MASTER OF COMPUTER APPLICATIONS

SEMESTER 1

DISCRETE MATHEMATICS AND GRAPH THEORY

Unit 6

Introduction to POSET

Table of Contents

1. (PRE-REQUISITES)

1.1 Objectives:

The objective of this topic is to make the students to:

- *Understand concept of partial order.*
- *Understand and implement properties of POSET.*

1.2. Cartesian Product

Let A and B be any two non-empty sets. The Cartesian product (or) cross product of A and B denoted by $A \times B$ is set of all ordered pairs (a, b) where a belongs to A and b belongs to B.

$$A \times B = \{(x, y): x \in A,\ y \in B\}.$$

Note:

If A and B are finite then the Cardinality of $|A \times B| = |A| \times |B|$.

Example: Let $A = \{1,2,3\}$ and B= $\{p, q, r\}$, then

$$A \times B = \{(1, p), (1, q),\ (1, r), (2, p), (2, q),\ (2, r), (3, p), (3, q),\ (3, r)\}$$

and $|A \times B| = |A| \times |B| = 3 \times 3 = 9$

1.3. Relations:

Let A and B be any two non-empty sets. The Cartesian product (or) cross product of A and B denoted by $A \times B$ is set of all ordered pairs (a, b) where a belongs to A and b belongs to B.

$\therefore A \times B = \{(a, b)/a \in A,\ b \in B\}$

Thus a relation R from a non-empty set A to a non-empty set B is a subset of a Cartesian product $A \times B$.

i.e., If $(a, b) \in R$, we say that a is related b and we write aRb

Note:

$|A| = m, |B| = n$ then total number of relations from A to B formed is given by 2^{mn}

When a relation is defined on A i.e., $(A\ to\ A)$ then the relation is called binary relation.

1.4. Different types of relations:

Reflexive relation:

A relation R defined on set A is called reflexive relation if $aRa, \ \forall\, a \in A$

Example

Let $A = \{1,2,3,4\}$ then

$R_1 = \{(1,1), (2,2), (3,3), (4,4)\}$ →Reflexive

$R_2 = \{(1,1), (2,2), (3,2), (4,4)\}$ → Non-reflexive

$R_3 = \{(1,2), (2,3), (3,4), (4,1)\}$ → Irreflexive

Note:

- A relation in which no element is related to itself is called Irreflexive relation.
- In the relation if at least one element is not related itself then it is called Non-reflexive.

Symmetric relation:

Let R be relation defined on set A, R is called symmetric relation $If\ aRb \Rightarrow bRa$

Example

Let $A = \{1,2,3,4\}$ then

$R_1 = \{(1,2), (2,1), (3,2), (2,3), (4,4)\}$ → Symmetric.

$R_2 = \{(1,2), (3,2), (2,3), (4,4)\}$ → Not symmetric (Asymmetric).

Note:

A relation which is not symmetric is called asymmetric relation.

Antisymmetric relation:

Let R be set defined on a set A. Then R is an antisymmetric relation

$If\ aRb\ and\ bRa \Rightarrow a = b$

Example

i. $a \leq b\ and\ b \leq a \Rightarrow a = b$.

ii. $a \geq b\ and\ b \geq a \Rightarrow a = b$.

iii. $A \subseteq B\ and\ B \subseteq A \Rightarrow A = B$.

Transitive relation:

Let R is a relation defined on a set A. Then R is called transitive relation $If\ aRb\ and\ bRc \Rightarrow aRc$

Example

Let $A = \{1,2,3,4,5\}$ then

$R_1 = \{(1,2),(2,3),(1,3),(3,4),(4,5),(3,5)\} \rightarrow$ Transitive.

$R_2 = \{(1,2),(2,3)\} \rightarrow$ Not Transitive.

Equivalence relation:

A relation R is defined on a set A. then R is called an equivalence relation. If it is reflexive, symmetric, and transitive.

1.5. Directed graph of a relation:

Let A relation R on set A $[R \subseteq A \times A]$ can be represented pictorially as follows

Step 1: Draw small circles representing the element of A. This circle is called **vertex or node.**

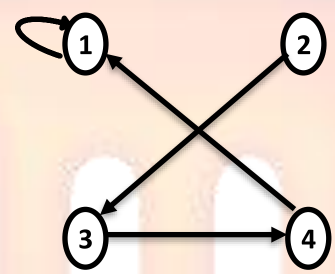
Step 2: Draw an arrow from vertex a_1 to vertex a_2 iff a_1 is related to a_2 . This arrow is called **edge.**

Note: If a_1 is related to itself then, we write the arrow which starts from vertex a_1 and ends at vertex a_1 and it is called loop

The resulting pictorial representation is known as directed graph of a relation.

Example

Let $A = \{1,2,3,4\}$ and $R = \{(1,1)(2,3),(3,4),(4,1)\}$ the digraph of the Relation is given by



2. INTRODUCTION

We often utilize relations to arrange some or all of the components in a set. For example, in the dictionary, we rank words using the relation comprising pairs of words (x, y), where x comes before y. We plan projects using the pairwise relation (x, y), where x and y are project activities that must be completed before y may begin. We use the relation containing the pairings (x, y) where x is smaller than y to arrange the set of numbers. When we add all of the pairings of the type (x, x) to these relations, we get a reflexive, antisymmetric, and transitive relation. These are the characteristics of the relations used to organize the components of sets.

3. DEFINITIONS AND EXAMPLES

3.1. Partial order:

A relation R is defined on a set A is called a partial order. If it is reflexive, antisymmetric, and transitive. $(R,\ A)$ is called partially ordered or POSET.

EXAMPLE: Show that the “greater than or equal” relation (≥) is a partial ordering on the set of integers.

Solution:

WKT $a \geq a$ for every integer a,

Therefore ≥ is reflexive.

If $a \geq b$ and $b \geq a$, then $a = b$.

Hence, ≥ is antisymmetric.

Finally, ≥ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$.

It follows that ≥ is a partial ordering on the set of integers and (**Z**, ≥) is a poset.

3.2. HASSE Diagram (or) POSET Diagram (or) Digraph of a partial order:

While drawing the directed graph of a partial order we use the following conventional conditions

1) As R is reflexive every element is related to itself, therefore self-loop need not be shown.

2) As R is transitive $aRb \text{ and } bRc \Rightarrow aRc$ we need not draw an edge from $a \text{ to } c$.

3) We represent element of a set A by dots.

4) The digraph drawn in such a way that there edges always upwards. We need not show direction.

The directed graph obtained using the above is known as POSET diagram or Hasse diagram.

EXAMPLE: Let R be a relation defined on set $\boldsymbol{A}$={**1**,**2**,**3**,**4**} "$\boldsymbol{xRy\ iff\ x\ divides\ y}$" show that R is a partial order and draw the Hasse diagram/Poset Diagram.

Solution

Reflexive:

We know every number divides itself

$\Rightarrow x\ divides\ x$

$\Rightarrow xRx$

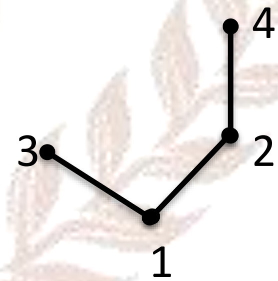
$\Rightarrow R\ is\ reflexive$

Antisymmetric:

wkt if $x\ divides\ y$ & $y\ divides\ x\ then\ x = y$

$\Rightarrow xRy$ & $yRx \Rightarrow x = y$

$\Rightarrow R\ is\ antisymmetric$



Transitive:

wkt if $x|y$ & $y|z$ then $x|z$

i.e., if xRy & yRz then xRz

$\Rightarrow R\ is\ Transitive$

$\therefore R$ is partial order

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

Example: *Draw the Hasse diagram representing the positive divisors of 36*

Solution

The set of all positive divisors of 36 is $D_{36} = \{1,2,3,4,6,9,12,18,36\}$

The relation R is a divisibility (that aRb if and only if $a\ divides\ b$) is a partial order on this set. The Hasse diagram for this partial order is required here

We note that, under R

1 is related all elements of D_{36}

2 is related to 2, 4, 6, 12, 18, 36

3 is related to $3,\ 6,\ 9,\ 12,\ 18,\ 36$

4 is related to $4,\ 12,\ ,36$

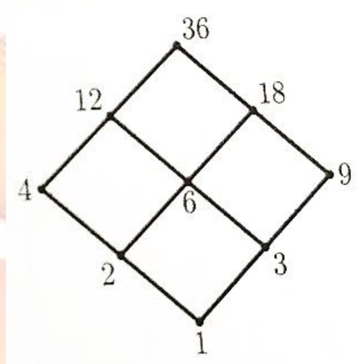
6 is related to 6, 12, 18, 36

9 is related to 9, 18, 36

12 is related to 12, 36

18 is related to 18, 36

36 s related 36



3.3. External element:

Maximal element: An element a is maximal element of A iff in the Hasse diagram of R no edge **starts** at a.

Minimal element: An element a is minimal element of A iff in the Hasse diagram of R no edge **terminates** at a.

Greatest element: An element $a \in A$ is called the greatest element if all elements are related to a.

Least element: An element $a \in A$ is called the least element if a related to all element of A.

Upper bound: Let B be a subset of A, an element $a \in A$ is called an upper bound of B if all the element of B are related to a.

Lower bound: Let B be a subset of A, an element $a \in A$ is called an upper bound of B if a related to all the element of B.

Least upper bound (Supremum): An element $a \in A$ is called as least upper bound of a subset B, if the following two conditions holds good

1) If a is an upper bound.

2) If a related to all upper bound.

Greatest lower bound (Infimum): An element $a \in A$ is called as greatest lower bound of a subset B, if the following two conditions holds good

i) If a is lower bound.

ii) All lower bounds are related to a.

Example

Find the maximal, minimal, greatest, and least element (if any) in the given Hasse diagram

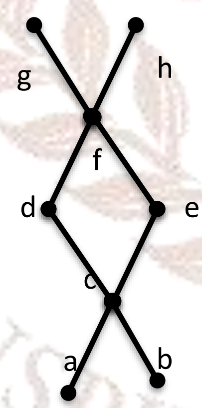


Fig 1

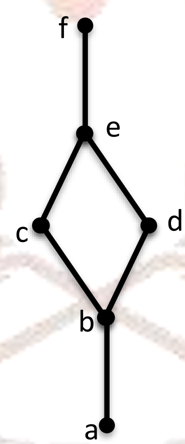


Fig 2

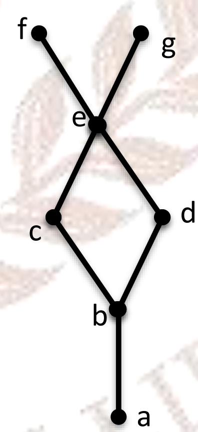


Fig 3

Solution

	Fig		
Element	1	2	3
Maximal	g & h	f	f & g
Minimal	a & b	a	a
Greatest	Nil	f	Nil
Least	Nil	a	a

4. COMPARABLE AND NON-COMPARABLE ELEMENTS

4.1. Comparable Elements:

Consider an ordered set A. Two elements a and b of set A are called comparable if $a \le b$ or $b \le a$

4.2. Non-Comparable Elements:

Consider an ordered set A. Two elements a and b of set A are called non-comparable if neither $a \le b$ nor $b \le a$.

Example: Consider A = {1, 2, 3, 5, 6, 10, 15, 30} is ordered by divisibility. Determine all the comparable and non-comparable pairs of elements of A.

Solution:

The comparable pairs of elements of A are:

{1, 2}, {1, 3}, {1, 5}, {1, 6}, {1, 10}, {1, 15}, {1, 30}

{2, 6}, {2, 10}, {2, 30}

{3, 6}, {3, 15}, {3, 30}

{5, 10}, {5, 15}, {5, 30}

{6, 30}, {10, 30}, {15, 30}

The non-comparable pair of elements of A are:

{2, 3}, {2, 5}, {2, 15}

{3, 5}, {3, 10}, {5, 6}, {6, 10}, {6, 15}, {10, 15}

4.3. Linearly Ordered Set:

Consider an ordered set A. The set A is called linearly ordered set or totally ordered set, if every pair of elements in A is comparable.

Example: The set of positive integers I+ with the usual order ≤ is a linearly ordered set.

Note: Let R be a parial order on set A. Then R is called a total order on A if for all $x, y \in A$ either $xRy \ or \ yRx$. In this case the poset (A, R) is called a totally ordered set.

5. SELF-ASSESSMENT QUESTIONS

SA1: *Let* $A = \{1,2,3,4,6,12\}$ *and on* A *define the relation* R *by* aRb *if "a divides b" prove that* R *is a partial order on* A *and Hasse Diagram of this relation.*

Solution

$$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,12), (2,2), (2,4), (2,6), (2,12),$$

$$(3,3), (3,6), (3,12), (4,4), (4,12), (6,6), (6,12), (12,12)\}$$

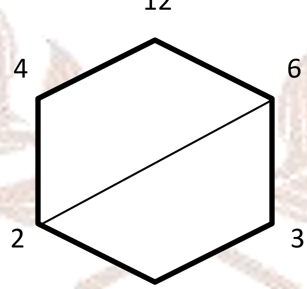
We observe the following.

$(x, x) \in R\ \forall x \in A \Rightarrow R\ is\ reflexive$

When $(x, y) \in R$ and $y \neq x,\ (y, x) \notin R \Rightarrow R\ is\ antisymmetric$

Also, we can see that when $(x, y)\ and\ (y, z) \in R$, then $(x, z) \in R$ which ensures that the transitive property.

Hence, we conclude that R is a partial order on A

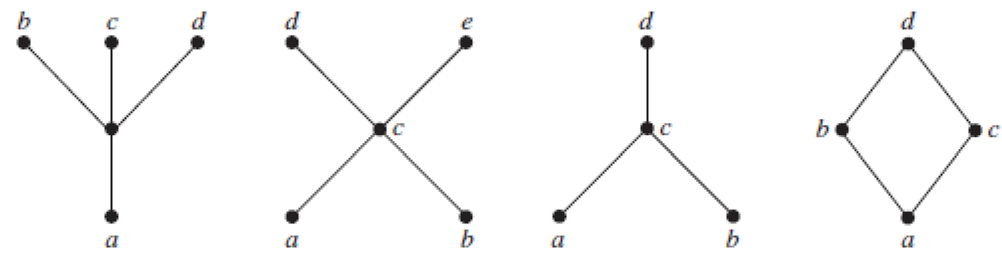


SA2: Let R be the relation on the set of people such that xRy if x and y are people and x is older than y. Show that R is not a partial ordering.

Solution:

Note that R is antisymmetric because if a person x is older than a person y, then y is not older than x. That is, if xRy, then $y\ Rx$. The relation R is transitive because if person x is older than person y and y is older than person z, then x is older than z. That is, if xRy and yRz, then xRz. However, R is not reflexive, because no person is older than himself or herself. That is, xRx for all people x. It follows that R is not a partial ordering.

SA3: Determine whether the posets represented by each of the Hasse diagrams in Figure have a greatest element and a least element.



(a) (b) (c) (d)

Solution:

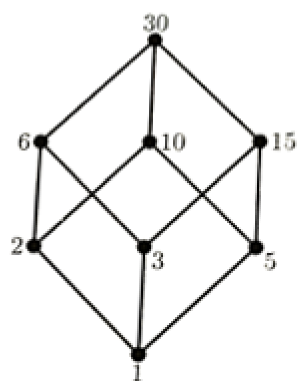
The least element of the poset with Hasse diagram (a) is a. This poset has no greatest element. The poset with Hasse diagram (b) has neither a least nor a greatest element. The poset with Hasse diagram (c) has no least element. Its greatest element is d. The poset with Hasse diagram (d) has least element a and greatest element d.

SA4: In the following cases consider the partial order of divisibility on the set A. Draw the Hasse diagram of poset and determine whether the poset is totally orderd or not

i) A = {1,2,3,5,6,10,15,30} [positive divisors of 30] and ii) A = {2,4,8,16,32}

Solution

(i)



(ii)

By examining the above Hasse diagrams we find that the given relation is totally ordered in case (ii) as it is evident that every member of A divies each one of the succeeding members of A. But is not totally ordered in case (i)

6. SUMMARY

This chapter includes concepts of Relations and its types. Here we discussed about reflexive, antisymmetric and transitive relation which are constitute the partial order or POSET. Also we discussed about Hasse diagram to represent the POSET along with its Properties.

7. REFERENCES

- Ralph P. Grimaldi (2019) Discrete and Combinatorial Mathematics: An Applied Introduction (5ed.) Pearson Publication.
- Kenneth H. Rosen (2012) Discrete Mathematics and Its Applications (7 th Edition) McGraw-Hill Publication.
- Dr. DSC (2016) Discrete Mathematical Structures (5th Edition), PRISM publication.
- https://www.geeksforgeeks.org/elements-of-poset/