MASTER OF COMPUTER APPLICATIONS

SEMESTER 1

DISCRETE MATHEMATICS AND GRAPH THEORY

Unit 2

Linear Equations

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1. INTRODUCTION

The foundation of linear algebra is systems of linear equations, which are used in this chapter to convey some of the key ideas in a clear and concise manner. A methodical approach to solving systems of linear equations is shown in this section. In social accounting, input-output tables, and the study of inter-industry economics, matrices are used by economists. Additionally, matrices are used in the study of electrical engineering's network analysis and communication theory.

1.1. Objectives:

At studying this unit, you should be able to:

- *Solve the system of equation by checking the consistency.*

2. PRE-REQUISITES

2.1. Echelon form of a matrix

A non – zero matrix **A** is said to be in *row echelon form* if the following conditions prevail:

i) All the zero rows are below non zero rows.

ii) The first non zero entry in any non zero row is 1 [and the entries below 1 in the same column are zero].

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note:

While reducing the given matrix to a row echelon form, we prefer to have the leading element non zero. If it is zero, we can interchange with any row having the leading element is nonzero. Then we use that leading nonzero element in every row to make all the elements in that column zero. The transformation in this process has to be performed for the entire row. Avoid fraction as far as possible during the process of elementary transformation.

2.2. Rank of a matrix

The rank of a matrix A in its echelon form is equal to the number of non zero rows. It is denoted by $\rho(A)$.

2.2.1. Steps to find the rank of a matrix

Step 1. In order to reduce the given matrix to a row echelon form we must prefer to have the leading entry (first entry in the first row) non zero, much preferably 1.

Step 2. In case this entry is zero, we can interchange with any suitable row to meet the requirement.

Step 3. We then focus on the leading non zero entry (starting from the first row) to make all the elements in that column zero. However the transformation has to be performed for the entire row.

Step 4. Row echelon form will be achieved first and we can instantly write down the rank, which being the number of zero rows.

NOTE:

- Elementary transformations do not change either the order or rank of a matrix.
- Further we can say that if r is the rank of a matrix A of order $m \times n$ $(r \leq m)$, r number of rows of the matrix are linearly independent.

Problem: Find the rank of the following matrix by reducing it to the row echelon form.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Solution

Solution:

Performing $R_2 \to -3R_1 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 \\ -3(1)+3 & -3(2)+6 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\rho(A) = 1$$

Problem: Find the rank of the following matrix by reducing it to the row echelon form. A =

$$\begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

Solution:

Let $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_1$$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to {}^{R_1}/_{2} \quad \text{and} \quad R_2 \to {}^{R_2}/_{2}$$

$$A \sim \begin{bmatrix} 1 & 3/2 & 5/2 & 2 \\ 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 2$$

Problem: Find the rank of the following matrix by reducing it to the row echelon form.

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Solution:

$$R_1 \leftrightarrow R_2$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$R_2 \to R_2 + 2R_1 \quad \text{and} \quad R_3 \to R_3 - R_1$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$R_2 \to R_4$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 3 & 3 & 1 \end{bmatrix}$$

$$R_3 \to R_3 + 2R_2 \quad \text{and} \quad R_4 \to R_4 - 3R_2$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$R_4 \to R_4 + R_3$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \to R_3/2$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 3$$

3. CONSISTENCY OF A SYSTEM OF LINEAR EQUATIONS

Consider a system of ‘m’ linear equations in ‘n’ unknowns as follows

$$\mathrm{a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1}$$

$$\mathrm{a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2}$$

……… ……..

……… ……..

$$\mathrm{a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n = b_m}$$

Where $\mathrm{a_i}$ & $\mathrm{b_i}$’s are constants

If b_i's are all zero, the system is said to be homogeneous. Otherwise, it is said to be Non-homogeneous.

The set of values $x_1, x_2, \dots, x_n$ which satisfy all the equations simultaneously is called a solution of the system of equations.

A system of linear equations is said to be **consistent** if it possess a solution. Otherwise it is said to be **inconsistent**.

The above system of equations can be written in the matrix equation A X = B, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} B = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_m \end{bmatrix}$$

$x_1 = x_2 = x_3 = \dots\dots\dots\dots = x_n = 0$ is a solution of the homogeneous system of equations and is called a **trivial solution.**

If at least one x_i, (i = 1, 2, …, n) is not equal to zero then it is called a **non trivial solution.**

The concept of the rank of a matrix helps us to conclude

(i). Whether the system is consistent or not.

(ii). Whether the system possess unique solution or many solution.

Condition for consistency and types of solution

Consider a system of m equations in n unknowns represented in the matrix form AX = B

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} B = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_m \end{bmatrix}$$

Here A is called the coefficient matrix.

The matrix formed by appending to A an extra column consistent of the elements of B is called the **augmented matrix** denoted by **[A: B]**

$$\text{That is,} \quad [\mathrm{A:B}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & : & b_2 \\ .. & .. & \dots & .. & \vdots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} & : & b_m \end{bmatrix}$$

The system of equations represented by the matrix equation AX = B is **consistent** if $\rho(A) = \rho(A:B)$

Suppose $\rho(A) = \rho(A:B) = r$, then the condition for two types of solution are as follows.

1. **Unique Solution:** If $\rho(A)=\rho(A:B) = r = n$, (where n is the number of unknowns).
2. **Infinite Solution:** $\rho(A)=\rho(A:B) = r < n$, In case (n - r) unknowns can take arbitrary value,
3. Obviously $\rho[A] \neq \rho[A:B]$ implies that the system in **inconsistent** (does not possess a solution).

Working procedure for problems

Step 1: We first form the augmented matrix [A: B] and we can clearly identify the portion of the coefficient matrix A in it.

Step 2: We reduce the matrix [A: B] to an echelon form by elementary row transformations. This will enable us to immediately write down the rank of A and also [A: B], with the result we can decide the consistency aspect of the system of equations.

Step 3: The echelon form of [A: B] is converted back to the equation form and the solution will emerge easily.

Problem: Test for consistency and solve $x + y + z = 6;\ x - y + 2z = 5;\ 3x + y + z = 8$

Solution:

$$[A:B] = \begin{bmatrix} 1 & 1 & 1: & 6 \\ 1 & -1 & 2: & 5 \\ 3 & 1 & 1: & 8 \end{bmatrix} \quad \text{is the augmented matrix.}$$

Perform $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 3R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & -2 & -2 & : & -10 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$[A:B]\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & -2 & 1 & : & -1\\ 0 & 0 & -3 & : & -9\end{bmatrix}$$

[Note: *We need not make the leading non zero entry in every row 1 as we can decide on the rank of the matrices A and [A: B] at this stage.*]

Both A and [A : B] matrices have all the three rows non zero.

Therefore $\rho[A]=3$ and $\rho[A:B]=3$ that is, r = 3.

Also, the number of independent variables n = 3.

Since $\rho[A]=\rho[A:B]=3$ ($r = n = 3$) the given system of equations is consistent and will have unique solution. Let us now convert the prevailing form of [A : B] into a set of equations as follows,

$$x+y+z=6\ \ldots\ldots(1)$$

$$-2y+z=-1\ \ldots\ldots(2)$$

$$-3z=-9\ldots\ldots\ldots\ldots(3)$$

From (3), z = 3,

Substitute this value in (2), we get y = 2.

Finally substituting these values in (1), we get x = 1.

Thus $x=1,\ y=2,\ z=3$ is the unique solution.

Problem: Test for consistency and solve

$$x-4y+7z=14$$
$$3x+8y-2z=13$$
$$7x-8y+26z=5$$

Solution:

$$[A:B]=\begin{bmatrix}1 & -4 & 7 & : & 14\\ 3 & 8 & -2 & : & 13\\ 7 & -8 & 26 & : & 5\end{bmatrix}$$ is the augmented matrix.

$\mathrm{R}_2 \to R_2 - 3R_1$ and $\mathrm{R}_3 \to R_3 - 7R_1$

$$[A:B] \sim \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 20 & -23 & : & -29 \\ 0 & 20 & -23 & : & -93 \end{bmatrix}$$

$\mathrm{R}_3 \to R_3 - R_2$

$$[A:B] \sim \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 20 & -23 & : & -29 \\ 0 & 0 & 0 & : & -64 \end{bmatrix}$$

Therefore $\rho[A] = 2$ and $\rho[A : B] = 3$.

Also, the number of independent variables $n = 3$.

Since $\rho[A] \neq \rho[A : B]$, the given system of equations is **inconsistent.**

Problem: Solve the following system of equations

$$x + y + z = 9, \quad x - 2y + 3z = 8, \quad 2x + y - z = 3$$

Solution:

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & :9 \\ 1 & -2 & 3 & :8 \\ 2 & 1 & -1 & :3 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & :9 \\ 0 & -3 & 2 & :-1 \\ 0 & -1 & -3 & :-15 \end{bmatrix}$$

$R_3 \to R_2 - 3R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & :9 \\ 0 & -3 & 2 & :-1 \\ 0 & 0 & 11 & :44 \end{bmatrix}$$

Hence we have

$$x + y + z = 9$$
$$-3y + 2z = -1$$
$$11z = 44$$
$$\therefore z = 4$$

By back substitution,

$y = 3, x = 2$

The solution is

$x = 2, \quad y = 3, \quad z = 4$

4. THE SOLUTION OF SYSTEM OF LINEAR EQUATIONS BY INVERSE OF A MATRIX

A solution for a system of linear Equations can be found by using the inverse of a matrix. Suppose we have the following system of equations

- $a_{11}\, x + a_{12}\, y + a_{13}\, z = b_1$
- $a_{21}\, x + a_{22}\, y + a_{23}\, z = b_2$
- $a_{31}\, x + a_{32}\, y + a_{33}\, z = b_3$

where, x, y, and z are the variables and a_{11}, a_{12}, … , a_{33} are the respective coefficients of the variables and b_1, b_2, and b_3 are the constants. We need to find the solution for the values of the variables in this system of equations.

Determinant as an Equation Solver

The above system of equations can be represented in the form of a square matrix as

$$A = \begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \& B = \begin{bmatrix} b1 \\ b2 \\ b3 \end{bmatrix}$$

i.e., AX = B or,

$$A = \begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \& B = \begin{bmatrix} b1 \\ b2 \\ b3 \end{bmatrix}$$

Here arise two cases

Case1

If A is a non-singular matrix i.e., |A| ≠ 0, then its inverse exists.

We have A X = B

or, A^{-1} (A X) = A^{-1} B (pre-multiplying by A^{-1})

or, (A^{-1} A) X = A^{-1} B

and, I X = A^{-1} B (I is the identity matrix)

or, X = A^{-1} B where, A^{-1} = (adj A) / |A|

This matrix equation provides a unique solution and is known as the Matrix Method.

Case2

If A is a singular matrix, then |A| = 0 then we calculate (adj A) B. If (adj A) B ≠ 0 (zero matrix), then the solution does not exist. The system of equations is inconsistent. Else, if (adj A) B = 0 then the system will either have infinitely many solutions (consistent system) or no solution (inconsistent system).

Problem: Solve the simultaneous equations $x + 2y = 4;\ 3x - 5y = 1$

Solution

We have already seen these equations in matrix form AX=B

$$\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

We need to calculate the inverse of A is

$$A^{-1} = \frac{1}{(1)(-5)-(2)(3)}\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-11}\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$$

Thus X is given by $X = A^{-1}B$

$$X = \frac{1}{-11}\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Hence $x = 2$ and $y = 1$.

Problem: Solve the given system of linear equation $x + 3y - 2z = 10; 2x - y + 6z = 3; x + y - 2z = 5$

Solution:

We can write the above equations in form of matrix as follows:

$$AX + B$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 6 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 5 \end{bmatrix}$$

To find the matrix X

we need to find the inverse of matrix such that
$X = A^{-1} B$

Determinant of the matrix A is:

|A| = 1(2 – 6) – 3(-4 – 6) – 2(2 + 1)

= -4 -3(-10) -2(3)

= -4 + 30 – 6

= 20

Let us find the adjoint of matrix A.

$$Adj(A) = \begin{bmatrix} -4 & 4 & 16 \\ 10 & 0 & -10 \\ 3 & 2 & -7 \end{bmatrix}$$

Therefore $A^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} & \frac{4}{5} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{3}{20} & \frac{1}{10} & -\frac{7}{20} \end{bmatrix}$

Hence $X = A^{-1}B = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} & \frac{4}{5} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{3}{20} & \frac{1}{10} & -\frac{7}{20} \end{bmatrix} \begin{bmatrix} 10 \\ 3 \\ 5 \end{bmatrix}$

$$\boldsymbol{X} = \begin{bmatrix} \frac{60}{23} \\ \frac{57}{23} \\ \frac{1}{23} \end{bmatrix}$$

Problem: There are three numbers whose sum is given as 3. If the second number is multiplied by 2 and the first number is added to it, 6 is obtained. If the third number is multiplied by 4 and the second number is added to it, 10 is obtained. Represent the given situation algebraically and find the numbers using the matrix method.

Solution

Consider that x, y, and z represent the first, second, and third numbers, respectively. Thus, according to the question,

x + y + z = 3

x + 2y = 6

y + 4z = 10

It can be written as AX = B where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 10 \\ 6 \end{bmatrix}$$

|A|= 1(8 – 0) – 1(4 – 0) + 1(1 – 0) = 8 – 4 + 1 = 5 ≠ 0.

Now, we need to find adj A.

$A_{11} = 8 - 0 = 8$, $A_{12} = -(4 - 0) = -4$, $A_{13} = 1 - 0 = 1$

$A_{21} = -(4 - 1) = -3$, $A_{22} = 4 - 0 = 4$, $A_{23} = -(1 - 0) = -1$

$A_{31} = 0 - 2 = -2$, $A_{32} = -(0 - 1) = 1$, $A_{33} = 2 - 1 = 1$

Adj. A will be $\begin{bmatrix} 8 & -3 & -2 \\ -4 & 4 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} Adj.of\,A = \frac{1}{5}\begin{bmatrix} 8 & -3 & -2 \\ -4 & 4 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$X = A^{-1}B$

$$X = \frac{1}{5}\begin{bmatrix} 8 & -3 & -2 \\ -4 & 4 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 10 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 24 - 30 - 12 \\ -12 + 40 + 6 \\ 3 - 10 + 6 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -18 \\ 34 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{-18}{5} \\ \frac{34}{5} \\ \frac{-1}{5} \end{bmatrix}$$

Hence,

X = -18/5

Y = 34/5

Z = -1/5

5. PROBLEM ON AGES

One application of linear equations is what are termed age problems. When solving age problems, generally the age of two different people (or objects) both now and in the future (or past) are compared. Which can be solved using bellow tips and table as mentioned bellow

Tip 1: If the current age is x, then n times the age is nx.

Tip 2:If the current age is x, then the age n years later/hence = x + n

Tip 3: If the current age is x, then age n years ago = x – n

Tip 4: The ages in a ratio a:b will be ax and bx

Tip 5: If the current age is x, then 1/n of the age is x/n

Person or Object	Current Age	Age Change

Problem: Joey is 20 years younger than Becky. In two years, Becky will be twice as old as Joey. Fill in the age problem chart and write the required equation.

Solution

The first sentence tells us that Joey is 20 years younger than Becky (this is the current age)

The second sentence tells us two things:

The age change for both Joey and Becky is plus two years

In two years, Becky will be twice the age of Joey in two years

Person or Object	Current Age	Age Change (+2)
Joey (J)	B – 20	B – 20 + 2 B – 18

Becky (B)	B	B = 2

Using this last statement gives us the equation to solve:

B + 2 = 2 (B − 18)

Problem: Carmen is 12 years older than David. Five years ago, the sum of their ages was 28. How old are they now?

Solution:

The first sentence tells us that Carmen is 12 years older than David (this is the current age)

The second sentence tells us the age change for both Carmen and David is five years ago (−5)

Filling in the chart gives us:

Person or Object	Current Age	Age Change (−5)
Carmen (C)	D + 12	D + 12 − 5 D + 7
David (D)	D	D − 5

The last statement gives us the equation to solve:

Five years ago, the sum of their ages was 28

$$(D+7)+(D-5)=28$$
$$2D+2=28$$
$$\underline{-2=-2}$$
$$2D=26D$$

$$D=\frac{26}{2}=13$$

Therefore, Carmen is David’s age (13) + 12 years = 25 years old.

Problem: The ratio of present age A and B is 13:10 after 2.5 years their ratio will be 32:25 then find the present age of A.

Solution: Let, present age of A = 13x and present age of B = 10x According to question:

$(13x + 2.5)/(10x + 2.5) = 32/25$

$\Rightarrow (13x + 2.5) \times 25 = (10x + 2.5) \times 32$

$\Rightarrow 325x + 62.5 = 320x + 80$

$\Rightarrow 5x = 17.5$

$\Rightarrow x = 3.5$

Problem: If 5 years ago, the ratio of age of Mradul and Love was 1 : 2 and after 15 years from present their ratio would be 5 : 6. Find the age of Love after 20 years.

Solution: Let, present age of Mradul be x and present age of Love be y.

Then, according to question $(x - 5)/(y - 5) = ½$

$\Rightarrow 2x - 10 = y - 5$

$\Rightarrow x = (y + 5)/2$———- (1)

Also, $(x + 15)/(y + 15) = 5/6$

$\Rightarrow 6x + 90 = 5y + 75$

$\Rightarrow 6x + 15 = 5y$

Putting value of x from equation 1, we get $3y + 15 + 15 = 5y$

$\Rightarrow 2y = 30$

$\Rightarrow y = 15$

$\therefore$ Age of Love after 20 years = 15 + 20 = 35 years.

Problem: The mean of the ages of father and his son is 27 years. After 18 years, the father will be twice as old as his son. Their present ages are?

Solution: Let the ages of father and son be x and y respectively,

Then the mean ages of father and son be $(x + y)/2$,

According to the given conditions

$(x + y)/2 = 27$

$\Rightarrow (x + y) = 54$

$\Rightarrow x + y = 54$ —eqn (1)

After 18 years

$x + 18 = 2(y + 18)$

$\Rightarrow x + 18 = 2y + 36$

$\Rightarrow x - 2y = 18$ —eqn (2)

Solving equations 1 & 2 we get

$x = 42$ & $y = 12$

Hence, age of father is 42 and son is 12

6. SELF-ASSESSMENT QUESTIONS

1. Test for consistency and solve $5x_1 + x_2 + 3x_3 = 20;\ 2x_1 + 5x_2 + 2x_3 = 18;\ 3x_1 + 2x_2 + x_3 = 14$
2. Show that the following system of equation does not possess any solution $5x + 3y + 7z = 5;\ 3x + 26y + 2z = 9;\ 7x + 2y + 10z = 5$
3. Suppose you have three numbers. The sum of the two numbers and the twice of the second equals 2. The sum of the second and third when subtracted from the twice of first gives 1. The difference of thrice of first and five times the third gives 5. Rewrite the statement in form of the system of equations. Solve it using Matrix Method as an equation solver.
4. Test for consistency and solve $x - 4y + 7z = 14;\ 3x + 8y - 2z = 13;\ 7x - 8y + 26z = 5$
5. The ages of A, B and C together is 65 years. B is 2/3 of A and C is 9 years older than A. Then, what is the ratio of the respective age of C, A and B?
6. The present age of Ram is 5 times the age of his son. After 12 years the age of Ram will be twice the age of his son, find the present age of son.

7. SUMMARY

In this chapter we studied about how to solve the system of equations by checking whether is consistent or not. Additionally we studied how to solve the system of equations with matrix inversion method and also by back substitution method.

8. ANSWERS TO SELF ASSESSMENT QUESTIONS

1. $[A:B] = \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 2 & 5 & 2 & : & 18 \\ 3 & 2 & 1 & : & 14 \end{bmatrix}$ is the augmented matrix.

$$R_2 \to 5R_2 - 2R_1 \quad \text{and} \quad R_3 \to 5R_3 - 3R_1$$

$$[A:B] \sim \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 0 & 23 & 4 & : & 50 \\ 0 & 7 & 4 & : & 10 \end{bmatrix}$$

$$R_3 \to 23R_3 + 7R_2$$

$$[A:B] \sim \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 0 & 23 & 4 & : & 50 \\ 0 & 0 & -120 & : & -120 \end{bmatrix}$$

Therefore $\rho[A] = 3$ and $\rho[A:B] = 3$ that is, r = 3.

Also, the number of independent variables n = 3.

Since $\rho[A] = \rho[A:B] = 3$ (r = n = 3) the given system of equations is consistent and will have unique solution.

Let us now convert the prevailing form of [A : B] into a set of equations as follows,

$$5x_1 + x_2 + 3x_3 = 20 \ldots \ldots (1)$$

$$23x_2 + 4x_3 = 50 \ldots \ldots (2)$$

$$-120x_3 = -120 \ldots\ldots\ldots\ldots (3)$$

From (3), $x_3 = 1$,

Substitute this value in (2), we get $x_2 = 2$.

Finally substituting these values in (1), we get $x_1 = 3$.

Thus $x_1 = 3,\ x_2 = 2,\ x_3 = 1$ is the unique solution.

2. $[A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$ is the augmented matrix.

$$R_2 \to 5R_2 - 3R_1 \quad \text{and} \quad R_3 \to 5R_3 - 7R_1$$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 0 & 121 & -11 & : & 30 \\ 0 & -11 & 1 & : & -10 \end{bmatrix}$$

$$R_3 \to R_2 + 11R_3$$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 0 & 121 & -11 & : & 30 \\ 0 & 0 & 0 & : & -80 \end{bmatrix}$$

Therefore $\rho[A] = 2$ and $\rho[A:B] = 3$.

Also, the number of independent variables n = 3.

Since $\rho[A] \neq \rho[A:B]$, the given system of equations is inconsistent.

3. Assume that x, y, and z are the three numbers. Rewriting the above statement we have the following system of equations

$$x + 2y + z = 2$$

$$2x - y - z = 1$$

$$3x - 5y = 5$$

In matrix notation, we have

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 3 & 0 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, and\ B = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

Here, the determinant of A = |A| = 1(5 – 0) – 2(–10 + 3) + 1(0 + 3) = 22 ≠ 0. Hence there exists a unique solution for X.

Calculating adj (A), we have $A_{ij} = (-1)^{(i+j)} M_{ij}$, where M_{ij} is the co-factor of a_{ij}

- A_{11} = 1(5 – 0) = 5, A_{12} = –1(–10 + 3) = 7, A_{13} = 1(0 + 3) = 3,
- A_{21} = –1(–10 –0) = 10, A_{22} = 1(–5 – 3) = –8, A_{23} = –1(0 – 6) = 6,
- A_{31} = 1(–2 + 1) = –1, A_{32} = –1(–1 – 2) = 3, A_{33} = 1(–1 – 4) = –5

$$adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 5 & 10 & -1 \\ 7 & -8 & 3 \\ 3 & 6 & -5 \end{bmatrix}$$

The inverse of the matrix A is A^{-1}.

$$A^{-1} = \frac{1}{|A|}(adj\ A) = \frac{1}{22}\begin{bmatrix} 5 & 10 & -1 \\ 7 & -8 & 3 \\ 3 & 6 & -5 \end{bmatrix}$$

Since $X = A^{-1} B$

$$X = \frac{1}{22}\begin{bmatrix} 5 & 10 & -1 \\ 7 & -8 & 3 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 10+10-5 \\ 14-8+15 \\ 6+6-25 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 15 \\ 21 \\ -13 \end{bmatrix}$$

Thus, x = 15/22, y = 21/22, and z = –13/22.

4. $[A:B] = \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 3 & 8 & -2 & : & 13 \\ 7 & -8 & 26 & : & 5 \end{bmatrix}$ is the augmented matrix.

$$R_2 \rightarrow R_2 - 3R_1 \text{ and } R_3 \rightarrow R_3 - 7R_1$$

$$[A:B] \sim \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 20 & -23 & : & -29 \\ 0 & 20 & -23 & : & -93 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$[A:B] \sim \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 20 & -23 & : & -29 \\ 0 & 0 & 0 & : & -64 \end{bmatrix}$$

Therefore $\rho[A] = 2$ and $\rho[A:B] = 3$.

Also, the number of independent variables $n = 3$.

Since $\rho[A] \neq \rho[A:B]$, the given system of equations is inconsistent.

5. The ages of A, B and C together = 65

Let age of A = x

Age of C = x + 9

Age of B = $2x/3$

Let the age of A be x years

The ages of A, B and C together = 65

According to the question,

$\Rightarrow x + x + 9 + 2x/3 = 65$

$\Rightarrow (3x + 3x + 27 + 2x)/3 = 65$

$\Rightarrow 8x + 27 = 65 \times 3$

$\Rightarrow 8x = 195 – 27$

$\Rightarrow x = 168/8$

$\Rightarrow x = 21$

A’s age = 21

B’ age = $2x/3$

$\Rightarrow 2 \times 21/3$

$\Rightarrow 14$

C’ age = $x + 9$

$\Rightarrow 21 + 9$

$\Rightarrow 30$

The ratio of the respective age of C, A and B = 30 : 21 : 14

∴ The ratio of the respective age of C, A and B are 30 : 21 : 14.

6. Let the present age of ram and his son be R years and S years respectively.

$R = 5S$

$R + 12 = 2 (S + 12)$

Also, $R + 12 = 2 (S + 12)$

$\Rightarrow R = 2S + 12$

Replacing R in terms of S we get,

$5S = 2S + 12$

$\Rightarrow S = 4$ years

∴ the present age of son = 4 years.

9. REFERENCES

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