MASTER OF COMPUTER APPLICATIONS

SEMESTER 1

DISCRETE MATHEMATICS AND GRAPH THEORY

Unit 5

Introduction to Functions

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1. INTRODUCTION

It is important to distinguish between a function's description and its execution. The function is the nebulous mathematical entity that, whether or not it is ever discussed, still exists. However, we need a means to explain the function when we do wish to discuss it. There are several methods to describe a certain function. The Rule of Four, which states that any function may be expressed in one of four different ways—algebraically (as a formula), numerically (as a table), graphically, or orally—is discussed in several calculus textbooks.

This Section examines a particular kind of connection known as a function. It is among the most important ideas in mathematics. A function may be thought of as a rule that creates new elements out of certain existing ones. The word "map" or "mapping" is one of several that are used to describe a function

1.1. Objectives

The objective of this topic is to make the students to

- ❖ *Understand basics of function and various types of function*.

2. PRE-REQUISITES

2.1. Cartesian Product

Let A and B be any two non-empty sets. The Cartesian product (or) cross product of A and B denoted by $A \times B$ is set of all ordered pairs (a, b) where a belongs to A and b belongs to B.

$$A \times B = \{(x, y): x \in A,\ y \in B\}.$$

Note:

If A and B are finite then the Cardinality of $|A \times B| = |A| \times |B|$.

Example: Let $A = \{1,2,3\}$ and B= $\{p, q, r\}$, then

$$A \times B = \{(1, p), (1, q),\ (1, r), (2, p), (2, q),\ (2, r), (3, p), (3, q),\ (3, r)\}$$

and $|A \times B| = |A| \times |B| = 3 \times 3 = 9$

3. RELATIONS

Let A and B be any two non-empty sets. The Cartesian product (or) cross product of A and B denoted by $A \times B$ is set of all ordered pairs (a, b) where a belongs to A and b belongs to B.

$\therefore A \times B = \{(a, b)/a \in A,\ b \in B\}$

Thus a relation R from a non-empty set A to a non-empty set B is a subset of a Cartesian product $A \times B$.

i.e., If $(a, b) \in R$, we say that a is related b and we write aRb

Relation: A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product AXB.

The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in AXB.

The second element is called the image of the first element.

Note:

$|A| = m$, $|B| = n$ then total number of relations from A to B formed is given by 2^{mn}

When a relation is defined on A i.e., $(A\ to\ A)$ then the relation is called binary relation.

3.1. Different types of relations:

Reflexive relation:

A relation R defined on set A is called reflexive relation if $aRa,\ \ \forall\, a \in A$

Example

Let $A = \{1,2,3,4\}$ then

$R_1 = \{(1,1), (2,2), (3,3), (4,4)\}$ →Reflexive

$R_2 = \{(1,1), (2,2), (3,2), (4,4)\}$ → Non-reflexive

$R_3 = \{(1,2), (2,3), (3,4), (4,1)\}$ → Irreflexive

Note:

- A relation in which no element is related to itself is called Irreflexive relation.

- In the relation if at least one element is not related itself then it is called Non-reflexive.

Symmetric relation:

Let R be relation defined on set A, R is called symmetric relation $If\ aRb \Rightarrow bRa$

Example

Let $A = \{1,2,3,4\}$ then

$R_1 = \{(1,2),(2,1),(3,2),(2,3),(4,4)\} \rightarrow$ Symmetric.

$R_2 = \{(1,2),(3,2),(2,3),(4,4)\} \rightarrow$ Not symmetric (Asymmetric).

Note:

A relation which is not symmetric is called asymmetric relation.

Antisymmetric relation:

Let R be set defined on a set A. Then R is an antisymmetric relation

$If\ aRb\ and\ bRa \Rightarrow a = b$

Example

i. $a \leq b\ and\ b \leq a \Rightarrow a = b.$
ii. $a \geq b\ and\ b \geq a \Rightarrow a = b.$
iii. $A \subseteq B\ and\ B \subseteq A \Rightarrow A = B.$

Transitive relation:

Let R is a relation defined on a set A. Then R is called transitive relation $If\ aRb\ and\ bRc \Rightarrow aRc$

Example

Let $A = \{1,2,3,4,5\}$ then

$R_1 = \{(1,2),(2,3),(1,3),(3,4),(4,5),(3,5)\} \rightarrow$ Transitive.

$R_2 = \{(1,2),(2,3)\} \rightarrow$ Not Transitive.

Equivalence relation:

A relation R is defined on a set A. then R is called an equivalence relation. If it is reflexive, symmetric, and transitive.

4. DEFINITIONS AND EXAMPLES

Domain: The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R.

Codomain: The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the codomain of the relation R. Note that range is subset of codomain.

4.1. Function:

A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B.

In other words, a function is a relation from a non-empty set A to a non-empty set B such that the domain of f is A and no two distinct ordered pairs in f have the same first element.

If f is a function from A to B and (a, b) e f. then f(a) = b, where b is called the image of a under fand a is called the preimage of b under f.

That is the function is denoted as $f: A \rightarrow B$ then $f(a) = b$

Example: Check whether the following are function or not and justify your answer

i. R= {(2,2),(2,4), (3,3), (4,4)}
ii. R={(1,2),(2,3), (3,4), (4,5), (5,6),(6,7)}

Solution

i. Since the same first element 2 corresponds to two different images 2 and 4, this relation is not a function.
ii. Since every element has one and only one image, this relation is a function.

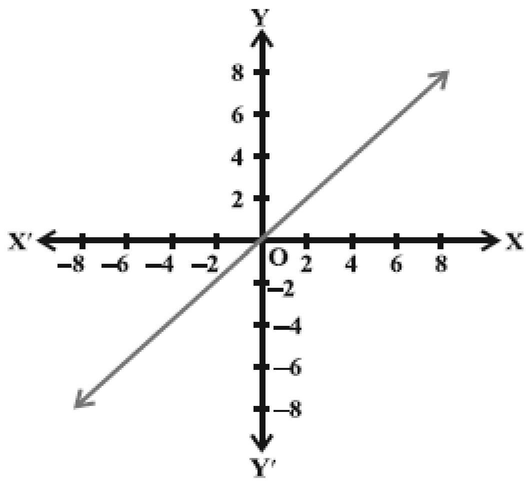
Real Function: A function which has either R or one of its subsets as its range is called a real valued function. Further, if its domain is also either R or a subset of R then it is called a real function.

Example: Let N be the set of natural numbers. Defined by a real valued function $f: N \rightarrow N$ by $f(x) = 3x + 2$, i.e., f={(1,5), (2,8),(3,11),(4,14),…}

5. TYPES OF FUNCTION

5.1. Identity Function:

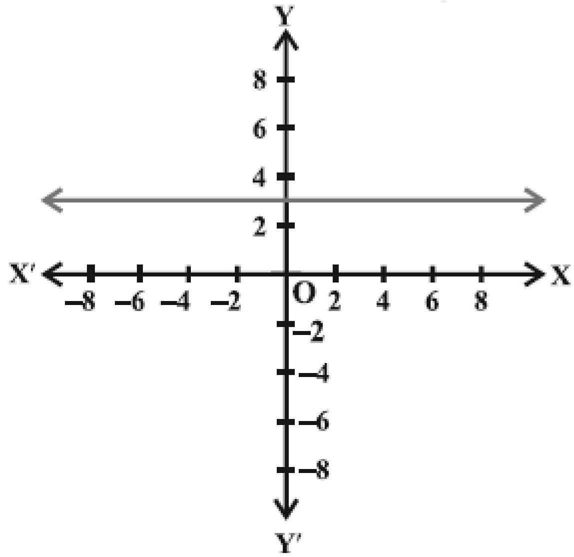
Let R be the set of real numbers. Define the real valued function $f: R \rightarrow R$ by $y = f(x) = x$ for each $x \in R$. Such a function is called the identity function. Here the domain and range of f are R. The graph is a straight line as shown in Figure It passes through the origin.



$f(x) = x$

5.2. Constant Function:

Define the function $f : R \rightarrow R$ by $y = f(x) = c,\ x \in R$ where c is a constant and each x in R Here domain of f is R and its range is {c}.



$f(x) = 3$

The graph is a line parallel to the x axis, for $y = f(x) = 3$

5.3. Polynomial Function:

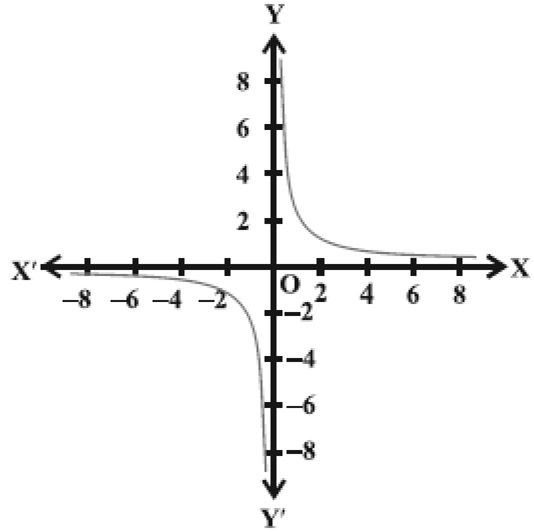
A function $f: R \to R$ is said to be polynomial function if for each x in R $y = f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where n is a non negative integer and $a_0, a_1, a_2, \dots, a_n \in Rs$

Example: $f: R \to R$ then $f(x) = x^6 + 3x^3 + 3$ and $g(x) = x^2 + \sqrt{2}$ are example of polynomial function but $h(x) = x^{\frac{2}{3}} + \sqrt{3}$ is not a polynomial function

5.4. Rational Functions

The functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in the domain, where $g(x) \neq 0$.

Example: The real valued function $f: R - \{0\} \to R$ defined by $f(x) = \frac{1}{x}$ $x \in R - \{0\}$

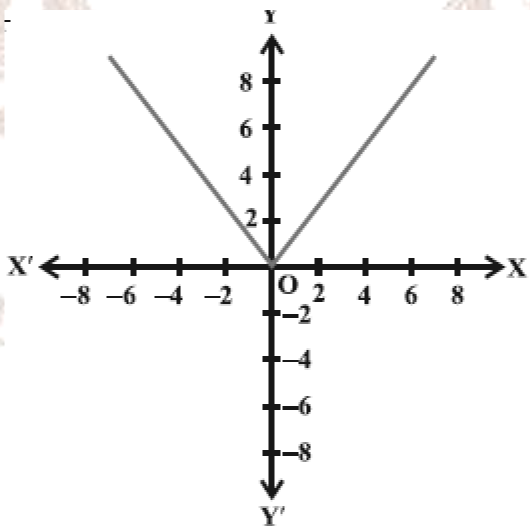


$f(x) = \frac{1}{x}$

5.5. The Modulus function:

A function $f: R \to R$ is defined by $f(x) = |x|$ for each $x \in R$ is called modulus function.

For each nonnegative values of x, $f(x)$ is equal to x. But for negative values of x, the values of f(x) is the negative of the values of x i.e., $f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$

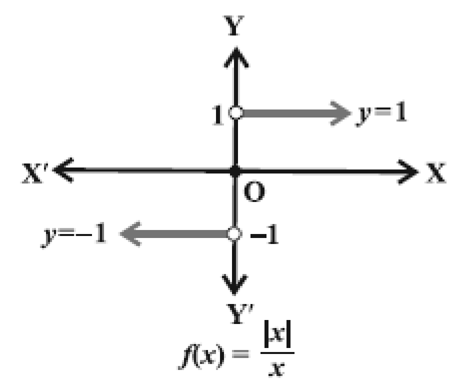


$f(x) = |x|$

5.6. Signum function

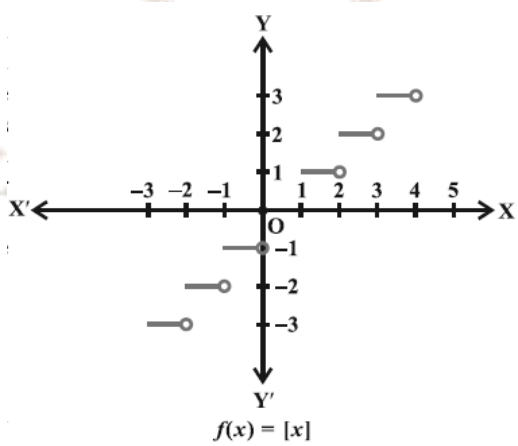
A function $f: R \to R$ is defined by $f(x) \begin{cases} 1, & if\ x > 0 \\ 0, & if\ x = 0 \\ -1, & if\ x < 0 \end{cases}$ is called signum function.

The domain of the signum function is R and the range is the set {-1,0,1}. The graph of the signum function is shown below.

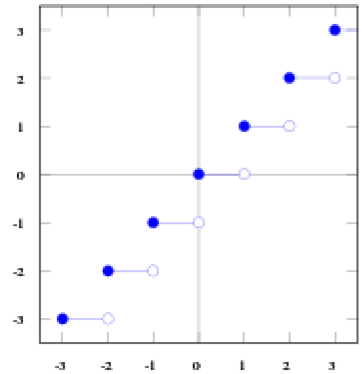


5.7. Greatest Integer Function

If a function f: R→ R is defined by f(x) = [x], x ∈ X. It round-off to the real number to the integer less than the number. Suppose, the given interval is in the form of (k, k+1), the value of greatest integer function is k which is an integer.

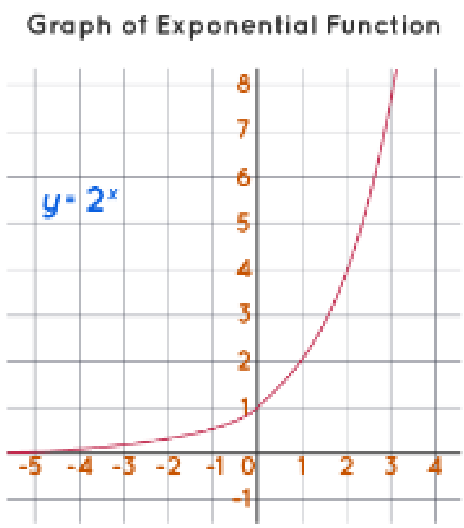


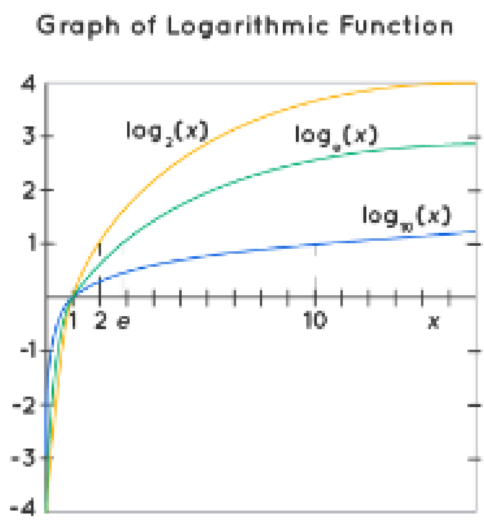
Example: [-21] = 21, [5.12] = 5. The graphical representation is



5.8. Logarithmic and Exponential Functions

Logarithmic functions have been derived from the exponential functions. The logarithmic functions are considered as the inverse of exponential functions. Logarithmic functions have a 'log' in the function and it has a base. The logarithmic function is of the form $y = \log_a x$. Here the domain value is the input value of 'x' and is calculated using the Napier logarithmic table. The logarithmic function gives the number of exponential times to which the base has raised to obtain the value of x. The same logarithmic function can be expressed as an exponential function as $x = a^y$.



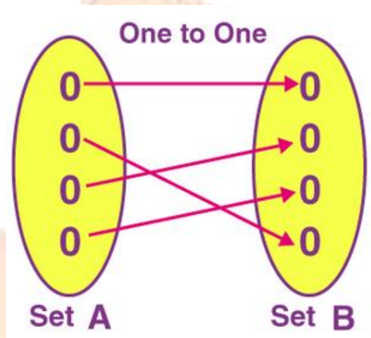


5.9. One-to-One function/Injective function

One-to-One function/Injective function define that each element of one set, say Set (A) is mapped with a unique element of another set, say, Set (B).

An injective function (injection) or one-to-one function is a function that maps distinct elements of its domain to distinct elements of its codomain.

Mathematically, it is stated as, if $f(x) = f(y)$ implies $x = y$, then f is one-to-one mapped, or f is 1-1 And equivalently, if $x \neq y$, then $f(x) \neq f(y)$.



Example: The function g(x) = x - 4 is a one to one function since it produces a different answer for every input. Also, the function g(x) = x^2 is NOT a one to one function since it produces 4 as the answer when the inputs are 2 and -2.

5.10. Onto function or Surjective function:

Consider two sets, Set A and Set B, which consist of elements. If for every element of B, there is at least one or more than one element matching with A, then the function is said to be **onto function or surjective function**.

Mathematically, a function $f:A \rightarrow B$ is onto if, for every element $b \in B$, there exists an element $a \in A$ such that $f(a)=b$.

Example:

Surjection

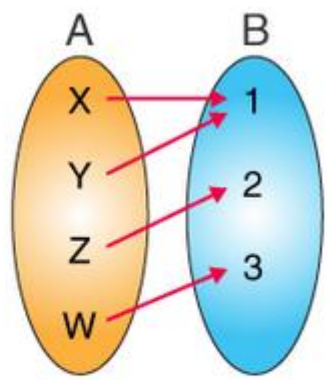


Fig.1

Not a surjection

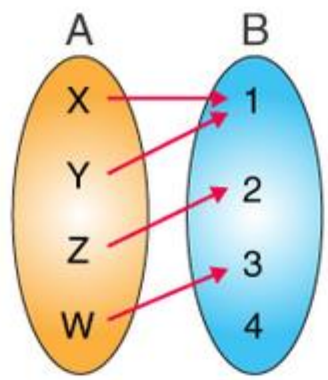


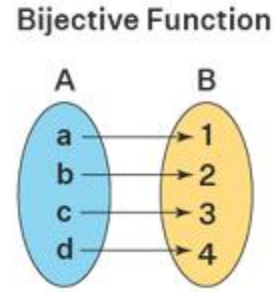
Fig.2

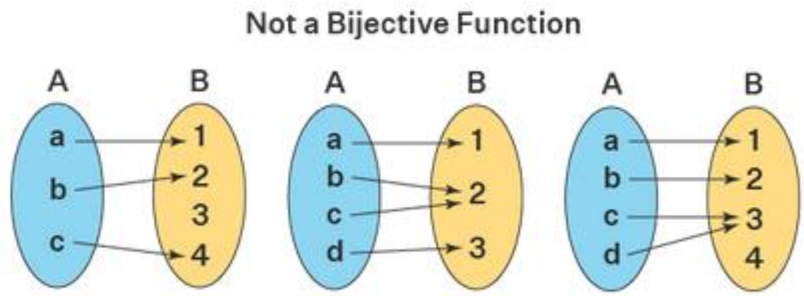
In the first figure, you can see that for each element of B, there is a pre-image or a matching element in Set A. Therefore, it is an onto function. But if you see in the second figure, one element in Set B is not mapped with any element of set A, so it's not an onto or surjective function.

5.11. Bijective Function:

A function is said to be bijective or bijection, if a function f: A → B satisfies both the injective (one-to-one function) and surjective function (onto function) properties.

Example:

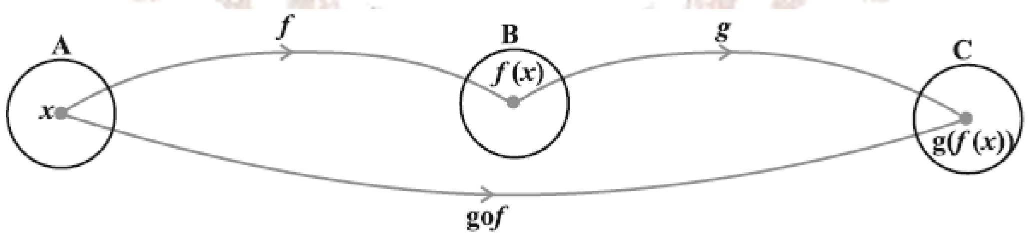




5.12. Composition of Functions

Suppose g : A→B and f: B→C, then the functions f and g, can be composed to obtain a function h: A→ C, denoted as follows,

h(x) = (f °g)(x) == f(g(x)) provided x ∈ A and g (x)∈ B.



Example:

Consider $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$ defined on on $f, g : R \to R$.
Show that $(g \circ f)(1) = 1, (g \circ f)(2) = 2, (g \circ f)(3) = 3$, and
$(g \circ f)(x) = x$

5.13. Inverse Functions

In mathematics a function, a, is said to be an inverse of another, b, if given the output of b a return the input value given to b. Additionally, this must hold true for every element in the domain co-domain(range) of b. In other words, assuming x and y are constants, if b(x) = y and a(y) = x then the function a is said to be an inverse of the function b.

Example

Consider the functions a(x) = 5x + 2 and b(y) = (y-2)/5. Here function b is an inverse function of a.

We can see this by inserting values into the functions.

For example when x is 1 the output of a is a(1) = 5(1) + 2 = 7.

Using this output as y in function b gives b(7) = (7-2)/5 = 1 which was the input value to function a.

5.14. A binary operation

A binary operation * on a set A is a function : A × A → A. We denote *(a, b) by a*b.

Example: Show that addition, subtraction and multiplication are binary operations on **R**, but division is not a binary operation on **R**. Further, show that division is a binary operation on the set **R** of nonzero real numbers.

Solution

6. SELF-ASSESSMENT QUESTIONS

SA1: Let A = {1, 5, 8, 9) and B {2, 4} And f={(1, 2), (5, 4), (8, 2), (9, 4)}. Then prove f is a onto function.

Solution:

From the question itself we get,

A={1, 5, 8, 9)

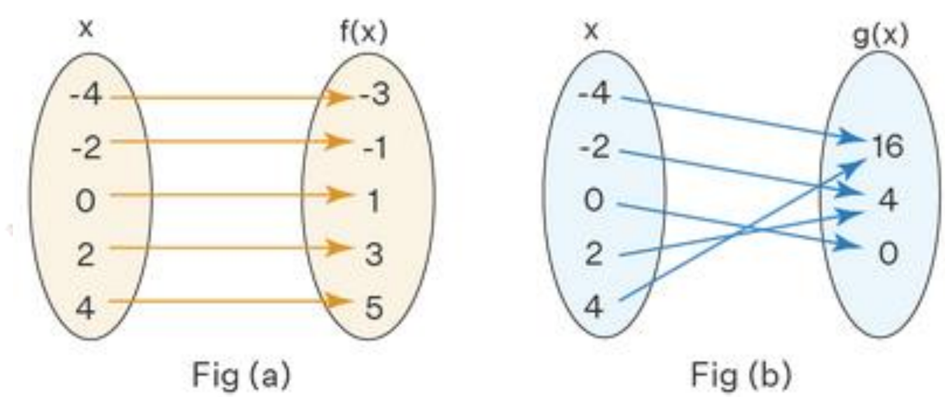
B{2, 4}

& f={(1, 2), (5, 4), (8, 2), (9, 4)}

So, all the element on B has a domain element on A or we can say element 1 and 8 & 5 and 9 has same range 2 & 4 respectively.

Therefore, f: A → B is a surjective function.

SA2: Which of the following is injunctive function?



Solution:

In Fig(a), for each x value, there is only one unique value of f(x) and thus, f(x) is one to one function.

In Fig (b), different values of x, 2, and -2 are mapped with a common g(x) value 4 and (also, the different x values -4 and 4 are mapped to a common value 16). Thus, g(x) is a function that is not a one to one function.

SA3:. A function f is defined by f(x)=2x+5. Find the values of f(x) at x=0, 7, -3.

Solution: It is given that f(x)=2x+5 then at x=0, then f(0)=5, f(7)=19, f(-3)=-1.

SA4: The Function 't' which maps temperature in degree Celsius into Temperature in degree Fahrenheit is defined by t(c)=9C/5+32 find (i) t(0), (ii) t(28), (iii) t(-10).

Solution

Given that t(c)=9C/5+32.

(i) t(0)=32.

(ii) t(28)=82.4

(iii) t(-10)=14

7. SUMMARY

In this chapter we studied about function and example. Later we discussed about various examples of functions such as constant function, identity function, rational function, modulus function, signum function. Also we discussed several examples.

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