of variables used in a Boolean expression.

2.5. Complement:

The complement is defined as the inverse of a variable, which is represented by a bar over the variable.

3. AXIOMS OF BOOLEAN ALGEBRA

3.1. AND Operator :

It performs logical multiplication and denoted by (.) dot. Let X and Y be two Boolean variables, then X.Y returns true only when X is true and Y is true, rest all other cases it returns false.

AXIOM	X	Y	X.Y
Axiom 1	0	0	0
Axiom 2	0	1	0
Axiom 3	1	0	0
Axiom 4	1	1	1

3.2. OR Operator :

It performs logical addition and denoted by (+) plus. Let X and Y be two Boolean variables, then X+Y returns true only when either X is true or Y is true, returns false, when both are false.

Axiom	X	Y	X+Y
Axiom 5	0	0	0
Axiom 6	0	1	1
Axiom 7	1	0	1
Axiom 8	1	1	1

3.3. NOT or INVERTER Operator :

It performs logical negation and denoted by (-) bar or (‘) prime symbols. It operates on single variable.

Axiom	X	$\overline{X}$ or X'
Axiom 9	0	1
Axiom 10	1	0

4. BASIC POSTULATES OR LAWS OF BOOLEAN ALGEBRA

Based on these axioms we can conclude many laws of Boolean Algebra which are listed below,

1. **Commutative Laws**

$$A + B = B + A$$

A	**B**	**A + B**
0	0	0
0	1	1
1	0	1
1	1	1

=

B	**A**	**B + A**
0	0	0
0	1	1
1	0	1
1	1	1

$$A.B = B.A$$

A	**B**	**A . B**
0	0	0
0	1	0
1	0	0
1	1	1

=

B	**A**	**B . A**
0	0	0
0	1	0
1	0	0
1	1	1

2. **Associative Laws**

$$(A + B) + C = A + (B + C)$$

A	**B**	**C**	**A + B**	**(A+ B)+ C**		**A**	**B**	**C**	**B + C**	**A+(B+ C)**
0	0	0	0	0		0	0	0	0	0
0	0	1	0	1	=	0	0	1	1	1
0	1	0	1	1		0	1	0	1	1
0	1	1	1	1		0	1	1	1	1
1	0	0	1	1		1	0	0	0	1
1	0	1	1	1		1	0	1	1	1
1	1	0	1	1		1	1	0	1	1

1	1	1	1	1		1	1	1	1	1

$$(A.B).C = A.(B.C)$$

A	B	C	AB	(AB) C		A	B	C	B.C	A(B.C)
0	0	0	0	0		0	0	0	0	0
0	0	1	0	0		0	0	1	0	0
0	1	0	0	0	=	0	1	0	0	0
0	1	1	0	0		0	1	1	1	0
1	0	0	0	0		1	0	0	0	0
1	0	1	0	0		1	0	1	0	0
1	1	0	1	0		1	1	0	0	0
1	1	1	1	1		1	1	1	1	1

3. AND Laws

$$A.0 = 0$$

$$A.1 = A$$

$$A.A = A$$

$$A.A = 0$$

4. OR Laws

$$A + 0 = A$$

$$A + 1 = 1$$

$$A + A = A$$

$$A + A = 0$$

5. Complementation Laws

$$If\ A = 0\ then\ \bar{A} = 1$$

$$If\ A = 1\ then\ \bar{A} = 0$$

$$\bar{\bar{A}} = A$$

6. **Distributive Laws**

$$A(B+C) = AB + AC$$

$$A + BC = (A+B).(A+C)$$

7. **Idempotence Law**

$\boldsymbol{A.A} = \boldsymbol{A}, If\ A = 1, then\ A.A = 1.1 = 1 = A\ and\ if\ A = 0, then\ A.A = 0.0 = 0 = A$

$\boldsymbol{A+A} = \boldsymbol{A}, If\ A = 1, then\ A + A = 1+1 = 1 = A\ and\ if\ A = 0, then\ A + A = 0+0 = 0 = A$

8. **Absorption Law**

$$A + A.B = A$$

$$A.(A+B) = A$$

9. **De-Morgan's Law**

$$\overline{A+B} = \bar{A}.\bar{B}$$

A	B	$\overline{A+B}$	$\bar{A}$	$\bar{B}$	$\bar{A}.\bar{B}$
0	0	1	1	1	1
0	1	0	1	0	0
1	0	0	0	1	0
1	1	0	0	0	0

$$\overline{A.B} = \bar{A} + \bar{B}$$

A	B	$\overline{AB}$	$\bar{A}$	$\bar{B}$	$\bar{A}+\bar{B}$
0	0	1	1	1	1
0	1	1	1	0	1
1	0	1	0	1	1
1	1	0	0	0	0

10. **Consensus Theorem**

$$AB + \bar{A}C + BC = AB + \bar{A}C$$

$$(A+B)(\bar{A} + C)(B+C) = (A+B)(\bar{A} + C)$$

Minimization of Boolean Algebra

Example: Using Boolean algebra techniques, simplify the following expression $AB + A(B + C) + B(B + C)$

Solution:

$$AB + A(B + C) + B(B + C) \Rightarrow AB + AB + AC + BB + BC[DistributiveLaw]$$
$$\Rightarrow AB + AB + AC + B + BC[IdempotentLaw]$$
$$\Rightarrow AB + AC + B + BC[IdempotentLaw] \Rightarrow AB + AC + B[AbsorptionLaw]$$
$$\Rightarrow B + BA + AC[CommutativeLaw] \Rightarrow B + AC[AbsorptionLaw]$$

4.1. Boolean Functions

A Boolean function is a special kind of mathematical function $f:X^n \rightarrow X$
of degree n, where X={0,1} is a Boolean domain and n is a non-negative integer. It describes the way how to derive Boolean output from Boolean inputs.

Example – Let, $F(A,B)=A'B'$ This is a function of degree 2 from the set of ordered pairs of Boolean variables to the set {0,1} where F(0,0)=1, F(0,1)=0, F(1,0)=0 and F(1,1)=0

5. SELF-ASSESSMENT QUESTIONS

SA1: *Using Boolean algebra techniques, simplify the following expression*

$$AB + A(B + C) + B(B + C)$$

Solution

$$\begin{aligned} AB + A(B + C) + B(B + C) &\Rightarrow AB + AB + AC + BB + BC[DistributiveLaw] \\ &\Rightarrow AB + AB + AC + B + BC[IdempotentLaw] \\ &\Rightarrow AB + AC + B + BC[IdempotentLaw] \\ &\Rightarrow AB + AC + B[AbsorptionLaw] \\ &\Rightarrow B + BA + AC[CommutativeLaw] \\ &\Rightarrow B + AC[AbsorptionLaw] \end{aligned}$$

SA2: Using Boolean algebra techniques, simplify the following expression $xyz + x\overline{y}z$

Solution:

$$\begin{aligned} xyz + x\overline{y}z &\Rightarrow yxz + \overline{y}xz[AssociativeLaw] \\ &\Rightarrow (y + \overline{y})(xz)[DistributiveLaw] \\ &\Rightarrow 1.(xz)[ComplementLaw] \\ &\Rightarrow xz \end{aligned}$$

SA3: Using Boolean algebra techniques, simplify the following expression $\overline{XY}Z + \overline{X}YZ + X\overline{Y}$.

Solution:

$$\begin{aligned} \overline{XY}Z + \overline{X}YZ + X\overline{Y} &\Rightarrow \overline{XY}Z + \overline{X}YZ + X\overline{Y} \\ &\Rightarrow \overline{X}Z\left(\overline{Y} + Y\right) + X\overline{Y}[AssociativeLaw] \\ &\Rightarrow \overline{X}Z(1) + X\overline{Y}[ComplementLaw] \\ &\Rightarrow \overline{X}Z + X\overline{Y} \end{aligned}$$

SA4: Using Boolean algebra techniques, prove **Let us prove A +BC = (A+B).(A+C)**

Solution

Let's try to simplify RHS of the expression:

(A+B).(A+C) = A.A + A.C + B.A + B.C

We know that A.A = A and B.A = A.B so the expression becomes:

A + AC + A.B + B.C

On further Simplifying A.(1+C) + A.B + B.C ,now 1+C = 1 also, A.1 = A, so the expression becomes:

A + A.B + B.C

A(1+B) + B.C = A.1 + B.C = **A + BC** (1+B = 1 and A.1 = 1)

7. SUMMARY

The axioms of Boolean algebra will be discussed in this section. These axioms/theorems are crucial because they are employed in many different areas of digital electronics, such as sequential circuit design and combinational circuit design. The foundation of digital electronics is comprised of these axioms.

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