MASTER OF COMPUTER APPLICATIONS

SEMESTER 1

DISCRETE MATHEMATICS AND GRAPH THEORY

Unit 1

Matrices and Determinants

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1. INTRODUCTION

Sylvester introduced the word "matrix" for the first time in 1850. He described a matrix as a word arrangement. A matrix algebra defining addition, multiplication, scalar multiplication, and inverses was described by Cayley in 1858. Understanding matrices is crucial as they are applicable to almost every area of mathematics. In social accounting, input-output tables, and the study of inter-industry economics, matrices are used by economists. Additionally, matrices are used in the study of electrical engineering's network analysis and communication theory.

Matrices organize numbers in rows and columns, crucial for solving equations and analyzing data. Determinants, derived from square matrices, reveal key properties like invertibility and scaling in transformations. Both are essential in various fields, enabling efficient problem-solving and understanding complex systems.

1.1. Objectives:

At studying this unit, you should be able to:

- *Understand basics of Matrices and types of matrices.*
- *Understand determinant and its properties.*
- *Apply the concepts of matrices to solve the system of equation.*

2. DEFINITIONS AND EXAMPLES

2.1. Matrix

A *matrix* is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix. We denote matrices by capital letters.

Example $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

2.2. Order of a matrix

A matrix having m rows and n columns is called a matrix of *order* $m \times n$ or simply $m \times n$ matrix (read as an m by n matrix)

So referring to the above example A as 3 × 3 matrix

In general, an $m \times n$ matrix has the following rectangular array

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Or $A = [a_{ij}]_{m\times n}, 1 \le i \le m, 1 \le j \le n;\ \ i, j \in N$

In general aij, is an element lying in the ith row and jth column. We can also call it as the (i, j)th element of A. The number of elements in an m × n matrix will be equal to mn.

Problem: Construct a 3 × 2 matrix whose entries are given by aij = i − 2j

Solution: The general 3 × 2 matrix is of the form

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \text{ where } i = 1, 2, 3 \text{ (rows)}, \quad j = 1, 2 \text{ (columns)}$$

It is given that $a_{ij} = i - 2j$

$a_{11} = 1 - 2 = -1 \quad a_{12} = 1 - 4 = -3$

$a_{21} = 2 - 2 = 0 \quad a_{22} = 2 - 4 = -2$

$a_{31} = 3 - 2 = 1 \quad a_{32} = 3 - 4 = -1$

$\therefore$ The required matrix is $A = \begin{bmatrix} -1 & -3 \\ 0 & -2 \\ 1 & -1 \end{bmatrix}$

3. TYPES OF MATRICES

3.1. Row matrix

A matrix having only one row is called a row matrix or a row vector.

Examples

(i) $A = [aij]_{1\times3}$ = [1 − 5 4] is a row matrix of order 1 × 3.

(ii) B = [bij]1 × 2 = [5 10] is a row matrix of order 1 × 2

3.2. Column matrix

A matrix having only one column is called a column matrix or a column vector.

Examples (i) $A = [aij]_{3\times1} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$

Note : Any matrix of order 1 × 1 can be treated as either a row matrix or a column matrix.

3.3. Square matrix

A square matrix is a matrix in which the number of rows and the number of columns are equal. A matrix of order n × n is also known as a square matrix of order n.

In a square matrix A of order n × n, the elements a11, a22, a33 … ann are called principal diagonal or leading diagonal or main diagonal elements.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Note: In general the number of elements in a square matrix of order n is n^2.

3.4. Diagonal Matrix

A square matrix $A = \left[a_{ij}\right]_{m\times n}$ is said to be a diagonal matrix if *aij=0* when *i ≠ j*

In a diagonal matrix all the entries except the entries along the main diagonal are zero.

Example: $A = [aij]_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix

3.5. Triangular matrix

A square matrix in which all the entries above the main diagonal are zero is called a lower triangular matrix. If all the entries below the main diagonal are zero, it is called an upper triangular matrix.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ is an upper triangular matrix and $B = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 2 & 0 \\ 6 & 8 & 3 \end{bmatrix}$ is a lower triangular matrix.

3.6. Scalar matrix

A square matrix $A = [a_{ij}]_{m\times n}$ is said to be scalar matrix if $a_{ij} = \begin{cases} a & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

i.e. A scalar matrix is a diagonal matrix in which all the entries along the main diagonal are equal.

Example: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

3.7. Identity matrix or unit matrix

A square matrix $A = [a_{ij}]_{m\times n}$ is said to be an identity matrix if $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

i.e. An identity matrix or a unit matrix is a scalar matrix in which entries along the main diagonal are equal to 1. We represent the identity matrix of order *n* as I*n*

Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3.8. Zero matrix or null matrix or void matrix

A matrix A is said to be a zero matrix or null matrix if all the entries are zero, and is denoted by O i.e. *aij* = 0 for all the values of *i*, *j*

Example: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3.9. Equality of Matrices

Two matrices A and B are said to be equal if

(i) both the matrices A and B are of the same order or size.

(ii) the corresponding entries in both the matrices A and B are equal.

Example: if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$ then find the values of *x, y, z, w.*

Solution:

Since the two matrices are equal, their corresponding entries are also equal. a=5, b=6, c=8, d=9

3.10. Transpose of a matrix

The matrix obtained from the given matrix A by interchanging its rows

into columns and its columns into rows is called the transpose of A and it is

denoted by A′ or A^T.

Example:

If $A = \begin{bmatrix} 4 & -3 \\ 2 & 0 \\ 1 & 5 \end{bmatrix}$ then $A^T = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 0 & 5 \end{bmatrix}$

Note that if A is of order $m \times n$ then A^T is order $n \times m$.

3.11. Symmetric matrix

A square matrix A = [*aij*] is said to be *symmetric* if A\`= A, that is, [*aij*] = [*aji*] for all possible values of *i* and *j*.

For example $A = \begin{bmatrix} -9 & 8 & -3 \\ 8 & 5 & 6 \\ -3 & 6 & \sqrt{2} \end{bmatrix}$ is a symmetric matrix as A\`= A

3.12. Skew Symmetric Matrix

A square matrix A = [*aij*] is said to be *skew symmetric* matrix if A\`= – A, that is *aji* = – *aij* for all possible values of *i* and *j*. Now, if we put *i* = *j*, we have *aii* = – *aii*. Therefore 2*aii* = 0 or *aii* = 0 for all *i*'s.

This means that all the diagonal elements of a skew symmetric matrix are zero.

For example, the matrix $A = \begin{bmatrix} 0 & 9 & a \\ -9 & 0 & -\sqrt{2} \\ -a & \sqrt{2} & 0 \end{bmatrix}$ is a skew symmetric matrix

4. OPERATIONS ON MATRICES

4.1. Addition and subtraction

Two matrices A and B can be added provided both the matrices are of the same order and their sum A + B is obtained by adding the corresponding entries of both the matrices A and B

$$\text{i.e. } A = [a_{ij}]_{m \times n} \text{ and } B = [b_{ij}]_{m \times n} \quad \text{then} \quad A + B = [a_{ij} + b_{ij}]_{m \times n}$$

$$\text{Similarly} \quad A - B = A + (-B) = [a_{ij}]_{m \times n} + [-b_{ij}]_{m \times n}$$

$$= [a_{ij} - b_{ij}]_{m \times n}$$

Note:

(1) The matrices A + B and A – B have same order equal to the order of A or B.

(2) Subtraction is treated as negative addition.

(3) The additive inverse of matrix A is – A.

Example:

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ & $B = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

Then

$$A+B=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}+\begin{bmatrix}6&5&4\\3&2&1\end{bmatrix}=\begin{bmatrix}1+6&2+5&3+4\\4+3&5+2&6+1\end{bmatrix}=\begin{bmatrix}7&7&7\\7&7&7\end{bmatrix}$$

$$A-B=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}-\begin{bmatrix}6&5&4\\3&2&1\end{bmatrix}=\begin{bmatrix}1-6&2-5&3-4\\4-3&5-2&6-1\end{bmatrix}=\begin{bmatrix}-5&-3&-1\\1&3&5\end{bmatrix}$$

4.1.1. Properties of matrix addition

The addition of matrices satisfy the following properties:

Commutative Law If A = [aij], B = [bij] are matrices of the same order, say m × n,

then A + B = B + A.

Associative Law For any three matrices A = [aij], B = [bij], C = [cij] of the same order, say m × n, (A + B) + C = A + (B + C).

Existence of additive identity Let A = [aij] be an m × n matrix and O be an m × n zero matrix, then A + O = O + A = A. In other words, O is the additive identity for matrix addition.

The existence of additive inverse Let A = [aij] m × n be any matrix, then we have another matrix as – A = [– aij] m × n such that A + (– A) = (– A) + A= O. So – A is the additive inverse of A or negative of A.

4.1.2. Multiplication of a matrix by a scalar

Let A be any matrix. Let k be any non-zero scalar. The matrix kA is obtained by multiplying all the entries of matrix A by the non zero scalar k.

i.e. $A = [aij]_{m\times n} \Rightarrow kA = [kaij]_{m\times n}$

This is called scalar multiplication of a matrix.

Note: If a matrix A is of order m × n then the matrix kA is also of the same

order m × n

Example:

Let $A=\begin{bmatrix}2&1&4\\7&3&6\end{bmatrix}_{2\times3}$ $then\ 3A=\begin{bmatrix}6&3&12\\21&9&18\end{bmatrix}_{2\times3}$

4.1.3. Matrix multiplication

Two matrices A and B are said to be conformable for multiplication if the number of columns of the first matrix A is equal to the number of rows of the second matrix B. The product matrix 'AB' is acquired by multiplying every row of matrix A with the corresponding elements of every column of matrix B element-wise and add the results. This procedure is known as row-by-column multiplication rule.

Let A be a matrix of order m × n and B be a matrix of order n × p then the product matrix AB will be of order m × p

i.e. order of A is m × n, order of B is n × p

Then the order of AB is m × p = number of rows of matrix A ×number of columns of matrix B

Example:

Let $A = \begin{bmatrix} 2 & 1 & 4 \\ 7 & 3 & 6 \end{bmatrix}_{2\times 3}$ & $B = \begin{bmatrix} 6 & 4 & 3 \\ 3 & 2 & 5 \\ 7 & 3 & 1 \end{bmatrix}_{3\times 3}$

It is to be noted that the number of columns of the first matrix A is equal to the number of rows of the second matrix B.

∴ Matrices A and B are conformable, i.e. the product matrix AB can be found.

$$AB = \begin{bmatrix} 2 & 1 & 4 \\ 7 & 3 & 6 \end{bmatrix} \begin{bmatrix} 6 & 4 & 3 \\ 3 & 2 & 5 \\ 7 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \begin{matrix} 2 & 1 & 4 \end{matrix} \begin{matrix} 6 \\ 3 \\ 7 \end{matrix} & \begin{matrix} 2 & 1 & 4 \end{matrix} \begin{matrix} 4 \\ 2 \\ 3 \end{matrix} & \begin{matrix} 2 & 1 & 4 \end{matrix} \begin{matrix} 3 \\ 5 \\ 1 \end{matrix} \\ \begin{matrix} 7 & 3 & 6 \end{matrix} \begin{matrix} 6 \\ 3 \\ 7 \end{matrix} & \begin{matrix} 7 & 3 & 6 \end{matrix} \begin{matrix} 4 \\ 2 \\ 3 \end{matrix} & \begin{matrix} 7 & 3 & 6 \end{matrix} \begin{matrix} 3 \\ 5 \\ 1 \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} (2)(6)+(1)(3)+(4)(7) & (2)(4)+(1)(2)+(4)(3) & (2)(3)+(1)(5)+(4)(1) \\ (7)(6)+(3)(3)+(6)(7) & (7)(4)+(3)(2)+(6)(3) & (7)(3)+(3)(5)+(6)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 12+3+28 & 8+2+12 & 6+5+4 \\ 42+9+42 & 28+6+18 & 21+15+6 \end{bmatrix} \qquad \therefore\ AB = \begin{bmatrix} 43 & 22 & 15 \\ 93 & 52 & 42 \end{bmatrix}$$

Problem:

Find AB, if $A = \begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{bmatrix}$

Solution

$$AB = \begin{bmatrix} 6(2) + 9(7) & 6(6) + 9(9) & 6(0) + 9(8) \\ 2(2) + 3(7) & 2(6) + 3(9) & 2(0) + 3(8) \end{bmatrix}$$

$$AB = \begin{bmatrix} 75 & 117 & 72 \\ 25 & 39 & 24 \end{bmatrix}$$

Note:

If AB is defined, then BA need not be defined. In the above example, AB is defined but BA is not defined because B has 3 column while A has only 2 (and not 3) rows. If A, B are, respectively m × n, k × l matrices, then both AB and BA are defined if and only if n = k and l = m. In particular, if both A and B are square matrices of the same order, then both AB and BA are defined.

5. NON-COMMUTATIVITY OF MULTIPLICATION OF MATRICES

Now, we shall see by an example that even if AB and BA are both defined, it is not necessary that AB = BA.

Example:

If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$ then find AB, BA. Show that AB ≠ BA.

Solution

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 - 8 + 6 & 3 - 10 + 3 \\ -8 + 8 + 10 & -12 + 10 + 5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 - 12 & -4 + 6 & 6 + 15 \\ 4 - 20 & -8 + 10 & 12 + 25 \\ 2 - 4 & -4 + 2 & 6 + 5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Clearly $AB \neq BA$

Note:

- In the above example both AB and BA are of different order and so AB ≠ BA. But one may think that perhaps AB and BA could be the same if they were of the same order. But it is not.
- If both matrix A & B are diagonal Matrix of same order then it satisfies the commutative property of multiplication. i.e., AB=BA

6. ZERO MATRIX AS THE PRODUCT OF TWO NON ZERO MATRICES

We know that, for real numbers a, b if ab = 0, then either a = 0 or b = 0. This need not be true for matrices, we will observe this through an example.

Example:

Find AB, if $A = \begin{bmatrix} 0 & -10 \\ 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$

Solution

We have $AB = \begin{bmatrix} 0 & -10 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus, if Thus, if the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix

6.1. Properties of multiplication of matrices

The multiplication of matrices possesses the following properties,

1. The associative law For any three matrices A, B and C. We have (AB) C = A (BC), whenever both sides of the equality are defined.

2. The distributive law For three matrices A, B and C.

(i) A (B+C) = AB + AC

(ii) (A+B) C = AC + BC, whenever both sides of equality are defined.

3. The existence of multiplicative identity For every square matrix A, there exist an identity matrix of same order such that IA = AI = A.

7. INVERTIBLE MATRICES

If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that AB = BA = I, then B is called the inverse matrix of A and it is denoted by A^{-1}. In that case A is said to be invertible.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \ \& \ B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Also

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus B is the inverse of A, in other words $B = A^{-1}$ and A is inverse of B, i.e., $A = B^{-1}$

Note:

1. A rectangular matrix does not possess inverse matrix, since for products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order.

2. If B is the inverse of A, then A is also the inverse of B.

7.1. Theorem: Uniqueness of inverse

Statement: Inverse of a square matrix, if it exists, is unique.

Proof

Let A = [aij] be a square matrix of order m. If possible, let B and C be two inverses of A. We shall show that B = C.

Since B is the inverse of A

AB = BA = I

Since C is also the inverse of A

AC = CA = I

Thus B = BI = B (AC) = (BA) C = IC = C

8. DETERMINANTS

8.1. Determinants

To every square matrix A of order n with entries as real or complex numbers, we can associate a number called determinant of matrix A and it is denoted by | A | or det (A) or ∆.

Thus, determinant formed by the elements of A is said to be the determinant of matrix A.

If $A = \begin{bmatrix} a11 & a12 \\ a21 & a22 \end{bmatrix}$the its $|A| = \begin{vmatrix} a11 & a12 \\ a21 & a22 \end{vmatrix} = a11a22 - a21a12$

To evaluate the determinant of order 3 or above we define minors and cofactors.

8.2. MINORS

Let | A | = |[aij]| be a determinant of order n. The minor of an arbitrary element aij is the determinant obtained by deleting the ith row and jth column in which the element aij stands. The minor of aij is denoted by Mij.

8.3. Cofactors

The cofactor is a signed minor. The cofactor of aij is denoted by Aij and is defined as Aij = (_1)$^{i+j}$ Mij.

The minors and cofactors of a11, a12, a13 of a third order determinant

$\begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix}$ are as follows

(i) Minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}\, a_{23}$.

Cofactor of a_{11} is $A_{11} = (-1)^{1+1}\ M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}\, a_{23}$

(ii)Minor of a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\, a_{33} - a_{31}a_{23}$

Cofactor of a_{12} is $A_{12} = (-1)^{1+2}\, M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -\,(a_{21}a_{33} - a_{23}\, a_{31})$

(iii) Minor of a_{13} is $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\, a_{32} - a_{31}\, a_{22}$

Cofactor of a_{13} is $A_{13} = (-1)^{1+3}\ M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\, a_{32} - a_{31}\, a_{22}$

Note: A determinant can be expanded using any row or column as given below:

$$A = \begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix}$$

$\Delta = a_{11}\ A_{11} + a_{12}\ A_{12} + a_{13}\ A_{13}$ or $a_{11}\ M_{11} - a_{12}\ M_{12} + a_{13}\ M_{13}$ (expanding by R_1)

$\Delta = a_{11}\ A_{11} + a_{21}A_{21} + a_{31}\ A_{31}$ or $a_{11}\ M_{11} - a_{21}\ M_{21} + a_{31}\ M_{31}$ (expanding by C_1)

$\Delta = a_{21}\ A_{21} + a_{22}\ A_{22} + a_{23}\ A_{23}$ or $-\,a_{21}\ M_{21} + a_{22}\ M_{22} - a_{23}\ M_{23}$ (expanding by R_2)

Problem

Find the minor and cofactor of each element of the determinant $\begin{vmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 5 & -2 & 6 \end{vmatrix}$

Solution

Minor of 3 is $M_{11} = \begin{vmatrix} -1 & 2 \\ -2 & 6 \end{vmatrix} = -6 + 4 = -2$

Minor of 4 is $M_{12} = \begin{vmatrix} 0 & 2 \\ 5 & 6 \end{vmatrix} = 0 - 10 = -10$

Minor of 1 is $M_{13} = \begin{vmatrix} 0 & -1 \\ 5 & -2 \end{vmatrix} = 0 + 5 = 5$

Minor of 0 is $M_{21} = \begin{vmatrix} 4 & 1 \\ -2 & 6 \end{vmatrix} = 24 + 2 = 26$

Minor of – 1 is $M_{22} = \begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix} = 18 - 5 = 13$

Minor of 2 is $M_{23} = \begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} = -6 - 20 = -26$

Minor of 5 is $M_{31} = \begin{vmatrix} 4 & 1 \\ -1 & 2 \end{vmatrix} = 8 + 1 = 9$

Minor of – 2 is $M_{32} = \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6$

Minor of 6 is $M_{33} = \begin{vmatrix} 3 & 4 \\ 0 & -1 \end{vmatrix} = -3 - 0 = -3$

Cofactor of 3 is $A_{11} = (-1)^{1+1} M_{11} = M_{11} = -2$

Cofactor of 4 is $A_{12} = (-1)^{1+2} M_{12} = -M_{12} = 10$

Cofactor of 1 is $A_{13} = (-1)^{1+3} M_{13} = M_{13} = 5$

Cofactor of 0 is $A_{21} = (-1)^{2+1} M_{21} = -M_{21} = -26$

Cofactor of – 1 is $A_{22} = (-1)^{2+2} M_{22} = M_{22} = 13$

Cofactor of 2 is $A_{23} = (-1)^{2+3} M_{23} = -M_{23} = 26$

Cofactor of 5 is $A_{31} = (-1)^{3+1} M_{31} = M_{31} = 9$

Cofactor of – 2 is $A_{32} = (-1)^{3+2} M_{32} = -M_{32} = -6$

Cofactor of 6 is $A_{33} = (-1)^{3+3} M_{33} = M_{33} = -3$

Note:

A square matrix A is said to be singular if $|A| = 0$

A square matrix A is said to be non-singular matrix, if $|A| \neq 0$.

Problem

Expand the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Solution

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45 - 48) - 2(36 - 42) + 3(32 - 35)$$

$$= -3 + 12 - 9 = 0$$

9. PROPERTIES OF DETERMINANTS

1. The value of a determinant is unaltered by interchanging its rows and columns.
2. If any two rows (columns) of a determinant are interchanged the determinant changes its sign but its numerical value is unaltered.

Note:

The sign of a determinant changes or does not change according as there is an odd or even number of interchanges among its rows (columns).

3. If two rows (columns) of a determinant are identical then the value of the determinant is zero.
4. If every element in a row (or column) of a determinant is multiplied by a constant "k" then the value of the determinant is multiplied by k.

Note:

1) Let A be any square matrix of order n. Then kA is also a square matrix which is obtained by multiplying every entry of the matrix A with the scalar k. But the determinant k |A| means every entry in a row (or a column) is multiplied by the scalar k.
2) Let A be any square matrix of order n then | kA | = kn| A |.
3) If two rows (columns) of a determinant are proportional i.e. one row (column) is a scalar multiple of other row (column) then its value is zero.

5. If every element in any row (column) can be expressed as the sum of two quantities then given determinant can be expressed as the sum of two determinants of the same order with the elements of the remaining rows (columns) of both being the same.

   Note: If we wish to add (or merge) two determinants of the same order we add corresponding entries of a particular row (column) provided the other entries in rows (columns) are the same.

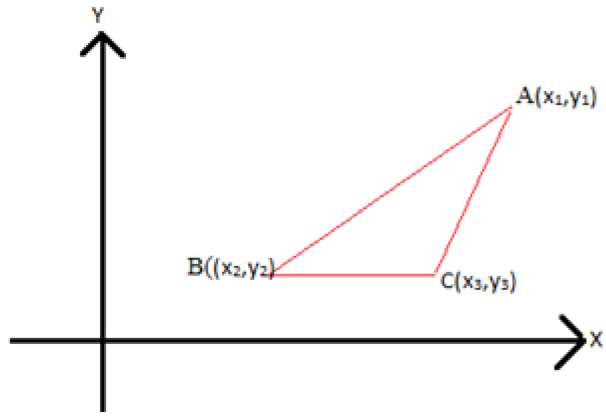
6. A determinant is unaltered when to each element of any row (column) is added to those of several other rows (columns) multiplied respectively by constant factors. i.e. A determinant is unaltered when to each element of any row (column) is added by the equimultiples of any parallel row (column).

   Example: A= $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

SI. No.	Elementary row transformation	Notation	Resultant of a matrix A
1	Interchange of first and second row	$R_1 \leftrightarrow R_2$	$\begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$
2	Multiplication of third row by a constant k	kR_3	$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ kc_1 & kc_2 & kc_3 \end{bmatrix}$
3	Addition to second row k times the first row	$R_2 \to kR_1 + R_2$	$\begin{bmatrix} a_1 & a_2 & a_3 \\ (ka_1 + b_1) & (ka_2 + b_2) & (ka_3 + b_3) \\ c_1 & c_2 & c_3 \end{bmatrix}$

9.1. Applications of determinants in finding the area of a triangle

Suppose we are given three points in the Cartesian plane as $(x_1, y_1), (x_2, y_2) \& (x_3, y_3)$ which represent the vertices of the triangle as shown below.



The area of the triangle obtained by joining these points is given by,

$$Area(\alpha) = ½\,[\, x_1\,(\,y_2 - y_3\,) + x_2\,(\,y_3 - y_1\,) + x_3\,(\,y_1 - y_2\,)\,]$$

The formula for finding area could be represented in the form of determinants as given below.

$$\alpha = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Example: Find the area of the triangle whose vertices are A (1, 1), B (4, 2), and C (3, 5)

Solution:

Area of triangle is given $\alpha = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$\alpha = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 3 & 5 & 1 \end{vmatrix}$$

α = ½ (1 (2 - 5) - 4 (4 - 3) + 3 (20 - 3))

α = ½ (1 (-3) -4 (1) + 3 (17))

α = ½ (- 3 - 4 + 51)

α = ½ (44)

α = 22 units.

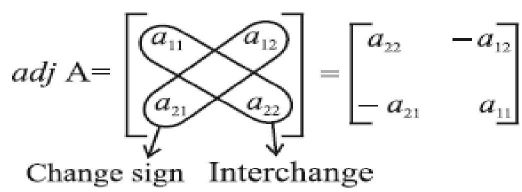
10. ADJOINT AND INVERSE OF A MATRIX

The adjoint of a square matrix A = $[a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by adj A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then adj A= transpose of $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Note: 1) For a square matrix of order 2, given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ The adj A can also be obtained by interchanging a_{11} and a_{22} and by changing signs of a_{12} and a_{21}, i.e.,



2) A (adj A) = (adj A) A =|A| I

Problem: Find Adj A for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

Problem: If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ then verify that adjA=|A|I. also find A^{-1}.

Solution

We have |A| = 1 (16 – 9) –3 (4 – 3) + 3 (3 – 4) = 1 ≠0

Now A11 = 7, A12 = –1, A13 = –1, A21 = –3, A22 = 1,A23 = 0, A31 = –3, A32 = 0,

A33 = 1

Therefore, adj A=$\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Now A.(adj A)= $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} . \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

A.(adj A)= $\begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix} =$

A.(adj A)= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 = |A|I$

$$|A|^{-1} = \frac{1}{|A|} adj\, A = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

11. SELF-ASSESSMENT QUESTIONS

1. Evaluate the determinant $A = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$
2. Using properties of determinant and without expanding prove that

$$\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$$

3. Find the area of a triangle by determinant method whose vertices are A (4, 9), B (- 3, 3), and C (6, 2)
4. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

12. SUMMARY

Matrices are arrays of numbers enclosed within brackets. They can be represented using concepts and notations such as order, equality, and types like zero and identity matrices. The transpose of a matrix involves switching its rows and columns. Symmetric matrices remain unchanged under transpose, while skew-symmetric matrices change sign. Matrices support operations like addition, multiplication, and scalar multiplication, but multiplication isn't commutative, and non-zero matrices can produce the zero matrix when multiplied, especially square matrices of order 2. Invertible matrices have unique inverses if they exist, crucial for solving systems of equations. Determinants, particularly for up to 3x3 matrices, signify scaling factors in linear transformations, with properties including the use of minors and cofactors. Determinants have applications in finding areas of triangles.

The adjoint of a matrix is obtained by taking the transpose of its cofactor matrix, useful in solving systems of linear equations and finding inverses. The inverse of a square matrix is found using the adjoint and determinant.

Understanding these concepts and operations is fundamental in various fields such as engineering, physics, and computer science. They underpin linear transformations, system solutions, and geometric calculations.

13. ANSWERS TO SELF ASSESSMENT QUESTIONS

1. It is observed that second row is two entries are zero thus by the properties of determinants we can expand along second row

   Thus, $|A| = -0\begin{vmatrix}-1 & -2\\ -5 & 0\end{vmatrix} + 0\begin{vmatrix}3 & -2\\ 3 & 0\end{vmatrix} - (-1)\begin{vmatrix}3 & -1\\ 3 & 5\end{vmatrix} = -12$

2. Applying R1 ->R1 + R2 to Δ, we get

   $$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

   Since the R1 and R3 elements are proportional $\Delta = 0$

3. Area of triangle is given $\alpha = \frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$

   $\alpha = \dfrac{1}{2}\begin{vmatrix}4 & 9 & 1\\ -3 & 3 & 1\\ 6 & 2 & 1\end{vmatrix}$

   α = ½ (4 (3 - 2) - 9 (-3 - 6) + 1 (- 6 - 18))

   α = ½ (4 (1) - 9 (- 9) +1 (- 24))

   α = ½ (4 + 81 - 24)

   α = ½ (61)

   α = 30.5 units

4.

   We have,

   $|A| = 1(-3-0) - 0 + 0 = -3$

   Now,

   $A_{11} = -3-0 = -3, A_{12} = -(-3-0) = 3, A_{13} = 6-15 = -9$

   $A_{21} = -(0-0) = 0, A_{22} = -1-0 = -1, A_{23} = -(2-0) = -2$

   $A_{31} = 0-0 = 0, A_{32} = -(0-0) = 0, A_{33} = 3-0 = 3$

   $$\therefore adjA = \begin{bmatrix}-3 & 0 & 0\\ 3 & -1 & 0\\ -9 & -2 & 3\end{bmatrix}$$

   $$\therefore A^{-1} = \frac{1}{|A|}adjA = -\frac{1}{3}\begin{bmatrix}-3 & 0 & 0\\ 3 & -1 & 0\\ -9 & -2 & 3\end{bmatrix}$$

14. REFERENCES

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